(P) Solve $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

clearly $\frac{a}{a'} = \frac{b}{b'} = \lambda = \frac{1}{2}$. $\Rightarrow$ $a = \frac{a'}{2}$, $b = \frac{b'}{2}$.

Let $x+y=t$

$\frac{dy}{dx} = \frac{x+y+1}{2(x+y)+3}$

$1+\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - 1 = \frac{t+1}{2t+3}$

$\boxed{\frac{dy}{dx} = \frac{dt}{dx} - 1}$

$$\frac{dt}{dx} = \frac{t+1}{2t+3} + 1 = \frac{t+1+2t+3}{2t+3} = \frac{3t+4}{2t+3}$$

$$\frac{dt}{dx} = \frac{3t+4}{2t+3}$$

$$\frac{2t+3}{3t+4}\,dt = dx$$

$$\int \frac{2t+3}{3t+4}\,dt = \int dx$$

$$\int \frac{3t+4-t-1}{3t+4}\,dt = x+c$$

$$\Rightarrow \int \left[1 - \frac{(t+1)}{3t+4}\right] dt = x+c$$

$$\Rightarrow t - \int \frac{t+1}{3t+4}\,dt = x+c$$

$$\Rightarrow t - \left[\int \left(\frac{t}{3t+4} + \frac{1}{3t+4}\right) dt\right] = x+c$$

$$\Rightarrow t - \left[\int \frac{t}{3t+4}\,dt + \int \frac{1}{3t+4}\,dt\right] = x+c$$

$$\Rightarrow t - \frac{1}{3}\log(3t+4) - \int \frac{t}{3t+4}\,dt = x+c$$

$$\Rightarrow t - \frac{1}{3}\log(3t+4) - \int \frac{t+4/3-4/3}{3(t+4/3)} = x+c$$

$$\Rightarrow t - \frac{1}{3}\log(3t+4) - \frac{1}{3}\left[\int 1 - \frac{4/3}{3t+4/3}\right] = x+c$$

$$\Rightarrow t - \frac{1}{3}\log(3t+4) - \frac{1}{3}\left[t - \frac{4}{3}\log\left(t+\frac{4}{3}\right)\right] = x+c$$

$$\Rightarrow t - \frac{1}{3}\log(3t+4) - \frac{1}{3}\left[t - \frac{4}{3}\log\left(\frac{3t+4}{3}\right)\right] = x+c$$

$$\Rightarrow x+y - \frac{1}{3}\log(3x+3y+4) - \frac{1}{3}\left[x+y - \frac{4}{3}\log\left(\frac{3x+3y+4}{3}\right)\right] = x+c$$

$$\Rightarrow 3x+3y - \log(3x+3y+4) - x - y - \frac{4}{3}\log\left(\frac{3x+3y+4}{3}\right) = 3x+3c$$

$$2y - x - \log(3x+2y+4) - \frac{4}{3}\log\left(\frac{3x+2y+4}{3}\right) = 3c$$

$$2y - x - \log(3x+2y+4)\left(1+\frac{4}{3}\right) = 3c$$

$$2y - x - \log(3x+2y+4)\left(\frac{7}{3}\right) = 3c$$

$$\Rightarrow \int \frac{3}{3t+4}\,dt + \int \frac{2t}{3t+4}\,dt = x+c$$

$$\Rightarrow \ln(3t+4) + \frac{2}{3}\int \frac{3t}{3t+4}\,dt = x+c$$

$$\Rightarrow \ln(3t+4) + \frac{2}{3}\int \frac{3t+4-4}{3t+4}\,dt = x+c$$

$$\Rightarrow \ln(3t+4) + \frac{2}{3}\left[\int 1\,dt - \int \frac{4}{3t+4}\,dt\right] = x+c$$

$$\Rightarrow \ln(3t+4) + \frac{2}{3}\left[t - \frac{4}{3}\int \frac{3}{3t+4}\,dt\right] = x+c$$

$$\Rightarrow \ln(3t+4) + \frac{2}{3}\left[t - \frac{4}{3}\ln(3t+4)\right] = x+c$$

$$\Rightarrow \ln(3t+4)\left[1 - \frac{8}{9}\right] + \frac{2t}{3} = x+c$$

$$\Rightarrow \ln(3t+4)\left[\frac{1}{9}\right] + \frac{2t}{3} = x+c$$

$$\frac{2(x+y)}{3} + \frac{1}{9}\ln(3(3x+3y)+4) = x+c$$

Case: III $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

If $a=b'$ and $b=a'$ (or) $a=-b'$ and $b=-a'$

$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+1}$ Here $\boxed{\begin{matrix} a=1, & b'=1 \\ b=2, & a'=2 \end{matrix}}$ case (III) is satisfied.

Apply C and II. (componendo and Dividendo)

$$\frac{dy+dx}{dy-dx} = \frac{x+2y+3+2x+y+1}{x+2y+3-2x-y-1} = \frac{3x+3y+4}{-x+y+2} = \frac{3(x+y)+4}{y-x+2}$$

$$\frac{dy+dx}{dy-dx} = \frac{3x+3y+4}{y-x+2}$$

$$\Rightarrow \frac{d(x+y)}{3(x+y)+4} = \frac{d(y-x)}{y-x+2}$$

Integrate on both sides

$$\int \frac{d(x+y)}{3(x+y)+4} = \int \frac{d(y-x)}{y-x+2}$$

$$\Rightarrow \frac{1}{3}\log(3(x+y)+4) = \log[y-x)+2]$$

$$\Rightarrow \log\frac{(3(x+y)+4)}{3} = \log\frac{[y-x+2]}{1}+c$$

$$\Rightarrow \boxed{\log(3(x+y)+4) = 3\log(y-x+2))+3c}$$

(P) $\frac{dy}{dx} = \frac{2x-3y+1}{3x-2y+2}$

$a = 2,\ b' = -2 \Rightarrow a = -b'$

& $b = -3,\ a' = 3$ $\Leftrightarrow b = -a'$ (iii) is satisfied

$$\frac{dy+dx}{dy-dx} = \frac{2x-3y+1+3x-2y+2}{2x-3y+1-3x+2y-2} = \frac{5x-5y+3}{-x-y-1}$$

$$\frac{d(x+y)}{d(y-x)} = \frac{5(x-y)+3}{-(x+y)-1}$$

$$\Rightarrow (-(x+y)-1)\, d(x+y) = (5(x-y)+3)\, d(y-x)$$

$$-\int (x+y+1)\, d(x+y) = \int 5(x-y)+3\ d(y-x)$$

$$-\left[\int (x+y)\, d(x+y) + \int d(x+y)\right] = -\int 5(y-x)\, d(y-x) + 3\int d(y-x)$$

$$\Rightarrow -\frac{(x+y)^2}{2} - (x+y) = -\frac{5(y-x)^2}{2} + 3(y-x) + c$$

$$\Rightarrow -\frac{(x+y)^2 - 2(x+y)}{2} = \frac{-5(y-x)^2 + 6(y-x)}{2} + c$$

$$\Rightarrow -x^2 - y^2 - 2xy - 2x - 2y = -5(y^2+x^2-2xy) + 6y - 6x + c$$

$$\Rightarrow -x^2 - y^2 - 2xy - 2x - 2y = -5y^2 - 5x^2 + 10xy + 6y - 6x$$

$$\Rightarrow -4y^2 - 4x^2 + 12xy - 4x + 8y = c$$

$$\Rightarrow \boxed{x^2 + y^2 - 3xy + x - 2y = c}$$