Linear D.E. of 1^{st} Order

Type-1:

Definition: A. DE of the form $\boxed{\frac{dy}{dx} + P(x).y = Q(x)}$ → (*)

where $P(x)$ and $Q(x)$ are function of x only is called linear differential equation (L.D.E) in y of first order and first degree.

Solution of Linear D.E:

To solve eq (*) we multiply both sides of it by $e^{\int P(x)dx}$ to get

$$\text{(*)} \Rightarrow \quad \frac{dy}{dx} \cdot e^{\int p(x)dx} + P(x).y.e^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

$$\Rightarrow \quad \frac{d}{dx}\left[y.e^{\int P(x)dx}\right] = Q(x).e^{\int P(x)dx}$$

integrating w.r.t 'x' on both sides.

$$\Rightarrow \quad \int \frac{d}{dx}\left[y.e^{\int P(x)dx}\right] = \int \left[Q(x).e^{\int P(x)dx}\right] dx$$

$$\Rightarrow \quad \boxed{y\, e^{\int P(x)dx} = \int Q.e^{\int P(x)dx} dx + C} \rightarrow (**)$$

where 'C' is constant.

Note: The factor $e^{\int pdx}$ is called an integrating factor (I.F) of the given L.D.E.

from (**), the general solution of the L.D.E is

$$\boxed{y\,(I.F) = \int [Q]\,(I.F).\,dx + C} \quad \text{where } I.F = e^{\int P(x)dx}$$

Type-2:

If the D.E of the form $\boxed{\frac{dx}{dy} + P(y).x = Q(y)}$ → (*)

where $P(y)$ and $Q(y)$ are functions of y only.

The general solution of (*) is

$$x.(I.F) = \int Q(y).(I.F)\,dy + C, \quad \text{where } \boxed{I.F = e^{\int P(y)dy}}$$

7 ① Solve $\frac{dy}{dx}+\frac{4x}{x^2+1}\cdot y=\frac{1}{(x^2+1)^2}$

x̄ Ⓢ Given $\frac{dy}{dx}+\frac{4x}{x^2+1}\cdot y=\frac{1}{(x^2+1)^2} \longrightarrow ①$

= eq ① is in the form of $\frac{dy}{dx}+P(x)y=Q(x)$. where

$P(x)=\frac{4x}{x^2+1}$, $Q(x)=\frac{1}{(x^2+1)^2}$

$\because \int \frac{f'(x)}{f(x)}dx = \log f(x)$

$\therefore I.F = e^{\int P(x)dx} = e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = e^{\log(x^2+1)^2}$

$\Rightarrow \boxed{I.F=(x^2+1)^2}$

The general solution of ① is

$y\cdot(I.F)=\int Q(I.F)\,dx+c$

$\Rightarrow y\cdot(x^2+1)^2=\int\frac{1}{(x^2+1)^2}\cdot(x^2+1)^2\,dx+c$

$\Rightarrow \boxed{y(x^2+1)^2=x+c}$

② Solve $\cos 3x\,\frac{dy}{dx}-3y=\cos 3x+\sin 3x$

Ⓢ $\frac{dy}{dx}-3\tan 3x\,y=\frac{\cos 3x+\sin 3x}{\cos 3x}$

$\Rightarrow \frac{dy}{dx}-3\tan 3x\cdot y=\sin 3x+\sin^2 3x\cdot\sec 3x \longrightarrow ①$

eq ① is of the form

$\frac{dy}{dx}+P(x)y=Q(x)$, where

$P(x)=-3\tan 3x$, $Q(x)=\sin 3x+\sin^2 3x\cdot\sec 3x$

Now $I.F=e^{\int P(x)dx}=e^{\int -3\tan 3x\,dx}$

$=e^{-3\int\tan 3x\,dx}=e^{-3\frac{\log(\sec 3x)}{3}}=e^{-\log(\sec 3x)}$

$$\Rightarrow \boxed{I \cdot F = \cos 3x}$$

The general solution of ① is

$$y \cdot (I \cdot F) = \int Q (I \cdot F)\, dx + c$$

$$\Rightarrow y \cdot \cos 3x = \int \left(\sin 3x + \sin^2 3x \cdot \overset{\sec 3x}{\left(\frac{1}{\cos 3x}\right)}\right)(\cos 3x)\, dx + c$$

$$\Rightarrow y \cos 3x = \int \left[(\sin 3x \cdot \cos 3x) + \underline{\sin^2 3x} \cdot \cancel{\cos^2 3x}\right] dx + c$$

$$\Rightarrow y \cos 3x = \frac{1}{2} \int 2 \sin 3x \cdot \cos 3x\, dx + \cancel{\frac{1}{4} \int (2 \sin 3x \cdot \cos 3x)^2\, dx + c}$$

$$= \frac{1}{2} \int \sin 6x\, dx + \cancel{\frac{1}{4} \int (\sin 6x)^2\, dx + c} \qquad \left[\because \sin 2A = 2 \sin A \cos A\right]$$

$$= \frac{1}{2}\left[-\frac{\cos 6x}{6}\right] + \frac{1}{4} \int \cancel{\sin^2 6x\, dx + c}$$

$$= \frac{-\cos 6x}{12} + \frac{1}{\cancel{4}} \int \left(\frac{1 - \cos 6x}{2}\right) dx + c \qquad \because \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$= \frac{-\cos 6x}{12} + \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + c$$

$$= -\frac{\cos 6x}{12} + \frac{x}{2} - \frac{\sin 6x}{12} + c$$

$$= \frac{1}{2}\left[x - \frac{\cos 6x}{6} - \frac{\sin 6x}{6}\right] + c$$

$$\Rightarrow \boxed{y \cos 3x = \frac{1}{2}\left(x - \frac{\cos 6x}{6} - \frac{\sin 6x}{6}\right) + c}$$

③ solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

Ⓢ $$\frac{dy}{dx} - \frac{(x-2)}{x(x-1)} y = \frac{x^3 (2x-1)}{x(x-1)}$$

$$\Longrightarrow \boxed{\frac{dy}{dx} - \frac{(x-2)}{x(x-1)} y = \frac{x^2(2x-1)}{x-1}} \longrightarrow ①$$

① is of the form $\frac{dy}{dx} + P(x) y = Q(x)$, where $P(x) = \frac{-(x-2)}{x(x-1)}$, $Q(x) = \frac{x^2(2x-1)}{x-1}$

Now $I.F = e^{\int P(x)\,dx}$

$$= e^{\int \frac{-(x-2)}{x(x-1)}dx}$$

$$= e^{\int \left(\frac{-2}{x} + \frac{1}{x-1}\right)dx}$$

$$= e^{\int -\frac{2}{x}dx + \int \frac{1}{x-1}dx}$$

$$= e^{-2\log x + \log(x-1)} = e^{\log x^{-2} + \log(x-1)}$$

$$= e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

$\because \frac{-(x-2)}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$-(x-2) = A(x-1) + Bx$

comparing: x coeff $\Rightarrow -1 = A+B$

const $\Rightarrow 2 = -A$

$\Rightarrow \boxed{A=-2}$

$\therefore \boxed{B=1}$

$\therefore \frac{-(x-2)}{x(x-1)} = -\frac{2}{x} + \frac{1}{x-1}$

$= -2x+2+x$

$= -x+2$

$= -(x-2)$

$\Rightarrow \boxed{I.F = \frac{x-1}{x^2}}$

$\therefore$ The general solution of ① is

$$y.(I.F) = \int Q.(I.F)\,dx + c$$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right) = \int \frac{\cancel{x^2}(2x-1)}{\cancel{x-1}}\cdot\left(\frac{\cancel{x-1}}{\cancel{x^2}}\right)dx + c$$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right) = \int (2x-1)\,dx + c$$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right) = x^2 - x + c$$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right) = x(x-1) + c$$

$$\Rightarrow y = \frac{x^3(x-1)}{x-1} + c\left(\frac{x^2}{x-1}\right)$$

$$\Rightarrow \boxed{y = x^3 + c\left(\frac{x^2}{x-1}\right)}$$

1) Solve the DE $\cos^2 x \frac{dy}{dx} + y\cos x = \sin x$

Given D.E $\cos^2 x \frac{dy}{dx} + y\cos x = \sin x$ → ①

$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \tan x \cdot \sec^2 x$

① in of the form $\frac{dy}{dx} + P(x)y = Q(x)$

where $\boxed{P(x) = \sec^2 x \quad Q(x) = \tan x \cdot \sec^2 x}$

Now $I.F = e^{\int P(x)dx} = e^{\int \sec^2 x \cdot dx} = e^{\tan x}$.

$\boxed{I.F = e^{\tan x}}$

∴ The general solution of ① is

$y \cdot (I.F) = \int Q(I.F)dx + c$

$\Rightarrow y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + C$

put $\tan x = t$

$\sec^2 x\, dx = dt$

$= \int t \cdot e^t \cdot dt + c$

$= te^t - \int e^t dt + c$

$y e^{\tan x} = te^t - e^t + c$

$\Rightarrow y e^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + c$

$\Rightarrow y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

$\Rightarrow \boxed{y = \tan x - 1 + c e^{-\tan x}}$

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(5) Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2 x}{x^3+1}$

(S) Given $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2 x}{x^3+1}$ $\longrightarrow (1)$

(1) is of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where $P(x)=\frac{3x^2}{1+x^3}$ $Q(x)=\frac{\sin^2 x}{x^3+1}$

Now $I.F = e^{\int P(x)dx} = e^{\int \frac{3x^2}{1+x^3}dx} = e^{\log(1+x^3)}$ $\quad \because \int \frac{f'(x)}{f(x)}dx = \log f(x)$

$\Rightarrow$ $\boxed{I.F = (1+x^3)}$

$\therefore$ The general solution of (1) is

$y.(I.F) = \int Q.(I.F)\,dx + c$

$\Rightarrow$ $y(1+x^3) = \int \frac{\sin^2 x}{\cancel{x^3+1}}.\cancel{(x^3+1)}\,dx+c$

$\Rightarrow$ $y(1+x^3) = \int \left[\frac{1-\cos 2x}{2}\right]dx+c$

$y(1+x^3) = \frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]+c$

$\Rightarrow$ $\boxed{y(1+x^3) = \frac{x}{2}-\frac{\sin 2x}{4}+c}$ //

(6) Solve the D.E $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$ and $y(0)=0$

(S) Given $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$ & $y(0)=0$

$\Rightarrow$ $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$ $\longrightarrow (1)$

(1) is of the form $\frac{dy}{dx}+P(x).y=Q(x)$

where $P(x)=\frac{2x}{1+x^2}$, $Q(x)=\frac{4x^2}{1+x^2}$

$I.F = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$

$\boxed{I.F = 1+x^2}$

$\therefore$ The general solution of ① is

$$y\cdot(I\cdot F) = \int Q\cdot(I\cdot F)\,dx + c$$

$$y(1+x^2) = \int\left(\frac{4x^2}{1+x^2}\right)(1+x^2)\,dx + c$$

$$y(1+x^2) = \int 4x^2\,dx + c$$

$$\boxed{y(1+x^2) = \frac{4}{3}x^3 + c} \longrightarrow ②$$

Given $\boxed{y(0)=0}$ substitute it in ②, we get

② $\Rightarrow$ $0(1+0) = 0 + c \Rightarrow \boxed{c=0}$

substitute c value in ②.

② $\Rightarrow$ $y(1+x^2) = \frac{4}{3}x^3$

$$\Rightarrow \boxed{\begin{array}{l} y = \frac{4}{3}\left(\frac{x^3}{1+x^2}\right) \\ \text{(or)} \\ 3y(1+x^2) = 4(x^3) \end{array}}$$

Type-2:

① Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

Ⓢ $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}} \longrightarrow ①$$

① is of the form $\frac{dx}{dy} + P(y)x = Q(y)$

where $P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{\tan^{-1}y}{1+y^2}$

$$I\cdot F = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}.$$

$\therefore$ The G.S of ① is

$$x(I\cdot F) = \int Q(y)\cdot(I\cdot F)\cdot dy + c$$

$$x\cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\cdot e^{\tan^{-1}y} + c$$

$$x e^{\tan^{-1}y} = \int t\cdot e^t\,dt + c \qquad \boxed{\begin{array}{l}\text{put } t = \tan^{-1}y \\ dt = \frac{1}{1+y^2}dy\end{array}}$$

$$x e^{\tan^{-1}y} = e^t(t-1) + c$$

$$\Rightarrow x e^{\tan^{-1}y} = e^{\tan^{-1}y}(t-1) + c$$

$$\Rightarrow \boxed{x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}}$$

② Solve $(x+2y^3)\frac{dy}{dx}=y$

Ⓢ Given $(x+2y^3)\frac{dy}{dx}=y$

$\Rightarrow \frac{x+2y^3}{y}=\frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+2y^2$

$\Rightarrow \boxed{\frac{dx}{dy}-\frac{1}{y}x=2y^2} \rightarrow ①$

① is in the form of $\frac{dx}{dy}+P(y)x=Q(y)$

where $P(y)=-\frac{1}{y}$, $Q(y)=2y^2$

Now, $I.F=e^{\int P(y)dy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}$

$\Rightarrow \boxed{I.F=\frac{1}{y}}$

$\therefore$ G.S of ① is

$x\cdot(I.F)=\int Q(y)(I.F)\,dy+c$

$\Rightarrow x\cdot\left(\frac{1}{y}\right)=\int(2y^2)\left(\frac{1}{y}\right)dy+c$ ①

$x\left(\frac{1}{y}\right)=2\int y\,dy+c$ ③

$=2\frac{y^2}{2}+c$

$\Rightarrow \frac{x}{y}=y^2+c$

$\Rightarrow \boxed{x=y^3+cy}$

③ Solve ~~$(x+2y^3)dy$~~ $(x+y+1)\frac{dy}{dx}=1$

Ⓢ Given $\frac{dx}{dy}=x+y+1$

$\Rightarrow \boxed{\frac{dx}{dy}-x=y+1} \rightarrow ①$

① is of the of $\frac{dx}{dy}+P(y)x=Q(y)$

where $P(y)=-1$, $Q(y)=y+1$

$\therefore I.F=e^{\int -1\,dy}=e^{-y} \Rightarrow \boxed{I.F=e^{-y}}$

$\therefore$ G.S of ① is

$x(I.F)=\int Q(y)(I.F)\,dy+c$

$\Rightarrow xe^{-y}=\int(y+1)(e^{-y})dy+c$

$xe^{-y}=\int ye^{-y}dy+\int e^{-y}dy+c$

$xe^{-y}=-ye^{-y}-e^{-y}+\frac{e^{-y}}{(-1)}+c$

$xe^{-y}=-ye^{-y}-2e^{-y}+c \Rightarrow x=-y-2+ce^{y}$

$\Rightarrow \boxed{x+y+2=ce^{y}}$

④ Solve $\frac{dx}{dy}+\frac{x}{y}x=\frac{1}{y^2}$

Ans: $2xy^3=y^2+c$

Additionals

① $y' = \cos^3 x + y\cot x$

③ Given $\frac{dy}{dx} = \cos^3 x + y\cot x$

$$\Rightarrow \boxed{\frac{dy}{dx} - \cot x\, y = \cos^3 x} \rightarrow ①$$

① is in the form of $\frac{dy}{dx} + P(x)y = Q(x)$

where $P(x) = -\cot x \quad Q(x) = \cos^3 x$

$$I.F = e^{\int P(x)dx} = e^{\int -\cot x\, dx} = e^{-\log \sin x} = \operatorname{cosec} x$$

$$\boxed{I.F = \operatorname{cosec} x}$$

The General solution of ① is

$$y\,(I.F) = \int Q\cdot(I.F)\,dx + c$$

$$\Rightarrow y\operatorname{cosec} x = \int \cos^3 x \cdot \operatorname{cosec} x\, dx + c$$

$$\Rightarrow y\operatorname{cosec} x = \int \frac{\cos^3 x}{\sin x}\, dx + c$$

$$= \int \frac{\cos x\cdot(1-\sin^2 x)}{\sin x}\, dx + c$$

$$= \int \left[\frac{\cos x}{\sin x} - \cos x\cdot\sin x\right] dx + c$$

$$= \int \cot x\, dx - \int \cos x\cdot\sin x\, dx + c$$

$$= \log \sin x - \frac{\sin^2 x}{2} + c \qquad \left[\because \int [f(x)]^n\cdot f'(x)\,dx = \frac{f^{n+1}}{n+1} + c\right]$$

$$\Rightarrow y\csc x = \log \sin x - \frac{1}{2}\sin^2 x$$

$$\boxed{y = \sin x \log \sin x - \frac{1}{2}\sin^3 x + C\sin x}$$

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$(1+x^2)y' + 2xy = x\sin x$

(S) Given $(1+x^2)\dfrac{dy}{dx} + 2xy = x\sin x$

$$\Rightarrow \boxed{\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x\sin x}{1+x^2}} \rightarrow (1)$$

(1) is in the form $\dfrac{dy}{dx} + P(x)y = Q(x)$,

where

$$P(x) = \frac{2x}{1+x^2} \quad Q(x) = \frac{x\sin x}{1+x^2}$$

$$I.F = e^{\int P(x)dx}$$

$$= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

$$\Rightarrow \boxed{I.F = 1+x^2}$$

∴ The general solution of (1) is

$$y(I.F) = \int Q\cdot(I.F)\,dx + c$$

$$y(1+x^2) = \int \frac{x\sin x}{1+x^2}\cdot(1+x^2)\,dx + c$$

$$\int uv = uv_1 - \int u'v_1\,dx$$

$$y(1+x^2) = \int x\sin x\,dx + c$$

$$y(1+x^2) = x(-\cos x) - \int(-\cos x)\,dx + c$$

$$= -x\cos x + \int \cos x\,dx + c$$

$$y(1+x^2) = -x\cos x + \sin x + c$$

$$\boxed{y = \frac{\sin x - x\cos x + c}{x^2+1}}$$

(3) $y' + 2xy = xe^{-x^2}$

(S) Given $\dfrac{dy}{dx} + 2xy = xe^{-x^2} \longrightarrow (1)$

(1) is of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$

where $P(x) = 2x \quad Q(x) = xe^{-x^2}$

$$IF = e^{\int P(x)dx} = e^{\int 2x\,dx} = e^{x^2} \Rightarrow \boxed{IF = e^{x^2}}$$

G.S of ① is

$y \cdot (I \cdot F) = \int Q \cdot (I \cdot F)\, dx + c$

$y(e^{x^2}) = \int x e^{-x^2} \cdot e^{x^2} dx + c$

$\Rightarrow \quad y(e^{x^2}) = \int x \cdot e^0 dx + c$

$\Rightarrow \quad y e^{x^2} = \frac{x^2}{2} + c$

$\Rightarrow \quad \boxed{y = \left[\frac{x^2}{2} + c\right] e^{-x^2}}$

(4) $x^2 y' + xy = 2x^2 e^{x^2}$

⑤ Given $x^2 \cdot \frac{dy}{dx} + xy = 2x^2 e^{x^2}$

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x} y = 2e^{x^2} \longrightarrow ①$

① is in the form of $\frac{dy}{dx} + Py = Q$

where $P(x) = \frac{1}{x}$ $Q(x) = 2e^{x^2}$

$\boxed{I \cdot F = x}$ and G.S: $\boxed{y = \frac{e^{x^2} + c}{x}}$

(5) $(x^2 - 2y) dx = x dy$

Ans: $\boxed{I \cdot F = x^2}$ & G.S: $\boxed{y = \frac{1}{x^2}\left[\frac{x^4}{4} + c\right]}$

(6) $xy' + (1+2x)y = 1 + x e^{-2x}$

Ans: $\boxed{I \cdot F = x \cdot e^{2x}}$ G.S: $\boxed{2xy e^{2x} = e^{2x} + x^2 + c}$

Clairaut's Equation:

An equation of the form –

$\boxed{y = px + f(p)}$ where $p = \frac{dy}{dx}$ or y' is known as clairaut's equation.

General Solution of clairaut's Equation:

To show that the general solution of clairaut's equation

$y = p(x) + f(p)$ is

$\boxed{y = Cx + f(C)}$, which is obtained by replacing p by C
'C' is an arbitrary const.

proof: Given clai/. equation is

$$\boxed{y = px + f(p)} \longrightarrow (1)$$

Diff/. (1) w.r.t 'x' and writing p for $\frac{dy}{dx}$

$$\frac{dy}{dx} = p(1) + x\cdot\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\Rightarrow p = p + x\cdot\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\Rightarrow x\cdot\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dp}{dx}\left(x + f'(p)\right) = 0}$$

omitting the factor $x + f'(p)$ which does involve $\frac{dp}{dx}$

$$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow \int dp = \int 0 \Rightarrow \boxed{p = C}$$

substituting the value of p in (1), we get

$$\boxed{y = Cx + f(C)}$$

Working Rule for Solving Clairaut's Equation

Replace p by c in $y = px + f(p)$ to obtain solution of ① where 'c' is const.

Singular Solution: If we eliminate p between $x + f'(p) = 0$, the given clairaut's equation $y = px + f(p)$

① Obtain the general solution and singular solution of the non-linear DE

$y = xy' + y'^2$

Ⓢ Given DE is $y = xy' + y'^2 \rightarrow ①$

$\Rightarrow y = px + p^2 \rightarrow ②$ $\because p = \frac{dy}{dx}$

$\therefore$ G.S of ② is $\boxed{y = cx + c^2}$ [$\because$ ② is clairaut's Eq]

For singular solution $x + f'(p) = 0$

$x + 2p = 0$

$x = -2p \Rightarrow \boxed{p = -\frac{x}{2}}$

from ② $y = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$

$\Rightarrow y = -\frac{x^2}{2} + \frac{x^2}{4} \Rightarrow 4y = x^2 - 2x^2$

$\Rightarrow 4y = -x^2 \Rightarrow \boxed{4y + x^2 = 0}$ is the required singular solution.

I. Find the general solution of the D.E

① $y = px + (1+p^2)^{1/2} \rightarrow ①$

Ⓢ ① is C.F (Clairaut's form). so replace p by c

$\boxed{y = cx + (1+c^2)^{1/2}}$

② $y = x\left(\frac{dy}{dx}\right) + e^{\frac{dy}{dx}} \rightarrow ①$

Ⓢ $y = xp + e^p \Rightarrow \boxed{y = cx + e^c}$

$\Rightarrow$ since in Clairaut's form replace p by c in ①

③ $\frac{dy}{dx} = \log(px - y)$

$\Rightarrow y' = \log(px - y)$

$\Rightarrow e^{y'} = px - y$

$\Rightarrow \boxed{y = px - e^p}$

$\Rightarrow \boxed{y = cx - e^c}$

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④ $p=\tan(px-y)$

Ⓢ $p=\tan(px-y)$

$\tan^{-1}p = px-y$

$\Rightarrow \boxed{y = px - \tan^{-1}p} \rightarrow ①$

replace p by 'c' in ① [∵ ① is C.F]

G.S of ① is

$\boxed{y = cx - \tan^{-1}c}$

⑤ $(y-px)^2 = a^2(1+p^2)$

$\Rightarrow y - px = a\sqrt{1+p^2}$

$\boxed{y = px + a\sqrt{1+p^2}} \rightarrow ①$

① is Clairaut's equation. so replace p by c to get G.S of ①

∴ G.S of ① is $\boxed{y = cx + a\sqrt{1+c^2}}$

⑥ Find the G.S and singular solution of the D.E $y = px - pa + ap^2 \rightarrow ①$ where $p = \frac{dy}{dx}$.

Ⓢ The G.S of the eq ① is

$\boxed{y = cx - ca + ac^2} \rightarrow ②$

since ① is Clairaut's equation

Now, Diff. ② w.r.t 'c' on both sides

$0 = x - a + 2ac$

$\boxed{c = \frac{a-x}{2a}}$

from ②;

$y = x\left(\frac{a-x}{2a}\right) - \left(\frac{a-x}{2a}\right)a + a\left(\frac{a-x}{2a}\right)^2$

$= \frac{a-x}{2a}\left[x - a + a\left(\frac{a-x}{2a}\right)\right]$

$= \frac{a-x}{2a}\left[x - a + \frac{a-x}{2}\right]$

$\boxed{y = \frac{a-x}{2a}\left[\frac{x-a}{2}\right] = -\frac{(x-a)^2}{4a}}$