1.1 System of Linear Equations

Definition: A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and the first power of a single variable.

The system of linear equations are non-homogeneous and Homogeneous

Non-homogeneous

First we write given equation in matrix as

$AX = B$

→ Matrix AB reduced to echelon form also called Gauss - Jordan Elimination. $\rho(A)$ and $\rho(AB)$

i, If $\rho(A) \neq \rho(AB)$

Then the system of linear equations has no solutions (inconsistant)

ii, If $\rho(A) = \rho(AB)$

then the system of linear equations has solutions (consistant)

iii, If $\rho(A) = \rho(AB) = n$ (no. of variables)

then the system of linear equation has unique solution.

iv, If $\rho(A) = \rho(AB) < n$

Then the system of equation have infinite solution

The above meathod is called guass elimination method.

1) Solve the system of linear equation

$x+y+z=4,\ 2x+5y-2z=3,\ x+7y-7z=5$

Sol: Given equations are written as

$AX = B$

$$\cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 7 \\ 1 & -2 & -7 \end{bmatrix}} \cancel{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix} \quad R_2 \to R_2 - 2R_1,\ R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

$\rho(A) = 2$ & $\rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

No solution, Inconsistant

2) Solve the equation $x+y+z=9;\ 2x+5y+7z=52$

$2x+y-z=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

Argumented matrix

$$AB = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 0 & -4 & -8 & -52 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix} \quad R_3 \to 3R_3 + 4R_2$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$$

$\rho(AB) = 3$, $\rho(A)$; $n = 3$

$\rho(AB) = \rho(A) = n$

∴ The given equations have unique solutions

Consistant.

$x + y + z = 9$

$3y + 5z = 34$

$-4z = -20$

$\boxed{z = 5}$

$3y + 25 = 34$

$3y = 9$

$\boxed{y = 3}$

$x + 3 + 5 = 9$

$x = 9 - 8$

$\boxed{x = 1}$

The solutions of the given ~~x+y~~ equations are $x = 1, y = 3, z = 5$

3) Show that the equation $x + y + z = 6$, $x + 2y + 3z = 14$, and $x + 4y + 7z = 20$

The given equations are written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Augmented matrix

$$AB = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(AB) = 2$; $\rho(A) = 2$; $n = 3$

$\rho(A) = \rho(AB) < n$

$\therefore$ The given equations are consistent and have infinite no. of solutions.

Let, $z = k$; $x + y + z = 6$

$y + 2z = 8$; $x = 6 - 8 + 2k - k$

$\boxed{x = -2 + k}$

$y = 8 - 2z = 8 - 2k$.

$\therefore$ The solutions are $x = -2 + k$; $y = 8 - 2k$; $z = k$

4) Discuss for what values of λ, μ let $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$, are having

a) no solution

b) unique solution

c) Infinite solution

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \quad R_3 \to R_3 - R_1 ; R_2 \to R_2 - R_1$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

Here $n = 3$

1, For λ values $\lambda = 3$ & $\mu \neq 10$ then $\rho(A) \neq \rho(AB)$ then the equations have no solutions.

2, If $\lambda \neq 3$, μ = Any value then

$\rho(A) = \rho(AB) = n = 3$

then the equations have unique solutions

3, If $a = 3$ and $\mu = 10$ then

$\rho(A) = \rho(AB) < n$

$2 = 2 < 3$

then the equations have infinite solution

Homogeneous linear equation

Homogeneous linear equations matrix form is $AX = 0$

Now matrix A is reduced to echelon form then we get $\rho(A) = r$ and we know that 'n' is no. of variables or unknows.

Number of linearly independent solutions is equal to "$n - r$"

→ If $n - r = 0$ $(n = r)$ then $AX = 0$ have zero solutions or trival solutions

→ If $AX = 0$ have non zero solutions then $|A| = 0$

5, Solve the system of equations $x + y - 3z + 2w = 0$; $2x - y + 2z \overset{-3w}{=} 0$; $3x - 2y + z - 4w = 0$ and $-4x + y - 3z + w = 0$

Sol: Let equations are written as matrix form

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 + 4R_1$;

$R_3 \to R_3 - 3R_1$;

$R_2 \to R_2 - 2R_1$;

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix} \quad R_4 \to R_4 + R_3$$

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -4 & 8 & -7 \\ 0 & -5 & 10 & 2 \\ 0 & 0 & -5 & 11 \end{bmatrix} \quad R_3 \to 3R_3 - 5R_2$$

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad R_4 \to 2R_3 - R_3$$

$\therefore n = r$

$x = 0,\ y = 0,\ z = 0,\ w = 0$

6) Solve completely system of equations

$x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$

Sol: Let equations are written in matrix form

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 \quad \begin{aligned} &R_4 \to R_4 - 5R_1 \\ &R_3 \to R_3 - 4R_1 \\ &R_2 \to R_2 - R_1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 \quad \begin{aligned} &R_4 \to R_4 - 4R_3 \\ &R_3 \to R_3 - R_2 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$\rho(A) = 2 = r$; $n = 4$

$n - r = 4 - 2 = 2$

$\therefore$ linear independent solutions are 2

$x+y-2z+3w=0$

$-3y+3z-4w=0$

Let, $w=k_1$, $z=k_2$

$-3y=-3z+4w$

$=-3k_2+4k_1$

$y=k_2-\frac{4k_1}{3}$

$x=-y+2z-3w$

$=-k_2+\frac{4k_1}{3}+2k_2-3k_1$

$=\frac{-3k_2+4k_1+6k_2-9k_1}{3}$

$=\frac{3k_2-5k_1}{3}$

$=k_2-\frac{5}{3}k_1$

$x=\frac{-5}{3}k_1+k_2$; $y=\frac{-4k_1}{3}+k_2$; $z=k_2$, $w=k_1$

$$X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-\frac{5}{3}k_1+1k_2\\-\frac{4}{3}k_1+1k_2\\0k_1+1k_2\\1k_1+0k_2\end{bmatrix}=k_1\underbrace{\begin{bmatrix}-5/3\\-4/3\\0\\1\end{bmatrix}}_{X_1}+k_2\underbrace{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}_{X_2}$$

$X_1=\begin{bmatrix}-5/3\\-4/3\\0\\1\end{bmatrix}$; $X_2=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

* Solve the system of equations $x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$

Sol:

$$x+3y+2z=0$$
$$2x-y+3z=0$$
$$3x-5y+4z=0$$
$$x+17y+4z=0$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - R_1$

5

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $r=2$ and $n=3$

No. of solutions $= n-r = 3-2 = 1$

$z = k$

$$x+3y+2z=0$$
$$-7y-z=0$$
$$-7y-k=0$$
$$-7y=k$$
$$\boxed{y = -k/7}$$

$$x+3y+2z=0$$
$$x+3(-k/7)+2k=0$$
$$x-\frac{3k}{7}+2k=0$$
$$x = \frac{3k-14k}{7} = \frac{-11k}{7}$$

$$x = \begin{bmatrix} -11/7\,k \\ -k/7 \\ k \end{bmatrix} = k\begin{bmatrix} -11/7 \\ -1/7 \\ 1 \end{bmatrix}$$

Eigen values (roots) and Eigen vectors

Let A is a square matrix and I is a unit matrix the characterstic equation of A is $|A-\lambda I|=0$

Here we get λ values $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots$ are called Eigen values (or) Eigen roots (or) characteristic roots of matrix A

Let 'x' is a Eigen vector corresponding to Eigen value $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$ then $(A-\lambda I)x=0$, this is homogeneous linear equations.

we know that how to solve $A, X=0$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

* Find Eigen values and Eigen vectors of a matrix

5 $$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} ; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

characterstic equation is

$$A - \lambda I = 0$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)\left[(1-\lambda)(0-\lambda)\right] - 2\left[(2)(-\lambda)-(-6)(-1)\right] - 3\left[(2)(-2)-(-1)(1-\lambda)\right]$$

$$\Rightarrow (-2-\lambda)\left[-\lambda+\lambda^2-12\right] - 2\left[-2\lambda-6\right] - 3\left[-4+1-\lambda\right] = 0$$

$$\Rightarrow 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 9 + 3\lambda = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$\lambda^3+\lambda^2-21\lambda-45=0$

$\lambda=5;\ (5)^3+(-5)^2-21(-5)-45=0$

$125+25-105-45$

$150 \quad -150$

$=0$

$$\lambda=5\ \begin{array}{|cccc}1 & 1 & -21 & -45\\ 0 & 5 & 30 & 45\\ \hline 1 & 6 & 9 & 0\end{array}$$

$(\lambda-5)(\lambda^2+6\lambda+9)=0$

$\lambda-5=0;\quad \lambda^2+6\lambda+9=0$

$\lambda=5;\quad \lambda^2+3\lambda+3\lambda+9=0$

$\lambda(\lambda+3)+3(\lambda+3)=0$

$(\lambda+3)(\lambda+3)=0$

$\lambda=5,-3,-3$

$(A-\lambda I)x=0$

$$\begin{bmatrix}-2-(-3) & 2 & -3\\ 2 & 1-(-3) & -6\\ 1 & -2 & -(-3)\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\qquad R_2\to R_2-2R_1,\ R_3\to R_3+R_1$$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$r=1;\ n=3$

no. of LI vectors $=n-r=3-1=2$

$x+2y-3z=0$

Let, $z=k_1;\ y=k_2$

$x=-2y+3z$

$=-2(k_2)+3(k_1)$

$=3k_1-2k_2$

$y = k_2 ; z = k_1$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3k_1 - 2k_2 \\ 0k_1 + 1k_2 \\ 1k_1 + 0k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Let, $x_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} ; x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Case 2:

$\lambda = 5$

$(A - \lambda I) X = 0$

$$\begin{bmatrix} (-2-5) & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 7R_2 + 2R_1 ;$
$R_3 \to 7 \cdot R_3 - R_1 ;$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16 & -32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{24} ; R_3 \to \frac{R_3}{-16}$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$r = 2 ; n = 3$

no. of LIV $= n - r = 3 - 2 = 1$

$-7x + 2y - 3z = 0$

Let, $z = k$

$y = -2z = -2k$

$-7x = -2y + 3z$

$= -2(-2k) + 3k$

$= 4k + 3k$

$-7x = 7k$

$x = k$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix};\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix};\ X_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Eigen value properties:

- Sum of the Eigen values of A = trace of (A)
- product of Eigen values of A = det (A)
- If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of matrix A, then $\lambda_1^m, \lambda_2^m, \lambda_3^m$ are eigen values of matrix A^m.
- If λ is Eigen value of A, then $\frac{1}{\lambda}, \lambda^{-1}$ is eigen value of A^{-1}.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$\lambda = -3, -3, 5$

Let, $\lambda_1 = -3,\ \lambda_2 = -3,\ \lambda_3 = 5$

Take $\lambda_1 \lambda_2 \lambda_3 = (-3)(-3)(5) = 45$

$|A| = -2(0-12) - 2(0-6) - 3(-4+1)$

$= 24 + 12 + 9 = 45$

$\lambda_1 \lambda_2 \lambda_3 = |A|$

Take $\lambda_1 + \lambda_2 + \lambda_3 = -3 - 3 + 5 = -1$

Trace of $A = -2 + 1 + 0 = -1$

$\lambda_1 + \lambda_2 + \lambda_3 =$ Trace of A.

* Find the sum and product of the eigen values of the given matrix without actually finding the Eigen values. $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Sum of the Eigen values = trace of A.

$= 3 - 2 + 3 = 4$

Product of Eigen values $= \det(A)$

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

$= 3(-6+4) - (-4)[3-4] + 4(-1+2)$

$= 3(-2) + 4(-1) + 4(1)$

$= -6 - 4 + 4$

$= -6$

find the eigen values of A^5 where,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}$$

$\lambda = 3, 4, 1$

Let, $\lambda_1 = 3$; $\lambda_2 = 4$; $\lambda_3 = 1$

Eigen values of $A^5 = \lambda_1^m, \lambda_2^m, \lambda_3^m$

$= 3^5, 4^5, 1^5$

Cayly Hamilton theorem

Every square matrix 'A' satisfies its own characteristic equation.

For matrix A, verify Cayly Hemilton theorem and find its inverse where.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$\Rightarrow (2-\lambda)((2-\lambda)($

$\Rightarrow 2-\lambda(4+\lambda^2$

$\Rightarrow 2-\lambda(\lambda^2-4$

$\Rightarrow 2\lambda^2-\lambda^3-$

$\Rightarrow -\lambda^3+6\lambda^2$

By the ca

satisfies

$-A^3+6A^2$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & - \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & - \\ -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+1+1 \\ -2-2-1 \\ 2+1+2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 \\ -5 & 6 \\ 5 & -5 \end{bmatrix}$$

$$A^2 \cdot A = \begin{bmatrix} 6 & -5 \\ -5 & 6 \\ 5 & -5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 12+5+ \\ -10-6- \\ 10+5+ \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$\Rightarrow (2-\lambda)((2-\lambda)(2-\lambda)-(1)) - ((-1)(-(2-\lambda)+1) + 1(1-(2-\lambda))$

$\Rightarrow 2-\lambda(4+\lambda^2-2\lambda-2\lambda-1)+1(-2+\lambda+1)+(1-2+\lambda)$

$\Rightarrow 2-\lambda(\lambda^2-4\lambda+3)-2+\lambda+1-1+\lambda$

$\Rightarrow 2\lambda^2-\lambda^3-8\lambda+4\lambda^2+6-3\lambda-2+\lambda$

$\Rightarrow -\lambda^3+6\lambda^2-9\lambda-4$

By the caley Hemilton theorem matrix satisfies characterstic matrix.

$-A^3+6A^2-9A-4I$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & +1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-12 & 5+5+12 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$-A^3+6A^2-9A-4I=0$$

$$-\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} - \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & 21 & -21 \\ 21 & -22 & 21 \\ -21 & 21 & -22 \end{bmatrix} + \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} -18 & 9 & -9 \\ 9 & -18 & 9 \\ -9 & 9 & -18 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -9 & 9 \\ -9 & 14 & -9 \\ 9 & -9 & 14 \end{bmatrix} + \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

$$A^3-6A^2+9A-4I=0$$

A^{-1} multiplying on both sides,

$$A^{-1}A^3 - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$A^2 - 6A + 9I = 4A^{-1}$$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 4A^{-1}$$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 1 & -1 \\ 1 & -6 & 1 \\ -1 & 1 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

2, verify ca

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Sol : $|A - \lambda I| = 0$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$\rightarrow (1-\lambda)[(3-\lambda)($

$\Rightarrow (1-\lambda)[-12+$

$\Rightarrow (1-\lambda)(\lambda^2+$

$\Rightarrow \lambda^2+\lambda-24$

$\Rightarrow -\lambda^3+20\lambda-$

$\lambda^3-20\lambda+$

By caley Ha

A^3-20A+

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ -2 & - \end{bmatrix}$$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 1 \\ -6 & 12 & -6 \\ 8 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 4A^{-1}$$

$$\begin{bmatrix} -6 & 1 & -1 \\ 1 & -6 & 1 \\ -1 & 1 & -6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 4A^{-1}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

2) verify cayley Hamiltion theorem of matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ and find its matrix.

so) : $|A - \lambda I| = 0$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{bmatrix}$$

$\Rightarrow (1-\lambda)[(3-\lambda)(-4-\lambda)-12] - 1[(-4-\lambda)-6] + 3[-4-(3-\lambda)(-2)]$

$\Rightarrow (1-\lambda)(-12+4\lambda-3\lambda+\lambda^2-12) - 1[-4-\lambda-6] + 3[-4-(-6+2\lambda)]$

$\Rightarrow (1-\lambda)(\lambda^2+\lambda-24) - 1[-\lambda-10] + 3[-2\lambda+2]$

$\Rightarrow \lambda^2+\lambda-24-\lambda^3-\lambda^2+24\lambda+\lambda+10-6\lambda+6$

$\Rightarrow -\lambda^3+20\lambda-8=0$

$\lambda^3-20\lambda+8=0$

By caley Hamition theorem matrix.

$A^3 - 20A + 8I = 0$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4-8+24 & -4-24+48 & -12+24+48 \\ 10+22-12 & 10+66-24 & 30-66-24 \\ 2+2-44 & 2+6-88 & 6-6-88 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} - \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 20A + 8I$

multiplying

$A^3 A^{-1} - 20A$

$A^2 - 20I$

$8A^{-1} = -A^2$

$8A^{-1} = -\begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 24 \\ -10 \\ -2 \end{bmatrix}$

$A^{-1} = \frac{1}{8} \begin{bmatrix} 24 \\ -10 \\ -2 \end{bmatrix}$

using calyhe

$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol: $|A - \lambda I|$

$\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - |$

$[(1-\lambda)(-1-\lambda$

$-1-\cancel{\lambda}+\cancel{\lambda}+$

$\lambda^2 - 5 =$

By Cayley

$A^2 - 5I =$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 20A + 8I = 0$

multiplying A^{-1} on both sides

$A^3A^{-1} - 20AA^{-1} + 8IA^{-1} = 0$

$A^2 - 20I + 8A^{-1} = 0$

$8A^{-1} = -A^2 + 20I$

$$8A^{-1} = -\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

\# using calyhamiltion theorem, find A^8 if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Sol: $|A - \lambda I|$

$$\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$$

$[(1-\lambda)(-1-\lambda)] - [2(2)]$

$-1 - \cancel{\lambda} + \cancel{\lambda} + \lambda^2 - 4$

$\lambda^2 - 5 = 0$

By Cayley Hemiton theorem;

$A^2 - 5I = 0$

$$A^2 = 5I$$

$$(A^2)^4 = (5I)^4$$

$$A^8 = 5^4 \cdot I^4$$

$$A^8 = 625 I$$

Verify caley Hamilton th

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

[Repeate

$$|A - \lambda I| = 0$$