D.E's Reducible to Exact

Type 1: $\boxed{\text{Homogeneous}}$

If the D.E $\boxed{Mdx+Ndy=0}$ ① is homogeneous i.e, $M(x,y)$ & $N(x,y)$ both are homogeneous functions of x and y of same degree and $Mx+Ny\neq 0$, then Integrating factor (I.F) of ① is

$$\boxed{I.F=\frac{1}{Mx+Ny}}$$

① Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Ⓢ Given $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0 \longrightarrow ①$

$M=x^2y-2xy^2 \qquad N=-(x^3-3x^2y)$

$\frac{\partial M}{\partial y}=x^2-4xy \qquad \frac{\partial N}{\partial x}=-(3x^2-6xy)$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

$\therefore$ ① is not exact D.E and

① is homogeneous function of degree 3.

$$\therefore I.F=\frac{1}{Mx+Ny}=\frac{1}{(x^2y-2xy^2)x+(-(x^3-3x^2y))y}$$

$$=\frac{1}{\cancel{x^3y}-2x^2y^2-\cancel{x^3y}+3x^2y^2}=\frac{1}{x^2y^2}$$

$$\boxed{I.F=\frac{1}{x^2y^2}}$$

multiply D.E ① with I.F.,

$$①\Rightarrow \frac{(x^2y-2xy^2)}{x^2y^2}dx-\frac{(x^3-3x^2y)}{x^2y^2}dy=0$$

$$\Rightarrow \left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0 \longrightarrow ②$$

$M_1=\frac{1}{y}-\frac{2}{x} \qquad N_1=\frac{3}{y}-\frac{x}{y^2}$

$\boxed{\frac{\partial M_1}{\partial y}=-\frac{1}{y^2}} \qquad \boxed{\frac{\partial N_1}{\partial x}=-\frac{1}{y^2}} \qquad \therefore \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

$\therefore$ The general solution of ② is

$$\int_{y\ \text{const.}} \left(\frac{1}{y}-\frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$\Rightarrow \quad \frac{x}{y} - 2\ln x + 3\ln y = \ln c$

$\Rightarrow \quad \frac{x}{y} + \ln x^{-2} + \ln y^3 = \ln c \Rightarrow \frac{x}{y} + \ln\frac{y^3}{x^2} = \ln c$

$\Rightarrow \quad \cancel{\frac{x}{y} + \ln x = \ln c} \qquad \Rightarrow \ln\frac{y^3}{x^2} - \ln c = \frac{-x}{y}$

$\Rightarrow \qquad\qquad \Rightarrow \ln\left(\frac{y^3}{cx^2}\right) = \frac{-x}{y}$

$\Rightarrow \quad \frac{y^3}{cx^2} = e^{-x/y} \quad \Rightarrow \boxed{y^3 = cx^2 e^{-x/y}}$ //

② Solve $x^2 y\,dx - (x^3+y^3)\,dy = 0$

Ⓢ Given $x^2 y\,dx - (x^3+y^3)dy = 0 \longrightarrow ①$

$M = x^2 y \qquad N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact But ① is homogeneous of degree 3.

$\therefore \quad IF = \frac{1}{Mx+Ny} = \frac{1}{(x^2y)x - (x^3+y^3)y} = \frac{1}{x^3y - x^3y - y^4}$

$\boxed{I.F = \frac{-1}{y^4}}$

multiplying ① with $IF = \frac{-1}{y^4}$, we get

$① \Rightarrow \frac{x^2y}{-y^4}\,dx - \left(\frac{x^3+y^3}{-y^4}\right)dy = 0$

$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \longrightarrow ②$

$M_1 = \frac{-x^2}{y^3} \qquad N_1 = \frac{x^3}{y^4} + \frac{1}{y}$

$\frac{\partial M_1}{\partial y} = +\frac{3x^2}{y^4} \qquad \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4} \qquad \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore$ The general solution of ② is

$$\int -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log y = c \quad \Rightarrow \quad \boxed{3y^3 \log y = 3cy^3 + x^3}$$

(3) Solve $xy\,dx - (x^2+2y^2)\,dy = 0 \longrightarrow (1)$

(S)

$$M = xy \qquad N = -(x^2+2y^2)$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = -2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not an Exact D.E. But the D.E (1) is homogeneous D.E.

$$\therefore I.F = \frac{1}{Mx+Ny} = \frac{1}{(xy)x + (-x^2-2y^2)y} = \frac{1}{x^2y - x^2y - 2y^3} = \frac{-1}{2y^3}$$

$$\boxed{I.F = \frac{-1}{2y^3}}$$

* By multiplying I.F with (1), we will get Exact D.E.

The general solution of (1) is

$$\int \frac{-xy}{2y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$\Rightarrow \int \frac{-x}{2y^2}\,dx + \log y = c \quad \Rightarrow \quad \boxed{-\frac{x^2}{4y^2} + \log y = c}$$

(4) Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0 \longrightarrow (1)$

(S)

$$M = 3xy^2 - y^3 \qquad N = -(2x^2y - xy^2)$$

$$\frac{\partial M}{\partial y} = 6xy - 3y^2 \qquad \frac{\partial N}{\partial x} = -4xy + y^2$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact D.E. But (1) is homogeneous D.E of degree 3

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{(3xy^2-y^3)x - (2x^2y - xy^2)y}$$

$$= \frac{1}{3x^2y^2 - xy^3 - 2x^2y^2 + xy^3} = \frac{1}{x^2y^2}$$

$$I.F = \frac{1}{x^2y^2}.$$

from ① $\int\left(\frac{3xy^2}{x^2y^2}-\frac{y^3}{x^2y^2}\right)dx-\int\left(\frac{2x^2y-xy^2}{x^2y^2}\right)dy=\int 0$ which becomes Exact.

$\Rightarrow \int\left(\frac{3}{x}-\frac{y}{x^2}\right)dx-\int\frac{2}{y}\,dy=\int 0$ $\longrightarrow$ (without 'x' term)

$\Rightarrow 3\log x+\frac{y}{x}-2\log y=\log c$

$\Rightarrow \log x^3+\frac{y}{x}-\log y^2=\log c \Rightarrow \boxed{\log\left(\frac{x^3}{y^2}\right)+\frac{y}{x}=\log c}$

⑤ Solve $(x^4+y^4)dx-xy^3dy=0 \longrightarrow$ ①

⑤ $M=x^4+y^4$ $\quad N=-xy^3$

$\frac{\partial M}{\partial y}=4y^3$ $\quad \frac{\partial N}{\partial x}=-y^3$

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

The given eq ① is not an exact. and is homogeneous

$I.F=\frac{1}{Mx+Ny}=\frac{1}{x^5+xy^4-xy^4}=\frac{1}{x^5}$

$\Rightarrow \boxed{I.F=\frac{1}{x^5}}$

multiplying ① by I.F,

$\left(\frac{x^4+y^4}{x^5}\right)dx-\frac{xy^3}{x^5}dy=0$

$\Rightarrow \left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}dy=0 \longrightarrow$ ② [It will becomes an Exact D.E]

$\therefore$ The general solution of ② is

$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx+\int 0\,dy=\int 0$

$\Rightarrow \log x-\frac{y^4}{4x^4}=c \Rightarrow \boxed{4x^4\log x-y^4=4cx^4}$

⑥ Solve $y^2dx+(x^2-xy)dy=0$. Ans: $-\frac{y}{x}+\log y=c$

⑦ Solve $y^2dx+(x^2-xy-y^2)dy=0$ Ans: $(x-y)y^2=c(x+y)$

⑧ Solve $y(x+y)dx-x^2dy=0$ Ans: $\frac{x}{y}+\log x=c$

Type-2:

If the D.E $\boxed{Mdx+Ndy=0}$ ① is of the form $\boxed{y f(xy)dx + x g(xy)dy = 0}$ then the integrating factor I.F of ① is $\boxed{\frac{1}{Mx-Ny} = I.F}$ provided $Mx - Ny \neq 0$

① Solve $y(1+xy)dx + x(1-xy)dy = 0$

Ⓢ Given $y(1+xy)dx + x(1-xy)dy = 0 \longrightarrow ①$

$M = y + xy^2$ $\qquad N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$ $\qquad \frac{\partial N}{\partial x} = 1 - 2xy$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ ① is not an exact D.E and ① is of the form

$y f(xy) dx + x g(xy) dy = 0$, then

$I.F = \frac{1}{Mx - Ny} = \frac{1}{(y+xy^2)x - (x-x^2y)y} = \frac{1}{xy + x^2y^2 - xy + x^2y^2}$

$$\boxed{I.F = \frac{1}{2x^2y^2}}$$

multiplying ① by I.F then ① becomes an exact

$① \Rightarrow \left(\frac{y+xy^2}{2x^2y^2}\right)dx + \left(\frac{x-x^2y}{2x^2y^2}\right)dy = 0$

$\Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \longrightarrow ②$

The general solution of ② is

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx - \int \frac{1}{2y}dy = c$$

$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

$\Rightarrow -\frac{1}{2xy} + \frac{1}{2}\left(\log\left(\frac{x}{y}\right)\right) = c \Rightarrow \boxed{\log\frac{x}{y} - \frac{1}{xy} = c}$

② Solve $y(x^4y^4+x^2y^2+xy)\,dx + x(x^4y^4-x^2y^2+xy)\,dy=0 \longrightarrow ①$

⑤ $M=x^4y^5+x^2y^3+xy^2 \qquad N=x^5y^4-x^3y^2+x^2y$

$$\frac{\partial M}{\partial y}=5x^4y^4+3x^2y^2+2xy \qquad \frac{\partial N}{\partial x}=5x^4y^3-3x^2y^2+2xy$$

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. $\therefore$ ① is not exact D.E and it is of the form $\cancel{y f(xy)}$ $y f(xy)\,dx + x g(xy)\,dy=0$ then

$$I.F=\frac{1}{Mx-Ny}=\frac{1}{\cancel{x^5y^5}+x^3y^3+\cancel{x^2y^2}-\cancel{x^5y^5}+x^3y^3-\cancel{x^2y^2}}=\frac{1}{2x^3y^3}$$

$\boxed{I.F=\frac{1}{2x^3y^3}}$. By multiplying D.E with I.F, Eq ① will becomes an exact D.E

$$\left(\frac{x^4y^5+y^3x^2+xy^2}{2x^3y^3}\right)dx+\left(\frac{x^5y^4-x^3y^2+x^2y}{2x^3y^3}\right)dy=0$$

General Solution of ① is

$$\Rightarrow \int\left(xy^2+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\int\left(-\frac{1}{y}\right)dy=0$$

$$\Rightarrow \frac{x^2y^2}{2}+\log x-\frac{1}{xy}-\log y=c$$

$$\Rightarrow \frac{x^2y^2}{2}+\log\left(\frac{x}{y}\right)-\frac{1}{xy}=c \Rightarrow \boxed{x^2y^2+2\log\left(\frac{x}{y}\right)-\frac{2}{xy}=c}$$

③ Solve $y(xy\sin xy+\cos xy)\,dx+x(xy\sin xy-\cos xy)\,dy=0 \longrightarrow ①$

⑤ $M=xy^2\sin xy+y\cos xy \qquad N=x^2y\sin xy-x\cos xy$.

$$\frac{\partial M}{\partial y}=2xy\sin xy+xy^2\cos xy\,(x)+\cos xy-y\sin xy\,(x)$$

$$\frac{\partial M}{\partial y}=2xy\sin xy+x^2y^2\cos xy+\cos xy-xy\sin xy$$

$$\frac{\partial N}{\partial x}=2xy\sin xy+x^2y^2\cos xy-\cos xy+xy\sin xy.$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

$\therefore$ ① is not an exact and it is of the form

$y f(xy)\,dx+x g(xy)\,dy=0$ then

$$I.F=\frac{1}{Mx-Ny}=\frac{1}{\cancel{x^2y^2\sin xy}+xy\cos xy-\cancel{x^2y^2\sin xy}+xy\cos xy}=\frac{1}{2xy\cos xy}.$$

multiplying ① by I.F and ① becomes an exact D.E

$$① \Rightarrow \left(\frac{xy^2 \sin xy + y\cos xy}{2xy\cos xy}\right)dx + \left(\frac{x^2y\sin xy - x\cos xy}{2xy\cos xy}\right)dy = 0$$

$$\Rightarrow \left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0 \longrightarrow ②$$

The General solution of ② is

$$\int\left(y\tan xy + \frac{1}{x}\right)dx - \int \frac{1}{y}\,dy = c$$

$$\Rightarrow y\int \tan xy\,dx + \log x - \log y = \log c \qquad \left[\because \int \tan x\,dx = \log \sec x + c\right]$$

$$\Rightarrow \cancel{y}\,\frac{\log(\sec xy)}{\cancel{y}} + \log\left(\frac{x}{y}\right) = \log c$$

$$\Rightarrow \log(\sec xy) + \log\left(\frac{x}{y}\right) = \log c$$

$$\Rightarrow \sec xy \cdot \left(\frac{x}{y}\right) = c \Rightarrow x \cdot \sec xy = yc$$

$$\Rightarrow \boxed{x = cy\cos xy}$$

④. $y(2xy+1)dx + x(1+2xy-x^3y^3)dy = 0$

Ans: $\frac{1}{x^2y^2} + \frac{1}{3x^3y^3} + \log y = c.$

Type-3 :

If $\boxed{\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)}$ i.e; function of 'x' only

then $\boxed{I.F = e^{\int f(x)dx}}$ of $Mdx + Ndy = 0$

① Solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

⑤ Given $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0 \longrightarrow ①$

$M = xy^2 - e^{1/x^3}$ $\qquad N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad \frac{\partial N}{\partial x} = -2xy$ $\Rightarrow \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$

$\therefore$ ① is not an exact D.E.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2y}\left(2xy - (-2xy)\right) = \frac{1}{-x^2y}(4xy)$

$= -\frac{4}{x} = f(x)$

$\Rightarrow \boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{4}{x} = f(x)}$

$\therefore \quad I.F = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}}$

$\Rightarrow \boxed{I.F = \frac{1}{x^4}}$

Multiplying ① with I.F., we get

① $\Rightarrow \quad \frac{1}{x^4}\left(xy^2 - e^{1/x^3}\right)dx - \frac{x^2y}{x^4}dy = 0$

$\Rightarrow \quad \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0 \longrightarrow ②$

$M' = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}, \quad N' = \frac{-y}{x^2}$

$\frac{\partial M'}{\partial y} = \frac{2y}{x^3} \qquad \frac{\partial N'}{\partial x} = \frac{2y}{x^3}$

$\Rightarrow \boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$

Thus, the General solution of ① is

$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\, dy = C$

$\frac{1}{x^3} = t \Rightarrow \frac{-3}{x^4}dx = dt$

$\Rightarrow \boxed{\frac{dx}{x^4} = \frac{-dt}{3}}$

$\int \frac{y^2}{x^3}dx - \int \frac{e^{1/x^3}}{x^4}dx = C$

$\frac{-y^2}{2x^2} - \int e^t\left(-\frac{dt}{3}\right) = C$

$\Rightarrow \boxed{\frac{-y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = C}$

(2) Solve $(x^3-2y^2)\,dx + 2xy\,dy = 0$

(S) $(x^3-2y^2)\,dx + 2xy\,dy = 0 \longrightarrow (1)$

$M = x^3 - 2y^2 \qquad N = 2xy$

$\frac{\partial M}{\partial y} = -4y \qquad \frac{\partial N}{\partial x} = 2y \Rightarrow \underline{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$

$\Rightarrow$ (1) is not an exact D.E.

find $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}\left(-4y-2y\right) = \frac{-6y}{2xy} = \frac{-3}{x}$

$\Rightarrow \boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-3}{x} = f(x)}$

$\therefore \quad I.F = e^{\int f(x)dx} = e^{\int \frac{-3}{x}dx} = e^{-3\log x} = \frac{1}{x^3}$

$\boxed{I.F = \frac{1}{x^3}}$

multiplying (1) with I.F,

$(1) \Rightarrow \frac{1}{x^3}(x^3-2y^2)\,dx + \frac{1}{x^3}(2xy)\,dy = 0$

$\Rightarrow \left(1 - \frac{2y^2}{x^3}\right)dx + \left(\frac{2y}{x^2}\right)dy = 0 \longrightarrow (2)$

$M_1 = 1 - \frac{2y^2}{x^3}, \qquad N_1 = \frac{2y}{x^2}$

$\frac{\partial M_1}{\partial y} = \frac{-4y}{x^3} \qquad \frac{\partial N_1}{\partial x} = \frac{-4y}{x^3} \Rightarrow \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$

Then, General solution of (1) is

$\int \left(1 - \frac{2y^2}{x^3}\right)dx + \int 0\,dy = c$

$x - 2y^2 \frac{1}{-2x^2} = c \Rightarrow x + \frac{y^2}{x^2} = c$

$\Rightarrow \boxed{x^3 + y^2 = cx^2}$

③ Solve $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0$

Given $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)\,dy=0 \longrightarrow ①$

$M = y+\frac{y^3}{3}+\frac{x^2}{2}$ $\qquad N=\frac{1}{4}(x+xy^2)$

$\frac{\partial M}{\partial y} = 1+\frac{3y^2}{3}$ $\qquad \frac{\partial N}{\partial x}=\frac{1}{4}+\frac{y^2}{4}$

$= 1+y^2$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ ① is not an exact D.E.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{\frac{x}{4}(1+y^2)}\left[(1+y^2)-\frac{1}{4}(1+y^2)\right]$

$= \frac{1}{\frac{x}{4}(1+y^2)}\left[(1+y^2)\left(1-\frac{1}{4}\right)\right]$

$= \frac{4}{x}\left(\frac{3}{4}\right) = \frac{3}{x} = f(x)$

$\Rightarrow \boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{3}{x} = f(x)}$

$\therefore \; I.F = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$.

$\boxed{I.F = x^3}$

multiplying ① with I.F,

①$\Rightarrow x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{x^3}{4}(x+xy^2)\,dy=0$

$M_1 = x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}$, $\qquad N_1=\frac{x^4}{4}+\frac{x^4y^2}{4}$

$\frac{\partial M_1}{\partial y} = x^3+x^3y^2$ $\qquad \frac{\partial N_1}{\partial x}=x^3+x^3y^2$

$\Rightarrow \boxed{\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}}$

Then, the general solution of ① is

$\int\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\int 0\,dy = c$

$\Rightarrow \quad y\frac{x^4}{4} + \frac{y^3}{3}\cdot\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^6}{6} = c$

$\Rightarrow \quad \frac{x^4 y}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} = c$

$\Rightarrow \quad \boxed{3x^4 y + x^4 y^3 + x^6 = 12c = C}$

(4) Solve $(x^2+y^2+1)dx - 2xy\,dy = 0$ Ans: $x^2 - y^2 = cx+1$

(5) Solve $(x^2+y^2)dx - 2xy\,dy = 0$ Ans: $x^2 - y^2 = C$

Type-4:

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of 'y' only i.e,

$\boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)}$ then $\boxed{I.F = e^{\int f(y)\,dy}}$

(1) Solve $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

(S) Given $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0 \longrightarrow (1)$

$M = xy^3+y \qquad N = 2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y} = 3xy^2+1 \qquad \frac{\partial N}{\partial x} = 4xy^2+2$

$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ (1) is not an exact D.E

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}\left(4xy^2+2-3xy^2-1\right)$

$= \frac{1}{y\cancel{(xy^2+1)}}\cancel{(xy^2+1)}$

$= \frac{1}{y} = f(y)$

$\Rightarrow \quad \boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y) = \frac{1}{y}.}$

$\therefore \quad I.F = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

$\boxed{I.F = y}$

multiplying ① with I.F.

$$① \Rightarrow \quad y(xy^3+y)dx + 2y(x^2y^2+x+y^4)dy = 0$$

$$\Rightarrow \quad (xy^4+y^2)dx + (2x^2y^3+2yx+2y^5)dy = 0$$

$$M' = xy^4+y^2 \qquad N' = 2x^2y^3+2yx+2y^5$$

$$\frac{\partial M'}{\partial y} = 4xy^3+2y \qquad \frac{\partial N'}{\partial x} = 4xy^3+2y$$

$$\Rightarrow \quad \boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$$

thus; General solution of ① is

$$\int (xy^4+y^2)dx + \int (2y^5)dy = c$$

$$\frac{x^2}{2}y^4 + y^2x + \frac{2y^6}{6} = c$$

$$\Rightarrow \quad \frac{x^2y^4+2xy^2}{2} + \frac{y^6}{3} = c \quad \Rightarrow \quad \boxed{3x^2y^4+6xy^2+2y^6 = 6c \text{ or } c}$$

2) Solve $3x^2y\,dx - (x^3+2y^4)dy = 0$

(S) Given $\quad 3x^2y\,dx - (x^3+2y^4)dy = 0 \longrightarrow ①$

$$M = 3x^2y \qquad N = -(x^3+2y^4)$$

$$\frac{\partial M}{\partial y} = 3x^2 \qquad \frac{\partial N}{\partial x} = -3x^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ ① is not an exact D.E.

Now $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{3x^2y}\left(-3x^2 - 3x^2\right) = \frac{-6x^2}{3x^2y} = \frac{-2}{y}$$

$$\boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{2}{y} = f(y)}$$

$$\therefore \quad I.F = e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = y^{-2}$$

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$$\boxed{I.F = y^{-2}}$$

multiplying ① with I.F

① ⟹ $y^{-2}(3x^2y)\,dx - y^{-2}(x^3+2y^4)\,dy = 0$

~~$M' = 3x^2y^{-1}$~~ ~~$N' = -x^3y^{-2} - 2y^6$~~

~~$\frac{\partial M'}{\partial y} = 9x^2y^{-1}$~~ ~~$\frac{\partial N'}{\partial x} =$~~

$M' = \frac{3x^2}{y}$ $\quad N' = -\frac{x^3}{y^2} - 2y^2$

$\frac{\partial M'}{\partial y} = \frac{-3x^2}{y^2}$, $\frac{\partial N'}{\partial x} = \frac{-3x^2}{y^2}$

$$\Rightarrow \boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$$

Then, the general solution of eq① is

$$\int\left(\frac{3x^2}{y}\right)dx + \int(-2y^2)\,dy = c$$

$$\Rightarrow \frac{3}{y}\cdot\left(\frac{x^3}{3}\right) - 2\left(\frac{y^3}{3}\right) = c$$

$$\Rightarrow \boxed{\frac{x^3}{y} - \frac{2y^3}{3} = c}$$

③ Solve $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

Ans: $xy + \frac{2x}{y^2} + y^2 = c$

④ Solve $y(x+y+1)\,dx + x(x+3y+2)\,dy = 0$

Ans: $x^2y^2 + 2xy^3 + 2xy^2 = c$

Type-5: If the given eq. $Mdx + Ndy = 0 \longrightarrow (1)$

can be expressed as

$$\boxed{x^a y^b [m y dx + n x dy] + x^{a'} y^{b'} [m' y dx + n' x dy] = 0} \longrightarrow (2)$$

where a, b, a', b', m, n and m', n' are constants.

then $\boxed{I.F = x^h y^k}$ where h & k are given by

$$\boxed{\frac{a+h+1}{m} = \frac{b+k+1}{n}}, \quad \boxed{\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}}$$

(P) $(y^2 + 2x^2 y) dx + (2x^3 - xy) dy = 0$

(S) Given $(y^2 + 2x^2 y) dx + (2x^3 - xy) dy = 0 \longrightarrow (1)$

$\Rightarrow \quad y^2 dx + 2x^2 y dx + 2x^3 dy - xy dy = 0$

$\Rightarrow \quad y(y dx - x dy) + x^2 (2y dx + 2x dy) = 0 \longrightarrow (2)$

Comparing (2) with standard form

$$x^a y^b (m y dx + n x dy) + x^{a'} y^{b'} (m' y dx + n' x dy) = 0$$

we get, $\quad a = 0, \ b = 1, \quad m = 1, \ n = -1$

$\qquad a' = 2, \ b' = 0 \quad m' = 2, \ n' = 2$

and $I.F = x^h y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}$$

$\Rightarrow \quad \dfrac{0+h+1}{1} = \dfrac{1+k+1}{-1}$

$\Rightarrow \quad -(h+1) = 2+k$

$\Rightarrow \quad \boxed{h+k+3=0} \longrightarrow (i)$

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$\Rightarrow \quad \dfrac{2+h+1}{2} = \dfrac{0+k+1}{2}$

$\Rightarrow \quad 2+h+1 = k+1$

$\Rightarrow \quad \boxed{h-k+2=0} \longrightarrow (ii)$

By solving (i) & (ii) we get $h = -5/2, \ k = -1/2$

$$\therefore \boxed{I.F = x^{-5/2} \cdot y^{-1/2}}$$

multiplying I.F with ①, we get

$$x^{-5/2}y^{-1/2}\left(y^2+2x^2y\right)dx+\left(2x^3-xy\right)x^{-5/2}y^{-1/2}dy=0$$

$$\Rightarrow \left(x^{-5/2}\cdot y^{3/2}+2x^{-1/2}\cdot y^{1/2}\right)dx+\left(2x^{1/2}\cdot y^{-1/2}-x^{-3/2}\cdot y^{1/2}\right)dy=0 \longrightarrow ③$$

$$M'=x^{-5/2}\cdot y^{3/2}+2x^{-1/2}y^{1/2} \qquad N'=2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2}$$

$$\frac{\partial M'}{\partial y}=\frac{3}{2}x^{-5/2}\cdot y^{1/2}+x^{-1/2}y^{-1/2} \qquad \frac{\partial N'}{\partial x}=x^{-1/2}y^{-1/2}+\frac{3}{2}x^{-5/2}y^{1/2}$$

$$\Rightarrow \boxed{\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}}$$

Thus, the G.S of ① is

$$\int\left(x^{-5/2}\cdot y^{3/2}+2x^{-1/2}\cdot y^{1/2}\right)dx+\int 0=c$$

$$\Rightarrow y^{3/2}\cdot\frac{x^{-5/2+1}}{-5/2+1}+2y^{1/2}\cdot\frac{x^{-1/2+1}}{-1/2+1}=c$$

$$\Rightarrow y^{3/2}\cdot\frac{x^{-3/2}}{(-3/2)}+2y^{1/2}\cdot\frac{x^{1/2}}{(1/2)}=c$$

$$\Rightarrow -\frac{2}{3}\frac{y^{3/2}}{x^{3/2}}+4\sqrt{xy}=c$$

$$\Rightarrow \boxed{-\frac{2}{3}\left(\frac{y}{x}\right)^{3/2}+4\sqrt{xy}=c}$$