3. Exact Differential Equations

Def: Exact D.E: A D.E is said to be an exact D.E if it can be expressed as perfect differential functions without any subsequent process i.e, multiplication, elimination, etc.

$$\because x\frac{dy}{dx}+y=0$$
$$x\,dy+y\,dx=0$$

Ex: 1) $x\,dy+y\,dx=0$

$$\Rightarrow \boxed{d(xy)=0}$$

2) $(x^2+y^2)\,dx+2xy\,dy=0$

$$\Rightarrow \boxed{d\left(\frac{x^3}{3}+y^2x\right)=0}$$

3) $y^2e^x\,dx+2ye^x\,dy=0$

$$\Rightarrow \boxed{d(e^x\cdot y^2)=0}$$

*** A necessary and sufficient condition (if and only if) for a D.E to be exact.

Let $\boxed{M\,dx+N\,dy=0}$ or $\boxed{M(x,y)\,dx+N(x,y)\,dy=0}$ be the D.E of first order and first degree with M and N. where M, and N are functions in x and y.

Then a necessary and sufficient condition for the D.E $M\,dx+N\,dy=0$ to be exact D.E is

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Let $du = Mdx + Ndy$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Mdx + Ndy$$

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

total derivative

$u = u(x, y)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

working rule for Solving an exact D.E

Compare the given equation with standard form

$\boxed{Mdx + Ndy = 0}$ $\rightarrow$ ① and find out M and N. then find $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$

* if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$, we conclude that the given equation is Exact.

If the equation is exact then

Step1: Integrate M w.r.t 'x' treating y as Constant

step2: Integrate w.r.t 'y' only those terms of N which does not contain 'x'.

step3: Equate the sum of these two integrals (Found in step① & ②) to an arbitrary constant and thus we obtain the required solution of exact eq. $Mdx + Ndy = 0$

i.e; The solution of ① is,

$$\boxed{\int_{y\ \text{const.}} M\,dx + \int (N \text{ without } x \text{ terms})\,dy = C}$$

Problems

1) Solve $(2x-y+1)\,dx+(2y-x+1)\,dy=0$

Ⓢ Given D.E $(2x-y+1)\,dx+(2y-x+1)\,dy=0 \longrightarrow ①$

Compare the given equation with standard form

$\boxed{M(x,y)\,dx+N(x,y)\,dy=0}$, we get

$M=2x-y+1, \qquad N=2y-x+1$

$\dfrac{\partial M}{\partial y}=-1, \qquad \dfrac{\partial N}{\partial x}=-1$

$\Rightarrow \boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$

$\therefore$ ① is an exact D.E

The solution of an exact D.E is, given by

$$\boxed{\int_{y\ \text{const.}} M\,dx+\int(N \text{ without '}x\text{' terms})\,dy=c}$$

So, solution of ① is

$$\int(2x-y+1)\,dx+\int(2y+1)\,dy=c$$

$\Rightarrow x^2-yx+x+y^2+y=c$

$\Rightarrow \boxed{x^2+y^2-xy+x+y=c}$

② Solve $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy=0$

Ⓢ Given $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy=0 \longrightarrow ①$

Compare the eq ① with $M\,dx+N\,dy=0$.

$\Rightarrow \quad M=x^2-4xy-2y^2, \qquad N=y^2-4xy-2x^2$

$\dfrac{\partial M}{\partial y}=-4x-4y \qquad \dfrac{\partial N}{\partial x}=-4y-4x$

clearly $\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$

$\Rightarrow$ eq ① is exact D.E.

then, the solution ① is given by

$$\boxed{\int_{y\text{ const.}} M\,dx + \int (N \text{ without 'x' term})\,dy = c}$$

$$\Rightarrow \int_{y\text{ const.}} (x^2-4xy-2y^2)\,dx + \int (y^2)\,dy = c$$

$$\Rightarrow \frac{x^3}{3} - 4y\cdot\frac{x^2}{2} - 2y^2(x) + \frac{y^3}{3} = c$$

$$\Rightarrow \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$$

$$\Rightarrow \boxed{x^3 - 6x^2y - 6xy^2 + y^3 = 3c \text{ or } C}$$

③ Solve $(ax+hy+g)dx + (hx+by+f)dy = 0 \longrightarrow ①$

Ⓢ $M = ax+hy+g \qquad N = hx+by+f$

$$\frac{\partial M}{\partial y} = h \qquad \frac{\partial N}{\partial x} = h$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Thus, ① is Exact D.E.

Now solution of ① is,

$$\int_{y\text{ const.}} (ax+hy+g)\,dx + \int (by+f)\,dy = c$$

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c$$

$$ax^2 + 2hxy + 2gx + 2fy + by^2 = c$$

$$\Rightarrow \boxed{ax^2+by^2+2gx+2hxy+2fy = C \text{ (or) } +C=0}$$

④ Solve $y\sin 2x\,dx - (1+y^2+\cos^2 x)\,dy = 0 \longrightarrow ①$

Ⓢ $M = y\sin 2x \qquad N = -(1+y^2+\cos^2 x)$

$\because \sin 2A = 2\sin A\cos A$

$$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = -(2\cos x(-\sin x))$$
$$= 2\sin x\cos x = \sin 2x$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Then, the eq^n ① is Exact D.E.

So, solution of ① is

$$\int_{y\text{ const.}} (y\sin 2x)\,dx + \int -(1+y^2)\,dy = c$$

$$\Rightarrow y\left(\frac{-\cos 2x}{2}\right) - \left(y+\frac{y^3}{3}\right) = c$$

$$\Rightarrow \frac{-y\cos 2x}{2} - y - \frac{y^3}{3} = c$$

$$\Rightarrow -3y\cos 2x - 6y - 2y^3 = 6c$$

$$\Rightarrow \boxed{3y\cos 2x + 6y + 2y^3 = C.} \quad \text{where } C = -6c$$

⑤ Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx + (x+\log x - x\sin y)\,dy = 0 \longrightarrow ①$

Ⓢ $M = y\left(1+\frac{1}{x}\right)+\cos y \qquad N = x+\log x - x\sin y$

$$\frac{\partial M}{\partial y} = 1+\frac{1}{x}+(-\sin y) \qquad \frac{\partial N}{\partial x} = 1+\frac{1}{x}-\sin y$$
$$= 1+\frac{1}{x}-\sin y$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Then, the eq^n ① is Exact D.E.

So, solution of ① is

$$\int_{y\text{ const.}} \left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx + \int (0)\,dy = C$$

$$\Rightarrow y\,[x+\log x] + x\cos y = c$$

$$\Rightarrow \boxed{xy + x\cos y + y\log x = c}$$

⑥ Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0 \longrightarrow ①$

Ⓢ $M = (e^y+1)\cos x, \qquad N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x \qquad \frac{\partial N}{\partial x} = e^y \cos x$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Thus, eq/ ① is Exact D.E.

so, the solution of ① is,

$$\int_{y\ \text{const}} [(e^y+1)\cos x]\,dx + \int 0\,dy = C$$

$$\Rightarrow (e^y+1)\sin x = C$$

$$\Rightarrow \boxed{e^y \sin x + \sin x = C}$$

⑦ Solve $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy e^{xy^2} - 3y^2)\,dy = 0 \longrightarrow ①$

$M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xy e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = y^2\left(e^{xy^2}\cdot(2xy)\right) + e^{xy^2}(2y) \qquad \frac{\partial N}{\partial x} = 2y\left[x\,(e^{xy^2})(y^2) + e^{xy^2}(1)\right]$$

$$= 2xy^3 e^{xy^2} + 2y\,e^{xy^2} \qquad \frac{\partial N}{\partial x} = 2xy^3 e^{xy^2} + 2y\,e^{xy^2}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Thus, eq/① is exact D.E,

so, the solution of ① is

$$\int_{y\ \text{const}} (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = C$$

$$\Rightarrow \cancel{y^2}\cdot\frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\frac{x^4}{\cancel{4}} + \left(\frac{-\cancel{3}y^3}{\cancel{3}}\right) = C$$

$$\Rightarrow \boxed{e^{xy^2} + x^4 - y^3 = C}$$

8) Solve $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0 \longrightarrow (1)$

(S)

$M = 1+e^{x/y}$ $\qquad N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = e^{x/y}\cdot\left(\frac{-x}{y^2}\right)$ $\qquad \frac{\partial N}{\partial x} = e^{x/y}\cdot\left(\frac{1}{y}\right) - \left[e^{x/y}\left(\frac{1}{y}\right) + \left(\frac{x}{y}\right)e^{x/y}\cdot\left(\frac{1}{y}\right)\right]$

$= \frac{-x}{y^2}e^{x/y}$ $\qquad = \cancel{e^{x/y}\left(\frac{1}{y}\right)} - \cancel{\left(\frac{1}{y}\right)e^{x/y}} - \frac{x}{y^2}\cdot e^{x/y}$

$\frac{\partial N}{\partial x} = -\frac{x}{y^2}\cdot e^{x/y}$

$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Thus, eq (1) is Exact D.E.

So, the solution of (1) is,

$\int_{y\ \text{const}} (1+e^{x/y})\,dx + \int 0\,dy = c$

$\Longrightarrow x + \frac{e^{x/y}}{\left(\frac{1}{y}\right)} = c \Longrightarrow \boxed{x + y e^{x/y} = c}$

9) Find the value of 'n' for which the following DE is exact

i) $(xy^2+nx^2y)\,dx + (x^3+x^2y)\,dy = 0.$

(S) Given $(xy^2+nx^2y)\,dx + (x^3+x^2y)\,dy = 0 \longrightarrow (1)$

is Exact D.E.

$M = xy^2+nx^2y$ $\qquad N = x^3+x^2y$

$\frac{\partial M}{\partial y} = 2xy + nx^2$ $\qquad \frac{\partial N}{\partial x} = 3x^2+2xy$

Given (1) is Exact, Then,

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\Longrightarrow 2xy + nx^2 = 3x^2 + 2xy$

$\Longrightarrow nx^2 = 3x^2$

$\boxed{n=3}$

(ii) $(x + y e^{2xy})\,dx + n x e^{2xy}\,dy = 0 \longrightarrow (1)$

$$M = x + y e^{2xy} \qquad N = n x e^{2xy}$$

$$\frac{\partial M}{\partial y} = y e^{2xy}(2x) + e^{2xy} \qquad \frac{\partial N}{\partial x} = n\left[x e^{2xy}\cdot(2y) + e^{2xy}(1)\right]$$

$$= 2xy\cdot e^{2xy} + e^{2xy} \qquad = n\,2xy\,e^{2xy} + n e^{2xy}$$

since (1) is Exact D.E.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow 2xy\cdot e^{2xy} + e^{2xy} = n\,2xy\,e^{2xy} + n e^{2xy}$$

$$\Longrightarrow \boxed{n = 1}$$

iii) $(2x e^{y} + 3y^2)\frac{dy}{dx} + (3x^2 + \underline{n} e^{y}) = 0$ Ans: $n = 2$

(10) Solve the following D.E

(i) $\sec^2 x \cdot \tan y\,dx + \sec^2 y \tan x\,dy = 0$. Ans: $\tan x \cdot \tan y = c$

(ii) $(\sin x \cdot \cos y + e^{2x})\,dx + (\cos x \cdot \sin y + \tan y)\,dy = 0$

Ans: $\frac{e^{2x}}{2} - \cos x \cos y + \log|\sec x| = c$.

(iii) $\frac{e^{y}}{x}\,dy - \frac{e^{y}}{x^2}\,dx = 0$

Ans: $e^{y} = cx$

(iv) $(r + \sin\theta - \cos\theta)\,dr + r(\sin\theta + \cos\theta)\,d\theta = 0$

Ans: $\frac{r^2}{2} + r(\sin\theta - \cos\theta) = c$. $\underline{(r, \theta) = (x, y)}$