2. linear Algebra

Rank of matrix :-

① reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelonform and hence find it's rank ?

A)
$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \\ \to R_4 \end{matrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ (inter change)

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$-11 + 8$

$R_4 \to R_4 - R_3$

$-4 - 0$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is Echelon form,

Rank of A = number of Non zero rows

$\rho(A) = 3$

② The matrix $A = \begin{bmatrix} 5 & 3 & 14 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$. Then find the rank ?

A)
$$A = \begin{bmatrix} 5 & 3 & 14 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

$R_3 \to 5R_3 - R_1$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -8 & -4 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 8R_2$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 12 & 8 \end{bmatrix} \to \text{This is echelon form.}$$

$\rho(A) = 3$

(3) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \to (2)$

(4) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(5) $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} - (2)$

(6) $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix} - (2)$

(3)

A) $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + R_1$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{This is echelon form}$$

$\rho(A) = 2$

(4) A)

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$\Rightarrow R_2 \to R_2 + R_1$
$R_3 \to R_3 + 2R_1$
$R_4 \to R_4 - R_1$

$$\Rightarrow \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$\Rightarrow R_3 \to 2R_3 - 11R_2$
$R_4 \to \cancel{2} R_4 + 2R_2$

$$\Rightarrow \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow R_4 \to 6R_4 + R_3$

$$\Rightarrow \begin{vmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

It is echelon form.

The Rank of the matrix is $\rho(A) = 4$

(5)

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$
$R_3 \to \cancel{2}R_3 + R_1$

$$\begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$\Rightarrow R_3 \to 3R_3 + R_2$
$R_4 \to 3R_4 - R_2$

$$\rightarrow \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

Inverse of Matrix A by using gauss-Jordan method

① ⇒ given that matrix A.

⇒ first we write $A = IA$

here I is unit matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4} \cdots \cdots$$

⇒ we apply elementary row operation on a [left side] and same operation apply on I.

⇒ we will do this until

$$\boxed{I = A^{-1}.A}$$

① Find the inverse of $A = \begin{vmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$ using elementary row operation?

An) $A = \begin{vmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

first $A = IA$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow 2R_3 + R_1$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_2$

$R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \rightarrow 2R_1 - R_3$

$R_2 \rightarrow 8R_2 - R_3$

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 = \frac{R_1}{(-4)}, \quad R_2 = \frac{R_2}{(-8)} \quad R_3 = \frac{R_3}{8}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{2}{4} \\ \frac{1}{8} & \frac{-3}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A$$

$I = A^{-1} A$

here $A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{2}{4} \\ \frac{1}{8} & \frac{-3}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix}$

(2) Find the inverse of $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$?

A) $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 2R_1 - 3R_2$, $R_3 \to 2R_3 - 11R_2$, $R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to 3R_2 + R_3$, $R_4 \to 6R_4 + R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -6 & 0 & -2 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -4 & -8 & 2 & 0 \\ -7 & -11 & 2 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to R_1 - R_4$, $R_2 \to R_2 + 2R_4$, $R_3 \to R_3 - R_4$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -6 & -6 & 6 & 12 \\ -6 & -12 & 0 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to \dfrac{R_1}{(-2)}$, $R_2 \to \dfrac{R_2}{(-6)}$, $R_3 \to \dfrac{R_3}{-6}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$I = A^{-1} A$

$$A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

(1)

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \Longleftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 2R_1 \qquad R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2 \qquad R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_3 \qquad R_4 \to 2R_4 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -3 & 2 \end{bmatrix} A$$

$R_1 \to 3R_1 + R_4$

$R_2 \to 3R_2 - R_4 \qquad R_3 \to R_3 - R_4$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 5 & -3 & 2 \\ 3 & -8 & 0 & -2 \\ 0 & -4 & 4 & -2 \\ 0 & 2 & -3 & 2 \end{bmatrix} A$$

$R_1 \to \frac{R_1}{3}, \; R_2 \to \frac{R_2}{3}, \; R_3 \to \frac{R_3}{-2}, \; R_4 \to \frac{R_4}{-3}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{5}{3} & -1 & \frac{2}{3} \\ 1 & \frac{-8}{3} & 0 & -\frac{2}{3} \\ 0 & 2 & -2 & 2 \\ 0 & -\frac{2}{3} & 1 & -\frac{2}{2} \end{bmatrix} A$$

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$$A^{-1} = \begin{bmatrix} -1 & \frac{5}{3} & -1 & \frac{2}{3} \\ -1 & -\frac{5}{3} & 0 & -\frac{2}{3} \\ 0 & 2 & -2 & 2 \\ 0 & -\frac{2}{3} & 1 & -2/3 \end{bmatrix}$$

System of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

Now this equations write matrix form is $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

here $x_1, x_2, x_3 \dots x_n$ is variables

$\Rightarrow$ matrix $AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} & b_2 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} & b_m \end{bmatrix}$

is called agumented matrix.

$\Rightarrow$ Now matrix AB reduced to echelon form

$\Rightarrow$ here find $\rho(A)$, $\rho(AB)$

here n = number of variables.

i) If $\rho(A) \neq \rho(AB)$ then system of linear equation has no solution or [not consistant]

ii) If $\rho(A) = \rho(AB)$ then system of linear equation has consistant or solution.

iii) If $\rho(A) = \rho(AB) = n$ linear equation has unique soution.

(5) If $\rho(A) = \rho(AB) < n$ then linear equation has infinite solutions.

→ This is called gauss Jordan elemination method.

(1) show that the equation $x+y+z=4$, $2x+5y-2z=3$, $x+7y-7z=5$ are not consistant!

A) $x+y+z=4$

$2x+5y-2z=3$

$x+7y-7z=5$

$Ax = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 6 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$\rho(A) = 2$, $\rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

system of linear equation has no solution.

(2) solve equations.

1) $x+y+z=9$

$2x+5y+7z=52$

$2x+y-z=0$

$Ax = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -18 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -20 \end{bmatrix}$$

$\rho(A) = 3$, $\rho(AB) = 3$, $n = 3$

$\rho(A) = \rho(AB) = n$

The system of linear equation has ~~solution~~ unique solution.

$x + y + z = 9$

$3y + 5z = 34$

$-4z = -20$

$z = \frac{-20}{-4} = 5$

$\boxed{z = 5}$

Take $3y + 5z = 34$

$3y = 34 - 25 \Rightarrow 3y = 9$

$\boxed{y = 3}$

Take $x + y + z = 9$

$x + 8 = 9$

$\boxed{x = 1}$

solution is $x = 1, y = 3, z = 5$.

(3) Show that the equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$ are consistant and solve them?

A) $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 24 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$\rho(A) = 2, \quad \rho(AB) = 2$

$\rho(A) = \rho(AB)$

linear equation has consistant (has solution)

here $n = 3$

$\rho(A) = \rho(AB) < n$

linear equation has infinite number of solutions

$x + y + z = 6$

$y + 2z = 8$

let $z = k$

$y = 8 - 2z = 8 - 2k$

take $x = 6 - y - z$

$= 6 - (8 - 2k) - k \quad = 6 - 8 + 2k - k$

$= k - 2$

The solutions is

$x = k - 2, \ y = 8 - 2k, \ z = k$

$x + y + 2z = 4$

$2x - \ldots$

discuss for what values of λ, μ. ~~this~~ The equation are $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) No solution (ii) A unique solution (iii) An infinite number of solution.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-6 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

i) if $\lambda = 3$, $\mu \neq 10$ then $\rho(A) \neq \rho(AB)$. Then the system of equation has no solutions

ii) If $\lambda \neq 3$, $\mu = \text{any value}$ then $\underset{(3)}{\rho(A)} = \underset{(3)}{\rho(AB)} = \underset{(3)}{n}$. Then the system of equation has unique solutions.

(or)

$\mu \neq 10$, $\lambda =$ any value

iii) If $\lambda = 3$, $\mu = 10$ then $\underset{(2)}{\rho(A)} = \underset{(2)}{\rho(AB)} < \underset{(3)}{n}$. Then the system of equation has infinite solutions.

If ... shows ~~such~~ ... [illegible]

Eigen values (roots) and Eigen vectors (or)

characteristic values and characteristic vectors

→ given matrix $A_{n\times n}$, let $I_{n\times n}$

→ characteristic ~~eqn~~ equation of matrix A is

$|A - \lambda I| = 0$

→ here we get λ values. This values are called Eigen values (or) roots.

Eigen vector :-

$X = x_1, x_2, x_3$ are Eigen vectors corresponding to Eigen values $\lambda_1, \lambda_2, \lambda_3$.

$\therefore (A - \lambda I)X = 0$

⇒ Here $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(1) Find Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$?

A) let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$\lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$|A - \lambda I| = 1-\lambda\left[(5-\lambda)(1-\lambda) - 1\right] - 1\left[(1-\lambda) - 3\right] + 3\left[1 - 3(5-\lambda)\right]$$

$$= 1-\lambda\left[5 - 6\lambda + \lambda^2 - 1\right] - \left[-2 - \lambda\right] + 3\left[1 - 15 + 3\lambda\right]$$

$$= 1-\lambda\left[\lambda^2 - 6\lambda + 4\right] + \left[\lambda + 2\right] - 42 + 9\lambda$$

$$= \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 - 42 + 9\lambda$$

$$= -\lambda^3 + 7\lambda^2 - 36$$

$|A - \lambda I| = 0$

$-\lambda^3 + 7\lambda^2 - 36 = 0$

$$\begin{array}{c|cccc} & \lambda^3 & \lambda^2 & \lambda & con \\ & -1 & 7 & 0 & -36 \\ \lambda = -2 & 0 & 2 & -18 & 36 \\ \hline & -1 & 9 & -18 & 0 \end{array}$$

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$(\lambda+2)(-\lambda^2+9\lambda-18) = 0$

$(\lambda+2)(-\lambda^2+6\lambda+3\lambda-18) = 0$

$-\lambda(\lambda-6)+3(\lambda-6) = 0$

$(\lambda-6)(\lambda-3) = 0$

$\lambda = -2, 6, 3$.

Eigen vector

case 1 $\lambda = -2$

$(A - \lambda I)x_1 = 0$

here $x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector.

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is echelon form. $n-2 = 3-2 = 1$

$3x + y + 3z = 0$

$+20y = 0 \Rightarrow y = 0$

let $z = k$

$3x + 0 + 3k = 0$

$3x = -3k \Rightarrow x = -k$.

$x = -k,\ y = 0,\ z = k$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} -k \\ 0k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case ② $\lambda = 3$

$(A-\lambda I)x_2 = 0$; $x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 1.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + 3R_1$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{-2x + y + 3z = 0}$

$\cancel{5y + 5z = 0}$

$\cancel{-20z = 0 \Rightarrow z = 0}$

$\cancel{5y + 0 = 0}$

$\cancel{y = 0}$

$-2x + y + 3z = 0$

$5y + 5z = 0 \Rightarrow y + z = 0$

$y = -z$

let $z = k$

$y = -k$

$-2x - k + 3k = 0 \Rightarrow -2x + 2k = 0$

$-x + k = 0$

$\boxed{x = k}$

$x = k, \; z = k, \; y = -k$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case-3

$\lambda = 6$

$(A-\lambda I)x_3 = 0$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 + R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \; R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x+y+3z=0$

$-4y+8z=0 \rightarrow -y+2z=0$

let $z=k$ $\quad -y+2k=0$

$\boxed{y=2k}$

$-5x+2x+3k=0$

$+5x=+5k$

$\boxed{x=k}$

$$X=\begin{bmatrix} x\\ y\\ z\end{bmatrix}\begin{bmatrix} k\\ 2k\\ k\end{bmatrix}=k\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$X=(-1,0,1)\ (1,-1,1)$ and $(1,2,1)$

---

① $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ ② $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ ③ $\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

cayley - Hemilton

every square matrix ~~sta~~ satisfy it's own chorectrstic equation.

$\Rightarrow$ here we find inverse of a matrix

① verify cayley - Hemilton theorem of matrix $\begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix}$ and find it's inverse.

A) $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 3-\lambda & -3\\ -2 & -4 & -4-\lambda\end{vmatrix}=0$$

$\Rightarrow 1-\lambda\left[(3-\lambda)(-4-\lambda)-12\right]-\left((-4-\lambda)-6\right)+3(-4+\cancel{}\ 6-2\lambda)$

$\Rightarrow 1-\lambda\left[-12-3\lambda+4\lambda+\lambda^2-12\right]+10+\lambda-12+18-6\lambda$

$\Rightarrow \left[-12+\lambda+\lambda^2-12+12\lambda-\lambda^2-\lambda^3+12\lambda-5\lambda+16\right]$

$\Rightarrow \left[-\lambda^3+20\lambda-8\right]=0$

$\lambda^3-20\lambda+8=0$

By cayley Hemilton theorem matrix A satisfies ~~stati~~ chorectastic equation $\lambda^3-20\lambda+8=0$.

That means $A^3-20A+8I=0$.

that means we prove that

$A^3 - 20A + 8I$ value is zero.

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A^2 \cdot A = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4-8+24 & -4-24+48 & -12+24+48 \\ 10+22-12 & 10+66-24 & 30-66-24 \\ 2+2-44 & 2+6-88 & 6-6-88 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & \cancel{22}\,-60 \\ -40 & -80 & -88 \end{bmatrix}$$

$A^3 - 20A + 8I$

$$\begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} - \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} = 0$$

$A^3 - 20A + 8I = 0$

Cayley-Hamilton theorem is proved.

Transpose matrix :-

Inter changeing ~~form~~ of rows and columns in given matrix A. It is denoted by A^T (or) A'.

ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

symetric matrix

~~If A^TA (or) $AA^T = I$~~ If $A^T = A$

when $A = A^T$ the given matrix A is symetric matrix

Skew symetric matrix

If $A^T = -A$

orthogonal matrix

$A^TA = I$ (or) $AA^T = I$.

conjugate matrix

The matrix uptain from any given matrix A, on replaceing it elements by the corresponding conjugate (x+jy) complex numbers is called the conjugate of A and it is denoted by $\bar{A}$. we read conjugate of A.

ex:- $A = \begin{bmatrix} 1+2i & -i & i-1 \\ 2 & -4 & 3i-2 \\ 6 & 7i & -4i \end{bmatrix}$

$i^2 = -1$

$z = x+jy$

$\bar{z} = \overline{x+iy}$

$= x-iy$

$\bar{A} = \begin{bmatrix} \overline{1+2i} & \overline{-i} & \overline{i-1} \\ \overline{2} & \overline{-4} & \overline{3i-2} \\ \overline{6} & \overline{7i} & \overline{-4i} \end{bmatrix}$

$z = x-iy$

$\bar{z} = \overline{x-iy}$

$= x+iy$

$\begin{bmatrix} 1-2i & i & -i-1 \\ 2 & -4 & -3i-2 \\ 6 & -7i & 4i \end{bmatrix}$

Transpose of conjugate matrix (or) conjugate of transpose matrix

$(\bar{A})^T$

let A is square matrix transpose of conjugate matrix of A is denoted by $(\bar{A})^T$

$\Rightarrow$ conjugate of transpose of matrix is $(\overline{A^T})$

$$(\bar{A})^T = (\overline{A^T})$$

It is denoted by A^θ (transpose of conjugate of matrix A)

$$(A)^\theta = (\bar{A})^T = (\overline{A^T})$$

Hermitian matrix

If $A^\theta = A$ it is Hermition matrix

Skew Hermition matrix

If $A^\theta = -A$

unitary matrix

If $A^\theta A = I$ (or) $AA^\theta = I$

① prove that $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

$$= \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^T A = I$

The given matrix is orthogonal matrix.

2) Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal?

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \qquad A^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$AA^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+1+1+1 & -1-1+1+1 & -1+1-1+1 & -1+1+1-1 \\ -1-1+1+1 & 1+1+1+1 & 1-1-1+1 & 1-1+1-1 \\ -1+1-1+1 & 1-1-1+1 & 1+1+1+1 & 1+1-1-1 \\ -1+1+1-1 & 1-1+1-1 & 1+1-1-1 & 1+1+1+1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{4}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= I$

$AA^T = I$. The given matrix is orthogonal.

③ show that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary?

A) $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} \bar{1} & \overline{1+i} \\ \overline{1-i} & \overline{-1} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

$(\bar{A})^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

Take $A^\theta \cdot A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1+(1+i)(1-i) & 1(1+i)+(1+i)(-1) \\ (1-i)1+(-1)(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1+1-\cancel{i}+\cancel{i}-i^2 & \cancel{1}+\cancel{i}-\cancel{1}-\cancel{i} \\ \cancel{1}-\cancel{i}-\cancel{1}+\cancel{i} & 1+\cancel{i}-\cancel{i}-i^2+1 \end{bmatrix}$,

$= \frac{1}{3}\begin{bmatrix} 2-(-1) & 0 \\ 0 & 2-(-1) \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

The given matrix is unitary.

④ P.T $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & \end{bmatrix}$ ... f homogeneous linear equation

① first we write give system of linear equation is
$AX=0$

② Now matrix A reduced to echolon form. Here we find rank of A that is $\rho(A)=r$.

③ here number of variables $= n$

④ number of solutions $= n-r$

Note
If $n-r=0$. Then system of linear equation have zero solutions.

① solve the systems of equations $x+y-3z+2w=0$, ~~$2x-y+2z$~~
$2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$

A) matrix form $AX=0$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-2R_1$, $R_3 \to R_3-3R_1$, $R_4 \to R_4+4R_1$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 3R_3-5R_2$, $R_4 \to 3R_4+5R_2$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & -5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to 2R_4-R_3$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & -21 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is Echelon form
Rank of $A = r = 4$, $n=4$.
number of variables :

solutions $= n-r$
$= 4-4 = 0$

$x+y-3z+2w=0$
$-3y+8z-9w=0$
$-16z+5w=0$
$-21w=0$

$w=0$, $-10z=0$, $z=0$, $-3y=0$, $y=0$, $x=0$

solutions is
$x=y=z=w=0$

Q) Solve completely system of equations $x+y-2z+3w=0$, $x-2y+z-w=0$, $4x+y-5z+8w=0$, $5x-7y+2z-w=0$! $(r=2)$
(solutions $4-2=2$, let $w=k_1$, $z=k_2$)

A)
$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_4$, $R_4 \to R_4 - 5R_1$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$r=2$, $n=4$

The number of solution $= n-r$
$= 4-2 = 2$

let $w=k_1$ and $z=k_2$

$x+y-2z+3w=0$
$-3y+3z-4w=0$

$$-3y+3z=4k_1$$

$$-3y=4k_1-3k_2$$

$$y=\frac{4k_1-3k_2}{-3}=-\frac{4}{3}k_1+k_2$$

$$x+y-2z+3w=0$$

$\frac{9}{3}+\frac{4}{3}$

$$x-\frac{4}{3}k_1+k_2-2k_2+3k_1=0$$

$$x-k_2+\frac{5}{3}k_1=0$$

$$x=-\frac{5}{3}k_1+k_2$$

$x=-\frac{5}{3}k_1+k_2$, $y=-\frac{4}{3}k_1+k_2$, $z=k_2$, $w=k_1$

(3) $x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z$

($r=2$, $n=3$, solutions $=1$, let $z=k$)

$$\begin{bmatrix}1&3&2\\2&-1&3\\3&-5&4\\1&17&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_2\to R_2-2R_1$, $R_3\to R_3-3R_1$, $R_4\to R_4-R_1$

$$\begin{bmatrix}1&3&2\\0&-7&-1\\0&-14&-2\\0&14&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_3\to R_3-2R_2$, $R_4\to R_4+2R_2$

$$\begin{bmatrix}1&3&2\\0&-7&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

here $r=2$ and $n=3$.

number of solutions $=n-r=3-2=1$

$z=k$

$$x+3y+2z=0$$

$$-7y-z=0$$

$$-7y-k=0$$

$$-7y=k \implies y=\frac{k}{-7}$$

$x + 3y + 2z = 0$

$x + 3\left(-\frac{k}{7}\right) + 2k = 0$

$x - \frac{3k}{7} + 2k = 0 \Rightarrow x = \frac{3k - 14k}{7} = \frac{-11k}{7}$

The solution is

$$x = \begin{bmatrix} -\frac{11}{7}k \\ -k/7 \\ k \end{bmatrix} = k \begin{bmatrix} -11/7 \\ 1/7 \\ 1 \end{bmatrix}.$$

Note

system of linear equations have non-zero solutions then $|A| = 0$.

Determine the values of λ for which the following set

~~note~~ -

every square matrix can be expressed as sum of symmetric matrix and skew-symmetric.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$P \qquad + \qquad q$$

here $p = \frac{1}{2}(A + A^T)$ and $q = \frac{1}{2}(A - A^T)$ here p is symmetric and q = skew symmetric that is we prove that $p^T = p$ $q^T = -q$.

2) every square matrix can be expressed as sum of hermition and skew-hermition.

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$$P \qquad + \qquad q$$

here $P = \frac{1}{2}(A + A^{\theta})$ and $q = \frac{1}{2}(A - A^{\theta})$ here p is hermition ~~symmetric~~ and q = skew hermition ~~symmetric~~ that is we prove that $A^{\theta} = A$ and $A^{\theta} = -A$.

Q) express the matrix

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$$

sum of hermition and skew hermition matrix?

A) matrix ~~A~~ $= p + q$, p is hermition and q is skew hermition matrix

here $p = \frac{1}{2}(A + A^{\theta})$, $q = \frac{1}{2}(A - A^{\theta})$

Now we ~~find~~ prove $p^{\theta} = p$ and $q^{\theta} = -q$

$$\bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix} \qquad (\bar{A})^T = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix} = A^{\theta}$$

$$A + A^{\theta} = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \overset{= A^{\theta}}{\begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}}$$

~~2 2-2i~~

$$= \begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix} = 2\begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$$P = \frac{1}{2}(A+A^\theta) = \frac{2}{2}\begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 7 \end{bmatrix} \quad (\bar{P})^T = P^\theta = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$P^\theta = P$ then P is hermition matrix

$$A - A^\theta = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} - \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 2+2i & 6-4i \\ 2i-2 & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix} = 2\begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

$$q = \frac{1}{2}(A-A^\theta) = \frac{2}{2}\begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix} = \begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

$$\bar{q} = \begin{bmatrix} -i & 1-i & 3+2i \\ -i-1 & -i & 4-i \\ -3+2i & -4-i & 0 \end{bmatrix} \quad (\bar{q})^T = \begin{bmatrix} -i & -i-1 & -3+2i \\ 1-i & -i & -4-i \\ 3+2i & 4-i & 0 \end{bmatrix}$$

$$q^\theta = -\begin{bmatrix} i & i+1 & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

$q^\theta = -q$

Then q is skew hermition matrix.

A = P + Q - Hermition + skew hermition matrix.

(9) express the matrix A as a sum of symmetric and skew symmetric.

where $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

A) $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$

Take $P = \frac{1}{2}\left[A + A^T\right] = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$

$P^T = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$

$P = P^T$

P is symetric matrix.

$A - A^T = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$

$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$

$Q^T = \frac{1}{2}\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$

$q^T = -q$

q is skew symetric matrix

$A = P + q$

⇒ A is the sum of symetric and skew symetric matrix.

Note

matrix A is square matrix $|A| = 0$ then matrix A is sigular matrix. If $|A| \neq 0$ then matrix A is called non-sigular matrix.

⇒ Non-sigular matrix express inverse matrix.