Integrating Factor

Def: Integrating factor: If an equation of the form $\boxed{Mdx + Ndy = 0}$ —(1) is not an exact, it can always be made exact by multiplying ~~by~~ some function of x and y. Such a multiplier is called an "Integrating factor".

Note:

* There is no general method for finding I.F.
* There ~~are~~ exists infinite integrating factors of eq(1).

Group of terms	I.F	Exact D.E.
1. $xdy - ydx$	i) $\frac{1}{x^2}$	$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
	ii) $\frac{1}{y^2}$	$-d\left(\frac{x}{y}\right) = d\left(\frac{-x}{y}\right) = \frac{-ydx + xdy}{y^2}$
	iii) $\frac{1}{xy}$	$d(\log(y/x)) = \frac{1}{(y/x)}\left[\frac{xdy - ydx}{x^2}\right] = \frac{x}{y}\left[\frac{xdy - ydx}{x^2}\right] = \left[\frac{xdy - ydx}{xy}\right]$
	iv) $\frac{1}{x^2+y^2}$	$d(\tan^{-1}(y/x)) = \frac{1}{1+(y/x)^2}\left[\frac{xdy - ydx}{x^2}\right] = \frac{x^2}{x^2+y^2}\left[\frac{xdy - ydx}{x^2}\right] = \frac{xdy - ydx}{x^2+y^2}$
2. $xdy + ydx$	$\rightarrow \frac{1}{(xy)^n}$	$= \begin{cases} d(\log(xy)) & \text{if } n = 1 \\ d\left(\frac{1}{(1-n)(xy)^{n-1}}\right) & \text{if } n \neq 1 \end{cases}$ for $n=1$: $d(\log(xy)) = \frac{1}{xy}(xdy + ydx) = \frac{xdy + ydx}{xy}$
3. $xdx + ydy$	$\frac{1}{(x^2+y^2)^n}$	$= \begin{cases} \frac{1}{2} d[\log(x^2+y^2)] & \text{if } n = 1 \\ \frac{1}{2} d\left[\frac{1}{(1-n)(x^2+y^2)^{n-1}}\right] & \text{if } n \neq 1 \end{cases}$

$\frac{1}{2} d(\log(x^2+y^2)) = \frac{1}{2}\left[\frac{1}{x^2+y^2}\right](2xdx + 2ydy)$

$= \frac{xdx + ydy}{x^2+y^2}$

Useful Results:

* $d(xy) = x\,dy + y\,dx$
* $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2y^2x\,dx}{x^4}$
* $d\left(\frac{e^x}{y}\right) = \frac{y e^x.dx - e^x dy}{y^2}$
* $d(\log(xy)) = \frac{x\,dy + y\,dx}{xy}$
* $d(y^2/x) = \frac{2xy\,dy - y^2\,dx}{x^2}$
* $d(y/x^2) = \frac{x^2dy - 2xy\,dx}{x^4}$

* Problems

1) Solve $y\,dx - x\,dy = 0$

S) Given $y\,dx - x\,dy = 0 \longrightarrow ①$

$M = y, \quad N = -x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ ① is not an exact D.E

if we multiply ① by $\frac{1}{y^2}$, we get

$① \Rightarrow \frac{y\,dx - x\,dy}{y^2} = 0$

$\Rightarrow d\left(\frac{x}{y}\right) = 0$

$\Rightarrow \int d\left(\frac{x}{y}\right) = \int 0$

$\Rightarrow \frac{x}{y} = c$

$\Rightarrow \boxed{x = yc}$

2) Solve $x\,dy - (y - x)dx = 0$

$x\,dy - y\,dx + x\,dx = 0$

$\Rightarrow$ ~~Dividing with x^2 on both sides~~ (or)

multiplying with $\frac{1}{x^2}$ on both sides

$\Rightarrow \frac{x\,dy - y\,dx + x\,dx}{x^2} = 0$

$\Rightarrow \frac{x\,dy - y\,dx}{x^2} + \frac{1}{x}dx = 0$

$\Rightarrow d(y/x) + \frac{1}{x}dx = 0$

$\Rightarrow \int d(y/x) + \int \frac{1}{x}dx = \int 0$

$\Rightarrow \boxed{\frac{y}{x} + \log x = c}$

③ $(x^2+y^2-2y)dy = 2x\,dx$

S) $(x^2+y^2-2y)\,dy = 2x\,dx$

$(x^2+y^2)dy - 2y\,dy = 2x\,dx$

$\Rightarrow (x^2+y^2)dy = 2x\,dx + 2y\,dy$

$\Rightarrow dy = \frac{2x\,dx + 2y\,dy}{x^2+y^2}$

$\Rightarrow dy = d(\log(x^2+y^2))$

$\Rightarrow \int dy = \int d(\log(x^2+y^2))$

$\Rightarrow \boxed{y = \log(x^2+y^2) + c}$

④ Solve $x\,dx + y\,dy + a^2\frac{y\,dx - x\,dy}{x^2+y^2} = 0$

S) $x\,dx + y\,dy + a^2\frac{(y\,dx - x\,dy)}{x^2+y^2} = 0$

$\Rightarrow$ $x\,dx + y\,dy$ ~~$d(xy)$~~ $+ a^2\, d(\tan^{-1}(x/y)) = 0$

$\Rightarrow$ $\int x\,dx + \int y\,dy$ ~~$\int d(xy)$~~ $+ a^2 \int d(\tan^{-1}(x/y)) = \int 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + a^2(\tan^{-1}(x/y)) = c$

$\boxed{\frac{x^2}{2} + \frac{y^2}{2} + a^2\tan^{-1}(x/y) = c}$

(5) Solve $y(2xy+e^x)dx = e^x dy$

(S) $y(2xy+e^x)dx = e^x dy$

$y(2xy\,dx + e^x dx) = e^x dy$

$2xy^2 dx + e^x y\,dx - e^x dy = 0$

multiplying with $\frac{1}{y^2}$ on both sides,

$2x\,dx + \frac{e^x y\,dx - e^x dy}{y^2} = 0$

$2x\,dx + d(e^x/y) = 0$

$\int 2x\,dx + \int d(e^x/y) = \int 0$

$\Rightarrow \boxed{x^2 + \frac{e^x}{y} = c}$

(6) Solve $(y-3x^2)dx - x(1-xy^2)dy = 0$

(S) $(y-3x^2)dx - x(1-xy^2)dy = 0$

$y\,dx - 3x^2 dx - x\,dy + x^2y^2 dy = 0$

$y\,dx - x\,dy + x^2y^2 dy - 3x^2 dx = 0$

$\Rightarrow x\,dy - y\,dx - x^2y^2 dy + 3x^2 dx = 0$

multiplying with $\frac{1}{x^2}$ on both sides

$\frac{x\,dy - y\,dx}{x^2} - y^2 dy + 3dx = 0$

$\int d(y/x) - \int y^2 dy + 3\int dx = \int 0$

$\boxed{\frac{y}{x} - \frac{y^3}{3} + 3x = c}$

(7) Solve $y\,dx - x\,dy + \log x\,dx = 0$

$y\,dx - x\,dy + \log x\,dx = 0$

$x\,dy - y\,dx - \log x\,dx = 0$

multiplying with $\frac{1}{x^2}$ on both sides

$\frac{x\,dy - y\,dx}{x^2} - \frac{\log x}{x^2}dx = 0$

$\int d(y/x) - \int \frac{\log x}{x^2}dx = \int 0$

$\int \underset{u}{\log x}\cdot \underset{v}{\frac{1}{x^2}}\,dx$ $\quad \boxed{\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx}$

$= \log x \cdot \int \frac{1}{x^2}dx - \int\left[\frac{d}{dx}(\log x)\int \frac{1}{x^2}dx\right]dx$

$= \log x\cdot\left(-\frac{1}{x}\right) - \int \frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx$

$= -\frac{\log x}{x} + \int x^{-2}dx$

$= -\frac{\log x}{x} - \frac{1}{x}$

Now $\int d(y/x) - \int \log x\cdot\frac{1}{x^2}dx = \int 0$

$\Rightarrow \frac{y}{x} - \left[\frac{\log x}{x} - \frac{1}{x}\right] = c$

$\Rightarrow \frac{y}{x} - \frac{\log x}{x} + \frac{1}{x} = c$

$\Rightarrow \boxed{y - \log x + 1 = cx}$

(8) Solve $x\,dy = [y + x\cos^2(y/x)]dx$

$x\,dy = y\,dx + x\cos^2(y/x)dx$

$x\,dy - y\,dx = x\cos^2(y/x)dx$

multiplying with $\frac{1}{x^2}$ on both sides

$\frac{x\,dy - y\,dx}{x^2} = \frac{x\cos^2(y/x)dx}{x^2}$

$\Rightarrow d(y/x) = \frac{\cos^2(y/x)}{x}dx$

$\Rightarrow \int \sec^2(y/x)\cdot d(y/x) = \int \frac{1}{x}dx$

$\Rightarrow \boxed{\tan(y/x) = \log x + c}$

(9) $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

$\Rightarrow y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

multiplying with $\frac{1}{y^2}$ on both sides

$\frac{y\,dx - x\,dy}{y^2} + 3x^2 e^{x^3}dx = 0$

$d(x/y) + 3x^2e^{x^3}dx = 0$

$\int d(x/y) + \int 3x^2\cdot e^{x^3}dx = \int 0$

$\boxed{\frac{x}{y} + e^{x^3} = c}$

10 Solve $y(2x^2y+e^x)dx-(e^x+y^3)dy=0$

5 $y(2x^2y+e^x)\,dx-(e^x+y^3)\,dy=0$

$\Rightarrow\quad 2x^2y^2\,dx+ye^x\,dx-e^x\,dy-y^3\,dy=0$

multiplying with $\frac{1}{y^2}$ on both sides

$$\frac{2x^2y^2\,dx}{y^2}+\frac{ye^x\,dx-e^x\,dy}{y^2}-\frac{y^3}{y^2}\,dy=0$$

$\Rightarrow\quad 2x^2\,dx+d\left(e^x/y\right)-y\,dy=0$

$\Rightarrow\quad \int 2x^2\,dx+\int d\left(e^x/y\right)-\int y\,dy=\int 0$

$\Rightarrow\quad \boxed{\frac{2x^3}{3}+\frac{e^x}{y}-\frac{y^2}{2}=C}$