Bernoulli's D.E

Definition (Bernoulli's DE form)

An equation of the form

$$\boxed{\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n} \longrightarrow (*)$$

where $P(x)$ and $Q(x)$ are function of x only and 'n' is a constant such that $n \neq 0$ and $n \neq 1$.

*) If $n=0$, the Bernoulli's DE becomes

$$(*) \Rightarrow \frac{dy}{dx} + P(x)\cdot y = Q(x) \longrightarrow \text{which is Linear D.E.}$$

*) If $n=1$, the Bernoulli's DE becomes

$$(*) \Rightarrow \frac{dy}{dx} + P(x)\cdot y = Q(x)\, y$$

$$\Rightarrow \frac{dy}{dx} + [P(x) - Q(x)]\, y = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} + R(x)\, y = 0} \quad \text{where } \boxed{R(x) = P(x) - Q(x)}$$

which is also Linear D.E.

Solution of Bernoulli's D.E

Bernoulli's DE is $\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n \longrightarrow (1)$

Dividing (1) by y^n, we get

$$(1) \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = Q(x)$$

$$\Rightarrow \boxed{y^{-n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = Q(x)} \longrightarrow (2)$$

Let $\boxed{y^{1-n} = z}$

$$(1-n)\, y^{1-n-1}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow (1-n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \boxed{y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dz}{dx}} \longrightarrow (3)$$

Substituting y^{-n} and (3) in (1), we get

$$(2) \Rightarrow \frac{1}{1-n}\frac{dz}{dx} + P(x)\,z = Q(x)$$

$$\Rightarrow \boxed{\frac{dz}{dx} + (1-n)\,P(x)\,z = Q(x)\,(1-n).} \longrightarrow (4)$$

clearly (4) is Linear D.E.

$$\therefore \boxed{I.F = e^{\int (1-n)P(x)\,dx}}$$ and

The general solution of (3) is

$$z\,(I.F) = \int Q(x)(1-n).(I.F)\,dx + c$$

$$\Rightarrow \boxed{z.(I.F) = (1-n)\int Q(x)(I.F)\,dx + c} \longrightarrow (5)$$

* Replacing z by y^{1-n} in eq (5) we get the required general solution of (1).

(1) Solve $\frac{dy}{dx} + y\tan x = \sec x\, y^3$

(S) Given $\frac{dy}{dx} + y\tan x = \sec x . y^3 \longrightarrow (1)$

Dividing (1) with y^3 on both sides

$$(1) \Rightarrow y^{-3}\frac{dy}{dx} + y^{-2}\tan x = \sec x \longrightarrow (2)$$

Let $\boxed{y^{-2} = z}$

$$\Rightarrow -2y^{-3}.\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \boxed{y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx}} \longrightarrow (3)$$

substitute z & (3) in eq (2), we get

$$(2) \Rightarrow -\frac{1}{2}\frac{dz}{dx} + z\tan x = \sec x$$

$\Rightarrow \quad -\frac{1}{2}\frac{dz}{dx} + \tan x . z = \sec x$

$\Rightarrow \quad -\frac{dz}{dx} + 2\tan x . z = 2\sec x$

$\Rightarrow \quad \boxed{\frac{dz}{dx} - 2\tan x . z = -2\sec x} \longrightarrow (4)$

clearly (4) is linear D.E

$\therefore \quad I.F = e^{\int -2\tan x \, dx} = e^{-2\log \sec x} = e^{\log \cos^2 x}$

$\Rightarrow \quad \boxed{I.F = \cos^2 x}$

$\therefore$ The general solution of (4) is

$z.(I.F) = -2\int \sec x.(I.F)\, dx + c$

$z.(\cos^2 x) = -2\int \sec x . \cos^2 x \, dx + c$

$\Rightarrow \quad z.\cos^2 x = -2\int \cos x \, dx + c$

$\Rightarrow \quad \boxed{z.\cos^2 x = -2\sin x + c} \longrightarrow (5)$

$\Rightarrow$ Replacing z by y^{-2} in (5), we get

$(5) \Rightarrow \quad y^{-2}\cos^2 x = -2\sin x + c$

$\Rightarrow \quad \boxed{\frac{\cos^2 x}{y^2} = -2\sin x + c} \Rightarrow$ Required G.S of eq (1)

$\cancel{\cos^2 x} =$

(2) Solve $x\frac{dy}{dx} + y = xy^3$

(S) $\quad x\frac{dy}{dx} + y = xy^3 \longrightarrow (1)$

Dividing (1) with x on both sides [which is Bernoulli's D.E]

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = y^3 \longrightarrow (1)$

Dividing (1) with y^3 on both sides.

$(1) \Rightarrow \quad y^{-3}\frac{dy}{dx} + \frac{1}{x}.y^{-2} = 1 \longrightarrow (2)$

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Let $\boxed{y^{-2} = z}$

$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \boxed{y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}} \longrightarrow (3)$

substitute y^{-2} & (3) in (2), we get

(2) $\Rightarrow -\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$

$\Rightarrow -\frac{dz}{dx} + \frac{2}{x}z = 2$

$\Rightarrow \boxed{\frac{dz}{dx} - \frac{2}{x}z = -2} \longrightarrow (4)$

Clearly (4) is L.D.E,

$I.F = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

$\Rightarrow \boxed{I.F = \frac{1}{x^2}}$

∴ The general solution of (4) is

$z(I.F) = \int Q\,(I.F)\,dx + C$

$\Rightarrow z\cdot\left(\frac{1}{x^2}\right) = \int(-2)\cdot\left(\frac{1}{x^2}\right)dx + c$

$\Rightarrow z\left(\frac{1}{x^2}\right) = -2\cdot\frac{x^{-2+1}}{-2+1} + c$

$\Rightarrow \frac{z}{x^2} = -2\cdot\frac{x^{-1}}{-1} + c$

$\Rightarrow \boxed{\frac{z}{x^2} = \frac{2}{x} + c} \longrightarrow (5)$

*) Replacing $z = y^{-2}$ in (5),

(5) $\Rightarrow \frac{y^{-2}}{x^2} = \frac{2}{x} + c$

$\Rightarrow \boxed{\frac{1}{x^2y^2} = \frac{2}{x} + c}$

Required G.S of (1)

(3) Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

(S) $\frac{dy}{dx} = e^{2x}\cdot e^{-y} - e^x$

$\Rightarrow \frac{dy}{dx} + e^x = e^{2x}\cdot e^{-y} \longrightarrow (1)$

dividing (1) with e^{-y} on both sides

$e^y\frac{dy}{dx} + e^x\cdot e^y = e^{2x} \longrightarrow (2)$

Let $\boxed{e^y = z}$

$\Rightarrow \boxed{e^y\frac{dy}{dx} = \frac{dz}{dx}} \longrightarrow (3)$

substitute e^y & (3) in (2).

(2) $\Rightarrow \boxed{\frac{dz}{dx} + e^x z = e^{2x}} \longrightarrow (4)$

clearly (4) is L.D.E

$I.F = e^{\int e^x dx} = e^{e^x}$

$\boxed{I.F = e^{e^x}}$

The General solution of (4) is

$z\cdot(I.F) = \int Q(x)\cdot(I.F)\,dx + c$

$\Rightarrow z\cdot e^{e^x} = \int e^{2x}\cdot e^{e^x}\cdot dx + c$

$= \int e^x\cdot e^x\cdot e^{e^x}dx + c$

$e^{e^x} = t$

$e^x\cdot e^{e^x} = \frac{dt}{dx}$

$\boxed{e^x\cdot e^{e^x}dx = dt}$

$= \int \ln t\cdot dt + c$

$= t[\ln t - 1] + c$

$= e^{e^x}[\ln e^{e^x} - 1] + c$ ∵ $\int t^n \ln t\,dt = \frac{t^{n+1}}{n+1}\left[\ln t - \frac{1}{n+1}\right]$

$z e^{e^x} = e^{e^x}[e^x - 1] + c$

$\Rightarrow \boxed{z\cdot e^{e^x} = e^{e^x}[e^x - 1] + c} \longrightarrow (5)$

*) Replacing $z = e^y$ in (5)

(5) $\Rightarrow e^y\cdot e^{e^x} = e^{e^x}[e^x - 1] + c$

$\Rightarrow \boxed{e^{e^x}(e^y - e^x + 1) = c}$

Required G.S of (1)

(4) Solve $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$

S: Given $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2 \longrightarrow (1)$

Dividing with $y(\log y)^2$ on both sides of (1)

$(1) \Rightarrow y^{-1}(\log y)^{-2}\frac{dy}{dx}+\frac{y}{x}\frac{(\log y)^{-1}}{y}=\frac{1}{x^2} \longrightarrow (2)$

put $\boxed{(\log y)^{-1}=z}$

$-(\log y)^{-2}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$

$\Rightarrow -y^{-1}(\log y)^{-2}\frac{dy}{dx}=\frac{dz}{dx}$

$\Rightarrow \boxed{y^{-1}(\log y)^{-2}\cdot\frac{dy}{dx}=-\frac{dz}{dx}} \longrightarrow (3)$

substituting (3) & $(\log y)^{-1}$ in (2),

$(2) \Rightarrow -\frac{dz}{dx}+\frac{1}{x}z=\frac{1}{x^2}$

$\Rightarrow \boxed{\frac{dz}{dx}-\frac{1}{x}z=\frac{-1}{x^2}} \longrightarrow (4)$

Clearly (4) is L.D.E

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\boxed{I.F=\frac{1}{x}}$

The general solution of (4) is

$z(I.F)=\int Q(x)(I.F)\,dx+c$

$\Rightarrow z\left(\frac{1}{x}\right)=\int\left(\frac{-1}{x^2}\right)\cdot\left(\frac{1}{x}\right)dx+c$

$\Rightarrow \frac{z}{x}=-\int x^{-3}dx+c$

$\Rightarrow \frac{z}{x}=-\frac{x^{-3+1}}{-3+1}+c$

$\Rightarrow \boxed{\frac{z}{x}=\frac{1}{2x^2}+c} \longrightarrow (5)$

* replacing $z=(\log y)^{-1}$ in (5)

$(5) \Rightarrow \frac{(\log y)^{-1}}{x}=\frac{1}{2x^2}+c$

$\Rightarrow \boxed{\frac{1}{x\log y}=\frac{1}{2x^2}+c}$

required G.S of (1)

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⑤ Solve $x\frac{dy}{dx}+y=y^2x^3\cos x$.

⑤ Given $x\frac{dy}{dx}+y=y^2x^3\cos x \longrightarrow ①$

Dividing ① with xy^2 on both sides,

$$① \Rightarrow \quad y^{-2}\frac{dy}{dx}+\frac{1}{x}\cdot y^{-1}=x^2\cos x \longrightarrow ②$$

put $y^{-1}=z$

$$-y^{-2}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \boxed{y^{-2}\frac{dy}{dx}=-\frac{dz}{dx}} \longrightarrow ③$$

substitute ③ in ②,

$$② \Rightarrow \quad -\frac{dz}{dx}+\frac{1}{x}\cdot z=x^2\cos x$$

$$\Rightarrow \boxed{\frac{dz}{dx}-\frac{1}{x}z=-x^2\cos x} \longrightarrow ④$$

Clearly ④ is L.D.E

$$I\cdot F=e^{\int -\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x} \Rightarrow \boxed{I\cdot F=\frac{1}{x}}$$

General solution of ④ is

$$z\cdot\left(\frac{1}{x}\right)=\int -x^2\cos x\cdot\left(\frac{1}{x}\right)dx+c$$

$$\Rightarrow \frac{z}{x}=-\int x\cos x\,dx+c \qquad \left[\because \int uv\,dx=u\int v\,dx-\int\left[\frac{du}{dx}\int v\,dx\right]dx\right.$$

Here $u=x,\ v=\cos x$

$$=-\left[x\sin x-\int \sin x\,dx\right]+c$$

$$\boxed{\frac{z}{x}=-\left[x\sin x+\cos x\right]+c} \longrightarrow ⑤$$

substitute $z=y^{-1}$ in ⑤,

$$⑤ \Rightarrow \quad \frac{y^{-1}}{x}=-\left[x\sin x+\cos x\right]+c$$

$$\boxed{\frac{1}{xy}+x\sin x+\cos x=c}$$

Additionals

1) Solve $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$, $y(1)=1$ $\Rightarrow$ Can be solve by converting into $Mdx+Ndy=0$ & taking $I.F = \frac{1}{x^2}$

S) $\frac{dy}{dx} - \frac{1}{2x}y = \frac{x^2+1}{2x} \cdot y^{-1} \rightarrow (1)$

$y\frac{dy}{dx} - \frac{1}{2x}y^2 = \frac{x^2+1}{2x} \rightarrow (2)$

$\boxed{y^2 = z} \Rightarrow \boxed{y \cdot \frac{dy}{dz} = \frac{1}{2}\frac{dz}{dx}} \rightarrow (3)$

$\boxed{\frac{dz}{dx} - \frac{1}{x}z = \frac{x^2+1}{x}} \rightarrow (4)$ $\boxed{I.f = \frac{1}{x}}$

G.S: $\boxed{z \cdot \frac{1}{x} = x - \frac{1}{x} + c} \rightarrow (5)$

G.S of (1) is $\boxed{y^2 = x^2 - 1 + cx} \rightarrow (6)$

Given $y(1)=1$ (6) $\Rightarrow$ $\boxed{1=c}$

$\therefore \boxed{y^2 = x^2 - 1 + x = x(x+1) - 1}$

2) Solve $\frac{dy}{dx} = 2y\tan x + y^2\tan^2 x$

S) $\frac{dy}{dx} - 2\tan x\, y = \tan^2 x \cdot y^2 \rightarrow (1)$

$y^{-2}\frac{dy}{dx} - 2\tan x\, y^{-1} = \tan^2 x \rightarrow (2)$

$\boxed{y^{-1} = z} \Rightarrow \boxed{y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}} \rightarrow (3)$

$\boxed{\frac{dz}{dx} + 2\tan x \cdot z = -\tan^2 x} \rightarrow (4)$

$\boxed{I.f = \sec^2 x}$

G.S of (4) is $\boxed{z\sec^2 x = \frac{-\tan^3 x}{3} + c} \rightarrow (5)$

G.S of (1) is $\boxed{\frac{\sec^2 x}{y} = \frac{-\tan^3 x}{3} + c}$

③ Solve $yy' + xy^2 = x$

Ⓢ

$$y \cdot \frac{dy}{dx} + xy^2 = x \longrightarrow ①$$

$\boxed{y^2 = z}$ $\boxed{y\frac{dy}{dx} = \frac{1}{2}\frac{dz}{dx}} \longrightarrow ②$

① ⟹

$$\boxed{\frac{dz}{dx} + 2xz = x} \longrightarrow ③$$

$\boxed{I.F = e^{x^2}}$ G.S of ③ is $\boxed{z \cdot e^{x^2} = e^{x^2} + c} \longrightarrow ④$

∴ G.S of ① is $\boxed{y^2 = 1 + Ce^{-x^2}}$

④ Solve $2xy' = 10x^3y^5 + y$

Ⓢ

$$2x\frac{dy}{dx} = 10x^3y^5 + y \Rightarrow \boxed{y^{-5}\frac{dy}{dx} - \frac{1}{2x} \cdot y^{-4} = 5x^2} \longrightarrow ①$$

$\boxed{y^{-4} = z} \Rightarrow \boxed{y^{-5}\frac{dy}{dx} = -\frac{1}{4}\frac{dz}{dx}} \longrightarrow ②$

① ⟹ $\boxed{\frac{dz}{dx} + \frac{2}{x} \cdot z = -20 \cdot x^2} \longrightarrow ③$

$\boxed{IF = x^2}$ G.S of ③ is $\boxed{z x^2 = -4x^5 + c} \longrightarrow ④$

∴ G.S of ① is $\boxed{\frac{x^2}{C - 4x^5} = y^4}$

⑤ Solve $xy' + y = y^2$

Ⓢ

$$x\frac{dy}{dx} + y = y^2$$

$\rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{1}{x} \longrightarrow ①$

$\rightarrow \boxed{y^{-1} = z}$ & $\boxed{y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}} \longrightarrow ②$

$\rightarrow \boxed{\frac{dz}{dx} - \frac{1}{x}z = \frac{-1}{x}} \longrightarrow ③$

$\rightarrow \boxed{I.f = \frac{1}{x}}$

$\rightarrow$ G.S of ③ is $\boxed{V = 1 + Cx} \longrightarrow ④$

∴ G.S of ① is $\boxed{y = \frac{1}{Cx+1}}$