hw1

1.（1）对 $\forall(x_{11},x_{21}),(x_{12},x_{22})\in S$，取$(x_{13},x_{23})$满足$x_{13}=\alpha x_{11}+(1-\alpha)x_{12},x_{23}=\alpha x_{21}+(1-\alpha)x_{22},\forall\alpha\in[0,1]$

由于$(x_{11},x_{21}),(x_{12},x_{22})\in S$，有$x_{11}+2x_{21}\geq 1,x_{11}-x_{21}\geq 1$与$x_{12}+2x_{22}\geq 1,x_{12}-x_{22}\geq 1$，故

$$\begin{aligned}x_{13}+2x_{23}&=\alpha x_{11}+(1-\alpha)x_{12}+2\left[\alpha x_{21}+(1-\alpha)x_{22}\right]\\&=\alpha(x_{11}+2x_{21})+(1-\alpha)(x_{12}+2x_{22})\\&\geq\alpha+(1-\alpha)\\&=1\end{aligned}$$

$$\begin{aligned}x_{13}-x_{23}&=\alpha x_{11}+(1-\alpha)x_{12}-\left[\alpha x_{21}+(1-\alpha)x_{22}\right]\\&=\alpha(x_{11}-x_{21})+(1-\alpha)(x_{12}-x_{22})\\&\geq\alpha+(1-\alpha)\\&=1\end{aligned}$$

所以$(x_{13},x_{23})\in S$，S是凸集。

（2）对 $\forall(x_{11},x_{21}),(x_{12},x_{22})\in S$，取$(x_{13},x_{23})$满足$x_{13}=\alpha x_{11}+(1-\alpha)x_{12},x_{23}=\alpha x_{21}+(1-\alpha)x_{22},\forall\alpha\in[0,1]$

由于$(x_{11},x_{21}),(x_{12},x_{22})\in S$，有$x_{21}\geq|x_{11}|$与$x_{22}\geq|x_{12}|$，故

$$\begin{aligned}x_{23}&=\alpha x_{21}+(1-\alpha)x_{22}\\&\geq\alpha|x_{11}|+(1-\alpha)|x_{12}|\\&\geq|\alpha x_{11}+(1-\alpha)x_{12}|\\&=|x_{13}|\end{aligned}$$

所以$(x_{13},x_{23})\in S$，S是凸集。

2．对$\forall x_1,x_2\in S$，令$x_3=\alpha x_1+(1-\alpha)x_2,\forall\alpha$。

由于$x_1,x_2\in S$，$\exists y_1,y_2,x_1=Ay_1,x_2=Ay_2,y_1\geq 0,y_2\geq 0$。令$y_3=\alpha y_1+(1-\alpha)y_2$，于是$y_3\geq 0$且$Ay_3=A\left[\alpha y_1+(1-\alpha)y_2\right]=\alpha Ay_1+(1-\alpha)Ay_2=\alpha x_1+(1-\alpha)x_2=x_3$ 。所以$x_3\in S$，S是凸集。

3．当$k=2$时，由凸集的定义知$\lambda_1x^{(1)}+\lambda_2x^{(2)}\in S$显然成立。

假设对于$k=m-1$，$\sum_{i=1}^{k}\lambda_ix^{(i)}\in S$成立。对于$k=m$，有

$$\sum_{i=1}^{m}\lambda_ix^{(i)}=\sum_{i=1}^{m-1}\lambda_ix^{(i)}+\lambda_mx^{(m)}$$

若$\lambda_m=1$，$\sum_{i=1}^{m}\lambda_ix^{(i)}=x^{(m)}\in S$。

若$0\leq\lambda_m<1$,

$$\begin{aligned}\sum_{i=1}^{m}\lambda_ix^{(i)}&=\sum_{i=1}^{m-1}\lambda_ix^{(i)}+\lambda_mx^{(m)}\\&=(1-\lambda_m)\sum_{i=1}^{m-1}\frac{\lambda_i}{1-\lambda_m}x^{(i)}+\lambda_mx^{(m)}\end{aligned}$$

由于$\sum_{i=0}^{m-1}\lambda_i=1-\lambda_m$，$\sum_{i=1}^{m-1}\frac{\lambda_i}{1-\lambda_m}=1$，由假设可知，$\sum_{i=1}^{m-1}\frac{\lambda_i}{1-\lambda_m}x^{(i)}\in S$。又因为$(1-\lambda_m)+\lambda_m=1$，根据凸集定义，$\sum_{i=1}^{m}\lambda_ix^{(i)}\in S$。

证毕。