Exercise 3

2.20

令$W = X - Y$,

$$F_W(w) = \begin{cases} 0 & w < -1 \\ \frac{1}{2}(1+w)^2 & -1 \le w \le 0 \\ 1 - \frac{1}{2}(1-w)^2 & 0 < w \le 1 \\ 1 & w > 1 \end{cases}$$

$$f_W(w) = \begin{cases} 0 & w < -1 \\ 1+w & -1 \le w \le 0 \\ 1-w & 0 < w \le 1 \\ 0 & w > 1 \end{cases}$$

令$V = X/Y$

$$F_V(v) = \begin{cases} 0 & v \le 0 \\ \frac{v}{2} & 0 < v \le 1 \\ 1 - \frac{1}{2v} & v > 1 \end{cases}$$

$$f_V(v) = \begin{cases} 0 & v \le 0 \\ \frac{1}{2} & 0 < v \le 1 \\ \frac{1}{2v^2} & v > 1 \end{cases}$$

3.3

$$F_{Y_n}(y) = \begin{cases} 0 & x < 0 \\ y^n & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$f_{Y_n}(y) = \begin{cases} 0 & x < 0 \\ ny^{n-1} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$\mathbb{E}(Y_n) = \int_0^1 y f_{Y_n}(y) dy = \frac{n}{n+1}$$

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n = 1:10000;
X = randn(length(n), 1)';
X_n = cumsum(X) ./ n;
X_cauchy = tan(pi * (rand(length(n), 1) - 0.5))';
X_n_cauchy = cumsum(X_cauchy) ./ n;
plot(n,[X_n;X_n_cauchy])
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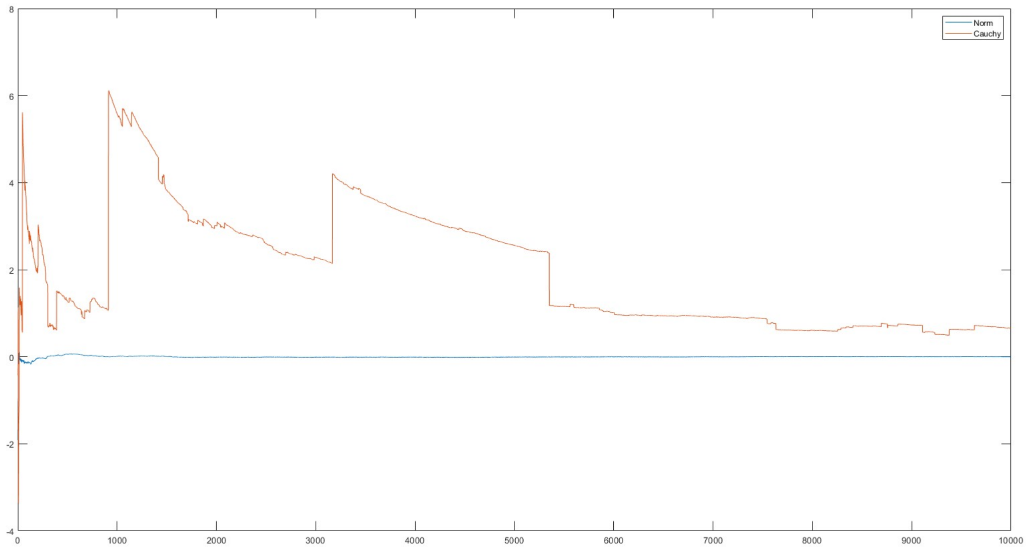

柯西分布的方差不收敛，所以中心极限定理不适用，其均值不收敛到期望。

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$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\begin{aligned}
\mathbb{E}(Y) &= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2x+1}{2}+\frac{1}{2}} dx \\
&= e^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx \\
&= e^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Y^2) &= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-4x+4}{2}+2} dx \\
&= e^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} dx \\
&= e^2
\end{aligned}$$

所以$\mathbb{V}(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y) = e^2 - e$

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(a)

$$\mathbb{E}(X) = \frac{1}{2}\int_0^1 x dx + \frac{1}{2}\int_3^4 x dx = 2$$

(b)

$$\mathbb{E}(X^2) = \frac{1}{2}\int_0^1 x^2 dx + \frac{1}{2}\int_3^4 x^2 dx = \frac{19}{3}$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{7}{3}$$

$$\mathrm{std}(X) = \frac{\sqrt{21}}{3}$$

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$$\begin{aligned}
\mathbb{E}(2X - 3Y + 8) &= \int_0^2 \int_0^1 \frac{1}{3}(x+y)(2x-3y+8)dxdy \\
&= \frac{1}{3}\int_0^2 \frac{2}{3} - \frac{1}{2}y + 4 + 8y - 3y^2 dy \\
&= \frac{49}{9}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[(2X - 3Y + 8)^2] &= \int_0^2 \int_0^1 \frac{1}{3}(x+y)(2x-3y+8)^2 dxdy \\
&= \frac{1}{3}\int_0^2 9y^3 - \frac{99\,y^2}{2} + \frac{160\,y}{3} + \frac{131}{3} dy \\
&= \frac{98}{3}
\end{aligned}$$

$$\mathbb{V}(2X - 3Y + 8) = \mathbb{E}[(2X - 3Y + 8)^2] - \mathbb{E}^2(2X - 3Y + 8) = \frac{245}{81}$$