Exercises 7

10.1

$$Z = \frac{\hat{\theta} - \theta_0}{\widehat{se}}$$

$$\begin{aligned}
\beta(\theta_*) &= P(|Z| > z_{\alpha/2}) \\
&= P\left(\frac{\hat{\theta} - \theta_0}{\widehat{se}} < -z_{\alpha/2}\right) + P\left(\frac{\hat{\theta} - \theta_0}{\widehat{se}} > z_{\alpha/2}\right) \\
&= P\left(\frac{\hat{\theta} - \theta_*}{\widehat{se}} < \frac{\theta_0 - \theta_*}{\widehat{se}} - z_{\alpha/2}\right) + P\left(\frac{\hat{\theta} - \theta_0}{\widehat{se}} > \frac{\theta_0 - \theta_*}{\widehat{se}} + z_{\alpha/2}\right)
\end{aligned}$$

由于$\frac{\theta_0 - \theta_*}{\widehat{se}} \sim N(0,1)$，有

$$\beta(\theta_*) = 1 - \Phi\left(\frac{\theta_0 - \theta_*}{\widehat{se}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_*}{\widehat{se}} - z_{\alpha/2}\right)$$

10.5

(a)

$$\begin{aligned}
\beta(c) &= P(Y > c) \\
&= \begin{cases} 0, \theta \le c \\ 1 - \left(\frac{c}{\theta}\right)^n, \theta > c \end{cases}
\end{aligned}$$

(b)

$$c = \frac{\sqrt[n]{0.95}}{2}$$

(c)

$$p = 1 - (0.48/0.5)^{20} = 0.56$$

所以应该接受 H_0

(d) 假如$\theta = 1/2$，一定有$Y \le \theta$，于是

$$p = 0$$

所以应该拒绝H_0

10.8

（a）

$$P(T > c) = P\left(\frac{T(x^n)}{1/\sqrt{n}} > c\sqrt{n}\right) = P(Z > c\sqrt{n}) = 1 - \Phi(c\sqrt{n}) = \Phi(-c\sqrt{n})$$

$$c = -\frac{\Phi^{-1}(\alpha)}{\sqrt{n}}$$

(b)

$$\begin{aligned}\beta(1) &= P\left(\frac{T(x^n)}{1/\sqrt{n}} > c\sqrt{n}\right)\\ &= P\left(\frac{T(x^n)-1}{1/\sqrt{n}} > (c-1)\sqrt{n}\right)\\ &= \Phi(-(c-1)\sqrt{n})\end{aligned}$$

（c）将c的值代入

$$\beta(1) = \Phi(\Phi^{-1}(\alpha) + \sqrt{n})$$

对于给定的$\alpha < 1$，$\Phi^{-1}(\alpha)$是一个有界常数，当$\beta(1) \to 1$时，$\Phi^{-1}(\alpha) + \sqrt{n} \to +\infty$，也就是$n \to +\infty$

10.9

使用10.1中证明的结论，有

$$\beta(\theta_1) = 1 - \Phi\left(\frac{\theta_0 - \theta_1}{\widehat{se}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_1}{\widehat{se}} - z_{\alpha/2}\right)$$

由于$\widehat{se} = \{nI(\hat{\theta})\}^{-1/2}$，当$n \to \infty$时，$-\Phi\left(\frac{\theta_0-\theta_1}{\widehat{se}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0-\theta_1}{\widehat{se}} - z_{\alpha/2}\right) \to 0$，$\beta(\theta_1) \to 1$

10.12

使用一阶矩作为λ的估计，也即

$$\widehat{\lambda} = n^{-1}\sum_{i=1}^{n} X_i$$

对于n足够大的情况，有

$$\widehat{se} = \sqrt{\frac{\lambda}{n}}$$

$$W = \frac{\widehat{\lambda} - \lambda_0}{\sqrt{n^{-1}\lambda_0}} \sim N(0,1)$$

拒绝域为$|W| > z_{\alpha/2}$，也就是

$$\{\widehat{\lambda} | \widehat{\lambda} < \lambda_0 - \sqrt{n^{-1}\lambda_0} z_{\alpha/2} 或 \widehat{\lambda} > \lambda_0 + \sqrt{n^{-1}\lambda_0} z_{\alpha/2}\}$$

（b）代码如下：

```
import numpy as np
from scipy.stats import norm

lambda_ = 1
n = 20

lower_bound = lambda_ - norm.ppf(1 - 0.05/2) * np.sqrt(lambda_/n)
upper_bound = lambda_ + norm.ppf(1 - 0.05/2) * np.sqrt(lambda_/n)

X = np.random.poisson(lambda_, (100000,n))

lambda_hat = np.mean(X, axis=1)

error_rate = np.mean((lambda_hat < lower_bound) | (lambda_hat > upper_bound))

print(error_rate)
```

进行十万次模拟，发生一类错误的频率为0.056。