Exersice 5

9.2

(a)

$$E(X) = \frac{a+b}{2}$$
$$Var(X) = E(X^2) - E^2(X) = \frac{(a-b)^2}{12}$$

解得

$$a = E(X) - \sqrt{3[E(X^2) - E^2(X)]}$$
$$b = E(X) + \sqrt{3[E(X^2) - E^2(X)]}$$

(b)

$$L(a) = \Pi(b-a)I(X_i \geq a)$$
$$L(b) = \Pi(b-a)I(X_i \leq a)$$

所以极大似然估计为

$$a = \min X_i$$
$$b = \max X_i$$

(c)

$$\tau = \int xdF(x) = \int_a^b xd\frac{x-a}{b-a} = \frac{a+b}{2}$$

所以最大似然估计为

$$\tau = \frac{\max X_i + \min X_i}{2}$$

(d)

$$E(\tilde{\tau}) = \frac{a+b}{2}$$

$$Var(\tilde{\tau}) = E(\tilde{\tau}^2) - E^2(\tilde{\tau}) = \frac{(b-a)^2}{12n}$$

$$\mathrm{MSE}(\tilde{\tau}) = Var(\tilde{\tau}) + Bias^2(\tilde{\tau}) = \frac{(b-a)^2}{12n} = 0.033$$

```python
import numpy as np
samples = np.random.uniform(1,3,[100000,10])
a = np.min(samples,axis=1)
b = np.max(samples,axis=1)
tau = (a+b)/2
MSE = np.mean((tau-2)**2)
```

$$\mathrm{MSE}(\widehat{\tau}) = 0.015$$

9.3

(a)

对数似然函数为

$$
\begin{aligned}
\mathcal{L} &= \sum_{n=1}^{N} \log\Big(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp^{-\frac{1}{2}(\frac{(x_n-\mu)^2}{\sigma^2})}\Big) \\
&= \sum_{n=1}^{N} \Big(\log\Big(\frac{1}{\sqrt{2\pi\sigma^2}}\Big) + \log\Big(\exp^{-\frac{1}{2}(\frac{(x_n-\mu)^2}{\sigma^2})}\Big)\Big) \\
&= -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mu} &= \frac{\partial}{\partial \mu}\Big(-\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2\Big) \\
&= \frac{\partial}{\partial \mu}\Big(-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2\Big) \\
&= \sum_{n=1}^{N}\frac{\partial}{\partial \mu}\Big(-\frac{1}{2\sigma^2}(X_n-\mu)^2\Big) \\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N}(X_n-\mu) \\
&= \frac{1}{\sigma^2}\left[-N\mu + \sum_{n=1}^{N}X_n\right]
\end{aligned}
$$

μ的极大似然估计是

$$
\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N}X_n
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma^2}\Big(-\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2\Big) \\
&= -\frac{N}{2}\cdot\frac{\partial}{\partial \sigma^2}\big(\log(2\pi\sigma^2)\big) + \frac{\partial}{\partial \sigma^2}\Big(-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2\Big) \\
&= \frac{1}{2\sigma^2}\Big(-N + \frac{1}{\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2\Big)
\end{aligned}
$$

σ^2的极大似然估计是

$$
\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n-\mu)^2
$$

记标准高斯分布的CDF为$\mathbf{F}$，τ的MLE$\hat{\tau}$满足

$$
\mathbf{F}\left(\frac{\hat{\tau}-\hat{\mu}}{\hat{\sigma}}\right) = 0.95
$$

于是

$$
\hat{\tau} = \hat{\sigma}\mathbf{F}^{-1}(0.95) + \hat{\mu} = \mathbf{F}^{-1}(0.95)\sqrt{\frac{1}{N}\sum_{n=1}^{N}(X_n-\mu)^2} + \frac{1}{N}\sum_{n=1}^{N}X_n
$$

(b)

$$
J_n = I_n(\mu,\sigma) = \frac{1}{n}\begin{bmatrix}\sigma^2 & \\ & \frac{\sigma^2}{2}\end{bmatrix}
$$

$$\nabla g = \begin{pmatrix} 1 \\ \mathbf{F}^{-1}(0.95) \end{pmatrix}$$

$$\widehat{\text{se}}(\hat{\tau}) = \sigma\sqrt{\frac{1}{n} + \frac{1}{2n}(\mathbf{F^{-1}}(0.95))^2}$$

记$1-\alpha$置信度的临界值为$z_{\alpha/2}$，置信区间为

$$\hat{\tau} \pm z_{\alpha/2}\sigma\sqrt{\frac{1}{n} + \frac{1}{2n}(\mathbf{F^{-1}}(0.95))^2}$$

9.5

对于泊松分布的随机变量X_i，其概率密度函数为

$$f(x_i) = \frac{\lambda^{x_i}e^{-\lambda}}{x_i!}$$

对于矩估计

$$\lambda = E(X_i) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

对于极大似然估计，首先给出对数似然函数

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{n} \log\left(\frac{\lambda^{X_i}e^{-\lambda}}{X_i!}\right) \\
&= \sum_{i=1}^{n}\Big(X_i \log\lambda - \lambda - \log X_i!\Big)
\end{aligned}$$

score function是极大似然估计的一阶导数

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\left(\sum_{i=1}^{n}\Big(X_i \log\lambda - \lambda - \log X_i!\Big)\right) \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial\lambda}\Big(X_i \log\lambda - \lambda - \log X_i!\Big) \\
&= \sum_{i=1}^{n}\left(\frac{X_i}{\lambda} - 1\right) \\
&= \frac{1}{\lambda}\sum_{i=1}^{n} X_i - n
\end{aligned}$$

所以极大似然估计为

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Fisher信息量为

$$\begin{aligned}
I_n(\lambda) &= -E\Big(\frac{\partial^2\mathcal{L}}{\partial\lambda^2}\Big) \\
&= -E\Big(\frac{\partial}{\partial\lambda}\Big(\frac{1}{\lambda}\sum_{i=1}^{n} X_i - n\Big)\Big) \\
&= -E\Big(-\frac{1}{\lambda^2}\sum_{i=1}^{n} X_i\Big) \\
&= \frac{n}{\lambda} \\
I(\lambda) &= \frac{1}{\lambda}
\end{aligned}$$

9.6

(a)

对于标准高斯分布$N(\theta,1)$，θ的极大W似然估计为

$$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

$$\mathbb{P}(Y_1=1)=\mathbb{P}(X_1>0)=\mathbb{P}(X_1-\theta<\theta)$$

$X_1-\theta$符合标准高斯分布，记标准高斯分布的CDF为$\mathbf{F}$

$$\hat{\Psi}=\mathbf{F}(\hat{\theta})$$

(b)

$$Var(\hat{\theta})=\frac{1}{n}$$

$$\begin{aligned}
se(\hat{\Psi})&=\sqrt{Var(\hat{\Psi})}\\
&=\sqrt{Var(\mathbf{F}(\hat{\theta}))}\\
&=\sqrt{\mathbf{F}'^2(\hat{\theta})Var(\hat{\theta})}\\
&=|f(\hat{\theta})|\sqrt{\frac{1}{n}}
\end{aligned}$$

所以95%置信区间为

$$\hat{\Psi}\pm 2se(\hat{\Psi})=\hat{\Psi}\pm 2|f(\hat{\theta})|\sqrt{\frac{1}{n}}$$

9.7

(a)

伯努利分布的概率密度函数为

$$f(x)=C_n^x p^x(1-p)^{n-x}$$

对数似然函数为

$$\begin{aligned}
\mathcal{L}&=\log\left(C_n^x p^x(1-p)^{n-x}\right)\\
&=\log C_n^x+x\log p+(n-x)\log(1-p)
\end{aligned}$$

关于p求导得

$$\frac{\partial\mathcal{L}}{\partial p}=\frac{x}{p}-\frac{n-x}{1-p}$$

所以极大似然估计为

$$\hat{p}=\frac{x}{n}$$

对于X_1,X_2，对应的估计为

$$\hat{p}_1=\frac{X_1}{n_1}$$
$$\hat{p}_2=\frac{X_2}{n_2}$$

$$\hat{\Psi} = \hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

（b）

$$I(p) = -E\left[\frac{\partial^2 \mathcal{L}}{\partial p^2}\right] = -E\left[\frac{x}{p^2} + \frac{n-x}{(1-p)^2}\right] = \frac{n}{p(1-p)}$$

$$I(p_1, p_2) = \begin{bmatrix} \frac{n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{n_2}{p_2(1-p_2)} \end{bmatrix}$$

9.8

高斯分布$N(N, \sigma^2)$的概率密度函数为

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

对数似然函数为

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma^2}\right)\right) \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \end{aligned}$$

求导

$$\frac{\partial \mathcal{L}}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\frac{\partial^2 \mathcal{L}}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \sigma} = -\frac{2}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)$$

求期望为

$$E\left[\frac{\partial^2 \mathcal{L}}{\partial \mu^2}\right] = -\frac{n}{\sigma^2}$$

$$E\left[\frac{\partial^2 \mathcal{L}}{\partial \sigma^2}\right] = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}E(X_i-\mu)^2 = -\frac{2n}{\sigma^2}$$

$$E\left[\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \sigma}\right] = -\frac{2}{\sigma^3}\sum_{i=1}^{n}E(X_i-\mu) = 0$$

所以Fisher信息量为

$$I(\mu, \sigma) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$