Exersice 8

10.13

$$L = -rac{n}{2} \mathrm{log}(2\pi) - rac{n}{2} \mathrm{log}(\sigma^2) - rac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

其中 μ 的极大似然估计为 $\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$,

$$\lambda = 2 \log \frac{L(\widehat{\mu})}{L(\mu_0)}$$

$$= \frac{1}{\sigma^2} \left(\sum_{i=1}^n (x_i - \mu_0)^2 - \sum_{i=1}^n (x_i - \widehat{\mu})^2 \right)$$

$$= \frac{n(\widehat{\mu} - \mu_0)^2}{\sigma^2}$$

Wald test 统计量为

$$W = rac{\hat{\mu} - \mu_0}{se(\hat{\mu})} = \sqrt{n} \left(rac{\hat{\mu} - \mu_0}{\sigma}
ight)$$

注意到

$$\lambda = W^2$$

又考虑到当n足够大的时候, $\lambda \sim \chi^2(1), W \sim N(0,1)$,也就是说在本题给定的情况下,这两种检验方式是等价的。

10.14

$$L = -rac{n}{2}\log(2\pi) - rac{n}{2}\log(\sigma^2) - rac{1}{2\sigma^2}\sum_{i=1}^n(x_i - \mu)^2$$
 $rac{\partial L}{\partial \sigma^2} = -rac{n}{2\sigma^2} + rac{1}{2\sigma^4}\sum_{i=1}^n(x_i - \mu)^2 = 0$ $\sigma^2 = rac{1}{n}\sum_{i=1}^n(x_i - \mu)^2$

likelihood ratio test 统计量为

$$egin{aligned} \lambda &= -n \log(\widehat{\sigma}^2) - rac{1}{\widehat{\sigma}^2} \sum_{i=1}^n (x_i - \mu)^2 + n \log(\sigma_0^2) + rac{1}{\sigma_0^2} \sum_{i=1}^n (x_i - \mu)^2 \ &= 2n \log\left(rac{\sigma_0}{\widehat{\sigma}}
ight) + \left(rac{1}{\sigma_0^2} - rac{1}{\widehat{\sigma}^2}
ight) \sum_{i=1}^n (x_i - \mu)^2 \ &= 2n \log\left(rac{\sigma_0}{\widehat{\sigma}}
ight) + n rac{\widehat{\sigma}^2 - \sigma_0^2}{\sigma_0^2} \end{aligned}$$

 σ 的 Fisher 信息量为

$$egin{align} I_n(\sigma) &= -rac{\partial^2 L}{\partial \sigma^2} = -rac{n}{\sigma^2} + rac{3}{\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 \ &= -rac{n}{\sigma^2} + rac{3n}{\sigma^4} \widehat{\sigma}^2 \ &\widehat{se} = rac{1}{\sqrt{I_n(\widehat{\sigma})}} = rac{\widehat{\sigma}}{\sqrt{2n}} \end{split}$$

Wald test 统计量为

$$W = rac{\hat{\sigma} - \sigma_0}{se(\hat{\sigma})} = \sqrt{2n} \left(rac{\hat{\sigma} - \sigma_0}{\hat{\sigma}}
ight)$$

 $\stackrel{{}_{\sim}}{=} n
ightarrow \infty$, $\stackrel{W^2}{\sim} \stackrel{P}{\sim} 1$.

10.15

$$L = \log \binom{n}{X} p^X (1-p)^{n-X} = \log \binom{n}{X} + X \log p + (n-X) \log(1-p)$$
$$\frac{\partial L}{\partial p} = \frac{X}{p} - \frac{n-X}{1-p} = 0$$
$$\widehat{p} = \frac{X}{n}$$

likelihood ratio test 统计量为

$$egin{aligned} \lambda &= 2X\lograc{\widehat{p}}{p_0} + (n-X)\lograc{1-\widehat{p}}{1-p_0} \ &= 2n\left(\widehat{p}\lograc{\widehat{p}}{p_0} + (1-\widehat{p})\lograc{1-\widehat{p}}{1-p_0}
ight) \end{aligned}$$

Wald test 统计量为

$$W=rac{\hat{p}-p_0}{se(\hat{p})}=\sqrt{n}\left(rac{\hat{p}-p_0}{\sqrt{\hat{p}(1-\hat{p})}}
ight)$$