Exersice 8

10.13

$$L=-\frac{n}{2}\log(2\pi)-\frac{n}{2}\log(\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

其中μ的极大似然估计为$\widehat{\mu}=\frac{1}{n}\sum_{i=1}^{n}x_i$，

$$\begin{aligned}\lambda&=2\log\frac{L(\widehat{\mu})}{L(\mu_0)}\\&=\frac{1}{\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu_0)^2-\sum_{i=1}^{n}(x_i-\widehat{\mu})^2\right)\\&=\frac{n(\widehat{\mu}-\mu_0)^2}{\sigma^2}\end{aligned}$$

Wald test 统计量为

$$W=\frac{\hat{\mu}-\mu_0}{se(\hat{\mu})}=\sqrt{n}\left(\frac{\hat{\mu}-\mu_0}{\sigma}\right)$$

注意到

$$\lambda=W^2$$

又考虑到当n足够大的时候，$\lambda\sim\chi^2(1),W\sim N(0,1)$，也就是说在本题给定的情况下，这两种检验方式是等价的。

10.14

$$L=-\frac{n}{2}\log(2\pi)-\frac{n}{2}\log(\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial L}{\partial\sigma^2}=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2=0$$

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

likelihood ratio test 统计量为

$$\begin{aligned}\lambda&=-n\log(\widehat{\sigma}^2)-\frac{1}{\widehat{\sigma}^2}\sum_{i=1}^{n}(x_i-\mu)^2+n\log(\sigma_0^2)+\frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2\\&=2n\log\left(\frac{\sigma_0}{\widehat{\sigma}}\right)+\left(\frac{1}{\sigma_0^2}-\frac{1}{\widehat{\sigma}^2}\right)\sum_{i=1}^{n}(x_i-\mu)^2\\&=2n\log\left(\frac{\sigma_0}{\widehat{\sigma}}\right)+n\frac{\widehat{\sigma}^2-\sigma_0^2}{\sigma_0^2}\end{aligned}$$

σ的 Fisher 信息量为

$$\begin{aligned}I_n(\sigma)=-\frac{\partial^2L}{\partial\sigma^2}&=-\frac{n}{\sigma^2}+\frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2\\&=-\frac{n}{\sigma^2}+\frac{3n}{\sigma^4}\widehat{\sigma}^2\end{aligned}$$

$$\widehat{se}=\frac{1}{\sqrt{I_n(\widehat{\sigma})}}=\frac{\widehat{\sigma}}{\sqrt{2n}}$$

Wald test 统计量为

$$W = \frac{\hat{\sigma} - \sigma_0}{se(\hat{\sigma})} = \sqrt{2n}\left(\frac{\hat{\sigma} - \sigma_0}{\hat{\sigma}}\right)$$

当$n \to \infty$，$\frac{W^2}{\lambda} \xrightarrow{P} 1$.

10.15

$$L = \log\binom{n}{X} p^X (1-p)^{n-X} = \log\binom{n}{X} + X\log p + (n-X)\log(1-p)$$

$$\frac{\partial L}{\partial p} = \frac{X}{p} - \frac{n-X}{1-p} = 0$$

$$\widehat{p} = \frac{X}{n}$$

likelihood ratio test 统计量为

$$\begin{aligned}\lambda &= 2X\log\frac{\widehat{p}}{p_0} + (n-X)\log\frac{1-\widehat{p}}{1-p_0} \\ &= 2n\left(\widehat{p}\log\frac{\widehat{p}}{p_0} + (1-\widehat{p})\log\frac{1-\widehat{p}}{1-p_0}\right)\end{aligned}$$

Wald test 统计量为

$$W = \frac{\hat{p} - p_0}{se(\hat{p})} = \sqrt{n}\left(\frac{\hat{p} - p_0}{\sqrt{\hat{p}(1-\hat{p})}}\right)$$

当$n \to \infty$，$\frac{W^2}{\lambda} \xrightarrow{P} 1$.