Exercise 2

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(a)

$$F_X(x)=\begin{cases}0 & x<0\\ \frac{x}{4} & 0\le x<1\\ \frac{1}{4} & 1\le x<3\\ \frac{3x-7}{8} & 3\le x<5\\ 1 & x\ge 5\end{cases}$$

(b)

$$F_Y(y)=1-F_X\left(\frac{1}{y}\right)=\begin{cases}0 & 0<y\le\frac{1}{5}\\ 1-\frac{\frac{3}{y}-7}{8} & \frac{1}{5}<y<\frac{1}{3}\\ \frac{3}{4} & \frac{1}{3}\le y\le 1\\ 1-\frac{1}{4y} & y>1\end{cases}$$

$$f_Y(y)=\frac{\partial F_Y(y)}{\partial y}=\begin{cases}0 & 0<y\le\frac{1}{5}\\ \frac{3}{8y^2} & \frac{1}{5}<y<\frac{1}{3}\\ 0 & \frac{1}{3}\le y\le 1\\ \frac{1}{4y^2} & y>1\end{cases}$$

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$$\begin{aligned}\mathbb{P}(Z>z)&=\mathbb{P}(X>z\cup Y>z)=\mathbb{P}(X>z)\mathbb{P}(Y>z)\\&=\begin{cases}1 & z<0\\ (1-z)^2 & 0\le z\le 1\\ 0 & z>1\end{cases}\end{aligned}$$

$$\begin{aligned}f_Z(z)&=\frac{\partial F_Z(z)}{\partial z}=\frac{-\partial\mathbb{P}(Z>z)}{\partial y}\\&=\begin{cases}0 & z<0\\ 2z(1-z) & 0\le z\le 1\\ 0 & z>1\end{cases}\end{aligned}$$

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由于$X\sim\text{Exp}(\beta)$，$f(x)=\beta e^{-\beta x},x>0$

$$F(x)=\begin{cases}0 & x<0\\ 1-e^{-\beta x} & x\ge 0\end{cases}$$

$$F^{-1}(q)=-\frac{\ln(q-1)}{\beta},0<q<1$$

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(a) $F_X(0)=1-p,F_Y(0)=p$而$F_{XY}(0,0)=0\neq F_X(0)F_Y(0)$，显然$X,Y$不是独立的随机变量。

(b)

$$
\begin{aligned}
P(X=i,Y=j) &= P(X=i,Y=j|N=i+j)P(N=i+j) \\
&= \tbinom{i+j}{i}p^i(1-p)^j\frac{\lambda^{i+j}e^{-\lambda}}{(i+j)!} \\
&= \frac{p^i\lambda^ie^{-p\lambda}}{i!}\frac{(1-p)^j\lambda^je^{-(1-p)\lambda}}{j!}
\end{aligned}
$$

而

$$
\begin{aligned}
P(X=i) &= \sum_{j=0}^{\infty}P(X=i,Y=j|N=i+j)P(N=i+j) \\
&= \sum_{j=0}^{\infty}\tbinom{i+j}{i}p^i(1-p)^j\frac{\lambda^{i+j}e^{-\lambda}}{(i+j)!} \\
&= \frac{p^i\lambda^ie^{-p\lambda}}{i!}\sum_{j=0}^{\infty}\frac{(1-p)^j\lambda^je^{-(1-p)\lambda}}{j!} \\
&= \frac{p^i\lambda^ie^{-p\lambda}}{i!}
\end{aligned}
$$

同理，有

$$
\begin{aligned}
P(Y=j) &= \sum_{i=0}^{\infty}P(X=i,Y=j|N=i+j)P(N=i+j) \\
&= \sum_{j=0}^{\infty}\tbinom{i+j}{i}p^i(1-p)^j\frac{\lambda^{i+j}e^{-\lambda}}{(i+j)!} \\
&= \frac{(1-p)^j\lambda^je^{-(1-p)\lambda}}{j!}\sum_{j=0}^{\infty}\frac{p^i\lambda^ie^{-p\lambda}}{i!} \\
&= \frac{(1-p)^j\lambda^je^{-(1-p)\lambda}}{j!}
\end{aligned}
$$

所以我们有

$$
P(X=i,Y=j)=P(X=i)P(Y=j)
$$

这说明了两个离散变量的独立性。

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当$0\le y\le 1$时

$$
F_Y(y)=P(Y\le y)=P(F(X)\le y)=P(X\le F^{-1}(y))=F(F^{-1}(y))=y
$$

所以，Y的概率累积函数为

$$
F_Y(y)=\begin{cases}0 & x<0\\ y & 0\le y\le 1\\ 1 & x>1\end{cases}
$$

概率密度函数为

$$
f_Y(y)=\begin{cases}0 & x<0\\ 1 & 0\le y\le 1\\ 0 & x>1\end{cases}
$$

$$
P(X\le x)=P(F^{-1}(U)\le x)=P(U\le F(x))=F(x)
$$

所以有$X\sim F$。

对于指数分布$X\sim \mathrm{Exp}(\beta)$，$F_X(x)=1-e^{-\beta x}(x\ge 0)$，那么$F^{-1}(u)=-\frac{\ln(1-u)}{\beta}$。也就是说令$x=-\frac{\ln(1-u)}{\beta}$，当$u\sim U(0,1)$时，$x$服从参数为$\beta$的指数分布。

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$$\int_0^1 \int_0^1 (x+y^2)\mathrm{d}x\mathrm{d}y = \frac{5}{6}$$

由于$\int\int f_{XY} dxdy = 1$，有$c = \frac{6}{5}$

$$P\left(X < \frac{1}{2} | Y = \frac{1}{2}\right) = \frac{\int_0^{0.5} c(x+0.5^2)\mathrm{d}x}{\int_0^1 c(x+0.5^2)\mathrm{d}x} = \frac{1}{3}$$