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VARIATION IN NORMS OF QUANTUM INPUT MODELS†

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Abstract. The main goal of this paper is to formulate a quantum data structure that prevents efficient classical counterparts of Quantum machine learning algorithms. Since such prevention have been an open problem in quantum machine learning theory, we provide some general properties regarding the problem, focusing on norm variation.

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1. Introcution

Quantum machine learning (QML) is a field of study subsequent to HHL algorithm [2] that approximately solves a system of linear equations in a logarithmic time. However, there have been critiques [3] on input model assumptions or quantum data structures utilized in QML algorithms. Furthermore, Tang [8] introduced *dequantization*, a method that provides efficient classical counterparts on classical data for QML algorithms by randomized-linear algebraic exploitations of quantum-advantageous assumptions. Currently, preventing dequantization is one of open problems [11] in QML theory.

As in ML, input data models are classical for QML usually. To make classical data compatible with QML algorithms, such data need to be efficiently transformed into quantum states. Simple solution is to assume the existence of Quantum Random Access Memory (QRAM), a quantum counterpart to classical RAM, capable of storing n bits of data and querying these data in superposition within a time complexity of polylog(n). Dequantization essentially involves

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mimicking QRAM for classical computers by assuming an input model of sampling and query access to a vector. This assumption allows for a fair comparison between quantum and classical machine learning methodologies.

The outcomes of dequantization establish a boundary for our comprehension of QML algorithms and their constraints. Consequently, a significant unresolved issue in QML pertains to identifying alternative methods for constructing data structures that prevent dequantization. This issue is the focal point, considering its fundamental unit termed 'Block-encoding.' The objective is to formally define an alternative data structure implicitly suggested by Kerenidis and Prakash, and Chakraborty, Gilyén, and Jeffery, and provide general exposition to the question of what properties let QML algorithms to be dequantized.

2. Nomenclature

3. Quantum accesibble classical data structure

Regarding QML and dequantization, notion such as Quantum accesibble classical memory or Quantum Read-Only Memory (QROM [5]) are termed simply 'QRAM' because most data used in ML are classical.

Definition 3.1 ([10] Definition 1)**.** For a table of data $T \in \{0,1\}^N$, QRAM is a collection of unitaries $U_Q(T)$, such that for all states $|i\rangle$ in the computational basis where $0 \leq i \leq N-1$,

$$U_Q(T)\,|i\rangle\,|0\rangle = |i\rangle\,|T_i\rangle\,.$$

Note that $U_Q(T)$ is unitary. By linearity, the number of qubits during access by superposition is $\lceil \log^N \rceil$. Hence, assume that query of the form $|i\rangle\,|0\rangle \to |i\rangle\,|T_i\rangle$ requires $\mathcal{O}(\text{polylog})$ time. Following is an input model or a data structure upon such QRAM, employed in quantum recommendation algorithm.

Theorem 3.2 ([4] Theorem 15)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Let (i, j, A_{ij}) be entries arriving in the system in a arbitrary order, and w be the number of entries already in the system. Then, there exists a data structre to store the matrix A with following properties:*

(1) *The size of the data structure is $\mathcal{O}(w\log^2(mn))$.*
(2) *The time to store a new entry (i, j, A_{ij}) is $\mathcal{O}(\log^2(mn))$.*
(3) *Corresponding to the rows of the matrix currently stored, a quantum algorithm that has quantum access to the data structure can perform the mapping*

$$\widetilde{U} : |i\rangle\,|0\rangle \to |i\rangle\,|A_i\rangle\,,$$

and for $\widetilde{A} \in \mathbb{R}^m$ with entries $\widetilde{A_i} = \|A_i\|$ and $j \in [n]$,

$$\widetilde{V} : |0\rangle\,|j\rangle \to |\widetilde{A}\rangle\,|j\rangle\,.$$

This quantum algorithm takes polylog(mn) *time.*

This data structure is an array of m binary trees. The value stored at the root is $\|A_i\|^2$ for $i \in [m]$, and depth of each tree is at most $\lceil \log n \rceil$. Here, dequantization questions the assumption of '*qauntum access* to the data structure that can efficiently handle classical inputs' and provides a fair comparison. Goal is to classically construct identical tree with only a polynomial slowdown.

Definition 3.3 ([11] Definition 4.1)**.** For all $i \in [n]$, if we can query for $v(i)$, we have *query access* to a vector $v \in \mathbb{C}^n$, denoted by $Q(v)$. For all $(i, j) \in [m] \times [n]$, if we can query for A_{ij}, we have $Q(A)$ to a matrix $A \in \mathbb{C}^{m \times n}$. Time cost of such query is denoted by $q(v)$ and $q(A)$, respectively.

Definition 3.4 ([11] Definition 4.2)**.** For a vector $v \in \mathbb{C}^n$, we have *sampling and query access* to v, denoted by $SQ(v)$, if we can:

(1) have query access to v;
(2) obtain independent samples $i \in [n]$ following the distribution $\mathcal{D}_v \in \mathbb{R}^n$ with $\mathcal{D}_v(i) := |v(i)|^2/\|v\|^2$;
(3) have query access to $\|v\|$.

Cost of entry querying, index sampling, norm querying, are denoted as $q(v), s(v)$, and $n(v)$, respectively. Also, let $sq(v) := \max(q(v), s(v), n(v))$.

Samples obtained from sampling and query access are analogue to the quantum state $|v\rangle := 1/\|v\| \sum v_i |i\rangle$ in the computational basis. Such sampling and query access may be generalized by some oversampling rate.

Definition 3.5. For $v \in \mathbb{C}^n$ and $\phi \geq 1$, we have $\widetilde{v} \in \mathbb{C}^n$ if $\|\widetilde{v}\|^2 = \phi\|v\|^2$ and $|\widetilde{v}_i|^2 \geq |v_i|^2$ for all $i \in [n]$.

Definition 3.6 ([11] Definition 4.3)**.** For $v \in \mathbb{C}^n$ and $\phi \geq 1$, if $Q(v)$ and $SQ(\widetilde{v})$ for $\widetilde{v} \in \mathbb{C}^n$, we have *$\phi$-oversampling and query access to* v or $SQ_\phi(v)$. Also,

$$s_\phi(v) := s(\widetilde{v}),\ q_\phi(v) := q(\widetilde{v}),\ n_\phi(v) := n(\widetilde{v}),\ sq_\phi(v) := \max(s_\phi(v), q_\phi(v), n_\phi(v)).$$

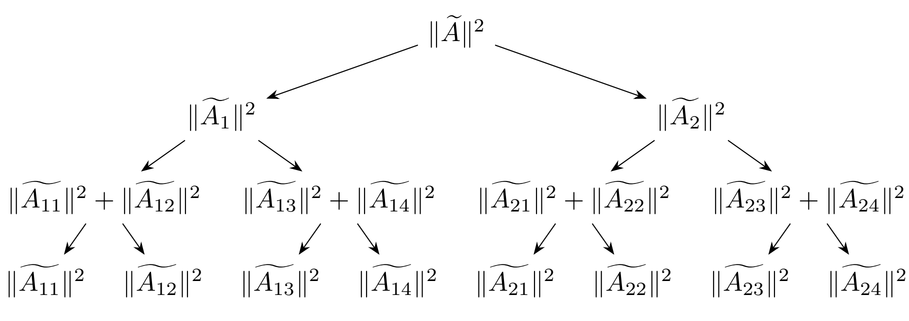

FIGURE 1. example of a ϕ-sampling and qeury access data structure

Note that $SQ_\phi(v)$ constructs an identical tree structure to the one from QRAM. So, we can dequantize whenever QML algorithm relies on QRAM based data structure.

4. Alternative Data Structures

However, there have been variations for such definitions of QRAM, which might prevent dequantization. We first present such variant with general approach, then with more concrete form of presentation by *block-encoding*. Theorem below constructs a more efficient variant of QRAM, where the memory requirement [9] is $\widetilde{\mathcal{O}}(mn)$ and time cost of update, insertion, or deletion for a single entry is $\mathcal{O}(\text{polylog}(n))$. of art.

Theorem 4.1 ([9] Theorem IV.2)**.** *Let $M = \max_{i\in[m]} \|a_i\|^2$ and $A \in \mathbb{R}^{m\times n}$. There eixsts an efficient QRAM data structure for storing matrix entries (i, j, a_{ij}) such that access to this data structure allows a quantum algorithm to implement following unitary in time $\widetilde{\mathcal{O}}(\log(mn))$.*

$$U\left|i, 0^{\lceil \log(n+1)\rceil}\right\rangle = |i\rangle \frac{1}{\sqrt{M}} \left(\sum_{j\in[n]} a_{ij} |j\rangle + (M - \|a_i\|^2)^{1/2} |n+1\rangle \right).k$$

Before heading to the proof, it is helpful to understand utility of M.

Definition 4.2. The normalized vector state corresponding to vector $x \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $\|x\|^2 \leq M$ is the quantum state

$$|\bar{x}\rangle = \frac{1}{\sqrt{M}} \left(\sum_{i\in[n]} x_i |i\rangle + (M - \|x\|^2)^{1/2} |n+1\rangle \right).$$

So the key variation is the norm in the input state.

Proof. m binary trees $B_i \longrightarrow$ use **rotation**

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□

Now we define the notion of block-encoding to clarify the prevention of result above. Block-encdoing was devised during a optimal solution of Hamiltonian simulation probem, [6] originally termed as 'qubitization'. Simply put, block-encoding is a technique of representing Hermitian or subnormalized matrix as the top-left block of a unitary matrix, that is;

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where $\cdot$ denotes arbitrary elements of U.

Definition 4.3 (Gilyén 2019)**.** For $A \in \mathbb{C}^{n\times m}$, $\alpha, \epsilon \in \mathbb{R}_+$ and $a \in \mathbb{N}$, $(s+a)$-qubit unitary U is an (α, a, ϵ)-block-encoding of A if

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I)\| \leq \epsilon.$$

For $n, m \leq 2^s$ we may define an embedding matrix $A_e \in \mathbb{C}^{2^s \times 2^s}$ such that the top-left block of A_e is A and all other entries are 0.

Theorem 4.4 (Chakraborty, 2019, Lemma 25)**.** *Let $A \in \mathbb{C}^{m \times n}$.*

(1) *Fix $p \in [0,1]$. If $A^{(p)}$ and $(A^{(1-p)})^\dagger$ are both stored in quantum-accessible data structures, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\mu_p(A), \lceil \log(n+m+1) \rceil, \epsilon)$-block-encoding of $\bar{A}$.*

(2) *On th other hand, if A is stored in a quantum-accessible data structure, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\|A\|_F, \lceil \log(m+n) \rceil, \epsilon)$-block-encoding of $\bar{A}$.*

Theorem 4.5. *Chakraborty2019*

Theorem 4.6. *main result*

5. Examples and Applications

Conflicts of interest : Declare conflicts of interest or state “The authors declare no conflict of interest.” Authors must identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

Data availability : In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study. If the study did not report any data, you might add “Not applicable” here.

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References

1. C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalities*, J. Wiley and Sons, New York, 1984.
2. A. W. Harrow, A. Hassidim and S. Lloyd, *Quantum algorithm for linear Systems of equations*, Phys. Rev. Lett **103** (2009), 150502.
3. S. Aaronson, *Read the fine print*, Nature Phys **11** (2015), 291-293.
4. I. Kerenidis and A. Prakash, *Quantum Recommendation Systems*, ITCS **67** (2017), 49:1–49:21.
5. R. Babbush, C. Gidney, B. Dominic, N. Weibe, J. McClean, A. Paler, A. Fowler and H. Neven, *Encoding Electronic Spectra in Quantum Circuits with Linear T Complexity* Phys. Rv. **8** (2018), 041015.
6. H. Low and I. Chuang, *Hamiltonian Simulation by Qubitization*, Quantum **3** (2019), 163.
7. Chakraborty, Gilyén and Jeffery. *The Power of Block-Encoded Matrix Powers*, ICALP **132** (2019), 33:1-33:14.

8. E. Tang. *A Quantum-Inspired Classical Algorithm for Recommendation Systems*, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing (2019), 217–228.
9. Kerenidis, Iordanis and Prakash, Anupam, *Quantum gradient descent for linear systems and leaste squares*, Phys.Rev.A **2** (2020), 022316.
10. S. Jaques and A. Rattew, *QRAM: A Survey and Critique*, arXiv: 2305.10310 [quant-ph] (2023).
11. E. Tang, *Quantum Machine Learning Without Any Quantum*, Ph.D. diss, University of Washington (2023).
12. D. Chan and J.S. Pang, *The generalized quasi variational inequality problems*, Math. Oper. Research **7** (1982), 211-222.
13. C. Belly, *Variational and Quasi Variational Inequalities*, J. Appl. Math. and Computing **6** (1999), 234-266.
14. D. Pang, *The generalized quasi variational inequality problems*, J. Appl. Math. and Computing **8** (2002), 123-245.

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