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CONSTRUCTIONS OF BLOCK-ENCLODING AND THEIR EFFECTS ON QUANTUM SPEEDUPS†

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Abstract. Abstract is here, not exceeding *160* words. It must contain *Main Facts.*

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1. Introcution

On claimed exponential speedups of quantum machine learning algorithms subsequent to HHL algorithm for solving linear system of equations, Tang presented classical counterparts for fair amount of QML algorithms by exploiting basic linear algebraic properties underlying data structures used during accessing matrices in a quantum-advantable way, and corresponding them to classical randomied numerical linear algebra methods. Such process is termed 'dequantization'.

All known linear algebraic QML techinques are captured by Quantum singluar vlaue transformation (QSVT), a unifying framework of quantum algorithms. QSVT can be classified by their input model assumptions, whether if inputs for QML are sparse or low-rank. Since sparse-access input models are known to give expnential speedup, dequantization attacks low-rank input models, where classical data without strong restrictions are appliable.

For classical data, QML algorithms must efficiently prepair them as qunatum states. So we assume the existence of quantum random access memory (QRAM), a qunatum device corresponding to classical RAM. QRAM stores n bits of data

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and query those data in superposition by a polylog(n) time. Dequantization is essentially the process of providing psuedo-QRAM to classical computers, by assumming a input model of sampeling and query access to a vector, which would lead to a fair comparison between quantum and classical machine learning.

Results of dequantization draw a border line for our understanding of QML algorithms and their limitations. Hence, one of the open problems of QML is whether there exist other ways to construct data structures that prevent dequantizaition. We focus on this matter by its basic unit, termed 'Block-encoding'. Our goal is to formally define two alternative data structure implicitly stated by Kerenidis and Prakash, and CHakraborty, Gilyén, and Jeffery, generelizing sparse-input model to QRAM-input model.

2. Nomenclature

3. Block-encoding

What lets an algorithm to be dequantized? Below is a QRAM data structure used for the recommendation algorithm, which is the first algorithm to be dequantized.

Theorem 3.1 (Kerenidis 2017, Theorem 15)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix with $A_{ij} \in \mathbb{R}$ being the entry of the i-th row and the j-th column. If w is the number of entries that have already arrived in the system, given the entries $(i, j, A_{ij}$ in an arbitrary order, there exists a data structure to store the matrix A with the following properties:*

(1) *The size of the data structure is $\mathcal{O}(w \log^2(mn))$.*
(2) *Given the entries (i, j, A_{ij}) in an arbitrary order, the time to store them is $\mathcal{O}(\log^2(mn))$.*
(3) *There exists a quantum algorithm that can perform the following maps in polylog(mn) time for $i \in [m]$ and $j \in [n]$:*

$$\widetilde{U} : |i\rangle\,|0\rangle \mapsto |i\rangle\,|A_i\rangle\,,$$
$$\widetilde{V} : |0\rangle\,|j\rangle \mapsto |\widetilde{A}\rangle\,|j\rangle\,,$$

where $\widetilde{A} \in \mathbb{R}^m$ has entries $\widetilde{A_i} = \|A_i\|$.

Proof. The data structure consists of an array of m binary trees $B_i, i \in [m]$. When a new entry $(i, j, A_{ij}$ arrives the leaf node j, in tree B_i is created if not present and updated otherwise. The depth of each tree B_i is at most $\lceil \log n\rceil$. An internal node v of B_i stores the sum of the values of all leaves in the subtree rooted at v, i.e. the sum of the square amplitudes of the entries of A_i in the subtree. Hence, the value stored at the root is $\|A_i\|^2$.

(1) The memory required for the data structure is $\mathcal{O}(w \log^2 mn)$ as for each entry $(i, j, A_{ij}$ at most $\lceil \log n\rceil$ new nodes are added, each node requiring $\mathcal{O}(\log mn)$ bits.

(2) The time required to store entry (i, j, A_{ij}) is $\mathcal{O}(\log^2 mn)$ as the insertion algorithm makes at most $\lceil \log n \rceil$ updates to the data structure and each update requires time $\mathcal{O}(\log mn)$ to retrieve the address of the updated node.
(3) The amplitudes stored in the internal nodes of B_i are used to apply a sequence of conditional rotations to the initial state $|0\rangle^{\lceil \log n \rceil}$ to obtain $|A_i\rangle$. Also, note that the amplitudes of the vector $\widetilde{A}$ are equanto $\|A_i\|$, and the values stored on the roots of the trees B_i are qual to $\|A_i\|^2$. Hence by a smilar construction for the m roots, we can perform the unitary $\widetilde{V}$ efficiently.

□

Theorem 3.2 ([2])**.** *Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $A_{ij} \in \mathbb{R}$ being the entry of the i-th row and the j-th column. If w is the number of non-zero entries of A, then there exits a data structure of size $\mathcal{O}\left(w \log^2(mn)\right)$ that, given the entries (i, j, A_{ij}) in an arbitrary order, stores them such that time taken to store each entry of A is $\mathcal{O}(\log(mn))$. Once this data structure has been initiated with all non-zero entries of A, there exists a quantum algorithm that can perform the following maps with ϵ-precision in $\mathcal{O}(\mathrm{polylog}(mn/\epsilon))$ time:*

$$\widetilde{U} : |i\rangle |0\rangle \mapsto \frac{1}{\|A_{i\cdot}\|} \sum_{j=1}^{n} A_{i,j} |j\rangle = |i, A_i\rangle ,$$
$$\widetilde{V} : |0\rangle |j\rangle \mapsto \frac{1}{\|A\|_F} \sum_{i=1}^{m} \|A_{i,\cdot}\| |i\rangle |j\rangle = |\widetilde{A}, j\rangle ,$$

where $|A_{i,\cdot}\rangle$ is the normalized quantum state corresponding to the i-th row of A and $|\widetilde{A}\rangle$ is a normalized quantum state such that $\langle i|\widetilde{A}\rangle = \|A_{i,\cdot}\|$, i.e. the norm of the i-th row of A.

In particular, given a vector $\boldsymbol{v} \in \mathbb{R}^{m \times 1}$ stored in this data structure, we can generate an ϵ-approximation of the superposition $\sum_{i=1}^{m} v_i |i\rangle / \|\boldsymbol{v}\|$ in complexity $\mathrm{polylog}(m/\epsilon)$*.*

The notion of block-encoding was devised as a solution to the problem of Hamiltonian simulation. Hamiltonian simulation, one of the original motivations for designing practical quantum computers, may stated as follows: For time evolution of the wave function $|\psi(t)\rangle$ governed by the Schrödinger equation, that is,

$$i\hbar \frac{\mathrm{d}}{\mathrm{dt}} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

the Hamiltonian, an operator with units of energy, is $H(t)$. Hamiltionian simulation is a problem of designing quantum circuit or unitary matrix U consisting of $\mathrm{poly}(n, t, 1/\epsilon)$ gates such that $\|U - e^{iHt}\| \leq \epsilon$. The cost of Hamiltonian simulation depends on the number of qubits n, evolution time t, target error ϵ, and access models of Hamiltonian H. While acheiving optimal Hamiltonian simulation by the process called 'Qubitization', Low and Chuang (2017) defined a *standard-form encoding*, a primitive statement of *block-encoding*. Basically, qubitization is a technique of representing Hermitian or subnormalized matrix as the top-left block of a unitary matrix, that is;

$$U = \begin{bmatrix} A/\alpha & \cdot \\ & \cdot & . \end{bmatrix}$$

where $\cdot$ denotes arbitrary elements of U.

Definition 3.3 (Block-encoding)**.** For $A \in \mathbb{C}^{n\times m}, \alpha, \epsilon \in \mathbb{R}_+$ and $a \in \mathbb{N}$, $(s+a)$-qubit unitary U is an (α, a, ϵ)-block-encoding of A if

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I)\| \leq \epsilon.$$

For $n, m \leq 2^s$ we may define an embedding matrix $A_e \in \mathbb{C}^{2^s\times 2^s}$ such that the top-left block of A_e is A and all other entries are 0.

Theorem 3.4 ([2])**.** *Let $A \in \mathbb{C}^{m\times n}$.*

(1) *Fix $p \in [0,1]$. If $A^{(p)}$ and $(A^{(1-p)})^\dagger$ are both stored in quantum-accessible data structures, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\mathrm{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\mu_p(A), \lceil \log(n+m+1)\rceil, \epsilon)$-block-encoding of $\bar{A}$.*
(2) *On th other hand, if A is stored in a quantum-accessible data structure, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\mathrm{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\|A\|_F, \lceil \log(m+n)\rceil, \epsilon)$-block-encoding of $\bar{A}$.*

Proof. For $j \in [m]$, we difne $|\psi_j\rangle$ and $|\phi_j\rangle$ as follows.

$$|\psi_j\rangle = \frac{\sum_{k\in[n]} A_{j,k}^p |j, m+k\rangle}{\sqrt{s_{2p}}(A)} + \sqrt{1 - \frac{\sum_{k\in[n]} A_{j,k}^{2p}}{s_{2p}(A)}}\, |j, n+m+1\rangle$$
$$|\phi_j\rangle =$$

□

Introduction is here

4. Main results

Main results are here

Theorem 4.1 (Pan and Zhang [1, p. 682 (1.5)])**.** *Theorem is here*

Proof. Proof is here [5, Proposition 1.4] □

Lemma 4.2 (Yun [7])**.** *Lemma is here*

Proof. Proof is here □

Corollary 4.3 ([7])**.** *Corollary is here*

Definition 4.4. Definition is here

Remark 4.1. Remark is here

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