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VARIATION IN NORMS OF QUANTUM INPUT MODELS†

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Abstract. The main goal of this paper is to formulate a quantum data structure that prevents the existence of efficient classical counterparts of Quantum machine learning algorithms based upon the data structure or input assumption. Such existence and prevention have been an open problem in the field of quantum machine learning theory, and provide some general properties regarding the problem.

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1. Introcution

Quantum machine learning (QML) is a interdisciplinary field of study subsequent to HHL algorithm [0] that approximately solves a system of linear equations in a logarithmic time. However, there have been serious critiques [0] on input model assumptions or quantum data structures utilized in QML algorithms. Furthermore, Tang [0] introduced *dequantization*, a method that provides efficient classical counterparts on classical data for QML algorithms by randomized-linear algebraic exploitations of quantum-advantageous assumptions. Since then, preventing dequantization have been one of urgent open problems [0] in the theory of QML.

All recognized QML techniques rooted in linear algebra fall under the umbrella of Quantum Singular Value Transformation (QSVT), serving as a unifying framework for quantum algorithms. QSVT is categorized based on the assumptions of the input models. These assumptions revolve around whether the inputs for QML are sparse or low-rank. While sparse-access input models are known for

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providing exponential speedups, dequantization targets low-rank input models where classical data, without strict limitations, can be applied.

To make classical data compatible with QML algorithms, they need to be efficiently transformed into quantum states. This necessitates assuming the existence of Quantum Random Access Memory (QRAM), a quantum counterpart to classical RAM, capable of storing n bits of data and querying these data in superposition within a time complexity of $\mathrm{polylog}(n)$. Dequantization essentially involves mimicking QRAM for classical computers by assuming an input model of sampling and query access to a vector. This assumption allows for a fair comparison between quantum and classical machine learning methodologies.

The outcomes of dequantization establish a boundary for our comprehension of QML algorithms and their constraints. Consequently, a significant unresolved issue in QML pertains to identifying alternative methods for constructing data structures that prevent dequantization. This issue is the focal point, considering its fundamental unit termed 'Block-encoding.' The objective is to formally define an alternative data structure, implicitly suggested by Kerenidis and Prakash, and Chakraborty, Gilyén, and Jeffery, and provided general exposition to the question of what properties let QML algorithms to be dequantized.

2. Nomenclature

The maximum of the p-th power of the ℓ_p norm for the row vectors:

$$s_p(A) := \max_{i\in[m]} \|a_i\|_p^p$$

The sparsity $s(A)$ is the maximu number of non-zero entries in a row of A.

3. Quantum accesibble classical data structure

Regarding QML and dequantization, notion such as Quantum accesibble classical memory or Quantum Read-Only Memory (QROM [0]) are termed simply 'QRAM' because most data used in ML are classical.

Definition 3.1 ([0] Definition 1)**.** For a table of data $T \in \{0,1\}^N$, QRAM is a collection of unitaries $U_Q(T)$, such that for all states $|i\rangle$ in the computational basis where $0 \le i \le N-1$,

$$U_Q(T)\,|i\rangle\,|0\rangle = |i\rangle\,|T_i\rangle\,.$$

Note that $U_Q(T)$ is unitary. By linearity, number of qubits during access by superposition is $\lceil \log^N \rceil$. Hence, assume that query of the form $|i\rangle\,|0\rangle \to |i\rangle\,|T_i\rangle$ requires $\mathcal{O}(\mathrm{polylog})$ time. Following is an input model or a data structure upon such QRAM, employed in quantum recommendation algorithm.

Theorem 3.2 ([0] Theorem 15)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Let (i, j, A_{ij}) be entries arriving in the system in a arbitrary order, and w be the number of entries already in the system. Then, there exists a data structre to store the matrix A with following properties:*

(1) *The size of the data structure is* $\mathcal{O}(w \log^2(mn))$.
(2) *The time to store a new entry* (i, j, A_{ij}) *is* $\mathcal{O}(\log^2(mn))$.
(3) *Corresponding to the rows of the matrix currently stored, a quantum algorithm that has quantum access to the data structure can perform the mapping*

$$\widetilde{U} : |i\rangle\,|0\rangle \to |i\rangle\,|A_i\rangle\,,$$

and for $\widetilde{A} \in \mathbb{R}^m$ *with entries* $\widetilde{A_i} = \|A_i\|$ *and* $j \in [n]$,

$$\widetilde{V} : |0\rangle\,|j\rangle \to |\widetilde{A}\rangle\,|j\rangle\,.$$

This quantum algorithm takes polylog(mn) *time.*

This data structure is an array of m binary trees. The value stored at the root is $\|A_i\|^2$ for $i \in [m]$, and depth of each tree is at most $\lceil \log n \rceil$. Here, dequantization questions the assumption of '*qauntum access* to the data structure that can efficiently handle classical inputs' and provides a comparison equalizer. Goal is to classically construct identical tree with only a polynomial slowdown.

Definition 3.3 ([0] Definition 4.1)**.** For all $i \in [n]$, if we can query for $v(i)$, we have *query access* to a vector $v \in \mathbb{C}^n$, denoted by $Q(v)$. For all $(i, j) \in [m] \times [n]$, if we can query for A_{ij}, we have $Q(A)$ to a matrix $A \in \mathbb{C}^{m \times n}$. Time cost of such query is denoted by $q(v)$ and $q(A)$, respectively.

Definition 3.4 ([0] Definition 4.2)**.** For a vector $v \in \mathbb{C}^n$, we have *sampling and query access* to v, denoted by $SQ(v)$, if we can:

(1) have query access to v;
(2) obtain independent samples $i \in [n]$ following the distribution $\mathcal{D}_v \in \mathbb{R}^n$ with $\mathcal{D}_v(i) := |v(i)|^2/\|v\|^2$;
(3) have query access to $\|v\|$.

Cost of entry querying, index sampling, norm querying, are denoted as $q(v)$, $s(v)$, and $n(v)$, respectively. Also, let $sq(v) := \max(q(v), s(v), n(v))$.

Samples obtained from sampling and query access are analogue to the quantum state $|v\rangle := 1/\|v\| \sum v_i\,|i\rangle$ in the computational basis. Such sampling and query access may be generalized by some oversampling rate.

Definition 3.5. For $v \in \mathbb{C}^n$ and $\phi \geq 1$, we have $\widetilde{v} \in \mathbb{C}^n$ if $\|\widetilde{v}\|^2 = \phi\|v\|^2$ and $|\widetilde{v}_i|^2 \geq |v_i|^2$ for all $i \in [n]$.

Definition 3.6 ([0] Definition 4.3)**.** For $v \in \mathbb{C}^n$ and $\phi \geq 1$, if $Q(v)$ and $SQ(\widetilde{v})$ for $v \in \mathbb{C}^n$, we have *ϕ-oversampling and query access to v* or $SQ_\phi(v)$. Also,

$$\begin{aligned} s_\phi(v) &:= s(\widetilde{v}), \\ q_\phi(v) &:= q(\widetilde{v}), \\ n_\phi(v) &:= n(\widetilde{v}), \\ sq_\phi(v) &:= \max(s_\phi(v), q_\phi(v), n_\phi(v)). \end{aligned}$$

Corollary 3.7. *For a QRAM* $U_Q(T)$, *we need* $\lceil \log^N \rceil$ *qubits.*

Proof. Note that $U_Q(T)$ is unitary. By linearity, access by superposition is possible. $\square$

4. QRAM, data structures and Block-encoding

Regarding QML and dequantization, 'QRAM' denotes a *quantum accesibble classical memory.*

Definition 4.1 ([0] Definition 1)**.** is a collection of unitaries $U_{\mathrm{QRAM}}(T)$, where $T \in \{0,1\}^n$ is a table of data, such that for all states $|i\rangle$ in the computational basis, $0 \leq i \leq n-1$,

$$U_{\mathrm{QRAM}}(T)\,|i\rangle\,|0\rangle = |i\rangle\,|T_i\rangle_.$$

The following is a data structure employed in the quantum recommendation algorithm, which happens to be the first QML algorithm to undergo the process of dequantization.

Theorem 4.2 (Kerenidis 2017, Theorem 15)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix with $A_{ij} \in \mathbb{R}$ representing the entry at the i-th row and the j-th column. Suppose w is the count of entries that have already been received in the system, where entries (i, j, A_{ij}) arrive in an arbitrary order. In such a scenario, there exists a data structure for storing the matrix A with the following properties:*

1. *The data structure's size is $\mathcal{O}(w \log^2(mn))$.*
2. *When provided with entries (i, j, A_{ij}) in an arbitrary order, the time required to store them is $\mathcal{O}(\log^2(mn))$.*
3. *A quantum algorithm exists that can perform the following mappings in $\mathrm{polylog}(mn)$ time for $i \in [m]$ and $j \in [n]$:*

$$\widetilde{U} : |i\rangle\,|0\rangle \mapsto |i\rangle\,|A_i\rangle\,,$$
$$\widetilde{V} : |0\rangle\,|j\rangle \mapsto |\widetilde{A}\rangle\,|j\rangle\,,$$

where $\widetilde{A} \in \mathbb{R}^m$ has entries $\widetilde{A_i} = \|A_i\|$.

Proof. The data structure comprises an array of m binary trees, denoted as B_i for $i \in [m]$. When a new entry (i, j, A_{ij}) arrives, a leaf node for j in tree B_i is created if it doesn't already exist, and updated if it does. The depth of each tree B_i is at most $\lceil \log n \rceil$. An internal node v within B_i stores the sum of the values of all the leaves in the subtree rooted at v, which is essentially the sum of the square amplitudes of the entries of A_i in that subtree. Consequently, the value stored at th e root of each tree is $\|A_i\|^2$.

1. The memory required for this data structure is bounded by $\mathcal{O}(w \log^2 mn)$. For each entry (i, j, A_{ij}), at most $\lceil \log n \rceil$ new nodes are added, and each node necessitates $\mathcal{O}(\log mn)$ bits.
2. The time required to store an entry (i, j, A_{ij}) is $\mathcal{O}(\log^2 mn)$. The insertion algorithm involves at most $\lceil \log n \rceil$ updates to the data structure, and each update takes $\mathcal{O}(\log mn)$ time to retrieve the address of the updated node.

(3) The amplitudes stored in the internal nodes of B_i are utilized to apply a sequence of conditional rotations to the initial state $|0\rangle^{\lceil \log n \rceil}$ to obtain $|A_i\rangle$. Furthermore, it's worth noting that the amplitudes of the vector $\widetilde{A}$ are equal to $\|A_i\|$, and the values stored in the roots of the trees B_i are equal to $\|A_i\|^2$. Consequently, a similar construction for the m roots allows us to efficiently perform the unitary $\widetilde{V}$.

□

Theorem 4.3 (Chakraborty, 2018, Theorem 1)**.** *Consider a matrix $A \in \mathbb{R}^{m\times n}$ with entries $A_{ij} \in \mathbb{R}$ representing the element at the i-th row and the j-th column. Let w be the count of non-zero entries in A. Then, there exists a data structure of size $\mathcal{O}(w \log^2(mn))$ that, when provided with entries (i, j, A_{ij}) in any order, stores them such that the time required to store each entry of A is $\mathcal{O}(\log(mn))$.*

Once this data structure has been initialized with all non-zero entries of A, there exists a quantum algorithm that can perform the following mappings with ϵ-precision in $\mathcal{O}(\mathrm{polylog}(mn/\epsilon))$ time:

$$\widetilde{U} : |i\rangle |0\rangle \mapsto \frac{1}{\|A_{i\cdot}\|} \sum_{j=1}^{n} A_{i,j} |j\rangle = |i, A_i\rangle ,$$

$$\widetilde{V} : |0\rangle |j\rangle \mapsto \frac{1}{\|A\|_F} \sum_{i=1}^{m} \|A_{i,\cdot}\| |i\rangle |j\rangle = |\widetilde{A}, j\rangle ,$$

where $|A_{i,\cdot}\rangle$ is the normalized quantum state corresponding to the i-th row of A, and $|\widetilde{A}\rangle$ is a normalized quantum state such that $\langle i|\widetilde{A}\rangle = \|A_{i,\cdot}\|$, which represents the norm of the i-th row of A.

Notably, when given a vector $\boldsymbol{v} \in \mathbb{R}^{m\times 1}$ stored in this data structure, it is possible to generate an ϵ-approximation of the superposition $\sum_{i=1}^{m} \frac{v_i}{\|\boldsymbol{v}\|} |i\rangle$ with a computational complexity of $\mathrm{polylog}(m/\epsilon)$.

Proof. □

Theorem 4.4 (Kerenidis, Prakash, 2020, Theorem IV.2)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $M = \max_{i\in[m]} \|a_i\|^2$. There is an efficient data structure for storing matrix entries (i, j, a_{ij}) such that access to this data structure allows a quantum algorithm to implement the following unitary in time $\widetilde{\mathcal{O}}(\log(mn))$.*

$$U |i, 0^{\lceil \log(n+1) \rceil}\rangle = |i\rangle \frac{1}{\sqrt{m}} \left(\sum_{j\in[n]} a_{ij} |j\rangle + (m - \|a_i\|^2)^{1/2} |n+1\rangle \right)$$

Proof. □

Theorem 4.5. *Thm on dequantizaion / or expansion of dat structs above.*

Proof. □

Corollary 4.6. ***Midway main Results***

Proof. □

5. Block-encoding

In order to appreciate powers of these alternatives, we introduce the notion of block-encoding, which was devised as a solution to the problem of Hamiltonian simulation. Hamiltonian simulation, one of the original motivations for designing practical quantum computers, may stated as follows: For time evolution of the wave function $|\psi(t)\rangle$ governed by the Schrödinger equation, that is,

$$i\hbar \frac{\mathrm{d}}{\mathrm{dt}} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

the Hamiltonian, an operator with units of energy, is $H(t)$. Hamiltionian simulation is a problem of designing quantum circuit or unitary matrix U consisting of poly$(n, t, 1/\epsilon)$ gates such that $\|U - e^{iHt}\| \leq \epsilon$. The cost of Hamiltonian simulation depends on the number of qubits n, evolution time t, target error ϵ, and access models of Hamiltonian H. While acheiving optimal Hamiltonian simulation by the process called 'Qubitization', Low and Chuang (2017) defined a *standard-form encoding*, a primitive statement of *block-encoding*. Basically, qubitization is a technique of representing Hermitian or subnormalized matrix as the top-left block of a unitary matrix, that is;

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where $\cdot$ denotes arbitrary elements of U.

Definition 5.1 (Block-encoding)**.** For $A \in \mathbb{C}^{n\times m}, \alpha, \epsilon \in \mathbb{R}_+$ and $a \in \mathbb{N}$, $(s+a)$-qubit unitary U is an (α, a, ϵ)-block-encoding of A if

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I)\| \leq \epsilon.$$

For $n, m \leq 2^s$ we may define an embedding matrix $A_e \in \mathbb{C}^{2^s \times 2^s}$ such that the top-left block of A_e is A and all other entries are 0.

Theorem 5.2 (Chakraborty, 2019, Lemma 25)**.** *Let $A \in \mathbb{C}^{m\times n}$.*

(1) *Fix $p \in [0, 1]$. If $A^{(p)}$ and $(A^{(1-p)})^\dagger$ are both stored in quantum-accessible data structures, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\mu_p(A), \lceil\log(n+m+1)\rceil, \epsilon)$-block-encoding of $\bar{A}$.*

(2) *On th other hand, if A is stored in a quantum-accessible data structure, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\|A\|_F, \lceil\log(m+n)\rceil, \epsilon)$-block-encoding of $\bar{A}$.*

Proof. For $j \in [m]$, we difne $|\psi_j\rangle$ and $|\phi_j\rangle$ as follows.

$$|\psi_j\rangle = \frac{\sum_{k\in[n]} A^p_{j,k} |j, m+k\rangle}{\sqrt{s_{2p}}(A)} + \sqrt{1 - \frac{\sum_{k\in[n]} A^{2p}_{j,k}}{s_{2p}(A)}} |j, n+m+1\rangle$$

$$|\phi_j\rangle = \cdots$$

□

Definition 5.3. singular value decomposition

Theorem 5.4 (Kerenidis, Prakash, 2020, Theorem IV.4)**.**

Proof. □

Theorem 5.5. ***Main Results***

Proof. □

6. Examples and Applications

Conflicts of interest : Declare conflicts of interest or state "The authors declare no conflict of interest." Authors must identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

Data availability : In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study. If the study did not report any data, you might add "Not applicable" here.

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