

## A SURVEY ON ALTERNATIVE QUANTUM INPUT MODELS<sup>†</sup>

HYEONMIN ROH\*, TAEWON KIM\*\*

**ABSTRACT.** This paper provides an expository expansion to an open problem in quantum machine learning; whether there are alternative data structures that prevent efficient classical counterparts for quantum machine learning algorithms.

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*Key words and phrases* :

### 1. Introcution

Quantum machine learning (QML) was activated by the HHL algorithm [3] that approximately solves a system of linear equations in a logarithmic time. However, as Aaronson [4] critiqued, HHL and other QML algorithms involved quantum advantageous *input assumptions* or *data structures*. Critically, Tang [9] introduced *dequantization*, a method of providing efficient classical counterparts for large amount of QML algorithms by randomized-linear algebraic exploitations of quantum-advantageous assumptions, hence demystifying claimed exponential speedups of QML algorithms. This paper focus on the problem of constructing data structrues that prevent dequantization, that is, an open problem stated in Tang’s thesis [12] with implicit examples by Chakraborty, Gilyén and Jeffry [8] and Kerenidis and Prakash [10].

### 2. Preliminaries

By practical or industrial reasons, the majority of data given for QML are classical.

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**Definition 2.1.** Classical data is an element of  $n$ -fold Cartesian product  $\{0, 1\}^n$ .

To utilize classical data within a quantum computer, there must be an efficient way or a structure to prepare those data inputs into quantum states, where quantum states are defined as follows.

**Definition 2.2.** For  $a_x \in \mathbb{C}$ , if  $\sum_x |a_x|^2 = 1$ , sum of  $\ell_2$ -normalized vectors in a complex vector space

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} a_x |x\rangle$$

is the state of  $n$  qubits.

We simply assume that such efficient preparation by supposing Quantum Random Access Memory (QRAM), a classical memory that can be accessed by quantum algorithms. Incidentally, QRAM is quite a broad term as noted in survey of Jaques and Rattew [11], but following abstraction might suffice for our discussion. Beforehand, it is useful to introduce following abbreviation

$$|\psi\rangle \otimes |\phi\rangle = |\psi\rangle |\phi\rangle = |\psi, \phi\rangle$$

since  $|\cdot\rangle$  denotes a column vector.

**Definition 2.3.** For classical data  $t_i \in \{0, 1\}^n$ , existence of QRAM is an assumption that there exist unitaries of the form

$$U |i, 0\rangle = |i, t_i\rangle$$

and such action of  $U$  requires  $\mathcal{O}(\text{polylog}(n))$  time.

Classical data  $t_i$  constitutes a classical memory  $T$ . Register of  $|i\rangle$  is called the address register. Time complexity of  $U$  expressed in big-oh notation means that there exists constants  $c$  and  $n_0$  both equal or greater than 0 such that  $T(n)$ , a time function of  $U$  for  $n$ , is equal or less than the polylogarithmic function of  $n$  multiplied by  $c$  for all  $n \geq n_0$ . Another useful notation in algorithmic analysis is  $\tilde{\mathcal{O}}(f(n))$ , shorthand for  $\mathcal{O}(g(n) \log^k g(n))$ .

### 3. Quantum accessible classical data structure

QRAM enables input models or *data structures* for processing matrix input in a quantum advantageous way. We restate such data structure utilized in the quantum recommendation algorithm by Kerenidis [5], first QML algorithm to be dequantized by Tang [9]. In advance, we check state preparation method by Grover and Rudolph [2].

**Corollary 3.1.** *Let a probability distribution  $\{p_i\}$  such that  $p_i^{(m)}$  is the probability for the random variable  $x$  to lie in the  $i$ th region, and  $p_s^{(m)}$  is the probability for  $x$  to lie in  $s$ th region. For an orthonormal state  $|i\rangle$ , let a  $m$ -qubit state as follows*

$$|\psi_m\rangle = \sum_{i=0}^{2^m-1} \sqrt{p_i^{(m)}} |i\rangle.$$

Then, we can get the mapping of

$$|\psi_{m+1}\rangle = \sum_{i=0}^{2^{m+1}-1} \sqrt{p_i^{(m+1)}} |i\rangle.$$

*Proof.* Let  $n = \log N$  for the total number of points  $N$  to discretize the distribution  $p(x)$ . That is, we divide the distribution with  $2^m$  of regions for some  $m$ . We subdivide these  $2^m$  regions to yield a  $2^{m+1}$  region discretization of  $p(x)$ .

First, we define the left and right boundaries of region  $i$  as  $x_L^i$  and  $x_R^i$  with the function

$$f(i) = \frac{\int_{x_L^i}^{(x_R^i - x_L^i)/2} p(x) dx}{\int_{x_L^i}^{x_R^i} p(x) dx}.$$

That is, for  $x$  in the region  $i$ , the probability of  $x$  also lying in the left half of this region. Since an integral on  $p(x)$  can be classically computed efficiently, we take an ancilla register initially in the state  $|0 \dots 0\rangle$  and construct a circuit which efficiently performs the computation

$$\sqrt{p_i^{(m)}} |i, 0 \dots 0\rangle \mapsto \sqrt{p_i^{(m)}} |i, \theta_i\rangle$$

where  $\theta_i \equiv \arccos \sqrt{f(i)}$ . Then, by a controlled rotation of angle  $\theta_i$  on the  $m+1$ 'th qubit, we have

$$\sqrt{p_i^{(m)}} |i, \theta_i, 0\rangle \mapsto \sqrt{p_i^{(m)}} |i, \theta_i\rangle (\cos \theta_i |0\rangle + \sin \theta_i |1\rangle)$$

where we can uncompute the register containing  $|\theta_i\rangle$  with

$$\sqrt{p_i^{(m)}} |i\rangle \mapsto \sqrt{\alpha_i} |i, 0\rangle + \sqrt{\beta_i} |i, 1\rangle$$

where  $\alpha$  and  $\beta$  are the probability for  $x$  to lie in the left or right half of the region  $i$ , respectively. We apply the mapping above for  $m$ -qubit states given as

$$|\psi_m\rangle = \sum_{i=0}^{2^m-1} \sqrt{p_i^{(m)}} |i\rangle.$$

Then, we get  $\sqrt{\psi_{m_1}}$  as desired.  $\square$

**Lemma 3.2.** *Given a certain probability distribution  $\{p_i\}$ , we can efficiently construct a superposition of*

$$|\psi(\{p_i\})\rangle = \sum_i \sqrt{p_i} |i\rangle.$$

*Proof.* By repeating the cororally above until  $m = n$ , we can efficiently construct a superposition as desired.  $\square$

This result, variable by construction details, is used in the data structure below to to efficiently perform  $\tilde{U}$  and  $\tilde{V}$ .

**Theorem 3.3** ([5] Theorem 15). *For a real matrix  $A \in \mathbb{R}^{m \times n}$ , let  $(i, j, A_{ij})$  be given data. If we assume QRAM, then, there exists a data structure representable by binary search trees to store the matrix  $A$  with following properties:*

- (1) *The size of the data structure is  $\mathcal{O}(w \log^2(mn))$  where  $w$  is the number of data entries already in the tree.*
- (2) *The time to store a new data  $(i, j, A_{ij})$  is  $\mathcal{O}(\log^2(mn))$ .*
- (3) *There exists a quantum algorithm that takes  $\text{polylog}(mn)$  time to perform the mapping*

$$\tilde{U} : |i, 0\rangle \rightarrow |i, A_i\rangle$$

and for  $\tilde{A} \in \mathbb{R}^m$  with entries  $\tilde{A}_i = \|A_i\|$ ,

$$\tilde{V} : |0, j\rangle \rightarrow |\tilde{A}, j\rangle.$$

*Proof.* The data structure consists of  $m$  binary trees  $B_i$ , where  $i \in [m]$ . Leaf node is created or updated by the arrival of new entry  $(i, j, A_{ij})$ , storing the value  $A_{ij}^2$ , where internal node  $v$  stores the sum of the values of all leaves in the subtree rooted at  $v$ . Hence, the value stored at the root is  $\|A_i\|^2$ .

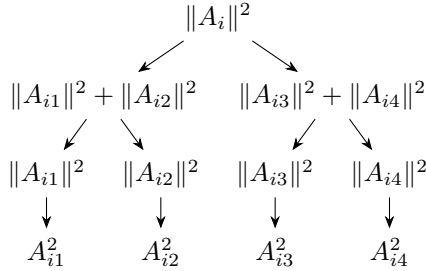


FIGURE 1. example for  $A^{i \times 4}$

The depth of the tree is at most  $\lceil \log n \rceil$  for most  $n$  leaves, which is the number of updates required for a new entry. If we store each tree sorted as ordered list, update node address retrieval time and bits required for memory would be  $\mathcal{O}(\log mn)$ . Thus, the size of the data structure is  $\mathcal{O}(w \log^2(mn))$  and the time to store a new entry is  $\mathcal{O}(\log^2(mn))$ .  $\square$

This data structure constitutes an array of  $m$  binary trees. The value stored at the root is  $\|A_i\|^2$  for  $i \in [m]$ , and depth of each tree is at most  $\lceil \log n \rceil$ . For a fair evaluation of quantum speedup provided by such QRAM-dependant data structure, dequantization starts from constructing equivalent binary search tree with only a polynomial slowdown.

**Definition 3.4** ([12] Definition 4.1). For all  $i \in [n]$ , if we can query for  $v(i)$ , we have *query access* to a vector  $v \in \mathbb{C}^n$ , denoted by  $Q(v)$ . For all  $(i, j) \in [m] \times [n]$ , if we can query for  $A_{ij}$ , we have  $Q(A)$  to a matrix  $A \in \mathbb{C}^{m \times n}$ . Time cost of such query is denoted by  $q(v)$  and  $q(A)$ , respectively.

**Definition 3.5** ([12] Definition 4.2). For a vector  $v \in \mathbb{C}^n$ , we have *sampling and query access* to  $v$ , denoted by  $SQ(v)$ , if we can:

- (1) have query access to  $v$ ;
- (2) obtain independent samples  $i \in [n]$  following the distribution  $\mathcal{D}_v \in \mathbb{R}^n$  with  $\mathcal{D}_v(i) := |v(i)|^2 / \|v\|^2$ ;
- (3) have query access to  $\|v\|$ .

Cost of entry querying, index sampling, norm querying, are denoted as  $q(v)$ ,  $s(v)$ , and  $n(v)$ , respectively. Also, let  $sq(v) := \max(q(v), s(v), n(v))$ .

Samples obtained from sampling and query access are analogue to the quantum state  $|v\rangle := 1/\|v\| \sum v_i |i\rangle$  in the computational basis. Such sampling and query access may be generalized by some oversampling rate.

**Definition 3.6.** For  $v \in \mathbb{C}^n$  and  $\phi \geq 1$ , we have  $\tilde{v} \in \mathbb{C}^n$  if  $\|\tilde{v}\|^2 = \phi\|v\|^2$  and  $|\tilde{v}_i|^2 \geq |v_i|^2$  for all  $i \in [n]$ .

**Definition 3.7** ([12] Definition 4.3). For  $v \in \mathbb{C}^n$  and  $\phi \geq 1$ , if  $Q(v)$  and  $SQ(\tilde{v})$  for  $\tilde{v} \in \mathbb{C}^n$ , we have  $\phi$ -oversampling and query access to  $v$  or  $SQ_\phi(v)$ . Also,  $s_\phi(v) := s(\tilde{v})$ ,  $q_\phi(v) := q(\tilde{v})$ ,  $n_\phi(v) := n(\tilde{v})$ ,  $sq_\phi(v) := \max(s_\phi(v), q_\phi(v), n_\phi(v))$ .

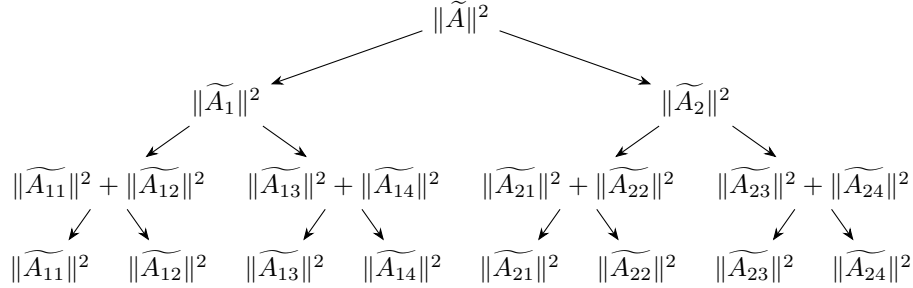


FIGURE 2. example of a  $\phi$ -sampling and query access data structure

Note that  $SQ_\phi(v)$  constructs an identical tree structure to the one from QRAM. So, we can dequantize whenever QML algorithm relies on QRAM based data structure.

#### 4. Alternative Data Structures

However, there have been variations for such definitions of QRAM, which might prevent dequantization. We first present such variant with general approach, then with more concrete form of presentation by *block-encoding*. Theorem below constructs a more efficient variant of QRAM, where the memory requirement [10] is  $\tilde{O}(mn)$  and time cost of update, insertion, or deletion for a single entry is  $\mathcal{O}(\text{polylog}(n))$ . of art.

**Theorem 4.1** ([10] Theorem IV.2). *Let  $M = \max_{i \in [m]} \|a_i\|^2$  and  $A \in \mathbb{R}^{m \times n}$ . There exists an efficient QRAM data structure for storing matrix entries  $(i, j, a_{ij})$  such that access to this data structure allows a quantum algorithm to implement following unitary in time  $\tilde{\mathcal{O}}(\log(mn))$ .*

$$U |i, 0^{\lceil \log(n+1) \rceil}\rangle = |i\rangle \frac{1}{\sqrt{M}} \left( \sum_{j \in [n]} a_{ij} |j\rangle + (M - \|a_i\|^2)^{1/2} |n+1\rangle \right).$$

Before heading to the proof, it is helpful to understand utility of  $M$ .

**Definition 4.2.** The normalized vector state corresponding to vector  $x \in \mathbb{R}^n$  and  $M \in \mathbb{R}$  such that  $\|x\|^2 \leq M$  is the quantum state

$$|\bar{x}\rangle = \frac{1}{\sqrt{M}} \left( \sum_{i \in [n]} x_i |i\rangle + (M - \|x\|^2)^{1/2} |n+1\rangle \right).$$

So the key variation is the norm in the input state.

*Proof.*  $m$  binary trees  $B_i \rightarrow$  use **rotation**

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•  
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□

Now we define the notion of block-encoding to clarify the prevention of result above. Block-encoding was devised during a optimal solution of Hamiltonian simulation problem, [7] originally termed as ‘qubitization’. Simply put, block-encoding is a technique of representing Hermitian or subnormalized matrix as the top-left block of a unitary matrix, that is;

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where  $\cdot$  denotes arbitrary elements of  $U$ .

**Definition 4.3** (Gilyén 2019). For  $A \in \mathbb{C}^{n \times m}$ ,  $\alpha, \epsilon \in \mathbb{R}_+$  and  $a \in \mathbb{N}$ ,  $(s+a)$ -qubit unitary  $U$  is an  $(\alpha, a, \epsilon)$ -block-encoding of  $A$  if

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I)\| \leq \epsilon.$$

For  $n, m \leq 2^s$  we may define an embedding matrix  $A_e \in \mathbb{C}^{2^s \times 2^s}$  such that the top-left block of  $A_e$  is  $A$  and all other entries are 0.

**Theorem 4.4** (Chakraborty, 2019, Lemma 25). *Let  $A \in \mathbb{C}^{m \times n}$ .*

- (1) *Fix  $p \in [0, 1]$ . If  $A^{(p)}$  and  $(A^{(1-p)})^\dagger$  are both stored in quantum-accessible data structures, then there exist unitaries  $U_R$  and  $U_L$  that can be implemented in time  $\mathcal{O}(\text{polylog}(mn/\epsilon))$  such that  $U_R^\dagger U_L$  is a  $(\mu_p(A), \lceil \log(n+m+1) \rceil, \epsilon)$ -block-encoding of  $\bar{A}$ .*

- (2) On the other hand, if  $A$  is stored in a quantum-accessible data structure, then there exist unitaries  $U_R$  and  $U_L$  that can be implemented in time  $\mathcal{O}(\text{polylog}(mn/\epsilon))$  such that  $U_R^\dagger U_L$  is a  $(\|A\|_F, \lceil \log(m+n) \rceil, \epsilon)$ -block-encoding of  $\bar{A}$ .

**Theorem 4.5.** *Chakraborty2019*

**Theorem 4.6.** *main result*

## 5. Examples and Applications

**Conflicts of interest :** Declare conflicts of interest or state “The authors declare no conflict of interest.” Authors must identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

**Data availability :** In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study. If the study did not report any data, you might add “Not applicable” here.

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**1st Author name** received M.Sc. from Seoul National University and Ph.D. at University of Minnesota. Since 1992 he has been at Chungnam National University. His research interests include numerical optimization and biological computation.

Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea.  
e-mail: `soh@cnu.ac.kr`

**2nd Author name** received M.Sc. from Kyungpook National University, and Ph.D. from Iowa State University. He is currently a professor at Chungbuk National University since 1991. His research interests are computational mathematics, iterative method and parallel computation.

Department of Mathematics, College of Natural Sciences, Chungbuk National University, Cheongju 361-763, Korea.  
e-mail: `gmjae@chungbuk.ac.kr`