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VARIATION IN NORMS OF QUANTUM INPUT MODELS†

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ABSTRACT. The main goal of this paper is to formulate a quantum data structure that prevents efficient classical counterparts of Quantum machine learning algorithms. Since such prevention have been an open problem in quantum machine learning theory, we provide some general properties regarding the problem, focusing on norm variation.

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1. Introcution

Quantum machine learning (QML) is a field of study after the HHL algorithm [2] that approximately solves a system of linear equations in a logarithmic time. However, there have been critiques [3] on input model assumptions or quantum data structures utilized in QML algorithms. Furthermore, Tang [8] introduced *dequantization*, a method that provides efficient classical counterparts on classical data for QML algorithms by randomized-linear algebraic exploitations of quantum-advantageous assumptions. Currently, preventing dequantization is one of the open problems [11] in QML theory.

Input data for QML algorithms are mostly classical as ML. For compatibllity of classical data with QML algorithms, such data get efficiently transformed into quantum states. A simple solution is to assume a quantum counterpart to classical RAM. That is *quantum random access memory* (QRAM) storing n bits of data and capable of querying these data in superposition within a time complexity of polylog(n). For a fair comparison between quantum and classical machine

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learning methodologies, dequantization assumes an input model of *sampling and query access* to a vector.

The outcomes of dequantization establish a boundary for our comprehension of QML algorithms and their constraints. Consequently, a significant unresolved issue in QML pertains to identifying alternative methods for constructing data structures that prevent dequantization. The objective is to provide general exposition to the question of what properties allow or prevent dequantization of QML algorithms.

2. Nomenclature

3. Quantum accesibble classical data structure

Most data given for QML are classical. So we use the term QRAM to denote notions such as Quantum accessible classical memory or Quantum Read-Only Memory (QROM [5]).

Definition 3.1 ([10] Definition 1)**.** Let a table of data $T \in \{0,1\}^N$. QRAM is a collection of unitaries $U_Q(T)$ such that for all states $|i\rangle$ in the computational basis with $0 \leq i \leq N-1$,

$$U_Q(T)\,|i\rangle\,|0\rangle = |i\rangle\,|T_i\rangle\,.$$

Or by mapping, query of the form $|i\rangle\,|0\rangle \to |i\rangle\,|T_i\rangle$ requires $\mathcal{O}(\text{polylog})$ time.

An *input model* or a *data structure* for the quantum recommendation algorithm uses such QRAM.

Theorem 3.2 ([4] Theorem 15)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Let (i, j, A_{ij}) be entries arriving in the system in an arbitrary order with w as the number of entries already in the system. Then, there exists a data structure to store the matrix A with the following properties:*

1. *The size of the data structure is $\mathcal{O}(w\log^2(mn))$.*
2. *The time to store a new entry (i, j, A_{ij}) is $\mathcal{O}(\log^2(mn))$.*
3. *Corresponding to the rows of the matrix currently stored, a quantum algorithm that has quantum access to the data structure can perform the mapping*

$$\widetilde{U} : |i\rangle\,|0\rangle \to |i\rangle\,|A_i\rangle\,,$$

and for $\widetilde{A} \in \mathbb{R}^m$ with entries $\widetilde{A_i} = \|A_i\|$ and $j \in [n]$,

$$\widetilde{V} : |0\rangle\,|j\rangle \to |\widetilde{A}\rangle\,|j\rangle\,.$$

This quantum algorithm takes polylog(mn) *time.*

This data structure is an array of m binary trees. The value stored at the root is $\|A_i\|^2$ for $i \in [m]$, and depth of each tree is at most $\lceil \log n \rceil$. Here, dequantization questions the assumption of '*qauntum access* to the data structure that can efficiently handle classical inputs' and provides a fair comparison. Goal is to classically construct identical tree with only a polynomial slowdown.

Definition 3.3 ([11] Definition 4.1)**.** For all $i \in [n]$, if we can query for $v(i)$, we have *query access* to a vector $v \in \mathbb{C}^n$, denoted by $Q(v)$. For all $(i, j) \in [m] \times [n]$, if we can query for A_{ij}, we have $Q(A)$ to a matrix $A \in \mathbb{C}^{m \times n}$. Time cost of such query is denoted by $q(v)$ and $q(A)$, respectively.

Definition 3.4 ([11] Definition 4.2)**.** For a vector $v \in \mathbb{C}^n$, we have *sampling and query access* to v, denoted by $SQ(v)$, if we can:

(1) have query access to v;
(2) obtain independent samples $i \in [n]$ following the distribution $\mathcal{D}_v \in \mathbb{R}^n$ with $\mathcal{D}_v(i) := |v(i)|^2/\|v\|^2$;
(3) have query access to $\|v\|$.

Cost of entry querying, index sampling, norm querying, are denoted as $q(v), s(v)$, and $n(v)$, respectively. Also, let $sq(v) := \max(q(v), s(v), n(v))$.

Samples obtained from sampling and query access are analogue to the quantum state $|v\rangle := 1/\|v\| \sum v_i |i\rangle$ in the computational basis. Such sampling and query access may be generalized by some oversampling rate.

Definition 3.5. For $v \in \mathbb{C}^n$ and $\phi \geq 1$, we have $\widetilde{v} \in \mathbb{C}^n$ if $\|\widetilde{v}\|^2 = \phi\|v\|^2$ and $|\widetilde{v}_i|^2 \geq |v_i|^2$ for all $i \in [n]$.

Definition 3.6 ([11] Definition 4.3)**.** For $v \in \mathbb{C}^n$ and $\phi \geq 1$, if $Q(v)$ and $SQ(\widetilde{v})$ for $\widetilde{v} \in \mathbb{C}^n$, we have *$\phi$-oversampling and query access to v* or $SQ_\phi(v)$. Also,

$$s_\phi(v) := s(\widetilde{v}),\ q_\phi(v) := q(\widetilde{v}),\ n_\phi(v) := n(\widetilde{v}),\ sq_\phi(v) := \max(s_\phi(v), q_\phi(v), n_\phi(v)).$$

$$\begin{tikzcd}
 & & & \|\widetilde{A}\|^2 \arrow[dll] \arrow[drr] & & & & \\
 & \|\widetilde{A_1}\|^2 \arrow[dl] \arrow[dr] & & & & \|\widetilde{A_2}\|^2 \arrow[dl] \arrow[dr] & & \\
\|\widetilde{A_{11}}\|^2 + \|\widetilde{A_{12}}\|^2 \arrow[d] \arrow[dr] & & \|\widetilde{A_{13}}\|^2 + \|\widetilde{A_{14}}\|^2 \arrow[d] \arrow[dr] & & \|\widetilde{A_{21}}\|^2 + \|\widetilde{A_{22}}\|^2 \arrow[d] \arrow[dr] & & \|\widetilde{A_{23}}\|^2 + \|\widetilde{A_{24}}\|^2 \arrow[d] \arrow[dr] & \\
\|\widetilde{A_{11}}\|^2 & \|\widetilde{A_{12}}\|^2 & \|\widetilde{A_{13}}\|^2 & \|\widetilde{A_{14}}\|^2 & \|\widetilde{A_{21}}\|^2 & \|\widetilde{A_{22}}\|^2 & \|\widetilde{A_{23}}\|^2 & \|\widetilde{A_{24}}\|^2
\end{tikzcd}$$

FIGURE 1. example of a ϕ-sampling and qeury access data structure

Note that $SQ_\phi(v)$ constructs an identical tree structure to the one from QRAM. So, we can dequantize whenever QML algorithm relies on QRAM based data structure.

4. Alternative Data Structures

However, there have been variations for such definitions of QRAM, which might prevent dequantization. We first present such variant with general approach, then with more concrete form of presentation by *block-encoding*. Theorem below constructs a more efficient variant of QRAM, where the memory requirement [9] is $\widetilde{\mathcal{O}}(mn)$ and time cost of update, insertion, or deletion for a single entry is $\mathcal{O}(\text{polylog}(n))$. of art.

Theorem 4.1 ([9] Theorem IV.2)**.** *Let $M = \max_{i\in[m]} \|a_i\|^2$ and $A \in \mathbb{R}^{m\times n}$. There eixsts an efficient QRAM data structure for storing matrix entries (i, j, a_{ij}) such that access to this data structure allows a quantum algorithm to implement following unitary in time $\widetilde{\mathcal{O}}(\log(mn))$.*

$$U\left|i, 0^{\lceil \log(n+1)\rceil}\right\rangle = |i\rangle \frac{1}{\sqrt{M}} \left(\sum_{j\in[n]} a_{ij} |j\rangle + (M - \|a_i\|^2)^{1/2} |n+1\rangle \right).k$$

Before heading to the proof, it is helpful to understand utility of M.

Definition 4.2. The normalized vector state corresponding to vector $x \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $\|x\|^2 \leq M$ is the quantum state

$$|\bar{x}\rangle = \frac{1}{\sqrt{M}} \left(\sum_{i\in[n]} x_i |i\rangle + (M - \|x\|^2)^{1/2} |n+1\rangle \right).$$

So the key variation is the norm in the input state.

Proof. m binary trees $B_i \longrightarrow$ use **rotation**

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□

Now we define the notion of block-encoding to clarify the prevention of result above. Block-encdoing was devised during a optimal solution of Hamiltonian simulation probem, [6] originally termed as 'qubitization'. Simply put, block-encoding is a technique of representing Hermitian or subnormalized matrix as the top-left block of a unitary matrix, that is;

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where $\cdot$ denotes arbitrary elements of U.

Definition 4.3 (Gilyén 2019)**.** For $A \in \mathbb{C}^{n\times m}$, $\alpha, \epsilon \in \mathbb{R}_+$ and $a \in \mathbb{N}$, $(s+a)$-qubit unitary U is an (α, a, ϵ)-block-encoding of A if

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I)U(|0\rangle^{\otimes a} \otimes I)\| \leq \epsilon.$$

For $n, m \leq 2^s$ we may define an embedding matrix $A_e \in \mathbb{C}^{2^s \times 2^s}$ such that the top-left block of A_e is A and all other entries are 0.

Theorem 4.4 (Chakraborty, 2019, Lemma 25)**.** *Let $A \in \mathbb{C}^{m \times n}$.*

(1) *Fix $p \in [0,1]$. If $A^{(p)}$ and $(A^{(1-p)})^\dagger$ are both stored in quantum-accessible data structures, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\mu_p(A), \lceil \log(n+m+1) \rceil, \epsilon)$-block-encoding of $\bar{A}$.*

(2) *On th other hand, if A is stored in a quantum-accessible data structure, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{polylog}(mn/\epsilon))$ such that $U_R^\dagger U_L$ is a $(\|A\|_F, \lceil \log(m+n) \rceil, \epsilon)$-block-encoding of $\bar{A}$.*

Theorem 4.5. *Chakraborty2019*

Theorem 4.6. *main result*

5. Examples and Applications

Conflicts of interest : Declare conflicts of interest or state "The authors declare no conflict of interest." Authors must identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

Data availability : In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study. If the study did not report any data, you might add "Not applicable" here.

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