ផ្ទៃក្រឡាត្រីកោណ

រូបមន្ត និងសម្រាយបញ្ជាក់

$$\text{ផ្ទៃក្រឡាត្រីកោណ} = \frac{\text{បាត} \times \text{កម្ពស់}}{2}$$

$$S_{\Delta ABC} = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

$$S_{\Delta ABC} = \frac{abc}{4R}$$

$$S = pr$$

$$S_{\Delta ABC} = \frac{a^2 \sin B \sin C}{2\sin(B+C)}$$

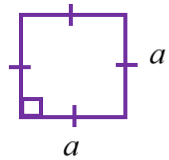
$$S_{\Delta ABC} = \sqrt{p(p-a)(p-b)(p-c)}$$

$$S_{\Delta ABC} = 2R^2 \sin A.\sin B.\sin C$$

អ្នករៀបរៀង៖ ភាច សំណាង និង សួន ច័ន្ទសុធី

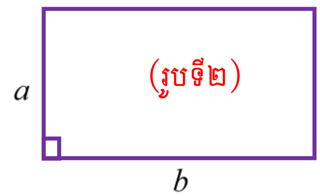
អ្នកត្រួតពិនិត្យ៖ យឹម ភក្តី និង សេក ចន្ថា

⇨ ផ្ទៃក្រឡាការេ $S=a\times a=a^2$



(រូបទី១)

⇨ ផ្ទៃក្រឡាចតុកោណកែង



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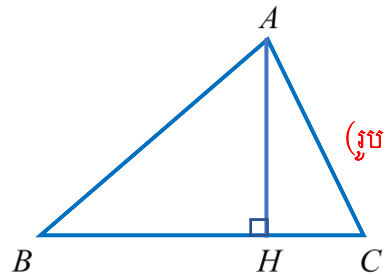
$S=a\times b$

⇨ ផ្ទៃក្រឡាត្រីកោណ $= \dfrac{\text{បាត} \times \text{កម្ពស់}}{2}$

$$S_{\Delta ABC}=\frac{1}{2}BC\times AH$$



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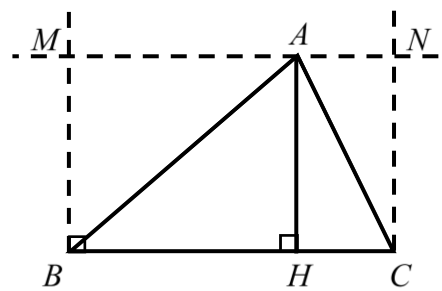
⇨ *សម្រាយបញ្ជាក់រូបមន្តផ្ទៃក្រឡាត្រីកោណ*

តាមរូបទី ៣ បន្ទាត់កែងទៅនឹង BC ត្រង់ B និង C ប្រសព្វនឹងបន្ទាត់កាត់តាម A ហើយស្រប នឹង BC ត្រង់ M និង N រៀងគ្នា



(រូបទី៤)

គេបាន MNCB ជាចតុកោណកែងដែលមាន

វិមាត្រ BC=MN និង BM = AH =CN ។

ម្យ៉ាងទៀត $\Delta ABM\cong\Delta BAH$ និង $\Delta ACN\cong\Delta CAH$

$$S_{MNCB}=S_{\Delta ABM}+S_{\Delta BAH}+S_{\Delta ACN}+S_{\Delta CAH}$$

$S_{MNCB}=2S_{\Delta BAH}+2S_{\Delta CAH}$　　*ព្រោះ* $\Delta ABM\cong\Delta BAH$ *និង* $\Delta ACN\cong\Delta CAH$

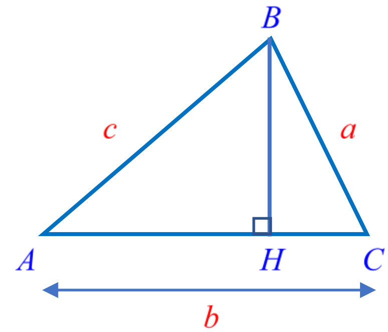
$$S_{MNCB}=2\left(S_{\Delta BAH}+S_{\Delta CAH}\right)$$

$S_{MNCB}=2S_{\Delta ABC}\quad\Rightarrow\quad S_{\Delta ABC}=\dfrac{1}{2}S_{MNCB}$　　*តែ* $S_{MNCB}=AH\times BC$

ដូច្នេះ $\boxed{S_{\Delta ABC}=\dfrac{1}{2}AH\times BC}$

⇨ *ស្រាយថា* $S_{\Delta ABC}=\dfrac{1}{2}bc\sin A$

យើងមាន $S_{\Delta ABC}=\frac{1}{2}BH\times AC=\frac{1}{2}BH\times b$ (1)



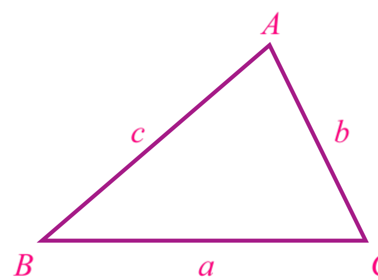
(រូបទី៥)

តែ $\sin A=\frac{BH}{c}\Rightarrow BH=c\sin A$ (2)

យក (2) ជំនួសក្នុង (1)

យើងបាន $S_{\Delta ABC}=\frac{1}{2}bc\sin A$

ស្រាយដូចគ្នាដែរ $\boxed{S_{\Delta ABC}=\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B=\frac{1}{2}ab\sin C}$ ។



(រូបទី៦)

⇨ ស្រាយថា $S_{\Delta ABC}=\frac{a^2\sin B\sin C}{2\sin\left(B+C\right)}$

យើងមាន $S_{\Delta ABC}=\frac{1}{2}ab\sin C$ (1)

តាម $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ (ទ្រឹស្តីបទស៊ីនុស)

$\frac{a}{\sin A}=\frac{b}{\sin B}\Rightarrow b=\frac{a\sin B}{\sin A}$ (2)

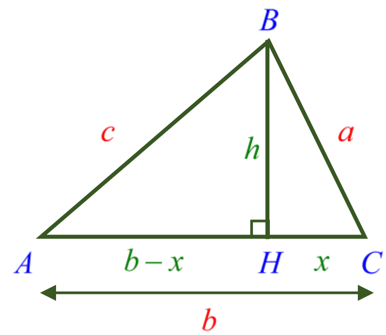
យក (2) ជំនួសក្នុង (1) គេបាន $S_{\Delta ABC}=\frac{a\times a\sin B\sin C}{2\sin A}$

$$S_{\Delta ABC}=\frac{a^2\sin B\sin C}{2\sin\left(B+C\right)}$$

ព្រោះ $\sin A=\sin\left(\pi-A\right)=\sin\left(A+B+C-A\right)=\sin\left(B+C\right)$

ដូចនេះ $\boxed{S_{\Delta ABC}=\frac{a^2\sin B\sin C}{2\sin\left(B+C\right)}}$ ($\angle B$ និង $\angle C$ ជាមុំអមនៃជ្រុងដែលមានរង្វាស់ a)

⇨ *ស្រាយថា* $S_{\Delta ABC}=\sqrt{p(p-a)(p-b)(p-c)}$



(រូបទី៧)

⇨ *របៀបទី១*

តាមពីតាគ័រគេបាន

$$\begin{cases} a^2=x^2+h^2 & (1) \\ c^2=(b-x)^2+h^2 & (2) \end{cases}$$

យក $(1)-(2)$ *គេបាន*

$$a^2-c^2=2bx-b^2 \qquad \Rightarrow \qquad x=\frac{a^2+b^2-c^2}{2b}$$

តាម (1) *គេបាន* $h=\sqrt{a^2-x^2}=\sqrt{a^2-\left(\dfrac{a^2+b^2-c^2}{2b}\right)^2}$

តាម $S_{\Delta ABC}=\dfrac{1}{2}bh=\dfrac{1}{2}b\times\sqrt{a^2-\left(\dfrac{a^2+b^2-c^2}{2b}\right)^2}=\sqrt{\dfrac{a^2b^2}{4}-\left(\dfrac{a^2+b^2-c^2}{4}\right)^2}$

$$=\sqrt{\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16}}$$

$$=\sqrt{\frac{\left(2ab-\left(a^2+b^2-c^2\right)\right)\left(2ab+\left(a^2+b^2-c^2\right)\right)}{16}}$$

$$=\sqrt{\frac{\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]}{16}}$$

$$=\sqrt{\frac{(b+c-a)(a+c-b)(a+b+c)(a+b-c)}{16}}$$

$$=\sqrt{\frac{(a+b+c)}{2}\times\frac{(a+b+c-2a)}{2}\times\frac{(a+b+c-2b)}{2}\times\frac{(a+b+c-2c)}{2}}$$

$$=\sqrt{\frac{a+b+c}{2}\times\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}$$

$$=\sqrt{p(p-a)(p-b)(p-c)} \qquad , \qquad (\text{តាង } p=\frac{a+b+c}{2})$$

ដូចនេះ: $\boxed{S_{\Delta ABC}=\sqrt{p(p-a)(p-b)(p-c)}}$ *(រូបមន្តហេរ៉ុង Heron’s Formula)*

⇨ *របៀបទី២*

តាង $p=\dfrac{a+b+c}{2}$

នោះ $p-a=\dfrac{b+c-a}{2}$, $p-b=\dfrac{a+c-b}{2}$, $p-c=\dfrac{a+b-c}{2}$

យើងមាន $S_{\Delta ABC}=\dfrac{1}{2}bc\sin A \quad \Rightarrow \quad (S_{\Delta ABC})^2=\dfrac{1}{4}b^2c^2\sin^2 A$

$$=\frac{1}{4}b^2c^2(1-\cos^2 A)$$

$$=\frac{1}{4}b^2c^2(1-\cos A)(1+\cos A) \qquad (1)$$

តាមទ្រឹស្តីបទកូស៊ីនុស $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

គេបាន $1-\cos A=1-\dfrac{b^2+c^2-a^2}{2bc}$

$$=\frac{2bc-(b^2+c^2-a^2)}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a-b+c)(a+b-c)}{2bc}$$

ម្យ៉ាងទៀត $1+\cos A=1+\dfrac{b^2+c^2-a^2}{2bc}$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c-a)(a+b+c)}{2bc}$$

តាម (1) គេបាន $(S_{ABC})^2=\dfrac{1}{4}b^2c^2\times\dfrac{(a+c-b)(a+b-c)}{2bc}\times\dfrac{(b+c-a)(a+b+c)}{2bc}$

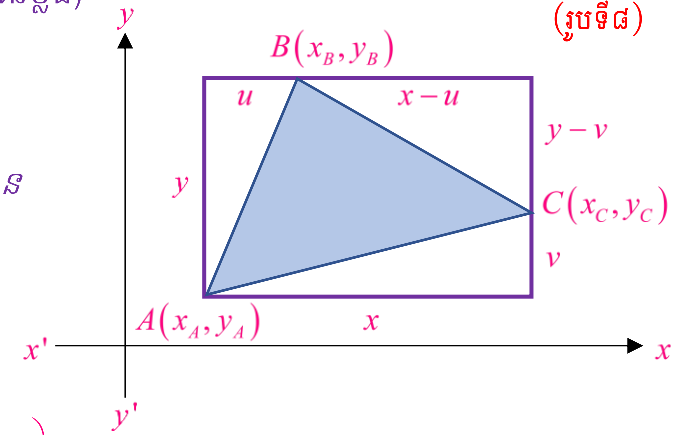
$$(S_{\Delta ABC})^2=\frac{a+b+c}{2}\times\frac{b+c-a}{2}\times\frac{a+c-b}{2}\times\frac{a+b-c}{2}$$

$$S_{\Delta ABC}=\sqrt{p(p-a)(p-b)(p-c)}$$

ដូចនេះ $\boxed{S_{\Delta ABC}=\sqrt{p(p-a)(p-b)(p-c)}}$

⇨ *ស្រាយថា* $S_{\Delta ABC}=\frac{1}{2}(xy-uv)$ (ប្រើចំពោះត្រីកោណដែលមានកំពូលស្ថិតក្នុងតម្រុយ អរតូណរមេនៃប្លង់)

(រូបទី៨)



ឧបមាថា ΔABC *មានកំពូល* $A(x_A,y_A)$, $B(x_B,y_B)$, $C(x_C,y_C)$ *សង់ចតុកោណកែងចារឹកក្រៅត្រីកោណ ដែលមានវិមាត្រស្របនឹងអ័ក្ស* $x'x$ *និង* $y'y$ *រៀងគ្នាដូចរូប*

គេបាន

$$
\begin{aligned}
S_{\Delta ABC} &= xy-\left(\frac{1}{2}xv+\frac{1}{2}yu+\frac{1}{2}(x-u)(y-v)\right)\\
&= xy-\left(\frac{1}{2}xv+\frac{1}{2}yu+\frac{1}{2}(xy-xv-yu+uv)\right)\\
&= \frac{1}{2}(xy-uv)
\end{aligned}
$$

ដូចនេះ: $\boxed{S_{\Delta ABC}=\frac{1}{2}(xy-uv)}$ *(តាមរូប* $x=x_C-x_A$, $y=y_B-y_A$, $u=x_B-x_A$, $v=y_C-y_A$*)*

⇨ *ស្រាយថា* $S_{\Delta ABC}=\frac{1}{2}\left|\det\begin{bmatrix}1&1&1\\x_A&x_B&x_C\\y_A&y_B&y_C\end{bmatrix}\right|$

$A(x_A,y_A)$, $B(x_B,y_B)$, $C(x_C,y_C)$ $\Rightarrow$ $\overrightarrow{AB}=(x_B-x_A,y_B-y_A)$, $\overrightarrow{AC}=(x_C-x_A,y_C-y_A)$

តាម $S_{\Delta ABC}=\frac{1}{2}bc\sin A \quad\Leftrightarrow\quad S_{\Delta ABC}=\frac{1}{2}AB.AC.\sin A$

នោះ:
$$
\begin{aligned}
S^2{}_{\Delta ABC} &= \frac{1}{4}AB^2.AC^2.\sin^2 A=\frac{1}{4}AB^2.AC^2\left(1-\cos^2 A\right)\\
&= \frac{1}{4}\left(AB^2.AC^2-AB^2.AC^2.\cos^2 A\right)=\frac{1}{4}\left(AB^2.AC^2-\left(\overrightarrow{AB}.\overrightarrow{AC}\right)^2\right)\\
&= \frac{1}{4}\left[\left((x_B-x_A)^2+(y_B-y_A)^2\right)\left((x_C-x_A)^2+(y_C-y_A)^2\right)\right]\\
&\quad -\frac{1}{4}\left[(x_B-x_A)(x_C-x_A)+(y_B-y_A)(y_C-y_A)\right]^2\\
&= \frac{1}{4}\left[(x_B-x_A)(y_C-y_A)-(y_B-y_A)(x_C-x_A)\right]^2
\end{aligned}
$$

$$S^2_{\Delta ABC} = \frac{1}{4}\left(x_B y_C - x_B y_A - x_A y_C + x_A y_A - x_C y_B + x_A y_B + x_C y_A - x_A y_A\right)^2$$

$$= \frac{1}{4}\left[\left(x_B y_C - x_C y_B\right) - \left(x_A y_C - x_C y_A\right) + \left(x_A y_B - x_B y_A\right)\right]^2$$

$$S^2_{\Delta ABC} = \frac{1}{4}\left(1\begin{vmatrix} x_B & x_C \\ y_B & y_C \end{vmatrix} - 1\begin{vmatrix} x_A & x_C \\ y_A & y_C \end{vmatrix} + 1\begin{vmatrix} x_A & x_B \\ y_A & y_B \end{vmatrix}\right)^2 = \frac{1}{4}\left(\det\begin{bmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{bmatrix}\right)^2$$

ដូចនេះ $\boxed{S_{\Delta ABC} = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{bmatrix}\right|}$ (ព្រោះ $S_{\Delta ABC} > 0$)

⇨ ស្រាយថា $S_{\Delta ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ (A, B, C ជាបីចំណុចក្នុងលំហ)

តាង $\overrightarrow{AB} = (x, y, z)$ $\overrightarrow{AC} = (x', y', z')$ នោះ

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \vec{i}\left(yz' - y'z\right) - \vec{j}\left(xz' - x'z\right) + \vec{k}\left(xy' - x'y\right)$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{\left(yz' - y'z\right)^2 + \left(xz' - x'z\right)^2 + \left(xy' - x'y\right)^2} \qquad (1)$$

យើងមាន $S_{\Delta ABC} = \frac{1}{2} AB \times AC \times \sin A = \frac{1}{2}\left|\overrightarrow{AB}\right|.\left|\overrightarrow{AC}\right|\sqrt{1 - \cos^2 A}$

$$= \frac{1}{2}\sqrt{\left|\overrightarrow{AB}\right|^2.\left|\overrightarrow{AC}\right|^2\left(1 - \frac{\left(\overrightarrow{AB}.\overrightarrow{AC}\right)^2}{\left|\overrightarrow{AB}\right|^2.\left|\overrightarrow{AC}\right|^2}\right)}$$

ព្រោះ $\cos A = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|.\left|\overrightarrow{AC}\right|}$

$$S_{\Delta ABC} = \frac{1}{2}\sqrt{\left|\overrightarrow{AB}\right|^2.\left|\overrightarrow{AC}\right|^2 - \left(\overrightarrow{AB}.\overrightarrow{AC}\right)^2}$$

$$= \frac{1}{2}\sqrt{\left(x^2 + y^2 + z^2\right)\left(x'^2 + y'^2 + z'^2\right) - \left(xx' + yy' + zz'\right)^2}$$

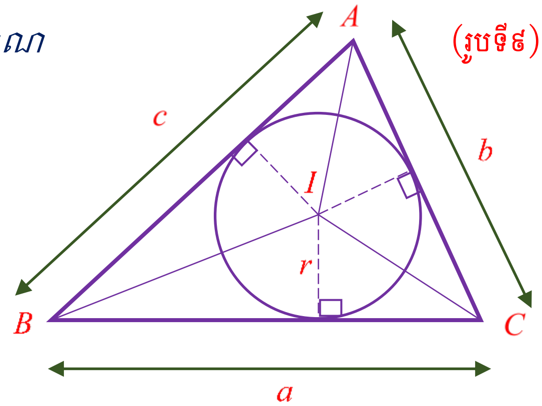
$$= \frac{1}{2}\sqrt{\left(xy'\right)^2 + \left(xz'\right)^2 + \left(yx'\right)^2 + \left(yz'\right)^2 + \left(zx'\right)^2 + \left(zy'\right)^2 - 2xx'yy' - 2yy'zz' - 2xx'zz'}$$

$$= \frac{1}{2}\sqrt{\left(yz' - y'z\right)^2 + \left(xz' - x'z\right)^2 + \left(xy' - x'y\right)^2} \qquad (2)$$

តាម (1) និង (2) គេបាន $\boxed{S_{\Delta ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$ ។

➩ ស្រាយបញ្ជាក់ថា $S_{\Delta ABC} = pr$

$p = \frac{1}{2}(a+b+c)$ 　　　 r : កាំរង្វង់ចារឹកក្នុងត្រីកោណ



$$\begin{aligned} S_{\Delta ABC} &= S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB} \\ &= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr \\ &= \frac{1}{2}(a+b+c)r \\ &= pr \end{aligned}$$

ដូចនេះ $\boxed{S_{\Delta ABC} = pr}$

➩ ស្រាយបញ្ជាក់ថា $S_{\Delta ABC} = \frac{abc}{4R}$ 　　 , R : កាំរង្វង់ចារឹកក្រៅត្រីកោណ

ទ្រឹស្តីបទស៊ីនុស $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

បើ $\frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{a}{2R}$

តាម $S_{\Delta ABC} = \frac{1}{2}bc\sin A = \frac{1}{2}bc \times \frac{a}{2R} = \frac{abc}{4R}$

ដូចនេះ $\boxed{S_{\Delta ABC} = \frac{abc}{4R}}$

➩ ស្រាយថា $S_{\Delta ABC} = 2R^2 \sin A.\sin B.\sin C$ 　　 , R : កាំរង្វង់ចារឹកក្រៅត្រីកោណ

យើងមាន $S_{\Delta ABC} = \frac{1}{2}bc\sin A$

ទ្រឹស្តីបទស៊ីនុស $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ 　　 គេបាន $b = 2R\sin B$, $c = 2R\sin C$

នោះ $S_{\Delta ABC} = \frac{1}{2}.2R\sin B.2R\sin C.\sin A = 2R^2 \sin A.\sin B.\sin C$

ដូចនេះ $\boxed{S_{\Delta ABC} = 2R^2 \sin A.\sin B.\sin C}$