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***Highlights**

A differential quadrature algorithm based on staggered grids is proposed
A type of non-uniform staggered sampling points is suggested
Different nodes are employed for the pore pressure and the displacement

A differential quadrature algorithm based on staggered grids for consolidation analysis of saturated soils

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Abstract: A staggered differential quadrature (SGDQ) algorithm is proposed for consolidation analysis of saturated soils. A type of non-uniform staggered grid points is suggested for space discretization and the implicit Euler scheme is applied for time stepping. Numerical tests are given for one-dimensional and two-dimensional consolidation problems. It is shown that non-physical pressure oscillations observed in the differential quadrature method (DQ) when strong pressure gradients appear are removed by the present formulation and therefore stability and accuracy is greatly enhanced.

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1 Introduction

It is a time-dependent process involving the dissipation of porous fluid pressure and the deformation of soil skeleton when the saturated soil is loaded externally which is termed as consolidation in soil mechanics. Terzaghi [1] presented the famous one-dimensional theory of soil consolidation in 1925. Biot [2] put forward the three-dimensional theory of consolidation which has become widely applicable in practice. Biot's theory consists of continuity of porous fluid flow and differential equilibrium equations of soil taking into account the coupling of dissipation of porous fluid pressure and deformation of soil skeleton. Closed form solutions are hardly available for the complexity of the actual problems. Nowadays, numerical tools such as finite element methods [3], finite difference methods [4], boundary element methods [5], meshfree methods [6], differential quadrature methods [7] and some other methods [8] have been successfully implemented in geotechnical engineering.

Biot's model is usually solved by the finite element method. Satisfactory results can be obtained when the approximated variable is smooth. However, if equal order interpolation is used for the displacement and the pore pressure, numerical oscillations will be induced in the pore pressure once strong pressure gradients appear. As is the case when a soil column is loaded suddenly by a uniform pressure and the used time step is very small. Two strategies are always utilized to alleviate the numerical instability: the first is to add a stability term into the differential equations of Biot's model [9] and the other is by using composite elements with interpolation of the displacement one order higher than that of the pore pressure [3]. Spurious oscillations have been

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greatly reduced by these strategies and the numerical accuracy is improved.

The finite difference method is another kind of numerical scheme for consolidation which deals with the strong form differential equations directly. Standard finite difference algorithms suffer of the same instable behavior mentioned above. Monotone finite difference procedures based on staggered grids have been established by Gaspar et al. [4, 10-12] for one- and two-dimensional quasi-static consolidation, dynamic consolidation and nonlinear consolidation respectively and it is noted that non-physical pressure oscillations can be removed completely by using different points for the displacement and the pore pressure.

The differential quadrature method (DQ) is a high order algorithm proposed by Bellman [13]. The essence of the differential quadrature analog is the expression of a function derivative in terms of weighted linear summation of function values at all grid points. DQ is popular for its easy implementation and high accuracy which has been used for analyzing one-dimensional [7, 14] and axisymmetric uncoupled consolidation [15] where the pore pressure is the sole variant. However, the numerical instability aforementioned is remarkable if DQ is adopted directly for consolidation analysis. To alleviate the numerical oscillations, a differential quadrature algorithm based on staggered grids (SGDQ) is proposed and a type of unequally distributed staggered points for the pore pressure and the displacement is suggested. The implicit Euler scheme is applied for time stepping. The structure of the remaining portion of this paper is organized as follows. In section 2, the governing equations for consolidation are introduced. Section 3 briefly describes the DQ method. The staggered sampling points, the formulations of SGDQ and the numerical tests for one- and two-dimensional consolidation problems are given in section 4 and section 5 respectively. Concluding remarks are made in section 6.

2 Governing equations

In Biot's theory, the governing equations consist of equilibrium of the soil and continuity of porous fluid flow, i.e.

$$-\mu\tilde{\Delta}\mathbf{u}-(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}+\operatorname{grad}p=\mathbf{0} \tag{1}$$

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{u})-\frac{\kappa}{\eta}\Delta p=0 \tag{2}$$

where λ and μ are the Lamé coefficients, $\mathbf{u}$ is the displacement vector, p is the pore pressure, $\tilde{\Delta}$ and Δ are the vectorial Laplace operator and the Laplace operator respectively, t is the time, κ is the permeability, and η is the dynamic viscosity of the porous fluid. The body force is neglected for simplicity. For elastoplastic problems, the stress in the equation of equilibrium is related to the strain through the constitutive relation and the strain consists of derivatives of the displacement which can be approximated using the differential quadrature analogue.

3 The differential quadrature method

In DQ, the derivative of the field variable at some grid point is approximated by weighted linear summation of the function values at all grid points:

$$\frac{\partial^r f(x)}{\partial x^r}\Big|_{x=x_i} = \sum_{j=1}^{N} A_{ij}^r f(x_j), \quad i = 1, 2, \ldots, N \tag{3}$$

where A_{ij}^r are the weighting coefficients for the rth order derivatives, x_j is the coordinate of the jth point and N is the number of grid points. Two keys in DQ are the determination of the weighting coefficients and the selection of the grid points.

The explicit way presented by Shu et al. [16] is employed for computation of the weighting coefficients with high-order polynomials taken as the basis functions. The coefficients for first order derivatives are computed as

$$\begin{aligned} A_{ij}^1 &= \frac{\Pi_{k=1,k\neq i}^N (x_i - x_k)}{(x_i - x_j)\Pi_{k=1,k\neq j}^N (x_j - x_k)}, \quad && i, j = 1, 2, \ldots, N \ (i \neq j) \\ A_{ii}^1 &= -\sum_{j=1, j\neq i}^{N} A_{ij}^1, && i = 1, 2, \ldots, N \end{aligned} \tag{4}$$

It is noted that, all the weighting coefficients are calculated algebraically. Higher order weighting coefficients can be obtained by a recurrence relationship:

$$\begin{aligned} A_{ij}^m &= m\left[A_{ii}^{m-1} A_{ij}^1 - \frac{A_{ij}^{m-1}}{(x_i - x_j)}\right], \quad && i, j = 1, 2, \ldots, N \ (i \neq j) \\ A_{ii}^m &= -\sum_{k=1,k\neq i}^{N} A_{ik}^m, && i = 1, 2, \ldots, N \end{aligned} \tag{5}$$

Both uniform and non-uniform grid points are applicable in DQ. It is well known that Lagrange interpolation polynomials based DQs suffer from severe oscillations when the order is high if uniform points are utilized and therefore the number of points is restricted to below a given value. This limitation can be, however, avoided by use of non-uniform points.

4 One-dimensional consolidation

In this section, a one-dimensional consolidation problem as shown in Fig. 1 is considered and the governing equations become

$$-(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0 \tag{6}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)-\frac{\kappa}{\eta}\frac{\partial^2 p}{\partial x^2}=0 \tag{7}$$

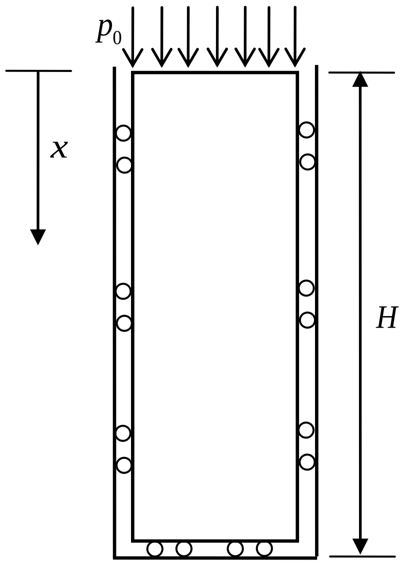


Fig. 1 One-dimensional consolidation

where u is the displacement. It is permeable at the top of the soil and impermeable at all the other boundaries. A uniform pressure is exerted suddenly on the top surface. Initial and boundary conditions for this specific case are:

$$u(x,0)=p(x,0)=0, \quad x\in[0,H] \tag{8}$$

$$u(H,t)=p(0,t)=0, \quad t\in[0,T] \tag{9}$$

$$(\lambda+2\mu)\frac{\partial u}{\partial x}(0,t)=-p_0, \quad \frac{\partial p}{\partial x}(H,t)=0, \quad t\in[0,T] \tag{10}$$

where T is the total time and all the other parameters are shown in Fig. 1.

4.1 Behavior of the differential quadrature method

4.1.1 Formulations

The Chebyshev-Gauss-Lobatto points are employed for space discretization which can be expressed as

$$x_i=\frac{1}{2}\left[1-\cos\left(\frac{i-1}{N-1}\pi\right)\right]H, \quad i=1,2,\cdots,N \tag{11}$$

Denote the coordinates of the grid points for the pore pressure and the displacement as x_i^p and x_i^u respectively which are taken as

$$x_i^u=x_i^p=x_i, \quad i=1,2,\cdots,N, \tag{12}$$

Introduction of Eq. (3) into Eqs. (6) and (7) yields

$$
\begin{aligned}
&-(\lambda+2\mu)\sum_{j=1}^{N} A_{ij}^{2} u_j + \sum_{j=1}^{N} A_{ij}^{1} p_j = 0, \quad i=2,\dots,N-1 \\
&\sum_{j=1}^{N} A_{ij}^{1} \dot{u}_j - \frac{\kappa}{\eta}\sum_{j=1}^{N} A_{ij}^{2} p_j = 0, \quad i=2,\dots,N-1
\end{aligned}
\tag{13}
$$

Boundary conditions are imposed at the two side points:

$$
u_N = p_1 = 0 \tag{14}
$$

$$
(\lambda+2\mu)\sum_{j=1}^{N} A_{1j}^{1} u_j = -p_0, \quad \sum_{j=1}^{N} A_{Nj}^{1} p_j = 0 \tag{15}
$$

The implicit Euler scheme is used for time discretization with step size of δt. Let u_j^n and p_j^n be the respective values at time $t_n = n\cdot\delta t$ and the discretized equations become

$$
\begin{aligned}
&-(\lambda+2\mu)\sum_{j=1}^{N} A_{ij}^{2} u_j^{n+1} + \sum_{j=1}^{N} A_{ij}^{1} p_j^{n+1} = 0, \quad i=2,\dots,N-1 \\
&\sum_{j=1}^{N} A_{ij}^{1} \frac{u_j^{n+1}-u_j^{n}}{\delta t} - \frac{\kappa}{\eta}\sum_{j=1}^{N} A_{ij}^{2} p_j^{n+1} = 0, \quad i=2,\dots,N-1 \\
&u_N^{n+1} = p_1^{n+1} = 0 \\
&(\lambda+2\mu)\sum_{j=1}^{N} A_{1j}^{1} u_j^{n+1} = -p_0, \quad \sum_{j=1}^{N} A_{Nj}^{1} p_j^{n+1} = 0
\end{aligned}
\tag{16}
$$

Denote $\hat{\mathbf{u}} = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}^{\mathrm{T}}$, $\hat{\mathbf{p}} = \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}^{\mathrm{T}}$ and Eq. (16) can be expressed in matrix form as

$$
\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}
\begin{bmatrix} \hat{\mathbf{u}}^{n+1} \\ \hat{\mathbf{p}}^{n+1} \end{bmatrix}
=
\begin{bmatrix} \mathbf{f}_1\left(\hat{\mathbf{u}}^{n}, \hat{\mathbf{p}}^{n}\right) \\ \mathbf{f}_2\left(\hat{\mathbf{u}}^{n}, \hat{\mathbf{p}}^{n}\right) \end{bmatrix}
\tag{17}
$$

The consolidation behavior of saturated soil can be modeled progressively by Eq. (17) with given initial and boundary conditions.

4.1.2 Numerical test

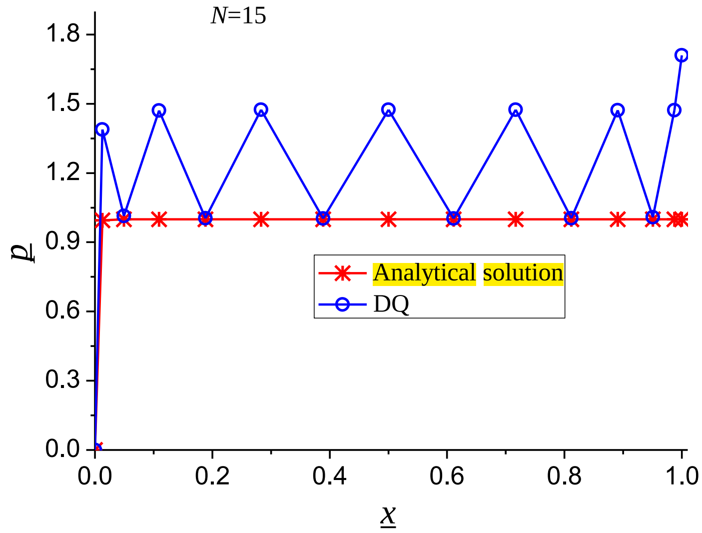
The obtained pore pressure is shown in Fig. 2 where the non-dimensional variables are defined as [4]

$$
\underline{x} = \frac{x}{H}, \quad \underline{t} = \frac{(\lambda+2\mu)\kappa t}{\eta H^2}, \quad \underline{p} = \frac{p}{p_0} \tag{18}
$$

and the analytical solution is:

$$
\underline{p}\left(\underline{x},\underline{t}\right)=\sum_{i=0}^{\infty}\frac{2}{M}\sin\left(M\underline{x}\right)\exp\left(-M^{2}\underline{t}\right)
$$
$$
M=\frac{\pi\left(2i+1\right)}{2} \tag{19}
$$

It is observed that, numerical oscillations appear in the pore pressure when the number of grid points is few (*N*=15), which can be mitigated by increase of the grid points (*N*=60).



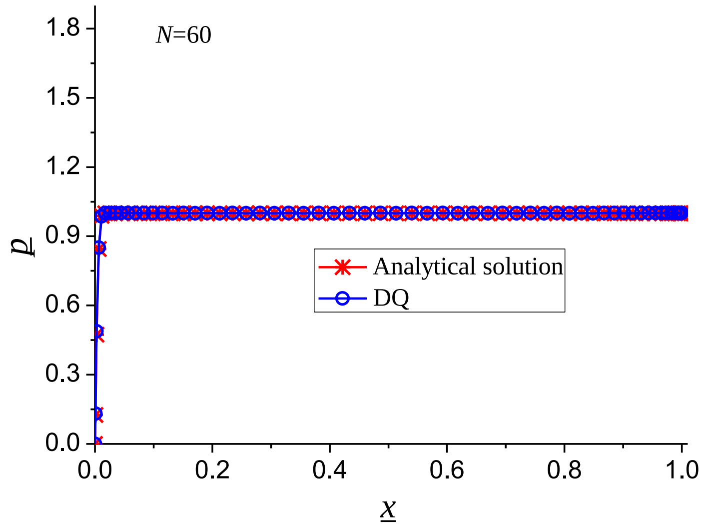


Fig. 2 Non-dimensional pore pressure obtained by DQ for $\underline{t}$=1e-5, δ*t*=1e-6

4.2 The staggered grid differential quadrature method

4.2.1 Formulations

To alleviate the oscillations of DQ for consolidation analysis, a differential quadrature algorithm based on staggered grids is proposed. Staggered grids are widely applied in the finite difference method which have not been considered in the framework of DQ to the best of the author's knowledge. Instead of the commonly used uniform grid points in the finite difference method, the more effective non-uniform staggered grids will be utilized in this formulation which are given as

$$
\begin{aligned}
x_i^u &= \frac{1}{2}\left[1-\cos\left(\frac{i-1}{N-1}\pi\right)\right]H, \quad i=1,\ 2,\cdots N \\
x_i^p &= \frac{1}{2}\left[1-\cos\left(\frac{i-1}{N-2}\pi\right)\right]H, \quad i=1,\ 2,\cdots N-1
\end{aligned}
\tag{20}
$$

It is seen that, different numbers of Chebyshev-Gauss-Lobatto points are employed for the pore pressure and the displacement. The grid points for N=15 are shown in Fig. 3.

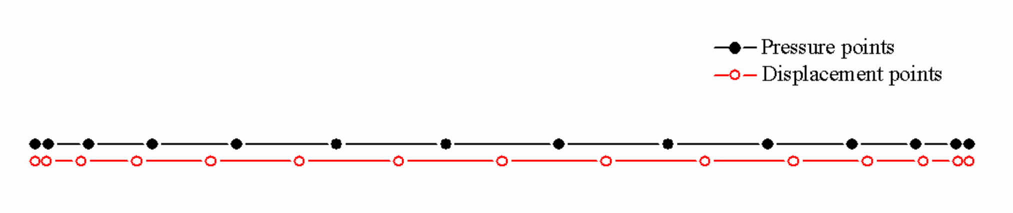


Fig. 3 Grid points for N=15

In SGDQ, the differential equations for equilibrium and continuity are discretized at the displacement and pressure points respectively:

$$
\begin{aligned}
&-(\lambda+2\mu)\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i^u} + \left.\frac{\partial p}{\partial x}\right|_{x=x_i^u} = 0, \quad i=2,\ldots N-1 \\
&\left.\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right|_{x=x_{i'}^p} - \frac{\kappa}{\eta}\left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_{i'}^p} = 0, \quad i' = 2,\ldots N-2
\end{aligned}
\tag{21}
$$

The differential quadrature analogues for the derivatives of the displacement and pore pressure at their corresponding grid points are expressed as:

$$
\begin{aligned}
\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=x_i^u} &= \sum_{j=1}^{N} A_{ij}^{u2} u_j \\
\left(\frac{\partial^2 p}{\partial x^2}\right)_{x=x_{i'}^p} &= \sum_{j'=1}^{N-1} A_{i'j'}^{p2} p_{j'}
\end{aligned}
\tag{22}
$$

where A_{ij}^{u2} and $A_{i'j'}^{p2}$ are the weighting coefficients for the second derivatives of the displacement and the pore pressure respectively. The first derivative of the displacement at some pressure point is firstly approximated by the polynomial interpolation of those at the displacement points and then by linear weighted summation of all the displacements at the displacement points using differential quadrature analogue, i.e.

$$
\begin{aligned}
\left(\frac{\partial u}{\partial x}\right)_{x=x_{i'}^p} &= \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x}\right)_{x=x_i^u} L_i\left(x_{i'}^p\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}^{u1} L_i\left(x_{i'}^p\right) u_j \\
&= \sum_{j=1}^{N}\left(\sum_{i=1}^{N} A_{ij}^{u1} L_i\left(x_{i'}^p\right)\right) u_j = \sum_{j=1}^{N} C_{i'j}^{u} u_j
\end{aligned}
\tag{23}
$$

The first derivative of the pressure at some displacement point is dealt with in the same way:

$$\left(\frac{\partial p}{\partial x}\right)_{x=x_i^u} = \sum_{i'=1}^{N-1}\left(\frac{\partial p}{\partial x}\right)_{x=x_{i'}^p} L_{i'}\left(x_i^u\right) = \sum_{i'=1}^{N-1}\sum_{j'=1}^{N-1} A_{i'j'}^{p1} L_{i'}\left(x_i^u\right) p_{j'}$$
$$= \sum_{j'=1}^{N-1}\left(\sum_{i'=1}^{N-1} A_{i'j'}^{p1} L_{i'}\left(x_i^u\right)\right) p_{j'} = \sum_{j'=1}^{N-1} C_{ij'}^{p} p_{j'} \tag{24}$$

Boundary conditions are imposed as

$$u_N = p_1 = 0,$$
$$\left(\lambda + 2\mu\right)\sum_{j=1}^{N} A_{1j}^{u1} u_j = -p_0, \quad \sum_{j'=1}^{N-1} A_{N-1j'}^{p1} p_{j'} = 0 \tag{25}$$

Introduction of Eqs. (22) - (25) into Eq. (21) yields the algebraic equations of SGDQ for consolidation analysis.

4.2.2 Numerical test

The pore pressure obtained by SGDQ is shown in Fig. 4. It is observed that, pressure oscillations have been eliminated completely by the present formulation. Therefore, numerical stability and accuracy is enhanced greatly by using different points for the pore pressure and the displacement. Pore pressures at different time are given in Fig. 5 where it can be seen that, high accuracy is reached in all the algorithms when a smooth variation is predicted.

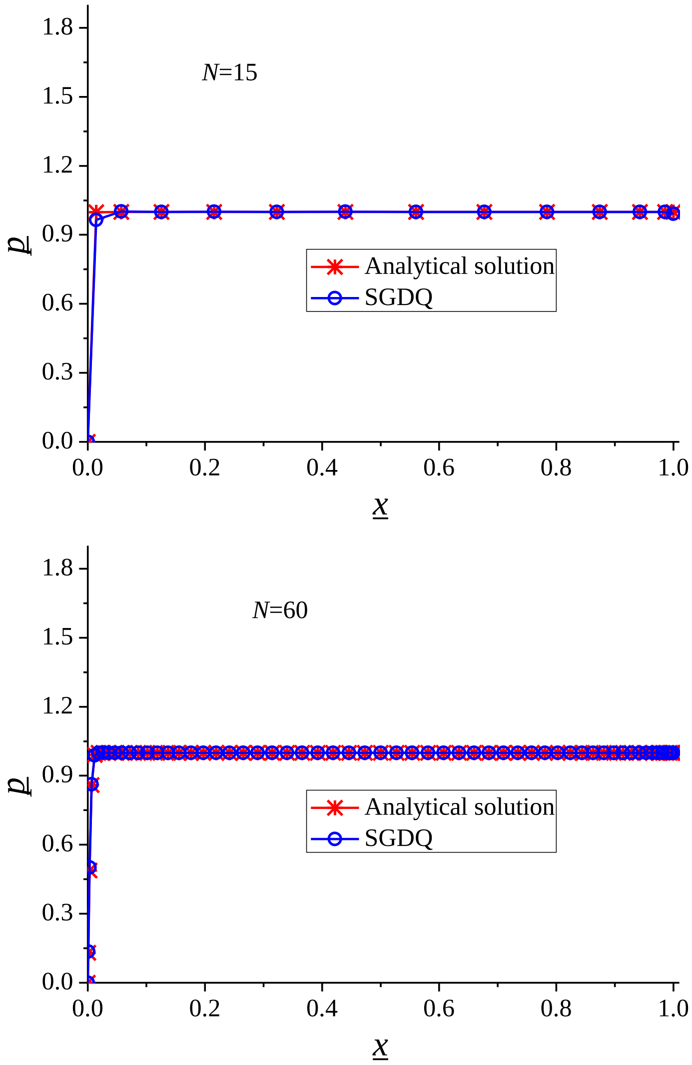
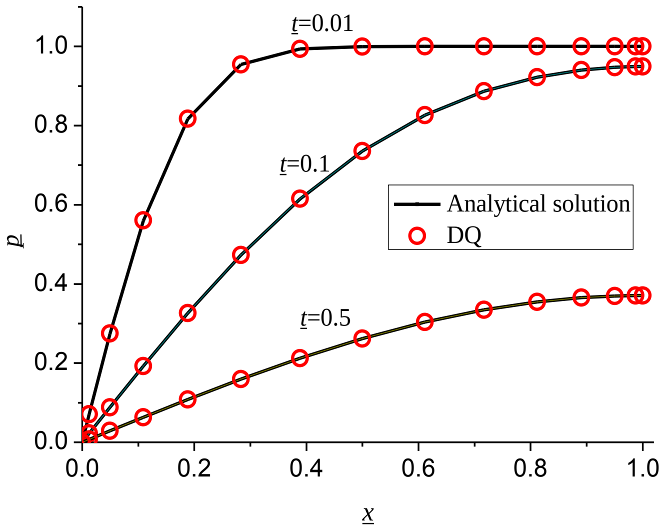


Fig. 4 Non-dimensional pore pressure obtained by SGDQ for t=1e-5, δt=1e-6



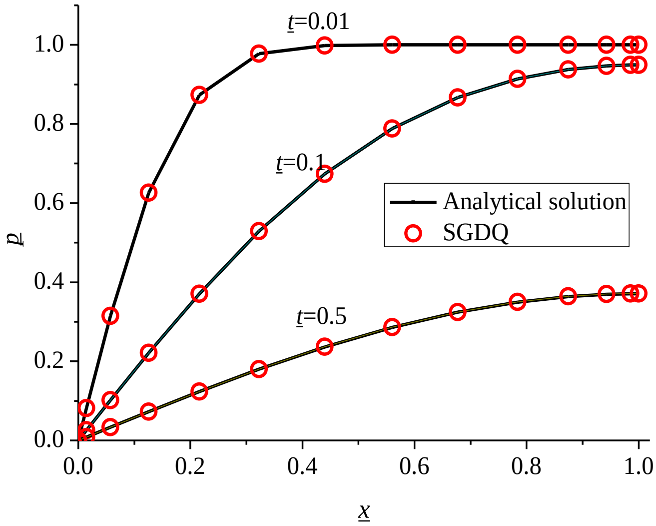


Fig. 5 Pressures obtained by different algorithms

5 Two-dimensional consolidation

5.1 Formulations

For simplicity, a rectangular domain $\Omega = \left(H_{1x}, H_{2x}\right) \times \left(H_{1y}, H_{2y}\right)$ is considered and the coordinates of the sampling points $\mathbf{x}_{ij}^{u} = \mathbf{x}_{ij}^{p} = \mathbf{x}_{ij} = \begin{bmatrix} x_{ij} & y_{ij} \end{bmatrix}^{T}$ in DQ can be given as

$$
\begin{aligned}
x_{ij} &= \frac{1}{2}\left[1-\cos\left(\frac{i-1}{N_x-1}\pi\right)\right]\left(H_{2x}-H_{1x}\right)+\frac{1}{2}\left(H_{2x}+H_{1x}\right) \\
y_{ij} &= \frac{1}{2}\left[1-\cos\left(\frac{j-1}{N_y-1}\pi\right)\right]\left(H_{2y}-H_{1y}\right)+\frac{1}{2}\left(H_{2y}+H_{1y}\right), \quad i=1,\cdots,N_x, \quad j=1,\cdots,N_y
\end{aligned}
\tag{26}
$$

where N_x and N_y are the respective number of points in *x*- and *y*-direction. Introduce Eq. (20) into Eq. (26) and the grid points for SGDQ can be defined accordingly. To deal with problems with discontinuities, the solution domain is usually needed to be divided into subdomains (elements) and the differential quadrature analogue is conducted in each element individually. Two elements of DQ and SGDQ are shown in Fig. 6 with 6×6 displacement points in each element. If the solution domain is not rectangular, the element can be transformed into a normalized square and the grid points can be obtained in a similar way.

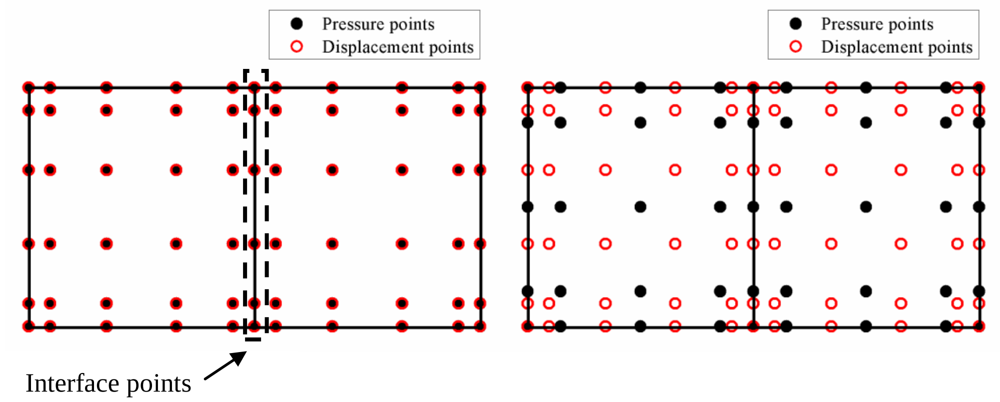


Fig. 6 Two-dimensional grid points, DQ (left), SGDQ (right)

Under the condition of plain strain, the differential equations of Biot's model become

$$\begin{aligned}
&-(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\mu\frac{\partial^2 u}{\partial y^2}-(\lambda+\mu)\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial p}{\partial x}=0\\
&-(\lambda+2\mu)\frac{\partial^2 v}{\partial y^2}-\mu\frac{\partial^2 v}{\partial x^2}-(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial p}{\partial y}=0\\
&\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\kappa}{\eta}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=0
\end{aligned} \tag{27}$$

where v is the displacement in y-direction. As in the one-dimensional case, the differential equations of equilibrium and continuity are discretized at the displacement points and the pressure points respectively:

$$\begin{aligned}
&-(\lambda+2\mu)\left.\frac{\partial^2 u}{\partial x^2}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}-\mu\left.\frac{\partial^2 u}{\partial y^2}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}-(\lambda+\mu)\left.\frac{\partial^2 v}{\partial x\partial y}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}+\left.\frac{\partial p}{\partial x}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}=0\\
&-(\lambda+2\mu)\left.\frac{\partial^2 v}{\partial y^2}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}-\mu\left.\frac{\partial^2 v}{\partial x^2}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}-(\lambda+\mu)\left.\frac{\partial^2 u}{\partial x\partial y}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}+\left.\frac{\partial p}{\partial y}\right|_{\mathbf{x}=\mathbf{x}_{ij}^u}=0\\
&\left.\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right|_{\mathbf{x}=\mathbf{x}_{i'j'}^p}-\left.\frac{\kappa}{\eta}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right|_{\mathbf{x}=\mathbf{x}_{i'j'}^p}=0
\end{aligned} \tag{28}$$

For brevity, the details of two-dimensional formulations are not given. It should be mentioned that, special care should be taken on the connectivity conditions at the interface points between adjacent elements (see Fig. 6 (left)). In the present study, the control conditions at these points are:

$$\begin{aligned}
&(\mathbf{n}\cdot\boldsymbol{\sigma})_i^{k_1}+(\mathbf{n}\cdot\boldsymbol{\sigma})_i^{k_2}=\mathbf{0}\\
&(\nabla p\cdot\mathbf{n})_{i'}^{k_1}+(\nabla p\cdot\mathbf{n})_{i'}^{k_2}=0
\end{aligned} \tag{29}$$

where $\mathbf{n}$ is the outward normal of the element along the interface, $\boldsymbol{\sigma}$ is the stress, and k_1 and k_2 denote the element number of adjacent elements. Eq. (29) means the equilibrium and continuity of normal velocity of porous fluid between adjacent elements.

5.2 Numerical examples

A two-dimensional problem as shown in Fig. 7 is studied in this example. The foundation is suddenly loaded by a uniform pressure $p_0 = 1\,\mathrm{MPa}$. Half of the domain is considered for the symmetry of the problem. It is pervious at the top surface and impervious at all the other boundaries. Horizontal displacement is fixed at the centerline and all displacements are constrained at the bottom and right surfaces. The material properties of the foundation are: κ = 1.0e-7 m/s, η = 1.0 Pa•s, λ = 0, μ = 5 MPa, and the time increment is taken as 1.0e-6 s. Two elements with 15×15 displacement nodes in each element are used. Contours of the pore pressure at *t*=1e-5 s are given in Fig. 8. It can be observed that, stability is greatly improved by SGDQ for two-dimensional problems. To verify the accuracy of those results, the commercial software ABAQUS is employed to solve the problem and two uniform meshes are adopted using 12000 and 72000 linear quadrilateral elements respectively. The pore pressure along the centerline of the foundation at 1.0e-5 s is shown in Fig. 9 where the agreement between those results is clearly seen. Large oscillations exist in the results of the finite element method although a very fine mesh is used.

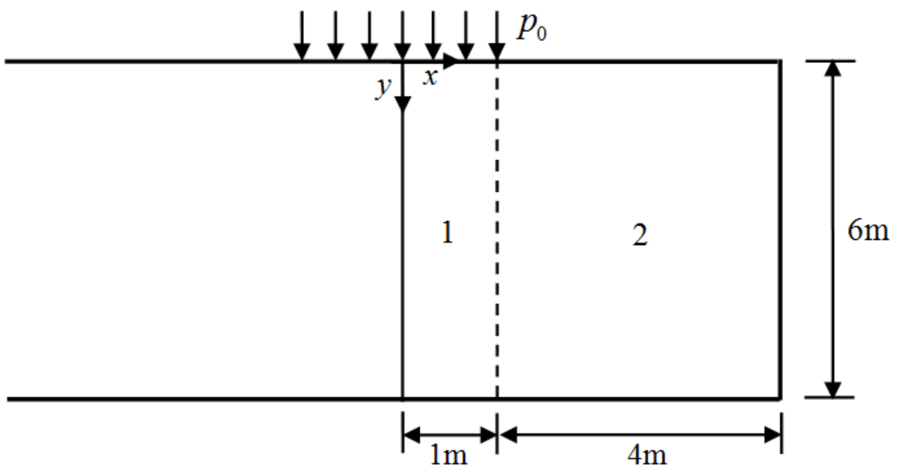


Fig. 7 Two-dimensional foundation loaded by uniform pressure

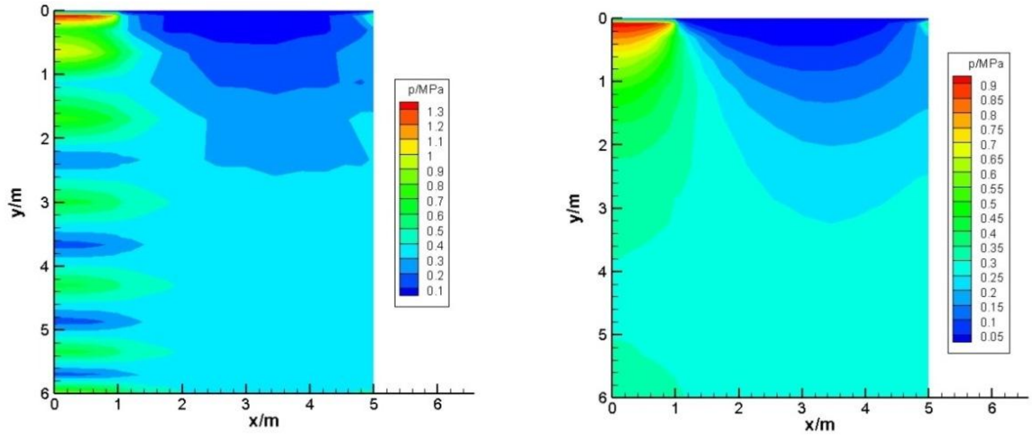


Fig. 8 Contours of pore pressure at t=1e-5 s (uniform pressure), DQ (left), SGDQ (right)

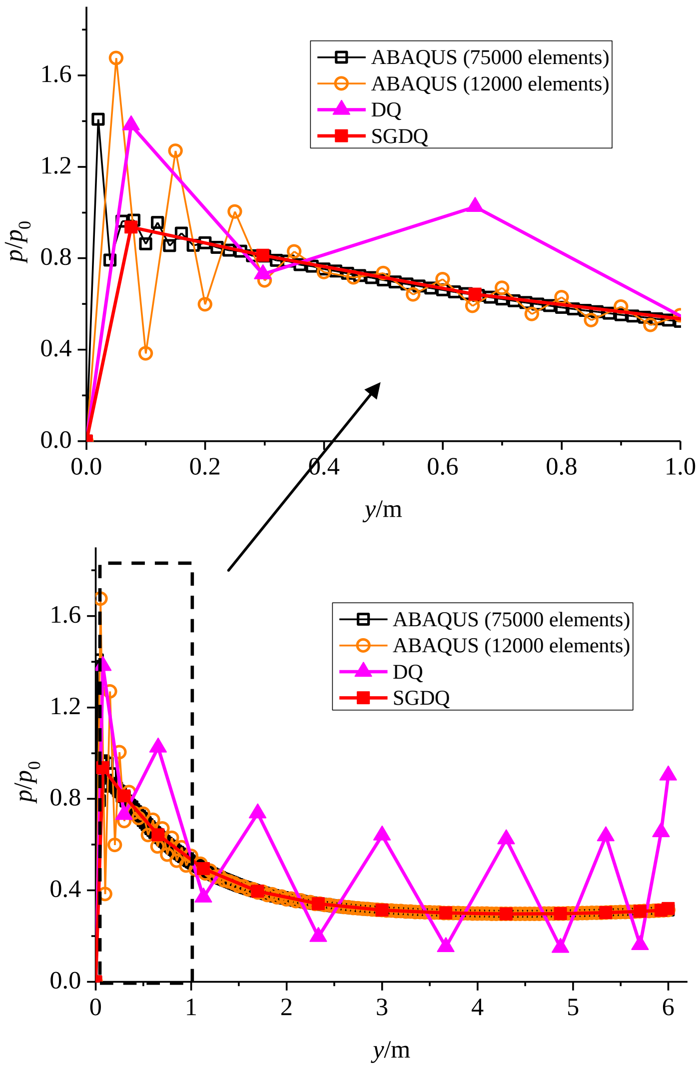


Fig. 9 Pore pressure along the centerline at t=1e-5 s (uniform pressure)

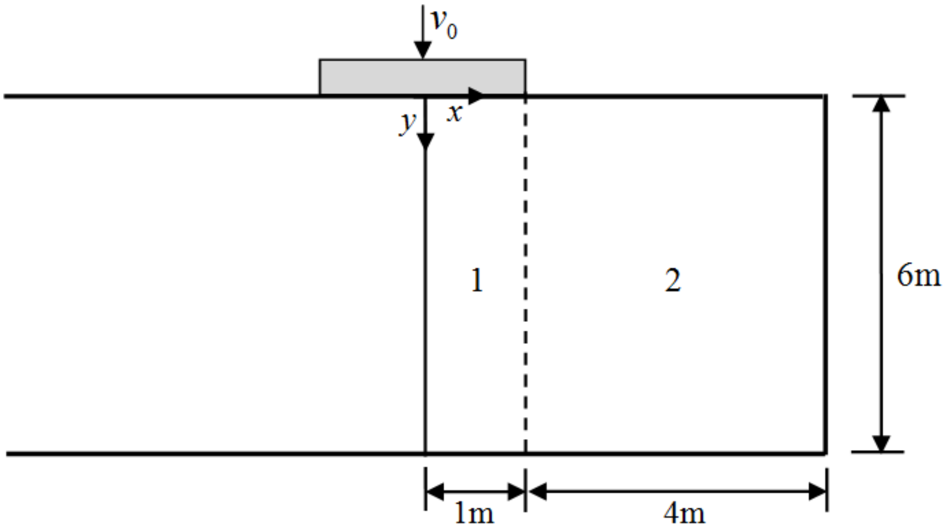


Fig. 10 Two-dimensional foundation loaded by given displacement

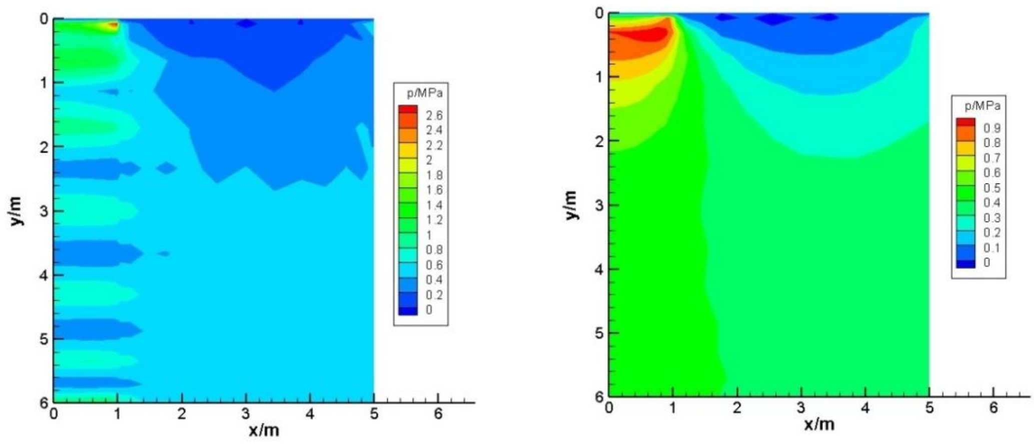


Fig. 11 Contours of pore pressure at t=1e-5 s (given displacement), DQ (left), SGDQ (right)

Then a rigid rough footing is simulated as shown in Fig. 10 and the displacement at the top surface is given as

$$u = 0, \quad v = 0.1\text{m}, \quad 0 \le x \le 1\text{m}, \quad y = 0$$

Contours of the pore pressure and the pore pressure at x=1m are given in Figs. 11 and 12 respectively. A similar conclusion can be drawn as the case loaded by uniform pressure.

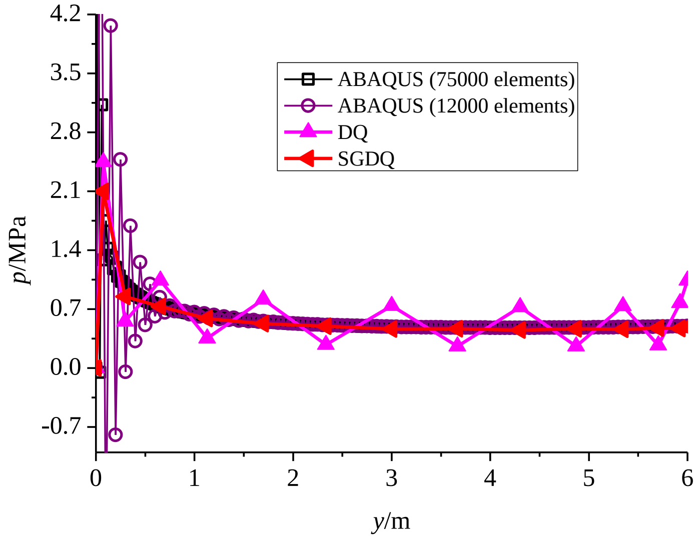


Fig. 12 Variation of pore pressure for x=1 m, t=1e-5 s (given displacement)

6 Conclusions

Numerical behavior of the DQ method for solution of Biot's consolidation model is studied. A differential quadrature algorithm based on staggered grids is proposed to alleviate the oscillations produced when strong pressure gradients appear and a type of non-uniform staggered grid points is suggested. Formulations are derived and one- and two-dimensional numerical tests are given. It is shown that, pressure oscillations can be avoided completely by SGDQ for all tests if different numbers of Chebyshev-Gauss-Lobatto points are utilized for the displacement and the pore pressure.

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Figure

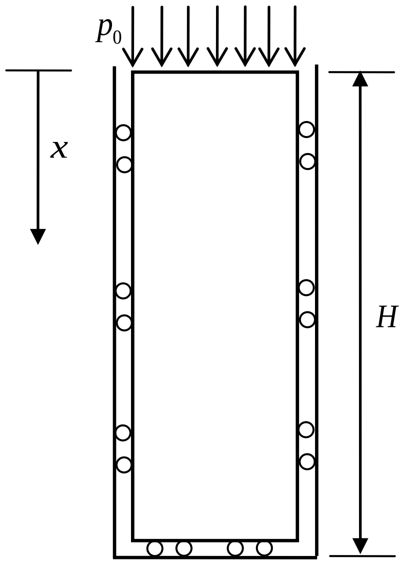
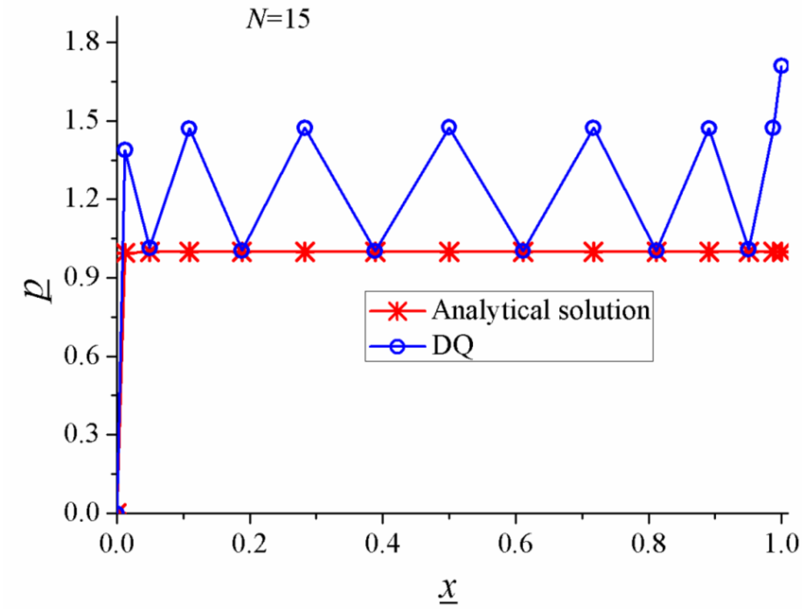
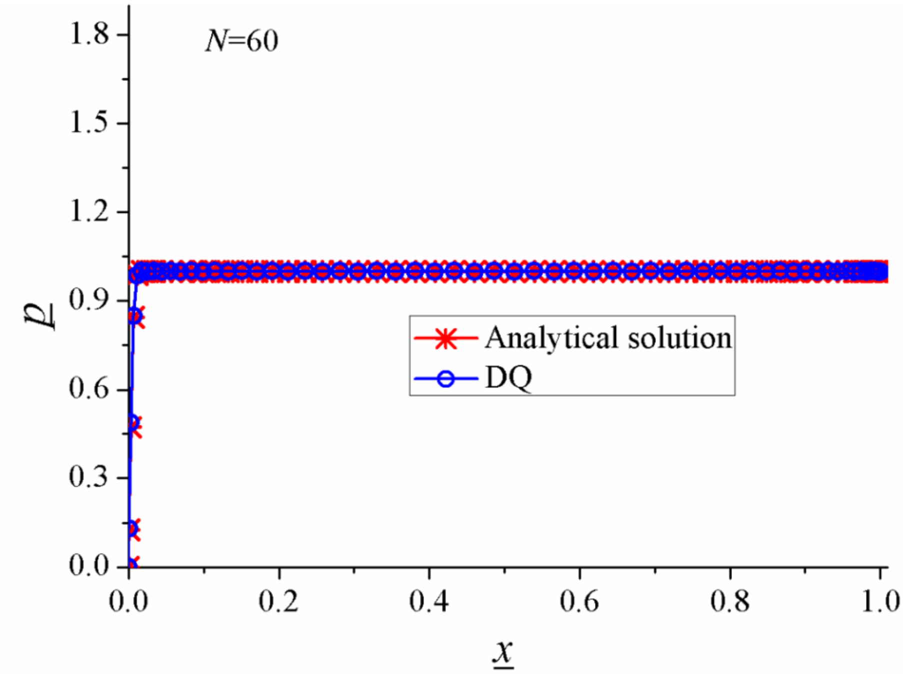


Fig. 1 One-dimensional consolidation

Figure


N=15
1.8
1.5
1.2
0.9
0.6
0.3
0.0
0.0
0.2
0.4
0.6
0.8
1.0
p
x
Analytical solution
DQ


N=60
1.8
1.5
1.2
0.9
0.6
0.3
0.0
0.0
0.2
0.4
0.6
0.8
1.0
p
x
Analytical solution
DQ

Figure
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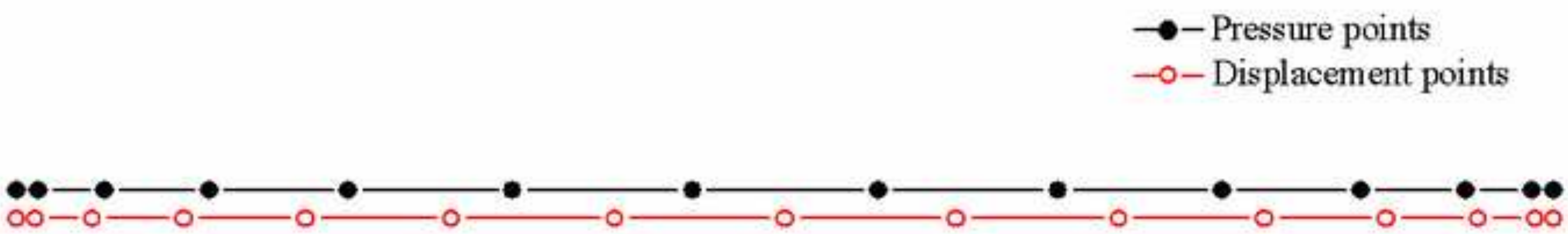
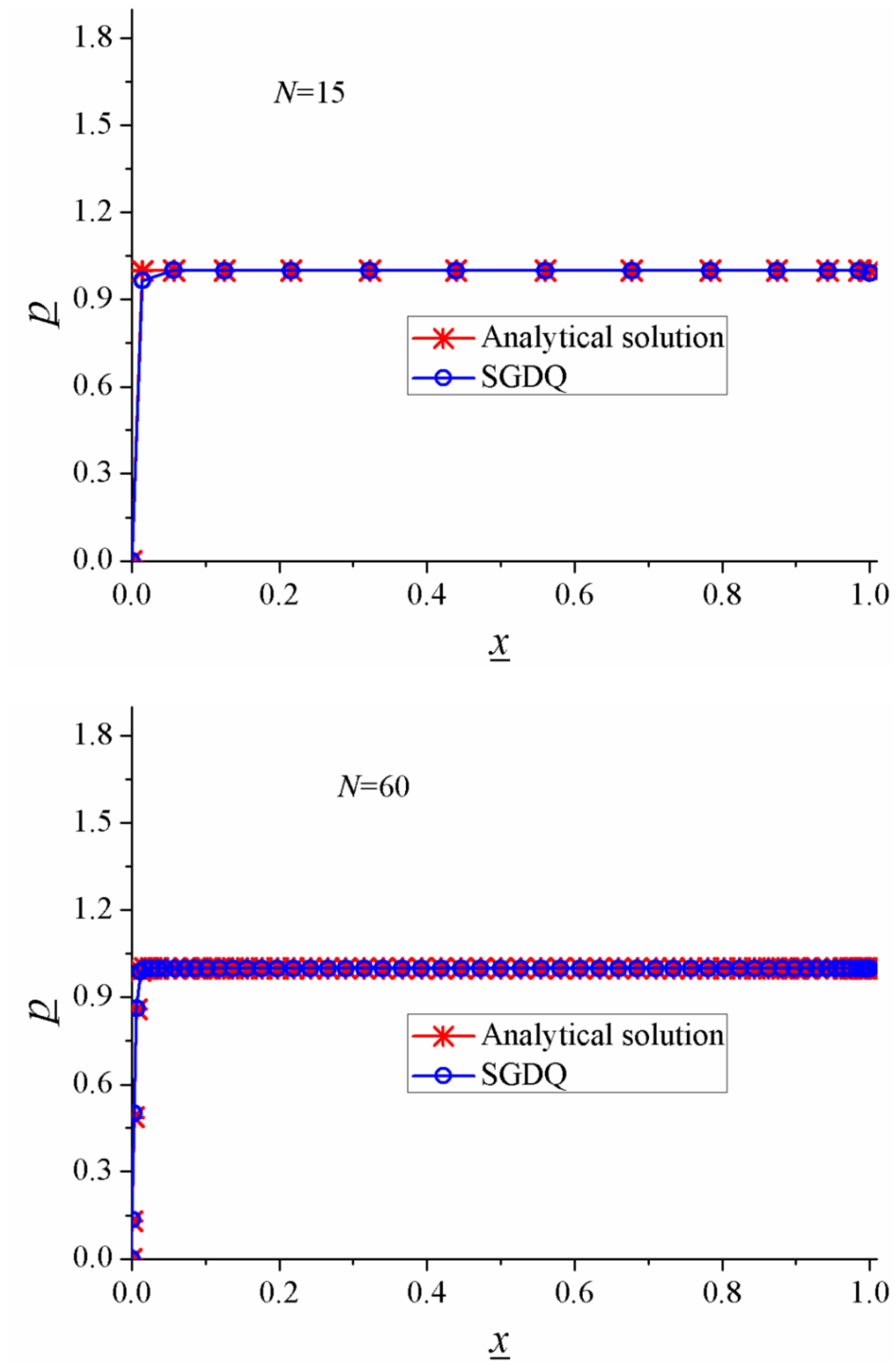


Figure



N=15
Analytical solution
SGDQ
p
x
1.8
1.5
1.2
0.9
0.6
0.3
0.0
0.0
0.2
0.4
0.6
0.8
1.0
N=60
Analytical solution
SGDQ
p
x

Figure

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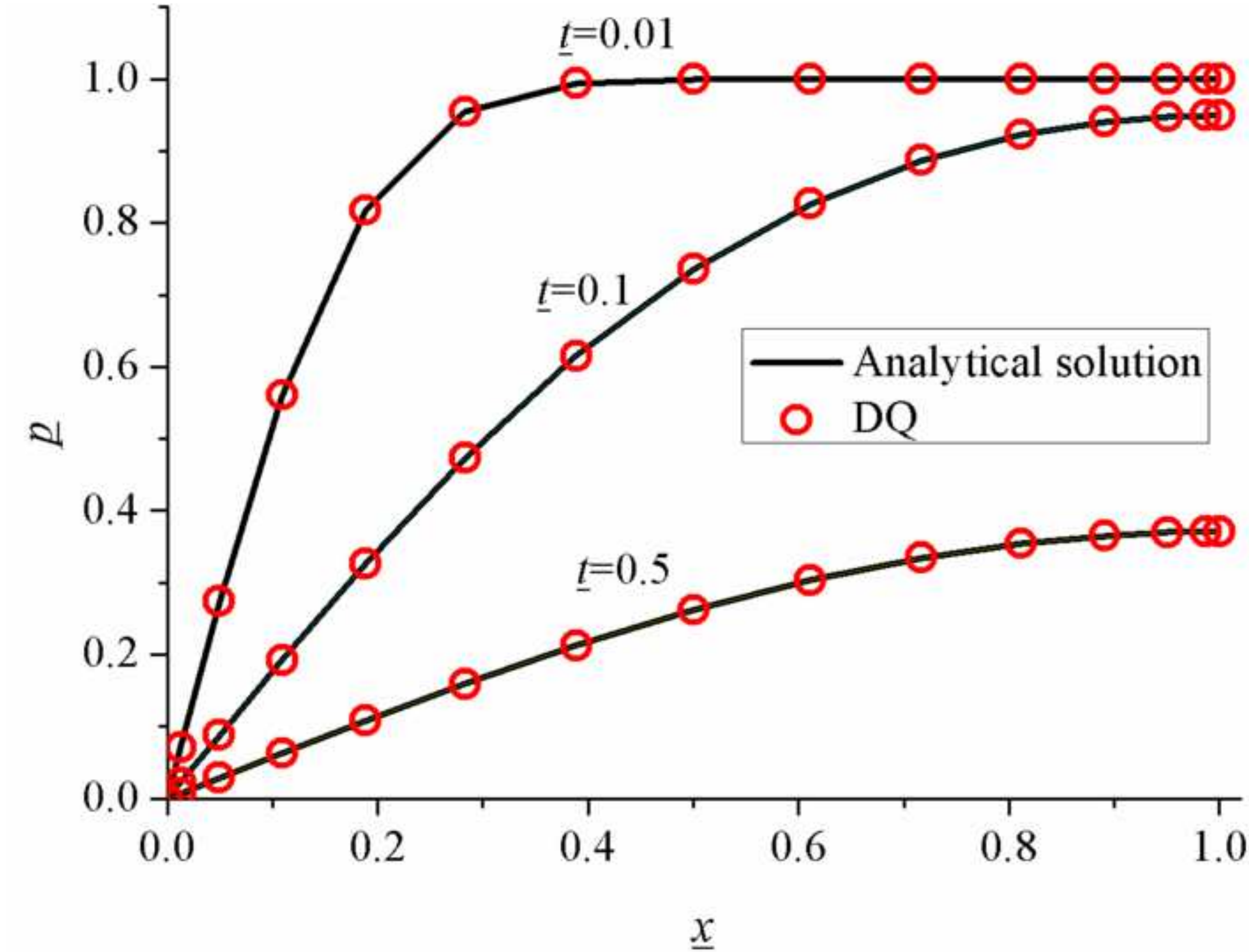

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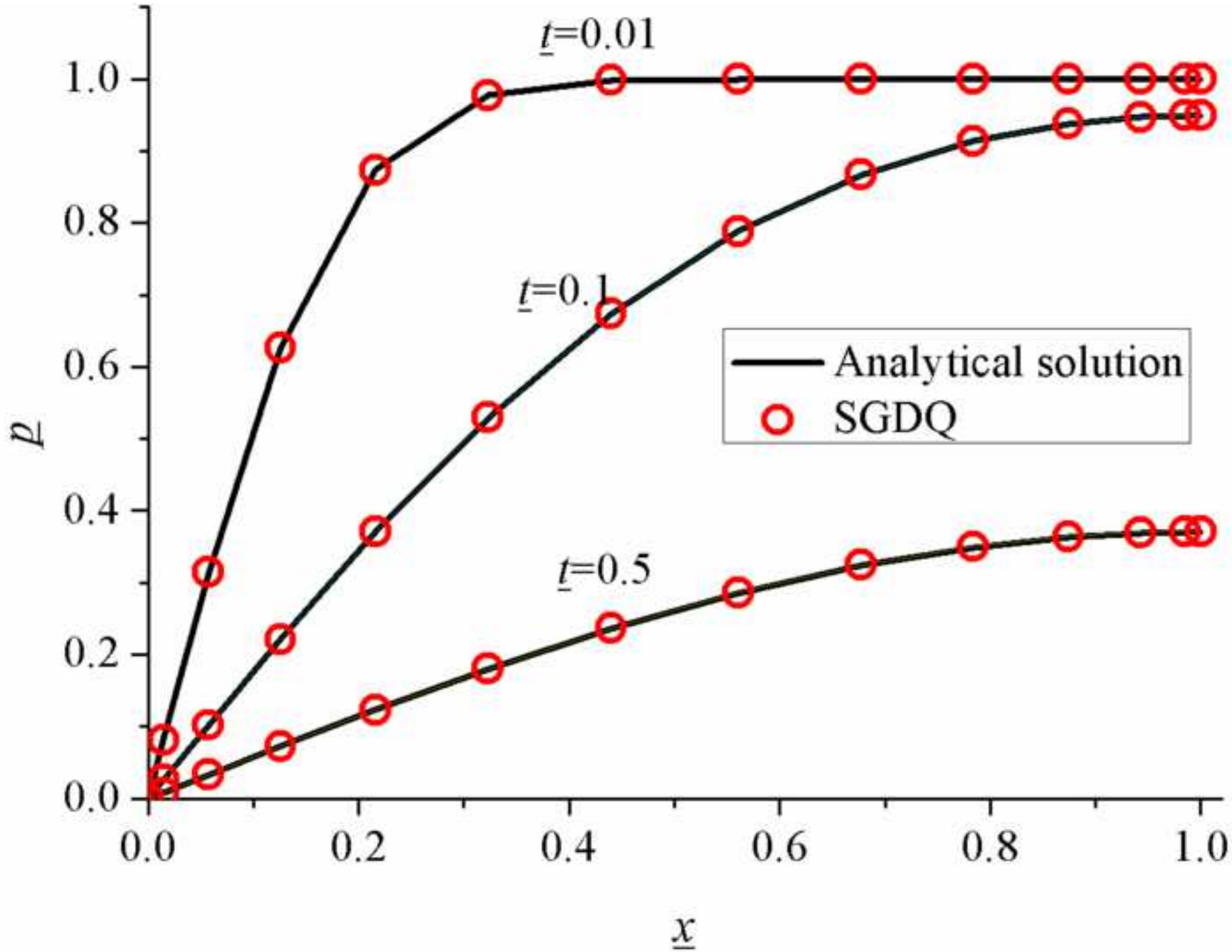

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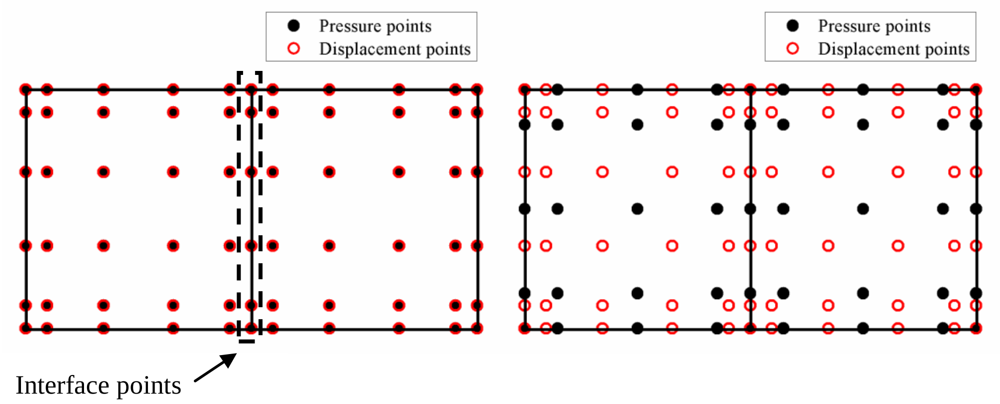

Fig. 6 Two-dimensional grid points, DQ (left), SGDQ (right)

Figure

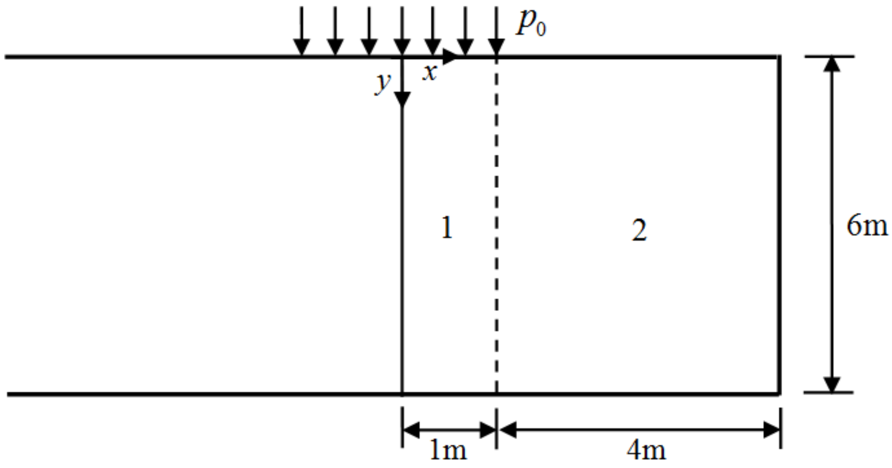

Fig. 7 Two-dimensional foundation loaded by uniform pressure

Figure
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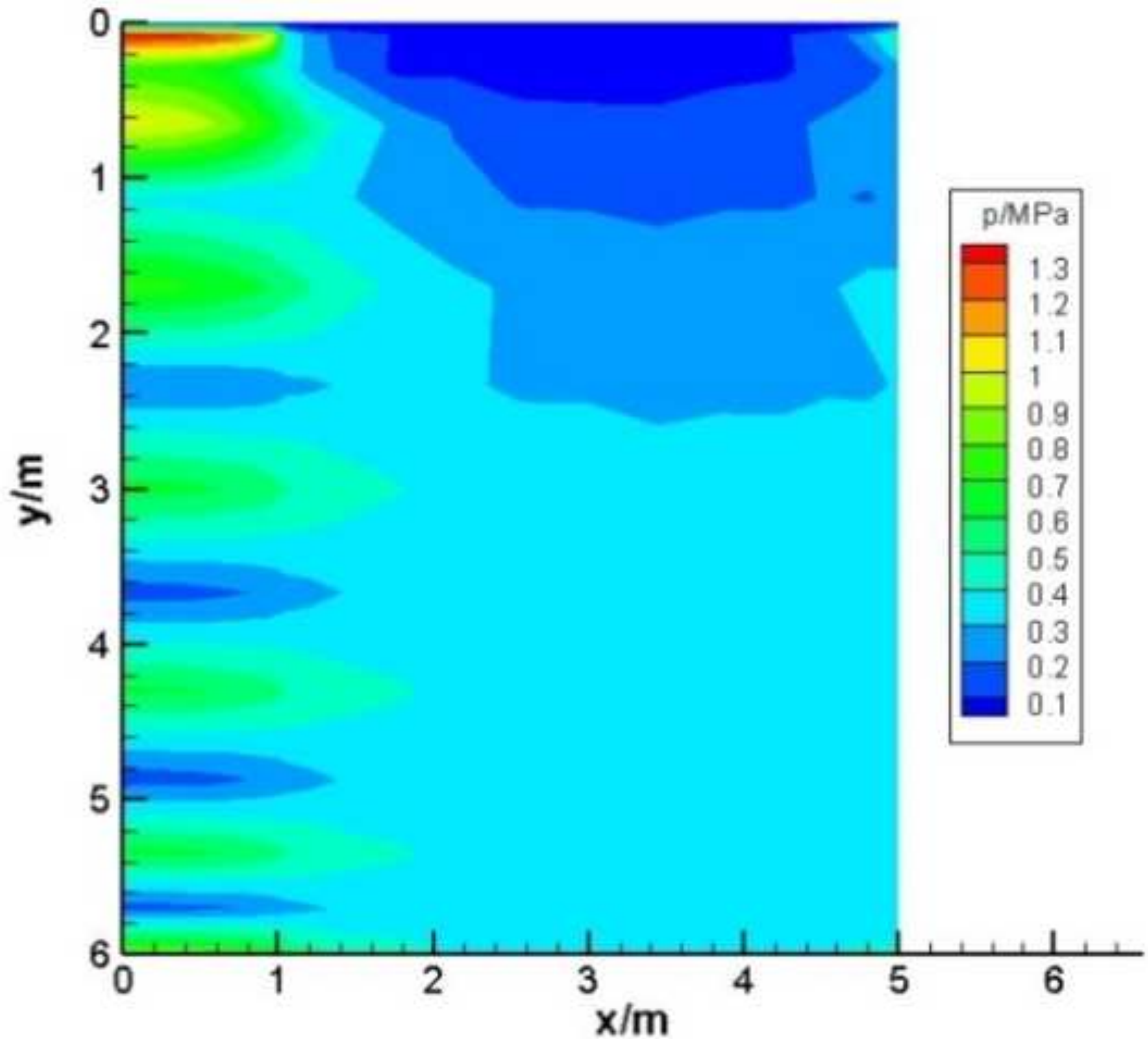

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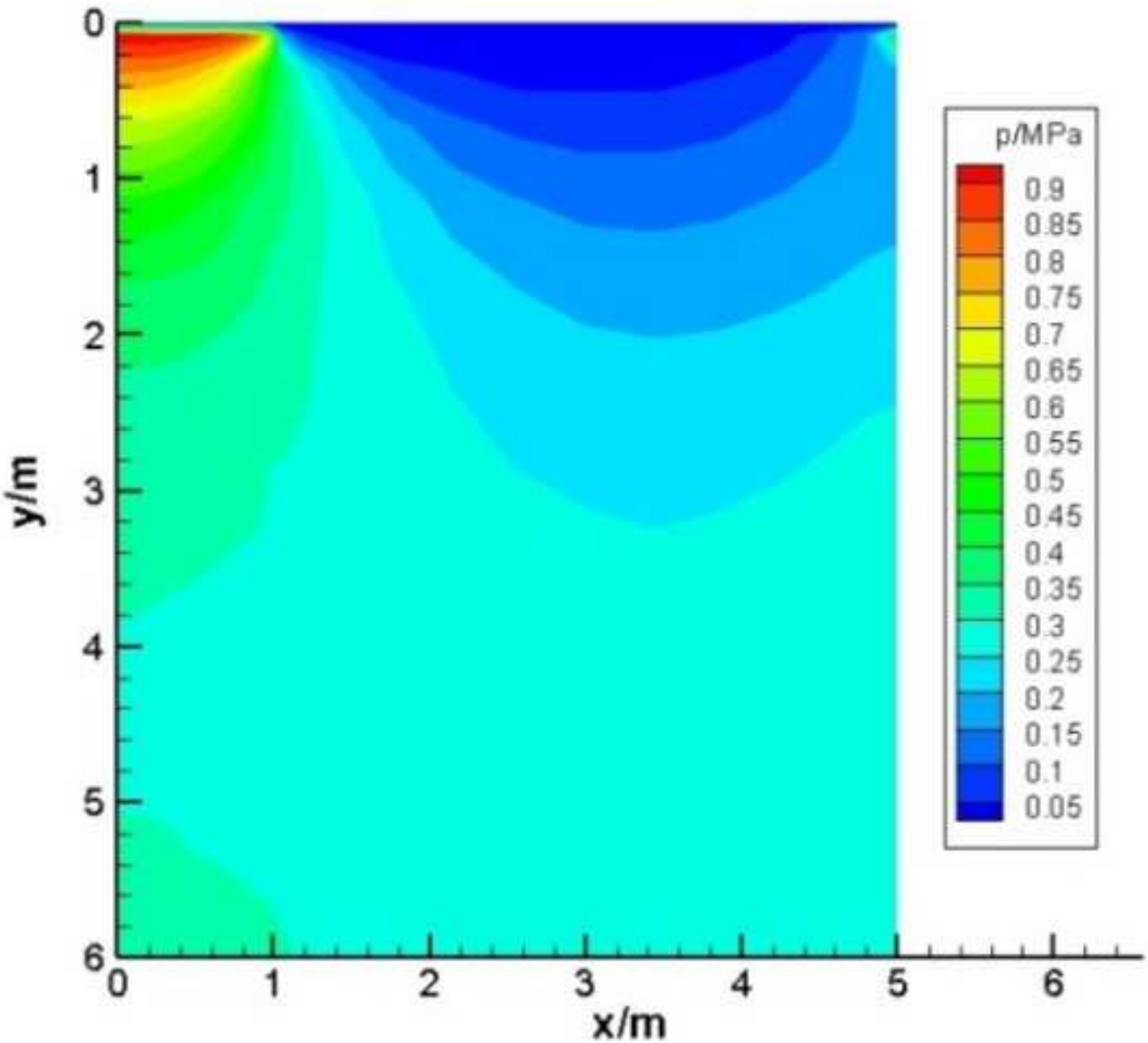

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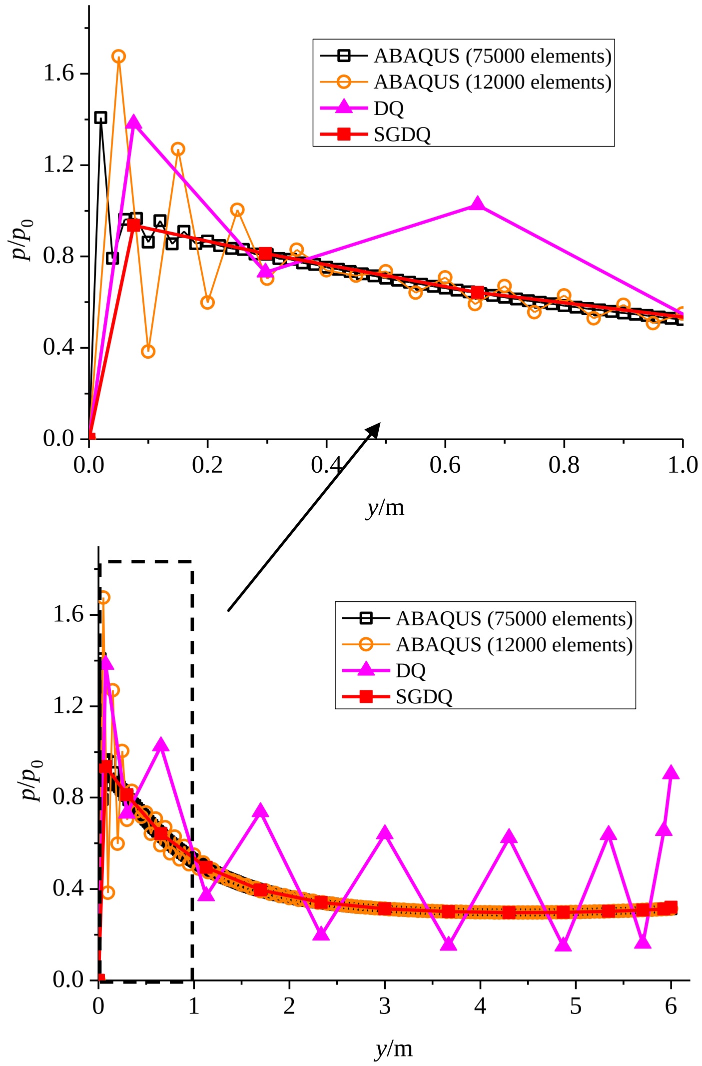

Fig. 9 Pore pressure along the centerline at t=1e-5 s (uniform pressure)

Figure

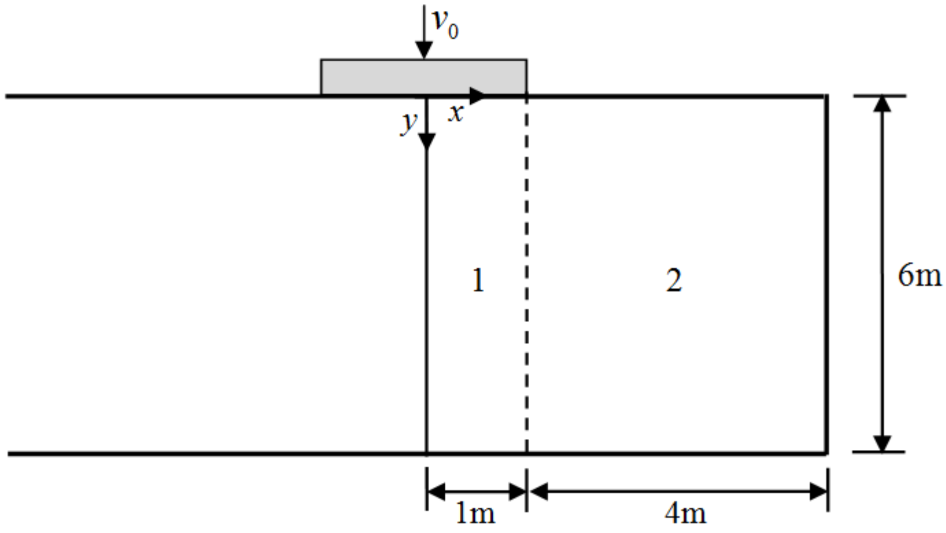

Fig. 10 Two-dimensional foundation loaded by given displacement

Figure
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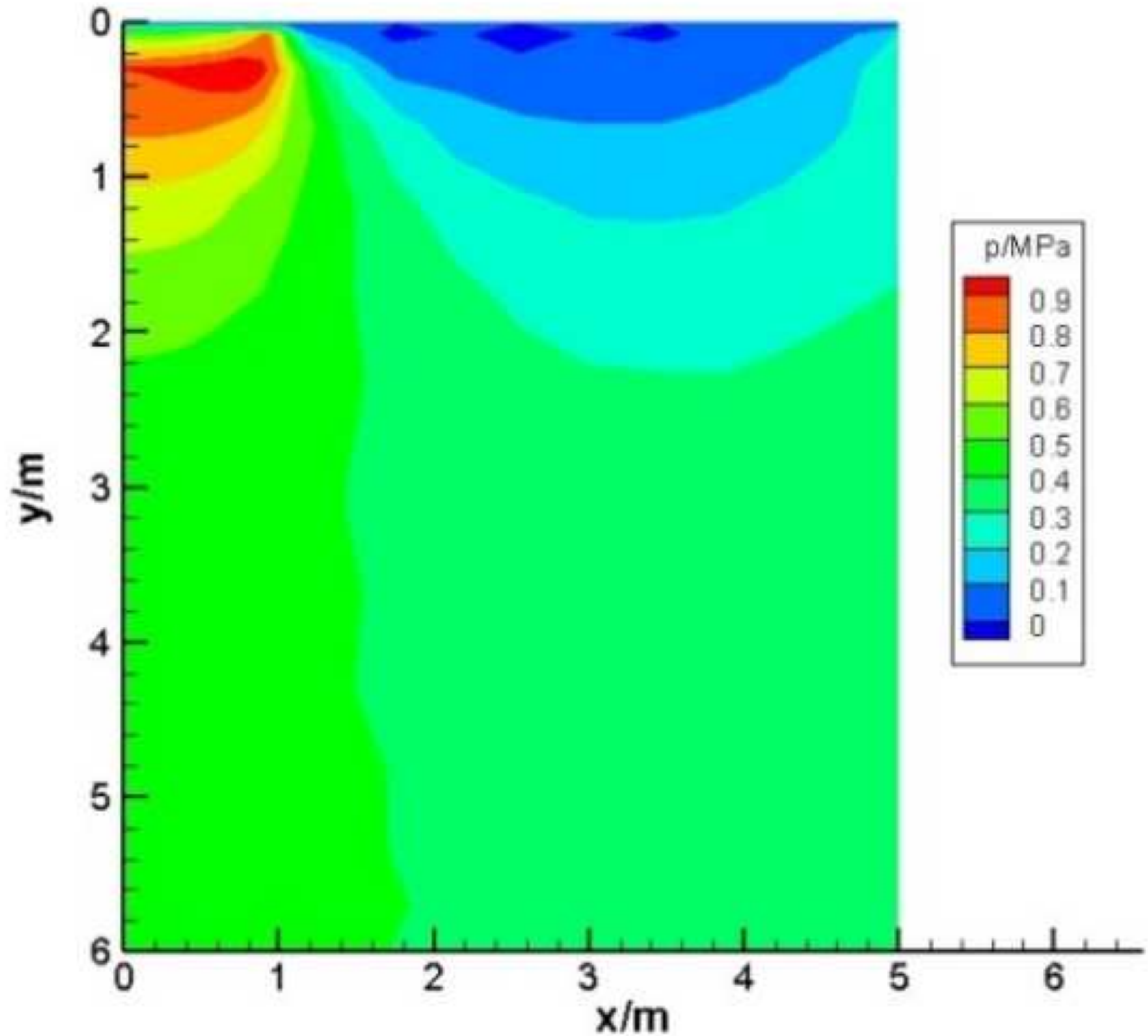

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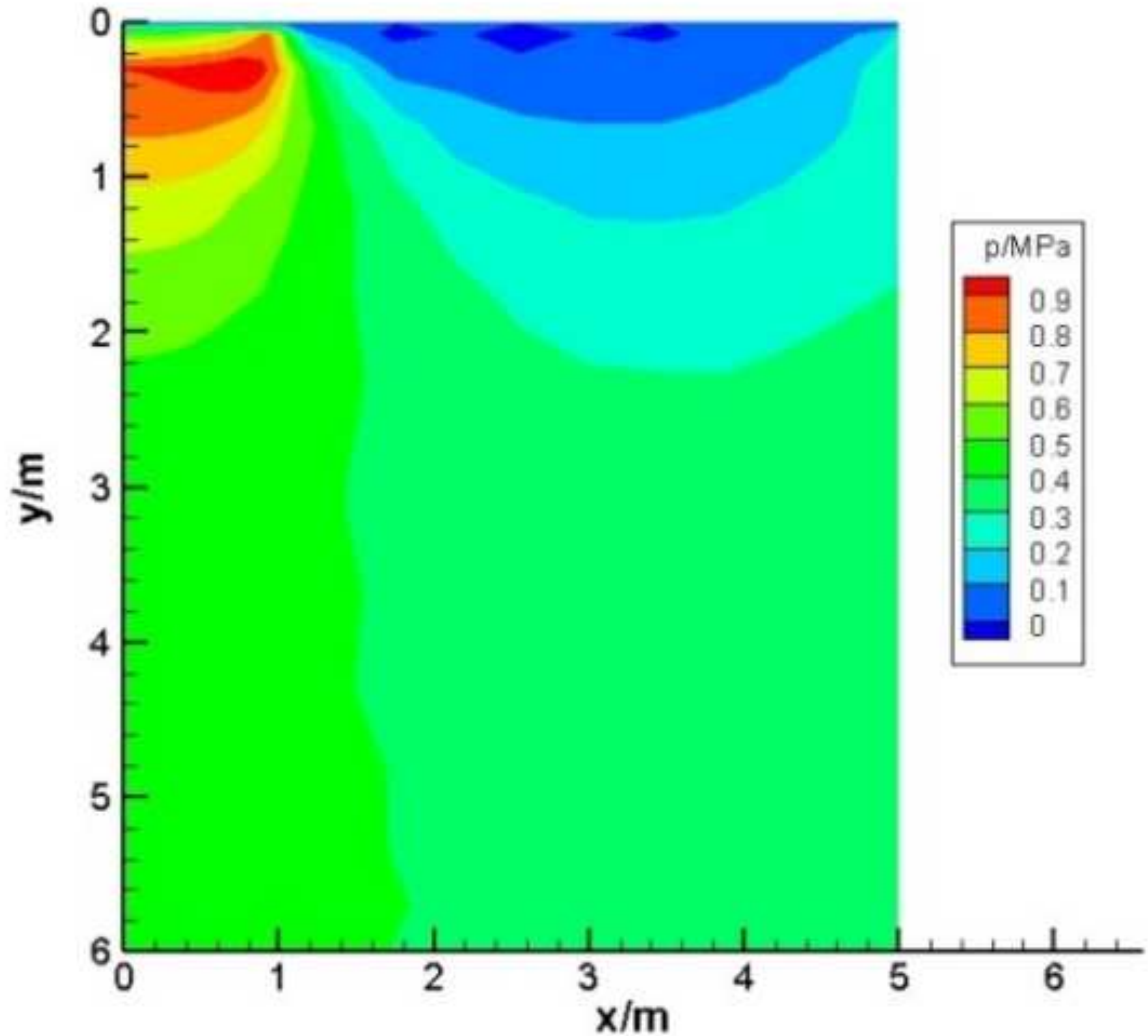

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