Foundations of Audio Signal Processing:

Exercise sheet 2

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Exercise 3.1.

Use:

$$a + bi \tag{1}$$

$$r = \sqrt{a^2 + b^2} \tag{2}$$

$$\cos(\varphi) = \frac{a}{r} \tag{3}$$

$$\sin(\varphi) = \frac{b}{r} \tag{4}$$

(a).

$$r = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16 + 16 * 3} = \sqrt{64} = 8 \tag{5}$$

$$\cos(\varphi) = \frac{4}{8} = \frac{1}{2} \tag{6}$$

$$\sin(\varphi) = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \tag{7}$$

Fulfilled by $\varphi = \frac{\pi}{3}$. Concluding:

$$4 + i4\sqrt{3} = 8e^{i\pi/3} \tag{8}$$

(b).

$$r = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{1 + 3} = \sqrt{4} = 2 \tag{9}$$

$$\cos(\varphi) = \frac{-1}{2} \tag{10}$$

$$\sin(\varphi) = \frac{\sqrt{3}}{2} \tag{11}$$

$$\tag{12}$$

Fulfilled by $\varphi = \frac{2\pi}{3}$. Concluding:

$$(-1+i\sqrt{3})^4 = (2e^{i2\pi/3})^4 = 16e^{i8\pi/3} \overset{\text{periodicity}}{=} 16e^{i2\pi/3} \tag{13}$$

(c).

$$\frac{(-1+i\sqrt{3})^4}{4++4\sqrt{3}} = \frac{8e^{i\pi/3}}{16e^{i2\pi/3}} = \frac{1}{2}e^{-i\pi/3} \overset{\text{periodicity}}{=} \frac{1}{2}e^{i5\pi/3} \tag{14}$$

(d).

Compute (1+i) first:

$$r = \sqrt{2} \tag{15}$$

$$\cos(\varphi) = \frac{1}{\sqrt{2}} \tag{16}$$

$$\sin(\varphi) = \frac{1}{\sqrt{2}} \tag{17}$$

Fulfilled by $\varphi = \frac{\pi}{4}$. So $(1+i) = \sqrt{2}e^{i\pi/4}$. Computing the original exercise:

$$2e^{i\pi/2} \cdot \sqrt{2}e^{i\pi/4} = 2\sqrt{2}e^{i3\pi/4} \tag{18}$$

Exercise 3.2.

(a).

The following figures show the requested functions.

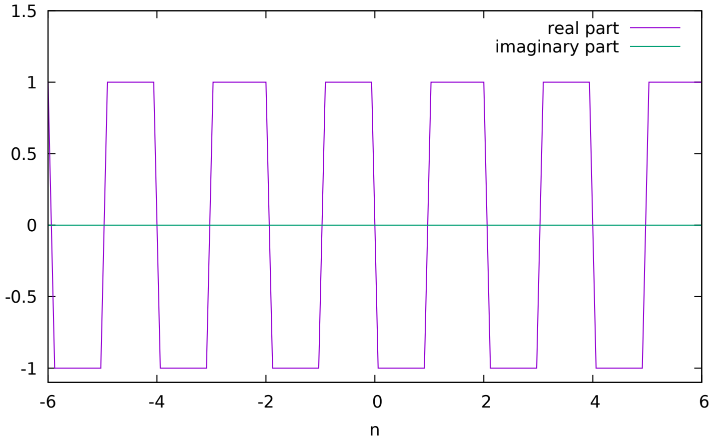

Abbildung 1: $f_{1/2}(n)$

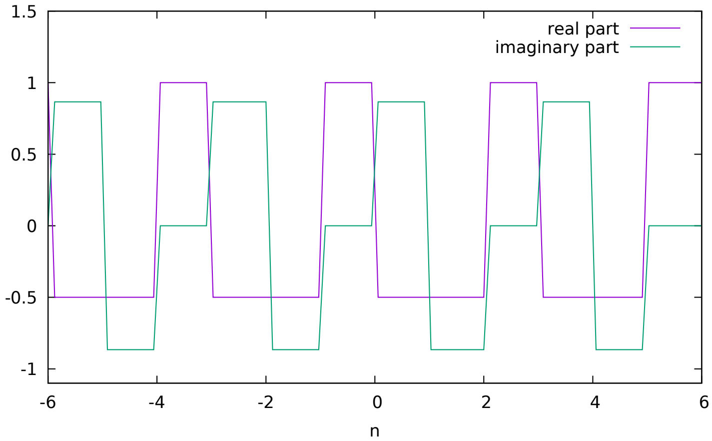

Abbildung 2: $f_{1/3}(n)$

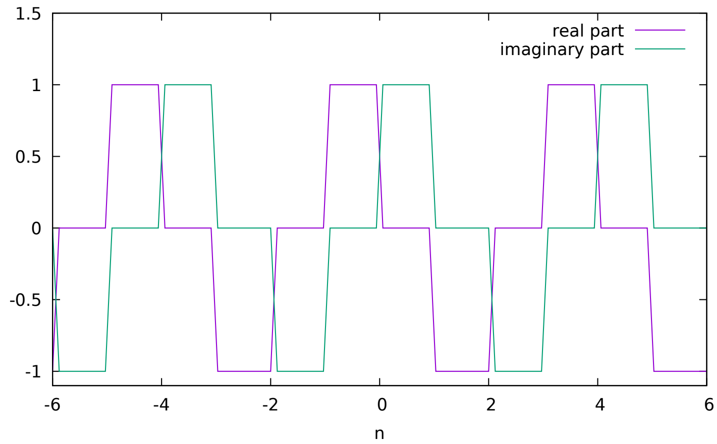

Abbildung 3: $f_{1/4}(n)$

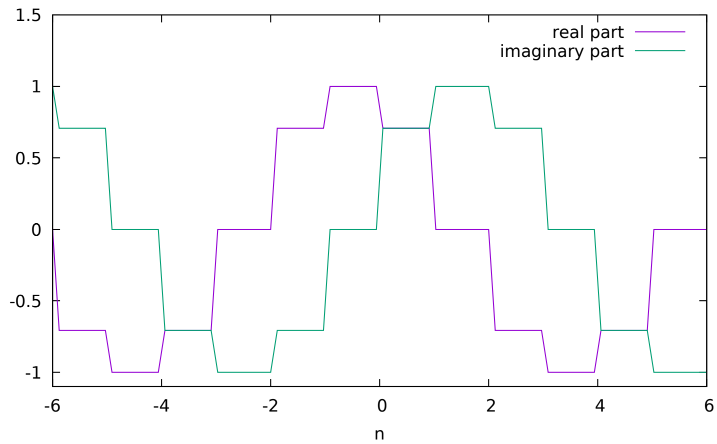

Abbildung 4: $f_{1/8}(n)$

(b).

„⇒“:

$$f_\omega \text{ periodic} \quad \Rightarrow \quad \exists p \in \mathbb{N} : f_\omega(n+p) = f_\omega(n) \quad \forall n \in \mathbb{Z} \tag{19}$$

$$\Rightarrow \quad e^{2\pi i \omega(n+p)} = e^{2\pi i \omega(n+p)} \tag{20}$$

$$\Rightarrow \quad \cos(2\pi\omega(n+p)) + i\sin(2\pi\omega(n+p)) = \cos(2\pi\omega n) + i\sin(2\pi\omega n) \tag{21}$$

$$\overset{\text{cos and sin are periodic with period } 2\pi}{\Rightarrow} \quad \exists k \in \mathbb{Z} : \quad 2\pi\omega(n+p) - 2\pi\omega n \overset{!}{=} k2\pi \tag{22}$$

$$\Rightarrow \quad \omega p = k \quad \Rightarrow \quad \omega = \frac{k}{p} \quad \Rightarrow \quad \omega \in \mathbb{Q} \tag{23}$$

„⇐“:

$$\omega \in \mathbb{Q} \quad \Rightarrow \quad \exists q \in \mathbb{Z}, r \in \mathbb{N} : \omega = \frac{q}{r} \quad \Rightarrow \quad \omega r = q \tag{24}$$

$$\Rightarrow \quad 2\pi\omega(n+r) - 2\pi\omega n = q2\pi \quad \forall n \in \mathbb{Z} \quad \overset{\text{cos and sin are periodic with period } 2\pi}{\Rightarrow} \tag{25}$$

$$\cos(2\pi\omega(n+r)) + i\sin(2\pi\omega(n+r)) = \cos(2\pi\omega n) + i\sin(2\pi\omega n) \tag{26}$$

$$\Rightarrow \quad e^{2\pi i\omega(n+r)} = e^{2\pi i\omega(n+r)} \tag{27}$$

$$f_\omega(n+p) = f_\omega(n) \quad \Rightarrow \quad f_\omega \text{ periodic} \tag{28}$$

Exercise 3.3.

$$\cos^3(x) = \frac{1}{8}(e^{ix} + e^{-ix})^3 = \frac{1}{8}(e^{ix} + e^{-ix})(e^{2ix} + 2 + e^{-2ix}) \tag{29}$$

$$= \frac{1}{8}(e^{3ix} + e^{ix} + 2e^{ix} + 2e{-}ix + e^{-ix} + e^{-3ix}) \tag{30}$$

$$= \frac{1}{4}\left[\frac{1}{2}(e^{i(3x)} + e^{-i(3x)})\right] + \frac{3}{4}\left[\frac{1}{2}(e^{ix} + e^{-ix})\right] \tag{31}$$

$$= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x) \tag{32}$$