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Computers & Operations Research 35 (2008) 1759 – 1770
computers & operations research
www.elsevier.com/locate/cor

Air defense missile-target allocation models for a naval task group

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Available online 17 October 2006

Abstract

In this study, we address the issue of allocating the air defense missiles to incoming air targets in order to maximize the air defense effectiveness of a naval task group. The shoot-look-shoot engagement policy for missile allocation is assumed to be used. Two integer linear programming models are developed and analysis of an example problem is presented. Computational results show that large instances of the proposed models can be solved within a few seconds optimally. The possible use and the extensions of the models are also discussed.

Keywords: Air defense; Weapon target allocation; Integer linear programming; Military operations research

1. Introduction

This study addresses the issue of allocating surface-to-air missiles (SAMs) to incoming air targets in a coordinated way within a naval task group (TG), which is a collection of naval combatants and auxiliaries that are grouped together for the accomplishment of one or more missions. The competing technologies of anti-ship missiles (ASMs) and defending SAM systems force the navies to update the systems and to develop new tactics continuously. All modern navies devote considerable resources to ASM defense systems [1]. The sinking of Israeli destroyer Elath by four Styx ASMs by the Egyptian Navy in 1967 was a first in naval history and the demonstration of the potential threat posed by ASMs. Six years later in 1973, 54 ASMs launched by the Syrian and Egyptian Navies failed to hit their intended targets due to defensive tactics developed by the Israeli Navy [1]. Exocet ASMs sank the British destroyer HMS Sheffield during the Falklands War in 1982. The ASM attack on the US Navy frigate Stark in Persian Gulf in 1987 is another example of the fragility of ASM defense.

Nations spend billions of dollars for their navies. However, it is still prohibitively expensive to equip all the platforms (ships) with adequate air defense systems. For many navies, equipping all the platforms with air defense systems is clearly not the best and cost effective solution. A number of navies acquire area air defense (AAD) platforms that can provide air defense support to the other ships that have limited or no effective air defense capability. Allocation of the capability of AAD ship(s) to other units in the TG is an important problem to be solved for efficient use of these platforms.

The aim of this study is to develop a model for TG air defense that captures the reality of ASM defense, generates an efficient allocation plan and measures the effectiveness of the air defense under a given scenario. A scenario is defined by the information on attacking ASMs, defending SAM systems and the formation of the TG.

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doi:10.1016/j.cor.2006.09.011

Proposed model can be categorized as a special type of weapon-target allocation (WTA) problem that can be used for evaluating the air defense effectiveness of a naval TG. WTA problem can be stated as the optimal allocation of existing weapons to a set of targets. Matlin [2], and Eckler and Burr [3] review the literature on the WTA problem. However, Matlin focuses on the problem from the attacker's perspective. Bracken and Brooks [4] argue that the WTA is not addressed much in the literature in an analytic sense after the 1972 Anti-Ballistic Missile Treaty.

The literature on WTA problem can be classified into three groups. Defense allocation models allocate defensive weapons to targets without taking into account the behavior of the opposing side. The actions of the opposing side and threat to defense are included in the scenario as a given input. Burr et al. [5], Shumate and Howard [6], Soland [7], Bertsekas et al. [8], Wacholder [9], and Jaiswal [10] consider WTA from the defenders point of view. Ahuja et al. [11] propose exact and heuristic approaches for WTA problem that minimizes the expected survival value of the targets.

The second class of approaches takes into account the opposing side's moves as well as the defensive moves. These models employ the two-person-zero-sum-game concept from game theory in the solution process. They reach the solution value of the game by assuming best defensive and offensive moves. The defense wants to minimize the maximum offense's return while the offense acts to maximize the minimum expected return. This approach is more suitable when the inventories of the opposing sides are known to some degree. Randolph and Swinson [12], Soland [13], O'Meara [14], O'Meara and Soland [15–17], Bracken et al. [18,19], and Soland [20] model both defender and attacker sides.

Rest of the literature addresses different aspects and questions within the WTA context. Simulation models and layered defense models are included in this category. Nguyen et al. [21] introduce the idea of using generating functions for evaluating the effectiveness of an air defense system. Nguyen and Reding [22,23] and Nguyen et al. [24] are the extensions of same approach. Nunn et al. [25], Orlin [26], and Mutairi et al. [27] analyze the effectiveness of layered defense systems. Interested readers are referred to Karasakal [28] for a recent survey of literature on WTA problem.

Proposed models introduce some extentions to generic WTA such as explicit resource coalescence under a shoot-look-shoot (SLS) engagement policy for maximizing the probability of no-leaker (i.e. shooting down all threat ASMs) for a naval TG. Proposed models can be seen as a customized extention of WTA problem for naval applications. Revised linear formulations can be solved using any ordinary solver for quick evaluation of air defense effectiveness of a naval TG. In this context, existing analysis methods, which can directly be used for evaluating the air defense effectiveness of a naval TG mainly consist of computer models that simulate the ASM defense. SEAROAD [29,30], JASMINE [31], SADM [32], and SAADS [33] are the examples of such models. According to our knowledge, the only analytical model that addresses resource allocation within a naval TG is Nguyen's study [34]. In his work, Nguyen studies the quantification of benefit from resource allocation for a naval TG having perfect coordination between its assets. SAMs are assumed to cover all the other ships of the TG and are capable of defending the ships within range. Other geometric and the defense system limitations such as distances of the ships, bearing and range of attacking ASM, effective range and speed of SAMs are not considered.

Griffiths et al. [35] considers a very restricted air defense scenario for a naval TG. They assume identical aircraft in line-ahead formation (i.e. aircrafts are flying towards the same bearing one after the other with a specified distance between each other) attacking a naval TG that is composed of warships with identical air defense weapons and obtain a difference equation for ship and aircraft damage. They report that their model has been used to approximate more complex scenarios as a screening process for detailed simulations. Beare [36] presents the utility of integer programming to determine the most effective mix of air defense systems. Proposed mathematical programming model functions as a selection tool to identify the most promising air defense weapon mix for scrutinized evaluation using simulation model.

The organization of this article is as follows. In Section 2, we give the problem description and explain the assumptions used. In Section 3, a detailed formulation of the problem is presented. Section 4 contains the implementation issues of the proposed models. Results are presented using a simple example. Computational results are given in Section 5. The last section contains some concluding remarks and possible use of the developed models.

2. Problem description

Consider a naval TG, composed of several ships with variable air defense capabilities, defending itself against an air attack. These ships may be either equipped with one or more SAM systems or none at all. The air defense capability may be limited to self-defense or may extend to area defense that the other ships within range can be defended. In a naval TG, the individual ships function together as a team to provide mutual support and defense against opposition to

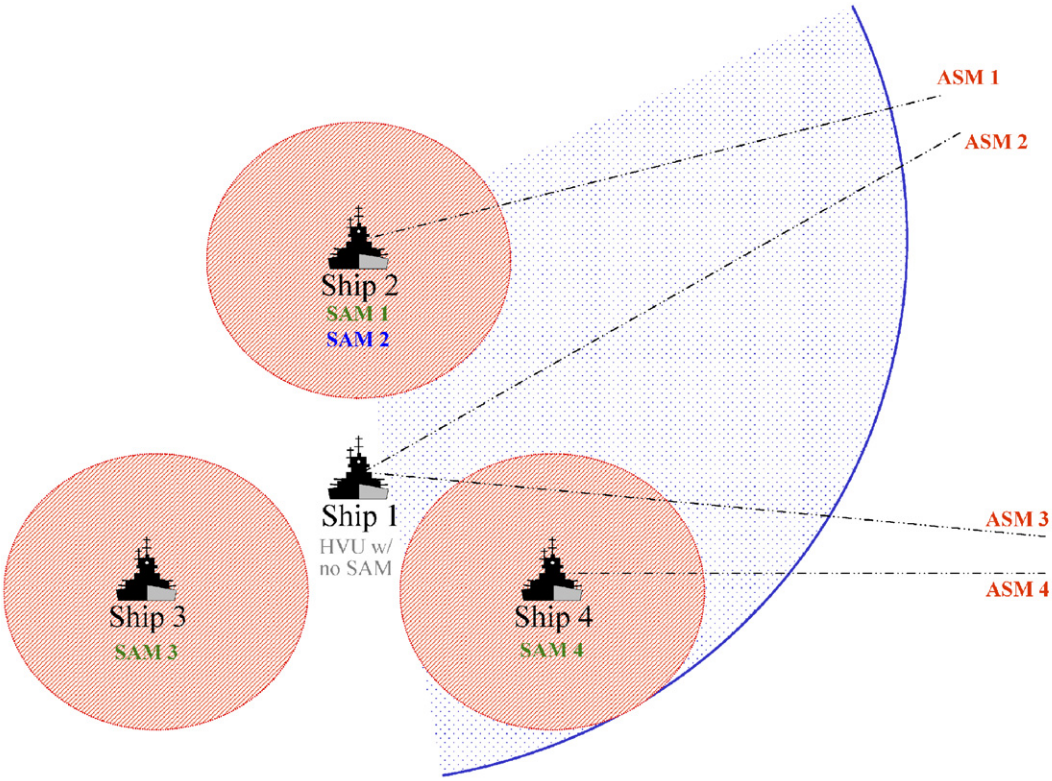

Fig. 1. Composition of a naval TG and an air attack scenario.

assigned missions. These ships are typically arrayed into a formation, called a screen, in which the most valuable and important units (termed high value units or HVUs) are surrounded and protected by the escorting vessels. Within the screen, the escort ships are stationed in sectors away from the HVU. Fig. 1 depicts a generic naval TG composition and an air attack scenario. In this scenario: a TG composed of four ships in formation, one HVU and three escort ships, is attacked by four ASMs. Ship 2 is targeted by ASM1, Ship 1 (HVU with no SAM system onboard) is targeted by ASM2 and ASM3, and Ship 4 is targeted by ASM4. There is no ASM threat to Ship 3. Ships 2, 3, and 4 have short-range self-defense SAM systems, SAM1, SAM3 and SAM4, respectively (such as NATO Sea Sparrow SAM) and the effective ranges are depicted by the circular areas around each ship. These short-range self-defense systems can only engage ASMs that are a direct threat to the ship. Ship 2 has an additional long-range area defense SAM system, SAM2 (such as SM-2 SAM) and part of its effective range is depicted by the arc drawn in bold. ASM1 can be engaged by both SAM1 and SAM2. ASM2 and ASM3 can be engaged by only SAM2. Note that SAM4 cannot engage ASM3 even if some part of the ASM3's flight path falls into the effective range of SAM4, since SAM4 is a self-defense system and can only engage the ASMs that are direct threat to it. ASM4 can be engaged by both SAM2 and SAM4.

Maximization of the probability of shooting down all incoming ASMs (i.e. the probability of no-leaker) is an important objective for TG air defense. However, saving the maximum possible number of the SAMs (for possible future attacks) from a limited number of SAMs in the magazines onboard of the ships, and the high price tag of each missile has to be considered accordingly. Several missile engagement tactics have been developed to achieve a balance between these competing objectives. One of the missile engagement tactics employed by navies is called SLS. The SLS tactic requires shooting at the target first, then looking to see if it was killed, and shooting again if necessary to achieve the kill. In this study we consider the case when the TG employs a SLS tactic and present models that maximize the probability of shooting down all incoming ASMs using that tactic. Solution procedures may be developed to include other firing policies such as shoot-shoot-look-shoot-shoot (SSLSS) and shoot-shoot-look-shoot (SSLS) policies. SLS firing policy has an implicit cost consideration. We only re-fire, if we have not already shot down the threatening ASM. Thus, we do not consume valuable SAM rounds, if it is not necessary.

An engagement process of a SAM system to an ASM can be divided into four phases. These are the tracking of the target illumination radar, the solution of the fire-control problem, the launch delay (i.e. the system delay between receiving the launch signal from the fire-control console and the missile leaving the launchers), and the flight time to the engagement. Note that this engagement process is for a generic semi-active SAM (i.e. the SAM is to be illuminated by the fire-control radar either throughout its flight or intermitted time frames during its flight). The engagement process for an active SAM (i.e., the SAM system does not need an illumination radar) may be considered only three phases excluding the tracking phase of the target illumination radar. Each engagement of both active and semi-active SAM systems takes a variable time for the last phase, which is the flight time to the engagement, and a constant time, call set-up time for the other three phases. Each engagement takes less time compared to the one before as the attacking ASM is approaching the TG. The maximum distance at which an ASM intercept can take place is determined by either the maximum effective range of the SAM system or the radar horizon of the fire-control radar against the incoming ASM, or the first detection range of the ASM if it is smaller than the above two.

When a SLS firing policy is used, there are few engagement opportunities (mostly less than 10) against each ASM. For example, an ASM with 300 m/s velocity, detected at 30 km distance can be attacked or reattacked at most four times by a SAM with 600 m/s velocity using a SLS tactic, given that the target illumination radar track time is 5 s, the fire-control solution time 2 s, and the launch delay is 2 s. Note that, we use a total of 9 s set-up time before each engagement, i.e. if we start the engagement at 30 km distance, the first intercept occurs at $30\,000\,\mathrm{m} - [9\,\mathrm{s} + (30\,000\,\mathrm{m} - 9\,\mathrm{s} \times 300\,\mathrm{m/s})/(300\,\mathrm{m/s} + 600\,\mathrm{m/s})] \times 300\,\mathrm{m/s} = 18\,200\,\mathrm{m}$. Second, third and forth intercepts occur at 10 333, 5089 and 1593 m, respectively. In reality, each engagement does not take the same set-up time, since the target illumination radar may already be on track, or the fire-control solution may already be solved. However, we use a conservative approach and consider that each engagement takes a constant set-up time.

Now, we state the basic assumptions that are needed to develop the model. The assumptions are:

- The TG sees all the air threats to intercept simultaneously. Thus, we investigate the case where the attack size is known.
- The ships in the TG are capable of coordinating the allocation of the air defense.
- The incoming ASMs are assumed to be classified such as subsonic sea-skimmer, supersonic high diver, etc. Thus, the single shot kill probability of each SAM against each ASM is known and does not change with the distance to the ASM.
- The relative positions of the ships within TG do not change as the air raid continues. The ships are thought to be stationary. This is a reasonable assumption since the speed of the ships is very low compared to the speed of the ASMs.
- Defense systems may predict the eventual target of attacking ASMs.
- The TG has both point and AAD missile systems. A point defense system may intercept the attackers that are attacking its own ship. An area defense system may intercept attackers within the area of its effective range.

We handle each SAM system separately even if they are of the same type as long as they are onboard different ships. This enables us to capture the geometric differences that need to be studied to develop the best stationing tactics for the TG.

3. Formulation of the problem

Indices:

i: Number of incoming ASMs, indexed $i \in N = \{1, \ldots, n\}$.

j: Number of SAM systems onboard the warships composing the naval TG, indexed $j \in M = \{1, \ldots, m\}$.

$v(i, j)$: Valid combinations of the ASM and the SAM systems (i.e. SAM system j can engage ASM i).

Parameters:

u_{ij}: The maximum number of missiles that can be launched from SAM system j against ASM i, $(i, j) \in v(i, j)$ using a SLS tactic. Time taken by each feasible engagement is determined by the speed of the attacking ASM, the speed of the defending SAM, the unavoidable delays between the detection of the attacking ASM and the launch of the SAM due to the detection sensors and fire-control system of the ship, and the initial distance of the attacking ASM at the beginning of the specific engagement. Thus, each engagement process takes a specified time according to the

ASM and SAM combination $v(i, j)$ and the start-time of the engagement. The SLS tactic requires us to ensure that the number of the allocated SAMs against each ASM can be scheduled in non-overlapping intervals.

p_{ij}: The single shot kill probability of SAM system j against ASM i, $0 < p_{ij} < 1$, $\quad (i, j) \in v(i, j)$.

d_j: The number of available rounds on SAM system j.

s_i: The maximum number of engagements that can be done against ASM i using a SLS tactic. This parameter will be discussed in detail after the model presentation.

Decision variables:

x_{ij}: The number of the missiles of SAM system j to be launched against ASM i, $(i, j) \in v(i, j)$.

Formulation:

The TG air defense problem, (P) can be formulated as follows:

$$\text{(P)}\ \max \quad \prod_{i \in N} \left[1 - \prod_{\{j \in M | (i,j) \in v(i,j)\}} (1 - p_{ij})^{x_{ij}} \right] \tag{1}$$

s.t.

$$\sum_{\{i \in N | (i,j) \in v(i,j)\}} x_{ij} \leqslant d_j, \quad \forall j \in M, \tag{2}$$

$$\sum_{\{j \in M | (i,j) \in v(i,j)\}} x_{ij} \leqslant s_i, \quad \forall i \in N, \tag{3}$$

$$0 \leqslant x_{ij} \leqslant u_{ij}, \quad \forall i, j \in v(i, j) \quad \text{and} \quad x_{ij} \text{ is integer.} \tag{4}$$

Objective function (1) maximizes the probability of shooting down all incoming ASMs. Note that in (1) we implicitly assume that air defense of each ship in TG is independent of each other. Self-defense SAM systems and other self-defense systems such as anti-air warfare gun batteries coincide with this assumption. Support provided by an area defense SAM system to an individual ship may depend on the support to another ship, when the number of available missiles in the area defense SAM system is the limiting factor. However, our assumption is reasonable considering the requirement for keeping the model mathematically tractable. Constraint set (2) reflects the restriction on the number of rounds available for each SAM system. Constraint set (3) limits the total number of engagements that can be done against each ASM. Constraint set (4) imposes integer restriction and lower and upper limits on the decision variables. The upper limit is determined by the maximum number of engagements that can be done during the engageability duration of each valid ASM and SAM combination using a SLS tactic.

Constraint set (3) actually employs a loose upper bound on the total number of engagements that can be done against each ASM, when more than one SAM system can engage the ASM along its flight path.

If more than one SAM system can engage an incoming ASM, calculating the maximum number of engagements against an ASM may be cumbersome. Fig. 2 depicts an example of such a situation. Both SAM1 and SAM2 can engage the ASM1 and their engageability durations are overlapping. Clearly $s_i \leqslant \sum_{\{j \in M | (i,j) \in v(i,j)\}} u_{ij}$. However, this upper bound will not be tight when the overlap in engageability durations is large. Developing a tight bound for s_i is required in order to be able to have a feasible SLS allocation. Let k_i denote the number of different SAM systems that can engage ASM i, $k_i = \sum_{\{j \in M | (i,j) \in v(i,j)\}} 1$. Then there are 2^{k_i} different combinations of SAM systems that can be used against ASM i. For a thorough control of the upper limit of the possible engagements in a SLS tactic, we need to determine the upper limit for each combination since the speeds of the SAMs vary. This would require $2^{k_i} - (k_i + 1)$ additional constraints. We already account for k_i combinations using the constraint set (4) and the other combination accounts for the no SAM engagement. Note that we impose the upper bounds of single combinations through u_{ij}. Instead of $2^{k_i} - (k_i + 1)$ constraints we develop an approximation with only one constraint. The engageable portion of the flight path of an ASM, denoted by l, can be divided into parts where the number of SAMs that can engage the ASM is different than the neighboring parts. For example, in Figure 2 the flight path of ASM1 is divided into three parts l_1, l_2, l_3. SAM2 is the only one that can engage ASM1 in part l_1. Both SAM1 and SAM2 can engage in part l_2. In the last part, l_3, only SAM1 can be used against ASM1. In this way, the speed of the fastest SAM for each part of the flight path can be used to calculate s_i for each ASM i. Assume that ASM1 with a speed of 280 m/s has been detected at 30 km distance of Ship 1. SAM1 with a speed of 1200 m/s has maximum and minimum effective ranges of 10 000 and 200 m, respectively. SAM2 with a speed of 600 m/s has 30 000 and 6000 m maximum and minimum effective ranges from

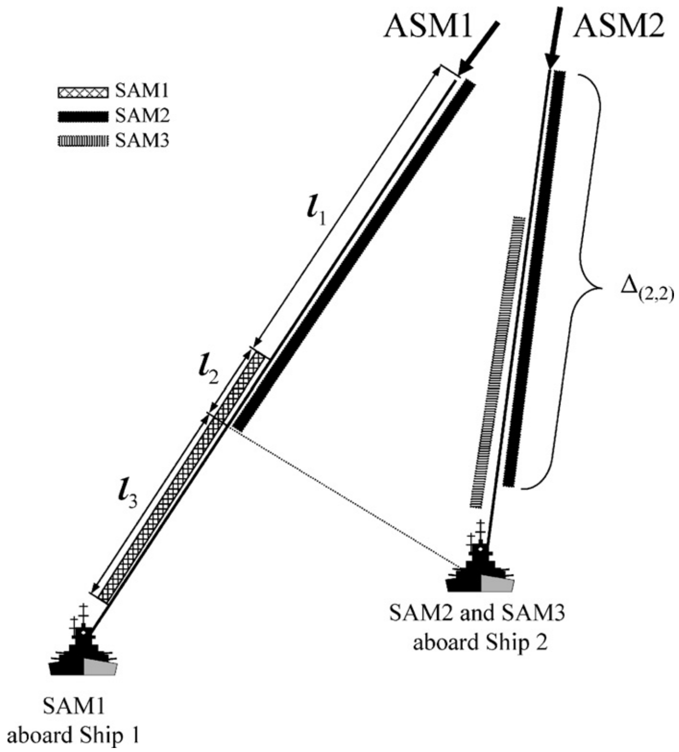

Fig. 2. An illustration of the ASM engageability durations for different SAM systems. Note that SAM2 is assumed to be an area air defense system and can engage any ASM within its effective range. SAM1 and SAM3 are assumed to be self-defense systems.

Ship 1, respectively. Then, in part l_1, SAM2 can intercept ASM1 two times at distances of 18 736 and 11 056 m away from Ship 1. Since SAM1 is faster than SAM2, SAM1 can engage in part l_2. Only one intercept occurs at a distance of 6064 m when the engagement starts at a distance of 10 000 m. In the last part, l_3, SAM1 can intercept ASM1 two times at distances of 2821 and 240 m. Thus, in this example the maximum number of engagements against ASM1 is five.

If there are two SAM systems that can engage an ASM, then this approximation is exact, i.e. we consider the case with one type of SAM system using the constraint set (4) and the case with two types of SAM systems using the constraint set (3). If more than two SAMs can engage one or more ASMs, then the allocation may require a feasibility check after solving the problem depending on the required precision of the results. Note that the proposed model is a screening tool to identify the promising TG composition and formation for scrutinized investigation.

Problem (P) is a non-linear integer-programming model. WTA problem with a convex objective function is shown to be NP-complete [37]. Our model has additional constraints and additional product form in the objective function compared to original WTA problem. Since non-linear programs are harder to solve than their linear counterparts and to make the problem more tractable, we eliminate the non-linearity from the model by applying transformations to (P).

We present the development of TG air defense models in two phases. In the first phase, we transform problem (P) to a more tractable form. This is an intermediate step in reaching the eventual formulations and is included for a better exposition of the idea. In the second phase, we derive two different models from the core formulation. Those models are slightly different from each other. Features and drawbacks of each model will be discussed in detail after its formulation.

Let h_i be the minimum desired probability of shooting-down ASM i, $0 < h_i < 1$, $i \in N$. Then we can add a new constraint set to the problem ensuring that the desired probability levels are met. Addition of new constraints to the problem does reduce the feasible solution space and may lead to the problem becoming infeasible. On the other hand, that modification opens way to develop efficient solution procedures for the problem. We define the modified problem (P′) as follows:

Objective function (to be discussed later)

$$\begin{aligned}&\text{s.t.}\\ &1-\prod_{\{j\in M|(i,j)\in v(i,j)\}}(1-p_{ij})^{x_{ij}}\geqslant h_i \quad \text{for all } i\in N, \qquad (5)\\ &(2), (3), \text{ and } (4).\end{aligned}$$

The constraint set (5) requires the allocation of enough SAMs that meet the desired probability of shooting-down each ASM. If the model gives a feasible solution, it shows that the desired probabilities set forth for each incoming ASM can be met within the limits of the defensive potential.

The non-linearity in the constraint set (5) can be transformed into linearity by using logarithms as in Kwon et al. [38]. Since $0<a\leqslant b$ if and only if $\ln(a)\leqslant\ln(b)$ where $a, b\in\Re$, taking the logarithms of Eq. (5) does not affect the solution. Then equation set (5) becomes: $\sum_{\{j\in M|(i,j)\in v(i,j)\}}\ln(1-p_{ij})x_{ij}\leqslant\ln(1-h_i)$ for all $i\in N$. We may further continue our linearization process by multiplying both sides with a large number, say β, and then rounding down. This approximation is reasonable from a practical point of view since the values of the coefficients in the inequalities come from probabilistic estimates. This gives an approximation of the feasible region with integer coefficients and transforms the problem from a non-linear integer linear programming model into an integer linear programming model.

Maximizing the air defense effectiveness of SAM systems might be a desirable objective function. However, proposed models use effectiveness in the constraints. Thus, we have been able to linearize the effectiveness term. We develop two different models that use the effectiveness idea in the objective function in a slightly different form below.

Assuming the objective function is constant, the resulting integer linear program after the above-described transformation is still not much of use in practical sense. It only gives us whether the desired probability levels are achievable or not. However, we can guarantee reaching a feasible solution by making a minor modification to the model. If we introduce an artificial SAM system that can engage every ASM and has a large inventory, then the model becomes a flexible one that reaches a feasible solution whatever the desired probability levels, h_i, are. Let j^* denote the artificial SAM system. We revise the set definitions, M and $v(i, j)$ including the artificial SAM system accordingly. If we penalize the use of the artificial SAMs in the objective function, and set the desired probability levels, h_i for all $i\in N$ very high (say $h_i=0.99$ for all $i\in N$) then the model will minimize the use of artificial SAMs and maximize the use of real SAMs to achieve the desired levels for the probability of shooting down each ASM. The artificial SAMs will meet the under-achievement.

The resulting integer linear programming model (P1) is:

$$\text{(P1)}\ \min\quad \sum_{i\in N}x_{ij^*} \qquad (6)$$

$$\text{s.t.}\quad \sum_{\{j\in M|(i,j)\in v(i,j)\}}a_{ij}x_{ij}\geqslant b_i \quad \text{for all } i\in N, \qquad (7)$$

$$(2), (3), \text{ and } (4),$$

where $a_{ij}=\lfloor-\beta\ln(1-p_{ij})\rfloor$ and $b_i=\lfloor-\beta\ln(1-h_i)\rfloor$.

In model (P1) x_{ij^*} is implicitly defined as an integer variable to present the idea clearly. However, we can define a continuous variable instead of x_{ij^*} depending on our preferences. We can define e_i as the deviation from the right-hand side value for ASM i in equation set (7) and minimize the sum of e_i. We may want to reduce the number of integer variables for a generic mixed integer-programming solver, or want to keep model pure integer for other solution procedures such as Lagrangean relaxation, polyhedral or heuristic approaches. In model (P1), determination of the value of p_{ij^*} should be made carefully. In order to minimize the chance of having degenerate solutions p_{ij^*} should be kept small.

An ideal weapon allocation solution is the one that maximizes the probability of shooting down each threat. Model (P1) does not guarantee to achieve the best solution for each threat ASM. However, it minimizes the *total* deviation from the desired probability levels. Thus, it does not distinguish each ASM, but considers all threats together.

Model (P1) minimizes the total number of artificial SAMs used to achieve the desired probability levels. However, (P1) is not very sensitive to the individual deviations for each ASM. Thus it is possible to have a larger deviation from the desired probability level of one ASM and very small or no deviations for the rest. This may lead us to a second formulation. We may easily convert the model (P1) to a model that minimizes the maximum deviation from the desired probability levels (or) indirectly minimizes the maximum number of artificial SAMs used. In this new model (P2), we define a single decision variable, e, instead of the artificial SAM of model (P1). Define the sets M and $v(i, j)$ as in the original definition of the sets excluding the artificial SAM. Then the model (P2) can be written as

$$\text{(P2) min} \quad e \tag{8}$$

$$\text{s.t.} \quad \sum_{\{j \in M \mid (i,j) \in v(i,j)\}} a_{ij}x_{ij} + e \geqslant b_i \quad \text{for all } i \in N, \tag{9}$$

$$e \geqslant 0, \tag{10}$$

$$(2), (3), \text{ and } (4),$$

where e is the decision variable that shows the maximum deviation from the right-hand side values of equation set (9).

Both models, (P1) and (P2), can be solved using a standard mathematical programming package for reasonable size problems. The application of the models and comparison of the solutions are presented in the next section.

4. Implementation of the models

Models, (P1) and (P2), have been implemented using GAMS (General Algebraic Modeling Language) mathematical programming package and solved using OSL Solver [39].

We show the results of the proposed models (P1) and (P2) on a simple example. The example is depicted in Fig. 3. Ship 1 has only self-defense SAM system, and Ship 2 has both self defense and area defense SAM systems (SAM2 is the area defense system). Assume that all necessary calculations for generating the input data have been performed.

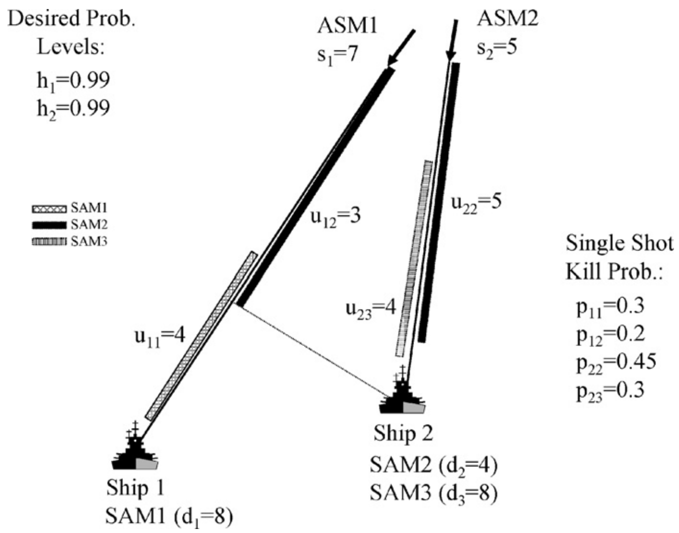

Fig. 3. Example of an air defense scenario.

Table 1
Interceptor allocation plans generated by the models

Model		SAM1	SAM2	SAM3
(P1)	ASM1	4	1	
	ASM2		3	2
(P2)	ASM1	4	3	
	ASM2		1	4

Table 2
A subset of possible allocation plans

Allocation plan no.	ASM1		ASM2		Probability of shooting		Probability of no-leaker
	SAM1	SAM2	SAM2	SAM3	ASM1	ASM2	
1	3	4	0	4	0.860	0.760	0.653
2	4	3	1	4	0.877	0.868	0.761
3	4	2	2	3	0.846	0.896	0.759
4	4	1	3	2	0.808	0.918	0.742
5	4	0	4	1	0.760	0.936	0.711

The SAM allocation plans generated by the models, (P1) and (P2), are reported in Table 1. An allocation plan shows which SAMs should engage which ASMs with how many missiles. For example, "4" in the last row and the last column of the Table1 means that SAM3 is to engage ASM2 with 4 missiles.

Since the example is small, we can write down a subset of possible allocations that ensure maximum number of SAMs allocated against each ASM while keeping the allocation plans feasible. Table 2 shows these allocation plans and their respective probability measures. All the other allocation plans for this problem will have less SAMs to allocate and will achieve lower levels for the probability of shooting-down the ASMs and the probability of no-leaker for the TG.

The SAM allocation plan no. 4 in Table 2 is the same as the plan generated by the model (P1). The model (P2) generates the plan no. 2, which has the highest probability of no-leaker for the TG.

Models found the optimal allocation plan and an allocation plan close to optimal for the sample problem in terms of the probability of no-leaker for the TG. However, model (P2) generated a more balanced allocation in the sense that the probability of shooting-down each ASM can differ reasonably (i.e. deviations from the right-hand side values of equation set (7) are 298 and 213 for (P1) and deviations of equation set (9) are 254 and 261 for (P2), where $\beta = 100$). Detailed experimental results for both models are given in the next section.

5. Computational results

We randomly generated test problems for specified numbers of ASMs, self-defense SAM systems and area defense SAM systems. We assume that an area defense SAM system can engage ASMs within its effective range, while a self-defense SAM system can only engage to the ASM that is a direct threat to itself.

We created a sample single shot kill probability matrix for ASM and SAM systems from a uniform distribution in the interval [0.05, 0.80]. The number of available rounds on SAM systems was generated from a discrete uniform distribution in the interval [2, 16]. These values may represent a number of real ASM and SAM systems according to the information found in open sources. The maximum numbers of engagements that can be done against ASMs were taken from a discrete uniform distribution in the interval [3, 7].

We generated 36 different problems with number of ASM and SAM systems ranging from 3 to 81. The number of self-defense SAM systems range from 3 to 45 and the number of area defense SAM systems range from 0 to 36. The number of area defense SAM systems were less than or equal to the number of self-defense SAM systems. The largest problem size (i.e. 81 ASMs and 81 SAM systems) is unrealistically large for a naval scenario. However, we generated large test problems in order to be able show the problem sizes that can be solved effectively. A realistic air defense scenario for a naval TG will probably have less than 10 ASMs and 10 SAM systems. We solved the generated problems using both models, (P1) and (P2). The results are shown in Table 3. Elapsed times for all of the 72 runs were less than 1 s. We compare the solutions of (P1) and (P2) in terms of probability of no-leaker for the whole TG. Model (P1) produced better results in 22 out of 36 problems. Model (P2) produced better results in 6 problems. Since the computational time is cheap for both models, we can solve both models and use the better one in terms of probability of no-leaker for evaluating the air defense effectiveness of a naval TG.

We have chosen five different problem sizes from Table 3 and carried out 10 runs for each problem size for evaluating the variability of elapsed time for solving the models. A total of 50 problems were solved using each model. All problems were generated using different random number streams. Minimum, average and maximum times required to solve the

Table 3
Results of randomly generated test problems

Number of ASMs	Number of SAMs		Probability of no-leaker		Elapsed time (s)[a]	
	Self-defense	Area defense	(P1)	(P2)	(P1)	(P2)
3	3	0	0.154	0.154	0.020	0.010
3	3	1	0.732	0.732	0.010	0.010
3	3	2	0.748	0.732	0.010	0.030
3	3	3	0.849	0.751	0.010	0.010
6	3	0	0.012	0.012	0.010	0.020
6	3	1	0.425	0.492	0.011	0.010
6	3	2	0.883	0.883	0.010	0.010
6	3	3	0.725	0.664	0.020	0.010
6	6	0	0.038	0.038	0.040	0.010
6	6	1	0.538	0.538	0.090	0.010
6	6	2	0.648	0.562	0.040	0.050
6	6	4	0.884	0.814	0.010	0.010
6	6	6	0.964	0.965	0.080	0.010
9	3	0	0.030	0.050	0.010	0.010
9	3	1	0.046	0.053	0.010	0.040
9	3	2	0.723	0.648	0.010	0.020
9	3	3	0.748	0.759	0.010	0.010
9	6	0	0.022	0.022	0.020	0.010
9	6	1	0.113	0.112	0.020	0.020
9	6	2	0.387	0.378	0.020	0.020
9	6	4	0.879	0.839	0.010	0.020
9	6	6	0.925	0.916	0.020	0.020
9	9	0	0.244	0.244	0.020	0.060
9	9	1	0.088	0.075	0.010	0.010
9	9	2	0.105	0.165	0.020	0.010
9	9	4	0.778	0.679	0.010	0.020
9	9	6	0.855	0.799	0.011	0.020
9	9	8	0.899	0.878	0.020	0.020
18	9	9	0.867	0.818	0.070	0.070
27	18	9	0.739	0.503	0.140	0.120
36	18	18	0.811	0.747	0.190	0.170
45	27	18	0.706	0.459	0.230	0.210
54	27	27	0.686	0.582	0.250	0.220
63	36	27	0.644	0.481	0.330	0.350
72	36	36	0.612	0.514	0.520	0.520
81	45	36	0.545	0.314	0.650	0.670

[a]CPU time on a personal computer with AMD Athlon 2000+ CPU and 256 MB of RAM.

Table 4
Summary of elapsed time for test problems (10 runs for eac size)

Number of ASMs	Number of SAMs		Elapsed time for P1 (s)[a]			Elapsed time for P2 (s)[a]		
	Self-defense	Area defense	Min	Average	Max	Min	Average	Max
18	9	9	0.05	0.07	0.09	0.06	0.08	0.09
27	18	9	0.12	0.13	0.16	0.12	0.12	0.14
45	27	18	0.20	0.24	0.34	0.19	0.22	0.32
63	36	27	0.33	23.05	226.99	0.31	0.41	0.74
81	45	36	0.59	0.63	0.72	0.61	0.82	2.37

[a]CPU time on a personal computer with AMD Athlon 2000+ CPU and 256 MB of RAM.

problems for each model type and problem size are depicted in Table 4. Only one run (i.e. model (P1) with 63 ASMs, 36 self-defense SAMs and 27 area defense SAMs) out of 100 runs took more than 3 s.

However, two instances of the problem with 27 ASMs, 18 self-defense SAMs and 9 area defense SAMs have not been able to be solved within 3600 s. These results show that we can rarely encounter problems that are computationally hard to solve. However, note that test problems are unrealistically large for real life scenarios.

6. Conclusion and future work

In this study, we develop realistic models for TG air defense problem. We make use of the formation information such as relative bearings and distances between ships as well as the specifics of attacking missiles. A generic engagement policy, shoot-look-shoot is assumed. Different types of attacking ASMs and different types of defending SAM systems are allowed. These assumptions are reasonable when TG operates in a formation and encounters an immediate air attack by ASMs. Considering the fact that a naval TG likely composes of less than 10 warships and a simultaneous ASM attack wave with less than 10 missiles, proposed models can be solved easily for realistic scenarios using any standard integer-programming solver.

The proposed models do not necessarily maximize the probability of no-leaker for TG, but provide approximate solutions for it by using an indirect approach. The proposed solution procedure is applied to randomly generated test problems. The quality of the results represents the potential value and the use of the models.

Air defense of a TG requires very quick reaction. The duration of an air attack might range from tens of seconds to a few minutes at most. Coordination and allocation of the air defense systems of the ships within the TG is utmost importance. Note that the speed of the ships is very small compared to the speed of the air attack. It may take from tens of minutes to several hours to change the formation from one to the other, while it takes tens of seconds from detection to time-on-target for an ASM. Thus, it is important to be in a suitable formation before a possible air attack. Proposed models may be used to investigate the effectiveness of the air defense formations under different scenarios in an exploratory analysis setting. We may improve the effectiveness of air defense of TG by predefining a robust formation against a perceived threat by evaluating the effectiveness of different TG formations against several probable or possible scenarios. Proposed models can be used to approximate realistic scenarios as a screening process for detailed simulations.

Composition of a naval TG and effectiveness of different SAM systems to protect the TG can also be investigated using proposed models.

Future work might involve developing solution methods (both exact like branch and bound or approximate like heuristic search) to compare both the computational time and quality of solution with the solver output. It could also be useful to investigate non-linear solution methodologies to the original formulation.

Acknowledgment

A part of this research was carried out while the author was visiting the Operational Research Division in the Canadian Department of National Defense. The author would like to thank the anonymous referees for their valuable comments and suggestions for improving the paper.

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