ASEN 3112

Spring 2020

Lecture 23

Whiteboard

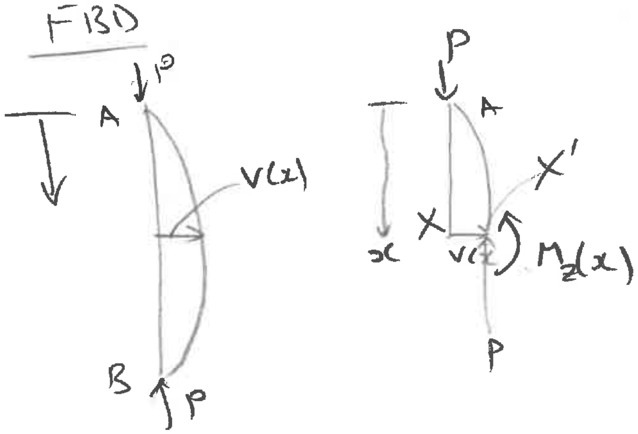
April 16, 2020

Stability (buckling) of continuous structures

Consider Euler Column (most simple case)

Similar to vibrations
- Buckling is governed by ODE
- Assume a form of solution.
- Key factor is BC's (analogous to IC's in free vibrations; also we have BC's in continuous)

FBD



$$\Sigma M_{X'} = 0$$

$$M_z(x) + P v(x) = 0$$

From beam theory: $M_z(x) = E I_{zz} v''(x) = E I v''(x)$

$$EI v''(x) + P v(x) = 0 \text{ —— ODE}$$

Dividing by EI

$$\boxed{v''(x) + \lambda^2 v(x) = 0} \leftarrow \text{Canonical ODE}$$

$$\lambda = \sqrt{\frac{P}{EI}}$$

General Solution

$$v(x) = A\cos\lambda x + B\sin\lambda x$$ — 2 constant

Apply BC's : $\underbrace{v(0) = 0 , \quad v(L) = 0}_{\text{2 BC's}}$

2 BC's
satisfactory for finding
2 constants

$$v(0) = A\underbrace{\cos\lambda(0)}_{1} + B\underbrace{\sin\lambda(0)}_{0} = 0$$

$$\Rightarrow \boxed{A = 0}$$ BC equation 1

$$v(x) = B\sin\lambda x$$

$$v(L) = \boxed{B\sin\lambda(L) = 0}$$ BC equation 2

$B = 0$ trivial solution $\Longrightarrow$ Equilibrium state
$B \neq 0$ buckling state

This implies $\underline{\sin\lambda L = 0}$ characteristic equation.

This type of characteristic equation is called
transcental equation in λ

$$\lambda L = n\pi, \quad n = 1, 2, \ldots \qquad \text{———} (*)$$

Recall $\lambda^2 = \dfrac{P}{EI}$

$\lambda^2 L^2 = n^2\pi^2$

$$\Rightarrow P_{cr,n} = \frac{n^2\pi^2 EI}{L^2}, \quad n = 1, 2 \ldots$$

Critical load (infinite of them)

Lowest is "the" critical load

$$P_{cr} = P_{cr,1} = \frac{\pi^2 EI}{L^2}$$

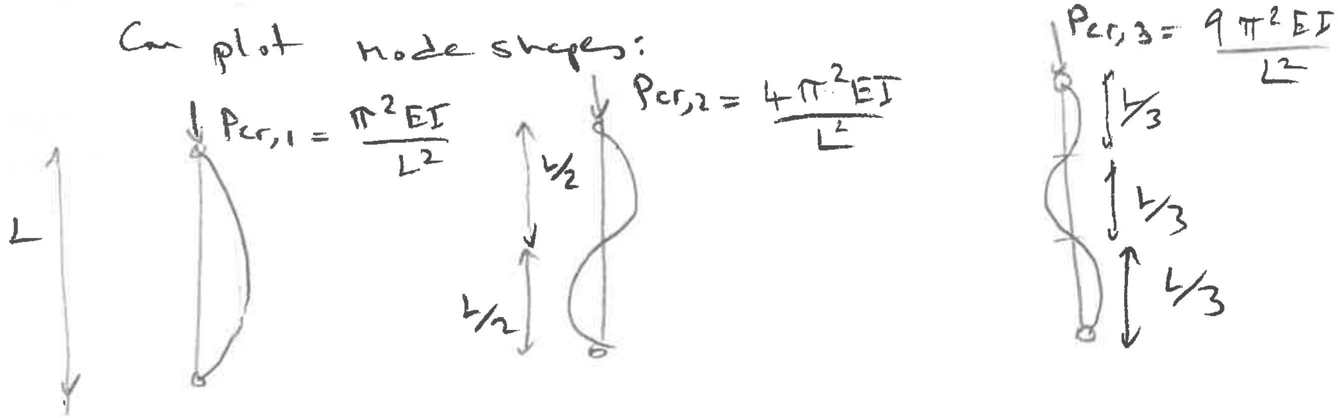
Corresponding mode shape for n^{th} critical load:

$$v_{cr,n}(x) = B \sin\Big(\underbrace{\frac{n\pi}{L}}_{\lambda} x\Big) \quad \text{[from Eq (*)]}$$

Select $n=1$, to find $v_{cr,1} = v_{cr}$

$$v_{cr}(x) = v_{cr,1}(x) = B \sin \frac{\pi}{L} x$$

Can plot mode shapes:



Alternative approach for getting characteristic eq.

Write BC equations in matrix form

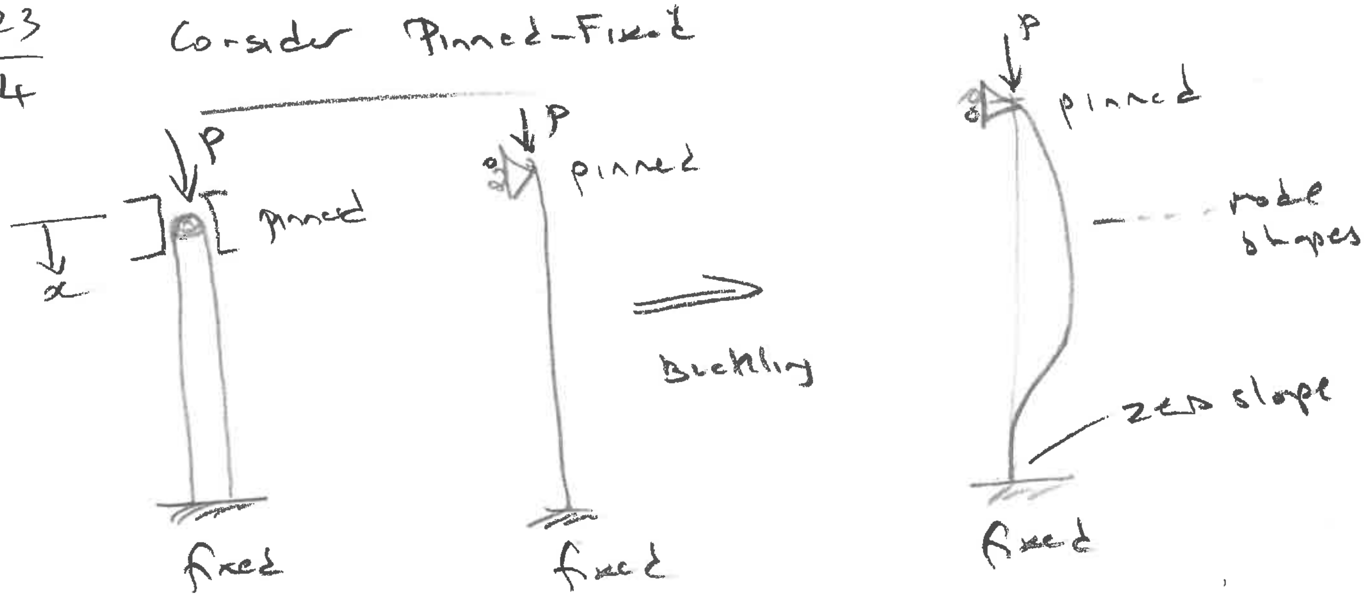
$$\begin{bmatrix} 1 & 0 \\ 0 & \sin\lambda L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Tends to be more useful approach for more complex BC

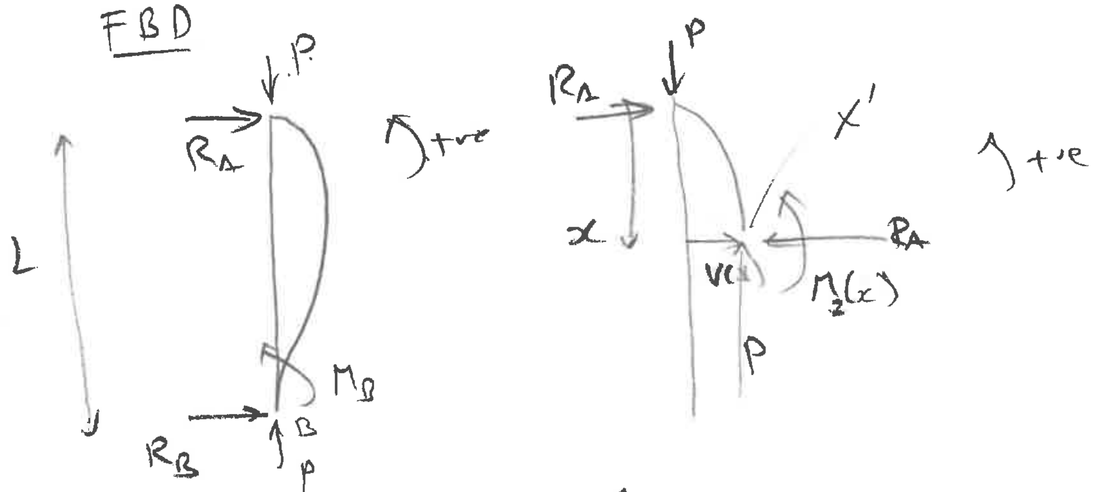
$$\Longrightarrow \begin{vmatrix} 1 & 0 \\ 0 & \sin\lambda L \end{vmatrix} = 0 \Longrightarrow \sin\lambda L = 0$$

L23/4

Consider Pinned-Fixed



FBD



$M_B = R_A L$

$R_A = \dfrac{M_B}{L}$

$\sum M_{x'} = 0$

$$\underbrace{M_z(x)}_{EI\,v''(x)} + P\,v(x) = R_A\, x = \frac{M_B}{L} x$$

$$\boxed{v''(x) + \lambda^2 v(x) = \frac{M_B}{EIL} x = \frac{\lambda^2 M_B}{PL} x}$$

Canonical ODE

Assume

$$v(x) = v_h(x) + v_p(x)$$

$$v(x) = \underbrace{A\cos\lambda x + B\sin\lambda x}_{v_h(x)} + \underbrace{Cx}_{v_p(x)}$$

Constants: A, B, C

$v_p(x) = Cx \implies$ plug into ODE

$$\lambda^2 Cx = \frac{\lambda^2 M_B}{PL} x \implies C = \frac{M_B}{PL}$$

$$v(x) = A\cos\lambda x + B\sin\lambda x + \frac{M_B}{PL} x$$

BCs

$x = 0 \quad v(0) = 0 \implies \boxed{A = 0}$

$x = L \quad v(L) = 0 \implies B\sin\lambda L + \frac{M_B}{P\cancel{L}}\cancel{L} = 0$

$$\boxed{B\sin\lambda + \frac{M_B}{P} = 0}$$

$x = L \quad v'(L) = 0$

$$\left(v'(x) = \frac{dv}{dx} \right) \implies \boxed{B\lambda\cos\lambda L + \frac{M_B}{P} = 0}$$

BC equations

Consider last 2 BC equations:

$$\frac{\sin\lambda L}{\lambda\cos\lambda L} = L \implies \boxed{\tan\lambda L = \lambda L}$$ characteristic Eq

smallest root of transcedental equation is $\lambda L = 4.493$

$$P_{cr} = \frac{20.19\,EI}{L^2}$$

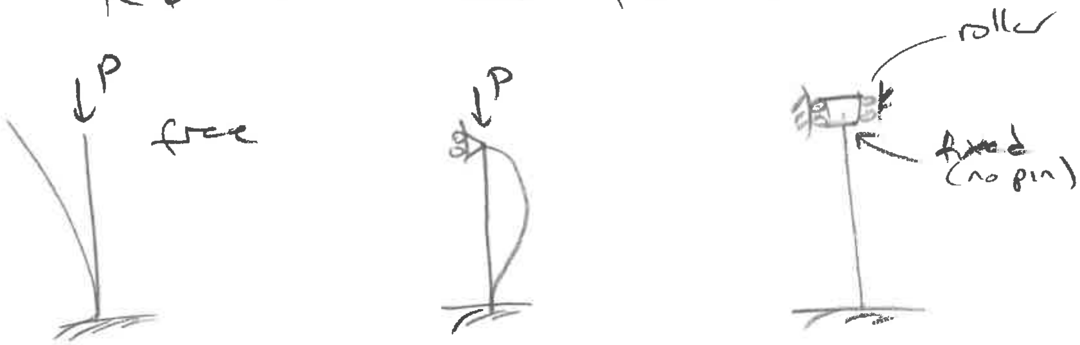
$2.019 \simeq 2.05\,\pi^2$
(twice critical load for pinned-pinned problem)
— less sensitive to buckling.

$\frac{L23}{6}$

$P_{cr} = \frac{\pi E I}{(KL)^2}$ $K = 0.7$ effective length

$K \uparrow \rightarrow P_{cr} \downarrow \Longrightarrow$ more unstable

$K \downarrow \rightarrow P_{cr} \uparrow \Longrightarrow$ More stable



P_{cr} < P_{cr} < P_c

$K = 2$ | $K = 0.7$ | $K = 0.5$

more unstable | | more stable

$\longrightarrow$

stability increase

Comment on friction (in practice)