ASEN 3112 - Structures - Spring 2020
Homework 11 Solutions

This is a practice homework; it is NOT TO BE HANDED IN

Problem 11.1: 2-DOF Discrete System

Two rigid-bar struts of equal length *a* are connected at the joints and at the bottom support by frictionless hinges as shown in Figure 10.1(a). The column is compressed by axial load *P*. The struts are maintained in vertically straight positions by torsional springs of the stiffnesses shown in the figure. Use linearized prebuckling (LPB) analysis (thus assuming very small displacements and rotations) throughout.

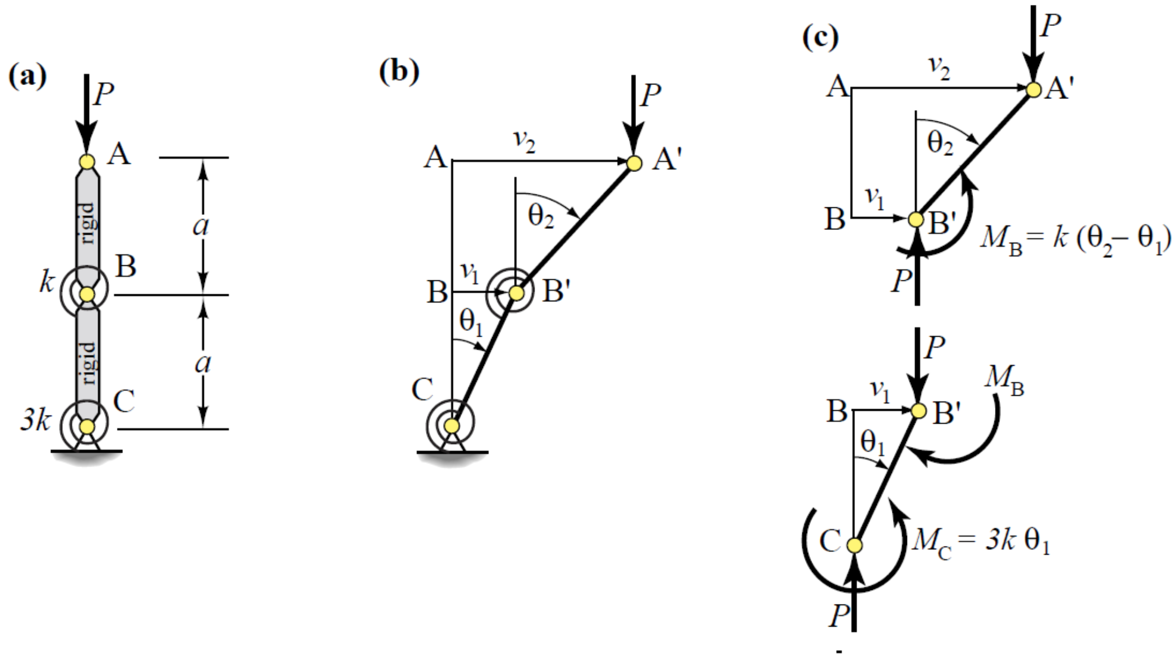


Figure 1: Column for Exercise 11.1: (a) configuration; (b) deflected shape and degrees of freedom, (c) FBDs of struts *AB* and *BC*. All deflections and tilt angles are actually very small (infinitesimal). Buckling shape is highly exaggerated for visibility.

(a) Derive the stability equation and extract the 2×2 stability matrix.

(b) Determine the two critical loads (eigenvalues of that matrix) in terms of *a* and *k*, and identify which one is the critical load.

(c) Compute the associated eigenvectors and use them to show the buckling shape sketches as separate diagrams.

Partial answer: $P_{cr} = \frac{1}{2}\left(5 - \sqrt{13}\right)\left(\frac{k}{a}\right) = 0.697\ k/a$

Solution. Draw an arbitrary buckling shape as in Figure 11.1(b) (the deflections are highly exaggerated for readability; they are actually infinitesimally small). This shape is defined by the lateral deflections v_A and v_B of A and B, positive to the right. These in turn define the tilt angles θ_1 and θ_2 shown. These two angles are taken to be positive CW. The necessary geometric relations, assuming small deflections, are

$$v_A = v_B + a\theta_2 = a(\theta_1 + \theta_2), \quad v_B = a\theta_1. \tag{1}$$

The two tilt angles are independent and are taken as DOFs. The restoring moments at joints B and C induced by the torsional springs are

$$M_B = k(\theta_2 - \theta_1), \quad M_C = 3k\theta_1 \tag{2}$$

with M_B and M_C positive as shown. The FBD analysis of the complete structure sketched in Figure 11.1(b) shows that horizontal reactions are zero because M_B is self equilibrating (that is, hinge moments act in action-reaction pairs, similarly to internal forces).

We will set up the stability eigensystem selecting θ_1 and θ_2 as independent variables. Two equilibrium equations, obtained by analyzing the displaced links shown in Figure 1(c), are required to set up the stability eigensystem. The FBD of the displaced link AB gives $P(v_A - v_B) - M_B = 0$, or $Pa\theta_2 - k(\theta_2 - \theta_1) = (Pa - k)\theta_2 + k\theta_1 = 0$. The FBD of the displaced link CB gives $Pv_B - M_C + M_B = 0$ or $(Pa - 4k)\theta_1 + k\theta_2 = 0$. Collecting these two relations into a matrix system gives the stability eigensystem:

$$\begin{bmatrix} Pa-k & k \\ k & Pa-4k \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} P-\kappa & \kappa \\ \kappa & P-4\kappa \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{with} \quad \kappa = k/a. \tag{3}$$

Buckling loads are obtained by seeking nontrivial solutions of (3). Solving $\det \begin{bmatrix} P-\kappa & \kappa \\ \kappa & P-4\kappa \end{bmatrix} = P^2 - 5\kappa + 3\kappa^2 = 0$ gives two roots: $P_{1,2} = \frac{1}{2}(5 \mp \sqrt{13})\kappa = \frac{1}{2}(5 \mp \sqrt{13})\,k/a$. The critical load is the smallest one:

$$\boxed{P_{cr} = P_1 = \frac{5 - \sqrt{13}}{2}\frac{k}{a} = 0.6972\frac{k}{a}} \tag{4}$$

The corresponding eigenvectors (normalized so that the largest component is one) are

$$\begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0.3028 \end{bmatrix}, \quad \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}_2 = \begin{bmatrix} -0.3028 \\ 1 \end{bmatrix}$$

These can be transformed to lateral deflections using (1). Normalizing so that the largest component is ± 1 we get

$$\boxed{P_1 = P_{cr} = 0.6972\frac{k}{a} \Rightarrow \begin{bmatrix} v_A \\ v_B \end{bmatrix}_1 = \begin{bmatrix} 1. \\ 0.2324 \end{bmatrix}, \quad P_2 = 4.3028\frac{k}{a} \Rightarrow \begin{bmatrix} v_A \\ v_B \end{bmatrix}_2 = \begin{bmatrix} 0.6972 \\ 1. \end{bmatrix}} \tag{5}$$

These eigenvectors are sketched in Figure 10.1A below. They provide the buckling mode shapes.

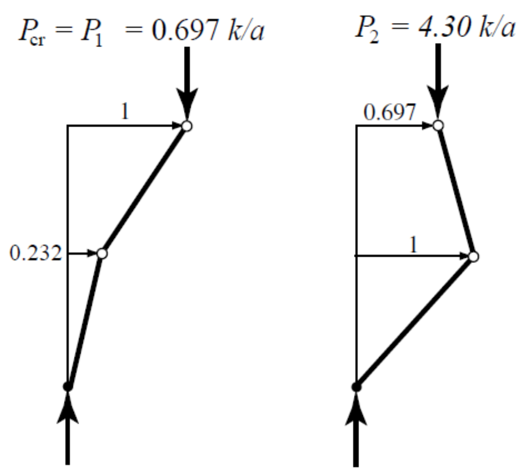


Figure 2: Normalized eigenvectors for problem of Exercise 11.1. These are the buckling mode shapes.

Problem 11.2: Buckling of a Fixed-Fixed Column

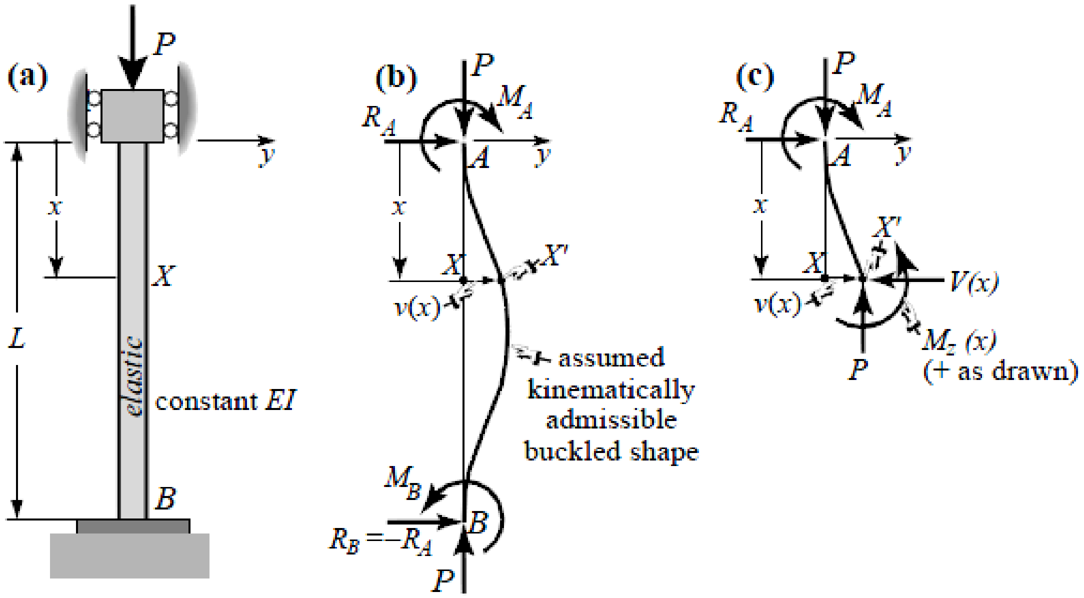


Figure.4 (a): fixed-fixed column for exercise 11.4. (b) FBD of complete column; (c) FBD at distance x from top.

An elastic column AB of length L is fixed at both ends (meaning that end rotations are precluded) and compressed by an axial load P at the top end A. Axis x origin at A, positive down. The elastic modulus is E and minimum second-moment of inertia of the cross section is I. See Figure.4(a). The kinematic boundary conditions are $v'(0) = 0$, $v(0) = 0$, $v'(L) = 0$, and $v(L) = 0$. Using LPB analysis, find the first *two* critical loads $P_{cr} = P_{cr1}$ and $P_{cr2} > P_{cr1}$, and verify that the effective length for the first critical load is $L_{eff} = \frac{1}{2}L$

Partial answers. ODE obtained via FBD of Figure.4(c): $EIv'' + Pv = R_A x + M_A$; characteristic equation: $\lambda[2(1 - cos\lambda L) - \lambda L\, sin\lambda L] = 0$; first critical load $P_{cr1} = \frac{4\pi^2 EI}{L^2} = 39.48\, EI/L^2$.

(Second one is not 4 times P_{cr1}, as would be the case for a pinned-pinned column.)

Solution. Conside the FBD of Figure 10.3(c). taking moments with respect to X' (that helps to get rid of two reactions there) we get $M_z(x) + P\,v(x) = R_A\,x + M_A$. Replacing $M_z(x) = EI\,v''(x)$ we arrive at the linear, nonhomogeneous, second-order ODE

$$EI\,v''(x) + P\,v(x) = R_A\,x + M_A, \tag{12}$$

whose canonical form is, upon dividing through by EI,

$$v''(x) + \lambda^2\,v(x) = \frac{1}{EI}(R_A\,x + M_A), \quad \text{with} \quad \lambda^2 = \frac{P}{EI}.$$

The homogeneous solution is $v(x)_H = A\cos\lambda x + B\sin\lambda x$. To get the particular solution, just divide the RHS by the coefficient λ^2 of v in the LHS: $v_P(x) = (1/EI)(R_A\,x + M_A)/\lambda^2 = R_A\,x/P + M_A/P$. The general solution is the sum of those two:

$$v(x) = v_H(x) + v_P(x) = A\cos\lambda x + B\sin\lambda x + \frac{R_A\,x}{P} + \frac{M_A}{P}. \tag{13}$$

Since the problem involves slope boundary conditions, we need the first x-derivative of this solution, which is

$$v'(x) = -\lambda\,A\sin\lambda x + \lambda\cos\lambda x + \frac{R_A}{P}. \tag{14}$$

Now apply the end conditions at A, which is fixed:

$$v(0) = A + \frac{M_A}{P} = 0, \quad v'(0) = \lambda B + \frac{R_A}{P} = 0, \tag{15}$$

whence $R_A = -\lambda\,B\,P$, and $M_A = -A\,P$. Replacing these into the foregoing $v(x)$ and $v'(x)$ gives

$$v(x) = A(\cos\lambda x - 1) + B(\sin\lambda x - \lambda\,x), \qquad v'(x) = -A\lambda\,\sin\lambda x + B(\lambda\cos\lambda - \lambda). \tag{16}$$

Apply now the end conditions at B: $v(L) = 0$ and $v'(L) = 0$, to get $A(\cos\lambda L - 1) + B(\sin\lambda L - \lambda\,L) = 0$ and $-A\lambda\,\sin\lambda L + B(\lambda\,\cos\lambda L - \lambda) = 0$. Rewriting these as a linear system in A and B we get the linearized pre-buckling (LPB) stability equation

$$\begin{bmatrix} \cos\lambda L - 1 & \sin\lambda L - \lambda L \\ -\lambda\sin\lambda L & \lambda\cos\lambda L - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

For a nontrivial solution (meaning that at least one of A and B is nonzero) the determinant of the coefficient matrix in 0 must vanish, which after simplifications yields the characteristic equation

$$d = \lambda\left[2(1-\cos\lambda L) - \lambda L\sin\lambda L\right] = 0.$$

The obvious root $\lambda = 0$ yields only the trivial solution $v(x) = 0$. Removing the λ factor we seek solutions of the trascendental equation

$$2(1-\cos\lambda L) - \lambda\,L\sin\lambda L = 0.$$

The smallest positive root[3]Actually $\lambda L = 2\,n\pi$, $n = 1, 2, \ldots$ gives an infinite set of roots, but that set does not exhaust them. of 0 is $\lambda L = 2\pi$, whence the smallest critical load is

$$\boxed{P_{cr} = P_{cr1} = \frac{4\pi^2 EI}{L^2} = \frac{39.478\,EI}{L^2}}$$

The next root of 1 is, to 4 places, $\lambda L = 8.987$, which is the first root of $\tan(\frac{1}{2}\lambda L) = \frac{1}{2}\lambda L$. Thus

$$\boxed{P_{cr2} = \frac{80.76\,EI}{L^2}}$$

ASEN 3112 - Structures –
Fall 2020

EXPOSITION IN HOMEWORKS

HW submission guidelines:

- **Write clearly your <u>name</u>, <u>student ID</u>, and <u>lab section ID</u> (011, 012) on <u>each</u> sheet you turn in.**
- **Restate the question so grader is sure which exercise you solved**
- **Do not fold your solution sheets.**

When writing out the homework, organize your solution to each assigned problem into four parts:

1. **Restate the question.** Short hand is OK if the question is long or elaborated. Restating makes sure the grader knows you are answering the right problem, and will help you to organize the subsequent exposition as your subconscious gets going.
2. **State the givens and unknowns to be found.** For givens always write down the physical units if those are stated in the problem.
3. **Draw key diagrams at the start of the solution.** For example, Free Body Diagrams (FBD) are essential part of many problems in statics.
4. **Write out the solution.** Be sure to **show work**. **Highlight** your answer(s) by a box, underline, arrow, or hi-lite marker. Also, do not forget to show the physical units. (This last item is also important in exams).

Example of Well Organized Homework Solution. Taken from the Solution Manual of M. Vable's *Mechanics of Materials*, Oxford, 1st ed.

1.40 An axial load is applied to a 1 inch diameter circular rod. The shear stress on section AA was found to be 20 ksi. The section AA is at 45° to the axis of the rod. Determine the applied force P and the average normal stress acting on section AA.

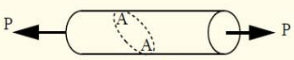


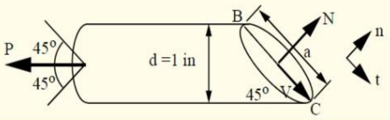
Fig. P1.40

1. Restate problem

Solution $d = 1$ in $\quad \tau_A = 20$ ksi $\quad P = ?$ $\quad \sigma_A = ?$

2. List givens & unknowns

By making an imaginary cut through AA we obtain the following free body diagram.



3. Draw key diagrams needed in solution

The length BC can be found as: $a = d/(\sin 45°) = 1.414$ in. The area of the elliptical inclined section can be found as: $A = (\pi/4)\,ad = (\pi/4)(1.414 \text{ in})(1 \text{ in}) = 1.1107 \text{ in}^2$

The shear force can be found as: $V = \tau_A A = (20 \text{ ksi})(1.107 \text{ in}^2) = 22.214$ kips

By equilibrium of forces in the t-direction we obtain: $P \sin 45° - V = 0$ or $P = V/(\sin 45°) = (22.214 \text{ kips})/(\sin 45°)$

$$\boxed{P = 31.4 \text{ kips}}$$

By equilibrium of forces in the n-direction we obtain: $N = P \cos 45° = (31.4 \text{ kips})(\cos 45°) = 22.214$ kips

The average normal stress on the inclined plane is: $\sigma_A = N/A = 22.214 \text{ kips} / 1.1107 \text{ in}^2 = 20 \text{ kips/in}^2$ $\boxed{\sigma_A = 20 \text{ ksi (T)}}$

4. Work out solution, step by step, & highlight or box requested answers