ASEN 3112
"Lecture" 16:
Finite Element Method Examples

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Announcements

- Talk to Dr. Johnson about
  - Homework 7 (FEM): due Friday, March 13
  - Exam 2 (beams, energy methods, FEM): Tuesday, March 17
  - Lab 2: due Thursday, April 2
- ANSYS Tutorial – Recitations on Thursday (March 12)
  - Thursday's recitations are for going through the ANSYS Tutorial
  - Section 011
    - Last names A to M: 8:30 am to 9:20 am.
    - Last names N to Z: 9:30 – 10:20 am
  - Section 012
    - Last names A to K: 2:30 – 3:20 pm
    - Last names L to Z: 3:30 – 4:20 pm
- Next week's recitations (Thursday, March 19) will be open office hours focused on Lab 2

Exam 2 Announcements

- Exam policies
  - Will have 1 hour and 15 minutes for the exam
  - 4 problems
  - Closed-book
  - Your crib sheet can be **one** 8.5” x 11” piece of paper with writing **on both sides**
  - Non-internet-enabled calculators are allow (no phones, laptops)
- Past exams will be posted ASAP

Exam 2 Announcements

- Did you watch the review **videos** I posted for Exam 1?
  a) Yes
  b) No

- Do you think you'll come to Monday's office hours (4–6 pm in AERO N240)?
  a) Definitely yes!
  b) Probably yes
  c) Probably no
  d) Definitely no

The Direct Stiffness Method

Breakdown

- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution

- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

conceptual steps

processing steps

post-processing steps

Member Stiffness Relations

- For each element, but not showing $^{(e)}$ superscript
- In local coordinates:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \bar{\boldsymbol{K}} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} \quad where \quad \bar{\boldsymbol{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- In global coordinates:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \boldsymbol{K} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \quad where \quad \boldsymbol{K} = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

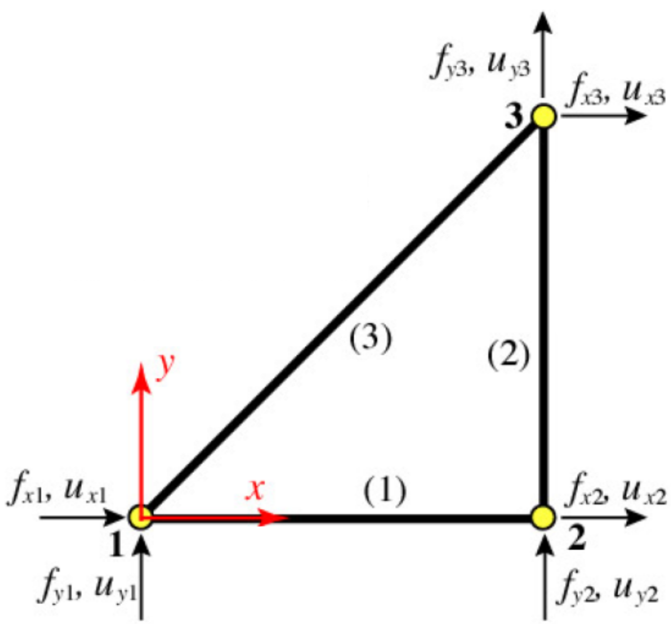
$$or \quad \boldsymbol{K} = \begin{bmatrix} [\hat{K}] & [-\hat{K}] \\ [-\hat{K}] & [\hat{K}] \end{bmatrix} \quad where \quad [\hat{K}] = \frac{EA}{L} \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

Assembly Process

- Assemble the $\left[\widehat{K}\right]$ matrices for each member into the global stiffness matrix.
- Remember our rules!
  1. Only have sums of $+\left[\widehat{K}\right]$s on the diagonal.
  2. Only have lone $-\left[\widehat{K}\right]$s off the diagonal.
  3. The number of $\left[\widehat{K}\right]$s in a diagonal cell shows the number of members at that node
  4. All diagonal cells must be filled
  5. A cell off the diagonal =0 means those two nodes aren't connected.

Assembly Process without Augmented Stiffness Matrices

$$\left[\hat{K}^{(e)}\right] = \frac{EA}{L}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$



$$\underline{K}^{(1)} = \begin{array}{cc} 1 & 2 \\ \end{array}$$

$$\underline{K}^{(1)} = \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{bmatrix} \left[\hat{K}^{(1)}\right] & \left[-\hat{K}^{(1)}\right] \\ \left[-\hat{K}^{(1)}\right] & \left[\hat{K}^{(1)}\right] \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$\underline{K}^{(2)} = \begin{array}{c} \begin{array}{cc} \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} \left[\hat{K}^{(2)}\right] & \left[-\hat{K}^{(2)}\right] \\ \left[-\hat{K}^{(2)}\right] & \left[\hat{K}^{(2)}\right] \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$$\underline{K}^{(3)} = \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} \left[\hat{K}^{(3)}\right] & \left[-\hat{K}^{(3)}\right] \\ \left[-\hat{K}^{(3)}\right] & \left[\hat{K}^{(3)}\right] \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 3 \end{array}$$

Assembly Process

3 x 3 matrix of 2 x 2 matrices

$$\boldsymbol{K} = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} [\hat{K}^{(1)}] + [\hat{K}^{(3)}] & [-\hat{K}^{(1)}] & [-\hat{K}^{(3)}] \\ [-\hat{K}^{(1)}] & [\hat{K}^{(1)}] + [\hat{K}^{(2)}] & [-\hat{K}^{(2)}] \\ [-\hat{K}^{(3)}] & [-\hat{K}^{(2)}] & [\hat{K}^{(2)}] + [\hat{K}^{(3)}] \end{bmatrix} \end{array}$$

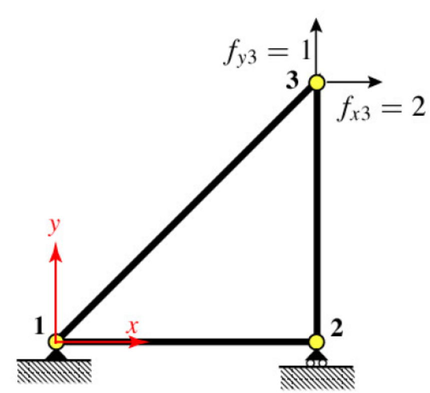
Reduced Global Stiffness Equation

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} = 0 \\ u_{y1} = 0 \\ u_{x2} = ? \\ u_{y2} = 0 \\ u_{x3} = ? \\ u_{y3} = ? \end{bmatrix} = \begin{bmatrix} f_{x1} = ? \\ f_{y1} = ? \\ f_{x2} = 0 \\ f_{y2} = ? \\ f_{x3} = 2 \\ f_{y3} = 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix}}_{\tilde{K}} \underbrace{\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix}}_{\tilde{u}} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{\tilde{f}}$$

Reduced Global Stiffness Equation (System): $\tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}}$

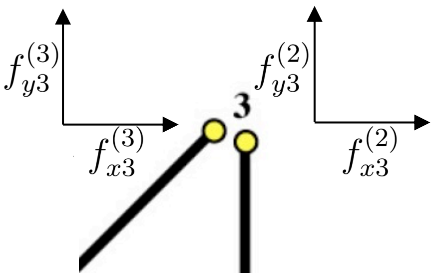
$$\tilde{u} = \tilde{K}^{-1} \tilde{f}$$



Solution: $$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix}$$

Recovery of Reaction Forces & Internal Force/Stress

Force Equilibrium at Nodes



Over all nodes

$$\underline{f} = \underline{f}^{(1)} + \underline{f}^{(2)} + \underline{f}^{(3)}$$

Recall that $\underline{f}^{e} = \underline{K}^{e}\underline{u}^{e}$

$$\underline{f} = \underline{K}^{(1)}\underline{u}^{(1)} + \underline{K}^{(2)}\underline{u}^{(2)} + \underline{K}^{(3)}\underline{u}^{(3)}$$

$$\boxed{\underline{f} = \underline{K}\,\underline{u} \quad \text{where} \quad \underline{K} = \underline{K}^{(1)} + \underline{K}^{(2)} + \underline{K}^{(3)}}$$

displacement of each node

external forces on each node

Recovery of Internal Force

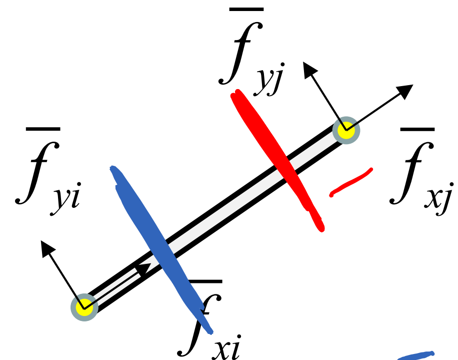
Through Forces

Recovery of Internal Force Through Forces

external forces on nodes in local coord.

Element Stiffness Equation in global CS:

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$$

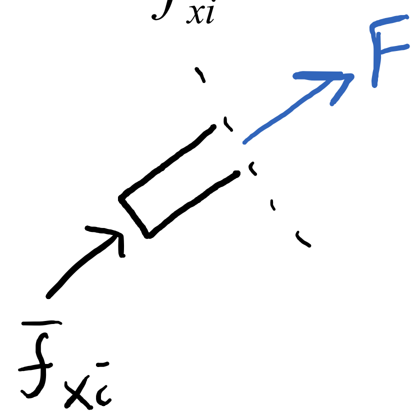


Note: for bar element

$$\bar{f}_{yi} = \bar{f}_{yj} = 0$$

Transformation from global to local CS:

$$\underbrace{\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}}_{\bar{\mathbf{f}}^e} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\mathbf{T}^e} \underbrace{\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix}}_{\mathbf{f}^e} \quad *$$



internal force

$$F = -\bar{f}_{xi} \quad *$$

$$F = +\bar{f}_{xj}$$

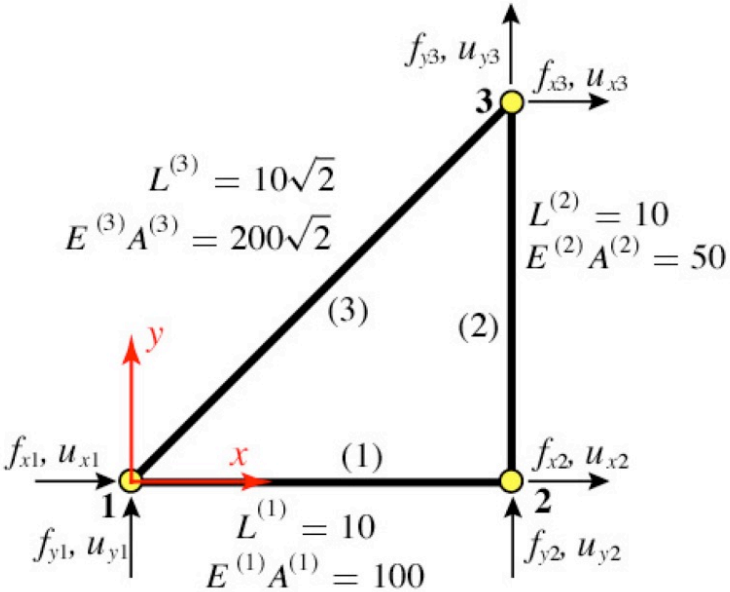
$$F = -\bar{f}_{xi} = -(\cos\varphi\, f_{xi} + \sin\varphi\, f_{yi})$$

$$F = +\bar{f}_{xj} = \cos\varphi\, f_{xj} + \sin\varphi\, f_{yj}$$

Example with Example Truss

Example: Bar 2 Using nodal solution for Bar 2:

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e \qquad \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \\ -1.0 \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.0 \\ 0.4 \\ -0.2 \end{bmatrix}$$



$$F = -\bar{f}_{xi} = -\left(c\ f_{xi} + s\ f_{yi}\right) \quad or \quad F = \bar{f}_{xj} = \left(c\ f_{xj} + s\ f_{yj}\right)$$

for $\cos 90 = 0 \quad \sin 90 = 1$

$$F = -\bar{f}_{xi} = -\left(0 \cdot 0 + 1 \cdot 1\right) = -1 \quad or \quad F = \bar{f}_{xj} = \left(0 \cdot 0 + 1 \cdot -1\right) = -1$$