ASEN 3112

Spring 2020

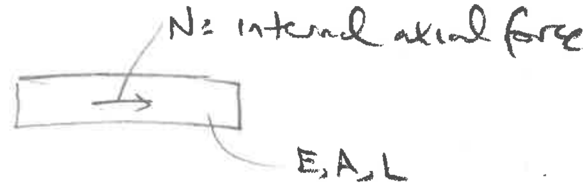
Lecture 12

Whiteboard

February 25, 2020

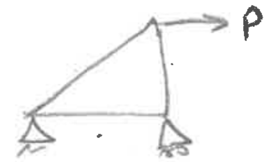
L12
1

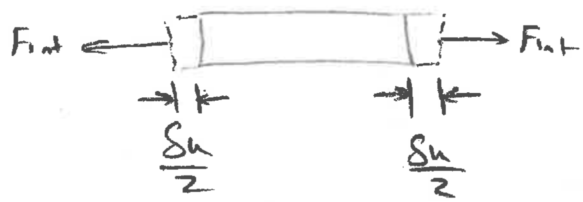
Principle of Virtual Work (PVW) on Rod (Bar)



A bar can also serve as part of a truss

e.g.





Virtual strain-displacement relation

$$\delta \epsilon_{xx} = \frac{\delta u}{L} \Longrightarrow \delta u = L\,\delta\epsilon_{xx}$$

Real stress-strain relation

$$\sigma_{xx} = E\,\epsilon_{xx} \qquad \left(\sigma_{xx} = \frac{N}{A}\right)$$

PVW: $\delta W_e = \delta W_{ie}$

$$\delta W_{ie} = N\,\delta u$$
$$= \sigma_{xx} A\, \delta\epsilon_{xx} L$$
$$= EA\,\epsilon_{xx}\,\delta\epsilon_{xx} L \qquad \left(\epsilon_{xx} = \frac{\Delta L}{L}\right)$$
$$= EA\,\frac{\Delta L}{L}\,\frac{\delta(\Delta L)}{\cancel{L}}\,\cancel{L}$$

$$\delta W_{ie} = \frac{EA}{L}\,\underbrace{\Delta L\,\delta(\Delta L)}_{\text{corresponds the "}(\Delta L)^2\text{"}}$$

$$\left[\text{Recall: } U_{rod} = \frac{1}{2}\,\frac{EA}{L}\,(\Delta L)^2\right]$$

PVW on a Beam

$$\delta W_{ie} = \int_L EI\,\underbrace{\kappa\,\delta\kappa}_{\text{correspond to "}\kappa^2\text{"}}\,dx \qquad \left[\text{Recall } U_{beam} = \frac{1}{2}\int EJ\,\kappa^2\,dx\right]$$

PVW on shaft

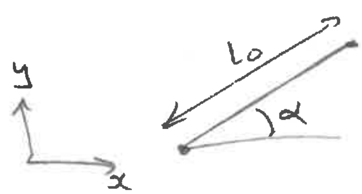
$$\delta W_{ie} = \int_L GJ\left(\frac{\partial\theta}{\partial x}\right)\delta\left(\frac{\partial\theta}{\partial x}\right)dx \qquad \left[\text{Recall } U_{shaft} = \frac{1}{2}\int GJ\,\theta'^2\,dx\right]$$

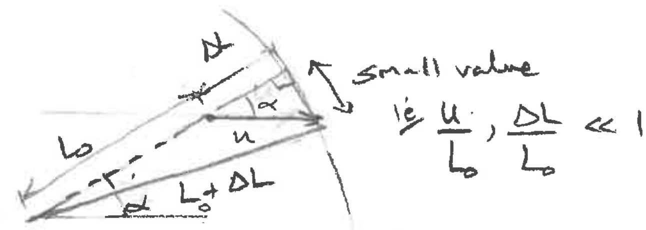
Example: bar

Truss: Elongation of a rod (bar) due to tip displacement

Undeformed

Deformed

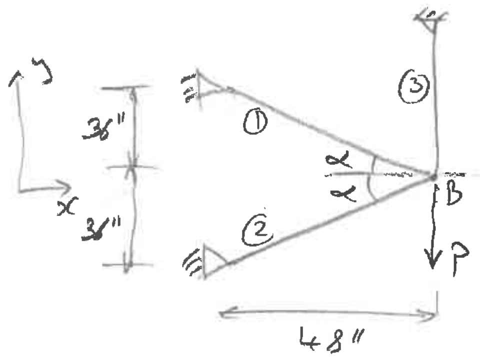




$\Delta L = u \cos\alpha$

Example

Three-bar truss



① E_1, A_1, L

② E_2, A_2, L

③ E_3, A_3, L_3

$$L_1 = L_2 = 60'' = L$$

$$\cos\alpha = \frac{48}{60} = \frac{4}{5}$$

$$\sin\alpha = \frac{36}{60} = \frac{3}{5}$$

Find at B: u_B, v_B

Unknown: u_B, v_B (real)

$\delta u_B, \delta v_B$ (virtual)

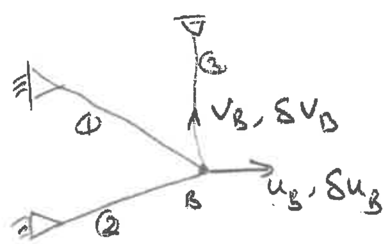
PVW $\boxed{\delta W_e = \delta W_{ie}}$

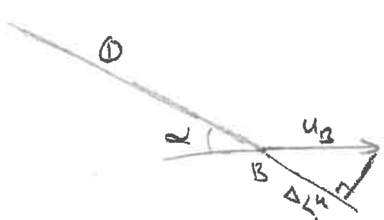
$$\delta W_e^u = \underbrace{0}_{\text{zero force in x-direction}} \delta u_B \longrightarrow \delta W_{ie}^u \ ?$$

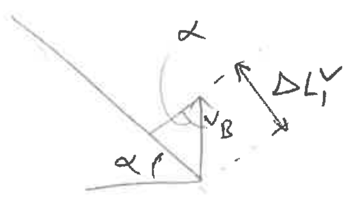
$$\delta W_e^v = -P\,\delta v_B \longrightarrow \delta W_{ie}^v \ ?$$

Recall: $\delta W_{ie_{rod}} = \frac{EA}{L} \Delta L\,(\delta\Delta L)$

$\delta\Delta L$ → $\delta\Delta L^u$, $\delta\Delta L^v$







$$\Delta L_1^u = \cos\alpha\, u_B = \frac{4}{5} u_B$$

$$\left(\delta\Delta L_1^u = \frac{4}{5}\delta u_B\right)$$

$$\Delta L_1^v = -\sin\alpha\, v_B = -\frac{3}{5} v_B$$

$$\left(\delta\Delta L_1^v = -\frac{3}{5}\delta v_B\right)$$

$$\longrightarrow \Delta L_1 = \Delta L_1^u + \Delta L_1^v = \underline{\frac{4}{5} u_B - \frac{3}{5} v_B}$$

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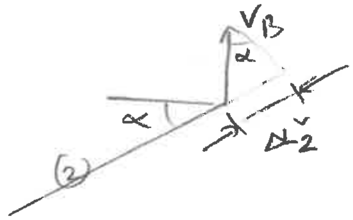
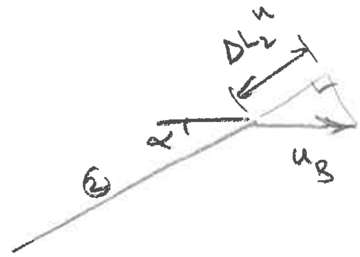
$$E_1 = E_2 = E$$





$$\Delta L_2^v = \sin\alpha\, v_B = \frac{3}{5} v_B$$

$$\left(\delta \Delta L_2^v = \frac{3}{5} \delta v_B \right)$$

$$\Delta L_2^u = \cos\alpha\, u_B = \frac{4}{5} u_B$$

$$\left(\delta \Delta L_2^u = \frac{4}{5} \delta u_B \right)$$

$$\longrightarrow \Delta L_2 = \Delta L_2^u + \Delta L_2^v = \frac{4}{5} u_B + \frac{3}{5} v_B$$

$$\delta W_{ie}^{rod} = \frac{EA}{L} \Delta L (\delta \Delta L)$$

$\boxed{u:}$

$$\delta \Delta L_1^u = \frac{4}{5} \delta u_B, \quad \delta \Delta L_2^u = \frac{4}{5} \delta u_B$$

(note: no ~~contribution for~~ from bar 3 because $\Delta L_3^u = 0 \Longrightarrow \delta L_3^u = 0$)

$$\delta W_{ie\,rod}^u = \frac{EA_1}{L} \underbrace{\left(\frac{4}{5} u_B - \frac{3}{5} v_B \right)}_{\Delta L_1} \underbrace{\left(\frac{4}{5} \delta u_B \right)}_{\delta \Delta L_1^u}$$

$$+ \frac{EA_2}{L} \underbrace{\left(\frac{4}{5} u_B + \frac{3}{5} v_B \right)}_{\Delta L_2} \underbrace{\left(\frac{4}{5} \delta u_B \right)}_{\delta \Delta L_2^u}$$

$$\delta W_{ie}^u = \frac{4E}{25L} \underbrace{\left[4(A_1 + A_2) u_B + 3(A_2 - A_1) v_B \right]}_{} \delta u_B = 0$$

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$$\delta \Delta L_1^v = -\frac{3}{5} v_B \ , \quad \delta \Delta L_2^v = \frac{3}{5} v_B$$

$\boxed{V:}$

$$\delta W_{ic_{\text{int}}}^v = \frac{EA_1}{L} \underbrace{\left(\frac{4}{5} u_B - \frac{3}{5} v_B\right)}_{\Delta L_1} \underbrace{\left(-\frac{3}{5}\delta v_B\right)}_{\delta \Delta L_1^v}$$

$$+ \frac{EA_2}{L} \underbrace{\left(\frac{4}{5} u_B + \frac{3}{5} v_B\right)}_{\Delta L_2} \underbrace{\left(\frac{3}{5}\delta v_B\right)}_{\delta \Delta L_2^v}$$

$$+ \frac{EA_3}{L} \left(-v_B\right)\left(-\delta v_B\right)$$

$$= \frac{3E}{25L}\left[4(A_2 - A_1) u_B + 3(A_1 + A_2) v_B\right] \delta v_B$$

$$+ \frac{EA_3}{L_3} v_B \, \delta v_B$$

$$\delta W_e^v = - \delta W_{ic}^v$$

$$-P \, \delta v_B = \frac{3E}{25L}\left[4(A_2 - A_1) u_B + 3(A_1 + A_2) v_B\right) + \frac{EA_3}{L_3} v_B$$

Combine u & v :

Two equations in two unknowns

$$4(A_1 + A_2) u_B + 3(A_2 - A_1) v_B = 0$$

$$4(A_2 - A_1) u_B + 3(A_1 + A_2) v_B + \frac{25L}{3L_3} A_3 v_B = -\frac{25PL}{3E}$$

Solve for u_B, v_B

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Note: The notes has treatment of virtual work.
by considering virtual force
— optional reading.

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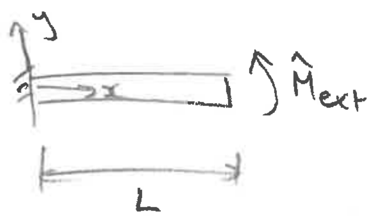
Example

Beam

$$\delta W_{ie}^{beam} = \int_L EI\, v'' \delta v''\, dx$$



Solve for $v(x)$

E, I constant

$$\boxed{\delta W_e = \delta W_{ie}} \quad (1)$$

Aside

Beam theory: $EIv'' = M$

$v(x) = ax^2 + bx + c$

BC: $v(x=0) = 0 \Longrightarrow c = 0$

$v'(x=0) = 0 \Longrightarrow b = 0$

(need to find "a")

$v(x) = ax^2 \qquad \delta v(x) = x^2 \delta a$

$v'(x) = 2ax \qquad \delta v'(x) = 2x\delta a$

$v''(x) = 2a \qquad \delta v''(x) = 2\delta a$

$$\delta W_e = \hat{M}_{ext}\, \delta v'\Big|_{x=L} = \hat{M}_{ext}(2x\delta a)\Big|_{x=L} = \hat{M}_{ext} 2 \delta a L$$

$$\delta W_{ie} = EI\, 2a\, 2\delta a L$$

Substitute into (1):

$$\hat{M}_{ext} 2 \cancel{\delta a} \cancel{L} = EI\, 4a \cancel{\delta a} \cancel{L} \Longrightarrow a = \frac{\hat{M}_{ext}}{2EI}$$

$$\boxed{v(x) = \frac{\hat{M}_{ext}}{2EI} x^2}$$

Clicker Question 1

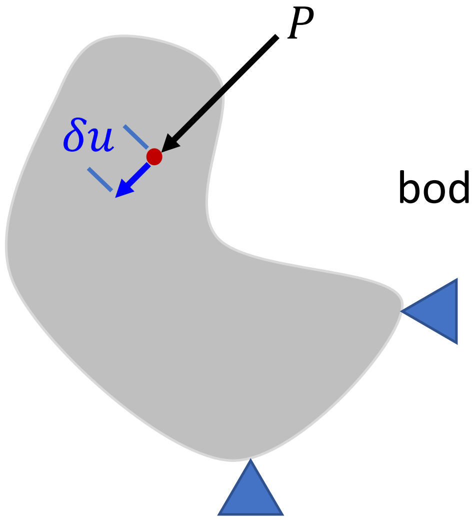
What is the correct expression of virtual work δW_e for the case below?

(a) $P\delta u$

(b) $u\delta P$

(c) $\frac{1}{2}P\delta u$

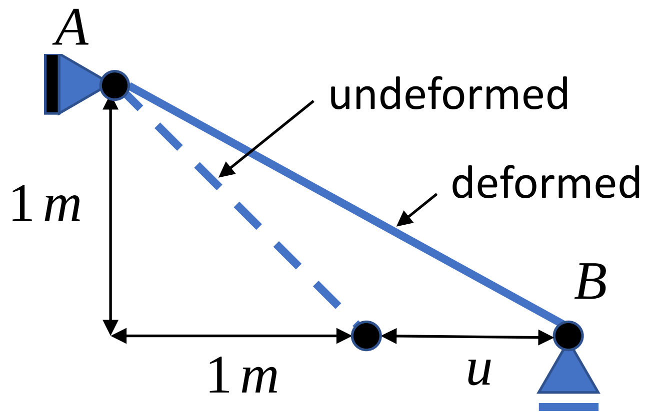
(d) $\frac{1}{2}u\delta P$



body in equilibrium

Clicker Question 2

What is the elongation of the bar shown below if a horizontal displacement u is applied to joint B?



(a) u

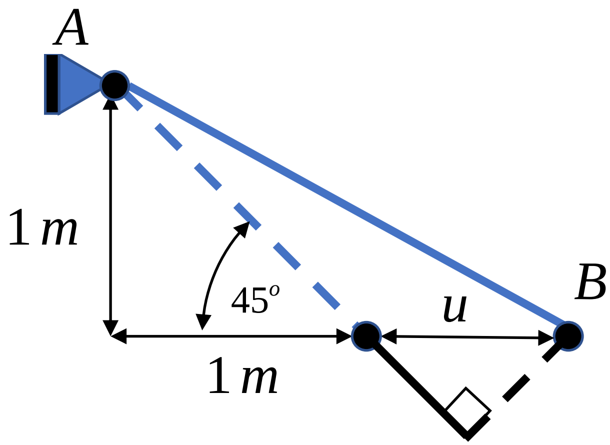
(b) $\sqrt{2}\,u$

(c) $\frac{\sqrt{2}}{2}\,u$

(d) none of the above

Clicker Question 2

What is the elongation of the bar shown below if a horizontal displacement is applied to joint B?



(a) u

(b) $\sqrt{2}\,u$

(c) $\frac{\sqrt{2}}{2}\,u$

(d) none of the above