ASEN 3112

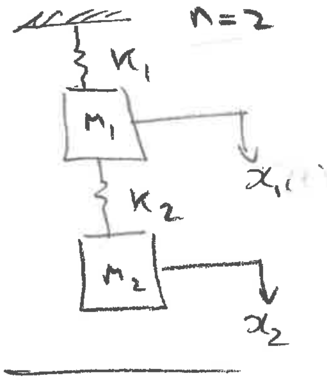
Spring 2020

Lecture 20

Whiteboard

April 2, 2020

MDOF Numerical Example (undamped)



$n=2$

- Obtain vibration characteristics
  - $\rightarrow$ natural frequencies (2 of them)
  - $\rightarrow$ mode shapes (2 of them)

$$\underline{M} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{kg}; \quad \underline{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{N/m}$$

(i) Form Eigenvalue problem:

$$\left[\underline{K} - \lambda \underline{M}\right] \underset{\sim}{u} = \underset{\sim}{0} \quad \text{or} \quad \underline{K}\,\underset{\sim}{u} = \lambda \underline{M}\,\underset{\sim}{u}$$

$$\begin{bmatrix} 1-2\lambda & -\lambda \\ -\lambda & 1-2\lambda \end{bmatrix} \underset{\sim}{u} = \underset{\sim}{0}$$

ii) Solve EVP. There are several ways to do so. If n is small, we set up the characteristic equation and use it to obtain the natural frequencies and mode shapes.

$$\begin{vmatrix} 1-2\lambda & -\lambda \\ -\lambda & 1-2\lambda \end{vmatrix} = 0$$

$$(1-2\lambda)(1-2\lambda) - (-\lambda)(-\lambda) = 0$$

$$1 - 2\lambda - 2\lambda + 4\lambda^2 - \lambda^2 = 0$$

$$\underbrace{3\lambda^2}_{a} \underbrace{-4\lambda}_{b} + \underbrace{1}_{c} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)(-4) - (4)(3)(1)}}{(2)(3)}$$

$$= \frac{4 \pm \sqrt{16-12}}{6} = 1, \frac{1}{3}$$

Tradition: Order in increasing size

$\lambda_1 = \frac{1}{3} \Rightarrow \omega_{n1} = \sqrt{1/3}$ rad/s — 1st natural frequency

$\lambda_2 = 1 \Rightarrow \omega_{n2} = \sqrt{1}$ rad/s — 2nd natural frequency

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Let $\underset{\sim}{u}_i = \begin{Bmatrix} u_{i1} \\ u_{i2} \end{Bmatrix}$, $i = 1,2$

that is

$$\underbrace{\underset{\sim}{u}_1}_{\text{1st mode shape}} = \begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix}, \quad \underbrace{\underset{\sim}{u}_2}_{\text{2nd mode shape}} = \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix}$$

So EVP: $\begin{bmatrix} 1-2\lambda_i & -\lambda_i \\ -\lambda_i & 1-2\lambda_i \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

that is

$$\begin{bmatrix} 1-2\lambda_1 & -\lambda_1 \\ -\lambda_1 & 1-2\lambda_1 \end{bmatrix} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}}_{\text{1st mode shape}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{1st } \underline{\text{mode}}$$

1st natural frequency

$$\begin{bmatrix} 1-2\lambda_2 & -\lambda_2 \\ -\lambda_2 & 1-2\lambda_2 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{2nd mode}$$

Use 1st Equation:

$$(1-2\lambda_i)u_{i1} - \lambda_i u_{i2} = 0 \qquad i = 1,2$$

$$\frac{(1-2\lambda_i)}{\lambda_i} = \frac{u_{i2}}{u_{i1}} \qquad \text{———} \quad *$$

1st mode shape:

$$\lambda_1 = \frac{1}{3} \Rightarrow \text{From } (*) \quad \frac{(1-2(\frac{1}{3}))}{\frac{1}{3}} = \frac{u_{12}}{u_{11}}$$

$$1 = \frac{u_{12}}{u_{11}}$$

$$\underset{\sim}{u_1} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \longleftarrow \text{1st mode shape}$$

2nd mode shape:

$$\lambda_2 = 1 \Longrightarrow \text{From } (*) \quad \frac{(1-2(1))}{1} = \frac{u_{22}}{u_{21}}$$

$$-1 = \frac{u_{22}}{u_{21}}$$

$$\underset{\sim}{u_2} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Eigensolution:
(Vibration characteristics)

$$\underbrace{\left[\frac{1}{3}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right]}_{\text{1st mode}}, \underbrace{\left[1, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\right]}_{\text{2nd mode}}$$

$\frac{L20}{4}$

Solving for "response" of MDOF System

— Undamped

— Unforced (Free)

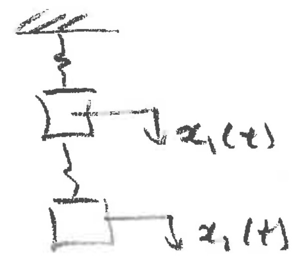
(no damping in MDOF systems in this section)

$\Longrightarrow$ By Modal Analysis

(Expansion Theorem)

<u>Recall</u>

eg



$$\underset{\sim}{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$\lambda_1 \rightarrow \omega_{n_1} \; ; \; \underset{\sim}{U}_1$$

$$\lambda_2 \rightarrow \omega_{n_2} \; ; \; \underset{\sim}{U}_2$$

$$\underset{\sim}{x}(t) = \beta_1(t)\,\underset{\sim}{U}_1 + \beta_2(t)\,\underset{\sim}{U}_2$$ — expansion of solution $\underset{\sim}{x}(t)$ in "modal space"

In matrix notation

$$\underset{\sim}{x}(t) = \underline{U}\;\underset{\sim}{\beta}$$ — Expansion

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \underbrace{\left[\begin{array}{c|c} \underset{\sim}{U}_1 & \underset{\sim}{U}_2 \end{array}\right]}_{\underline{U}} \underbrace{\begin{Bmatrix} \beta_1(t) \\ \beta_2(t) \end{Bmatrix}}$$

$\underline{U}$ Modal matrix

vector of generalized coordinates (principle coordinates)

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} U_{11} & U_{21} \\ U_{12} & U_{22} \end{bmatrix} \begin{Bmatrix} \beta_1(t) \\ \beta_2(t) \end{Bmatrix}$$

L20

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Physical space

$x_1(t) = U_{11}\,\beta_1(t) + U_{21}\,\beta_2(t)$ —— 1st equation motion of mass 1

$x_2(t) = U_{21}\,\beta_1(t) + U_{22}\,\beta_2(t)$ —— 2nd equation motion of mass 2

Back to EOM

$$\underline{M}\,\ddot{\underset{\sim}{x}} + \underline{K}\,\underset{\sim}{x} = \underset{\sim}{0}$$

IC

$\underset{\sim}{x_0} = \underset{\sim}{x}(t=0)$

$\dot{\underset{\sim}{x}}_0 = \underset{\sim}{V_0} = \dot{\underset{\sim}{x}}(t=0)$

Replace $\underset{\sim}{x}$ with $\underset{\sim}{x} =$

$$\underline{M}\,\underline{U}\,\ddot{\underset{\sim}{\beta}} + \underline{K}\,\underline{U}\,\underset{\sim}{\beta} = \underset{\sim}{0}$$

Premultiply by $\underline{U}^T$

$$\underline{U}^T\underline{M}\,\underline{U}\,\ddot{\underset{\sim}{\beta}} + \underline{U}^T\underline{K}\,\underline{U}\,\underset{\sim}{\beta} = 0$$

We ~~show~~ will show later that this process "uncoupled" the two equations!

eg $n=2$

$D_1^M\,\ddot{\beta}_1(t) + D_1^K\,\beta_1(t) = 0$ —— Eq 1

$D_2^M\,\ddot{\beta}_2(t) + D_2^K\,\beta_2(t) = 0$ —— Eq 2

in modal space

Two uncoupled ODE's.

Solve as two separate 1DOF problems to obtain

modal space

$\beta_1(t) =$ ______

$\beta_2(t) =$ ______

Once you have $\beta_1(t)$, $\beta_2(t)$, retransform back to $\underset{\sim}{x}$

$$\underset{\sim}{x} = \underline{U}\,\underset{\sim}{\beta}$$

Normalization

let $\underset{\sim}{V_i} = \alpha_i \underset{\sim}{U_i} \qquad i = 1,2$

Form $\underbrace{\underset{\sim}{V_i}^T \underline{M} \underset{\sim}{V_i}} = \alpha_i^2 \underset{\sim}{U_i}^T \underline{M} \underset{\sim}{U_i} = 1$ ——— (*)

Choose α_i such that (*) is satisfied

Thus $\alpha_i = \dfrac{1}{\sqrt{\underline{U_i}^T \underline{M} \underline{U_i}}}$

This way, we enforce $\underset{\sim}{V_i}^T \underline{M} \underset{\sim}{V_i} = 1$

We write normalized modal matrix

$$\underline{V} = \left[\underset{\sim}{V_1} \;\vdots\; \underset{\sim}{V_2} \right] = \begin{bmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{bmatrix}$$

$$\underline{V}^T \underline{M} \, \underline{V} = \underline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{V}^T \underline{K} \, \underline{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

This is an intermediate, but very useful.

Apply Modal with normalization:

$$\underset{\sim}{x} = \beta_1(t)\,\underset{\sim}{V_1} + \beta_2\,\underset{\sim}{V_2}$$

$$\underset{\sim}{x} = \underline{V}\,\underset{\sim}{\beta}$$

$\left.\begin{matrix}\underset{\sim}{\hat{V}_1} \\ \underset{\sim}{\hat{V}_2}\end{matrix}\right\}$ normalized mode shapes

Back to EOM:

$$\underline{M}\,\ddot{\underset{\sim}{x}} + \underline{K}\,\underset{\sim}{x} = \underset{\sim}{0}$$

Replace $\underset{\sim}{x} = \underline{V}\,\underset{\sim}{\beta}$

$$\underline{M}\,\underline{V}\,\ddot{\underset{\sim}{\beta}} + \underline{K}\,\underline{V}\,\underset{\sim}{\beta} = \underset{\sim}{0}$$

Premultiply by $\underline{V}^T$:

$$\underbrace{\underline{V}^T\underline{M}\,\underline{V}}_{\underline{I}}\,\ddot{\underset{\sim}{\beta}} + \underbrace{\underline{V}^T\underline{K}\,\underline{V}}_{\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}}\,\underset{\sim}{\beta} = \underset{\sim}{0}$$

Modal EOM:

$$\ddot{\beta}_j + \lambda_j \beta_j = 0 \qquad j = 1,2$$

Analogous to 1DOF $\ddot{x} + \omega_n^2 x = 0$

$$\left[\begin{matrix}\ddot{\beta}_1 + \lambda_1 \beta_1 = 0 & \longrightarrow \text{EOM \#1 in modal space} \\ \ddot{\beta}_2 + \lambda_2 \beta_2 = 0 & \longrightarrow \text{EOM \# 2 in modal space}\end{matrix}\right.$$

$\boxed{\text{Effectively we have converted our 2DOF coupled ODE's to two 1DOF uncoupled ODES}}$

$\frac{L20}{8}$

Initial Conditions also need to be converted to modal space:

$$\underset{\sim}{x_0} = \underline{V}\, \underset{\sim}{\beta_0} \quad , \quad \underset{\sim}{\dot{x}_0} = \underset{\sim}{V}\, \underset{\sim}{\dot{\beta}_0}$$

Recall $\underline{V}^T \underline{M}\, \underline{V} = \underline{I}$

Post multiply by $\underline{V}^{-1}$

$$\underline{V}^T \underline{M} \underbrace{\underline{V}\, \underline{V}^{-1}}_{\underline{I}} = \underline{I}\, \underline{V}^{-1}$$

$$\underline{V}^T \underline{M} = \underline{V}^{-1}$$

$$\underbrace{\underline{V}^{-1}}_{\underline{V}^T \underline{M}} \underset{\sim}{x_0} = \underbrace{\underline{V}^{-1} \underline{V}}_{\underline{I}} \underset{\sim}{\beta_0}$$

$$\Rightarrow \underset{\sim}{\beta_0} = \underline{V}^T \underline{M}\, \underset{\sim}{x_0}$$

similarly

$$\Rightarrow \dot{\beta}_0 = \underline{V}^T \underline{M}\, \underset{\sim}{V_0}$$

Final set in modal space

$$\ddot{\beta}_1 + \lambda_1 \beta_1 = 0$$
$$\ddot{\beta}_2 + \lambda_2 \beta_2 = 0$$

; IC'

$$\beta_0 = \underline{V}^T \underline{M}\, x_0$$
$$\dot{\beta}_0 = \underline{V}^T \underline{M}\, V_0$$

→ Solve for $\beta_1(t)$, $\beta_2(t)$

→ Transform back to $\underset{\sim}{x} = \underline{V}\, \underset{\sim}{\beta}$ <u>DONE</u>