Recitation 9

ASEN 3112 – Spring 2020

Problem: Forced Spring-Mass System

The 2-DOF spring-mass system of Figure 2 has the following matrix equation of motion

$$m\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{bmatrix} + k\begin{bmatrix}6 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}F_0\cos\Omega t\\ 0\end{bmatrix}.$$

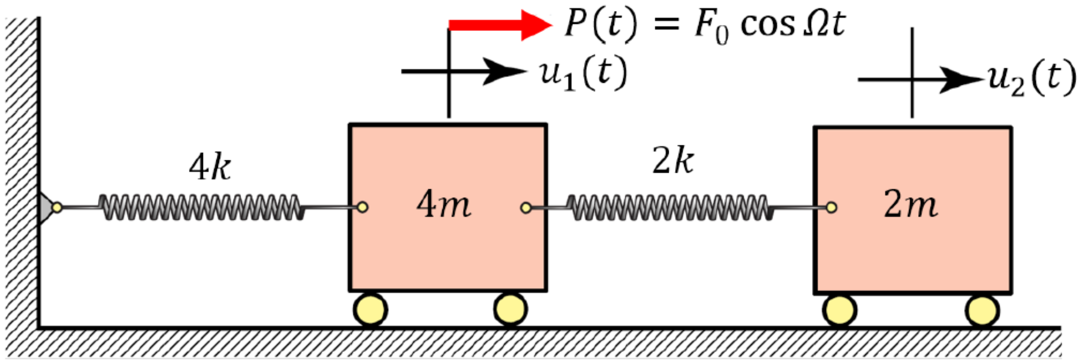

Figure 3: Two-DOF spring-mass system for Problem 2.

a. Solve for the *natural frequencies* and *mode shapes*. Scale each mode so that its largest component is 1.

b. Normalize the mode shape vectors (eigen vectors) with respect to the mass matrix. Verify that this normalization produces a mass normalized modal matrix by checking the generalized mass and stiffness matrices.

c. The system is forced by a function *P*(*t*). Use the mass normalized modal matrix to perform modal analysis and determine the total (homogeneous and steady-state) response of the system. Assume the system is at complete rest at $t = 0$.

a)

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{U} = \boldsymbol{0} \;\rightarrow\; \left(k\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} - \omega^2 m \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = \frac{\omega^2 m}{k}, \qquad \begin{bmatrix} 6-4\lambda & -2 \\ -2 & 2-2\lambda \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$8\lambda^2 - 20\lambda + 8 = 0 \;\rightarrow\; 2\lambda^2 - 5\lambda + 2 = 0, \qquad \lambda = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix}$$

$$\omega_1^2 = \lambda_1 \frac{k}{m} = \frac{k}{2m}, \qquad \omega_2^2 = \lambda_2 \frac{k}{m} = \frac{2k}{m}$$

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Let $U_1^{(r)} = 1$ for $r = 1, 2$:

$$U_2^{(r)} = \frac{1}{1 - \lambda_r}, \qquad U_2^{(1)} = \frac{1}{1 - \lambda_1} = 2, \qquad U_2^{(2)} = \frac{1}{1 - \lambda_2} = -1$$

$$U_1^{(r)} = \frac{1}{6 - 4\lambda_r}, \qquad U_1^{(1)} = \frac{2\ x\ 2}{6 - 4\lambda_1} = 1, \qquad U_1^{(2)} = \frac{2\ x\ (-1)}{6 - 4\lambda_2} = 1, \qquad \Phi = \begin{bmatrix} 0.5 & 1.0 \\ 1.0 & -1.0 \end{bmatrix}$$

b)

$$M_r = \Phi_r^T \boldsymbol{M} \Phi_r, \qquad K_r = \Phi_r^T \boldsymbol{K} \Phi_r,$$

$$M_1 = \Phi_1^T \boldsymbol{M} \Phi_1 = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 3m, \qquad M_2 = 6m, \qquad K_1 = 1.5k, \qquad K_2 = 12k$$

$$\Phi = \begin{bmatrix} \dfrac{1}{2\sqrt{3m}} & \dfrac{1}{\sqrt{6m}} \\ \dfrac{1}{\sqrt{3m}} & -\dfrac{1}{\sqrt{6m}} \end{bmatrix}$$

Recheck generalized masses and stifnesses

$$M_1 = \Phi_1^T \boldsymbol{M} \Phi_1 = \begin{bmatrix} \dfrac{1}{2\sqrt{3m}} & \dfrac{1}{\sqrt{3m}} \end{bmatrix} \begin{bmatrix} 4m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 1/2\sqrt{3m} \\ 1/\sqrt{3m} \end{bmatrix} = 1, \qquad M_2 = 1, \qquad K_1 = \frac{k}{2m}, \qquad K_2 = \frac{2k}{m}$$

c)

Forced Undamped System EOM:

$$\boldsymbol{M}_g\ddot{\eta} + \boldsymbol{K}_g\eta = \boldsymbol{f}(t)$$

$$\boldsymbol{M}_g = \Phi^T\boldsymbol{M}\Phi = \boldsymbol{I}, \qquad \boldsymbol{K}_g = \Phi^T\boldsymbol{K}\Phi = \boldsymbol{diag}(\omega^2)$$

Modal Force Vector:

$$\boldsymbol{f}(t) = \Phi^T\boldsymbol{P}(t) = \begin{bmatrix} \frac{1}{2\sqrt{3m}} & \frac{1}{\sqrt{3m}} \\ \frac{1}{\sqrt{6m}} & -\frac{1}{\sqrt{6m}} \end{bmatrix} \begin{bmatrix} F_0 cos\Omega t \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{F_0}{2\sqrt{3m}}\cos\Omega t \\ \frac{F_0}{\sqrt{6m}}\cos\Omega t \end{bmatrix}$$

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Modal EOM in matrix form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1(t) \\ \ddot{\eta}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{k}{2m} & 0 \\ 0 & \frac{2k}{m} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} \frac{F_0}{2\sqrt{3m}}\cos\Omega t \\ \frac{F_0}{\sqrt{6m}}\cos\Omega t \end{bmatrix}$$

Solution is sum of homogeneous and particular solution:

$$\eta = \eta_p + \eta_h$$

Particular Solution:

$$\ddot{\eta} + \omega^2\eta = A\cos\Omega t$$

$$\eta_{guess} = B\cos\Omega t\,, \qquad \dot{\eta}_{guess} = -B\Omega\sin\Omega t\,, \qquad \ddot{\eta}_{guess} = -B\Omega^2\cos\Omega t$$

$$B = \frac{A}{\omega^2 - \Omega^2}\,, \qquad \eta_p = \frac{A}{\omega^2 - \Omega^2}\cos\Omega t$$

Homogeneous Solution:

$$\eta_h = C\cos\omega t + D\sin\omega t\,, \qquad \eta = \frac{A}{\omega^2 - \Omega^2}\cos\Omega t + C\cos\omega t + D\sin\omega t$$

$$\eta(0) = 0 = \frac{A}{\omega^2 - \Omega^2} + C, \qquad C = -\frac{A}{\omega^2 - \Omega^2}, \qquad \dot{\eta}(0) = 0 = D$$

$$\eta = \frac{A}{\omega^2 - \Omega^2}(\cos\Omega t - \cos\omega t)$$

$$\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}\dfrac{F_0}{2\sqrt{3m}(\omega_1^2 - \Omega^2)}(\cos\Omega t - \cos\omega_1 t)\\ \dfrac{F_0}{\sqrt{6m}(\omega_2^2 - \Omega^2)}(\cos\Omega t - \cos\omega_2 t)\end{bmatrix}$$

$$\boldsymbol{u}(t) = \Phi\eta(t) = \begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}\dfrac{1}{2\sqrt{3m}} & \dfrac{1}{\sqrt{6m}}\\ \dfrac{1}{\sqrt{3m}} & -\dfrac{1}{\sqrt{6m}}\end{bmatrix}\begin{bmatrix}\dfrac{F_0}{2\sqrt{3m}(\omega_1^2 - \Omega^2)}(\cos\Omega t - \cos\omega_1 t)\\ \dfrac{F_0}{\sqrt{6m}(\omega_2^2 - \Omega^2)}(\cos\Omega t - \cos\omega_2 t)\end{bmatrix}$$