ASEN 3112 – Spring 2020
Homework 8 Solutions

Problem 8.1:

Consider a viscously damped SDOF spring-mass oscillator, described by the EOM

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$

and subject to the following initial conditions:

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 = v_0.$$

(a) Derive the response $x(t)$ for the *underdamped* case, where $\zeta < 1$, in terms of x_0 and v_0.

(b) Derive the response $x(t)$ for the *overdamped* case, where $\zeta > 1$, in terms of x_0 and v_0.

For both parts (a) and (b) show your derivations in detail.

EOM

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

ICs

$$x(0) = x_0 \,,\quad \dot{x}(0) = \dot{x}_0 = V_0$$

Assume Solution

$$x(t) = A e^{\lambda t}$$
$$\dot{x}(t) = A\lambda e^{\lambda t} = \lambda x$$
$$\ddot{x}(t) = A\lambda^2 e^{\lambda t} = \lambda^2 x$$

Substitute into EOM and solve for λ

$$\left(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\right)x = 0$$

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Underdamped Case : $\zeta < 1$

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm i\omega_d t \,,\qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$x(t) = A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_2 e^{(-\zeta\omega_n - i\omega_d t)}$$
$$= e^{-\zeta\omega_n t}\left(B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right)$$

at $t=0$

$$x(0) = x_0 = B_1$$

$$\dot{x}(t) = -\zeta\omega_n\, x(t) + e^{-\zeta\omega_n t}\,\omega_d\left(-B_1\sin(\omega_d t) + B_2\cos(\omega_d t)\right)$$

$$\dot{x}(0) = V_0 = -\zeta\omega_n x_0 + \omega_d B_2$$

$$\longrightarrow B_2 = \frac{V_0 + \zeta\omega_n x_0}{\omega_d}$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left[x_0\cos(\omega_d t) + \left(\frac{V_0 + \zeta\omega_n x_0}{\omega_d}\right)\sin(\omega_d t)\right]}$$

Overdamped Case $\zeta > 1$

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$x(t) = e^{-\zeta\omega_n t}\left(A_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} + A_2 e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

$$\dot{x}(t) = -\zeta\omega_n\, x(t)$$
$$+ e^{-\zeta\omega_n t}\left(\omega_n\sqrt{\zeta^2-1}\right)\left(A_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} - A_2 e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

at $t=0$

$$x(0) = x_0 = A_1 + A_2$$
$$\dot{x}(0) = V_0 = -\zeta\omega_n x_0 + \left(\omega_n\sqrt{\zeta^2-1}\right)(A_1 - A_2)$$

$$A_1 = x_0 - A_2$$

$$\longrightarrow V_0 = -\zeta\omega_n x_0 + \left(\omega_n\sqrt{\zeta^2-1}\right)(x_0 - 2A_2)$$

$$\longrightarrow A_2 = \left[\frac{V_0 + \zeta\omega_n x_0}{\omega_n\sqrt{\zeta^2-1}} - x_0\right]\left(\frac{-1}{2}\right)$$

$$A_2 = \frac{x_0}{2} - \frac{v_0 + \zeta\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$A_1 = x_0 - A_2 = \frac{x_0}{2} + \frac{v_0 + \zeta\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

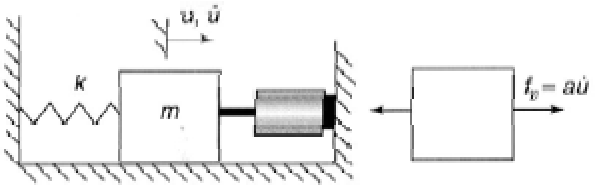
$$\boxed{x(t) = e^{-\zeta\omega_n t}\left[\left(\frac{x_0}{2} + \frac{v_0 + \zeta\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}\right)e^{\omega_n\sqrt{\zeta^2-1}\,t} + \left(\frac{x_0}{2} - \frac{v_0 + \zeta\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}\right)e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right]}$$

Problem 8.2:

The spring-mass oscillator in the figure has a velocity-feedback force generator that exerts a force f_v on the mass that is proportional to the velocity of the mass. The sign of the force can be either positive or negative. For a particular setup of this SDPF system, the spring, mass, and feedback force parameters lead to the following differential equation of motion:

$$\ddot{u} - 2\dot{u} + 5u = 0$$

where u is the displacement of the mass in inches.



(a) If the initial conditions are

$$u(0) = 0, \quad \dot{u}(0) = 0.10\,\text{in./sec},$$

determine the motion function $u(t)$.

Note that here we are using the notation of $u(t)$ instead of $x(t)$ to describe the vibratory motion. This is just a different notation; the approach is exactly as followed in class.

(a) From Eq. 3.40, the equation of motion of this system has the form

$\ddot{u} + a\dot{u} + bu = 0$

where $a = -2$ and $b = 5$. From Eq. 3.43, the solution will have the form

$u(t) = \overline{C}_1 e^{\bar{s}_1 t} + \overline{C}_2 e^{\bar{s}_2 t}$

where, from Eq. 3.42,

$$\left.\begin{array}{l}\bar{s}_1\\ \bar{s}_2\end{array}\right\} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} = 1 \pm \sqrt{-4} = 1 \pm 2i$$

Since $\bar{s}_1$ and $\bar{s}_2$ are complex conjugates, the solution can be written in the alternative form

$u(t) = e^t\left[A_1 \cos(2t) + A_2 \sin(2t)\right]$

Then,

$\dot{u} = e^t\left[A_1 \cos(2t) + A_2 \sin(2t) - 2A_1 \sin(2t) + 2A_2 \cos(2t)\right]$

With the given initial conditions,

$u(0) = A_1 = 0, \quad \dot{u}(0) = 2(1/\text{sec})A_2(\text{in.}) = 0.10\ \text{in./sec} \ \rightarrow A_2 = 0.05\ \text{in.}$

Finally,

$u(t) = 0.05 e^t \sin(2t)\,\text{in.}$ **Ans. (a)**

Problem 8.3:

Consider a spring-mass-damper system with k = 4000 N/m, m = 10 kg, and c = 40 N-s/m.

Find the steady-state and total responses of the system under the harmonic force $F(t)$ = 200 cos(10t) N and the initial conditions $x_0 = 0.1$ m and $\dot{x}_0 = 0$.

$k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$,
$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$ m

$\zeta = c/c_c = \left(c/2\sqrt{km}\right) = \left(40/2\sqrt{4000(10)}\right) = 0.1$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.1)^2}\,(20) = 19.899749$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$

$$X = \delta_{st}\Big/\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \frac{0.05}{\left\{(1-0.5^2)^2 + (2(0.1)(0.5))^2\right\}^{\frac{1}{2}}}$$

$= 0.066082$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 * 0.1 * 0.5}{1 - 0.5^2}\right) = 0.132552 \text{ rad}$$

steady state response, Eq. (3.25):

$x_p(t) = X \cos(\omega t - \phi)$

$\quad = 0.066082 \cos(10t - 0.132552)$ m

Total response, Eq. (3.35):

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \qquad (E.2)$$

or $$X_0 \cos\phi_0 = x_0 - X \cos\phi \qquad (E.3)$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 + \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \qquad (E.4)$$

or $$X_0 \sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\right\} \qquad (E.5)$$

For known values, Eqs. (E.3) and (E.5) yield

$X_0 \cos\phi_0 = 0.034498$, $\quad X_0 \sin\phi_0 = -0.000922$

Hence

$$X_0 = \left\{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.034510$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0\sin\phi_0}{X_0\cos\phi_0}\right) = -0.026710$$

Thus the total response, Eq. (E.1), will be

$$x(t) = 0.034510\, e^{-2t} \cos(19.899749\, t + 0.026710)$$
$$+ 0.066082 \cos(10t - 0.132552) \text{ m} \qquad (E.6)$$

NOTE: The following is a practice problem that SHOULD NOT be handed in. We will provide you with its solution along with the solutions of HW 8.

Practice Problem 8.4:

The equation of motion of a spring-mass-damper system subjected to a harmonic force can be expressed as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = f_0 \cos(\omega t),$$

where $f_0 = F_0/m$, $\omega_n = \sqrt{k/m}$, and $\zeta = c/(2m\omega_n)$.

i. Find the steady-state response (particular solution) of the system in the form

$x_p(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

ii. Find the total response of the system in the form

$x(t) = x_h(t) + x_p(t) = A \cos(\omega_d t) + B \sin(\omega_d t) + C_1 \cos(\omega t) + C_2 \sin(\omega t)$

Assume the initial conditions of the system as $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$.

Equation of motion:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos\omega t \qquad (1)$$

(i) <u>steady state response:</u>

$$x_s(t) = C_1 \cos\omega t + C_2 \sin\omega t \qquad (2)$$

$$\dot{x}_s(t) = -\omega C_1 \sin\omega t + \omega C_2 \cos\omega t \qquad (3)$$

$$\ddot{x}_s(t) = -\omega^2 C_1 \cos\omega t - \omega^2 C_2 \sin\omega t \qquad (4)$$

Substitute Eqs. (2) – (4) in Eq. (1):

$$\left(-\omega^2 C_1 + 2\zeta\omega_n \omega C_2 + \omega_n^2 C_1 - f_0\right)\cos\omega t$$

$$+ \left(-\omega^2 C_2 - 2\zeta\omega_n\omega C_1 + \omega_n^2 C_2\right)\sin\omega t = 0 \qquad (5)$$

Since Eq. (5) is valid for all time t, the coefficients of $\cos\omega t$ and $\sin\omega t$ must be zero so that

$$C_1\left(-\omega^2 + \omega_n^2\right) + C_2\left(2\zeta\omega_n\omega\right) = f_0 \qquad (6)$$

$$C_1\left(-2\zeta\omega_n\omega\right) + C_2\left(-\omega^2 + \omega_n^2\right) = 0 \qquad (7)$$

The solution of Eqs. (6) and (7) is given by

$$C_1 = \frac{\left(\omega_n^2 - \omega^2\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2} \qquad (8)$$

$$C_2 = \frac{\left(2\zeta\omega\omega_n\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2} \qquad (9)$$

By substituting Eqs. (8) and (9) in Eq. (2), we obtain

$$x_s(t) = \frac{\left(\omega_n^2 - \omega^2\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2}\cos\omega t$$

$$+ \frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2} \sin\omega t \qquad (10)$$

(ii) Total response of the system:

$$x(t) = x_h(t) + x_s(t)$$

$$= A\cos\omega_n t + B\sin\omega_n t + C_1\cos\omega t + C_2\sin\omega t \qquad (11)$$

and hence

$$\dot{x}(t) = -A\omega_n\sin\omega_n t + \omega_n B\cos\omega_n t$$
$$- C_1\omega\sin\omega t + C_2\omega\cos\omega t \qquad (12)$$

Let the initial conditions be

$$x(0) = x_0 \qquad (13)$$

and

$$\dot{x}(0) = \dot{x}_0 \qquad (14)$$

Eqs. (13) and (11) give:

$$A + C_1 = x_0 \qquad (15)$$

Eqs. (14) and (12) give:

$$\omega_n B + \omega C_2 = \dot{x}_0 \qquad (16)$$

Eqs. (15) and (16) give

$$A = x_0 - C_1 \qquad (17)$$

$$B = (\dot{x}_0 - \omega C_2)/\omega_n \qquad (18)$$

or

$$A = x_0 - \frac{(\omega_n^2-\omega^2) f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2} \qquad (19)$$

$$B = \frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n}\left\{\frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\} \qquad (20)$$

Thus the total response of the system can be expressed as (Eq. (11)):

$$x(t) = \left[x_0 - \left\{ \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega \omega_n)^2} \right\} \right] \cos \omega t$$

$$+ \left[\frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n} \left\{ \frac{(2 \zeta \omega \omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega \omega_n)^2} \right\} \right] \sin \omega t$$

$$+ \left\{ \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega \omega_n)^2} \right\} \cos \omega t$$

$$+ \left\{ \frac{(2 \zeta \omega \omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega \omega_n)^2} \right\} \sin \omega t \qquad (21)$$