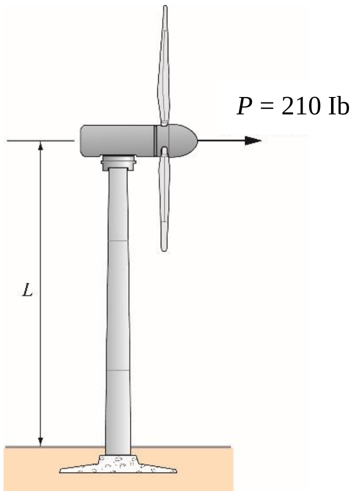
ASEN 3112 – Spring 2020
Homework 9 Solutions

Problem 9.1:



A static force of P = 210 lb applied as shown in the figure displaces a wind-turbine mass 1.5 inches in the horizontal direction. The mass is released at this point and at the end of two complete oscillation cycles that cover 1.25 sec, the displacement is 0.85 in. Determine: (a) the undamped natural frequency ω_n, (b) the effective stiffness k, (c) the effective mass m, and (d) the effective damping factor ζ.

(b) From the static deflection equation we get

$$\boxed{k = \frac{P}{\delta_{st}} = \frac{210\ \text{lb}}{1.5\ \text{in}} = 140\ \text{lb/in.}}$$

From the oscillation envelope curve, $u(2T_d) = u(0)e^{-\zeta\omega_n(2T_d)}$, so

$$\ln\left[\frac{u(2T_d)}{u(0)}\right] = \frac{-2\zeta\omega_n(2\pi)}{\omega_n\sqrt{1-\zeta^2}} \approx -4\pi\zeta \quad \text{for small } \zeta\ ,$$

(d) Consequently

$$\boxed{\zeta \approx -\frac{1}{4\pi}\ln\left[\frac{u(2T_d)}{u(0)}\right] = -\frac{1}{4\pi}\ln\left[\frac{0.85}{1.5}\right] = 0.0452.}$$

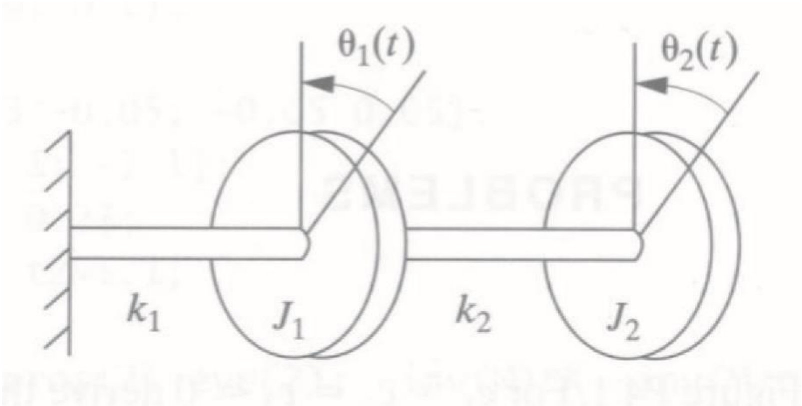
(a) From the equation for the damped natural frequency, $\omega_d = 2\pi/T_d = \omega_n\sqrt{1-\zeta^2}$ so

$$\boxed{\omega_n = \frac{2\pi}{T_d\sqrt{1-\zeta^2}} = \frac{2\pi}{(1.25/2)\sqrt{1-0.0452^2}} = 10.0634\ \text{rad/sec.}}$$

(c) From the expression $\omega_n^2 = k/m$ for the undamped natural frequency we get

$$\boxed{m = \frac{k}{\omega_n^2} = \frac{210\ \text{lb/in}}{(10.0634\ \text{rad/sec})^2} = 1.382\ \text{lb-sec}^2/\text{in.}}$$

Problem 9.2:



Torsional system with two disks (two degrees of freedom)

Determine the equation of motion in matrix form, then calculate the natural frequencies and mode shapes of the torsional system shown in the figure. Assume that the torsional stiffness values provided by the shaft are equal ($k_1 = k_2$) and that disk 1 has three times the inertia as that of disk 2 ($J_1 = 3J_2$). Assume that both rods have polar moments of inertia J_p.

Hint: Model this as a 2DOF spring-mass system, where the inertia J is analogous to a mass m, and where each spring represents the torsional stiffness of the corresponding portion of the shaft denoted by k_1 and k_2, respectively.

Solution. The equations of motion are

$$J_1\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k\theta_1 + k\theta = 0$$

Let $k = k_1 = k_2$ and $J_1 = 3J_2$. In matrix form the equation of motions would be

$$J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

Calculate the natural frequencies:

$$det(-\omega^2 J + K) = \begin{vmatrix} -3\omega^2 J_2 + 2k & -k \\ -k & -\omega^2 J_2 + k \end{vmatrix} = 0$$

$$\omega_1 = 0.482\sqrt{\frac{k}{J_2}}$$

$$\omega_2 = 1.198\sqrt{\frac{k}{J_2}}$$

Calculate the mode shapes: Mode shape 1:

$$\begin{bmatrix} -3(0.2324)k + 2k & -k \\ -k & -(0.2324)k + k \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$$

$$u_{11} = 0.7676u_{12}$$

$$\mathbf{u_1} = \begin{bmatrix} 0.7676 \\ 1 \end{bmatrix}$$

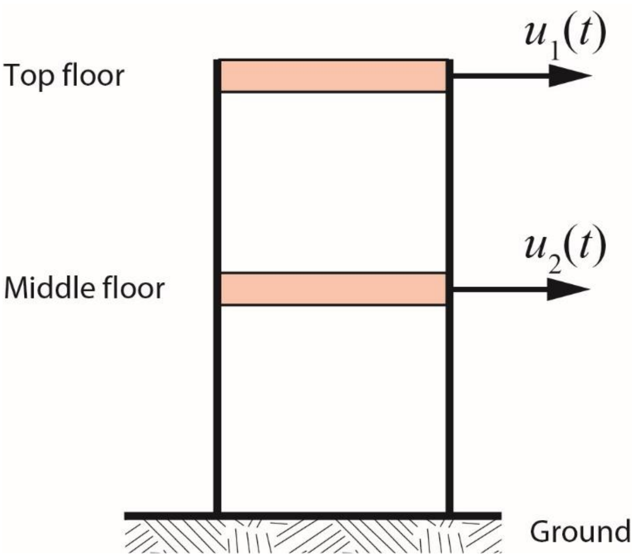
Mode shape 2:

$$\begin{bmatrix} -3(1.434)k + 2k & -k \\ -k & -(1.434)k + k \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$$

$$u_{21} = -0.434 u_{22}$$

$$\mathbf{u_2} = \begin{bmatrix} -0.434 \\ 1 \end{bmatrix}$$

Problem 9.3:



The stiffness and mass matrices of the two-story building shown in Figure 9.3 (the model is undamped) are

$$\mathbf{K} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \text{ kips/in,} \qquad \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{ kip-sec}^2\text{/in.} \tag{1}$$

a. Determine the two natural frequencies in both rad/sec (ω_1 and ω_2), and in Hertz (f_1 and f_2), as well as the periods T_1 and T_2 in seconds.

b. Get the two corresponding eigenvectors (= mode shapes) and scale them so that the largest entry (in absolute value) is one. Then *rescale* these modes into ϕ_1 and ϕ_2 so that the generalized masses are one. Check that the generalized masses are unity whereas the generalized stiffnesses are ω_1^2 and ω_2^2. Also check that ϕ_1 and ϕ_2 are *orthogonal* with respect to both **M** and **K**.

c. The structure is unforced. If the initial conditions (IC) are $u_1(0) = 2$ in, $u_2(0) = 1$ in, $\dot{u}_1(0) = 0$, and $\dot{u}_2(0) = 0$, determine the expressions for the responses $u_1(t)$ and $u_2(t)$ in terms of sines and cosines, using modal analysis. Plot the responses $u_1(t)$ and $u_2(t)$ for $0 \le t \le 2$ sec, in the same graph.

Notes:

- Consider this system as a 2 degrees of freedom undamped mass-spring system.
- You do not need to model and derive the equations of motion; the stiffness and mass matrices are already given to you.
- Here ϕ_1 and ϕ_2 denote the normalized eigenvectors, which are V_1 and V_2 in the class notes.

Solution.

a. The vibration eigenproblem is

$$\mathbf{K}\mathbf{U} = \omega^2 \mathbf{M}\mathbf{U}, \quad \text{or} \quad \left(\mathbf{K} - \omega^2 \mathbf{M}\right)\mathbf{U} = 0. \tag{18}$$

Replacing numbers:

$$\left(\begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{19}$$

For convenience let $\lambda = \omega^2/1000$, then equation (19) becomes

$$\begin{bmatrix} 1-2\lambda & -1 \\ -1 & 2-3\lambda \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{20}$$

For nontrivial solutions the determinant of the matrix in equation (20) must vanish, whence

$$\det \begin{bmatrix} 1-2\lambda & -1 \\ -1 & 2-3\lambda \end{bmatrix} = (1-2\lambda)(2-3\lambda) - (-1)^2 = 6\lambda^2 - 7\lambda + 1 = 0. \tag{21}$$

Solving this quadratic equation gives

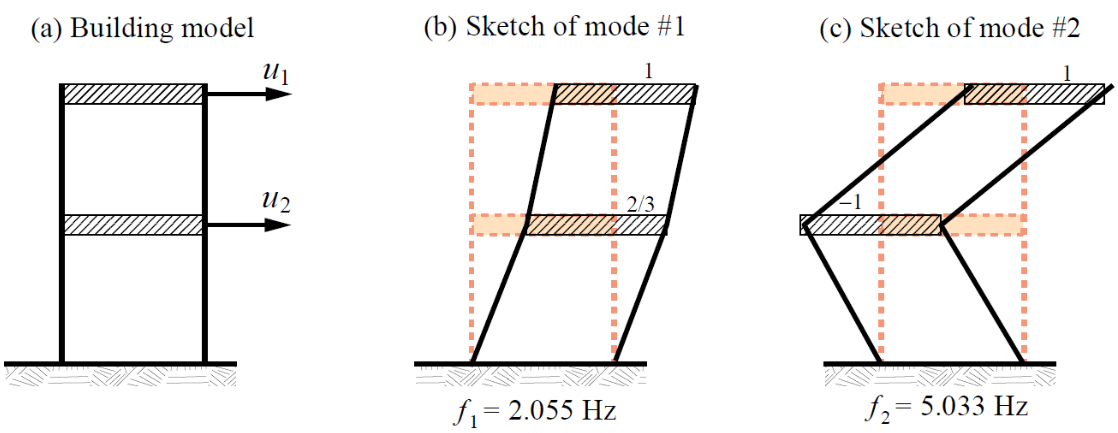
$$\begin{aligned} \lambda_{1,2} = \frac{7 \pm \sqrt{49-24}}{12} \;\Rightarrow\; \lambda_1 = \frac{1}{6},\ \lambda_2 = 1,\ \Rightarrow\ \omega_1^2 &= 1000\lambda_1 = 500/3\ (\text{rad/sec})^2, \\ \omega_2^2 &= 1000\lambda_2 = 1000\ (\text{rad/sec})^2. \end{aligned} \tag{22}$$

Summarizing, the natural frequencies in rad/sec and Hz, and periods in sec, are

$$\begin{aligned} \omega_1 &= 12.91\ \text{rad/sec}, & \omega_2 &= 31.62\ \text{rad/sec}, \\ f_1 &= \frac{12.91}{2\pi} = 2.055\ \text{Hz}, & f_2 &= \frac{31.62}{2\pi} = 5.033\ \text{Hz}, \\ T_1 &= \frac{1}{f_1} = 0.4867\ \text{sec}, & T_2 &= \frac{1}{f_2} = 0.1987\ \text{sec}. \end{aligned} \tag{23}$$

b. Mode shapes, normalized to +1 as largest entry:

$$\boxed{\boldsymbol{\phi}_1 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, \qquad \boldsymbol{\phi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \tag{24}$$



These two shapes are sketched in Figure 9.3(b,c). Note that these sketches are not required as part of the assignment.

Generalized masses and stiffnesses:

$$\begin{aligned} M_1 = {\phi_1}^T \mathbf{M} \phi_1 &= \begin{bmatrix} 1 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} = 10/3, \\ M_2 = {\phi_2}^T \mathbf{M} \phi_2 &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5, \\ K_1 = {\phi_1}^T \mathbf{K} \phi_1 &= \begin{bmatrix} 1 & 2/3 \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} = 5000/9, \\ K_2 = {\phi_2}^T \mathbf{K} \phi_2 &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5000. \end{aligned} \tag{25}$$

Orthogonality checks:

$$ {\phi_1}^T \mathbf{M} \phi_2 = \begin{bmatrix} 1 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0, {\phi_1}^T \mathbf{K} \phi_2 = \begin{bmatrix} 1 & 2/3 \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0. \tag{26}$$

Vibration modes orthonormalized with respect to mass matrix:

$$\boxed{\phi_1 = \sqrt{\frac{3}{10}} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0.547723 \\ 0.365148 \end{bmatrix}, \quad \phi_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.447214 \\ -0.447214 \end{bmatrix}.} \tag{27}$$

Recheck generalized masses and stiffnesses:

$$\begin{aligned} M_1 = {\phi_1}^T \mathbf{M} \phi_1 &= \begin{bmatrix} \sqrt{3/10} & 2/3\sqrt{3/10} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{3/10} \\ 2/3\sqrt{3/10} \end{bmatrix} = 1, \\ M_2 = {\phi_2}^T \mathbf{M} \phi_2 &= \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = 1, \\ K_1 = {\phi_1}^T \mathbf{K} \phi_1 &= \begin{bmatrix} \sqrt{3/10} & 2/3\sqrt{3/10} \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} \sqrt{3/10} \\ 2/3\sqrt{3/10} \end{bmatrix} = 500/3 = \omega_1^2, \\ K_2 = {\phi_2}^T \mathbf{K} \phi_2 &= \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = 1000 = \omega_2^2. \end{aligned} \tag{28}$$

Check orthogonality with respect to mass matrix:

$$ {\phi_1}^T \mathbf{M} \phi_2 = \begin{bmatrix} \sqrt{3/10} & 2/3\sqrt{3/10} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = 0. \tag{29}$$

Check orthogonality with respect to stiffness matrix:

$$ {\phi_1}^T \mathbf{M} \phi_2 = \begin{bmatrix} \sqrt{3/10} & 2/3\sqrt{3/10} \end{bmatrix} \begin{bmatrix} 1000 & -1000 \\ -1000 & 2000 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = 0. \tag{30}$$

c. Denote the modal amplitudes by $\eta_1(t)$ and $\eta_2(t)$, which are collected in vector $\eta(t) = \begin{bmatrix} \eta_1(t) & \eta_2(t) \end{bmatrix}^T$. Since the system is undamped and unforced, the uncoupled modal equations are

$$\mathbf{M}_g \ddot{\eta} + \mathbf{K}_g \eta = 0, \tag{31}$$

in which $\mathbf{M}_g = \phi^T \mathbf{M} \phi$ and $\mathbf{K}_g = \phi^T \mathbf{K} \phi$ are the generalized mass and stiffness matrix respectively, and $\phi = [\boldsymbol{\phi_1} \quad \boldsymbol{\phi_2}]$ is the modal matrix. If this matrix is constructed with the mass-orthonormalized vibration modes (28), that is

$$\phi = \begin{bmatrix} \sqrt{\frac{3}{10}} & \sqrt{\frac{1}{5}} \\ \frac{2}{3}\sqrt{\frac{3}{10}} & -\sqrt{\frac{1}{5}} \end{bmatrix} = \begin{bmatrix} 0.547723 & 0.447214 \\ 0.365148 & -0.447214 \end{bmatrix}, \tag{32}$$

then $\mathbf{M}_g = \mathbf{I}$ (the identity matrix), whereas $\mathbf{K}_g = \mathbf{diag}[\omega_i^2]$, a diagonal matrix of squared frequencies stacked along the diagonal. Consequently (**??**) reduces to

$$\ddot{\eta} + \mathbf{diag}[\omega_i^2] = 0. \tag{33}$$

Replacing values for ω_1^2 and ω_2^2 we obtain

$$\ddot{\eta}_1 + \frac{500}{3}\eta_1 = 0, \quad \ddot{\eta}_2 + 1000\eta_2 = 0. \tag{34}$$

These are two canonical, undamped, unforced second-order ODE. Their solutions contain only the homogeneous component:

$$\begin{aligned} \eta_1(t) &= A_1 \cos\omega_1 t + B_1 \sin\omega_1 t & \eta_2(t) &= A_2 \cos\omega_2 t + B_2 \sin\omega_2 t, \\ \dot{\eta}_1(t) &= -\omega_1 A_1 \sin\omega_1 t + \omega_1 B_1 \cos\omega_1 t, & \dot{\eta}_2(t) &= -\omega_2 A_2 \sin\omega_2 t + \omega_2 B_2 \cos\omega_2 t. \end{aligned} \tag{35}$$

To determine the coefficients we need to express the IC in terms of modal amplitudes. Recall that $\eta(0) = \phi^T \mathbf{M}\mathbf{u}(0)$ and $\dot{\eta}(0) = \phi^T \mathbf{M}\dot{\mathbf{u}}(0)$. Replacing numbers:

$$\begin{aligned} \eta(0) = \begin{bmatrix} \eta_1(0) \\ \eta_2(0) \end{bmatrix} = \phi^T \mathbf{M} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{10}} & \frac{2}{3}\sqrt{\frac{3}{10}} \\ \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6\sqrt{\frac{3}{10}} \\ \sqrt{\frac{1}{5}} \end{bmatrix}, \\ \dot{\eta}(0) = \begin{bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{bmatrix} = \phi^T \mathbf{M} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{36}$$

Evaluating (**??**) at $t = 0$ and comparing to (**??**) provides four conditions: $\eta_1(0) = A_1 = 6\sqrt{3/10}$, $\eta_2(0) = A_2 = \sqrt{1/5}$, $\dot{\eta}_1(0) = -\omega_1 B_1 = 0$, $\dot{\eta}_2(0) = -\omega_2 B_2 = 0$. The last two give $B_1 = B_2 = 0$, whereas the first two yield A_1 and A_2 directly. It follows that the modal solutions are $\eta_1(t) = 6\sqrt{3/10}\cos\sqrt{500/3}\,t$ and $\eta_2(t) = \sqrt{1/5}\cos\sqrt{1000}\,t$. Combining these through the modal matrix as $\mathbf{u}(t) = \phi\eta(t)$ gives the physical response

$$\boxed{u_1(t) = \frac{9}{5}\cos\left(\sqrt{\frac{500}{3}}\,t\right) + \frac{1}{5}\cos\left(\sqrt{1000}\,t\right) \text{ in.}, u_2(t) = \frac{6}{5}\cos\left(\sqrt{\frac{500}{3}}\,t\right) - \frac{1}{5}\cos\left(\sqrt{1000}\,t\right) \text{ in.}} \tag{37}$$

ASEN 3112 - Structures – Spring 2020 – Homework Submission Guidelines

HW submission guidelines:

- **Write clearly your <u>name</u>, <u>student ID</u>, and <u>lab section ID</u> (011, 012, 013, or 014) on <u>each</u> sheet you turn in.**
- **Begin answering a new exercise on a new page to facilitate grading.**
- **Restate the questions so the grader is sure which exercise you solved.**
- **Do no fold your solution sheets.**

When writing out the homework, organize your solution to each assigned problem into four parts:

- **Restate the question.** Short hand is OK if the question is long or elaborated. Restating makes sure the grader knows you are answering the right problem, and will help you to organize the subsequent exposition as your subconscious gets going.
- **State the givens and unknowns to be found.** For givens always write down the physical units if those are stated in the problem.
- **Draw key diagrams at the start of the solution.** For example, Free Body Diagrams (FBD) are essential part of many problems in statics.
- **Write out the solution.** Be sure to **show work**. **Highlight** your answer(s) by a box, underline, arrow, or hi-lite marker. Also, do not forget to show the physical units. (This last item is also important in exams).

Example of Well Organized Homework Solution: Taken from the Solution Manual of M. Vable's *Mechanics of Materials*, Oxford, 1st ed.

1.40 An axial load is applied to a 1 inch diameter circular rod. The shear stress on section AA was found to be 20 ksi. The section AA is at 45° to the axis of the rod. Determine the applied force P and the average normal stress acting on section AA.

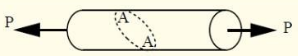


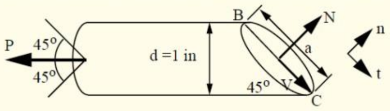
Fig. P1.40

1. Restate problem

Solution $d = 1$ in $\tau_A = 20$ ksi $P = ?$ $\sigma_A = ?$

2. List givens & unknowns

By making an imaginary cut through AA we obtain the following free body diagram.



3. Draw key diagrams needed in solution

The length BC can be found as: $a = d/(\sin 45^\circ) = 1.414$ in. The area of the elliptical inclined section can be found as: $A = (\pi/4)\,ad = (\pi/4)(1.414 \text{ in})(1 \text{ in}) = 1.1107 \text{ in}^2$

The shear force can be found as: $V = \tau_A A = (20 \text{ ksi})(1.107 \text{ in}^2) = 22.214$ kips

By equilibrium of forces in the t-direction we obtain: $P \sin 45^\circ - V = 0$ or $P = V/(\sin 45^\circ) = (22.214 \text{ kips})/(\sin 45^\circ)$

$$\boxed{P = 31.4 \text{ kips}}$$

By equilibrium of forces in the n-direction we obtain: $N = P \cos 45^\circ = (31.4 \text{ kips})(\cos 45^\circ) = 22.214$ kips

The average normal stress on the inclined plane is: $\sigma_A = N/A = 22.214 \text{ kips} / 1.1107 \text{ in}^2 = 20 \text{ kips/in}^2$ $\boxed{\sigma_A = 20 \text{ ksi (T)}}$

4. Work out solution, step by step, & highlight or box requested answers