ASEN 3112
“Lecture” 16:
Finite Element Method Examples

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Announcements

- Talk to Dr. Johnson about
  - Homework 7 (FEM): due Friday, March 13
  - Exam 2 (beams, energy methods, FEM): Tuesday, March 17
  - Lab 2: due Thursday, April 2
- ANSYS Tutorial – Recitations on Thursday (March 12)
  - Thursday's recitations are for going through the ANSYS Tutorial
  - Section 011
    - Last names A to M: 8:30 am to 9:20 am.
    - Last names N to Z: 9:30 – 10:20 am
  - Section 012
    - Last names A to K: 2:30 – 3:20 pm
    - Last names L to Z: 3:30 – 4:20 pm
- Next week's recitations (Thursday, March 19) will be open office hours focused on Lab 2

Exam 2 Announcements

- Exam policies
  - Will have 1 hour and 15 minutes for the exam
  - 4 problems
  - Closed-book
  - Your crib sheet can be **one** 8.5” x 11” piece of paper with writing **on both sides**
  - Non-internet-enabled calculators are allow (no phones, laptops)
- Past exams will be posted ASAP

Exam 2 Announcements

- Did you watch the review **videos** I posted for Exam 1?
  a) Yes 42
  b) No 30

- Do you think you'll come to Monday's office hours (4–6 pm in AERO N240)?
  a) Definitely yes! 3
  b) Probably yes 16
  c) Probably no 27
  d) Definitely no 25

The Direct Stiffness Method

Breakdown
- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution
- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution** – unknown displacements
- **Recovery of Derived Quantities**

conceptual steps

processing steps

post-processing steps

Member Stiffness Relations

- For each element, but not showing $^{(e)}$ superscript
- In local coordinates:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \bar{\boldsymbol{K}} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} \quad where \quad \bar{\boldsymbol{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- In global coordinates:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \boldsymbol{K} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \quad where \quad \boldsymbol{K} = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

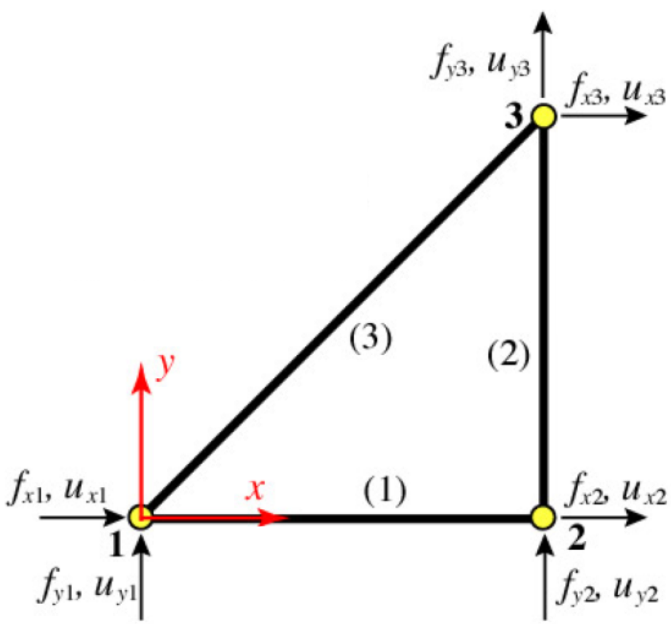
$$or \quad \boldsymbol{K} = \begin{bmatrix} [\hat{K}] & [-\hat{K}] \\ [-\hat{K}] & [\hat{K}] \end{bmatrix} \quad where \quad [\hat{K}] = \frac{EA}{L} \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

Assembly Process

- Assemble the $\left[\widehat{K}\right]$ matrices for each member into the global stiffness matrix.
- Remember our rules!
  1. Only have sums of $+\left[\widehat{K}\right]$s on the diagonal.
  2. Only have lone $-\left[\widehat{K}\right]$s off the diagonal.
  3. The number of $\left[\widehat{K}\right]$s in a diagonal cell shows the number of members at that node
  4. All diagonal cells must be filled
  5. A cell off the diagonal =0 means those two nodes aren't connected.

Assembly Process without Augmented Stiffness Matrices

$$[\hat{K}^{(e)}] = \frac{EA}{L}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$



$$\underline{K}^{(1)} = \begin{array}{cc} \begin{matrix} \quad 1 & \qquad\qquad 2 \end{matrix} & \\ \begin{bmatrix} [\hat{K}^{(1)}] & [-\hat{K}^{(1)}] \\ [-\hat{K}^{(1)}] & [\hat{K}^{(1)}] \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

$$\underline{K}^{(2)} = \begin{array}{cc} \begin{matrix} \quad 2 & \qquad\qquad 3 \end{matrix} & \\ \begin{bmatrix} [\hat{K}^{(2)}] & [-\hat{K}^{(2)}] \\ [-\hat{K}^{(2)}] & [\hat{K}^{(2)}] \end{bmatrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{array}$$

$$\underline{K}^{(3)} = \begin{array}{cc} \begin{matrix} \quad 1 & \qquad\qquad 3 \end{matrix} & \\ \begin{bmatrix} [\hat{K}^{(3)}] & [-\hat{K}^{(3)}] \\ [-\hat{K}^{(3)}] & [\hat{K}^{(3)}] \end{bmatrix} & \begin{matrix} 1 \\ 3 \end{matrix} \end{array}$$

Assembly Process

3x3 matrix of 2x2 matrices

$$
\begin{array}{cc}
 & \begin{array}{ccc} \quad 1 \quad & \quad\quad 2 \quad\quad & \quad 3 \quad \end{array} \\
\boldsymbol{K} = & \left[\begin{array}{c|c|c}
[\hat{K}^{(1)}]+[\hat{K}^{(3)}] & [-\hat{K}^{(1)}] & [-\hat{K}^{(3)}] \\ \hline
[-\hat{K}^{(1)}] & [\hat{K}^{(1)}]+[\hat{K}^{(2)}] & [-\hat{K}^{(2)}] \\ \hline
[-\hat{K}^{(3)}] & [-\hat{K}^{(2)}] & [\hat{K}^{(2)}]+[\hat{K}^{(3)}]
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

Reduced Global Stiffness Equation

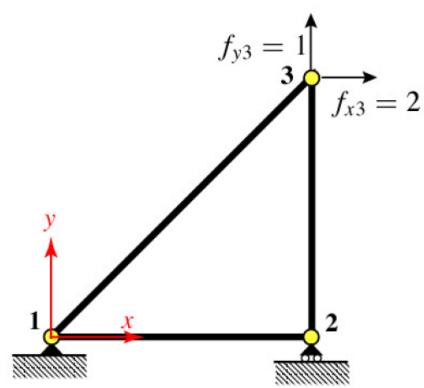
$$
\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix}
\begin{bmatrix} u_{x1} = 0 \\ u_{y1} = 0 \\ u_{x2} = ? \\ u_{y2} = 0 \\ u_{x3} = ? \\ u_{y3} = ? \end{bmatrix}
=
\begin{bmatrix} f_{x1} = ? \\ f_{y1} = ? \\ f_{x2} = 0 \\ f_{y2} = ? \\ f_{x3} = 2 \\ f_{y3} = 1 \end{bmatrix}
$$

$$
\underbrace{\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix}}_{\tilde{K}}
\underbrace{\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix}}_{\tilde{u}}
=
\underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{\tilde{f}}
$$

Reduced Global Stiffness Equation (System): $\tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}}$

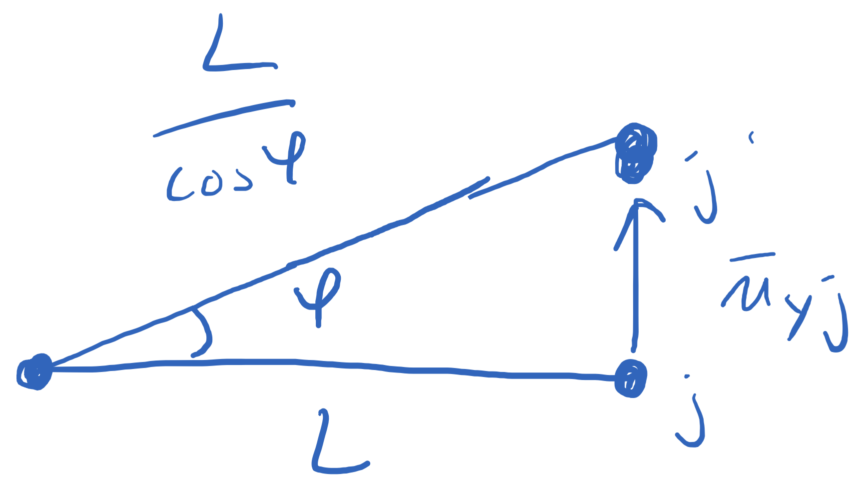
$$\tilde{u} = \tilde{K}^{-1}\tilde{f}$$

Solution:
$$
\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix}
$$



Recovery of Reaction Forces & Internal Force/Stress

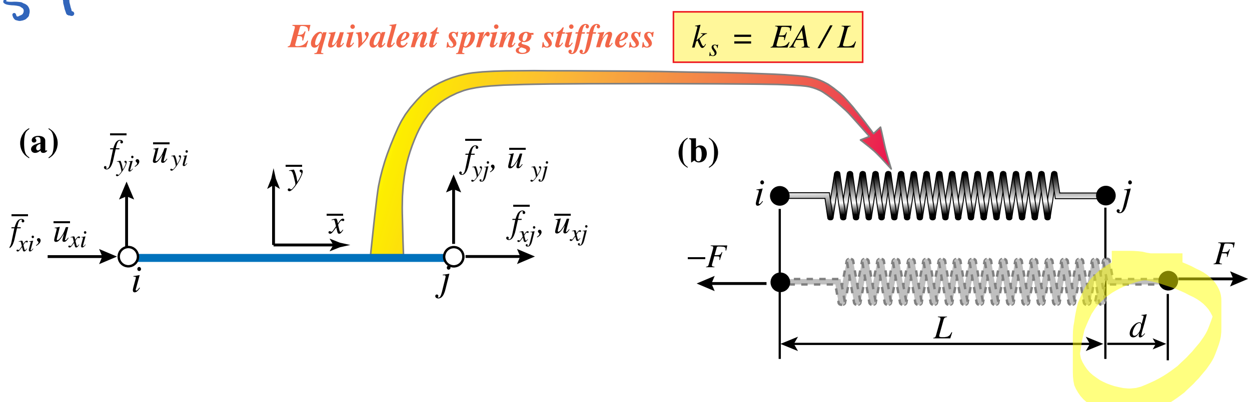
$$\sigma = \frac{E}{L}\left(\bar{u}_{xj} - \bar{u}_{xi}\right)$$



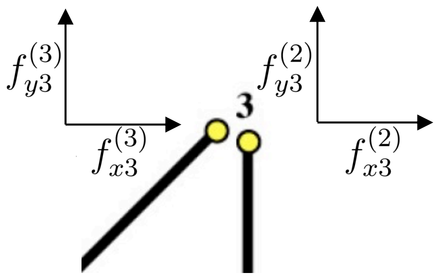
small angle approx

$$\cos\varphi \approx 1$$

$$\frac{L}{\cos\varphi} \approx L$$



Force Equilibrium at Nodes



Over all nodes

$$\underline{f} = \underline{f}^{(1)} + \underline{f}^{(2)} + \underline{f}^{(3)}$$

Recall that $\underline{f}^e = \underline{K}^e \underline{u}^e$

$$\underline{f} = \underline{K}^{(1)} \underline{u}^{(1)} + \underline{K}^{(2)} \underline{u}^{(2)} + \underline{K}^{(3)} \underline{u}^{(3)}$$

$$\underline{f} = \underline{K}\,\underline{u} \quad \text{where} \quad \underline{K} = \underline{K}^{(1)} + \underline{K}^{(2)} + \underline{K}^{(3)}$$

displacement of each node

external forces on each node

Recovery of Internal Force Through Forces

$$\underline{f} = f^{(1)} + f^{(2)} + f^{(3)}$$

in global coord.

external forces on nodes

force of members on nodes = force of node of members

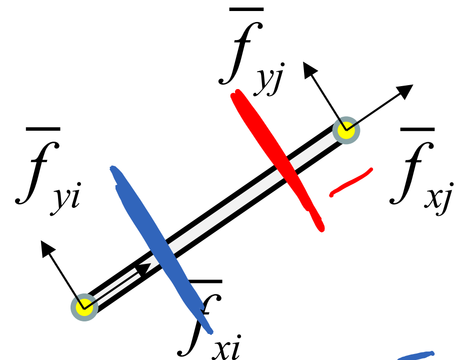
$\underline{f} = \underline{K}\,\underline{u} \rightarrow$ gives $\underline{f}$, external forces on nodes

To find F_{int} or σ, need $\underline{f}^{(e)}$

force of node on member

Recovery of Internal Force Through Forces

external forces on nodes in local coord.



Note: for bar element

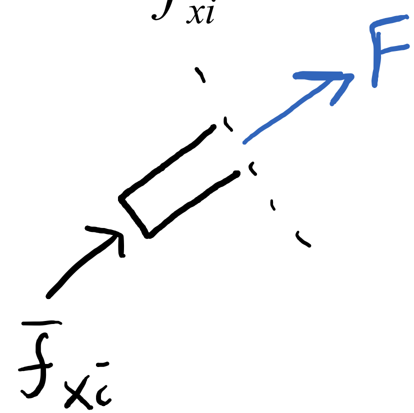
$$\bar{f}_{yi} = \bar{f}_{yj} = 0$$

Element Stiffness Equation in global CS:

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$$

Transformation from global to local CS:

$$\underbrace{\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}}_{\bar{\mathbf{f}}^e} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\mathbf{T}^e} \underbrace{\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix}}_{\mathbf{f}^e} \quad *$$



internal force

$$F = -\bar{f}_{xi} \quad *$$

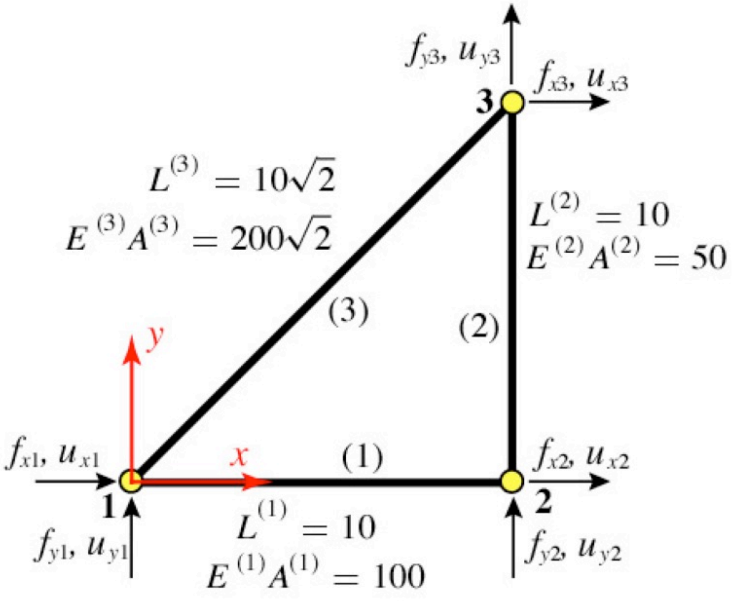
$$F = +\bar{f}_{xj}$$

$$F = -\bar{f}_{xi} = -(\cos\varphi\, f_{xi} + \sin\varphi\, f_{yi})$$

$$F = +\bar{f}_{xj} = \cos\varphi\, f_{xj} + \sin\varphi\, f_{yj}$$

Example with Example Truss

Example: Bar 2 Using nodal solution for Bar 2:



$$\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e \qquad \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \\ -1.0 \end{bmatrix} = 5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.0 \\ 0.0 \\ 0.4 \\ -0.2 \end{bmatrix}$$

$$F = -\bar{f}_{xi} = -\left(c\ f_{xi} + s\ f_{yi}\right) \quad or \quad F = \bar{f}_{xj} = \left(c\ f_{xj} + s\ f_{yj}\right)$$

for $\cos 90 = 0 \quad \sin 90 = 1$

$$F = -\bar{f}_{xi} = -\left(0 \cdot 0 + 1 \cdot 1\right) = -1 \quad or \quad F = \bar{f}_{xj} = \left(0 \cdot 0 + 1 \cdot -1\right) = -1$$

0. Calculate $\underset{\sim}{u}$

1. Calculate force of nodes i & j on bar (e)

$$\underset{\sim}{f}^{(e)} = \underset{\sim}{K}^{(e)} \underset{\sim}{u}^{(e)}$$

2. Then do translation to local coord.

$$F_{int} = -\bar{f}_{xi}^{(e)} = -(\cos\varphi \, f_{xi}^{(e)} + \sin\varphi \, f_{yi}^{(e)})$$

$$F_{int} = +\bar{f}_{xj}^{(e)} = \cos\varphi \, f_{xj}^{(e)} + \sin\varphi \, f_{yj}^{(e)}$$

3. $\sigma = \frac{F_{int}}{A}$

=