ASEN 3112
Lecture 8:
Beam Differential Equations

Dr. Johnson

Prof. Hussein

Department of Aerospace Engineering Sciences

University of Colorado, Boulder

Exam 1 Announcements

- Syllabus has been updated with new exam make up policy
- In class next Tuesday, Feb. 11
- Covers Ch. 1-9 of textbook
- If you have an accommodation, respond to my e-mail or send me an e-mail if you didn’t get one

Exam 1 Announcements

- Exam policies
  - Will have 1 hour and 15 minutes for the exam
  - 3-4 problems?
  - Closed-book
  - Your crib sheet can be one 8.5” x 11” piece of paper with writing **on both sides**
  - Non-internet-enabled calculators are allow (no phones, laptops)
- Past exams posted
- Will post a review video ASAP

Announcements

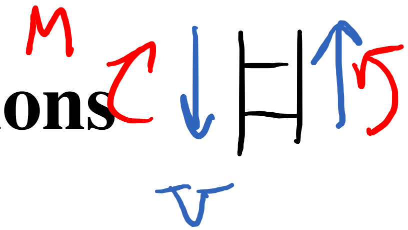
- Lecture capture videos are posted to Canvas
  - We trust that you won't just skip class
- Lab 1 video and the associated individual quiz will be posted later today
- Homework 2 is graded and posted to Canvas and Gradescope
- Pre-exam office hours
  - Monday, 4:00 – 6:00 pm in **AERO N240**
  - Tuesday, 9:00 – 10:00 am in AERO 302

This Week's Outline

- Applying beam bending differential equations (Ch. 11)
  - Finding deflection with 2^{nd} and 4^{th} order ODEs
  - Boundary conditions
  - Matching conditions
  - Solving statically indeterminate problems with superposition

- Note: Ch. 12 is not covered in this course.

Beam Notation & Sign Conventions



Quantity	Symbol	Sign convention(s)
Problem specific load	varies	You pick'em
Generic load for ODE work	$p(x)$	+ if up
Transverse shear force	$V_y(x)$	+ if up on +x face
Bending moment	$M_z(x)$	+ if it produces compression on top face
Slope of deflection curve	$dv(x)/dx = v'(x)$	+ if positive slope, or cross-section rotates CCW
Deflection curve	$v(x)$	+ if beam cross-section moves upward

Note 1: Some textbooks (e.g. Vable and Beer-Johnson-DeWolf) use $V = -V_y$ as alternative transverse shear force symbol. This has the advantage of eliminating the minus sign in two of the ODEs listed on the next slide. V will only be used occsionally in this course.

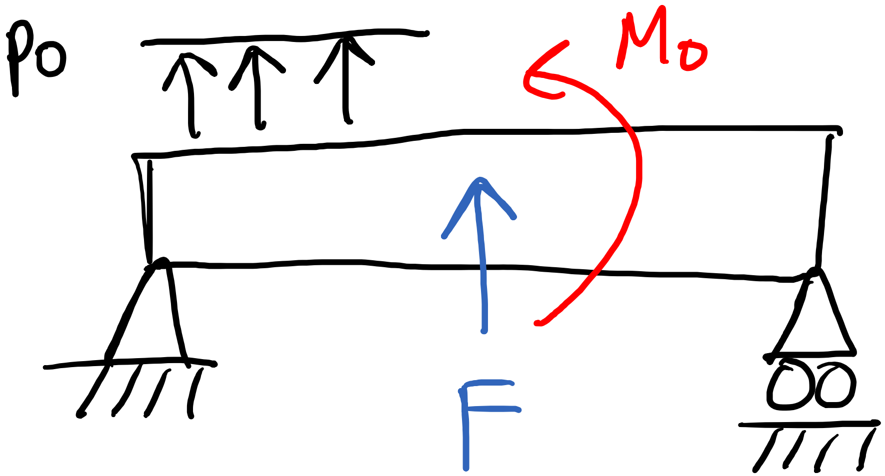
Note 2. In our beam model, the slope $v'(x) = dv(x)/dx$ is equal to the rotation $\theta(x)$ of the cross section

Beam Differential Equations

Connected quantities	Ordinary Differential Equations (ODEs)
From load to transverse shear force	$\dfrac{dV_y}{dx} = -p$ or $p = -V_y' = V'$
From transverse shear to bending moment	$\dfrac{dM_z}{dx} = -V_y$ or $M_z' = -V_y = V$
From bending moment to deflection	$E\,I_{zz}\,v'' = M_z$ *or* $v'' = \dfrac{M_z}{E\,I_{zz}}$
From load to moment	$M_z'' = p$
From load to deflection	$E\,I_{zz}\,v^{IV} = p$

Ungraded Clicker Question

- Right now, I am confident that I could find V(x), M(x), and the max bending stress for this beam?



a) Strongly agree
b) Agree
c) Disagree
d) Strongly disagree

$$
\begin{tikzcd}
p(x) \arrow[r, bend right, "-p(x) = V'(x)"'] & V(x) \arrow[r, bend right, "-V(x) = M'(x)"'] & M(x) \arrow[r, bend right, "M(x) = EI\theta'(x)"'] & \theta(x) \arrow[r, bend right, "\theta(x) = v'(x)"'] & v(x)
\end{tikzcd}
$$

We defined $K = \frac{d\theta}{ds_0} = \frac{d\theta}{dx} = \theta'(x) = v''(x)$

$$\boxed{K = v''(x)}$$

We also defined $M = EIK$

$$M(x) = EI\,v''(x)$$
$$V(x) = -EI\,v'''(x)$$
$$p(x) = EI\,v''''(x)$$

$$\boxed{\sigma = \frac{-My}{I}}$$

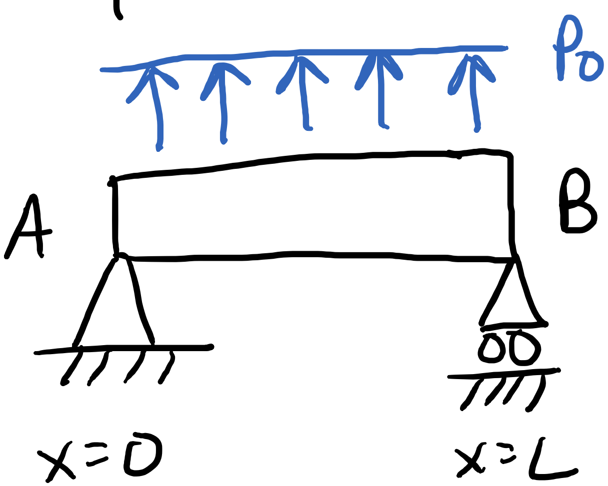
Typically $p(x)$ known

$$\int p(x) \Rightarrow -V(x)$$
$$\iint p(x) \Rightarrow M(x)$$
$$\frac{\iiiint p(x)}{EI} \Rightarrow v(x)$$

V & M for statically det. beams $\neq f(EI)$

$$v = f(EI)$$

Ex. 1



Given p_0, L, E, I

Find $v(x)$

4th order ODE method

$$p(x) = p_0 = EIv''''(x)$$

$$p_0 x + C_1 = EIv'''(x) = -V(x)$$

$$\frac{1}{2} p_0 x^2 + C_1 x + C_2 = EIv''(x) = M(x)$$

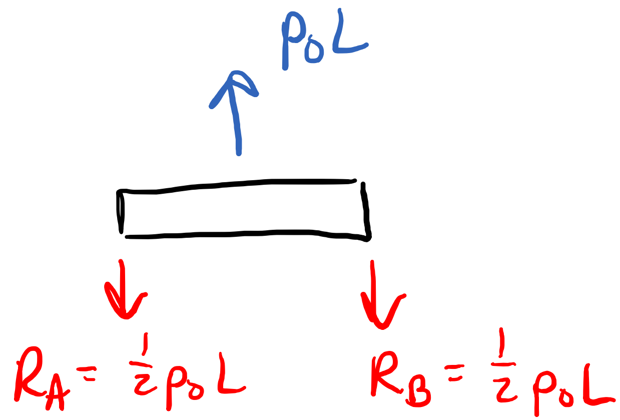
$$\frac{1}{6} p_0 x^3 + \frac{1}{2} C_1 x^2 + C_2 x + C_3 = EIv'(x)$$

$$1/24\, p_0 x^4 + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4 = EIv(x)$$

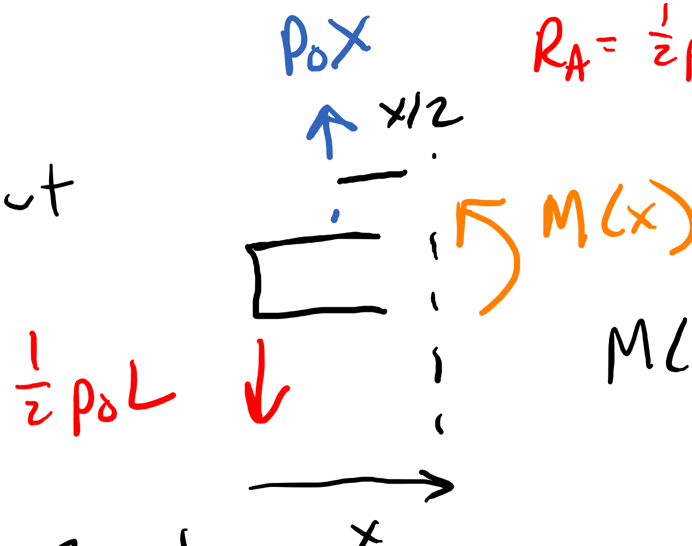
2nd order ODE method

Find $M(x)$ by cuts.

1. Global FBD



2. Cut



$\Sigma M \rightarrow$

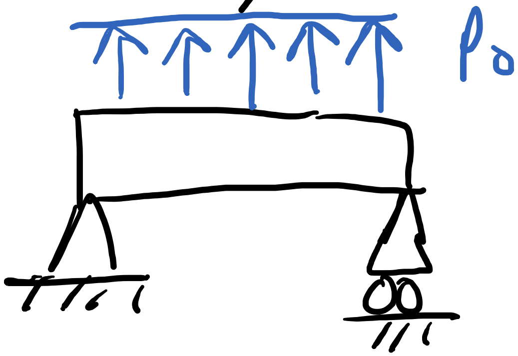
$$M(x) = \frac{1}{2} p_o x^2 - \frac{1}{2} p_o L x$$

$$M(x) = \frac{1}{2} p_o x^2 - \frac{1}{2} p_o L x = EI v''(x)$$

$$\frac{1}{6} p_o x^3 - \frac{1}{4} p_o L x^2 + C_3 = EI v'(x)$$

$$\frac{1}{24} p_o x^4 - \frac{1}{12} p_o L x^3 + C_3 x + C_4 = EI v(x)$$

Boundary Conditions

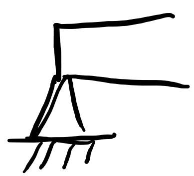
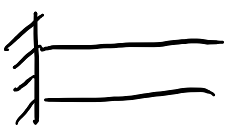
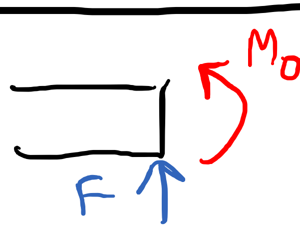


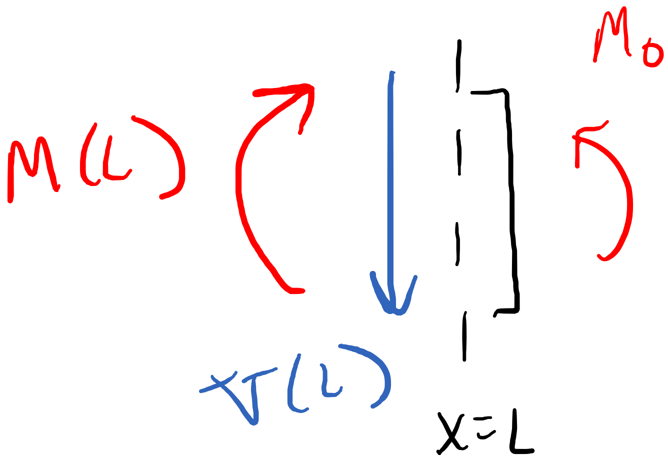
2nd order

$v(0)=0$	$v(L)=0$
$M(0)=0$	$M(L)=0$

pinned joint @ A | @ B 4th order

Do we easily know

Joint	Image	v	v'	V	M
Pinned / Roller		$=0$	?	?	$=0$
Clamped		$=0$	$=0$	?	?
Free		?	?	F	M_o



$\Sigma F \rightarrow \quad V(L) = 0$

$\Sigma M \rightarrow \quad M(L) = M_o$

Assume so small $(dx \rightarrow 0)$

$p(x)$ has no effect

forces don't cause moments

$$EI\,v(x) = \frac{1}{24} p_0 x^4 - \frac{1}{12} p_0 L x^3 + C_3 x + C_4$$

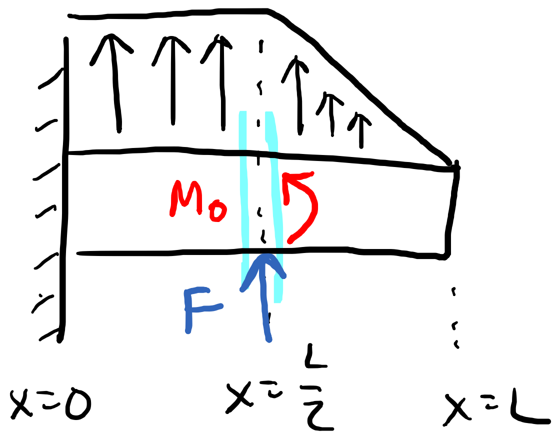
① $v(0) = 0$ $\qquad 0 = 0 + 0 + 0 = C_4$

② $v(L) = 0$ $\qquad EI\,v(L) = 0 = \frac{1}{24} p_0 L^4 - \frac{1}{12} p_0 L^4 + C_3 L + 0$

$$C_3 = \frac{1}{24} p_0 L^3$$

C

Ex. 2

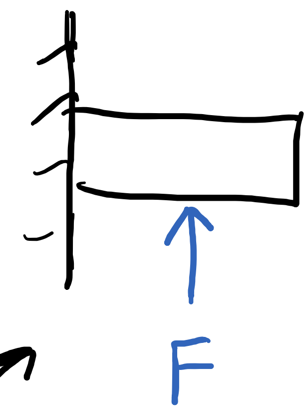


Critical points

1. External force
2. External moment
3. Change in p(x)

At critical points, we have a piecewise function **INTEGRATE EACH PIECE**

$$p(x) = \begin{cases} p_1(x) & 0 \le x \le \frac{L}{2} \\ p_2(x) & \frac{L}{2} \le x \le L \end{cases}$$

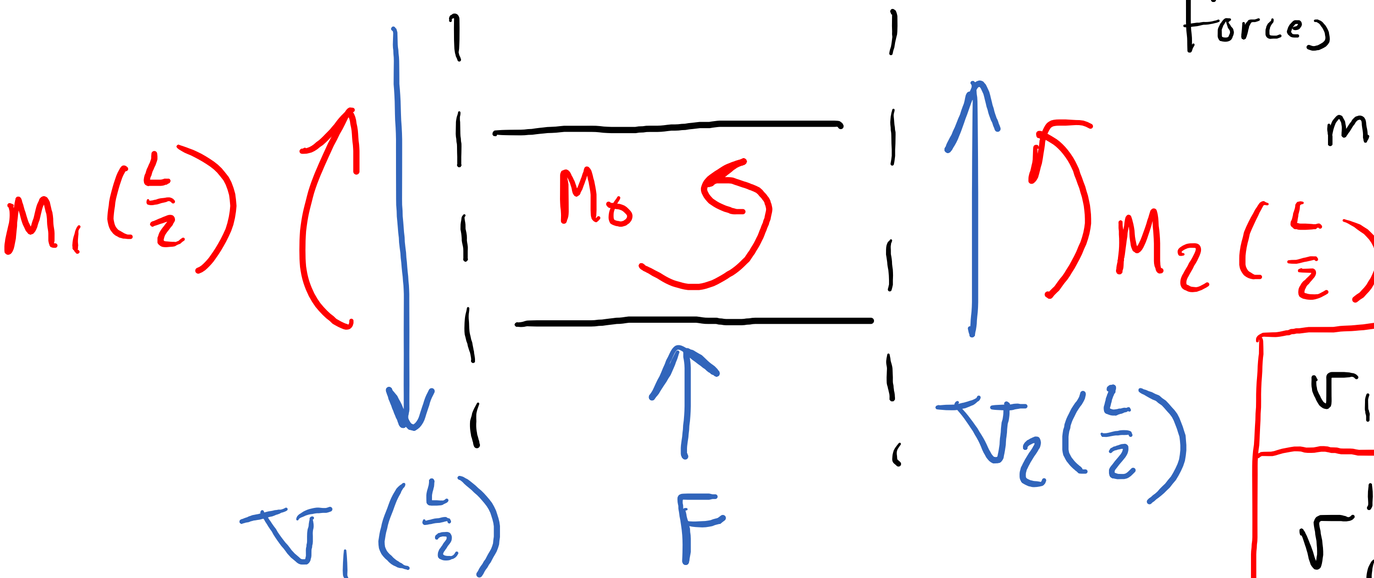


$$p(x) = \begin{cases} 0 & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases}$$

Matching Conditions

p(x) doesn't act

Forces don't cause moments



ALWAYS

$$\boxed{v_1(\tfrac{L}{2}) = v_2(\tfrac{L}{2})}$$

$$\boxed{v_1'(\tfrac{L}{2}) = v_2'(\tfrac{L}{2})}$$

$$\Sigma F_y = 0 = V_2(\tfrac{L}{2}) - V_1(\tfrac{L}{2}) + F$$

$$\boxed{V_2(\tfrac{L}{2}) - V_1(\tfrac{L}{2}) = -F}$$

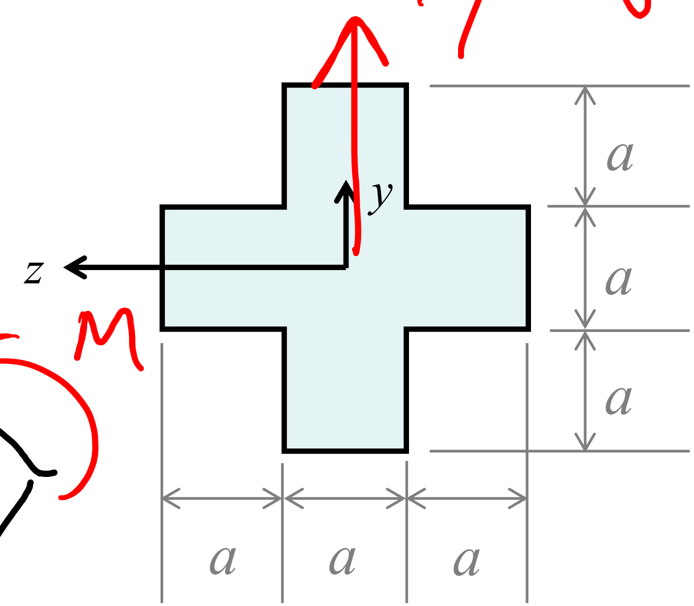
$$\Sigma M = 0 = M_2(\tfrac{L}{2}) - M_1(\tfrac{L}{2}) + M_0$$

$$\boxed{M_2(\tfrac{L}{2}) - M_1(\tfrac{L}{2}) = -M_0}$$

Question 1

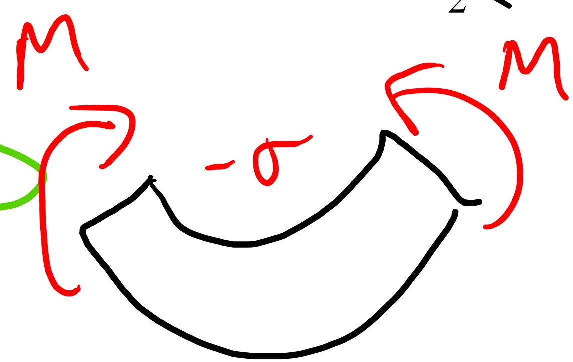
$\sigma = \frac{-My}{I}$

- A beam has the following cross-section. The centroid of the cross-section is right in the middle. At this particular point along the length of the beam, the shear force across the cross-section is V = -40 kN and the bending moment is $M = +{}^{100}/_{3} = +33.\overline{3}$ kN-m.



$a = 0.05$ m

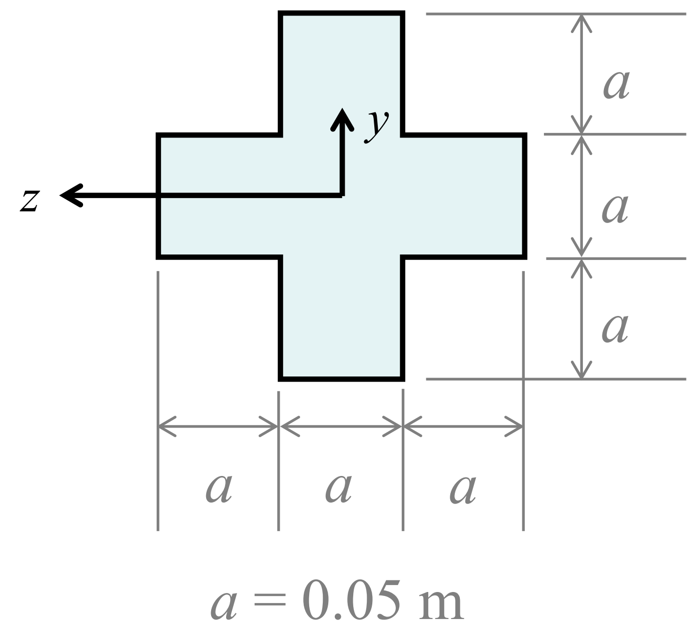
- Is the top of the beam in tension or compression?
  a) Tension
  b) Compression



Question 2

- A beam has the following cross-section. The centroid of the cross-section is right in the middle. At this particular point along the length of the beam, the shear force across the cross-section is V = -40 kN and the bending moment is M = $+\,{}^{100}/_{3} = +33.\overline{3}$ kN-m.

- What is the relationship between the maximum tensile and maximum compressive bending stresses?
  a) Tensile is larger
  b) Compressive is larger
  c) They are the same



Question 4

$\sigma = \frac{-My}{I}$

- A cantilever (fixed) beam is loaded with a point load P. If we know the material will fail in COMPRESSION, which point is more likely to fail first: