ASEN 3112

Spring 2020

Lecture 9

Whiteboard

February 13, 2020

L9
1

2/13/20

Mechanics of materials is concerned with the relationship between forces and deformation

General treatment of these two quantities are facilitated by:

- stress (σ_{ij}) $\quad i,j = x, y, z$
- strain (ε_{ij})

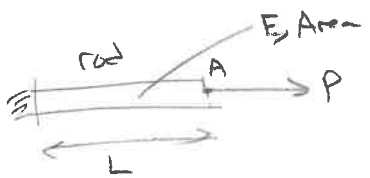
- work (W) and strain energy (U)

$\sigma_{ij}, \varepsilon_{ij}$ — tensors — coordinate dependent

W, U — scalars — $\underbrace{\text{coordinate invariant}}_{\text{advantage of energy method}}$

Basic Principle

Force—displacement



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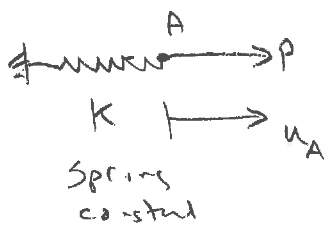
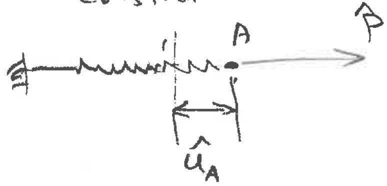
Arbitrary



$dP = K\, du_A$ infinitesimal treatment

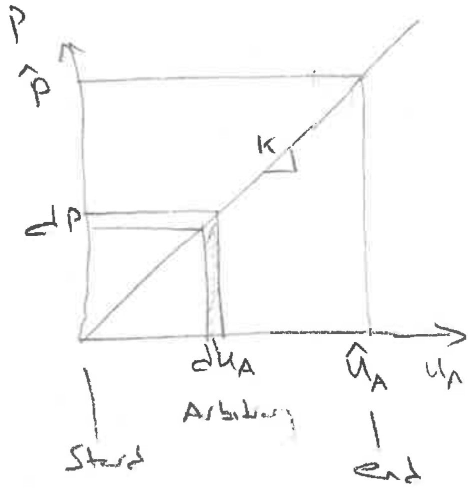
End



$\hat{P} = K\, \hat{u}_A$

> Conservation of Energy
> - No work is lost
> - External work is stored as internal energy

Consider relationship $\overset{\text{variable}}{P} = P(u_A) = \overset{\text{constant}}{K}\ \overset{\text{variable}}{u_A}$



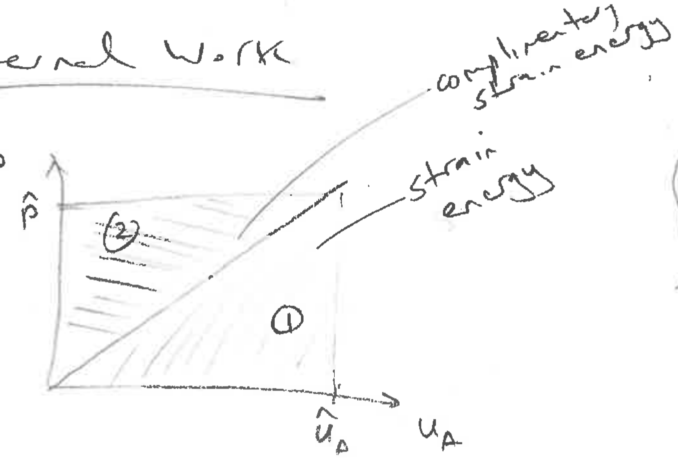
linear system

$$dP = K\,du_A$$

$$\int_0^{\hat{P}} dP = \int_0^{\hat{u}_A} K\,du_A$$

$$\hat{P} = K u_A \Big]_0^{\hat{u}_A} = K\hat{u}_A$$

External Work



Area ① = Area ② for linear system

$$\int dW_e = \int_0^{\hat{u}_A} \overset{\text{variable}}{P}\,du_A$$

$$W_e = \int_0^{\hat{u}_A} K u_A\,du_A$$

$$W_e = \frac{1}{2}K\hat{u}_A^2$$

$$\int dW_e = \int_0^{\hat{P}} \overset{\text{variable}}{u_A}\,dP$$

$$W_e = \int_0^{\hat{P}} \frac{1}{K}P\,dP$$

$$W_e = \frac{1}{2K}\hat{P}^2$$

$$W_e = \frac{1}{2}K\hat{u}_A^2$$

$$\left(\hat{P} = K\hat{u}_A \Rightarrow \hat{u}_A = \frac{\hat{P}}{K}\right)$$

$$= \frac{1}{2}\cancel{K}\hat{u}_A\left(\frac{\hat{P}}{\cancel{K}}\right)$$

$$\boxed{W_e = \frac{1}{2}\hat{P}\hat{u}_A}$$

L9
3

— Work done on body $=$ Δ in energy stored in body

charge

strain energy U

for non-dissipative systems

$$W_e = U$$

— Conservation of Energy

$$W_e + W_i = 0$$

internal work

$$\therefore W_i = -U$$

Recorder loaded spring

Free body diagram

internal point experiencing internal force

$$W_e = \frac{1}{2}\hat{P}\hat{u}_A$$

$$W_i = \frac{1}{2}\hat{F}_{int}\hat{u}_A$$

$$W_e + W_i = 0$$

$$\frac{1}{2}\hat{P}\hat{u}_A + \frac{1}{2}\hat{F}_{int}\hat{u}_A = 0$$

$$\Longrightarrow \hat{F}_{int} = -P = \underline{-k\,u_A}$$

Strain strain

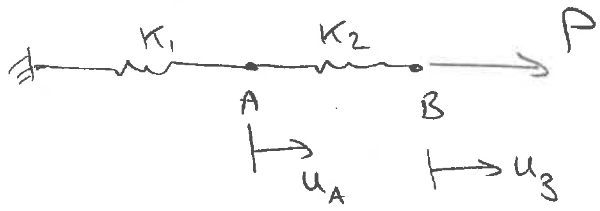
$$U = -W_i = -\frac{1}{2}\hat{F}_{int}\hat{u}_A$$

$$U = \frac{1}{2}k\hat{u}_A^2 = \frac{1}{2}\frac{1}{k}(F_{int})^2$$

Notation clarity $\hat{u}_A$: maximum displacement of point A (not vector)

[Side note: $\vec{u}_A = u_{Ax}\vec{i} + u_{Ay}\vec{j}$]

Example



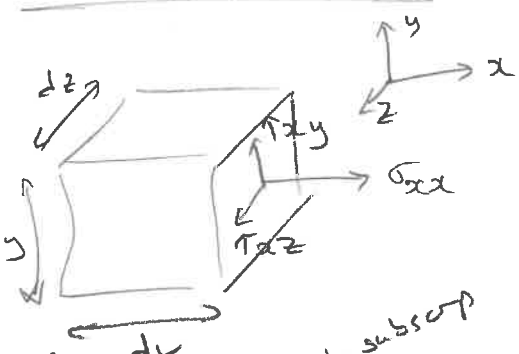
Write an expression for strain energy U for this system

(load until $u_A \to \hat{u}_A$
$u_B \to \hat{u}_B$)

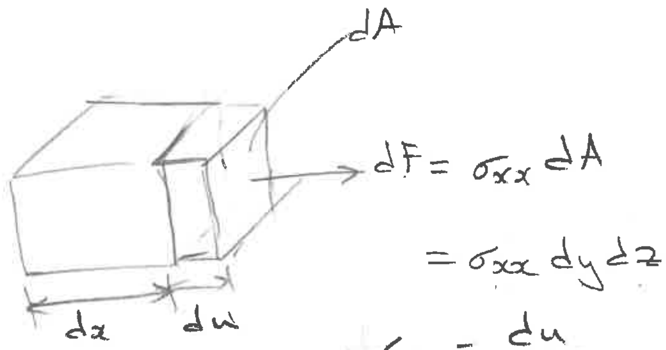
$$U = \underbrace{\tfrac{1}{2}K_1 \hat{u}_A^2}_{\text{Strain energy in Spring 1}} + \underbrace{\tfrac{1}{2}K_2(\hat{u}_B - \hat{u}_A)^2}_{\text{strain energy in Spring 2}}$$

Now, introduce σ_{ij}, ε_{ij} into energy formulation

Elastic Strain Energy



du (subscript omitted)



$dF = \sigma_{xx}\, dA$

$= \sigma_{xx}\, dy\, dz$

$\varepsilon_{xx} = \dfrac{du}{dx}$

$du = \varepsilon_{xx}\, dx$

$$\underset{\text{omit subscript}}{dW} = \tfrac{1}{2}\underset{dP}{dF}\, du = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx}\underbrace{dx\,dy\,dz}_{dV}$$

$\quad ||$

$dU =$

$$\frac{dU}{dV} = U_0\Big|_{xx} = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx}$$

strain energy density

L9
5

Generalize to 3D

$$U_0 = \frac{1}{2}\Big(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz} \Big)$$

Elastic strain energy U

$$U = \int_V dU = \iiint_{xyz} U_0 \, dx\,dy\,dz$$

Clicker 2

$$W_e = mg(\Delta h), \quad h: \text{height}$$

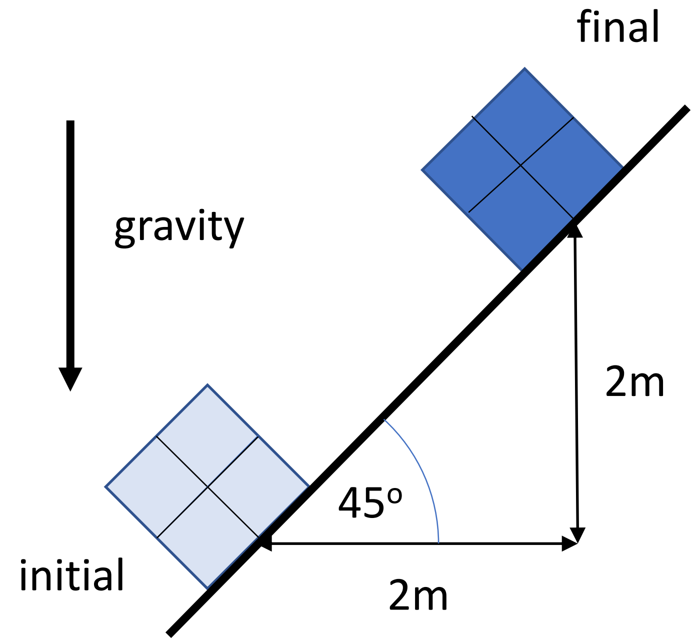
$$= (1\,\text{kN})(2\,\text{m})$$

$$= 2\,\text{kNm}$$

$\longrightarrow$ (b)

Clicker Question 1

You need to push the rigid mass with a weight of 1kN up an inclined plane. What work do you need to perform? Ignore friction.



(a) $\sqrt{2}kNm$

(b) $2kNm$

(c) $1kNm$

(d) none of the above

Clicker Question 2

You slowly place a sphere with a weight of 10 kN at the center of an elastic beam. At the end of this quasi-static loading process the beam deflects by 0.1 m. How much external work has been performed on the beam and is stored in the beam?

(a) $1kNm$

(b) $2kNm$

(c) $0.5kNm$

(d) none of the above