ASEN 3112

Spring 2020

Lecture 22

April 9, 2020

Stability of Structures: Basic Concepts

Objective

This Lecture

(1) presents basic concepts & terminology on structural stability

(2) describes conceptual procedures for testing stability

(3) classifies models and analysis methods

(4) concludes by contrasting exact versus linearized determination of critical loads

Terminology Related to Mechanical Systems: Table 23.1 of Lecture 23

Term	*Definition*
System	A functionally related set of components regarded as a physical entity.
Configuration	The relative disposition or arrangement of system components.
State	The condition of the system as regards its form, structure or constitution.
Degrees of Freedom	A set of **state variables** that uniquely characterizes the state. Abbrv.: **DOF**
Kinematic DOF	A DOF that is directly linked to the system geometry; e.g., a displacement.
Input	The set of all actions that can influence the state of a system, or component.
Output	The set of all quantities that characterize the state of a system, or component.
Model	A mathematical idealization of a physical system.
Discrete Model	A model with a finite # of DOF. Often expressed as vector equations.
Continuous Model	A model with an infinite # of DOF. often expressed as a ODE or PDE.
Event	A change in the state produced by an agent.
Behavior	A pattern of events.
Reference State	A state of the system adopted as base or origin to measure relative changes. Often the same as the **undeformed state** (cf. Table 23.2).
Motion	The change in system geometry, as measured from a reference state.
Kinematics	The study of system motion independently of force agents.
Kinetics	The study of forces as action agents, and their effect on the system state.
Kinematic Constraint	Any condition that restricts the system motion, usually expressed in terms of kinematic DOF. Also simply called **constraint**.
Environment	A set of entities that do not belong to the system, but can influence its behavior.
Open System	A system that is influenced by entities outside the system (its environment).
Closed System	A system that is not affected by entities outside the system.
Interaction	The mutual effect of a system component, or group of such components, on other components.
Forces	The action agents through which effects are transmitted between system components, or between environment entities and system components.
Internal Forces	Forces that act between system components.
External Forces	Forces that act between environment entities and system components.
Constraint Force	A force manifested by removing a constraint while keeping it enforced.
Reaction Force	A constraint force that is an external force.
Applied Load	An external force specified as data. Also simply called **load**.

Terminology Related to Static Stability Analysis: Table 23.2 of Lecture 23

Term	*Definition*
Reference Loads	A set of applied loads taken as reference for application of a load factor.
Load Factor	A scalar, denoted by ,, which scales reference loads to get the actual applied loads. Also called **load parameter** and **load multiplier**.
System Response	Values of the DOF, or subset thereof, expressed as function of the load factor, or of the load level if only one load is applied. Also simply called **response**.
Equilibrium State	A state in which internal and external forces are in equilibrium. The associated configuration is called an **equilibrium configuration**.
Undeformed State	The equilibrium state under zero applied loads, or, equivalently, , D 0. The associated configuration is called an **undeformed configuration**.
Equilibrium Response	A system response in which all states are equilibrium states.
State Space	A RCC frame with a DOF subset as axes.
Response Space	A RCC frame with the load factor as one axis, and a DOF subset as the others.
Response Plot	A display of the system response in response space.
Equilibrium Path	An equilibrium response viewed in response space.
Perturbation	An externally imposed disturbance of an equilibrium state while actual loads are kept fixed. It may involve application of forces or motions.
Allowed Perturbation	A perturbation that satisfies kinematic constraints. Also called **admissible perturbation**, and (in the sense of variational calculus) **virtual variation**.
Stability	The ability of a system to recover an equilibrium state upon being disturbed by any allowed perturbations.
Instability	The inability of a system to recover an equilibrium state upon being disturbed by at least one allowed perturbation.
Stable	Qualifier for an equilibrium state, or configuration, at which stability holds.
Unstable	Qualifier for an equilibrium state, or configuration, at which instability occurs.
Neutrally Stable	Qualifier for an equilibrium state, or configuration, at which transition between stability and instability occurs.
Critical	A qualifier that flags the occurrence of neutral stability. Applicable to **state**, **configuration**, **load**, and **load factor**. For example: **critical load**.
Critical Point	In a equilibrium response plot, a location where a critical state occurs.
Bifurcation Point	A critical point at which two or more equilibrium paths cross.
Limit Point	A critical point at which the load factor reaches a maximum or minimum.
Buckling	Name used by structural engineers for the occurrence of a bifurcation point.
Snapping	Name used by structural engineers for the occurrence of a limit point. Also called **snap-through**, **snap buckling**, and **snap-through buckling**.

Two Stability Scenarios

Static Stability

Stability of static equilibrium configurations of a mechanical system (e.g., structures)

Dynamic Stability

Stability of the motion of a dynamic system (e.g., a vehicle trajectory)

In this course we consider only static stability

Testing Static Stability: Varying Applied Loads

Scale applied loads by a parameter λ - this is called the load factor or the loading parameter

If $\lambda = 0$, structure is under zero load, and takes up a reference undeformed configuration $\mathcal{C}_0$

We assume that the structure is stable at $\lambda = 0$

Now apply loads by varying λ monotonically away from zero. Structure assumes equilibrium configurations $\mathcal{C}(\lambda)$

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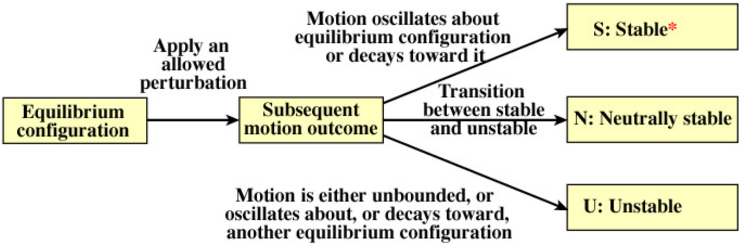
Testing Static Stability: Perturb and Release

To test stability of a specific configuration $\mathcal{C}(\lambda_d)$, freeze $\lambda = \lambda_d$. Apply a small perturbation (for example, a tiny deflection) and let it go!

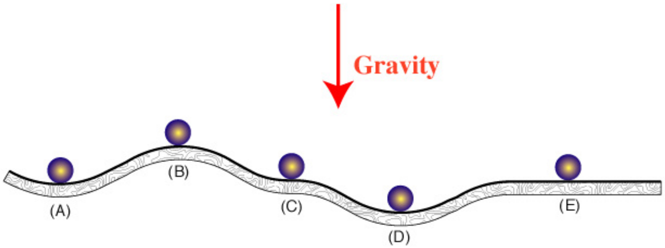
The structure is set in motion. Three possible outcomes are pictured in the next slide.

Testing Static Stability: Perturbation Outcomes



* Strictly speaking, S requires stability for all possible admissible perturbations

Ex. - Identify Stability Outcomes for 5 Heavy Rollers Shown Below - All 5 Are in Static Equilibrium Configurations Under Gravity



Testing Static Stability: Critical Load

For sufficiently small $\lambda = \lambda_d$ assume that the structure is stable (S).

It becomes neutrally stable (N) at a value $\lambda = \lambda_{cr}$ This value is called a critical load parameter or critical load factor. The associated load is called the critical load (or loads, if more than one).

The determination of the critical load(s) is a key goal of the stability analysis.

Testing Static Stability: Clarifications

As described, the stability test is dynamic since

there is a before event: apply perturbation
there is an after event: what happens upon release

Thus time is involved. However, under certain conditions (next slide) time may be factored out. If so, a dynamic response analysis is no longer needed, which greatly simplifies test procedures. Those are called static stability criteria.

Another question: how small is a "small perturbation"? In examples later, they are supposed to be infinitesimal.

Assumptions That Allow Use of Static Stability Criteria

Linearly elastic material

displacements and rotations, however, are not necessarily small

Loads are conservative (= derivable from potential)

gravity and hydrostatic loads are conservative
aerodynamic and propulsion loads are generally not

Setting Up Stability Equations for Static Stability Criteria

Equilibrium Method

use FBD in perturbed equilibrium configuration, look for nontrivial solutions as function of λ

Energy Method

set up total potential energy of system, analyze whether its Hessian is positive definite (S), nonnegative definite (N), or indefinite (U), as function of λ

Only the Equilibrium Method will be used in this course

Models To Predict Critical Loads Of Structures

Discrete Models

Finite # of DOF
Lead to discrete set of equations
and algebraic eigenproblems for critical loads

Continuous Models

Infinite # of DOF
Lead to ordinary or partial differential equations in space
and trascendental eigenproblems for critical loads

Stability Model Subclassification

Discrete Models

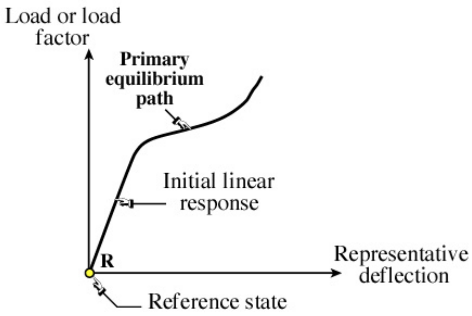
Lumped parameter
Finite element discretizations

Continuous Models

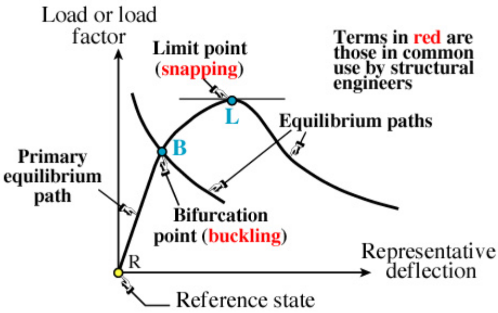
Ordinary Differential Equations (ODE) for 1D models
Partial Differential Equations (PDE) for 2D and 3D models

Only those in red are considered in these Lectures

Load-Deflection Response Plot



Critical Points Displayed in Response Plot

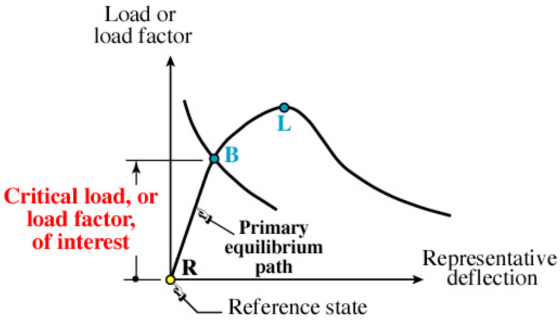


Importance of Critical Points in Structural Design

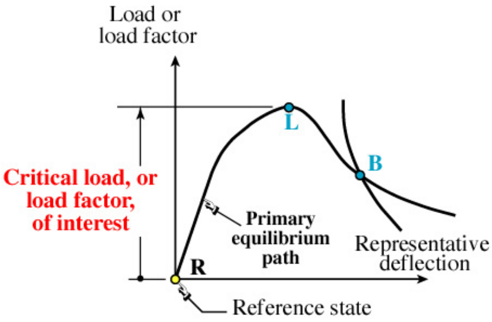
General property in static stability:

Transition from stability to instability always occurs at a critical point

First Critical Point Found Along Primary Equilibrium Path is That of Interest in Design



First Critical Point Found Along Primary Equilibrium Path is That of Interest in Design - 2



Geometrically Exact Versus Linearized Prebuckling (LPB) Stability Analysis

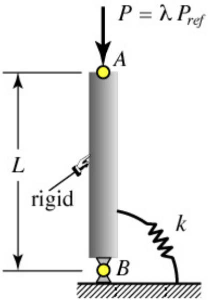
Geometrically Exact Analysis

Exact geometry of deflected structure taken into account
Can handle both bifurcation and limit points
Loading must be conservative (to stay within statics)
Inelastic behavior and imperfections may be accommodated

Linearized Prebuckling (LPB) Analysis

Deformations prior to buckling are neglected
Perturbations of the reference configuration involve only
infinitesimal displacements and rotations
Only linear elastic behavior and conservative loading
No geometric or loading imperfections
The critical state must be a bifurcation point

Geometrically Exact Analysis Example 1: The Hinged Rigid Cantilever (HRC) Column



Load P **remains vertical**, even if the column tilts
This is necessary so the applied loads are **conservative**

Geometrically Exact Stability Analysis of HRC Column: FBD

Perturb column by a tilt angle θ, which is **arbitrary** (not necessarily small). Taking moments with respect to hinge B yields

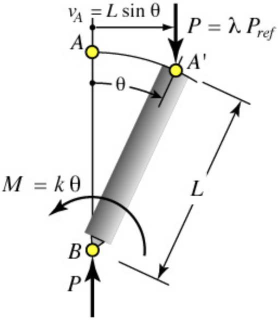
$$k\,\theta = P\,v_A = \lambda\,P_{ref}\,L\sin\theta$$

$$\Rightarrow\quad k\,\theta - \lambda\,P_{ref}\,L\sin\theta = 0$$

This equilibrium equation has **two solutions**:

$$\boxed{\theta = 0 \quad \text{for any } \lambda} \qquad \boxed{\lambda = \frac{k}{P_{ref}\,L}\,\frac{\theta}{\sin\theta}}$$

These define the equilibrium paths for the **untilted** (vertical) and **tilted** column, respectively



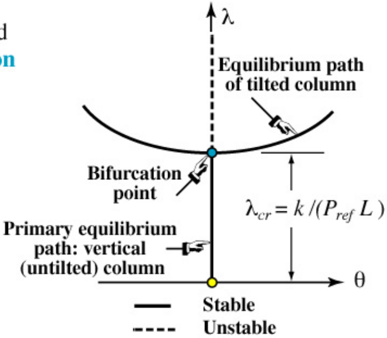
Geometrically Exact Stability Analysis of HRC Column: Bifurcation Point

The two equilibrium paths, plotted at right, intersect at the **bifurcation point** $\theta = 0$, $\lambda = \lambda_{cr}$, in which

$$\boxed{\lambda_{cr} = \frac{k}{P_{ref}\ L}}$$

whence the critical load is

$$\boxed{P_{cr} = \lambda_{cr} P_{ref} = \frac{k}{L}}$$

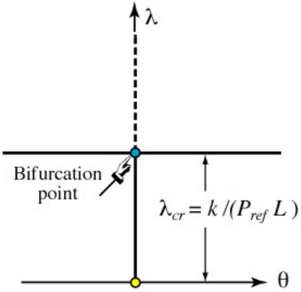


Note that the column keeps taking **additional load** after buckling, which suggests a **safe** configuration from the standpoint of postbuckling reserve strength

Linearized Prebuckling (LPB) Stability Analysis of HRC Column

Assume tilt angle is very small: $\theta \ll 1$ so that $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Replacing into the exact equilibrium expression yields the **LPB stability equation**

$$(k - \lambda P_{ref} L)\,\theta = 0$$

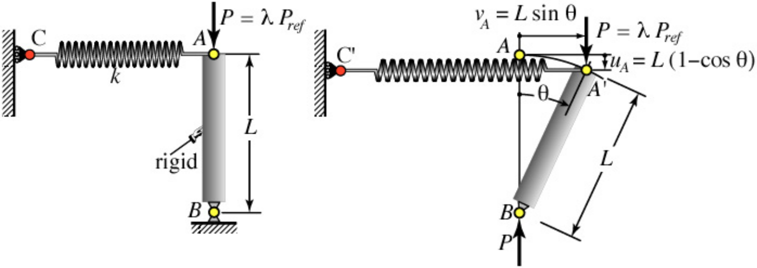


One solution is $\theta = 0$ (untilted column). For a nonzero θ the **expression in parentheses must vanish**, whence

$$\lambda_{cr} = \frac{k}{P_{ref} L} \qquad P_{cr} = \lambda_{cr} P_{ref} = \frac{k}{L}$$

This is the **same result** given by the geometrically exact analysis for the **critical load**. But note that the LPB analysis **does not provide any information on postbuckling** behavior. See diagram above

Geometrically Exact Analysis Example 2: The Propped Rigid Cantilever (PRC) Column



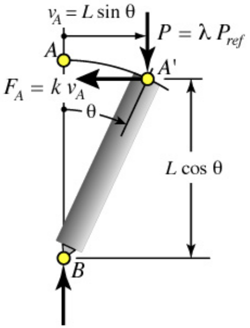
Load P **remains vertical**, even if the column tilts
As the column tilts, spring remains **horizontal** - see right Figure

Geometrically Exact Stability Analysis of PRC Column: FBD

Perturb column by a tilt angle θ, which is **arbitrary** (not necessarily small). Taking moments with respect to hinge B yields

$$k\, v_A\, L \cos\theta \;=\; P\, v_A$$

$$\Rightarrow\quad k\, L^2 \sin\theta \cos\theta - \lambda\, P_{ref}\, L \sin\theta = 0$$



Important: do not cancel out $\sin\theta$ yet!
As in the previous example, this equilibrium equation has **two solutions**:

$$\boxed{\theta = 0 \quad \text{for any } \lambda} \qquad \boxed{\lambda = \frac{k\,L}{P_{ref}} \cos\theta}$$

These define the equilibrium paths for the **untilted** (vertical) and **tilted** column, respectively

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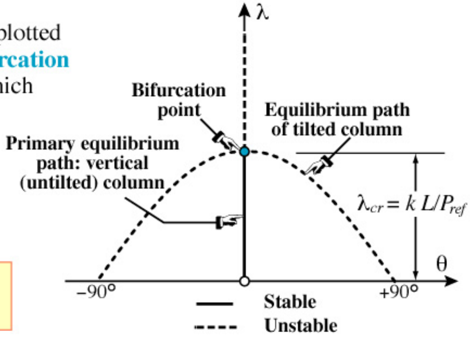
The two equilibrium paths, plotted at right, intersect at the **bifurcation point** $\theta = 0,\ \lambda = \lambda_{cr}$, in which

$$\boxed{\lambda_{cr} = \frac{k\,L}{P_{ref}}}$$

whence the critical load is

$$\boxed{P_{cr} = \lambda_{cr} P_{ref} = k\,L}$$



Note that the **load capacity decreases** after buckling, which suggests an **unsafe** configuration from the standpoint of postbuckling reserve strength

Stability of Structures:
Discrete Models

Objective

This Lecture cover discrete models for structural stability. Examples illustrated use of Linearized Prebuckling (LPB) analysis to obtain critical loads and buckling mode shapes from the solution of an algebraic eigenproblem.

We will focus on simple lumped-parameter models of columns that can be easily solved by hand. The stability equations are obtained by the equilibrium method, which relies on doing FBD in a slightly perturbed configuration.

Some Theory is Left Out

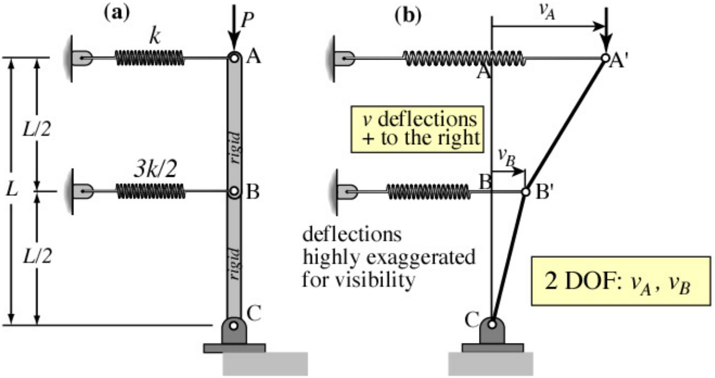
Several of the statements in Lecture 23, such as

"the static stability analysis method requires conservative loads to be valid"
"this equibrium patch (or branch) is unstable"

are left unproven.

Proofs require more advanced math techniques, such as variational calculus and nonconservative dynamics. Those are given in ASEN 6107: Nonlinear Finite Element Methods

Problem 1. Two-Rigid-Strut Cantilevered Column, Propped by Extensional Springs



Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Choosing FBD

The lateral deflections: v_A and v_B, are taken as independent degrees of freedom (DOF). Consequently, **two equilibrium equations** from FBD are required. Three combinations of two FBD are possible:

(c1-c2) **Strut AB and strut BC**
(c1-c3) **Strut AB and whole column ABC**
(c2-c3) **Strut BC and whole column ABC**

Now in all FBD translational force equilibrium holds identically by construction (see previous slide). On the other hand, the remaining moment equilibrium conditions become part of the stability equations.

Since the three FBDs are linearly dependent [check the previous slide: merging (c1) and (c2) yields (c3)], the **results of the stability analysis will be exactly the same** as long it is done correctly. On the other hand, the stability matrix equations will be different. For the ensuing derivations **we pick (c1-c3)**.

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Stability Equations

For strut AB, taking moments with respect to the displaced hinge point B' positive CW, gives

$$\sum M_{B'} = P\,(v_A - v_B) - F_A\,(L/2) = P\,(v_A - v_B) - k\,L\,v_A/2 = 0$$

For the whole column ABC, taking moments with respect to the hinge C, positive CW, gives

$$\sum M_C = P\,v_A - F_A\,L - F_B\,L/2 = P\,v_A - k\,L\,v_A - 3k\,L\,v_B/4 = 0$$

Rewriting these two equations in matrix form gives

$$\begin{bmatrix} P - \frac{1}{2}kL & -P \\ P - kL & -\frac{3}{4}kL \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \mathbf{A}\mathbf{v} = \mathbf{0}$$

This is called the **matrix stability equation**. Matrix **A** is called the **stability matrix**.

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Stability Equations

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Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Critical Loads

The linear algebraic system $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ has two kinds of solutions:

(1) The **trivial solution** $\mathbf{v} = \mathbf{0}$, or $v_A = v_B = 0$, corresponds to the **undeflected** (vertical) column. This solution is valid for *any* P. It is the **primary equilibrium path**.

(2) To find **critical loads** we must have $\mathbf{v} \neq \mathbf{0}$; i.e., at least one of its components v_A or v_B must be nonzero. Thus $\mathbf{A}$ must be **singular** or, equivalently, have a **zero determinant**. Since the entries of $\mathbf{A}$ depend on P , this becomes an **algebraic eigenproblem** in P. Those eigenvalues are the solution of the **characteristic equation**, which is obtained by setting the determinant of $\mathbf{A}$ to zero:

$$\boxed{\det(\mathbf{A}) = 0 \;\;\Rightarrow\;\; P^2 - \tfrac{7}{4}\,k\,L\,P + \tfrac{3}{8}\,k^2 L^2 = 0}$$

Solving this quadratic in P gives the **two critical loads** as its roots:

$$\boxed{P_{cr1} = \tfrac{1}{4}\,k\,L = 0.25\,k\,L, \quad P_{cr2} = \tfrac{3}{2}\,k\,L \;=\; 1.50\,k\,L}$$

The smallest one is *the* **critical load**:

$$\boxed{P_{cr} \;=\; P_{cr1} = \tfrac{1}{4}\,k\,L \;=\; 0.25\,k\,L}$$

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Buckling Mode Shapes

Replacing $P_{cr1} = k\,L/4$ into the eigenproblem $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ gives the following equation to determine the associated eigenvector $\mathbf{v}$:

$$\frac{kL}{4}\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_{A1} \\ v_{B1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Rightarrow\quad \mathbf{v}_1 = \begin{bmatrix} v_{A1} \\ v_{B1} \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Likewise, replacing $P_{cr2} = 3\,k\,L/2$ into $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ gives the following equation to determine the eigenvector $\mathbf{v}$:

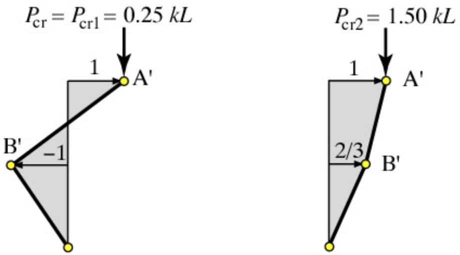
$$\frac{kL}{4}\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Rightarrow\quad \mathbf{v}_2 = \begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = c_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

In the foregoing eigenvector expressions, c_1 and c_2 are arbitrary nonzero scaling factors. A common eigenvector normalization condition is to make their largest components equal to +1. This is done for $\mathbf{v}_1$ by taking $c_1 = 1$ or $c_1 = -1$ (either one works), whereas for $\mathbf{v}_2$ we take $c_2 = 2/3$. The two eigenvectors normalized as per that criterion are

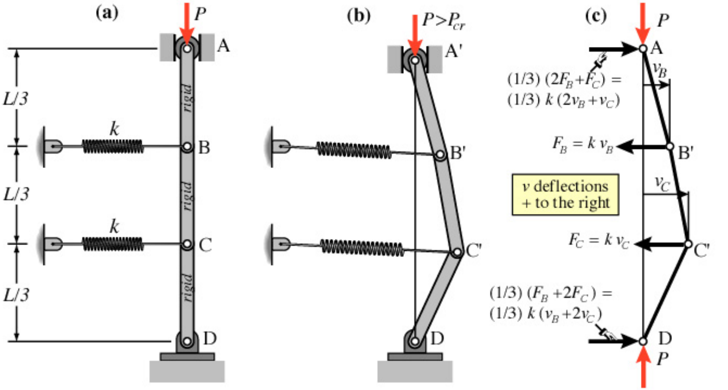
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for } P_{cr1} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} \text{ for } P_{cr2}$$

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Result Summary

The normalized eigenvectors are pictured below along with the associated critical loads.

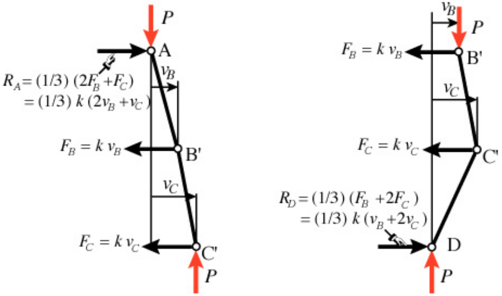


Problem 2: Pinned-Pinned Column Made of Three Struts Propped by Extensional Springs



Problem worked out in Lecture 24

Problem 2: Pinned-Pinned Column Made of Three Struts Propped by Extensional Springs: FBD



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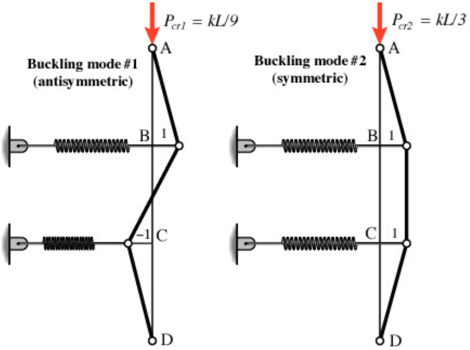
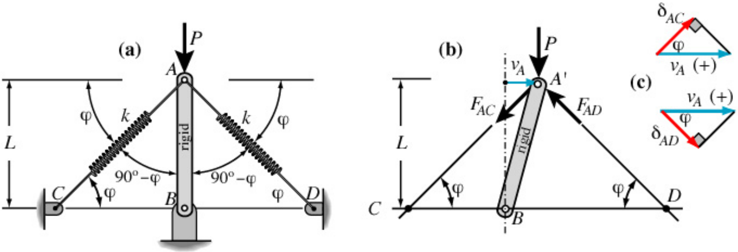


Problem 3. Cantilevered Rigid Column Propped by Inclined Springs



Worked out in Lecture 24, along with the following design problem.

Practical design problem: if AC and BC are **elastic cables of given cross section** A, find the **optimal rise angle** φ in the sense of maximizing the critical load for a given cable volume $V = A\,L/\sin\varphi$
More precisely, for which φ is $dP_{cr}/d\varphi = 0$ if V is fixed?

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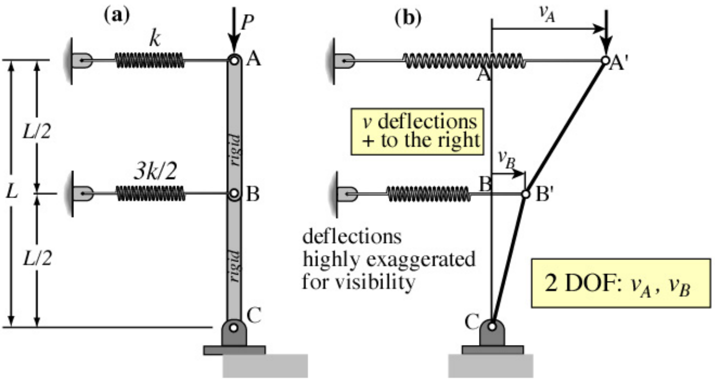

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(c2-c3) **Strut BC and whole column ABC**

Now in all FBD translational force equilibrium holds identically by construction (see previous slide). On the other hand, the remaining moment equilibrium conditions become part of the stability equations.

Since the three FBDs are linearly dependent [check the previous slide: merging (c1) and (c2) yields (c3)], the **results of the stability analysis will be exactly the same** as long it is done correctly. On the other hand, the stability matrix equations will be different. For the ensuing derivations **we pick** **(c1-c3)**.

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Stability Equations

For strut AB, taking moments with respect to the displaced hinge point B' positive CW, gives

$$\sum M_{B'} = P\,(v_A - v_B) - F_A\,(L/2) = P\,(v_A - v_B) - k\,L\,v_A/2 = 0$$

For the whole column ABC, taking moments with respect to the hinge C, positive CW, gives

$$\sum M_C = P\,v_A - F_A\,L - F_B\,L/2 = P\,v_A - k\,L\,v_A - 3k\,L\,v_B/4 = 0$$

Rewriting these two equations in matrix form gives

$$\begin{bmatrix} P - \frac{1}{2}kL & -P \\ P - kL & -\frac{3}{4}kL \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \mathbf{A}\mathbf{v} = \mathbf{0}$$

This is called the **matrix stability equation**. Matrix **A** is called the **stability matrix**.

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Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Critical Loads

The linear algebraic system $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ has two kinds of solutions:

(1) The **trivial solution** $\mathbf{v} = \mathbf{0}$, or $v_A = v_B = 0$, corresponds to the **undeflected** (vertical) column. This solution is valid for *any* P. It is the **primary equilibrium path**.

(2) To find **critical loads** we must have $\mathbf{v} \neq \mathbf{0}$; i.e., at least one of its components v_A or v_B must be nonzero. Thus $\mathbf{A}$ must be **singular** or, equivalently, have a **zero determinant**. Since the entries of $\mathbf{A}$ depend on P , this becomes an **algebraic eigenproblem** in P. Those eigenvalues are the solution of the **characteristic equation**, which is obtained by setting the determinant of $\mathbf{A}$ to zero:

$$\boxed{\det(\mathbf{A}) = 0 \quad \Rightarrow \quad P^2 - \tfrac{7}{4}\,k\,L\,P + \tfrac{3}{8}\,k^2 L^2 = 0}$$

Solving this quadratic in P gives the **two critical loads** as its roots:

$$\boxed{P_{cr1} = \tfrac{1}{4}\,k\,L = 0.25\,k\,L, \quad P_{cr2} = \tfrac{3}{2}\,k\,L = 1.50\,k\,L}$$

The smallest one is *the* **critical load**:

$$\boxed{P_{cr} = P_{cr1} = \tfrac{1}{4}\,k\,L = 0.25\,k\,L}$$

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Buckling Mode Shapes

Replacing $P_{cr1} = k\,L/4$ into the eigenproblem $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ gives the following equation to determine the associated eigenvector $\mathbf{v}$:

$$\frac{kL}{4}\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_{A1} \\ v_{B1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} v_{A1} \\ v_{B1} \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Likewise, replacing $P_{cr2} = 3\,k\,L/2$ into $\mathbf{A}\,\mathbf{v} = \mathbf{0}$ gives the following equation to determine the eigenvector $\mathbf{v}$:

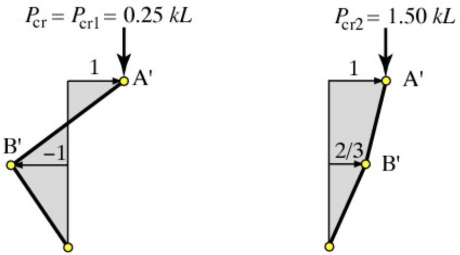
$$\frac{kL}{4}\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = c_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

In the foregoing eigenvector expressions, c_1 and c_2 are arbitrary nonzero scaling factors. A common eigenvector normalization condition is to make their largest components equal to +1. This is done for $\mathbf{v}_1$ by taking $c_1 = 1$ or $c_1 = -1$ (either one works), whereas for $\mathbf{v}_2$ we take $c_2 = 2/3$. The two eigenvectors normalized as per that criterion are

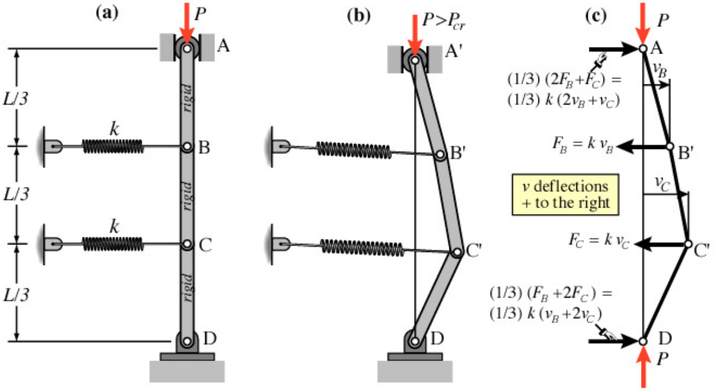
$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for } P_{cr1} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} \text{ for } P_{cr2}}$$

Problem 1. Two-Rigid-Strut Cantilevered Column Propped by Extensional Springs: Result Summary

The normalized eigenvectors are pictured below along with the associated critical loads.

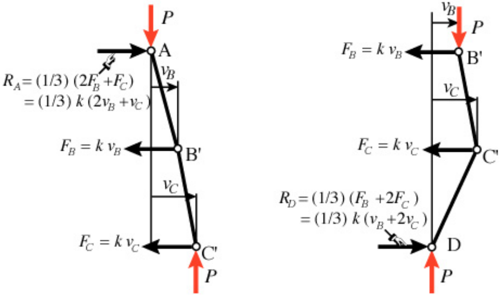


Problem 2: Pinned-Pinned Column Made of Three Struts Propped by Extensional Springs

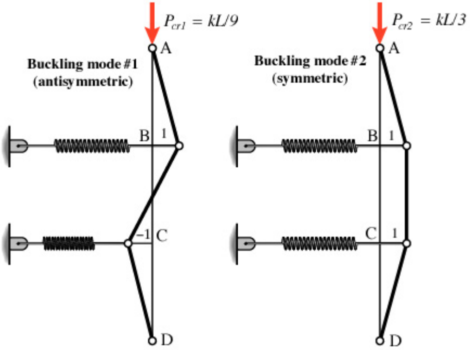


Problem worked out in Lecture 24

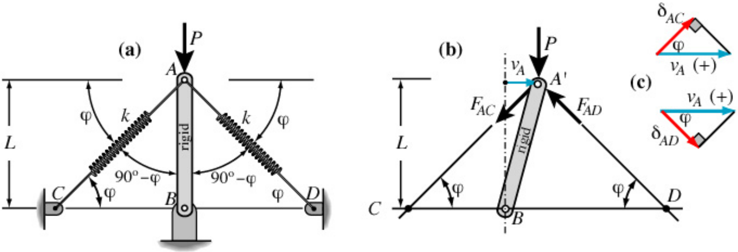
Problem 2: Pinned-Pinned Column Made of Three Struts Propped by Extensional Springs: FBD



Problem 2. Pinned-Pinned Column Made of Three Struts Propped by Extensional Springs: Result Summary



Problem 3. Cantilevered Rigid Column Propped by Inclined Springs



Worked out in Lecture 24, along with the following design problem.

Practical design problem: if AC and BC are **elastic cables of given cross section** A, find the **optimal rise angle** φ in the sense of maximizing the critical load for a given cable volume $V = A\, L/\sin\varphi$ More precisely, for which φ is $dP_{cr}/d\varphi = 0$ if V is fixed?