ASEN 3112 – Spring 2020
Homework 6 Solution

Exercise 6.1: Use the **Virtual Displacement Method** and solve the following problems:

a) First assume that member BA is removed from the truss in Figure 6.1. For the remaining two-bar truss, determine the horizontal and vertical displacement at point B and the forces in members BD and BC.

b) Now consider the entire, three-bar truss. Find the horizontal and vertical displacement at point B and the forces in all three members.

Assume that the elastic modulus is E = 200 GPa and areas of the members are $A_{BD} = 0.1m^2$, $A_{BC} = 0.15m^2$, and $A_{BA} = 0.125m^2$.

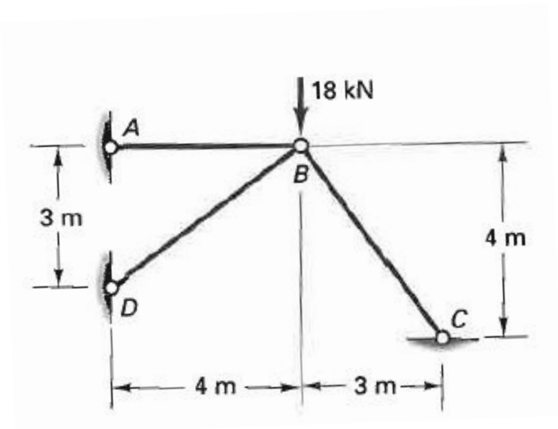


Figure 6.1

Solution:

a)

From observation of Figure 6.1:

$$L_{BC} = L_{BD} = 5m$$

$$\Delta L^u_{BC} = -\frac{3}{5}u_B, \qquad \partial\Delta L^u_{BC} = -\frac{3}{5}\partial u_B, \qquad \Delta L^u_{BD} = \frac{4}{5}u_B, \qquad \partial\Delta L^u_{BD} = \frac{4}{5}\partial u_B$$

$$\Delta L^v_{BC} = \frac{4}{5}v_B, \qquad \partial\Delta L^v_{BC} = \frac{4}{5}\partial v_B, \qquad \Delta L^v_{BD} = \frac{3}{5}v_B, \qquad \partial\Delta L^v_{BC} = \frac{3}{5}\partial v_B$$

Now create the virtual work equations for the horizontal and vertical directions using $\partial W_{ie} = \frac{EA}{L}(\Delta L)(\partial\Delta L)$ in which $\Delta L = \Delta L^u + \Delta L^v$.

$$\partial W^u_{ie} = \frac{EA_{BC}}{L_{BC}}\left(-\frac{3}{5}u_B + \frac{4}{5}v_B\right)\left(-\frac{3}{5}\partial u_B\right) + \frac{EA_{BD}}{L_{BD}}\left(\frac{4}{5}u_B + \frac{3}{5}u_B\right)\left(\frac{4}{5}\partial u_B\right)$$

$$\partial W^v_{ie} = \frac{EA_{BC}}{L_{BC}}\left(-\frac{3}{5}u_B + \frac{4}{5}v_B\right)\left(\frac{4}{5}\partial v_B\right) + \frac{EA_{BD}}{L_{BD}}\left(\frac{4}{5}u_B + \frac{3}{5}u_B\right)\left(\frac{3}{5}\partial v_B\right)$$

Equate the external work in the horizontal and vertical directions to the virtual work

$$\partial W_e^u = 0, \qquad \partial W_e^v = -18kN\,\partial v_B$$

Equating the external and virtual work for both directions yields

$$\partial W_e \quad = \partial W_{ie}$$

$$\partial W_{ie}^u = \frac{E}{L}\partial u_B\left(0.15\left(\frac{9}{25}u_B - \frac{12}{25}v_B\right) + 0.1\left(\frac{16}{25}u_B + \frac{12}{25}v_B\right)\right)$$

$$\partial W_{ie}^u = \frac{E}{L}\partial u_B\left(\frac{2.95}{25}u_B - \frac{0.6}{25}v_B\right) = 0$$

$$\partial W_{ie}^v = \frac{E}{L}\partial v_B\left(0.15\left(-\frac{12}{25}u_B + \frac{16}{25}v_B\right) + 0.1\left(\frac{12}{25}u_B + \frac{9}{25}v_B\right)\right)$$

$$\partial W_{ie}^v = \frac{E}{L}\partial v_B\left(-\frac{0.6}{25}u_B + \frac{3.3}{25}v_B\right) = \ -18kN\,\partial v_B$$

The ∂u_B and ∂v_B will cancel out in each equation leaving the following matrix.

$$\frac{E}{L}\begin{bmatrix} \frac{2.95}{25} & -\frac{0.6}{25} \\ -\frac{0.6}{25} & \frac{3.3}{25} \end{bmatrix}\begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} 0 \\ -18000\ N \end{bmatrix}$$

Solving for the two unknowns will yield

$$u_B = \ -0.72\mu m, \qquad v_B = \ -3.5\mu m$$

To calculate the forces in each member, use the equation $F = EA\left(\frac{\Delta L}{L}\right)$.

$$\Delta L_{BC} = \ -\frac{3}{5}(-0.72\mu m) + \frac{4}{5}(-3.5\mu m) = -2.37\mu m$$

$$\Delta L_{BD} = \frac{4}{5}(-0.72\mu m) + \frac{3}{5}(-3.5\mu m) = -2.68\mu m$$

$$F_{BC} = EA_{BC}\left(\frac{\Delta L_{BC}}{L_{BC}}\right) = 200 * 10^9\frac{N}{m^2}(0.15m^2)\left(\frac{-2.37 * 10^{-6}m}{5m}\right) = -14.2\ kN$$

$$F_{BD} = EA_{BD}\left(\frac{\Delta L_{BD}}{L_{BD}}\right) = 200 * 10^9\frac{N}{m^2}(0.1m^2)\left(\frac{-2.68 * 10^{-6}m}{5m}\right) = -10.7\ kN$$

b)
The change in length is needed for the new member BA.

$$\Delta L_{BA}^{u} = u_B, \qquad \partial\Delta L_{BA}^{u} = \partial u_B, \qquad \Delta L_{BA}^{v} = 0, \qquad \partial\Delta L_{BA}^{v} = \partial v_B$$

Due to the member BA being horizontal, the virtual work in the vertical direction is the same as in part a, but the horizontal direction has the following equation

$$\partial W_{ie}^{u} = \frac{EA_{BC}}{L_{BC}}\left(-\frac{3}{5}u_B + \frac{4}{5}v_B\right)\left(-\frac{3}{5}\partial u_B\right) + \frac{EA_{BD}}{L_{BD}}\left(\frac{4}{5}u_B + \frac{3}{5}u_B\right)\left(\frac{4}{5}\partial u_B\right) + \frac{EA_{BA}}{L_{BA}}(u_B)\partial u_B$$

The external work also remains the same therefore

$$\partial W_{ie}^{u} = E\partial u_B\left(\frac{0.15}{5}\left(\frac{9}{25}u_B - \frac{12}{25}v_B\right) + \frac{0.1}{5}\left(\frac{16}{25}u_B + \frac{12}{25}v_B\right) + \frac{0.125}{4}u_B\right) = 0$$

$$\partial W_{ie}^{u} = E\partial u_B\left(\frac{13.7125}{250}u_B - \frac{0.6}{125}v_B\right) = 0$$

The matrix generated from the equations is now

$$\begin{bmatrix} \frac{13.7125E}{250} & -\frac{0.6E}{125} \\ -\frac{0.6}{25}\left(\frac{E}{L}\right) & \frac{3.3}{25}\left(\frac{E}{L}\right) \end{bmatrix} \begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} 0 \\ -18000\ N \end{bmatrix}$$

Solving for the system of equations yields

$$u_B = -0.3\mu m, \qquad v_B = -3.5\mu m$$

Calculating the forces is the same as in part a

$$\Delta L_{BC} = -\frac{3}{5}(-0.3\mu m) + \frac{4}{5}(-3.5\mu m) = -2.62\mu m$$

$$\Delta L_{BD} = \frac{4}{5}(-0.3\mu m) + \frac{3}{5}(-3.5\mu m) = -2.34\mu m$$

$$\Delta L_{BA} = (-0.3\mu m) + 0 = -0.3\mu m$$

$$F_{BC} = EA_{BC}\left(\frac{\Delta L_{BC}}{L_{BC}}\right) = 200 * 10^9 \frac{N}{m^2}(0.15m^2)\left(\frac{-2.62 * 10^{-6}m}{5m}\right) = -15.7\ kN$$

$$F_{BD} = EA_{BD}\left(\frac{\Delta L_{BD}}{L_{BD}}\right) = 200 * 10^9 \frac{N}{m^2}(0.1m^2)\left(\frac{-2.34 * 10^{-6}m}{5m}\right) = -9.4\ kN$$

$$F_{BA} = EA_{BA}\left(\frac{\Delta L_{BA}}{L_{BA}}\right) = 200 * 10^9 \frac{N}{m^2}(0.125m^2)\left(\frac{-0.3 * 10^{-6}m}{4m}\right) = -1.9\ kN$$

Exercise 6.2: Use the **Virtual Displacement Method** and solve the following problems:

a) Determine the horizontal and vertical displacements at point A and the forces in members AC and AD for the two-bar truss shown in Figure 6.2(a).

b) Check the results for the forces in the members by comparing them against the forces found by the methods for joints.

c) In Figure 6.2(b) there have been two members added: AB and AE. For all four members (AB, AC, AD, and AE) find the horizontal and vertical displacements at point A and the forces in each member.

The relative bar length to cross sectional area values, A_i/L_i, for each bar "i" are as follows: 0.04 for bar AB and bar AD, 0.02 for bar AC, and 0.08 for bar AE. Assume the elastic modulus is E = 200 GPa.

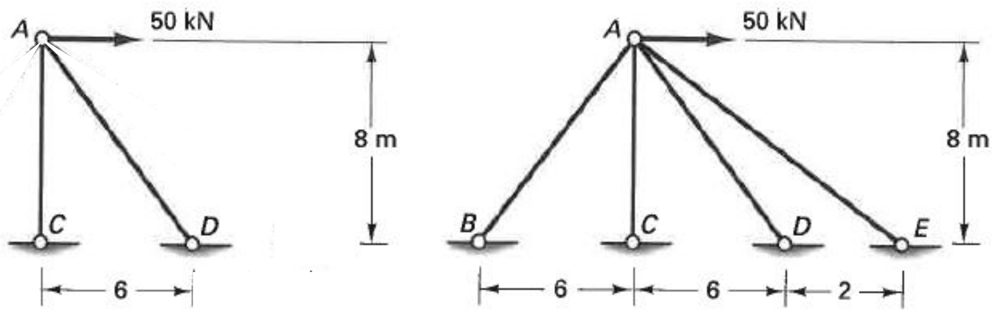


Figure 6.2

Solution:

a)

From observation of Figure 6.2a:

$$L_{AD} = 10m, \qquad L_{AC} = 8m$$

$$\Delta L^u_{AC} = 0, \qquad \partial\Delta L^u_{AC} = 0, \qquad \Delta L^u_{AD} = -\frac{3}{5}u_A, \qquad \partial\Delta L^u_{AD} = -\frac{3}{5}\partial u_A$$

$$\Delta L^v_{AC} = v_A, \qquad \partial\Delta L^v_{AC} = \partial v_A, \qquad \Delta L^v_{AD} = \frac{4}{5}v_A, \qquad \partial\Delta L^v_{AC} = \frac{4}{5}\partial v_A$$

Now create the virtual work equations for the horizontal and vertical directions using $\partial W_{ie} = \frac{EA}{L}(\Delta L)(\partial\Delta L)$ in which $\Delta L = \Delta L^u + \Delta L^v$.

$$\partial W^u_{ie} = \frac{EA_{AC}}{L_{AC}}(0 + v_A)(0) + \frac{EA_{AD}}{L_{AD}}\left(-\frac{3}{5}u_A + \frac{4}{5}u_A\right)\left(-\frac{3}{5}\partial u_A\right)$$

$$\partial W^v_{ie} = \frac{EA_{AC}}{L_{AC}}(0 + v_A)(\partial v_A) + \frac{EA_{AD}}{L_{AD}}\left(-\frac{3}{5}u_A + \frac{4}{5}u_A\right)\left(\frac{4}{5}\partial v_A\right)$$

Use external work in the horizontal and vertical directions and equate to the virtual work

$$\partial W^u_e = 50kN\ \partial u_B, \qquad \partial W^v_e = 0$$

Equating the external and virtual work for both directions yields

$$\partial W_e = \partial W_{ie}$$

$$\partial W_{ie}^{u} = 0.04E\partial u_A\left(\frac{9}{25}u_A - \frac{12}{25}v_A\right) = E\left(\frac{0.36}{25}u_A - \frac{0.48}{25}v_A\right)\partial u_A = 50kN\ \partial u_A$$

$$\partial W_{ie}^{u} = E(0.36u_A - 0.48v_A)\partial u_A = 25 * 50kN\ \partial u_A$$

$$\partial W_{ie}^{v} = 0.02Ev_A\partial v_A + 0.04E\left(-\frac{12}{25}u_A + \frac{16}{25}v_A\right)\partial v_A = E\left(-\frac{0.48}{25}u_A + \frac{1.14}{25}v_A\right)\partial v_A = 0$$

$$\partial W_{ie}^{v} = E(-0.48u_A + 1.14v_A)\partial v_A = 0$$

The ∂u_B and ∂v_B will cancel out in each equation leaving the following matrix.

$$\begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 1.14 \end{bmatrix}\begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} \frac{50000(25)}{E} \\ 0 \end{bmatrix}$$

Solving for the two unknowns will yield

$$u_A = 39.6\mu m, \qquad v_A = 16.7\mu m$$

To calculate the forces in each member, use the equation $F = EA\left(\frac{\Delta L}{L}\right)$.

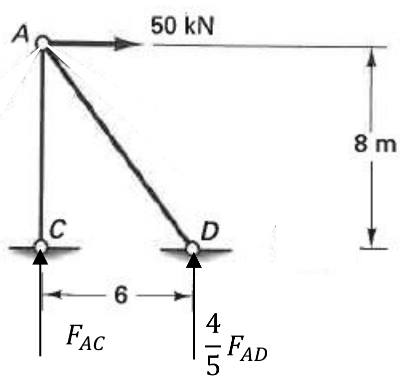
$$\Delta L_{AC} = 0 + (16.7\mu m) = 16.7\mu m$$

$$\Delta L_{AD} = -\frac{3}{5}(39.6\mu m) + \frac{4}{5}(16.7\mu m) = -10.4\mu m$$

$$F_{AC} = EA_{AC}\left(\frac{\Delta L_{AC}}{L_{AC}}\right) = 200 * 10^9\frac{N}{m^2}(0.02m)(16.7 * 10^{-6}m) = 66.7\ kN$$

$$F_{AD} = EA_{AD}\left(\frac{\Delta L_{AD}}{L_{AD}}\right) = 200 * 10^9\frac{N}{m^2}(0.04m)(-10.4 * 10^{-6}m) = -83.3\ kN$$

b)



For determining the forces through the method for joints. Take the moment about point B with forces rotating in the counter clockwise direction as positive. Then, sum the forces in the y-direction.

$$\sum M_B = 0 = \frac{4}{5}F_{AD}(6m) - 50kN(8m) \rightarrow F_{AD} = 83.3kN = 83.3kN(Compression)$$

$$\sum F_y = 0 = \frac{4}{5}F_{AD} + F_{AC} \rightarrow F_{AC} = -66.7kN = 66.7kN(Tension)$$

c)

From observation of Figure 6.2a:

$$L_{AD} = 10m, \qquad L_{AC} = 8m, \qquad L_{AB} = 10m, \qquad L_{AE} = 8\sqrt{2}$$

$$\Delta L_{AB}^{u} = \frac{3}{5}u_A, \qquad \partial\Delta L_{AB}^{u} = \frac{3}{5}\partial u_A, \qquad \Delta L_{AE}^{u} = -\frac{1}{\sqrt{2}}u_A, \qquad \partial\Delta L_{AE}^{u} = -\frac{1}{\sqrt{2}}\partial u_A$$

$$\Delta L_{AB}^{v} = \frac{4}{5}v_A, \qquad \partial\Delta L_{AB}^{v} = \frac{4}{5}\partial v_A, \qquad \Delta L_{AE}^{v} = \frac{1}{\sqrt{2}}v_A, \qquad \partial\Delta L_{AE}^{v} = \frac{1}{\sqrt{2}}\partial v_A$$

The new virtual work equations are as follows

$$\partial W_{ie}^{u} = \frac{EA_{AC}}{L_{AC}}(0 + v_A)(0) + \frac{EA_{AD}}{L_{AD}}\left(-\frac{3}{5}u_A + \frac{4}{5}u_A\right)\left(-\frac{3}{5}\partial u_A\right) + \frac{EA_{AB}}{L_{AB}}\left(\frac{3}{4}u_A + \frac{4}{5}v_A\right)\left(\frac{3}{5}\partial u_A\right) + \frac{EA_{AE}}{L_{AE}}\left(-\frac{1}{\sqrt{2}}u_A + \frac{1}{\sqrt{2}}v_A\right)\left(-\frac{1}{\sqrt{2}}\partial v_A\right)$$

$$\partial W_{ie}^{v} = \frac{EA_{AC}}{L_{AC}}(0 + v_A)(\partial v_A) + \frac{EA_{AD}}{L_{AD}}\left(-\frac{3}{5}u_A + \frac{4}{5}u_A\right)\left(\frac{4}{5}\partial v_A\right) + \frac{EA_{AB}}{L_{AB}}\left(\frac{3}{4}u_A + \frac{4}{5}v_A\right)\left(\frac{4}{5}\partial u_A\right) + \frac{EA_{AE}}{L_{AE}}\left(-\frac{1}{\sqrt{2}}u_A + \frac{1}{\sqrt{2}}v_A\right)\left(\frac{1}{\sqrt{2}}\partial v_A\right)$$

External work is the same, therefore; equating it with virtual work leads to the following

$$\partial W_{ie}^{u} = 0.04E\partial u_A\left(\frac{9}{25}u_A - \frac{12}{25}v_A\right) + 0.04E\partial u_A\left(\frac{9}{25}u_A + \frac{12}{25}v_A\right) + 0.08E\partial u_A\left(\frac{1}{2}u_A - \frac{1}{2}v_A\right) = E\left(\frac{3.44}{50}u_A - \frac{2}{50}v_A\right)\partial u_A = 50kN\ \partial u_A$$

$$\partial W_{ie}^{u} = E(3.44u_A - 2v_A)\partial u_A = 50 * 50kN\ \partial u_A$$

$$\partial W_{ie}^{v} = 0.02Ev_A\partial v_A + 0.04E\left(-\frac{12}{25}u_A + \frac{16}{25}v_A\right)\partial v_A + 0.04E\left(\frac{12}{25}u_A + \frac{16}{25}v_A\right)\partial v_A + 0.08\left(-\frac{1}{2}u_A + \frac{1}{2}v_A\right)\partial v_A = E\left(-\frac{2}{50}u_A + \frac{5.56}{50}v_A\right)\partial v_A = 0$$

$$\partial W_{ie}^{v} = E(-2u_A + 5.56v_A)\partial v_A = 0$$

The matrix for these equations is as follows

$$\begin{bmatrix} 3.44 & -2 \\ -2 & 5.56 \end{bmatrix} \begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} \frac{50000(50)}{E} \\ 0 \end{bmatrix}$$

The calculated displacements are

$$u_A = 4.6\mu m, \qquad v_A = 1.65\mu m$$

To calculate the forces in each member, use the equation $F = EA\left(\frac{\Delta L}{L}\right)$.

$$\Delta L_{AB} = \frac{3}{5}(4.6\mu m) + \frac{4}{5}(1.65\mu m) = 4.08\mu m$$

$$\Delta L_{AC} = 0 + (1.65\mu m) = 1.65\mu m$$

$$\Delta L_{AD} = -\frac{3}{5}(4.6\mu m) + \frac{4}{5}(1.65\mu m) = -1.44\mu m$$

$$\Delta L_{AE} = -\frac{1}{\sqrt{2}}(4.6\mu m) + \frac{1}{\sqrt{2}}(1.65\mu m) = -2.09\mu m$$

$$F_{AB} = EA_{AB}\left(\frac{\Delta L_{AB}}{L_{AB}}\right) = 200 * 10^9 \frac{N}{m^2}(0.04m)(4.08 * 10^{-6}m) = 32.6\ kN$$

$$F_{AC} = EA_{AC}\left(\frac{\Delta L_{AC}}{L_{AC}}\right) = 200 * 10^9 \frac{N}{m^2}(0.02m)(1.65 * 10^{-6}m) = 6.6\ kN$$

$$F_{AD} = EA_{AD}\left(\frac{\Delta L_{AD}}{L_{AD}}\right) = 200 * 10^9 \frac{N}{m^2}(0.04m)(-1.44 * 10^{-6}m) = -11.5\ kN$$

$$F_{AE} = EA_{AE}\left(\frac{\Delta L_{AE}}{L_{AE}}\right) = 200 * 10^9 \frac{N}{m^2}(0.08m)(-2.09 * 10^{-6}m) = -33.4\ kN$$

Exercise 6.3: Consider the cantilevered beam shown in Figure 6.3. The beam is subject to a point load P at $x = \frac{L}{2}$, with $P = 10kN$. The length of the beam is $L = 0.5m$ and the product of Young's modulus and 2nd area moment of inertia is $EI = 10^6\ Nm^2$.

Assume that the deflection of a cantilevered beam, $v(x)$, can be described *approximately* by the following function:

$$v(x) = \left(3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right) v_B + \left(\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right) L\ \varphi_B$$

where v_B is the unknown vertical deflection at point B and φ_B the unknown rotation at point B. Note that the function $v(x)$ describes the kinematics of the beam.

1. Using the **Virtual Displacement Method**, compute the displacement v_B and the rotation φ_B at point B.
2. Plot the displacements $v(x)$ and rotation $v'(x)$ along the beam.
3. Plot the internal bending moment $M(x) = EI\ v''(x)$. Discuss whether this bending moment is exact or is an approximation.

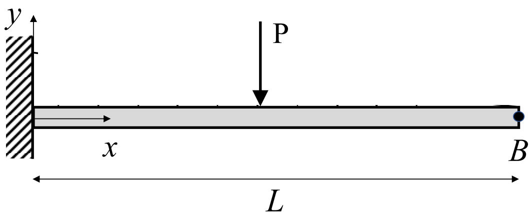


Figure 6.3

Hints:

1. The virtual internal work of a beam is:

$$\delta W_{ie} = \int_0^L E\ I\ \kappa\ \delta\kappa\ dx$$

where κ is the curvature, i.e. $\kappa = v''$, of the real displacement and $\delta\kappa$ is the curvature of the virtual displacement $\delta v(x)$.

2. The virtual displacement $\delta v(x)$ can be generated by considering an arbitrary value for v_B, i.e. consider $\delta v(x)$ for some non-zero value of δv_B and $\delta\varphi_B$=0. The virtual displacement $\delta v(x)$ can be also generated by considering an arbitrary value for φ_B, i.e. consider $\delta v(x)$ for some non-zero value of $\delta\varphi_B$ and δv_B=0.

3. Apply the Principle of Virtual Work:

$$\delta W_e = \delta W_{ie}$$

once for considering a virtual displacement δv_B and once for considering a virtual rotation $\delta\varphi_B$. This will yield two equations with two unknowns, that are the displacement v_B and the rotation φ_B.

Solution:

a)

First, take the derivative of the given equation $v(x)$.

$$v'(x) = \left(\frac{6x}{L^2} - \frac{6x^2}{L^3}\right) v_B + \left(\frac{3x^2}{L^2} - \frac{2x}{L}\right) \varphi_B$$

$$v''(x) = \left(\frac{6}{L^2} - \frac{12x}{L^3}\right) v_B + \left(\frac{6x}{L^2} - \frac{2}{L}\right) \varphi_B$$

The equation for virtual work is given. Substituting the proper values for κ yields

$$\delta W_{ie}^{\varphi} = \int_0^L EI \left(\left(\frac{6}{L^2} - \frac{12x}{L^3}\right) v_B + \left(\frac{6x}{L^2} - \frac{2}{L}\right) \varphi_B\right)\left(\left(\frac{6x}{L^2} - \frac{2}{L}\right) \partial\varphi_B\right) dx$$

$$\delta W_{ie}^{\varphi} = EI \int_0^L \left(\left(-\frac{72x^2}{L^5} + \frac{36x}{L^4} + \frac{24x}{L^4} - \frac{12}{L^3}\right) v_B + \left(\frac{36x^2}{L^4} - \frac{24x}{L^3} + \frac{4}{L^2}\right) \varphi_B\right) \partial\varphi_B dx$$

$\delta W_{ie}^{\varphi} = EI\left(\left(-\frac{24x^3}{L^5} + \frac{18x^2}{L^4} + \frac{12x^2}{L^4} - \frac{12x}{L^3}\right) v_B + \left(\frac{12x^3}{L^4} - \frac{12x^2}{L^3} + \frac{4x}{L^2}\right) \varphi_B\right) \partial\varphi_B$(evaluate at L NOT L/2 because energy is stored in entire beam)

$$\delta W_{ie}^{\varphi} = EI\left(-\frac{6}{L^2} v_B + \frac{4}{L} \varphi_B\right) \partial\varphi_B$$

or

$$\delta W_{ie}^{\varphi} = (-24E6 * v_B + 8E6 * \varphi_B)\partial\varphi_B$$

$$\delta W_{ie}^{v} = \int_0^L EI \left(\left(\frac{6}{L^2} - \frac{12x}{L^3}\right) v_B + \left(\frac{6x}{L^2} - \frac{2}{L}\right) \varphi_B\right)\left(\left(\frac{6}{L^2} - \frac{12x}{L^3}\right) \partial v_B\right) dx$$

$$\delta W_{ie}^{v} = EI \int_0^L \left(\left(\frac{144x^2}{L^6} - \frac{144x}{L^5} + \frac{36}{L^4}\right) v_B + \left(-\frac{72x^2}{L^5} + \frac{36x}{L^4} + \frac{24x}{L^4} - \frac{12}{L^3}\right) \varphi_B\right) \partial v_B dx$$

$\delta W_{ie}^{v} = EI\left(\left(\frac{48x^3}{L^6} - \frac{72x}{L^5} + \frac{36x}{L^4}\right) v_B + \left(-\frac{24x^3}{L^5} + \frac{18x^2}{L^4} + \frac{12x^2}{L^4} - \frac{12x}{L^3}\right) \varphi_B\right) \partial v_B$(evaluate at L)

$$\delta W_{ie}^{v} = EI\left(+\frac{12}{L^3} v_B - \frac{6}{L^2} \varphi_B\right) \partial v_B$$

or

$$\delta W_{ie}^{v} = (+96E6 * v_B - 24E6 * \varphi_B)\partial v_B$$

For the external work use the equation $\partial W_e = \int_0^L p\partial v(x)dx$. However, because p is a point load at $\frac{L}{2}$ evaluate the load at that point and the equation becomes $\partial W_e = p\partial v\left(\frac{L}{2}\right)$.

$$\partial W_e^v = p\left(3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3\right)\partial v_B = \frac{1}{2}p\partial v_B$$

$$\partial W_e^\varphi = p\left(\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2\right)L\partial\varphi_B = -\frac{1}{8}pL\partial\varphi_B$$

Equating the external and virtual work yields

$$\delta W_{ie}^v = EI\left(-\frac{12}{L^3}v_B + \frac{6}{L^2}\varphi_B\right)\partial v_B = \frac{1}{2}p\partial v_B$$

$$\delta W_{ie}^\varphi = EI\left(-\frac{6}{L^2}v_B + \frac{4}{L}\varphi_B\right)\partial\varphi_B = -\frac{1}{8}pL\partial\varphi_B$$

$$\begin{bmatrix} \frac{12}{L^3} & -\frac{6}{L^3} \\ -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}\begin{bmatrix} v_B \\ \varphi_B \end{bmatrix} = \begin{bmatrix} \frac{p}{2EI} \\ -\frac{pL}{8EI} \end{bmatrix}$$

v_B and φ_B can be obtained from this system of equations

$$v_B = -1.30 * 10^{-4}m, \qquad \varphi_B = -3.125 * 10^{-4}rads$$

b)

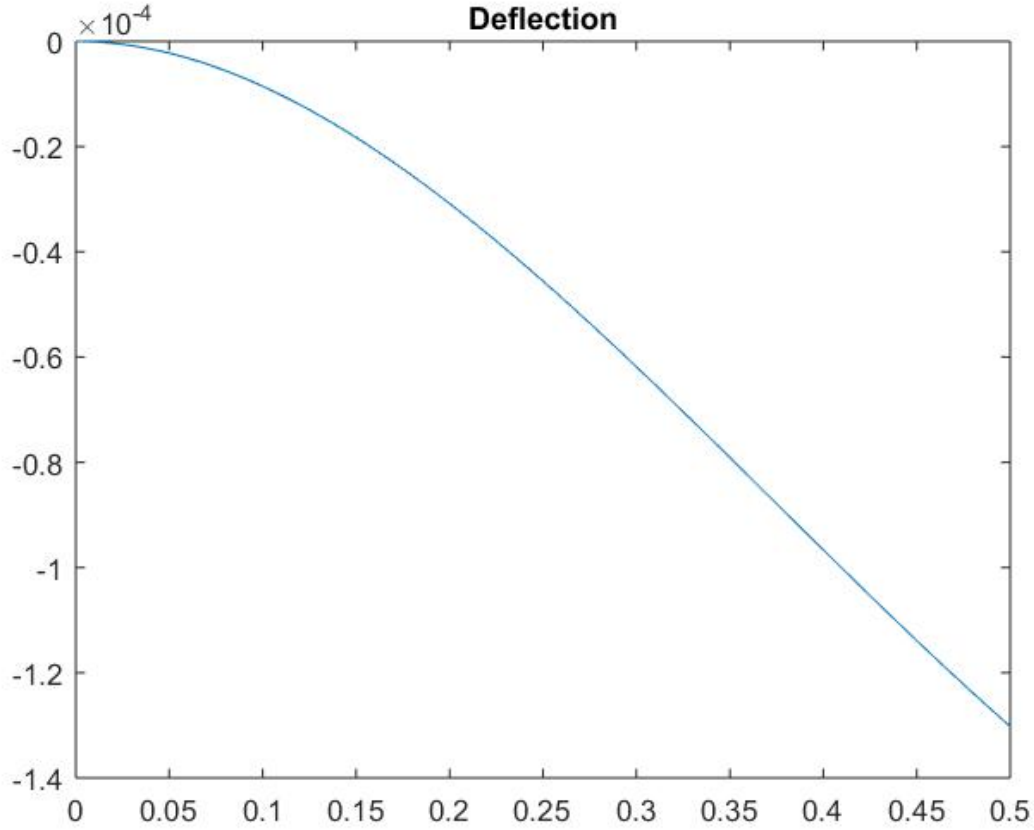
Substituting the values obtained for v_B and φ_B we obtain

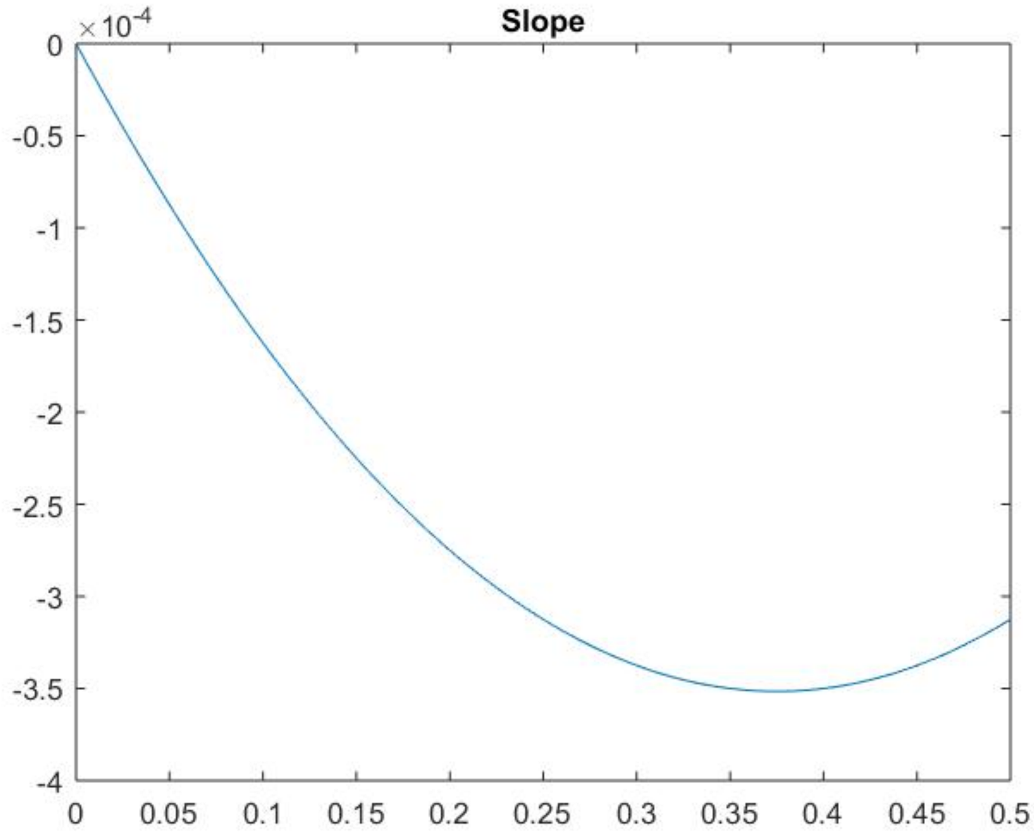
$$v(x) = \left(3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right)(-1.30 * 10^{-4}m) + \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)(-3.125 * 10^{-4}rads)$$

$$v(x) = 8.3 * 10^{-4}x^3 - 9.35 * 10^{-4}x^2, \qquad v(0) = 0, \qquad v(L) = -1.30 * 10^{-4}m$$

$$v'^{(x)} = 24.9 * 10^{-4}x^2 - 18.7 * 10^{-4}x, \qquad v'(0) = 0, \qquad v'(L) = -3.125 * 10^{-4}rads$$

The resulting plots from Matlab are shown below


Deflection
×10^-4
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


Slope
×10^-4
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

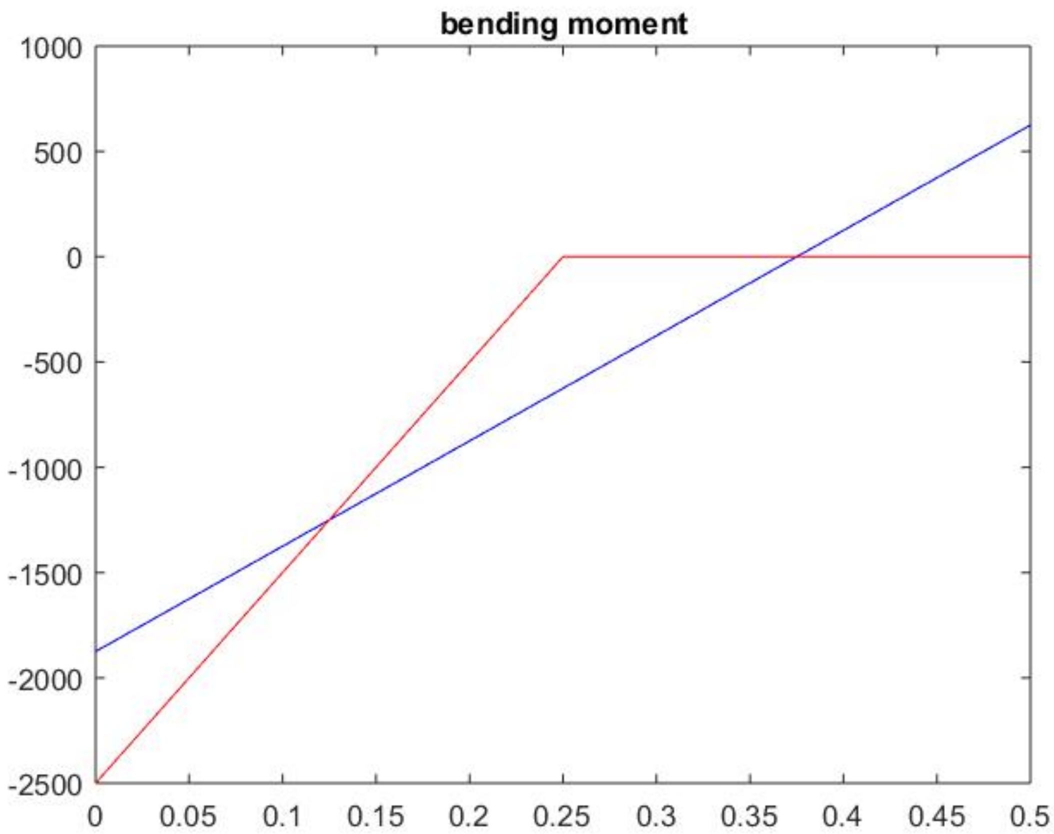
c)

$$v''(x) = 49.8 * 10^{-4}x - 18.7 * 10^{-4}, \qquad v''(0) = 0, \qquad v(L) = -18.7 * 10^{-4}m^{-1}$$

Using the given moment equation yields

$$M(x) = EI(49.8 * 10^{-4}x - 18.7 * 10^{-4}), \qquad M(0) = -1870Nm, \qquad M(L) = 620NM$$

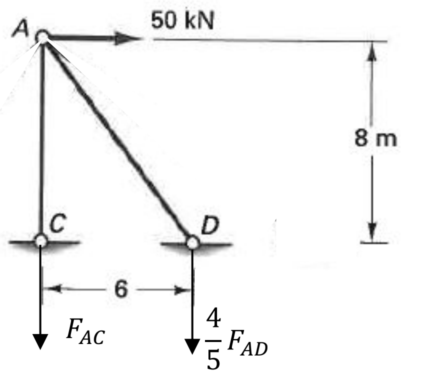
The resulting graph in Matlab is shown below



In this graph, the blue line represents the equation obtained for moment, the red line is what would be expect from an exact solution. In an exact solution, the moment is expected to be highest at the fixed end and zero from $\frac{L}{2}$ (the location of the point load) and beyond. However, this equation does not do that because it is an approximation and not an exact solution.

Exercise 6.4: Use **the virtual force method** and compute the displacements at joint A of the two-bar truss shown in Figure 6.2(a). Use the material and geometric properties specified in Exercise 6.2.

a)



From Exercise 6.2 the forces in both members AD and AC are known.

$$F_{AD} = -83.3kN, \qquad F_{AC} = 66.7kN$$

Apply a dummy horizontal load of 1kN in the positive x-direction and solve for bar forces n_{AC}^{u} and n_{AD}^{u}.

$$\sum M_D = 0 = F_{AC}(6m) - 1kN(8m) \rightarrow F_{AC} = \frac{4}{3} = n_{AC}^{u}$$

$$\sum F_y = 0 = \frac{4}{5}F_{AD} + F_{AC} \rightarrow F_{AD} = -\frac{5}{3} = n_{AD}^{u}$$

Apply a dummy vertical load of 1kN in the positive y-direction and solve for bar forces n_{AC}^{v} and n_{AD}^{v}.

$$\sum M_D = 0 = F_{AC}(6m) - 1kN(6m) \rightarrow F_{AC} = 1 = n_{AC}^{v}$$

$$\sum F_y = 0 = 1 - F_{AC} - \frac{4}{5}F_{AD} \rightarrow F_{AD} = 0 = n_{AD}^{v}$$

To find displacement use the equations below

$$\partial u = \sum_{i=1}^{2} \frac{N_i n_i^u L_i}{E_i A_i} = \frac{F_{AC} n_{AC}^u L_{AC}}{EA_{AC}} + \frac{F_{AD} n_{AD}^u L_{AD}}{EA_{AD}}$$

$$\partial u = \frac{50m^{-1}(66.7 * 10^3 N)\left(\frac{4}{3}\right)}{200 * 10^9 \frac{N}{m^2}} + \frac{25m^{-1}(-83.3 * 10^3 N)\left(-\frac{5}{3}\right)}{200 * 10^9 \frac{N}{m^2}} = 39.5\mu m$$

$$\partial v = \sum_{i=1}^{2} \frac{N_i n_i^v L_i}{E_i A_i} = \frac{F_{AC} n_{AC}^v L_{AC}}{E A_{AC}} + \frac{F_{AD} n_{AD}^v L_{AD}}{E A_{AD}}$$

$$\partial v = \frac{50m^{-1}(66.7 * 10^3 N)(1)}{200 * 10^9 \frac{N}{m^2}} + 0 = 16.7\mu m$$