ASEN 3112 – Spring 2020
Homework 7 Solutions: FEM

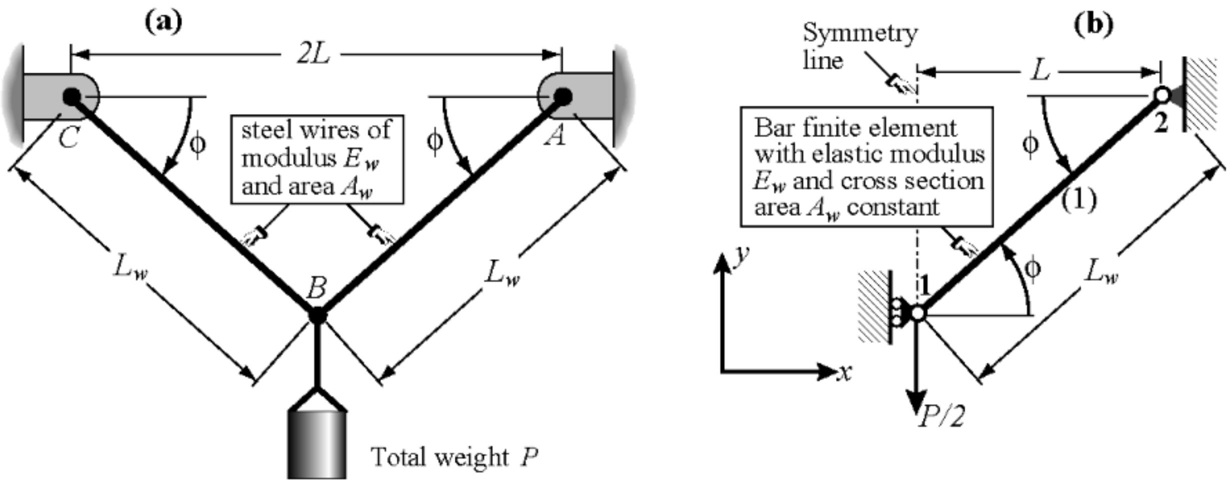

Figure 7. 1: (a) the original structure, (b) one-element FEM model of the structure right half.

Exercise 7.1: Figure 7.1 (a) shows a lifting system. A load P hangs out from a high strength steel wire. The weight of the wire is negligible compared to P, and as a result the two wire segments under load can be considered straight lines. The elastic modulus and cross section area of the wire are E_W and A_W, respectively. Both the half-span L and the total weight P in Figure 7.1 (a) are considered fixed data (that is, are not part of the design). The area A_W and elevation angle ϕ are design parameters to be determined by the conditions stated below. The angle is within the range $0^\circ < \phi < 90^\circ$. Note that $L_W = L/\cos\phi$.

Use the FEM to analyze the displacements of the lifting system. Because of the symmetry of the structure and loading, it is possible to consider only half of the structure, say the right half, for the FEM analysis as illustrated in Figure 7.1(b). This 2D FEM model contains only one bar element and two nodes, with the support conditions shown there. It is loaded at node 1 by the downward force $P/2$. Denote the downward motion of the load point B as Δ, positive down. This will be provided by the FEM solution as $\Delta = -u_{y1}$, which is the only unknown displacement DOF. The design condition is $\Delta \le \Delta_{max}$, where Δ_{max} is a given maximum deflection.

(a) Use the master stiffness equation. Note that since there is only one element the merge process is trivial. Then apply the 3 support conditions shown in Figure 7.1 (b). Solve for $\Delta = -u_{y1}$ as function of P, E_W, A_W, L and ϕ.

(b) From the above, solve for the wire area A_W that makes $\Delta = \Delta_{max}$, in terms of P, E_W, L, Δ_{max} and ϕ.

(c) Using the area A_W found in (b), express the total volume of the wire $V_W = 2A_W L_W$ in terms of P, E_W, L, Δ_{max} and ϕ. Finally, determine the angle ϕ in the range $\phi \in [0^\circ, 90^\circ]$ that *minimizes* this volume. Does this angle depend on P, E_W, L or Δ_{max}? A result of this kind is called an *optimal design under deflection constraints.*

Solution:

The design condition is $\Delta \leq \delta_{max}$, where Δ_{max} is a prescribed maximum deflection. The objective of this problem is to find the angle φ that minimizes the wire volume when the deflection constraint is activated.

(a) Form the master stiffness equations, apply BCs and solve for the vertical deflection of joint B. The orientation angle of the only element (1), which connects nodes 1 and 2, is $\phi^{(1)} = \phi$.

$$\mathbf{K}^{(1)}\mathbf{u}^{(1)} = \frac{E_w A_w}{L_w}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}\begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = \mathbf{f}^{(1)}.$$

in which $c = \cos\phi^e = \cos\phi$ and $s = \sin\phi^e = \sin\phi$. Forming the master stiffness equations is trivial since there is only one element: just remove all element superscripts. Also L_W is replaced by $L/\cos\phi$:

$$\mathbf{K}\mathbf{u} = \frac{E_w A_w c}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \mathbf{f}.$$

The load and displacement BCs are $u_{x1} = u_{x2} = u_{y2} = 0$ and $f_{y1} = -\frac{1}{2}P$.

Deleting the first, third and fourth equations of the above equation provides the reduced stiffness equation, which solved for u_{y1} gives

$$u_{y1} = -\frac{P\,L}{2\,E_W\,A_W\,c\,s^2} \Rightarrow \boxed{\Delta = -u_{y1} = \frac{P\,L}{2\,E_W\,A_W\,c\,s^2}}$$

(b) The wire area A_W that makes $\Delta = \Delta_{max}$ is obtained by inserting $\delta = \delta_{max}$ in the above Equation and solving for A_W:

$$A_W = \frac{P\,L}{2\,E_W\,\Delta_{max}\,c\,s^2} \tag{1}$$

(c) The total volume of the wire is

$$V_W = 2A_W L_W = \frac{P\,L^2}{E_W\,\Delta_{max}\,c^2\,s^2} \tag{2}$$

The only design variable left in Equation 2 is the angle ϕ. The wire volume V_W is minimized if $s^2c^2 = (\sin\phi\cos\phi)^2$ is maximized. For $0 \le \phi \le 90^\circ$, this obviously happens for $\phi = 45^\circ$.

This optimal angle does not depend on P, L, E_W, A_W or Δ_{max}.

Exercise 7.2

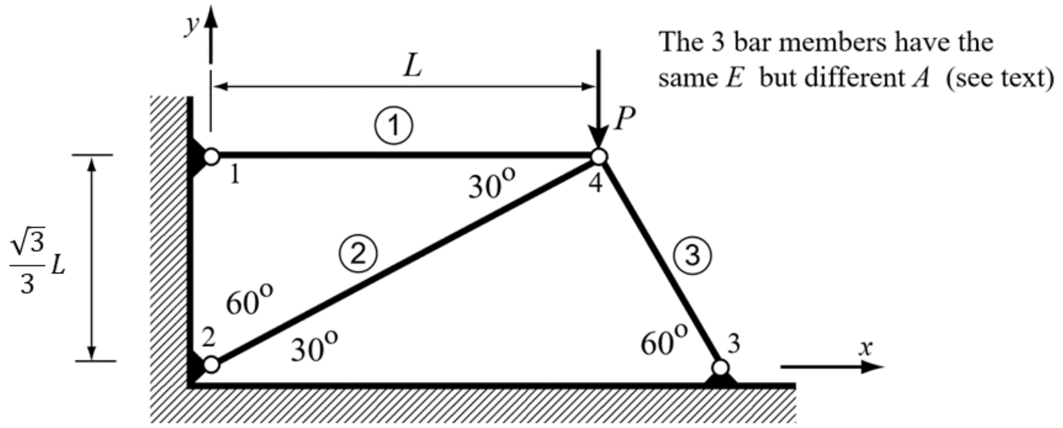

Figure 2: Problem for Exercise 7.2.

Using the Direct Stiffness Method version of FEM, analyze the three-bar, pin-jointed, statically indeterminate truss show in Figure 2. As numerical properties take $E = 3000$ for all bars, $L = 30$, $P = 200$, bar cross section areas $A^{(1)} = 2$, $A^{(2)} = 4$, $A^{(3)} = 3$. (Units are unimportant in this particular exercise).

Assume that disconnection, localization and local element stiffness construction have been done (the first two are primarily conceptual steps, used to explain the DSM). The next six steps are as follows.

Step 1. Globalization. Form the element stiffness equations of the three elements in the *global* frame (x, y), using equation (17.14) of Lecture 17.

Member (1), end nodes 1-4, $E^{(1)} = 3000$, $A^{(1)} = 2$, $L^{(1)} = L = 30$, angle $\varphi^{(1)} = 0°$:

$$\mathbf{K}^{(1)}\mathbf{u}^{(1)} = \begin{bmatrix} 200 & 0 & -200 & 0 \\ 0 & 0 & 0 & 0 \\ -200 & 0 & 200 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x4}^{(1)} \\ u_{y4}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = \mathbf{f}^{(1)} \tag{1}$$

Member (2), end nodes 2-4, $E^{(2)} = 3000$, $A^{(2)} = 4$, length $L^{(2)} = 2L/\sqrt{3} = 20\sqrt{3}$, angle $\varphi^{(2)} = 30°$:

$$\mathbf{K}^{(2)}\mathbf{u}^{(2)} = \begin{bmatrix} 150\sqrt{3} & 150 & -150\sqrt{3} & -150 \\ 150 & 50\sqrt{3} & -150 & -50\sqrt{3} \\ -150\sqrt{3} & -150 & 150\sqrt{3} & 150 \\ -150 & -50\sqrt{3} & 150 & 50\sqrt{3} \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x4}^{(2)} \\ u_{y4}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x4}^{(2)} \\ f_{y4}^{(2)} \end{bmatrix} = \mathbf{f}^{(2)} \tag{2}$$

Member (3), end nodes 3-4, $E^{(3)} = 3000$, $A^{(3)} = 3$, length $L^{(3)} = 2L/3 = 20$, angle $\varphi^{(3)} = 120°$:

$$\mathbf{K}^{(3)}\mathbf{u}^{(3)} = \frac{1}{2}\begin{bmatrix} 225 & -225\sqrt{3} & -225 & 225\sqrt{3} \\ -225\sqrt{3} & 675 & 225\sqrt{3} & -675 \\ -225 & 225\sqrt{3} & 225 & -225\sqrt{3} \\ 225\sqrt{3} & -675 & -225\sqrt{3} & 675 \end{bmatrix} \begin{bmatrix} u_{x3}^{(3)} \\ u_{y3}^{(3)} \\ u_{x4}^{(3)} \\ u_{y4}^{(3)} \end{bmatrix} = \begin{bmatrix} f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{bmatrix} = \mathbf{f}^{(3)} \tag{3}$$

Converting the last two matrices to floating point, with $\sqrt{3} = 1.732\ldots$

$$\mathbf{K}^{(2)} = \begin{bmatrix} 259.808 & 150. & -259.808 & -150. \\ 150. & 86.6025 & -150. & -86.6025 \\ -259.808 & -150. & 259.808 & 150. \\ -150. & -86.6025 & 150. & 86.6025 \end{bmatrix} \tag{4}$$

$$\mathbf{K}^{(3)} = \begin{bmatrix} 112.5 & -194.856 & -112.5 & 194.856 \\ -194.856 & 337.5 & 194.856 & -337.5 \\ -112.5 & 194.856 & 112.5 & -194.856 \\ 194.856 & -337.5 & -194.856 & 337.5 \end{bmatrix} \tag{5}$$

Step 2. Merge. Assemble the free-free master stiffness matrix equations $\mathbf{Ku} = \mathbf{f}$. The result, using any method (such as augment-and-add), is

$$\begin{bmatrix} 200. & 0. & 0. & 0. & 0. & 0. & -200. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 259.808 & 150. & 0. & 0. & -259.808 & -150. \\ 0. & 0. & 150. & 86.6025 & 0. & 0. & -150. & -86.6025 \\ 0. & 0. & 0. & 0. & 112.5 & -194.856 & -112.5 & 194.856 \\ 0. & 0. & 0. & 0. & -194.856 & 337.5 & 194.856 & -337.5 \\ -200. & 0. & -259.808 & -150. & -112.5 & 194.856 & 572.308 & -44.8557 \\ 0. & 0. & -150. & -86.6025 & 194.856 & -337.5 & -44.8557 & 424.103 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} \tag{6}$$

Note that $\mathbf{K}$ is symmetric.

Step 3. Boundary Conditions. Apply the node displacement BCs $u_{x1} = u_{y1} = u_{x2} = u_{y2} = u_{x3} = u_{y3} = 0$ and the node force BCs $f_{x4} = 0$ and $f_{y4} = -P = -200$ to reduce (6) to a linear system of 2 equations with 2 unknowns:

$$\begin{bmatrix} 572.308 & -44.8557 \\ -44.8557 & 424.103 \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ -200. \end{bmatrix} \tag{7}$$

Step 4. Displacement Solution. Solve the reduced stiffness system (7) by any method to get u_{x4} and u_{y4}. The solution is

$$u_{x4} = -0.0372703, \qquad u_{y4} = -0.475526 \tag{8}$$

Completing this with the known displacements yields the complete displacement vector:

$$\boxed{\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.0372703 & -0.475526 \end{bmatrix}^T} \tag{9}$$

(Note: this is written as the transpose of a row vector to save some space because column vectors take up a lot of room; likewise with $\mathbf{f}$ below.)

Step 5. Recovery of Reactions. Inserting $\mathbf{u}$ into the master stiffness equations $\mathbf{f} = \mathbf{Ku}$ given by 6 and carrying out the matrix multiplication yields

$$\boxed{\mathbf{f} = \begin{bmatrix} 7.45405 & 0 & 81.012 & 46.7723 & -88.4661 & 153.228 & 0 & -200. \end{bmatrix}^T} \tag{10}$$

The first six entries are the reactions $f_{x1} = 7.45405$, $f_{y1} = 0$, $f_{x2} = 81.012$, $f_{y2} = 46.7723$, $f_{x3} = -88.4661$, $f_{y3} = 153.228$. The last two entries reproduce the known forces applied at node 4.

Step 6. Recovery of Internal Forces. Find the axial forces $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ in the 3 members. For each element (e) of nodes i, j and node coordinates x_i, y_i, x_j, y_j , compute $c = \cos\varphi^{(e)} = (x_j - x_i)/L^{(e)}$, $s = \sin\varphi^{(e)} = (y_j - y_i)/L^{(e)}$. Grab the element node displacements from $\mathbf{u}$ and place in $\mathbf{u}^{(e)}$. The displacements in the element local system are recovered from the transformation $\bar{\mathbf{u}}^{(e)} = \mathbf{T}_d \mathbf{u}^{(e)}$. Then $d^{(e)} = \bar{u}_{xj} - \bar{u}_{xi}$ and finally $F^{(e)} = (E^{(e)} A^{(e)}/L^{(e)}) d^{(e)}$. Here are the detailed calculations for $e = 3$, for which $i = 3$, $j = 4$, $E^{(3)} = 3000$, $A^{(3)} = 3$, $L^{(3)} = (2/3)L = 20$, $\varphi^{(3)} = 120°$, $c = \cos 120° = -1/2 = -0.5$ and $s = \sin 120° = \sqrt{3}/2 = 0.866025$:

$$
\begin{aligned}
\begin{bmatrix} \bar{u}_{x3} \\ \bar{u}_{y3} \\ \bar{u}_{x4} \\ \bar{u}_{y4} \end{bmatrix} &= \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \\
&= \begin{bmatrix} -0.5 & 0.866025 & 0 & 0 \\ -0.866025 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.866025 \\ 0 & 0 & -0.866025 & -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.0372703 \\ -0.475526 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.393182 \\ 0.27004 \end{bmatrix}.
\end{aligned}
\tag{11}
$$

Hence $d^{(3)} = -0.393182 - 0 = -0.393182$ and $F^{(3)} = (3000 \times 3/20) \times (-0.393182) = 450 \times (-0.393182) = -176.932$, which is compression (C). Doing this operation for the three elements gives

$$
\boxed{F^{(1)} = -7.45405 \text{ (C)}, \qquad F^{(2)} = -93.5446 \text{ (C)}, \qquad F^{(3)} = -176.932 \text{ (C)}.}
\tag{12}
$$

where (T) means tension and (C) compression. As can be expected from the configuration of Figure 1, all members are under compression.
