ASEN 3112
Lecture 14:
Finite Element Method 2

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Announcements

- Upcoming due dates
  - Homework 6 (energy methods): **this Friday**, March 6
  - Homework 7 (FEM): Friday, March 13
  - Exam 2 (beams, energy methods, FEM): Tuesday, March 17
- Note that the FEM recitation is after HW7 and Exam 2

- My Office Hours (all in AERO 302)
  - Thursday, March 5, 11:30 am – 12:30 pm
  - Tuesday, March 10, 9:00 – 10:00 am
  - Thursday, March 12, 11:30 am – 12:30 pm
  - *Then by appointment*

Finite Element Methods Outline

- Last class (Ch. 16 & 17)
  - Member stiffness equations
- **Today (Ch. 17 & 18)**
  - **Transforming from local to global coordinates**
  - **Understanding the global stiffness matrix**
- Thursday, March 5 (Ch. 18)
  - Assembling the global stiffness matrix
  - Applying boundary conditions to solve
- Tuesday, March 10
  - Examples
  - Exam 2 review?

The Direct Stiffness Method

Breakdown

- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution

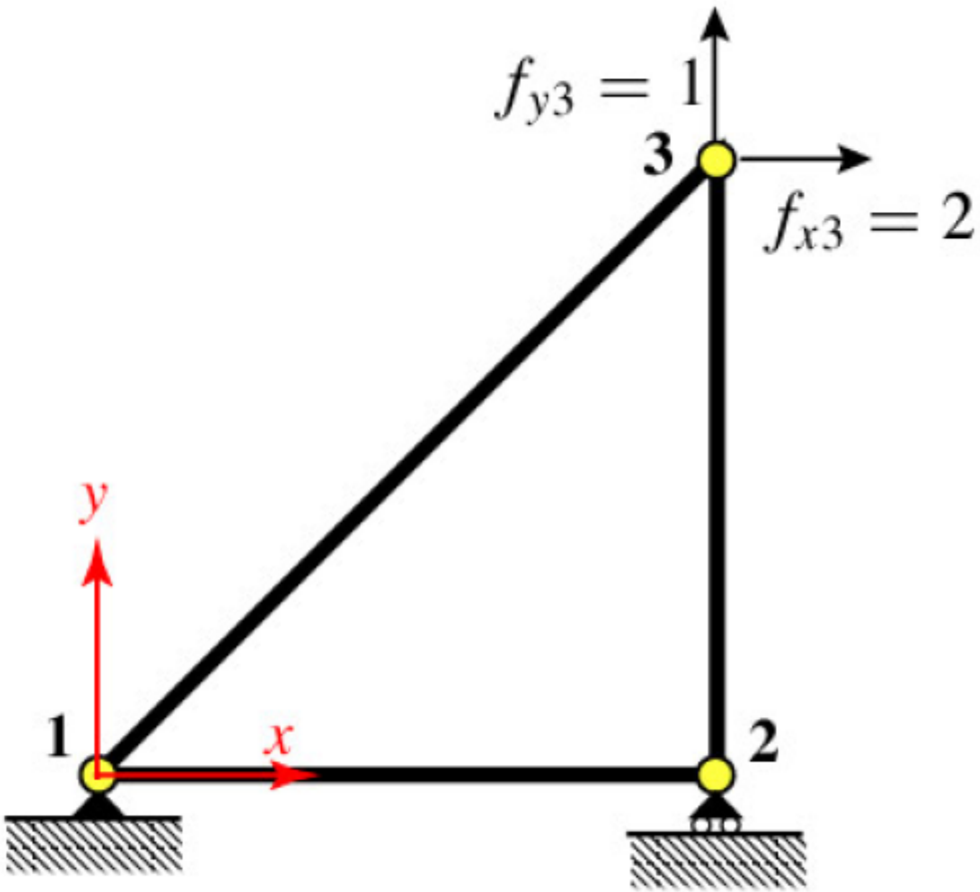
- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

conceptual steps

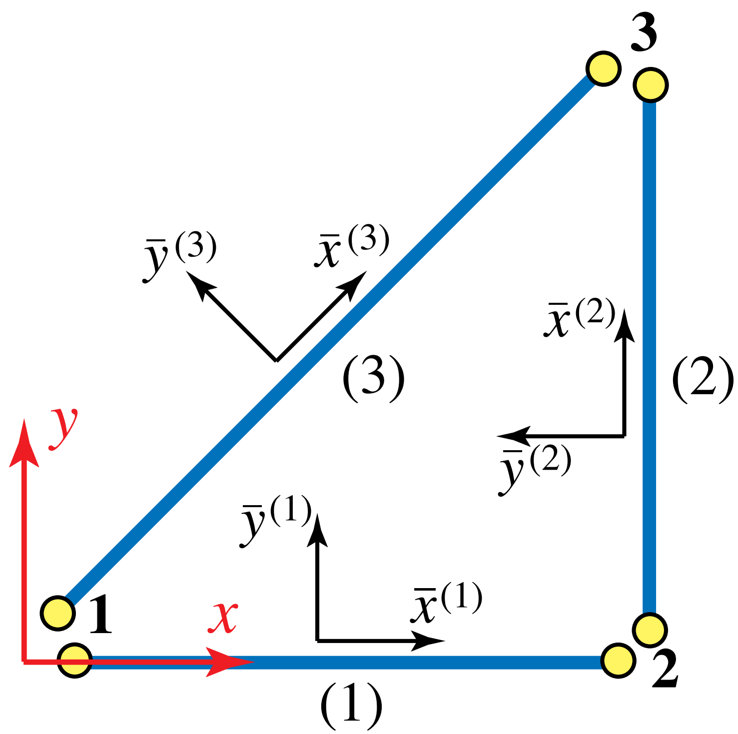
processing steps

post-processing steps

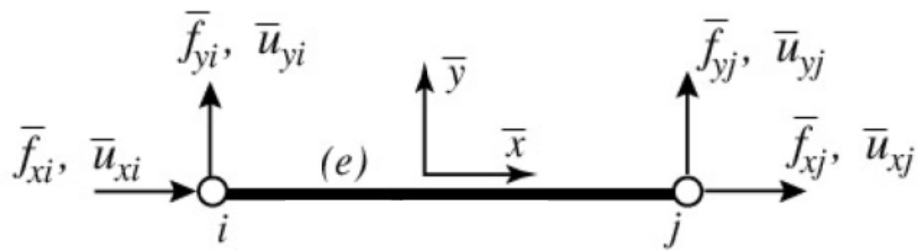
Our Example Truss



Localization

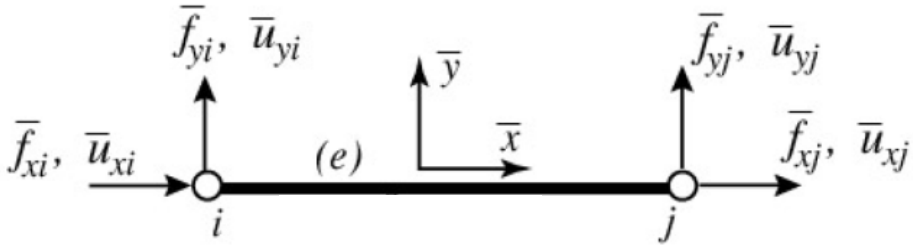


Member Stiffness Relation:

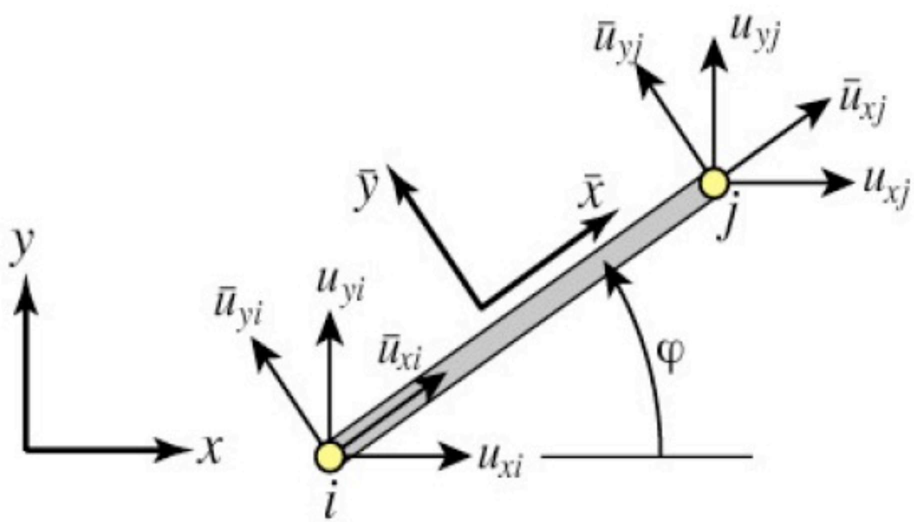


$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

Globalization (Ch. 17)



$$\bar{\boldsymbol{f}}^{\boldsymbol{e}} = \bar{\boldsymbol{K}}^{\boldsymbol{e}}\ \bar{\boldsymbol{u}}^{\boldsymbol{e}}$$



$$\boldsymbol{f}^{\boldsymbol{e}} = \boldsymbol{K}^{\boldsymbol{e}}\ \boldsymbol{u}^{\boldsymbol{e}}$$

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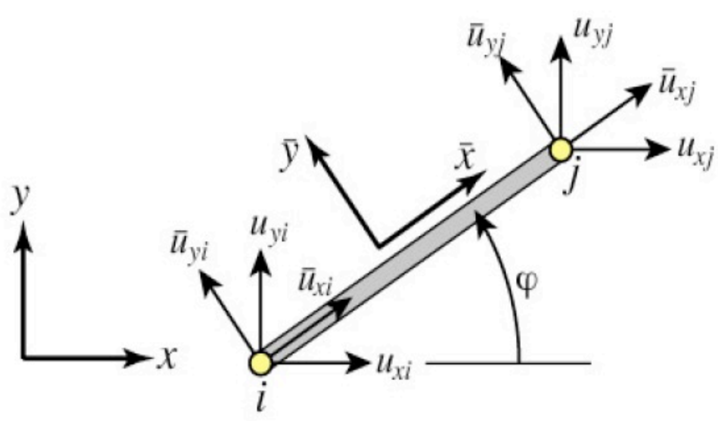
Displacement and Force Transformations

Displacement Transformation

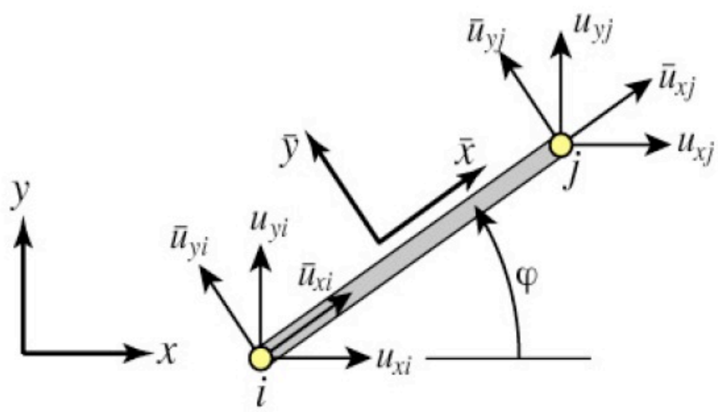


Node displacements transform as

$$\bar{u}_{xi} = u_{xi}c + u_{yi}s,$$
$$\bar{u}_{xj} = u_{xj}c + u_{yj}s,$$

in which $c = \cos\varphi \quad s = \sin\varphi$

Displacement Transformation



Node displacements transform as

$$\bar{u}_{xi} = u_{xi}c + u_{yi}s, \qquad \bar{u}_{yi} = -u_{xi}s + u_{yi}c$$
$$\bar{u}_{xj} = u_{xj}c + u_{yj}s, \qquad \bar{u}_{yj} = -u_{xj}s + u_{yj}c$$

in which $c = \cos\varphi \quad s = \sin\varphi$

In matrix form

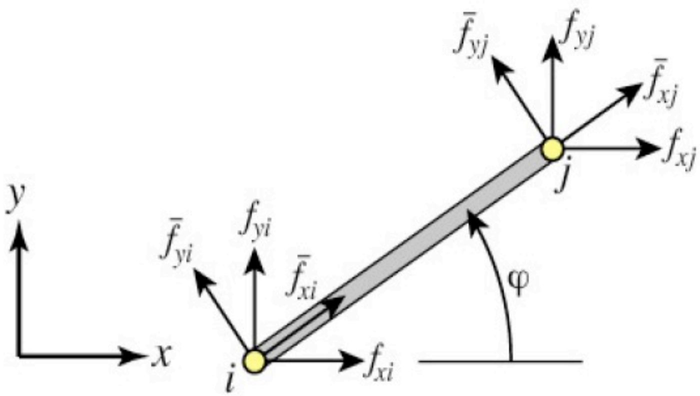
$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

or

$$\boxed{\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e}$$

Note:
global on RHS,
local on LHS

Force Transformation

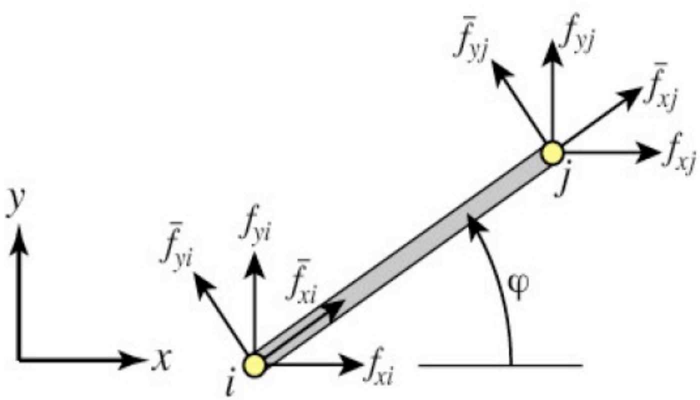


Node forces transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Note:
global on LHS,
local on RHS

Force Transformation



Node forces transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Note:
global on LHS,
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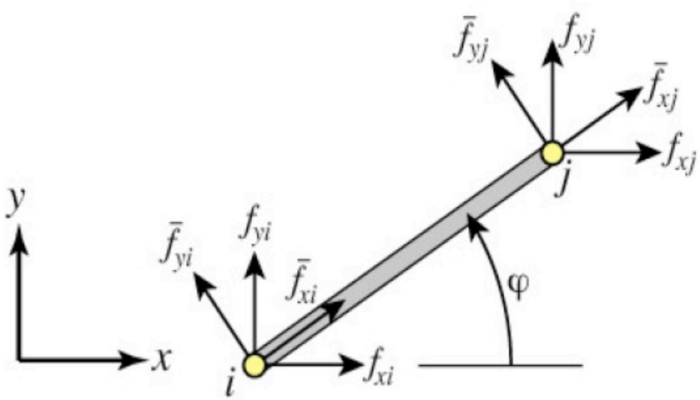
Force Transformation



Note: $\mathbf{T}^e$ is a orthonormal matrix:

$$\left(T^e\right)^T = \left(T^e\right)^{-1}$$

Node forces transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Note:
global on LHS,
local on RHS

or

$$\boxed{\mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e}$$

Elemental Stiffness Matrix in Global Coordinate System

$$\bar{\mathbf{K}}^e \, \mathbf{u}^e = \bar{\mathbf{f}}^e$$

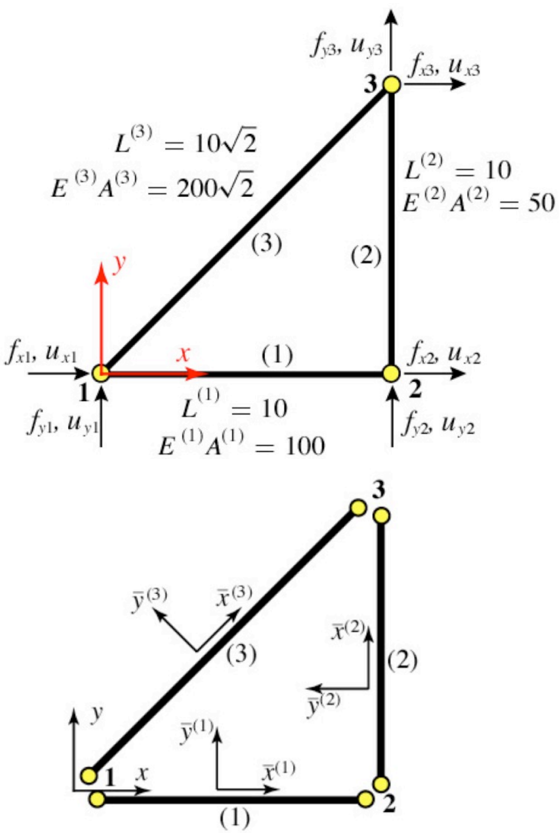
$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e \qquad \mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e$$

$$\boxed{\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e} \qquad \boxed{\mathbf{K}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{K}}^e \, \mathbf{T}^e}$$

Element stiffness equation in global CS

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Element Stiffness Equations



Bar 1:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \end{matrix}\right\} \text{node i} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{node j} \end{matrix}$$

Bar 2:

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \end{matrix}\right\} \text{node i} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{node j} \end{matrix}$$

Bar 3:

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \end{matrix}\right\} \text{node i} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{node j} \end{matrix}$$

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Now starting Ch. 18

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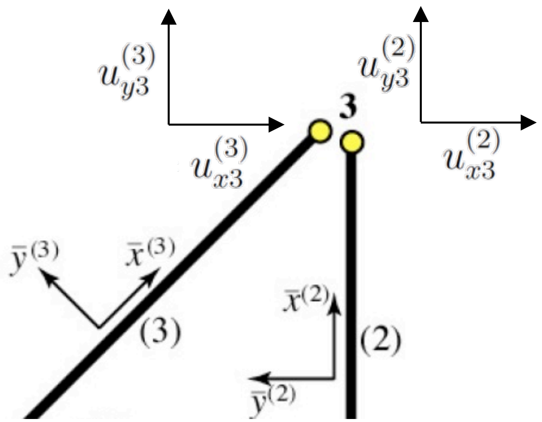
Compatibility of Node Displacements

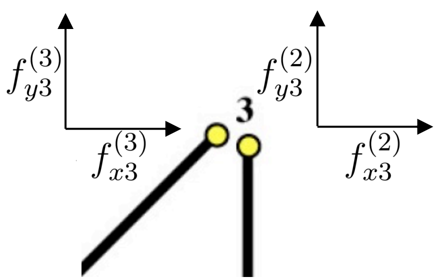


Compatibility of Nodal Displacements:

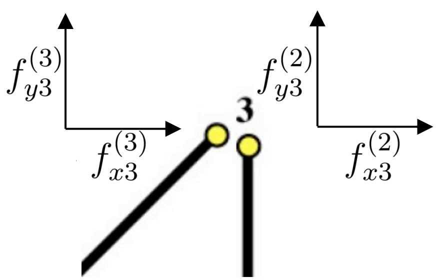
$$\underbrace{u_{x3}^{(2)} = u_{x3}^{(3)}}_{u_{x3}}, \quad \underbrace{u_{y3}^{(2)} = u_{y3}^{(3)}}_{u_{y3}}$$

drop bar superscript
for displacement

Force Equilibrium at Nodes



Force Equilibrium at Nodes



Global Static Equilibrium