ASEN 3112
Lecture 7:
Beam Differential Equations

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Announcements

- Lab 1
  - Download **your group's data** from Canvas
  - Individual video quiz will be posted by Thursday
    - Must be completed before 1:00 pm on Thursday, Feb. 20
- Homework 3 due Friday, Feb. 7 at 11:59 pm MST

- I will have office hours tomorrow (Feb. 5) from 11:30 am – 12:30 pm in the 3rd floor lobby

Change to Exam Make Up Policy

- “There will be no unexcused exam makeups provided. If you miss an exam, course instructors will evaluate each case on an individual basis based on the context and information available to make a determination if a makeup exam will be provided. Students are encouraged to provide as much documentation as possible to enable an informed decision.”

Exam 1 Announcements

- In class next Tuesday, Feb. 11
- Covers Ch. 1-9 of textbook
- If you have an accommodation, respond to my e-mail or send me an e-mail if you didn't get one
- Exam policies
  - Will have 1 hour and 15 minutes for the exam
  - 3-4 problems?
  - Closed-book
  - Your crib sheet can be one 8.5" x 11" piece of paper with writing **on both sides**
  - Non-internet-enabled calculators are allow (no phones, laptops)
- Will post past exams ASAP
- Will post a review video ASAP

This Week's Outline

- Torsion clicker questions

- Beam bending differential equations
  - Today: Deriving them (Ch. 10)
  - Thursday: Applying them (Ch. 10 & 11)

Coefficients α and β for Single Rectangular Cross Section as Functions of Aspect Ratio

b/t	1.0	1.2	1.5	2.0	2.5	3.0	4.0	5.0	6.0	10.0	∞
α	0.208	0.219	0.231	0.246	0.258	0.267	0.282	0.291	0.299	0.312	1/3
β	0.141	0.166	0.196	0.229	0.249	0.263	0.281	0.291	0.299	0.312	1/3

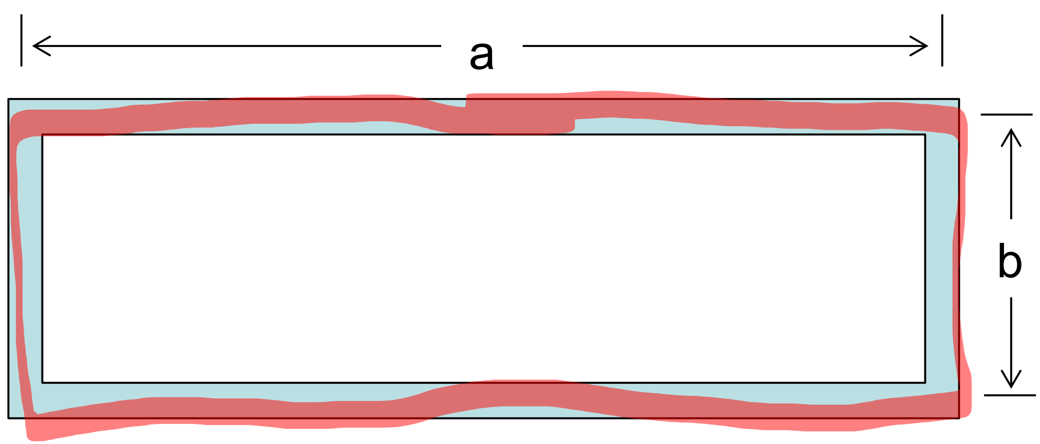
$J_\alpha = \alpha b t$

Interpolation formulas valid for all aspect ratios are given in the Lecture 8 Notes. If $b/t > 3$, $\alpha \sim \beta$ within 1%

If the section is sufficiently thin so that $b/t > 5$ (say) one can take $\alpha = \beta \sim 1/3$, which is easy to remember.

Question 1

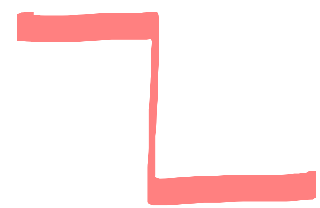
- What theory would be best to use for the following cross-section?
  a) Exact theory
  b) Open thin-walled theory – rectifying into one rectangle
  c) Open thin-walled theory – decomposing into multiple rectangles
  d) Closed thin-walled theory
  e) A hybrid of open and closed thin-walled theory

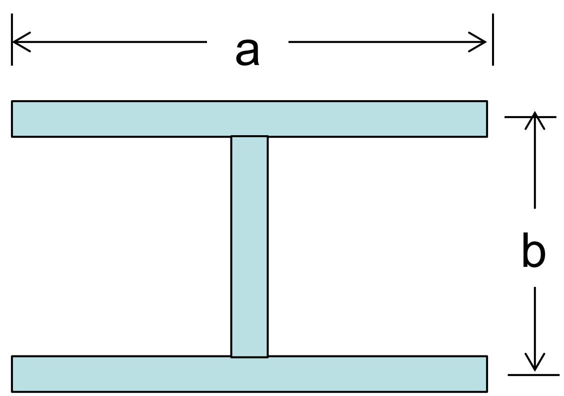


$t << a$
$t << b$

Question 2

- What theory would be best to use for the following cross-section?
  a) Exact theory
  b) Open thin-walled theory – rectifying into one rectangle
  c) Open thin-walled theory – decomposing into multiple rectangles
  d) Closed thin-walled theory
  e) A hybrid of open and closed thin-walled theory

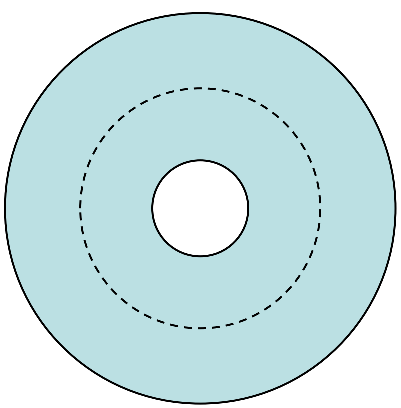




Constant t
$t << a$
$t << b$

Question 3

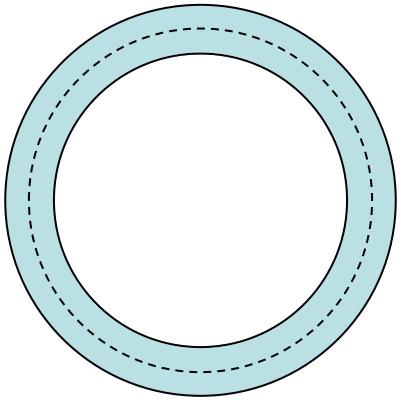
- What theory would be best to use for the following cross-section?
  a) Exact theory
  b) Open thin-walled theory – rectifying into one rectangle
  c) Open thin-walled theory – decomposing into multiple rectangles
  d) Closed thin-walled theory
  e) A hybrid of open and closed thin-walled theory



R/t = 1

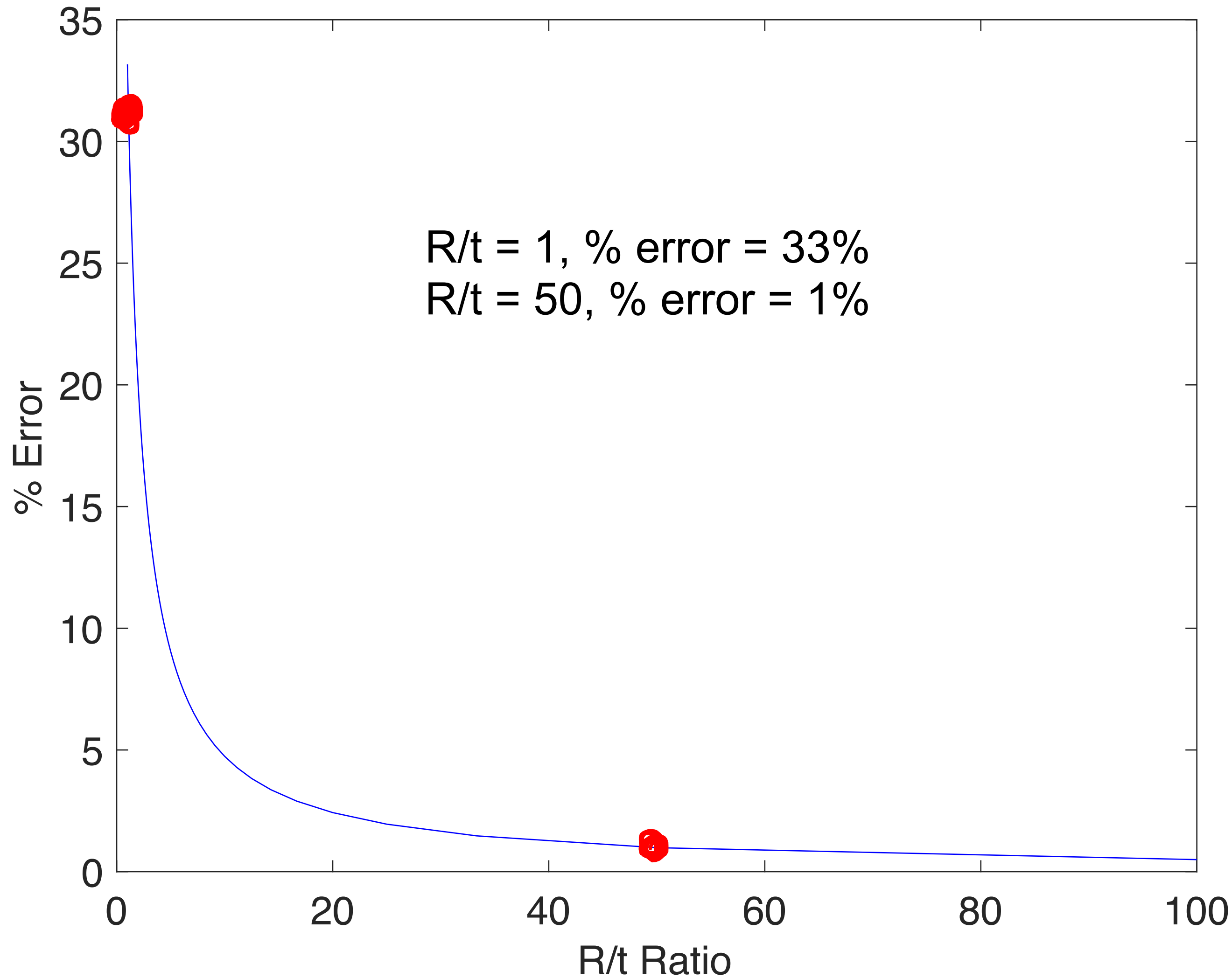
Question 4

- What theory would be best to use for the following cross-section?
  a) Exact theory
  b) Open thin-walled theory – rectifying into one rectangle
  c) Open thin-walled theory – decomposing into multiple rectangles
  d) Closed thin-walled theory
  e) A hybrid of open and closed thin-walled theory



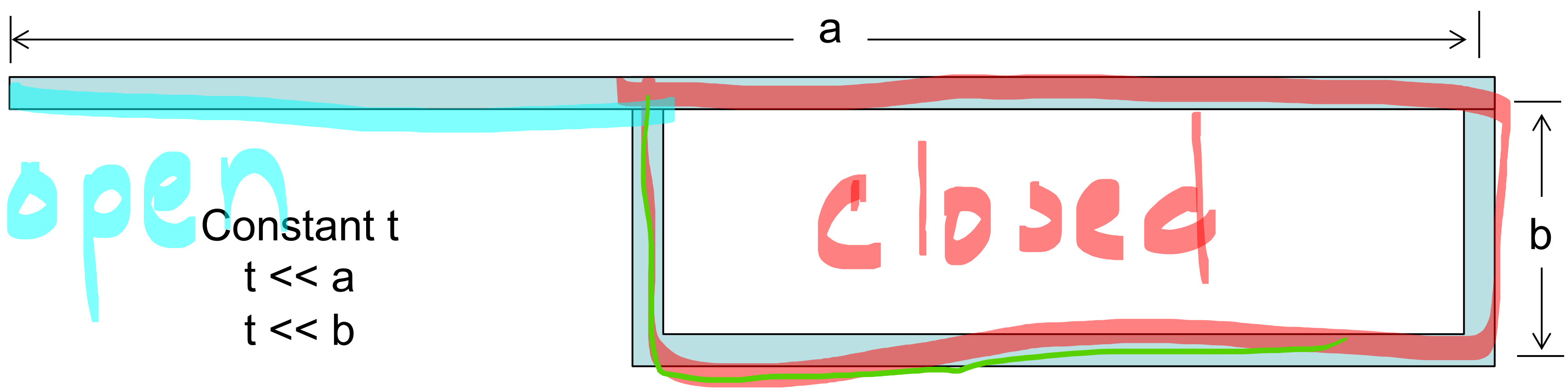
$R/t = 50$

% Error from Exact Theory vs. R/t Ratio



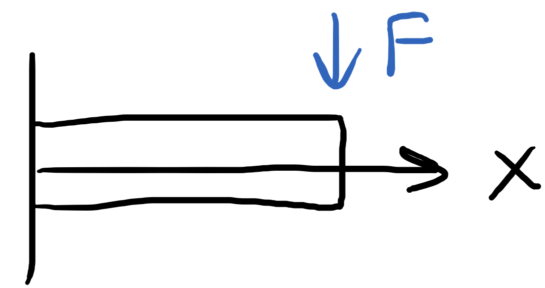
Question 5

- What theory would be best to use for the following cross-section?
  a) Exact theory
  b) Open thin-walled theory – rectifying into one rectangle
  c) Open thin-walled theory – decomposing into multiple rectangles
  d) Closed thin-walled theory
  e) A hybrid of open and closed thin-walled theory



Beam Bending Differential Equations

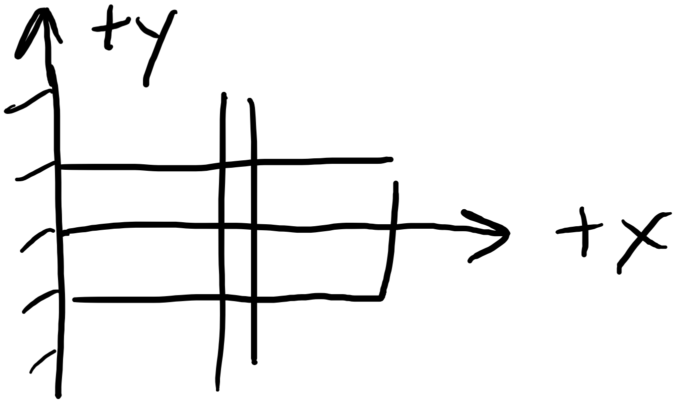
Beams are 1D elements that take transverse loads

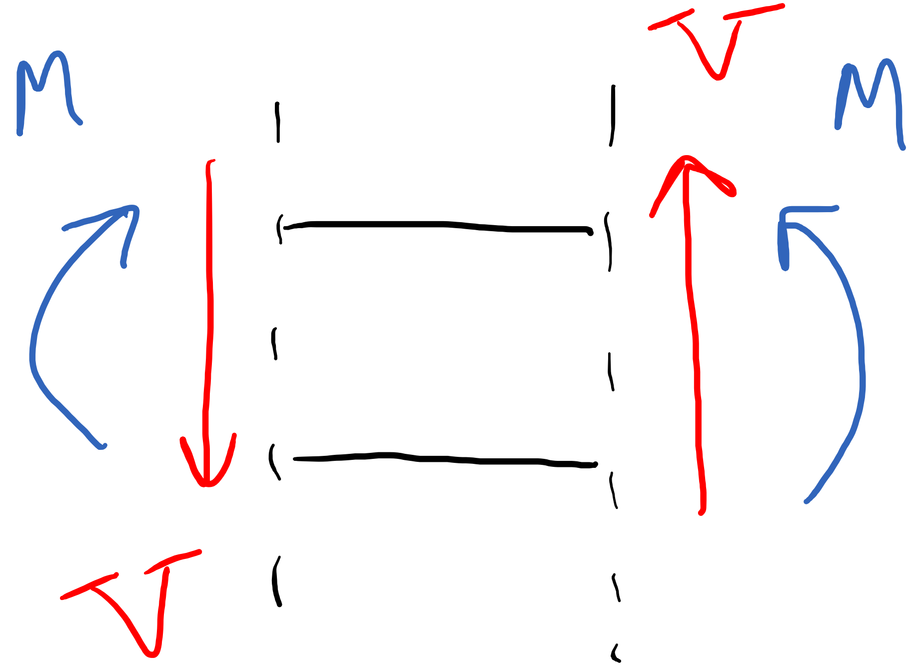


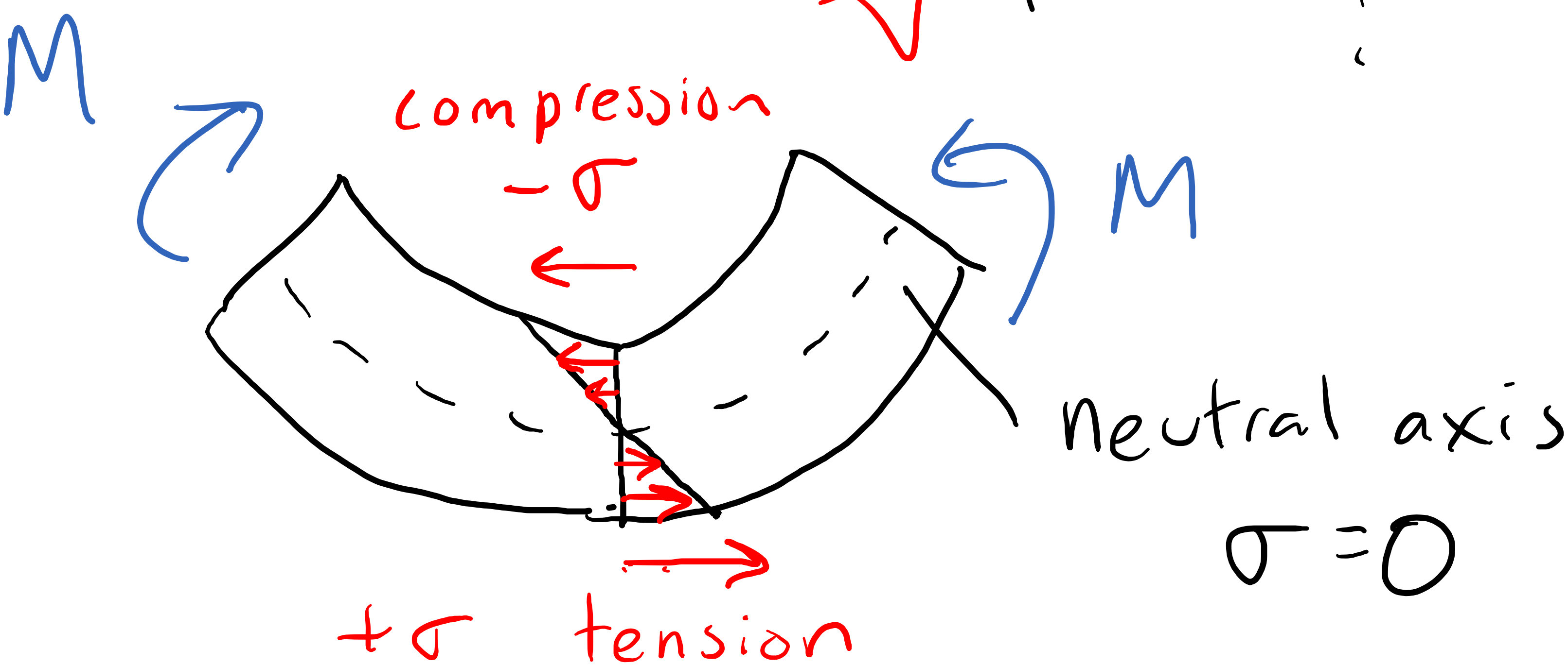
We will be using classical Bernoulli – Euler beam theory. Assumptions:

- straight
- prismatic (or varies smoothly)
- loading & deformation all in 1 plane
- small deformations (allowes us to linearize)
- material is elastic & isotropic

Sign convention







Beam Notation & Sign Conventions

Quantity	Symbol	Sign convention(s)
Problem specific load	varies	You pick'em
Generic load for ODE work	$p(x)$	+ if up
Transverse shear force	$V_y\,(x)$	+ if up on +x face
Bending moment	$M_z\,(x)$	+ if it produces compression on top face
Slope of deflection curve	$dv(x)/dx = v'(x)$	+ if positive slope, or cross-section rotates CCW
Deflection curve	$v(x)$	+ if beam cross-section moves upward

Note 1: Some textbooks (e.g. Vable and Beer-Johnson-DeWolf) use $V = -\,V_y$ as alternative transverse shear force symbol. This has the advantage of eliminating the minus sign in two of the ODEs listed on the next slide. V will only be used occsionally in this course.

Note 2. In our beam model, the slope $v'(x) = dv(x)/dx$ is equal to the rotation $\theta(x)$ of the cross section

Derive ODEs that relate:

M, V - shear force

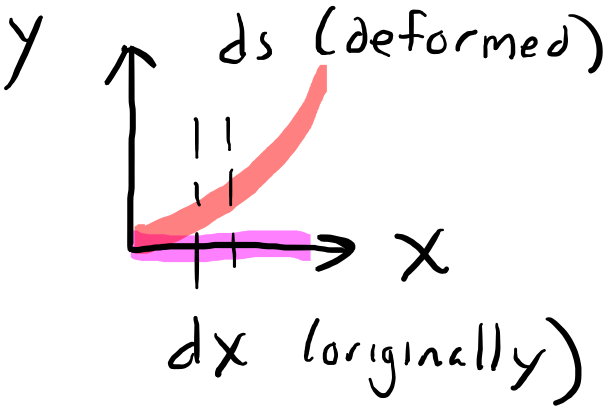
Beam Differential Equations

p, v ~ deflection

Connected quantities	Ordinary Differential Equations (ODEs)
From load to transverse shear force	$\dfrac{dV_y}{dx} = -p$ or $p = -V_y' = V'$
From transverse shear to bending moment	$\dfrac{dM_z}{dx} = -V_y$ or $M_z' = -V_y = V$
From bending moment to deflection	$E\,I_{zz} v'' = M_z$ *or* $v'' = \dfrac{M_z}{E\,I_{zz}}$
From load to moment	$M_z'' = p$
From load to deflection	$E\,I_{zz}\, v^{IV} = p$

Deformations → Strain → Stress → Internal Load

Look at an infinitesimal slice of beam

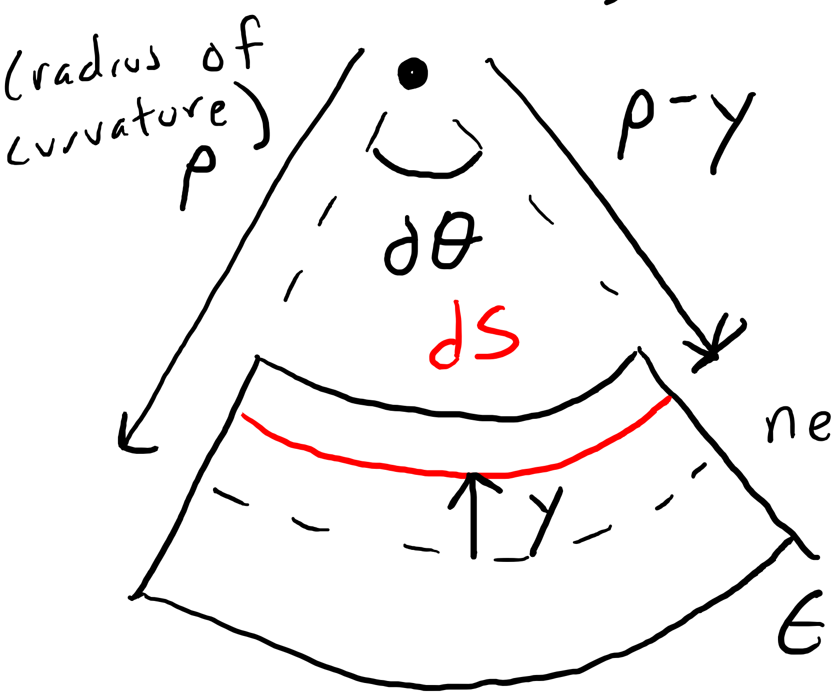


∂s = arc length

$ds_0 = \rho\, d\theta \qquad ds = (\rho - y)\, d\theta$

$$\epsilon_x = \frac{\partial s - \partial s_0}{\partial s_0} = \frac{\cancel{\rho \partial\theta} - y\partial\theta - \cancel{\rho\partial\theta}}{\rho\, d\theta}$$

$$\epsilon_x = \frac{-y}{\rho}\frac{\cancel{\partial\theta}}{\cancel{\partial\theta}} \qquad \epsilon_x = \frac{-y}{\rho}$$



neutral axis

$ds_0 = dx$

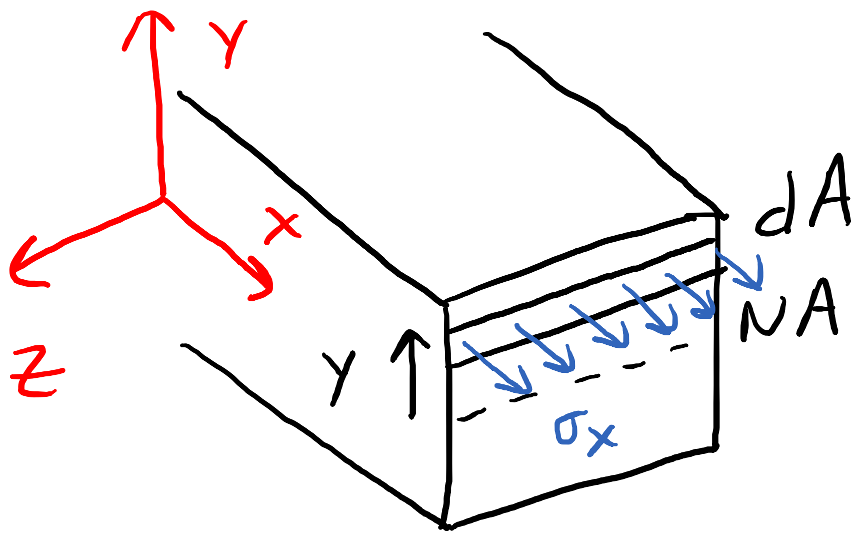
$\epsilon_x = 0, \ \sigma_x = 0$

Define curvature $\kappa = \frac{1}{\rho} = \frac{d\theta}{ds_0}$

$$\boxed{\epsilon_x = -y\kappa}$$

$$\boxed{\sigma_x = -Ey\kappa}$$

Flexure formula



M due to σ = bending M

$$dM_z = -y\,\sigma\,dA$$

- sign b/c moment in -z direction

$$M_z = \int_A -y\sigma\, dA$$

$$M_z = \int_A -y(-EyK)\,dA = +EK \int_A y^2 dA$$

moment of inertia I_z

rectangular cross-section: $I_z = \frac{bh^3}{12}$

$M_z = E K I_z$ For each slice of the beam

$$M(x) = E I K(x)$$

Axial: $F = E A \frac{du}{dx}$

Torsion: $T = G J \frac{d\phi}{dx}$

Bending: $M = E I K$

load

material properties

geometry

deformation

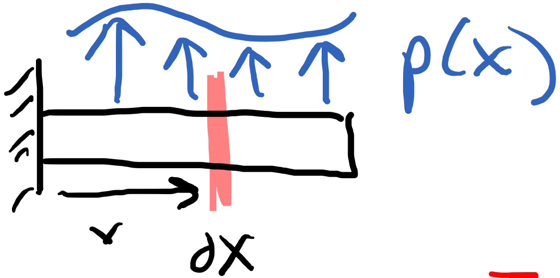
Flexure formula:

$$\boxed{M_z = E I_z K}$$

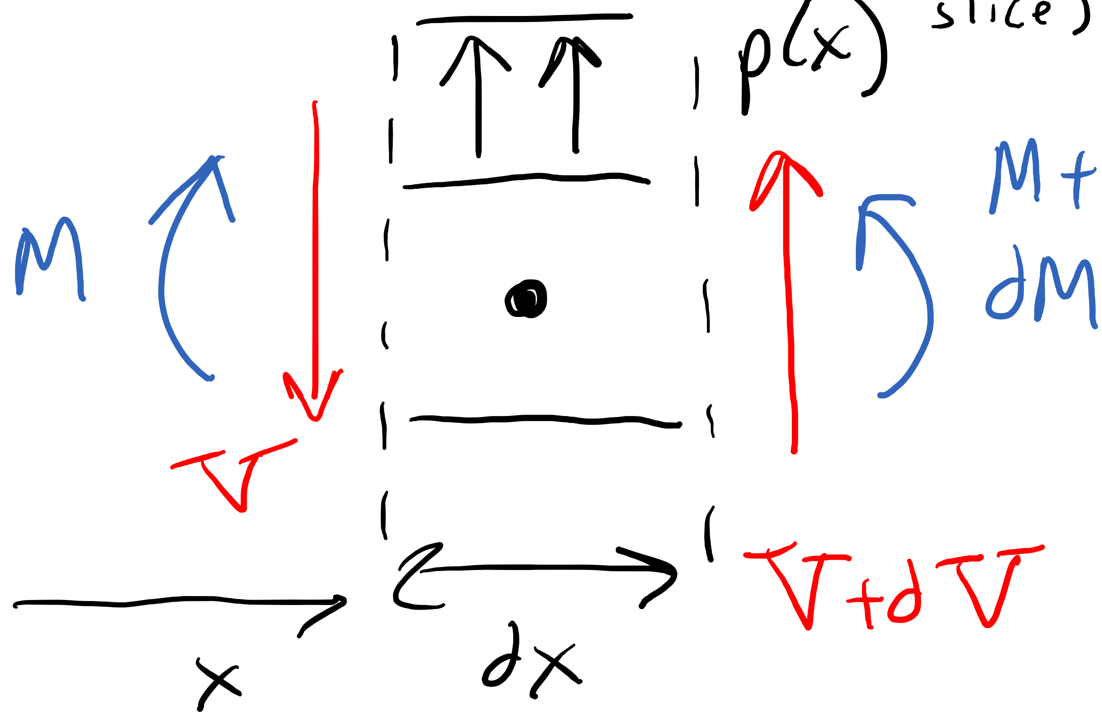
sub. in $K = \dfrac{-\sigma_x}{E y}$

$$\boxed{\sigma_x = \frac{-M_z y}{I_z}}$$

Relate M to external forces (p) (const. over slice)



$$\boxed{V'(x) = -p(x)}$$



$$\Sigma M = 0 = (V+dV)\frac{dx}{2} + V\frac{dx}{2} + (\cancel{M} + dM) - \cancel{M}$$

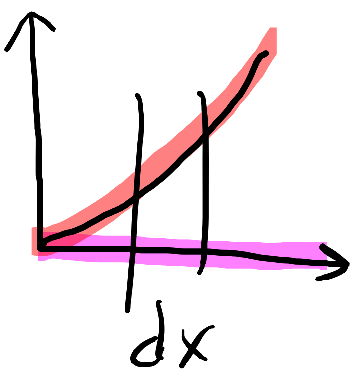
linearize

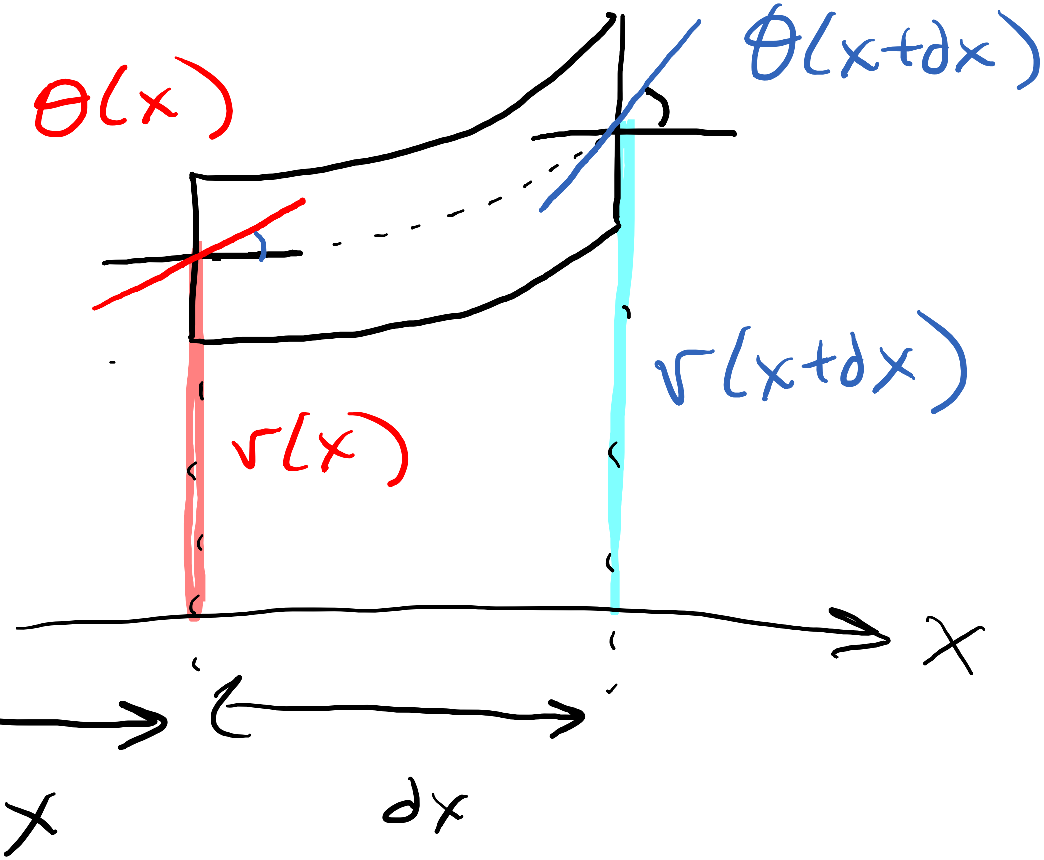
$$V\,dx + \cancel{\frac{dV\,dx}{2}} + dM = 0$$

$$\frac{dM}{dx} = -V \qquad M'(x) = -V(x) \qquad M''(x) = p(x)$$

$$V'(x) = -p(x) \qquad M'(x) = -V(x)$$

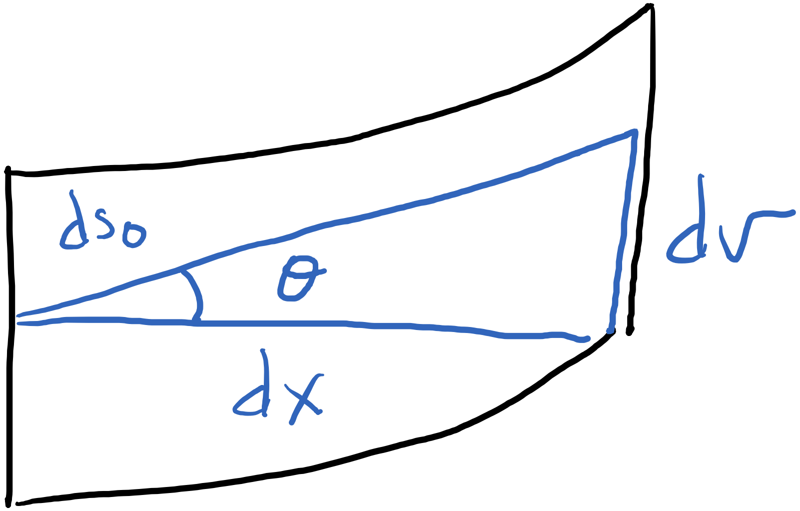
Now relate everything to deflection (v)





v = deflection of NA from x-axis

θ = angle between NA + x-axis



$$\tan\theta = \frac{\partial v}{\partial x}$$

small θ approx.

$$\theta = \frac{\partial v}{\partial x}$$

$$\boxed{\theta(x) = v'(x)}$$

$$p \to V \to M \to \theta \to v$$