ASEN 3112:
Spring 2020
Exam 3

Date: April 23, 2020
Time: 1-2:15 pm

Name: ____

Student ID: ____________________________________

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this exam.

Name: ____

Signature: ____

Date: ____

Please label your work with the part of the problem you are working on and circle your final answer for each part.

1. This exam is open-book, open-notes.

2. Solve all three problems.

3. Total time for the exam is 1 hour and 15 minutes.

4. Do not redefine the problem; carefully read the problem statement and answer the questions that are asked.

5. Make sure that one can follow your analysis; describe briefly what you are doing.

6. You must show how you got to your solution. Simply showing the final results will lead to point reduction.

7. You must cross out all work you do not want graded. Any work that is not crossed out is fair game for grading.

8. Include units on final answers whenever applicable.

9. Write your solution on the exam sheets using the space provided for each question.

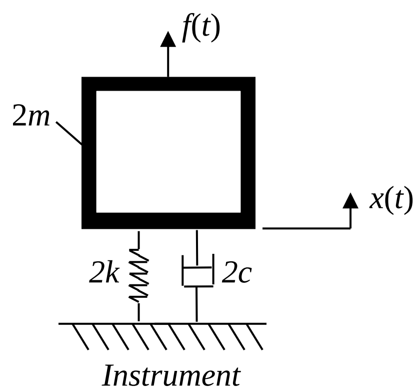
10. You can either print the exam and write the answers in the empty parts (and add additional sheets as needed), or you can write all your answers in empty sheets and number them and order them properly.

11. One you finish your exam, scan it, please scan it and upload a single pdf file of your complete exam to Gradescope. Make sure all pages are numbered and included in the proper order.

Question 1. 30 points

A hollow spacecraft crate of mass $2m$ is shown in the figure below. The crate (represented by solid black in the figure) is resting on a viscoelastic suspension. The coordinate $x(t)$ denotes the absolute position of the crate as a function of time. Assume the crate is a rigid structure of mass $2m$, where m = 1 kg. The spring constant is $2k$ where k = 1 N/m. The spacecraft is operating at zero gravity.

To examine the damping properties of this system, the crate is initially displaced by 3 cm and left to vibrate naturally until it reached rest. During this time, the crate experienced numerous oscillations, but the amplitude was observed to clearly decrease with time due to the damping provided by the viscoelastic suspension. The displacement amplitudes corresponding to the third and fifth peaks were measured as 1.8 cm and 1.6 cm, respectively.



Answer the following and <u>make sure to show your working out, formulas used, etc.</u>

(a) Calculate the natural frequency of the system.
(b) Determine the damping ratio ξ of the system.
Hint: In your derivation, you can make an approximation that we typically make if ξ is very small (i.e., $\xi << 1$).
(c) Classify the level of damping; i.e., is this system undamped, underdamped, critically damped, or overdamped.
(d) Calculate the period of oscillations of this system.
(e) How many total peaks of oscillations will the crate experience (starting from the very initial motion) until it reaches a peak displacement of less than 0.1 cm?

Write work for Problem 1 here:

ASEN3112 _ Midterm exam 3 _ Question One "1" Solution

a)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\ N/m}{2\ kg}} = \boxed{1\ rad/sec}$$

b)

$$\delta = \frac{1}{n} \ln \frac{x(t)}{x(t+nT)} \longrightarrow \delta = \frac{1}{2} \ln \frac{1.8\ cm}{1.6\ cm} = 0.0589$$

$$\xi = \frac{1}{\sqrt{1+\left(\frac{2\pi}{\delta}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{2\pi}{0.0589}\right)^2}} = 0.0094 \Rightarrow \boxed{\xi = 0.0094}$$

c)

$\xi = 0.0094 \longrightarrow \xi < 1 \Rightarrow$ system is underdamped

d)

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}} \simeq 6.31\ sec$$

e)

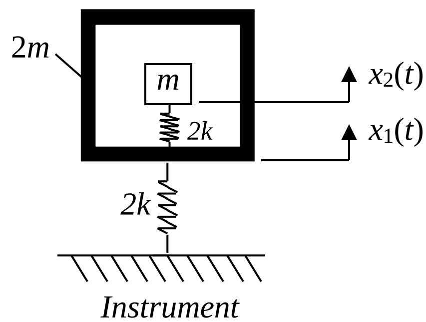
$$\frac{0.001}{0.03} = e^{-\xi\omega_n t^*} \longrightarrow \ln\left(\frac{0.001}{0.03}\right) = -\xi t^* \omega_n = -0.0094\, t^*$$

$\Rightarrow t^* \simeq 361.83\ sec$

$$\text{Number of peaks} = \frac{t^*}{T_d} = \frac{361.83\ sec}{6.31\ sec} \simeq 53 \longrightarrow n = \begin{cases} 58 & \text{(if first peak wasn't count)} \\ 59 \end{cases}$$

Question 2. 35 points

A new version of the spacecraft crate considered in Question 1 was installed. The new system (shown below) has extremely low damping in the suspension, and as such we will assume this new system to be undamped. The crate now is used to package a delicate instrument of mass m. The instrument itself is supported by an internal suspension of stiffness $2k$, as illustrated in the figure below. As in Question 1, m = 1 kg and k = 1 N/m. In this new system, the coordinates x_1 and x_2 denote the absolute position of the crate and instrument, respectively.



The equations of motion governing $x_1(t)$ and $x_2(t)$ are given as:

$$\begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}_{\mathbf{1}} \\ \ddot{\boldsymbol{x}}_{\mathbf{2}} \end{Bmatrix} + (k/m) \begin{bmatrix} \mathbf{4} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{2} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_{\mathbf{1}} \\ \boldsymbol{x}_{\mathbf{2}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \mathbf{0}$$

These equations of motion are given to you, please <u>do not re-derive them</u>.

(a) Calculate the natural frequencies and express them in rad/s. Express your answer to the fourth decimal number.
(b) Calculate the mode shapes, $\mathbf{U}_1$ and $\mathbf{U}_2$.
(c) Normalize the eigenvectors with respect to the mass matrix, create a mass-normalized modal matrix, and verify that this mass-normalized modal matrix uncouples the problem and gives you an identity matrix and a diagonal matrix of eigenvalues.

Write work for Problem 2 here:

res/ds) b) Eigen problem:

$$K\,U = \omega^2 M\,U$$

$$\frac{k}{m}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\Rightarrow \left[\frac{k}{m}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 2\omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}\right]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{bmatrix} 4k/m & -2k/m \\ -2k/m & 2k/m \end{bmatrix} - \begin{bmatrix} 2\omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}\right]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4k/m - 2\omega^2 & -2k/m \\ -2k/m & 2k/m - \omega^2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic polynomial :

$$\det\begin{bmatrix} 4k/m - 2\omega^2 & -2k/m \\ -2k/m & 2k/m - \omega^2 \end{bmatrix} = 0$$

$$\Rightarrow \left(\frac{4k}{m} - 2\omega^2\right)\left(\frac{2k}{m} - \omega^2\right) - \frac{4k^2}{m^2} = 0$$

$$\Rightarrow \frac{8k^2}{m^2} - 4\frac{\omega^2 k}{m} - \frac{4\omega^2 k}{m} + 2\omega^4 - \frac{4k^2}{m^2} = 0$$

$$\Rightarrow \frac{4k^2}{m^2} - 8\frac{\omega^2 k}{m} + 2\omega^4 = 0$$

$$\Rightarrow \omega^4 - 4\omega^2\left(\frac{k}{m}\right) + 2\frac{k^2}{m^2} = 0$$

$$\Rightarrow \omega^4 - \left(\frac{4k}{m}\right)\omega^2 + \frac{2k^2}{m^2} = 0$$

$$\Rightarrow \omega^2 = \frac{\frac{4k}{m} \pm \sqrt{\left(\frac{4k}{m}\right)^2 - 4\left(\frac{2k^2}{m^2}\right)}}{2}$$

$$\Rightarrow \omega^2 = \frac{1}{2}\left[\frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{8k^2}{m^2}}\right]$$

$$= \frac{1}{2}\left[\frac{4k}{m} \pm \sqrt{\frac{8k^2}{m^2}}\right]$$

$$= \frac{1}{2}\left[\frac{4k}{m} \pm 2\sqrt{2}\frac{k}{m}\right]$$

$$= \frac{2k}{m} \pm \sqrt{2}\frac{k}{m}$$

$$\boxed{\omega^2 = \frac{k}{m}\left[2 \pm \sqrt{2}\right]}$$

$$\boxed{\omega_1^2 = 3.4142\frac{k}{m}} \quad \boxed{\omega_2^2 = 0.5858\frac{k}{m}}$$

b) Mode shapes:

Mode shape 1:

$$\begin{bmatrix} 4k/m - 2\omega_1^2 & -2k/m \\ -2k/m & 2k/m - \omega_1^2 \end{bmatrix} \begin{Bmatrix} U_{11} \\ U_{12} \end{Bmatrix} = 0$$

$$\begin{bmatrix} 4k/m - 2(3.4142)\frac{k}{m} & -2k/m \\ -2k/m & 2k/m - 3.4142\frac{k}{m} \end{bmatrix} \begin{Bmatrix} U_{11} \\ U_{12} \end{Bmatrix} = 0$$

$$\frac{k}{m}\begin{bmatrix} -2.8284 & -2 \\ -2 & -1.4142 \end{bmatrix} \begin{Bmatrix} U_{11} \\ U_{12} \end{Bmatrix} = 0$$

Set $U_{11} = 1$

$$-2.8284(1) - 2(U_{12}) = 0$$

$$U_{12} = \frac{-2.8284}{2} = -1.4142$$

$$\begin{bmatrix} U_{11} \\ U_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ -1.4142 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.7071 \\ 1 \end{bmatrix}$$

Similarly,

Mode shape 2

$$\frac{k}{m}\begin{bmatrix} 4 - 2(0.5858) & -2 \\ -2 & 2 - 0.5858 \end{bmatrix} \begin{bmatrix} U_{21} \\ U_{22} \end{bmatrix} = 0$$

$$\frac{k}{m}\begin{bmatrix} 2.8284 & -2 \\ -2 & 1.4142 \end{bmatrix} \begin{bmatrix} U_{21} \\ U_{22} \end{bmatrix} = 0.$$

Set $u_{21} = 1$

$$2.8284(1) - 2u_{22} = 0$$

$$u_{22} = \frac{2.8284}{2} = 1.4142$$

$$\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.4142 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 1 \end{bmatrix}$$

Part c: $\boxed{\text{15 minutes 50 seconds.}}$

$$M_1 = \phi_1^T M \phi_1 = 2$$

$$m_2 = \phi_2^T M \phi_2 = 2$$

$$K_1 = 6.8284 \frac{k}{m}$$

$$K_2 = 1.716 \frac{k}{m}$$

Check:

$$\omega_1^2 = \frac{K_1}{M_1} = \frac{6.8284}{2} \frac{k}{m} = 3.4142 \frac{k}{m}$$

$$\omega_2^2 = \frac{K_2}{M_2} = \frac{1.1716}{2} \frac{k}{m} = 0.5858 \frac{k}{m}$$

Eigenvector Mass Orthonormalization:

let $\tilde{\phi}_1 = c_1 \phi_1$

$$(\tilde{\phi}_1)^T M \tilde{\phi}_1 = 1$$

$$c_1^2 \left[(\phi_1)^T M \phi_1 \right] = 1$$

$$c_1 = \frac{1}{\sqrt{\phi_1^T M \phi_1}} = \frac{1}{\sqrt{2}}$$

$\tilde{\phi}_2 = c_2 \phi_2$

$$(\tilde{\phi}_2)^T M \tilde{\phi}_2 = 1$$

$$c_2^2 \left[\phi_2^T M \phi_2 \right] = 1$$

$$c_2 = \frac{1}{\sqrt{\phi_2^T M \phi_2}} = \frac{1}{\sqrt{2}}$$

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -0.7071 \\ 1 \end{bmatrix}$$

$$\phi_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0.7071 \\ 1 \end{bmatrix}$$

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -0.7071 & 0.7071 \\ 1 & 1 \end{bmatrix}$$

$$v^T M v = M_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v^T K v = K_g = \begin{bmatrix} 3.4142\,k/m & 0 \\ 0 & 0.5858\,k/m \end{bmatrix}$$

New modal equations are given by:

$$M_g \ddot{\eta} + k_g \eta = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\eta} + \begin{bmatrix} 3.4142\,k/m & 0 \\ 0 & 0.5858\,k/m \end{bmatrix} = 0$$

$$\Rightarrow \ddot{\eta} + 3.4142\frac{k}{m} = 0$$

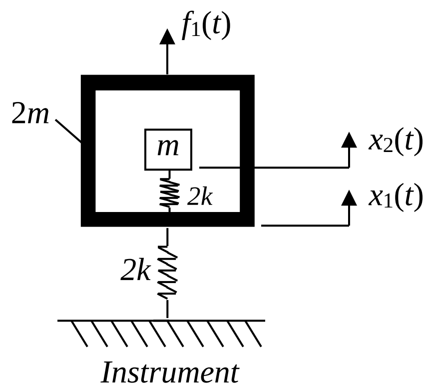
$$\ddot{\eta} + 0.5858\frac{k}{m} = 0$$

Question 3. 35 points

The new system considered in Question 2 is now subjected to a steady state force of

$$f_1(t) = Fcos(\Omega t),$$

where $\Omega = 1$ rad/s and $F = 0.01$ N. As in the previous questions, $m = 1$ kg. However, due to some material deterioration, the value of k has now changed to $k = 1/2$ N/m.



The equations of motion governing $x_1(t)$ and $x_2(t)$ are given as:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + (k/m) \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t)/m \\ 0 \end{Bmatrix}$$

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}$$

Once again, these equations of motion are given to you, please <u>do not re-derive them</u>.

The eigensolution of this problem is as follows:

$\omega_{n1} = 0.5412$ rad/s; $\mathbf{V}_1 = \begin{Bmatrix} 1/2 \\ \sqrt{2}/2 \end{Bmatrix}$; $\omega_{n2} = 1.3066$ rad/s; $\mathbf{V}_2 = \begin{Bmatrix} 1/2 \\ -\sqrt{2}/2 \end{Bmatrix}$

Note, the $\mathbf{V}_1$ and $\mathbf{V}_2$ vectors provided above are already normalized with respect to the mass matrix.

<u>Do not recalculate this eigensolution</u>; please simply use the values we are giving you.

Apply modal analysis and find <u>only</u> the steady-state response of both $x_1(t)$ and $x_2(t)$, i.e., neglect the homogeneous part of the total solution and only provide the particular solution. Show all your working out.

Write work for Problem 3 here:

Question 3

April 23, 2020

The uncoupled system of equations for a MDOF system will be as following:

$$\begin{cases} \ddot{\beta}_1 + \omega_{n1}^2 \beta_1 = \gamma_1 \\ \ddot{\beta}_2 + \omega_{n2}^2 \beta_2 = \gamma_2 \end{cases} \tag{1}$$

where

$$\begin{aligned} \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = V^T f = \begin{bmatrix} 1/2 & \sqrt{2}/2 \\ 1/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0.01 \cos t \\ 0 \end{bmatrix} \\ = \begin{bmatrix} 0.005 \cos(t) \\ 0.005 \cos(t) \end{bmatrix} \end{aligned} \tag{2}$$

Substituting equation (2) into (1) will yield:

$$\begin{cases} \ddot{\beta}_1 + (0.5412)^2 \beta_1 = 0.005 \cos(t) \\ \ddot{\beta}_2 + (1.3066)^2 \beta_2 = 0.005 \cos(t) \end{cases} \tag{3}$$

By assuming a solution of the form $\beta = A \cos(t) \Longrightarrow \ddot{\beta} = -A \cos(t)$

$$\begin{cases} -A_1 \cos(t) + A_1 (0.5412)^2 \cos(t) = 0.005 \cos(t) \\ -A_2 \cos(t) + A_2 (1.3066)^2 \cos(t) = 0.005 \cos(t) \end{cases} \tag{4}$$

The solution of this system is:

$$\begin{aligned} A_1 = \tfrac{0.005}{(0.5412)^2 - 1} = -0.007 \\ A_2 = \tfrac{0.005}{(1.3066)^2 - 1} = 0.007 \end{aligned} \tag{5}$$

So modal responses will be:

$$\begin{cases} \beta_1 = -0.007 \cos(t) \\ \beta_2 = 0.007 \cos(t) \end{cases} \tag{6}$$

Now we need to transfer the modal coordinates to the physical coordinates.

$$\begin{aligned} \boldsymbol{x} = \boldsymbol{V} \boldsymbol{\beta} \\ = \begin{bmatrix} 1/2 & 1/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} (\beta_1 + \beta_2)/2 \\ \frac{\sqrt{2}}{2}\beta_1 - \frac{\sqrt{2}}{2}\beta_2 \end{bmatrix} \end{aligned} \tag{7}$$

plugging (6) into (7):

$$\boldsymbol{x} = \left[\begin{array}{c} 0 \\ \sqrt{2}(-0.007)\cos(t) \end{array} \right] \tag{8}$$