ASEN 3112

Spring 2020

Lecture 17

March 12, 2020

Structural Dynamics Notation

Structural dynamics studies the **motion** of structures under time-dependent forces. The time will always be denoted by t. Derivatives with respect to t will be denoted by superposed dots. For example, if $u = u(t)$ is a **scalar motion**, the associated **velocity** and **acceleration** are compactly written

$$\boxed{\dot{u} \equiv \frac{du(t)}{dt} \qquad \ddot{u} \equiv \frac{d^2u(t)}{dt^2}}$$

Sometimes we denote velocity by v and acceleration by a where appropriate.

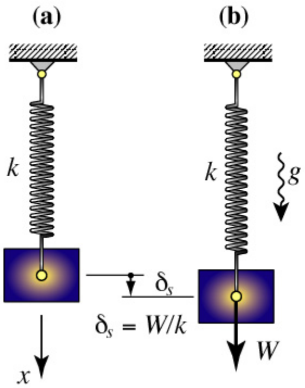
Note that partial differentiation with respect to t is not needed: the ordinary differentiation symbol d suffices.

Free Undamped SDOF Oscillator: Configuration

A weight of mass $m > 0$ hangs from an extensional spring of stiffness $k > 0$ under gravity acceleration g directed along the spring (x) direction. There is no damping. The weight force is $W = m\,g$, and the **static deflection** of the mass is

$$\boxed{\delta_s = W/k = m\,g\,/k}$$

This configuration is called a **Single Degree of Freedom Oscillator**, or SDOF Oscillator for short. See Figures.

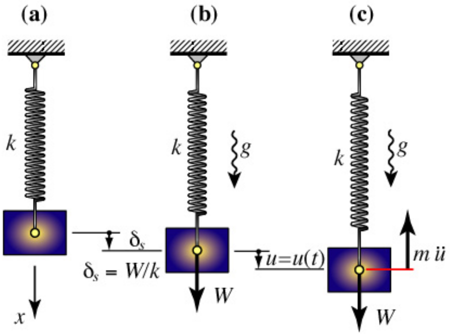


Free Undamped SDOF Oscillator: Initial Conditions

Apply now an **initial displacement** u_0 and **initial velocity** v_0 to the mass at $t = 0$, so that

$$u(0) = u_0, \quad \dot{u}(0) = v_0$$

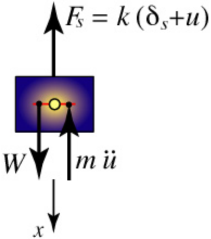
These are called the **initial conditions**, or ICs. Now release the mass and **do not apply any force** for $t > 0$. The oscillator is set in motion and a **free vibrations** response ensues.



Let $u(t)$ be the **deviation** from the static equilibrium position, so that the total motion is $\delta_s + u(t)$, as shown in Figure (c). The function $u(t)$ is called the **dynamic response**. It is the **only degree of freedom** (DOF) of this system.

Free Undamped SDOF Oscillator: EOM

The **equation of motion** (EOM) of the undamped SDOF oscillator is obtained from the **Dynamic Free Body Diagram** (DFBD) shown on the right. The spring is replaced by a force F_s that is linear in the static deflection δ_s and deviation from equilibrium u. The **inertia force** $m\,\ddot{u}$ acts **opposite** to the acceleration $\ddot{u}$ (because it resists it). Force equilibrium along x gives



$$m\,\ddot{u} = W - k\,(\delta_s + u) = m\,g - k\,\delta_s - k\,u$$

But the **static equilibrium terms cancel**: $m\,g - k\,\delta_s = 0$, which leaves

$$\boxed{m\,\ddot{u} + k\,u = 0}$$

This second-order ODE can be converted to **canonical form** as worked out in the next slide.

Free Undamped SDOF Oscillator: Canonical EOM

The **equation of motion** (EOM) of the free undamped SDOF oscillator derived in the previous slide is

$$\boxed{m\,\ddot{u} + k\,u = 0}$$

This second-order ODE can be converted to **canonical form** by dividing through by m (recall that $m > 0$) and introducing $\omega_n^2 = k/m$:

$$\boxed{\ddot{u} + \omega_n^2\,u = 0}$$

The parameter

$$\boxed{\omega_n = +\sqrt{\frac{k}{m}}}$$

is called the **undamped circular natural frequency**, or **natural frequency** for short. Its units are radians per second (rad/s)

Free Undamped SDOF Oscillator: Canonical EOM

The **equation of motion** (EOM) of the free undamped SDOF oscillator derived in the previous slide is

$$\boxed{m\,\ddot{u} + k\,u = 0}$$

This second-order ODE can be converted to **canonical form** by dividing through by m (recall that $m > 0$) and introducing $\omega_n^2 = k/m$:

$$\boxed{\ddot{u} + \omega_n^2\,u = 0}$$

The parameter

$$\boxed{\omega_n = +\sqrt{\frac{k}{m}}}$$

is called the **undamped circular natural frequency**, or **natural frequency** for short. Its units are radians per second (rad/s)

Free Undamped SDOF Oscillator: EOM Features

The **canonical equation of motion** (EOM) of the free undamped SDOF oscillator

$$\boxed{\ddot{u} + \omega_n^2 u = 0}$$

displays the following key features:

EOM is **linear**: superposition applies

EOM has **constant coefficients**: solutions are elementary circular functions

EOM is of **second order in time**: **two constants of integration =>** **two initial conditions** required

Free Undamped SDOF Oscillator: Solution in Terms of Complex Exponentials

Assume $u(t) = C \exp(\lambda\, t)$, where C is nonzero and generally complex. Since $\ddot{u} = \lambda^2 C \exp(\lambda\, t)$, substitution into the canonical EOM requires

$$(\lambda^2 + \omega_n^2)\, C\, e^{\lambda\, t} = 0$$

But since C is nonzero and the exponential never vanishes, the expression in parenthesis must be zero. This gives the characteristic equation

$$\boxed{\lambda^2 + \omega_n^2 = 0} \;\Rightarrow\; \lambda_{1,2} = \pm\, i\, \omega_n \quad \text{in which} \quad i = \sqrt{-1}$$

Since both solutions $\exp(i\, \omega_n t)$ and $\exp(-i\, \omega_n t)$ satisfy the linear ODE, so does any **linear combination** of them. We thus arrive at the **general solution** in terms of complex exponentials

$$\boxed{u(t) = C_1\, e^{i\omega_n t} + C_2\, e^{-i\omega_n t}}$$

in which C_1 and C_2 are generally **complex** numbers. We know, however, that $u(t)$ is real, so a rewrite in terms of real terms is convenient.

Free Undamped SDOF Oscillator: Solution in Terms of Trigonometric Functions

To express the response $u(t)$ in real form, we use Euler's relation

$$e^{\pm i\,\theta} = \cos\theta \pm i\sin\theta$$

Inserting into the solution derived in the previous slide we obtain

$$u(t) = (C_1 + C_2)\,\cos\omega_n t + i(C_1 - C_2)\sin\omega_n t$$

Introducing $A_1 = C_1 + C_2$ and $A_2 = i\,(C_1 - C_2)$ for convenience, we can compactly express the response in terms of trigonometric functions as

$$\boxed{u(t) = A_1\,\cos\omega_n t + A_2\sin\omega_n t}$$

Here A_1 and A_2 are real constants that can be directly determined from the initial conditions as $u(0) = u_0 = A_1$, and $u(0) = v_0 = A_2\omega_n$. Replacing yields

$$\boxed{u(t) = u_0\,\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t}$$

This expression gives the SDOF oscillator response directly in terms of ICs.

Free Undamped SDOF Oscillator: Phased Response

Reproducing the previous solution for convenience:

$$\boxed{u(t) = u_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t}$$

This is the **free vibration response** of the undamped SDOF oscillator expressed in terms of trigonometric functions and initial conditions. Although this result only holds for free vibrations, this solution will be later used as the **homogeneous** portion of the response of a harmonically **forced** SDOF oscillator (also called the **transient response**). Another common version of this result is the **phased response form**, also known as **phase-shifted response form**:

$$\boxed{u(t) = U \cos(\omega_n t - \alpha)}$$

Here the amplitude U and the phase angle α are linked to the initial conditions by

$$\boxed{U = \sqrt{u_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \qquad \tan\alpha = \frac{v_0}{\omega_n\, u_0}}$$

Free Undamped SDOF Oscillator: Effect of ICs

To study the effect of initial conditions, consider first the case where the mass is displaced from its static equilibrium position by u_0 and released. Then $v_0 = 0$, so the response is

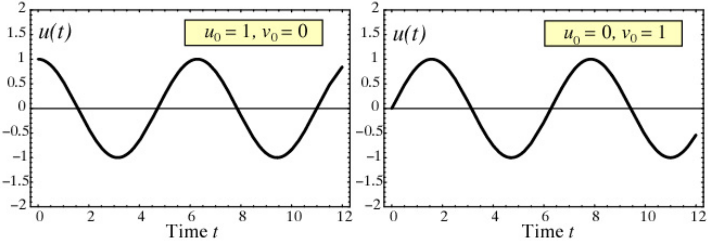
$$\boxed{u(t) = u_0 \cos \omega_n t}$$

This is called a **simple harmonic motion** with amplitude u_0, undamped **natural frequency** f_n and undamped **natural period** T_n given by

$$\boxed{f_n = \frac{\omega_n}{2\pi} \qquad T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n}}$$

The frequency f_n is expressed in **cycles per second or Hertz** (abbr. Hz; 1 Hz = cycle/s). The period is given in seconds per cycle, or simply seconds (s). It is easily verified that for this case the phased response has $U = u_0$ and $\alpha = 0$. If $u_0 = 0$ but v_0 is nonzero the response is $u(t) = v_0 \sin \omega_n t/\omega_n$. This has the same frequency but amplitude v_0/ω_n and starts at zero. Plots for a unit initial displacement and initial velocity are shown on next slide.

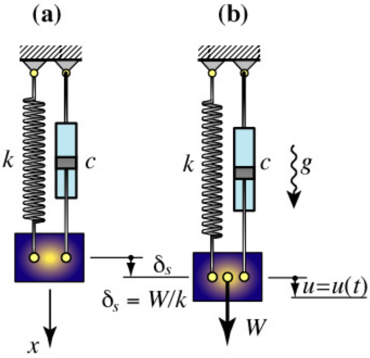
Free Undamped SDOF Oscillator: Responses for Unit Initial Displacement and Initial Velocity



Free Damped SDOF Oscillator: Configuration

The undamped SDOF oscillator is generalized by adding a **viscous damper** that exerts a resisting force $c\,\dot{u}$ proportional to the velocity $\dot{u}$ of the moving mass. Here c is called the **damping coefficient**. Note that the **static deflection** of the mass does not change:

$$\delta_s = W/k = m\,g\,/k$$



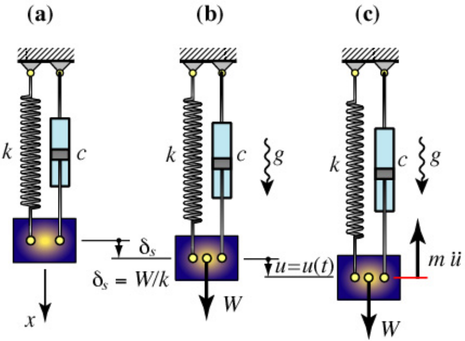
This configuration is called a **Damped Single Degree of Freedom Oscillator**, or Damped SDOF Oscillator for short.
(Viscous damping will be assumed unless the contrary is stated.) See Figures.

Free Damped SDOF Oscillator: ICs

Apply now an **initial displacement** u_0 and **initial velocity** v_0 to the mass at $t = 0$, so that

$$u(0) = u_0, \quad \dot{u}(0) = v_0$$

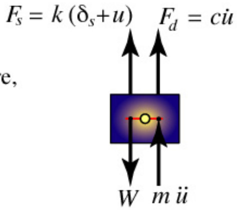
These are called the **initial conditions** or ICs. Now release the mass and **do not apply any force** for $t > 0$. The oscillator is set in motion and a **damped response** ensues.



Let $u(t)$ be the **deviation** from the static equilibrium position, so that the total motion is $\delta_s + u(t)$, as shown in Figure (c). The function $u(t)$ is called the **dynamic response**. It is the **only degree of freedom** (DOF) of this system.

Free, Damped SDOF Oscillator: EOM From DFBD

The **equation of motion** (EOM) of the damped SDOF oscillator is obtained from the **Dynamic Free Body Diagram** (DFBD) shown in the Figure. The spring is replaced by a force F_s as before, whereas the damper is replaced by a **damping force** $F_d = c\,\dot{u}$. The **inertia force** $m\,\ddot{u}$ again acts **opposite** to the acceleration $\ddot{u}$ (because it resists it). Force equilibrium along x gives



$$m\,\ddot{u} = W - k\,(\delta_s + u) - c\,\dot{u} = m\,g - k\,\delta_s - k\,u - c\,\dot{u}$$

The static equilibrium terms cancel: $m\,g - k\,\delta_s = 0$, which leaves

$$\boxed{m\,\ddot{u} + c\,\dot{u} + k\,u = 0}$$

This second-order ODE can be converted to **canonical form** as described in the next slide.

Free, Damped SDOF Oscillator: EOM

From the DFBD of the previous slide we have obtained the physical form of the EOM

$$\boxed{m\,\ddot{u} + c\,\dot{u} + k\,u = 0}$$

To convert to canonical form, divide through by the mass m and denote

$$\omega_n^2 = \frac{k}{m} \qquad \omega_n = +\sqrt{\frac{k}{m}} \qquad c = 2\xi\,\omega_n\,m \qquad \xi = \frac{c}{2\omega_n m}$$

Here ω_n is the undamped natural frequency introduced previously, while ξ is the **viscous damping factor**, also called **damping ratio** and **damping coefficient** (qualifier "viscous" is usually omitted). Using these definitions, we can compactly write the canonical form of the EOM as

$$\boxed{\ddot{u} + 2\xi\,\omega_n\,\dot{u} + \omega_n^2\,u = 0}$$

This is again a **second order, linear** ODE with constant coefficients. The only difference with respect to the undamped case is the presence of the velocity dependent term $2\,\xi\,\omega_n\,\dot{u}$ in the LHS.

Free, Damped SDOF Oscillator: Char Equation

As usual in solving second-order linear ODE with the canonical EOM form

$$\boxed{\ddot{u} + 2\xi\,\omega_n\,\dot{u} + \omega_n^2\,u = 0}$$

assume an exponential solution

$$u(t) = A\,e^{\lambda\,t} \qquad A \text{ nonzero}$$

in which A and λ are generally complex quantities to be determined. Inserting into the EOM we obtain the **characteristic equation**

$$\boxed{\lambda^2 + 2\xi\,\omega_n\lambda + \omega_n^2 = 0}$$

which is **quadratic** in λ. Its two roots are given by

$$\boxed{\lambda_{1,2} = -\xi\omega_n \pm \omega_n\,\sqrt{\xi^2 - 1}}$$

Free Damped SDOF Oscillator: Three Response Types

For the undamped case: $\xi = 0$, the characteristic roots reduce to $\pm\, i\ \omega_n$, as found previously. The magnitude of the **damping factor** ξ compared to unity can be used to distinguish three cases:

$\xi < 1$ **Underdamped case**. Damping is **subcritical**. The characteristic roots are **complex conjugate**. The motion is **oscillatory** with decreasing amplitude. This is the most common case in typical structures.

$\xi > 1$ **Overdamped case**. Damping is **overcritical**. The characteristic roots are **negative real and distinct**. The motion is **non-oscillatory**. Its amplitude decreases monotonically except possibly for one 0-crossing.

$\xi = 1$ **Critically damped case**. Damping is **critical**. The characteristic roots are **negative real and coalesce**. The motion is **non-oscillatory**. Its amplitude decreases monotonically except possibly for one 0-crossing.

We now proceed to study these three cases, with emphasis on the first one.

Free Damped SDOF Oscillator: Underdamped Case

For the **underdamped case** $\xi < 1$, the roots of the characteristic equation can be written

$$\boxed{\lambda_{1,2} = -\xi\,\omega_n \pm i\,\omega_d}$$

in which ω_d denotes the **damped circular natural frequency**, given by

$$\boxed{\omega_d = \omega_n \sqrt{1 - \xi^2}}$$

Like ω_n, this is expressed in radians per second (rad/s). The corresponding **damped period** is

$$\boxed{T_d = \frac{2\pi}{\omega_d}}$$

With the help of these definitions and Euler's formula, the **general solution** can be compactly expressed in terms of trigonometric functions as

$$\boxed{u(t) = e^{-\xi\,\omega_n t}\,(\,A_1\ \cos\omega_d t + A_2 \sin\omega_d\, t)}$$

Free Damped SDOF Oscillator: Underdamped Case (cnt'd)

As in the undamped case, the coefficients of the **general solution** can be directly expressed in terms of the **initial conditions**: $u(0) = u_0$ and $u(0) = v_0$. We obtain $A_1 = u_0$ and $A_2 = (v_0 + \xi\,\omega_d u_0)/\omega_n$, which substituted give

$$\boxed{u(t) = e^{-\xi\,\omega_n t}\left(u_0 \cos\omega_d t + \frac{v_0 + \xi\,\omega_n u_0}{\omega_d}\sin\omega_d t\right)}$$

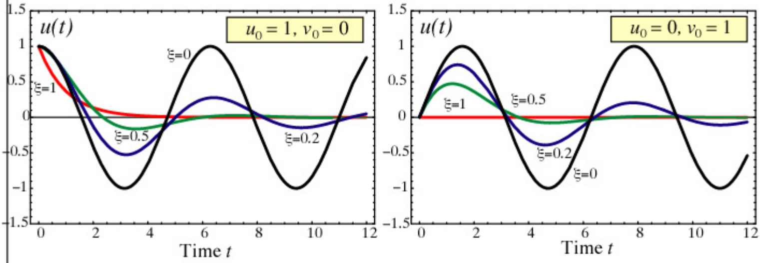
This equation may in turn be rewritten in the **phased form**

$$\boxed{u(t) = U\,e^{-\xi\,\omega_n t}\cos(\omega_d t - \alpha)}$$

in which

$$\boxed{U = \sqrt{u_0^2 + \left(\frac{v_0 + \xi\,\omega_n u_0}{\omega_d}\right)^2} \qquad \tan\alpha = \frac{v_0 + \xi\,\omega_n u_0}{\omega_d u_0}}$$

Free Damped SDOF Oscillator: Underdamped Response Plots



Free Damped SDOF Oscillator: Critically Damped Case

If $\xi = 1$ the characteristic roots coalesce: $\lambda_1 = \lambda_2 = -\omega_n$. The theory of second-order linear ODE says that the solution is

$$u(t) = (A_1 + A_2 t)\; e^{-\omega_n t}$$

in which coefficients A_1 and A_2 depend on **initial conditions**. Introducing these we arrive at the final expression for the response

$$\boxed{u(t) = [u_0 + (v_0 + \omega_n u_0)\, t\,]\; e^{-\omega_n t}}$$

Typical pictures of the response are shown on the last slide together with the overdamped case. The critically damped case response curves are plotted in red.

Free Damped SDOF Oscillator: Overdamped Case

If $\xi > 1$ the characteristic equation has two distinct negative real roots. For convenience introduce the pseudo frequency

$$\omega^* = \omega_n \sqrt{\xi^2 - 1}$$

Then the solution may be compactly expressed in terms of the hyperbolic sine and cosine as

$$u(t) = e^{-\xi\,\omega_n t}(\,A_1\,\cosh\omega^* t + A_2\,\sinh\omega^* t\,)$$

in which coefficients A_1 and A_2 depend on **initial conditions**. Introducing these we arrive at the final expression for the response

$$\boxed{u(t) = e^{-\xi\,\omega_n t}\left[u_0 \cosh\omega^* t + \frac{v_0 + \xi\,\omega_n\,u_0}{\omega^*}\sinh\omega^* t\right]}$$

The effect of the damping factor ξ on the response of an overdamped SDOF oscillator may be observed in the pictures of the next slide.

Free Damped SDOF Oscillator: Critically-Damped & Overdamped Response Plots