ASEN 3112
Lecture 15:
Finite Element Method 3

Dr. Johnson
Prof. Hussein

Department of Aerospace Engineering Sciences
University of Colorado Boulder

Announcements

- Upcoming due dates
  - Homework 6 (energy methods): **tomorrow, Friday**, March 6
  - Homework 7 (FEM): Friday, March 13
  - Exam 2 (beams, energy methods, FEM): Tuesday, March 17
- Note that the FEM recitation is after HW7 and Exam 2

- My Office Hours (all in AERO 302)
  - Tuesday, March 10, 9:00 – 10:00 am
  - Thursday, March 12, 11:30 am – 12:30 pm
  - *Then by appointment*

Finite Element Methods Outline

- Last class (Ch. 16 & 17)
  – Member stiffness equations
- Today (Ch. 17 & 18)
  – Transforming from local to global coordinates
  – Understanding the global stiffness matrix
- **Thursday, March 5 (Ch. 18)**
  – **Assembling the global stiffness matrix**
  – **Applying boundary conditions to solve**
- Tuesday, March 10
  – Examples
  – Exam 2 review?

The Direct Stiffness Method

Breakdown

- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution

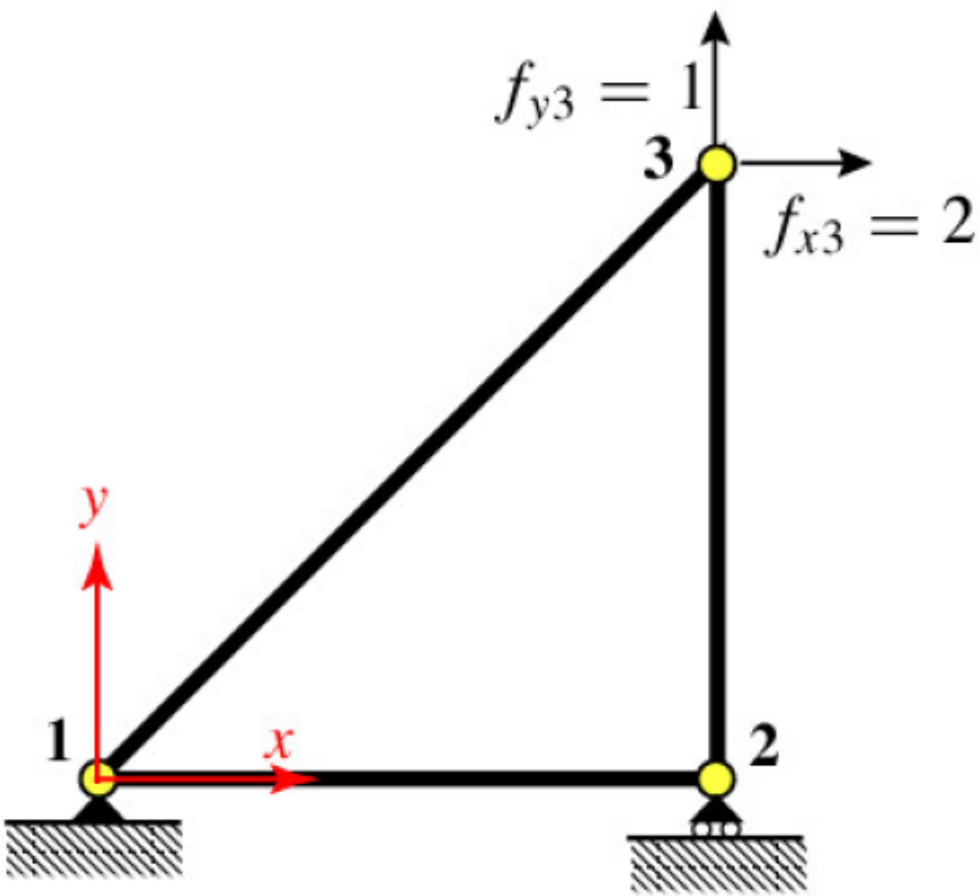
- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

conceptual steps

processing steps

post-processing steps

Our Example Truss



Member Stiffness Relations

- For each element, but not showing $^{(e)}$ superscript
- In local coordinates:

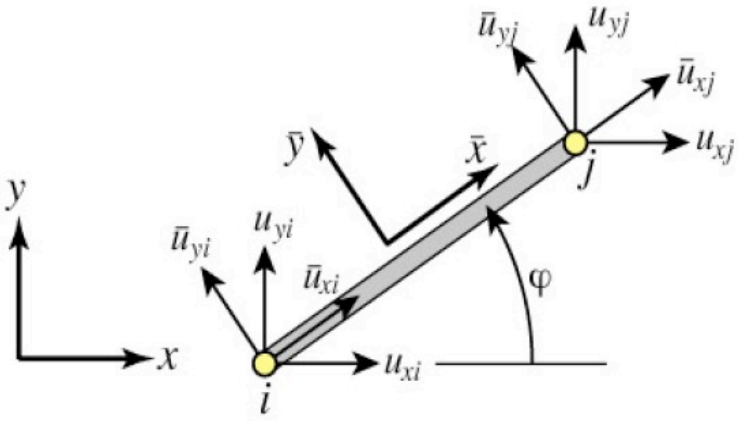
$$\begin{bmatrix}\bar{f}_{xi}\\ \bar{f}_{yi}\\ \bar{f}_{xj}\\ \bar{f}_{yj}\end{bmatrix} = \bar{\boldsymbol{K}}\begin{bmatrix}\bar{u}_{xi}\\ \bar{u}_{yi}\\ \bar{u}_{xj}\\ \bar{u}_{yj}\end{bmatrix} \quad where \quad \bar{\boldsymbol{K}} = \frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

- In global coordinates:

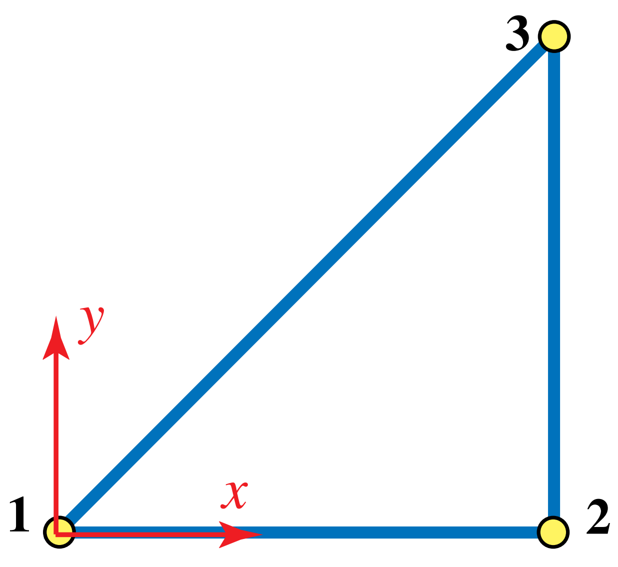
$$\begin{bmatrix}f_{xi}\\ f_{yi}\\ f_{xj}\\ f_{yj}\end{bmatrix} = \boldsymbol{K}\begin{bmatrix}u_{xi}\\ u_{yi}\\ u_{xj}\\ u_{yj}\end{bmatrix} \quad where \quad \boldsymbol{K} = \frac{EA}{L}\begin{bmatrix}c^2 & sc & -c^2 & -sc\\ sc & s^2 & -sc & -s^2\\ -c^2 & -sc & c^2 & sc\\ -sc & -s^2 & sc & s^2\end{bmatrix}$$

$$or \quad \boldsymbol{K} = \begin{bmatrix}[\hat{K}] & [-\hat{K}]\\ [-\hat{K}] & [\hat{K}]\end{bmatrix} \quad where \quad [\hat{K}] = \frac{EA}{L}\begin{bmatrix}c^2 & sc\\ sc & s^2\end{bmatrix}$$

Global Stiffness Relations

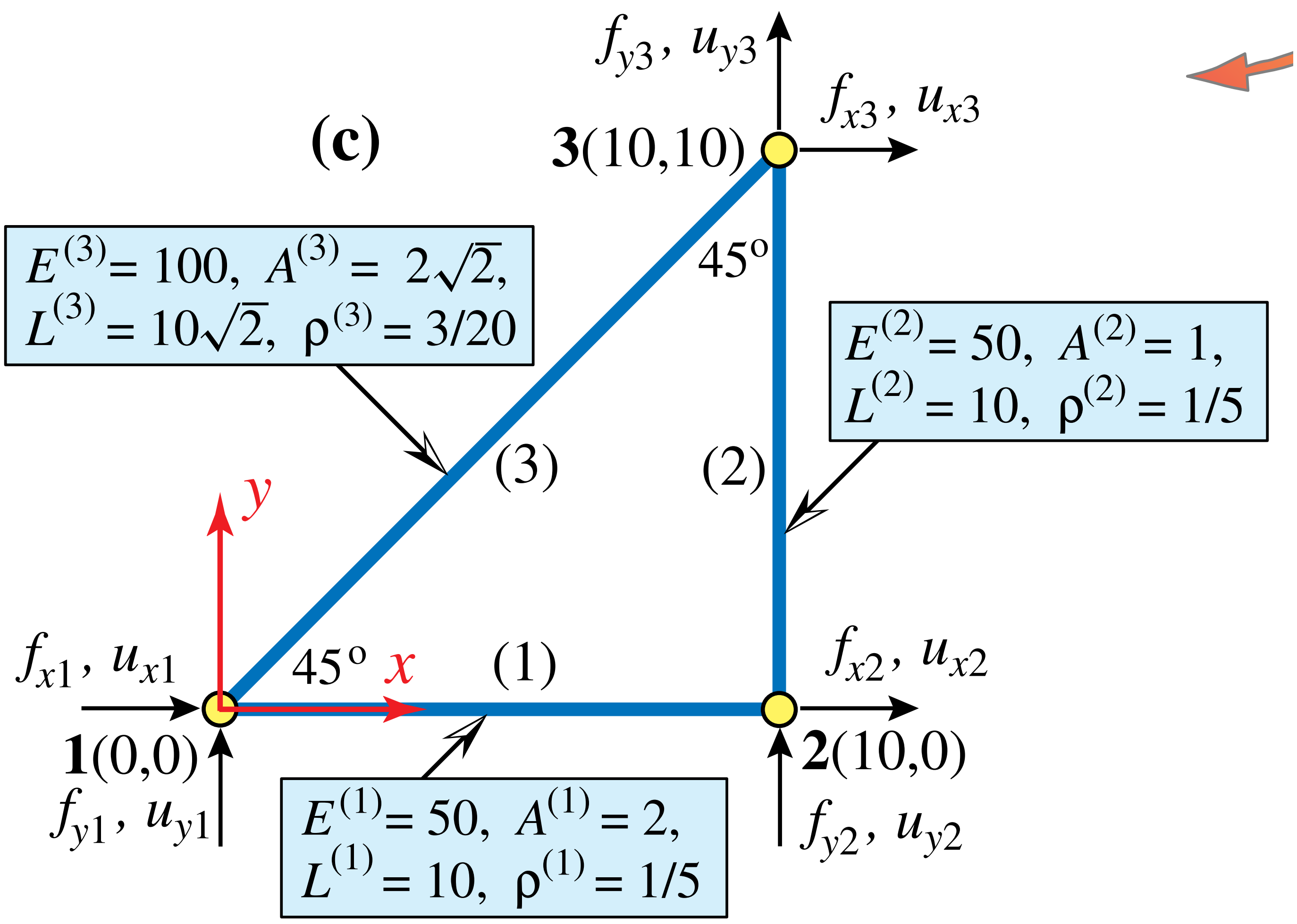


$$\boldsymbol{f^e} = \boldsymbol{K^e}\ \boldsymbol{u^e}$$



$$\boldsymbol{f} = \sum \boldsymbol{K}\ \boldsymbol{u}$$

Member Properties in Our Example Truss

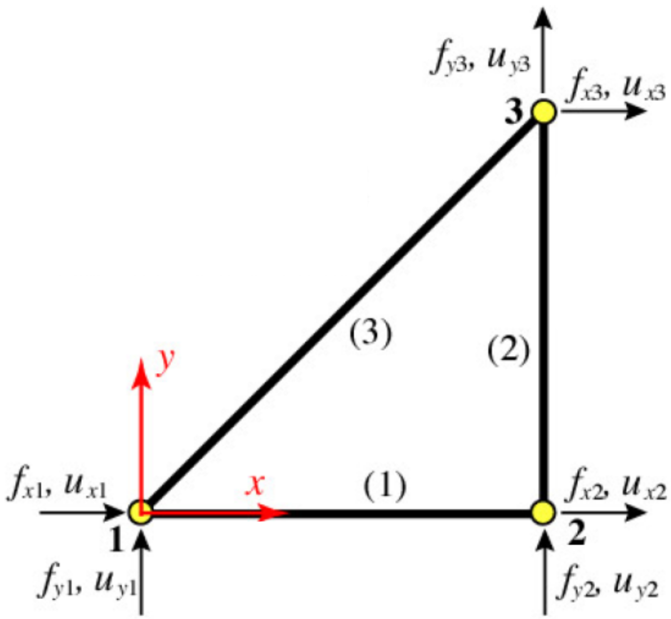


Augmented Stiffness Matrices

- Element stiffness equation augmented by "missing" DOFs to yield 6 equations
- **Only done for illustration purposes; step not explicitly needed**

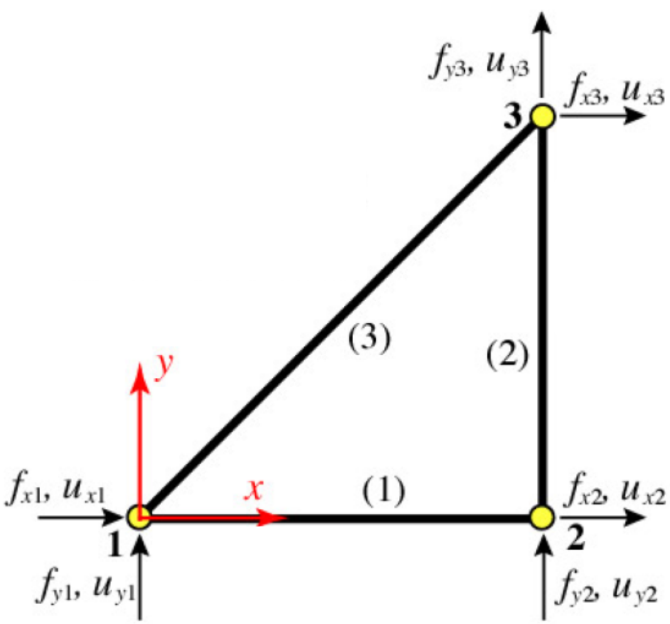
Bar 1:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$



Augmented Stiffness Matrices

- Element stiffness equation augmented by "missing" DOFs to yield 6 equations
- **Only done for illustration purposes; step not explicitly needed**



Bar 1:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Bar 2:

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Bar 3:

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Global Stiffness Equation

Bar 1:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

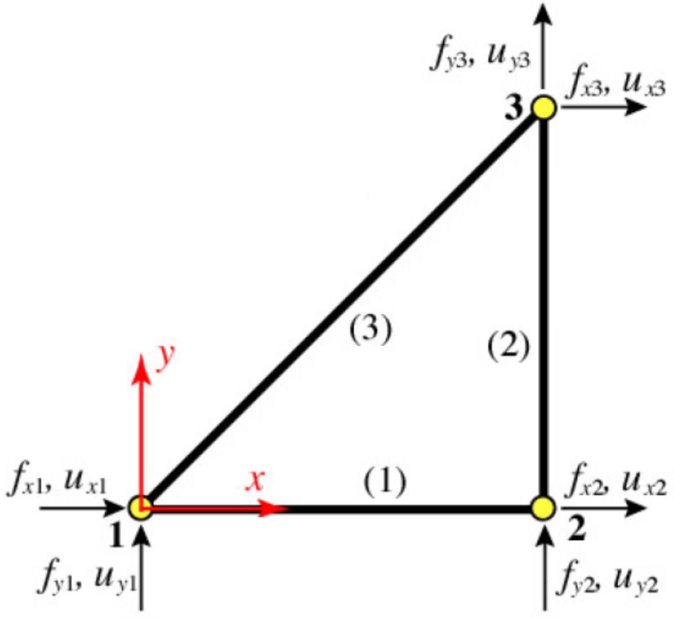
Bar 2:

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Bar 3:

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Assembly Process without Augmented Stiffness Matrices



Assembly Process

$$\boldsymbol{K} = \left[\begin{array}{c:c:c} \quad & \quad & \quad \\ \hdashline \quad & \quad & \quad \\ \hdashline \quad & \quad & \quad \end{array}\right]$$

The Direct Stiffness Method

Breakdown

- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution

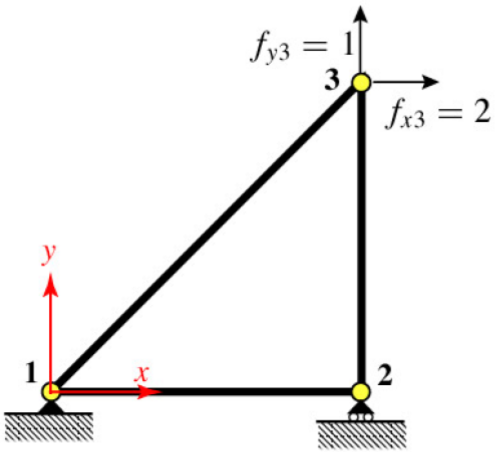
- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

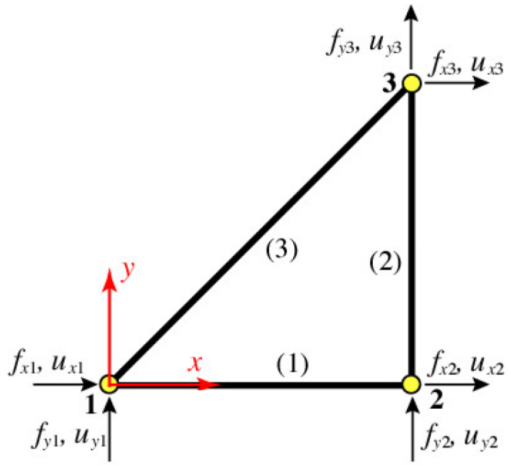
conceptual steps

processing steps

post-processing steps

Applying Boundary Conditions





Displacement BCs:

$$u_{x1} = u_{y1} = u_{y2} = 0$$

Force BCs:

$$f_{x2} = 0, \quad f_{x3} = 2, \quad f_{y3} = 1$$

Global stiffness equation:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

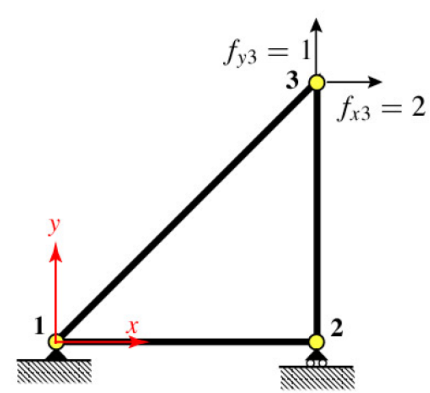
Reduced Global Stiffness Equation

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} = 0 \\ u_{y1} = 0 \\ u_{x2} = ? \\ u_{y2} = 0 \\ u_{x3} = ? \\ u_{y3} = ? \end{bmatrix} = \begin{bmatrix} f_{x1} = ? \\ f_{y1} = ? \\ f_{x2} = 0 \\ f_{y2} = ? \\ f_{x3} = 2 \\ f_{y3} = 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Reduced Global Stiffness Equation (System): $\tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}}$

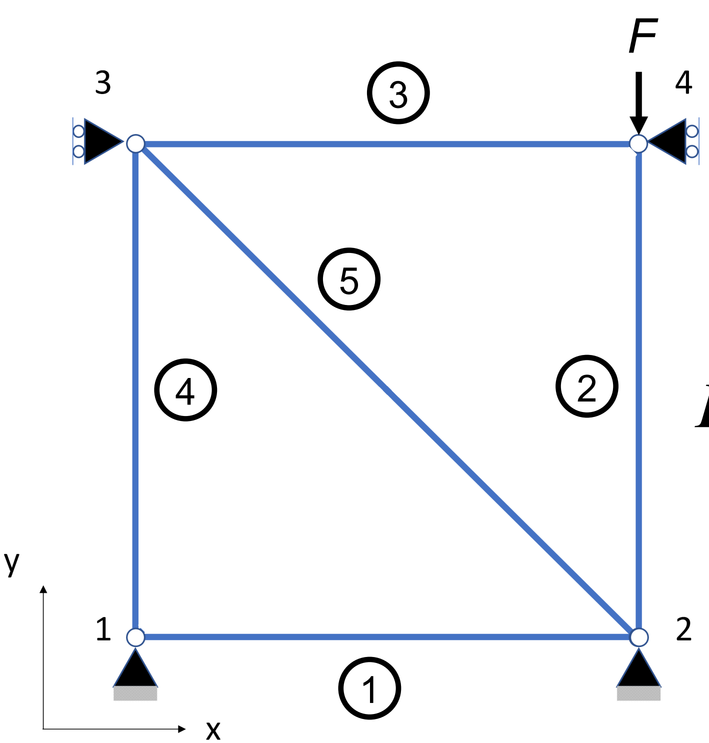
Solution: $$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix}$$



Clicker Question 1

- Consider the 6-bar truss structure shown below. Nodes 1 and 2 are pinned and Nodes 3 and 4 are on "rollers" allowing a motion in the y-direction. At Node 4 a load is applied in the vertical direction.

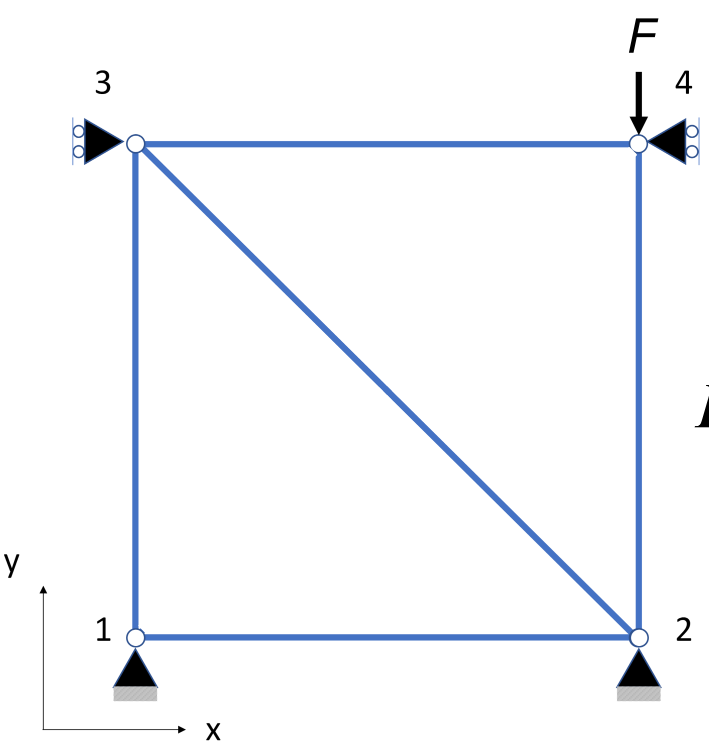
In which cell in the global stiffness matrix $\boldsymbol{K}$ would you put $\left[-\widehat{K}\right]$ for bar 5?



$$\boldsymbol{K} = \begin{bmatrix} a & & & b \\ & & c & \\ & & d & \\ & e & & \end{bmatrix}$$

Clicker Question 2

- Consider the 5-bar truss structure shown below. Nodes 1 and 2 are pinned and Nodes 3 and 4 are on "rollers" allowing a motion in the y-direction. At Node 4 a load is applied in the vertical direction.



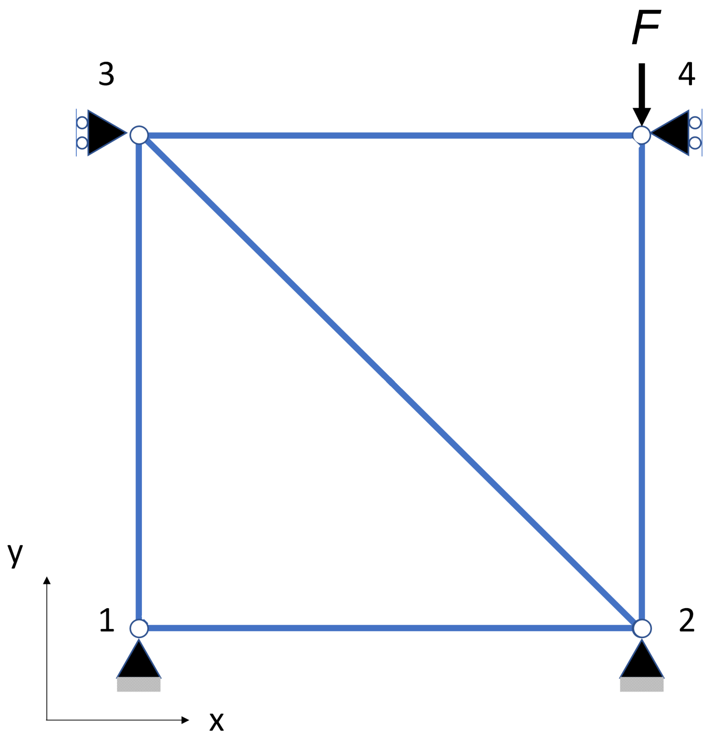
Which of the cells in the global stiffness matrix $\boldsymbol{K}$ are 0?

$$\boldsymbol{K} = \begin{bmatrix} a & & & b \\ & & & \\ & & d & \\ & e & & \end{bmatrix}$$

c. All four cells shown are non-zero.

Clicker Question 3

- Consider the 6-bar truss structure shown below. Nodes 1 and 2 are pinned and Nodes 3 and 4 are on "rollers" allowing a motion in the y-direction. At Node 4 a load is applied in the vertical direction.



What is the size of the Reduced Global Stiffness Equation $\widetilde{\boldsymbol{K}}$?

(a) 1 x 1

(b) 2 x 2

(c) 3 x 3

(d) more than 3 x 3

The Direct Stiffness Method

Breakdown
- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution
- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

conceptual steps

processing steps

post-processing steps

Recovery of Reaction Forces

Complete Displacement Solution:

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix} \qquad \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{bmatrix}$$

Displacement BCs:

$$\boxed{u_{x1} = u_{y1} = u_{y2} = 0}$$

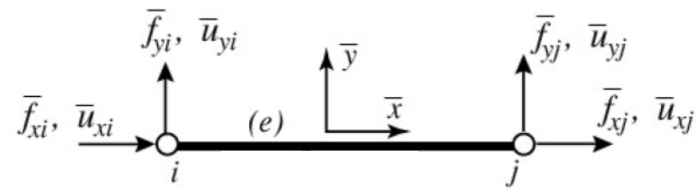
Complete Force Vector:

$$\mathbf{f} = \mathbf{K}\mathbf{u} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

support reactions

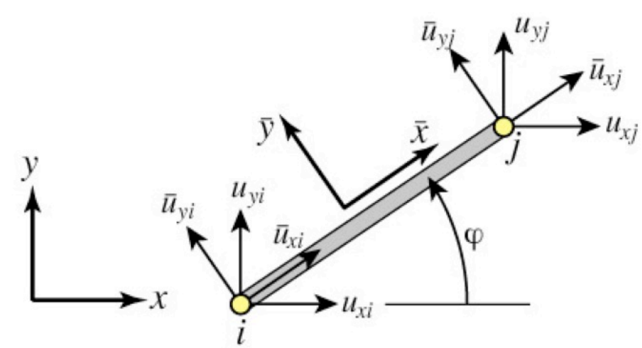
recovers applied external forces

Recovery of Internal Stress Through Displacements



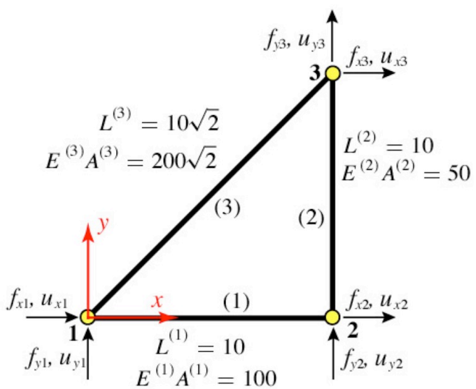
Transformation from global to local CS:

$$\underbrace{\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}}_{\bar{\mathbf{u}}^e} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\mathbf{T}^e} \underbrace{\begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}}_{\mathbf{u}^e}$$



Recovery of Internal Stress Through Displacements

Example: Bar 2



$$\begin{bmatrix} \bar{u}_{x2} \\ \bar{u}_{y2} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ & 0 & 0 \\ -\sin 90^\circ & \cos 90^\circ & 0 & 0 \\ 0 & 0 & \cos 90^\circ & \sin 90^\circ \\ 0 & 0 & -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{u}_{x2} \\ \bar{u}_{y2} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.4 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.2 \\ -0.4 \end{bmatrix}$$

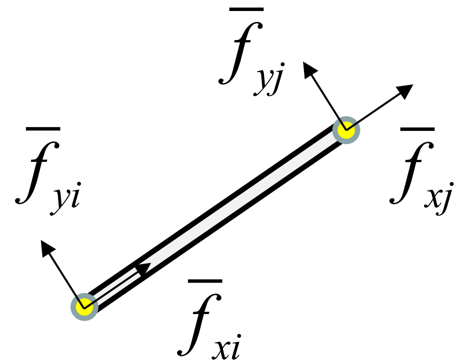
$$\Delta L^{(2)} = \bar{u}_{x3} - \bar{u}_{x2} = -0.2$$

$$F^{(2)} = \frac{(EA)^{(2)}}{L^{(2)}} \Delta L^{(2)} = -\frac{50}{10} 0.2 = -1$$

Recovery of Internal Stress Through Forces

Element Stiffness Equation in global CS:

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$$



Note: for bar element

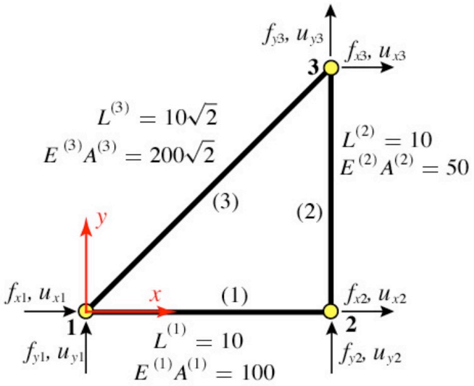
$$\bar{f}_{yi} = \bar{f}_{yj} = 0$$

Transformation from global to local CS:

$$\underbrace{\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}}_{\bar{\mathbf{f}}^e} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\mathbf{T}^e} \underbrace{\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix}}_{\mathbf{f}^e}$$

Recovery of Internal Stress Through Forces

Example: Bar 2 Using nodal solution for Bar 2:



$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e \qquad \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \\ -1.0 \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.0 \\ 0.4 \\ -0.2 \end{bmatrix}$$

$$F = -\bar{f}_{xi} = -\left(c\, f_{xi} + s\, f_{yi}\right) \quad or \quad F = \bar{f}_{xj} = \left(c\, f_{xj} + s\, f_{yj}\right)$$

for $\cos 90 = 0 \quad \sin 90 = 1$

$$F = -\bar{f}_{xi} = -\left(0 \cdot 0 + 1 \cdot 1\right) = -1 \quad or \quad F = \bar{f}_{xj} = \left(0 \cdot 0 + 1 \cdot -1\right) = -1$$

The Direct Stiffness Method

Breakdown

- **Disconnection**
- **Localization**
- **Member (Element) Formation**

Assembly & Solution

- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

conceptual steps

processing steps

post-processing steps