Recitation 6

ASEN 3112 – Fall 2020

Problem 1: Analysis of a 2-Bar Truss using Virtual Displacement and Virtual Force Method (~ 30 minutes)

First, consider the 2-bar truss shown in Figure 1. A force of magnitude P is acting in horizontal direction at joint C. Joints A and B are pinned. The dimensions and geometric properties are given in the figure. Both bars are made of an isotropic material with Young's modulus E. In your calculations, keep the length L, the load P, the cross-sectional area A_0, and the Young's modulus E as symbols. Assume a linear elastic response, infinitesimal strains, and small displacements and rotations.

- Compute the forces in the bars.
- Compute the elastic strain energy stored in the truss.
- Compute the displacement of joint C in **horizontal direction** by the **Conservation of Energy Principle**.
- Compute the displacement of joint C in **vertical direction** by the **Virtual Displacement Method**. "***Hint***: *You may use the result from Part 3 to solve this problem. This will reduce the number of unknowns in Part 4 from two to one. Thus, you only need one equation to solve for the remaining unknown. If you have not solved Part 3, assume that the horizontal displacement at joint C is u_C=(3PL)/(EA0).*"
- ~~Verify your answer for Part 4 by computing the **vertical displacement** of joint C with the **Virtual Force Method**.~~

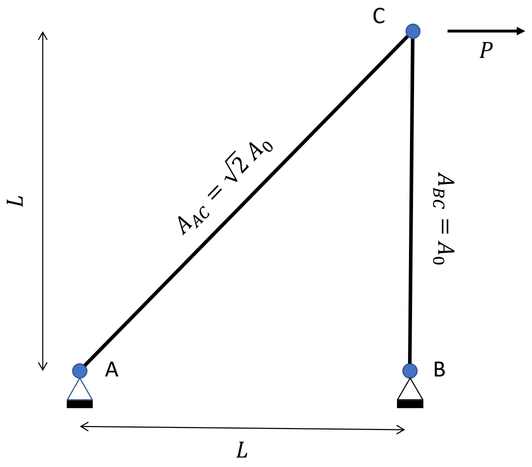


Figure 1: Two-bar truss

Now consider the three-bar truss in **Figure 2** which is constructed by adding the horizontal bar DC to the two-bar truss of Figure 1. Joint D is pinned. The additional bar is made of the same material as the other two bars. If the joint C has a vertical displacement of $v_C = -(PL)/(4EA_0)$, give the displacement of joint C in **horizontal direction**:

- Using the **Virtual Displacement Method**. "***Hint***: *As v_C is given, the number of unknowns is reduced from two to one. Thus, you only need one equation to solve for the remaining unknown.*"

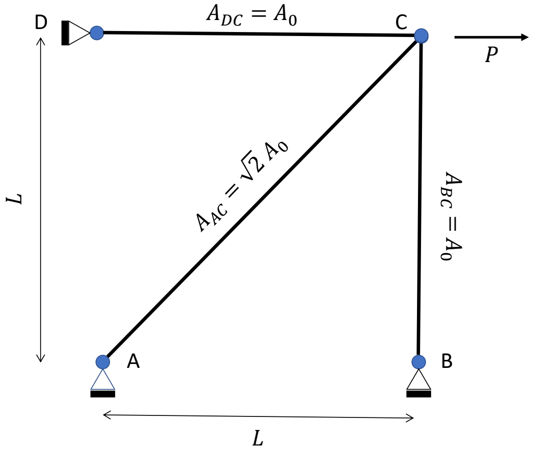
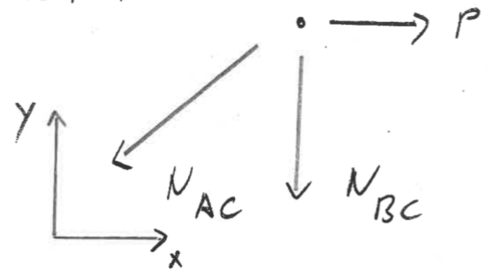


Figure 2: Three-bar truss

Solution 1

1. FBD:



$$\Sigma F_x = 0 \qquad P - \frac{\sqrt{2}}{2} N_{AC} = 0 \; ; \; \boxed{N_{AC} = \sqrt{2}\, P}$$

$$\Sigma F_y = 0 \qquad -\frac{\sqrt{2}}{2} N_{AC} - N_{BC} = 0 \quad \boxed{N_{BC} = -P}$$

2.

$$U_{truss} = \frac{1}{2} \sum_{i=1}^{2} \frac{N_i^2 L_i}{E A_i}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2} P^2 \cancel{\sqrt{2}} L}{E \cancel{\sqrt{2}} A_0} + \frac{1}{2} \frac{P^2 L}{E A_0}$$

$$= \frac{3}{2} \frac{P^2 L}{E A_0}$$

3.

$$W_c = U \qquad \frac{1}{2} P u_c = \frac{3}{2} \frac{P^2 L}{E A_0}$$

$$u_c = \frac{3PL}{EA_0}$$

4. Since we already know u_c, so we only need to construct one virtual work equation kinematics:

Kinematics:

$$\Delta L_{AC} = \frac{\sqrt{2}}{2}(u_c + v_c)$$

$$\Delta L_{BC} = v_c$$

$$\delta W_{ic}^{v_c} = \frac{E A_0 \sqrt{2}}{\sqrt{2} L} \frac{\sqrt{2}}{2}(u_c + v_c) \frac{\sqrt{2}}{2} \delta u_c$$

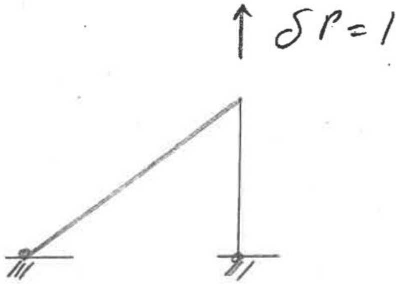
$$\delta W_c^{v_c} = P \delta u_c$$

$$\delta W_{ic}^{v_c} = \delta W_c^{v_c}$$

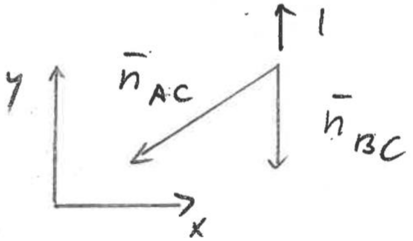
$$\frac{E A_0}{2L}(u_c + v_c) \cancel{\delta u_c} = P \delta u_c$$

$$v_c = \frac{2PL}{EA_0} - \frac{3PL}{EA_0} = \boxed{-\frac{PL}{EA_0}}$$

5.



FBD:



$\sum F_x = 0 \quad \bar{n}_{AC} = 0$

$\sum F_y = 0 \quad \bar{n}_{BC} = 1$

$$v_c = \sum_{i=1}^{2} \frac{N_i \cdot \bar{n}_i \, L_i}{E A_i} = \frac{-P \cdot 1 \cdot L}{E A_0}$$

$$v_c = \frac{-PL}{EA_0}$$ same result as in Part 4

6. Elongation in additional bar DC:

$$\Delta L_{DC} = u_c$$

$$\delta W_{ic}^{u_c} = \frac{E\sqrt{2}A_0}{\cancel{\sqrt{2}}L} \frac{\sqrt{2}}{2}(u_c + v_c)\frac{\sqrt{2}}{2}\delta u_c + \frac{EA_0}{L} u_c \, \delta u_c$$

$$\delta W_e = P \, \delta u_c$$

$$\frac{EA_0}{L}\left\{\left(1 + \frac{1}{2}\right)u_c + \frac{1}{2} v_c\right\}\delta u_c = P\,\delta u_c$$

$$3u_c + v_c = \frac{2PL}{EA_0}$$

For $v_c = \dfrac{-PL}{4EA_0}$

$$u_c = \frac{1}{3}\left(2 + \frac{1}{4}\right)\frac{PL}{EA_0} = \frac{3}{4}\frac{PL}{EA_0}$$

Problem 2: Analysis of Beam-Bar Structures using Virtual Displacement Method (~ 30 minutes)

The beam AB is fixed at point A and connected to the bars CD and BD with pins at C and B, respectively. The bars are pinned at point D. A uniform distributed load of magnitude $w = 10e4\ N/m$ acts along the length of the beam. The elastic modulus $E_{beam} = 7e10\ N/m^2$ area moment of inertia $I_{beam} = 1.0e\text{-}4\ m^4$ and $L = 1.5\ m$.

The bars are made of the same material as the beam (i.e. $E_{beam} = E_{bar}$) and have cross-sectional area $A_{bar} = 0.01\ m^2$. The angle $\alpha = 30$ degrees.

Assume that the beam deflection, i.e. displacement in y-direction is:

$v(x) = ax^3/L^3 + bx^2/L^2$

where the parameters a and b are to be determined.

- Determine the slope and deflection of the beam at point *F* using the virtual displacement method.

"***Hint***: *Feel free to use MATLAB to evaluate integrals, numerical expressions, and to solve for the unknown parameters a and b.*"

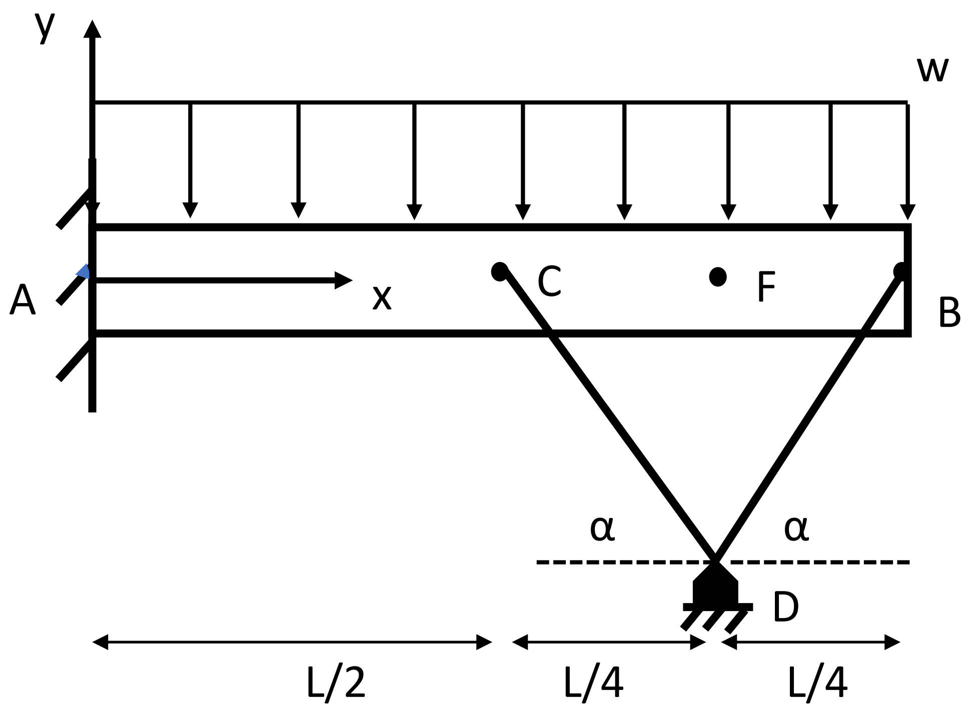


Figure 3: Beam-bar structure

Solution 2

Virtual Displacement Method: $\delta W_e = \delta W_{ie}$

where $\delta W_{ie} = \int_0^L EI\varkappa\, \delta\varkappa\, dx + \sum_{i=1}^{2} \frac{E_i A_i}{L_i} \Delta L_i\, \delta \Delta L_i$

Approximation: $v(x) = ax^3 + bx^2$

Note: parameters a and b are unknown.

We obtain two equations by considering the balance of virtual work for δv due to δa and δv due to δb.

$\varkappa$: $v''(x) = \frac{2b}{L_{beam}^2} + \frac{6ax}{L_{beam}^3}$

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ΔL_{CD}: $\Delta L_{CD} = v\left(x = \frac{L_{beam}}{2}\right) \sin\alpha$

$$= \left(\frac{a}{8} + \frac{b}{4}\right)\frac{1}{2}$$

ΔL_{Bb}: $\Delta L_{Bb} = v\left(x = L_{beam}\right) \sin\alpha$

$$= (a+b)\frac{1}{2}$$

Using the above expressions for $\varkappa$ and ΔL_i

we can evaluate δW_{ie} for δa and δb

$$\delta W_{ie}^{a} = \frac{6EI}{L_{beam}^{3}}(2a+b) + \frac{AE}{256 L_{bar}}(65a+66b)$$

$$\delta W_{ie}^{b} = \frac{2EI}{L_{beam}^{3}}(3a+2b) + \frac{AE}{128 L_{bar}}(33a+34b)$$

The external virtual work for δa and δb are

$$\delta W_{e}^{a} = \int -w\,\delta v^{a}\,dx = -\frac{L_{beam}\,w}{4}$$

$$\delta W_{e}^{b} = \int -w\,\delta v^{b}\,dx = -\frac{L_{beam}\,w}{3}$$

Using the numerical values given and computing $L_{bar} = \frac{L}{4\cos\alpha}$, we can express the work balance equations as follows:

$$10^{8}\cdot\begin{bmatrix} 4.3535 & 4.2922 \\ 4.2922 & 4.3770 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3750 \\ -5000 \end{bmatrix}$$

Solving the above equation yields:

$$a = 7.982\cdot 10^{-5} \qquad b = -8.970\cdot 10^{-5}$$

Computing $v(x = \frac{3}{4}L_{beam})$ and $v'(x=\frac{3}{4}L_{beam})$ for point F yields:

$$v_F = -1.678\cdot 10^{-5}\ m$$

$$v'_F = 1.006\cdot 10^{-7}$$

Problem 3: Analysis of Beam-Bar Structures using Virtual Force Method (~ 30 minutes)

The beam AB is pinned at point A and connected to the bar CB with a pin at point B.

A uniform distributed load of magnitude *w* acts along the length of the beam.

E and *I* are constant along the beam. The bar has an elastic modulus $E_{bar} = E$ and an area $A_{bar} = 100\, I$.

Using the Virtual Force Method

- Determine the deflection of the beam at point D.
- Determine the slope (rotation) of the beam at point D.

"**Hint**: Consider the internal virtual work in the beam due to bending moments and in the bar due to normal forces."

"**Hint**: Feel free to use MATLAB (symbolic) to evaluate integrals."

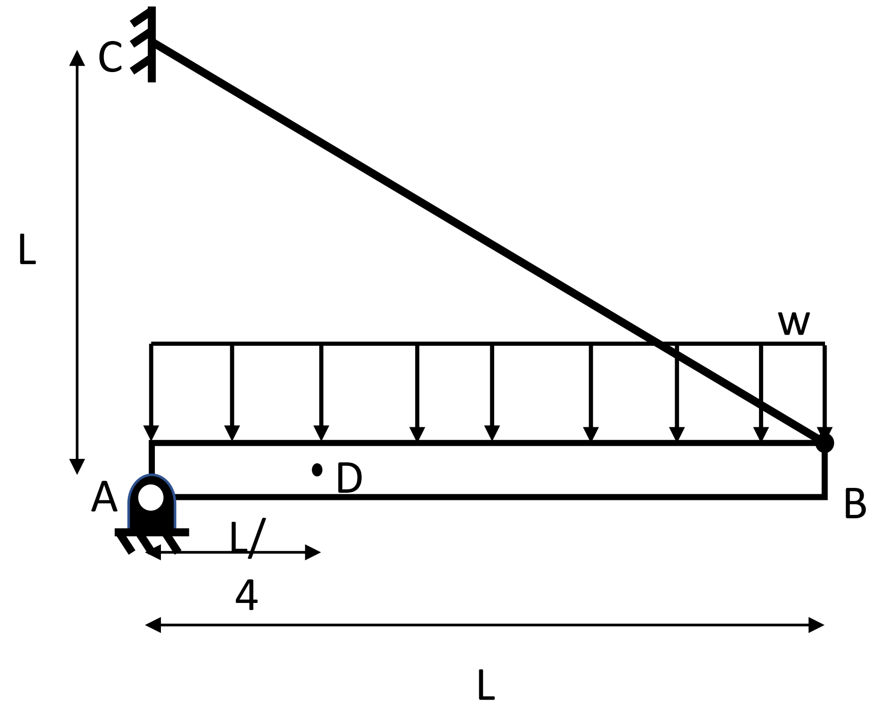


Figure 4: Beam-bar structure

Hint: The internal virtual work in a bar and a beam are:

$$\delta W^*_{ie,bar} = \frac{L\ N\ \bar{n}}{E\ A} \qquad \bar{n}: \text{ force due to dummy load}$$

$$\delta W^*_{ie,beam} = \int_L \frac{M\ \bar{m}}{E\ I}\, dx \qquad \bar{m}: \text{ moment due to dummy load}$$

Solution 3

Virtual Force Method: $\delta W_e^* = \delta W_{ie}^*$

where $\delta W_{ie}^* = \int_{L_{beam}} \frac{M \bar{m}}{EI} + \frac{N \bar{n} L_{bar}}{E_{bar} A_{bar}}$ (1)

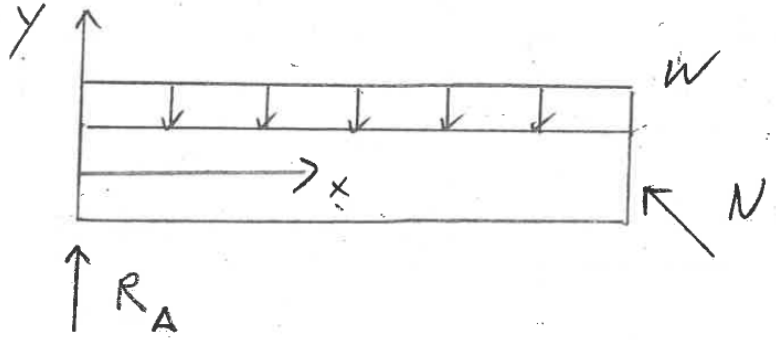
M, N: caused by real load

$\bar{m}, \bar{n}$: caused by dummy load

To compute deflection at point D, apply dummy force at D

To compute slope (rotation) at point D, apply dummy moment at D.

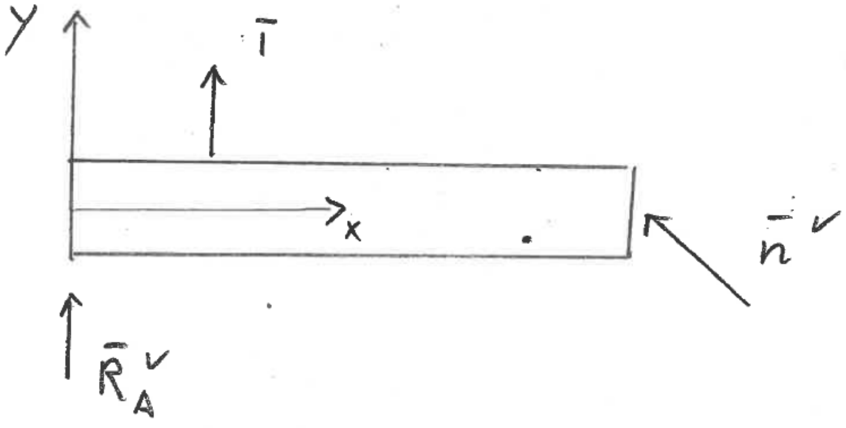
a) M and N due to real load



$$R_A = \frac{wL}{2} \qquad N = \frac{\sqrt{2}\, wL}{2}$$

$$M(x) = \frac{wL}{2}x - \frac{wx^2}{2} = \frac{w}{2}x(L-x)$$

b) $\bar{m}^v$ and $\bar{n}^v$ due to dummy force

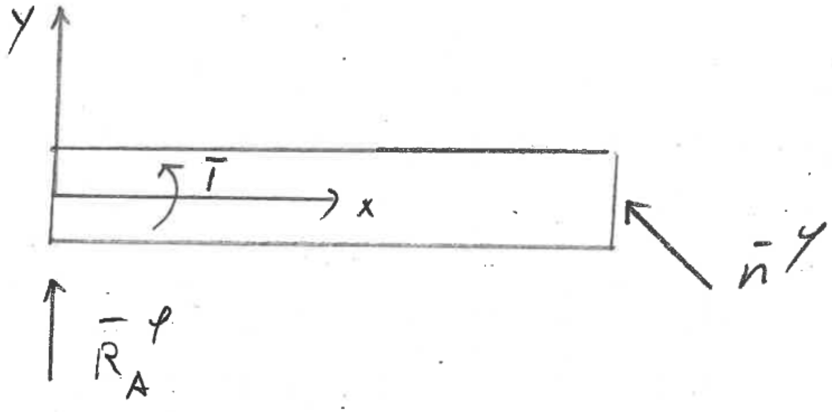


$$\bar{R}^v_A = -\frac{3}{4} \qquad \bar{n}^v = -\frac{\sqrt{2}}{4}$$

$$\bar{m}^v(x) = -\frac{3}{4}x \qquad 0 \le x \le \frac{L}{4}$$

$$\bar{m}^v(x) = \frac{x-L}{4} \qquad \frac{L}{4} \le x \le L$$

c) $\bar{m}^\varphi$ and $\bar{n}^\varphi$ due to dummy moment



$$R_A^\varphi = \frac{1}{L} \qquad \bar{n}^\varphi = -\frac{\sqrt{2}}{L}$$

$$\bar{m}^\varphi = \frac{1}{L}x \qquad 0 \le x \le \frac{L}{4}$$

$$\bar{m}^\varphi = \frac{1}{L}x - 1 \qquad \frac{L}{4} \le x \le L$$

d) Using the results of a) and b) to evaluate (1) yields

$$V_D = \frac{w}{EI} \int_0^{L/4} \frac{x}{2} (L-x) \left(-\frac{3}{4} x\right) dx$$

$$+ \frac{w}{EI} \int_{L/4}^{L} \frac{x}{2} (L-x) \left(\frac{x-L}{4}\right) dx$$

$$+ \frac{\sqrt{2}\, w L}{2 E_{bar} A_{bar}} \left(-\frac{\sqrt{2}}{4}\right) \sqrt{2} L$$

$$V_D = \frac{-wL^2 (475 L^2 + 128\sqrt{2})}{51200\, EI}$$

Using the results of a) and c) to evaluate (1) yields

$$V_p' = \frac{w}{EI} \int_0^{L/4} \frac{x}{2} (L-x) \frac{x}{L}\, dx$$

$$+ \frac{w}{EI} \int_{L/4}^{L} \frac{x}{2} (L-x) \left(\frac{x}{L} - 1\right) dx$$

$$+ \frac{\sqrt{2}\, w L}{2 E_{bar} A_{bar}} \left(-\frac{\sqrt{2}}{L}\right) \sqrt{2} L$$

$$V_D' = \frac{-wL (275 L^2 + 96\sqrt{2})}{9600\, EI}$$