Exercise 4.1

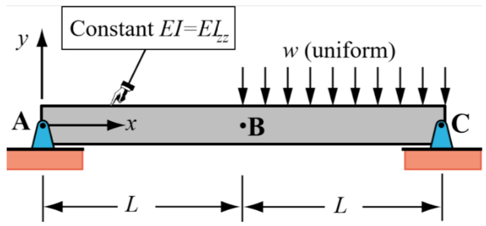

Figure 1: Simply supported beam with uniform load over right halfspan.

The beam of Figure 1 is simply supported (SS) at both ends. Find:(a)statically determinant or indeterminate (b) expression of the deflection function $v(x)$ using the second order method, and (c) deflection at midspan $x = L$; both in terms of the data L, E and I_{zz}.

Partial result: $vB = v(L) = \frac{-5wL^4}{48EI_{zz}}$. This is the same whether using the deflection function expression over AB, or over BC, thus providing a useful derivations check.

Exercise 4.1 Solution

(a) The beam is statically determinant because we have 2 unknowns (R_A and R_C) and 2 equations (ΣF_y and ΣM)
The reactions R_A and R_C can be obtained from a FBD analysis of the whole beam with the applied distributed load replaced by a point load wL acting downward at $x = 3L/2$. This gives

$$R_A = \frac{wL}{4}, \qquad R_C = \frac{3wL}{4} \tag{1}$$

The bending moment function $M_z(x)$ can be determined (using two Rigid Body Diagrams, one over AB and one over BC) to be

$$M_z = R_A x = \frac{wLx}{4} \text{ for } 0 \leq x \leq L, \qquad M_z = \frac{wLx}{4} - \frac{w}{2}(x-L)^2 \text{ for } L \leq x \leq 2L. \tag{2}$$

Quick checks: $M_{zB} = M_z(L) = wL^2/4$ from either expression, and $M_{zC} = M_z(2L) = 0$ from the second one. The deflection function will be denoted by $v_1(x)$ over AB, and $v(x) = v_2(x)$ over BC. Integrating the moments once yields

$$EIv_1' = \frac{1}{8}wLx^2 + C_1, \quad EIv_2'(x) = \frac{1}{8}wLx^2 - \frac{w}{6}(x-L)^3 + C_2 \tag{3}$$

The slope continuity condition at $x = L$ is $v_1'(L) = v_2'(L)$. Replacing in the foregoing equations gives $wL^3/8 + C_1 = wL^3/8 + C_2$ whence $C_1 = C_2$. Integrating once more

$$EIv_1 = \frac{1}{24}wLx^3 + C_1x + C_3, \quad EIv_2(x) = \frac{1}{24}wLx^3 - \frac{w}{24}(x-L)^4 + C_1x + C_4. \tag{4}$$

The deflection continuity condition at $x = L$ is $v_1(L) = v_2(L)$. Replacing in the above expressions gives $wL^4/24 + C_1L + C_2 = wL^4/24 + C_1L + C_4$ whence $C_3 = C_4$.

At this point we have only two constant of integration left: C_1 and C_3, and we can apply the simple support kinematic conditions $v_A = v_1(0) = 0$ and $v_C = v_2(2L) = 0$. Substituting $v_1(0)$ in $v_1(x)$ yields $C_1 = 0$ whereas substituting $v_2(2L) = 0$ into $v_2(x)$ yields $C_3 = -7wL^3/48$. Replacing these constants and moving EI to the RHS we obtain the beam deflection function as

$$\boxed{v(x) = v_1(x) = \frac{wLx}{48EI_{zz}}(2x^2 - 7L^2) \qquad\qquad \text{for } 0 \leq x \leq L,}$$

$$\boxed{v(x) = v_2(x) = \frac{wLx}{48EI_{zz}}(2x^2 - 7L^2) - \frac{w}{24EI_{zz}}(x-L)^4 \quad \text{for } L \leq x \leq 2L,}$$

Substituting $x = L$ we obtain the midspan deflection in terms of the data as

$$\boxed{v_B = v(L) = -\frac{5wL^4}{48EI_{zz}}}$$

This value should be the same whether one uses v_1 or v_2, thus providing a check that the deflection continuity condition was correctly applied.

Exercise 4.2

(see *Hints*)

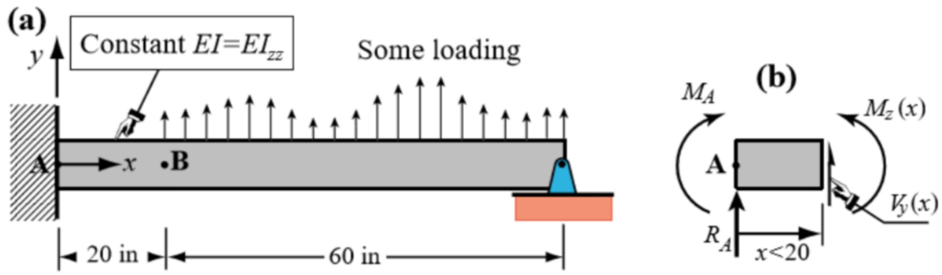

Figure 2: Beam for an "inverse" problem: given deflections, find forces.

The beam shown in Figure 2(a) is fixed at the left and SS at the right. The deflection curve over segment AB, which is free of applied loads, was experimentally found (upon data fitting) to be $v(x) = (20x^3 - 40x^2) \times\ 10^6$ in. The bending rigidity EI_{zz}, is $130 \times 10^6\ lb - in^2$. Determine the reaction force $R_A = -V_y(0)$ and the reaction moment $M_A = M_z(0)$.

Exercise 4.2 Solution

(a) The beam is statically indeterminate because we have 3 unknowns (R_A and M_A R_C) and 2 equations (ΣF_y and ΣM)
Make a FBD at $x < 20$ in., as pictured in Figure 2(b). By inspection

$$R_A = -V_y(x) \ \text{ for any } x \text{ in AB,} \quad M_A = M_z + R_A\, x \ \text{ whence } M_A = M_z(0). \tag{5}$$

The moment and transverse shear force can be obtained from the given $v(x)$ by differentiation:

$$M_z(x) = EI_{zz}\, v''(x) = EI_{zz}(120x - 80) \times 10^{-6} \text{ in-lb,} \tag{6}$$

$$V_y(x) = -M_z'(x) = -EI_{zz}v'''(x) = -120EI_{zz} \times 10^{-6} \text{ lb.} \tag{7}$$

The foregoing expressions are valid for $0 \le x \le 20$. To get reactions at the fixed end A, set $x = 0$ to get

$$\boxed{R_A = -V_y(0) = -120EI_{zz} \times 10^{-6} = -120 \times 130 \times 10^6 \times 10^{-6} = -15600 \ \text{ lb,}}$$

$$\boxed{M_A = M_z(0) = -80EI_{zz} \times 10^{-6} = -80 \times 130 \times 10^6 \times 10^{-6} = -10400 \ \text{ in-lb.}}$$

The negative signs for R_A and M_A mean that both of the computed reactions act in the opposite sense of those drawn in Figure 2(b).

Exercise 4.3

Continuous beams are multispan, seamless beam members with intermediate supports. This configuration is commonly used for reinforced concrete bridges. This Exercise deals with a two-span, prismatic continuous beam typical of a freeway overpass. See Figure 3.

Figure 3: Two-span freeway overpass.

The beam configuration is defined in Figure 4 below.

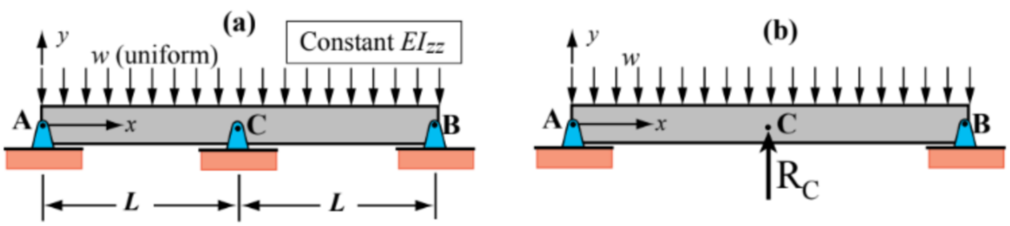

Figure 4: Two-span continuous beam model of freeway overpass.

The structure is statically indeterminate. The reaction R_C at the midspan is taken as a redundant force. The objective is to find R_C in terms of w and L as a prelude to drawing the moment diagram $M_z(x)$. The solution method is sketched below.

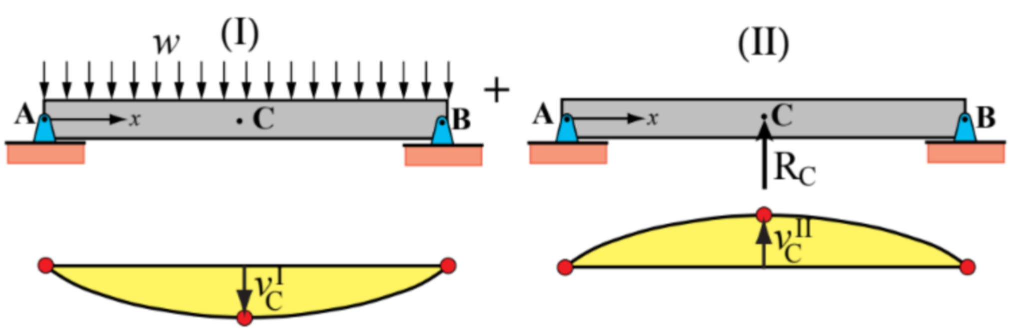

Figure 5: Two-span continuous beam: solution by superposition and table lookup.

Express the center deflection v_c^I and v_c^{II} in terms of the data. Solve for R_C as a function of w and L from $v_c^I + v_c^{II} = 0$. (Be careful with signs; R_C will not depend on EI_{zz}.) With this information, recover the end reactions $R_A = R_B$ from statics and sketch the bending moment diagram over ACB.
Partial answer: $M_{zC} = \frac{-wL^2}{8}$.

Exercise 4.3 Solution

(a) The beam is statically indeterminate because we have 3 unknowns (R_A and R_B R_C) and 2 equations (ΣF_y and ΣM)
Decompose the problem of Figure 4 into the two load cases (I) and (II) illustrated in Figure 5. The boundary conditions for both load cases are:

$$v(0) = 0, \qquad v'(L) = 0 \tag{8}$$

Distributed loading case: Due to symmetry $R_A = R_B$, therefore $R_A = wL$. Sum of the moments about point x gives $M_x = wLx - \frac{1}{2}wx^2$. Integrating this twice gives the slope and displacement functions:

$$EIv_1' = \frac{1}{2}wLx^2 - \frac{1}{6}wx^3 + C_1 \tag{9}$$

$$EIv_1 = \frac{1}{6}wLx^3 - \frac{1}{24}wx^4 + C_1x + C_2 \tag{10}$$

Applying the boundary conditions gives the constants of integration:

$$C_1 = -\frac{1}{3}wL^3, \qquad C_2 = 0 \tag{11}$$

Point load at C case: From symmetry $R_A = R_B$, therefore $R_A = -\frac{1}{2}R_C$. Sum of the moments about point x gives $M_x = -\frac{1}{2}R_Cx$. Integrating this twice gives the slope and displacement functions:

$$EIv_2' = -\frac{1}{4}R_Cx^2 + C_3 \tag{12}$$

$$EIv_2 = -\frac{1}{12}R_Cx^3 + C_3x + C_4 \tag{13}$$

Applying the boundary conditions gives the constants of integration:

$$C_3 = \frac{1}{4}R_CL^2, \qquad C_4 = 0 \tag{14}$$

Given that point C is simply supported we know that the sum of the deflections at point C from the distributed loading and R_C loading cases is zero, so $v_1(L) + v_2(L) = 0$. Therefore, R_C can be calculated as:

$$\boxed{R_C = \frac{5}{4}wL}$$

From statics we know that $R_A + R_B + R_C = wL$ and $R_A = R_B$. Therefore, we can find that:

$$\boxed{R_\mathrm{A} = R_\mathrm{B} = \frac{3}{8}wL}$$

Find Bending Moment and Draw Diagram. The bending moment can be found by superposition for the two cases. That is:

$$M_{xAC} = M_{x1} + M_{x2} \tag{15}$$

$$M_{xAC} = \frac{3}{8}wLx - \frac{1}{2}wx^2 \tag{16}$$

$$M_{xAC} = wx(\frac{3}{8}L - \frac{1}{2}x). \tag{17}$$

This varies parabolically, is zero at $x = 0$ and $x = 3L/4$, positive for $0 \leq x \leq 3L/4$, negative for $x > 3L/4$ and $M_{zC} = -\frac{1}{8}wL^2$ at beam midspan. The variation over CB is the mirror image of that over AC as may be expected from symmetry. The complete bending moment diagram is plotted in the Figure 6 below for $w = 1$ and $L = 1$.

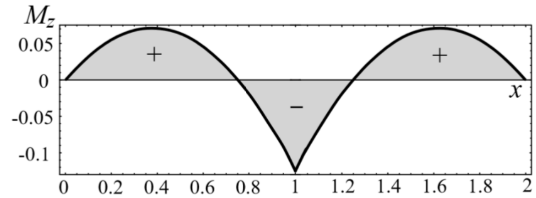

Figure 6: Bending moment diagram extending from $x = 0$ to $x = 2L$.

Exercise 4.4

A beam with length L is welded to two walls at A and B. The beam is loaded with a linearly-decreasing distributed load with magnitude w(0) = w 0 at x = 0 and w(L) = 0 at x = L. The beam has a constant bending rigidity EI.Discuss its determinancy and obtain the deflection curve by 4th order method.

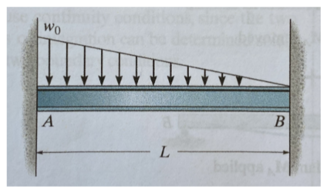

Figure 7: Beam fixed on both sides

Exercise 4.4 Solution

(a) The beam is statically indeterminant. The right wall is redundant. There are 4 support reactions(A_y , M_A, B_y and M_B) and 2 equilibrium equations (ΣF_y and ΣM)

(b)

$$|w(x)| = \frac{-w_0}{L}x + w_0 \tag{18}$$

and since $w(x)$ is down:

$$\begin{aligned} EIv''''(x) &= -\omega(x) \\ &= \frac{w_0}{L}x - w_0 \end{aligned} \tag{19}$$

(c) B.C are :

$$\begin{aligned} v(0) = 0, v(l) = 0 \\ v'(0) = 0, v'(l) = 0 \end{aligned} \tag{20}$$

(d) No matching conditions are necessary, because $v(x)$, $M(x)$ and $V(x)$ are all continuous (one piece).

(e)

$$\begin{aligned} &EIv''''(x) = \tfrac{w_0}{L}x - w_0 \\ &EIv'''(x) = \tfrac{w_0}{2L}x^2 - w_0x + C_1 \\ &EIv''(x) = \tfrac{w_0}{6L}x^3 - \tfrac{1}{2}w_0x^2 + C_1x + C_2 \\ &EIv'(x) = \tfrac{w_0}{24L}x^4 - \tfrac{1}{6}w_0x^3 + \tfrac{1}{2}C_{1x} + C_2x + C_3 \\ &EIv(x) = \tfrac{w_0}{120L}x^5 - \tfrac{1}{24}w_0x^4 + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4 \end{aligned} \tag{21}$$

Apply B.C to the system:

$$\begin{array}{r}1.\ v(0)=0\rightarrow c_4=0\\ 2.\ v'(0)=0\rightarrow C_3=0\\ 3.\ v(L)=0\rightarrow 0=\frac{\omega_0}{120}L^4-\frac{1}{24}\omega_0 L^4+\frac{1}{6}L^3C_1+\frac{1}{2}L^2C_2\\ 4\cdot v'(L)=0\rightarrow 0=\frac{w_0}{24}L^3-\frac{1}{6}\omega_0 L^3+\frac{1}{2}L^2C_1+LC_2\end{array} \tag{22}$$

Solve the 2 equations for C_1 and C_2:

$$\begin{array}{l}C_1=\frac{7}{20}w_0 L\\ C_2=\frac{-1}{20}w_0 L^2\end{array} \tag{23}$$

Therefore:

$$\begin{array}{l}v(x)=\frac{1}{EI}\left[\frac{\omega_0}{120L}x^5-\frac{1}{24}\omega_0 x^4+\frac{1}{6}\left(\frac{7}{20}\omega_0 L\right)x^3+\frac{1}{2}\left(-\frac{1}{20}\omega_0 L^2\right)x^2\right]\\ \boxed{v(x)=\frac{w_0}{120EI}\left[\frac{x^5}{L}-5x^4+7Lx^3-3L^2x^2\right]}\end{array} \tag{24}$$

Hints for exercises 4.1 and 4.2

4.1

Divide beam into two segments: AB and BC, and find the bending moment $M_z(x)$ for each one using the FBD. Then, integrate in x over each part twice. This will produce two deflection curves, say $v_1(x)$ over $0 \le x \le L$ and $v_2(x)$ over $L \le x \le 2L$, with a grand total of four integration constants. Find those using:

1. the two end simple support conditions: $v_1(0) = v_2(2L) = 0$
2. the two kinematic continuity conditions at midspan: $v_1'(L) = v_2'(L)$ and $v_1(L) = v_2(L)$

Final solution check: $v_B' = v_1'(L) = v_2'(L)$, $v_B = v_1(L) = v_2(L)$. The last one provides the answer to (b).

4.2

This is an "inverse" engineering problem: deflections are given (over AB), find forces. Link unknown reactions R_A and M_A to transverse shear force and bending moment through the FBD sketched in Figure 2(b). Differentiate $v(x)$ as necessary to get bending moment and shear forces, from which you can get the reactions. Interesting feature of this problem is that the requested results do not depend on which loading acts over $20in \le x \le 80in$, so that information is not necessary.