ASEN 3112

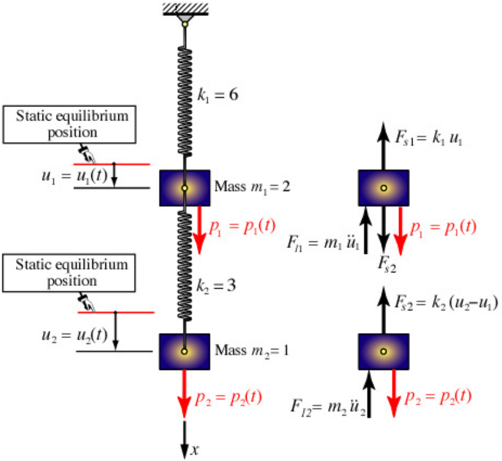
Spring 2020

Lecture 21

April 7, 2020

Modal Analysis of MDOF Forced Undamped Systems

2-DOF, Forced, Undamped Mass-Spring Example System



Matrix Equations of Motion of Example System

$$m_1 = 2, \quad m_2 = 1, \quad c_1 = c_2 = 0, \quad k_1 = 6, \quad k_2 = 3,$$

$$p_1 = p_1(t), \quad p_2 = p_2(t)$$

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}}$$

$$\boxed{\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{p}(t)}$$

Frequency & Modal Analysis Results of Unforced System Can be Reused

$$\omega_1^2 = \frac{3}{2} = 1.5 \qquad \omega_2^2 = 6$$

$$\phi_1 = \tfrac{1}{2}\,\mathbf{U}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \qquad \phi_2 = \mathbf{U}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\phi_1 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4088 \\ 0.8165 \end{bmatrix}, \qquad \phi_2 = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5773 \\ -0.5773 \end{bmatrix}$$

$$\mathbf{\Phi} = [\,\phi_1 \quad \phi_2\,] = \begin{bmatrix} \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{2}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0.4082 & 0.5773 \\ 0.8165 & -0.5773 \end{bmatrix}$$

Modal Forces (a.k.a. Generalized Forces)

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{p}(t) \qquad \mathbf{u}(t) = \mathbf{\Phi}\,\boldsymbol{\eta}(t)$$

$$\mathbf{\Phi}^T\mathbf{M}\,\mathbf{\Phi}\,\ddot{\boldsymbol{\eta}}(t) + \mathbf{\Phi}^T\mathbf{K}\,\mathbf{\Phi}\,\boldsymbol{\eta}(t) = \mathbf{\Phi}^T\mathbf{p}(t)$$

$$\boxed{\mathbf{f}(t) = \mathbf{\Phi}^T\,\mathbf{p}(t)}$$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \mathbf{\Phi}^T\mathbf{p}(t) = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \frac{p_1(t) + 2\,p_2(t)}{\sqrt{6}} \\ \frac{p_1(t) - p_2(t)}{\sqrt{3}} \end{bmatrix}$$

Modal Equations of Motion

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1(t) \\ \ddot{\eta}_2(t) \end{bmatrix} + \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\boxed{\begin{aligned} \ddot{\eta}_1(t) + (3/2)\,\eta_1(t) &= f_1(t) = \big(p_1(t) + 2\,p_2(t)\big)/\sqrt{6} \\ \ddot{\eta}_2(t) + 6\,\eta_2(t) &= f_2(t) = \big(p_1(t) - p_2(t)\big)/\sqrt{3} \end{aligned}}$$

Modal Equations of Motion

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1(t) \\ \ddot{\eta}_2(t) \end{bmatrix} + \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\boxed{\begin{aligned} \ddot{\eta}_1(t) + (3/2)\,\eta_1(t) &= f_1(t) = \big(p_1(t) + 2\,p_2(t)\big)/\sqrt{6} \\ \ddot{\eta}_2(t) + 6\,\eta_2(t) &= f_2(t) = \big(p_1(t) - p_2(t)\big)/\sqrt{3} \end{aligned}}$$

Forced Response Example

Force excitation

IC: "cooked up" to get all responses proportional to $\cos \Omega t$

$$\mathbf{p}(t) = \begin{bmatrix} 0 \\ F_2 \cos \Omega t \end{bmatrix}, \qquad \mathbf{u}_0 = \begin{bmatrix} * \\ * \end{bmatrix}, \qquad \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\ddot{\eta}_1(t) + (3/2)\,\eta_1(t) = (2/\sqrt{6})\,F_2 \cos \Omega t$$

$$\ddot{\eta}_2(t) + 6\,\eta_2(t) = -(1/\sqrt{3})\,F_2 \cos \Omega t$$

$$\eta_1(t) = \frac{(2/\sqrt{6})\,F_2}{\omega_1^2 - \Omega^2} \cos \Omega t \qquad \eta_2(t) = -\frac{(1/\sqrt{3})\,F_2}{\omega_2^2 - \Omega^2} \cos \Omega t$$

"Cooked up" steady state (particular) solutions

Physical Response Obtained Through Modal Matrix

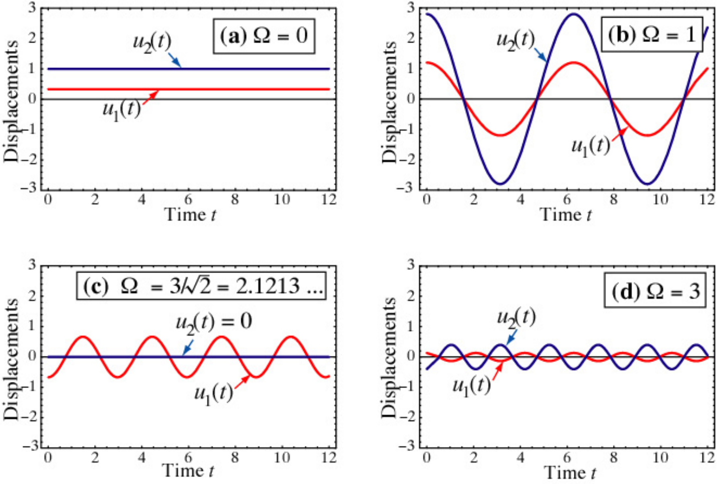
$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$$

$$= \ \tfrac{1}{3} F_2 \cos \Omega t \begin{bmatrix} \dfrac{1}{\omega_1^2 - \Omega^2} - \dfrac{1}{\omega_2^2 - \Omega^2} \\ \dfrac{2}{\omega_1^2 - \Omega^2} + \dfrac{1}{\omega_2^2 - \Omega^2} \end{bmatrix}$$

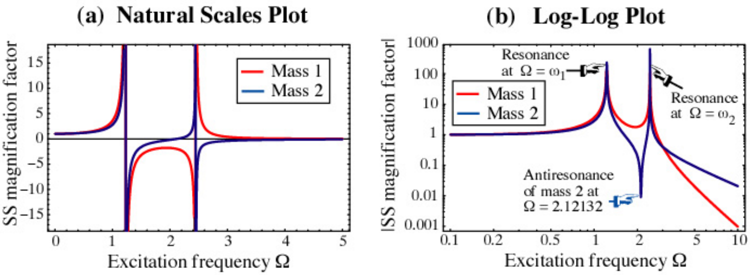
$$= \frac{F_2 \cos \Omega t}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} \begin{bmatrix} \omega_2^2 - \omega_1^2 \\ \omega_1^2 + 2\omega_2^2 - 3\Omega^2 \end{bmatrix}$$

$$\boxed{= \frac{F_2 \cos \Omega t}{(6 - \Omega^2)(3 - 2\Omega^2)} \begin{bmatrix} 6 \\ 2(9 - 2\Omega^2) \end{bmatrix}}$$

Physical Responses for Four Values of Force Excitation Frequency Ω

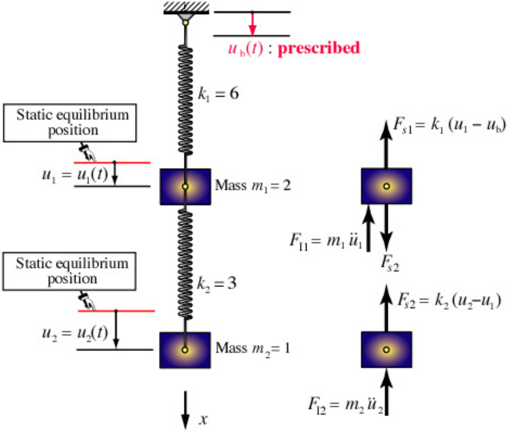


Frequency Response Plots

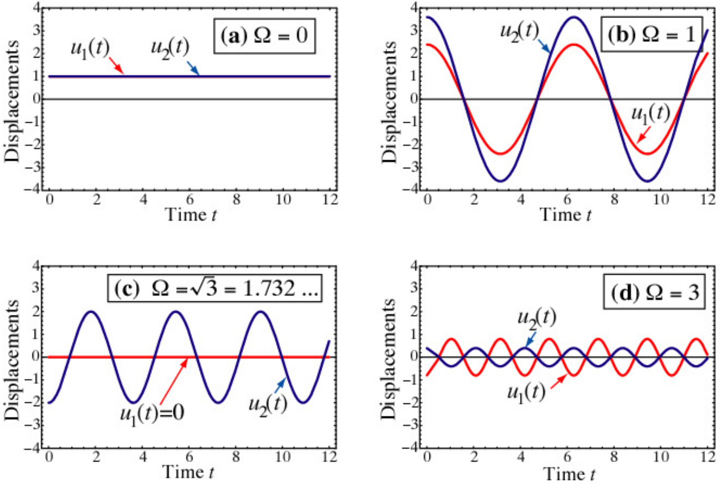


Log-Log version pinpoints resonances and antiresonances better

Second Example in Lecture 24 Notes: Base-Motion Excitation

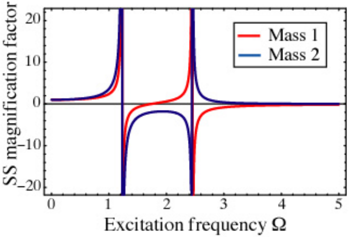


Physical Responses for Four Values of Base Excitation Frequency Ω

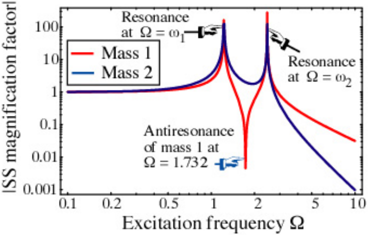


Frequency Response Plots

(a) Natural Scales Plot



(b) Log-Log Plot



Log-Log version pinpoints resonances and antiresonances better

Passive Vibration Isolation Devices

Passive Isolation Type	Applications	Typical Frequency Range
Air isolators	Large industrial equipment,, optical instruments	1.5-3 Hz
Spring & spring dampers	Heavy loads, pumps, compressors	3-9 Hz
Elastometer or cork pads	Systems/devices experiencing high frequency noise	3-40 Hz
Elastomer mounts	Machinery, instruments, vehicles, aircraft	1-20 Hz
Negative stiffness isolators	Sensitive instruments, optics & laser systems, cryogenics	0.15-2.5 Hz
Wire rope isolators	Machinery, instruments, vehicles, aircraft	10-40 Hz
Base isolators	Buildings, bridges	seismic frequencies
Tuned mass dampers	Buildings, aerospace	any frequency, but usually low