Recitation 7

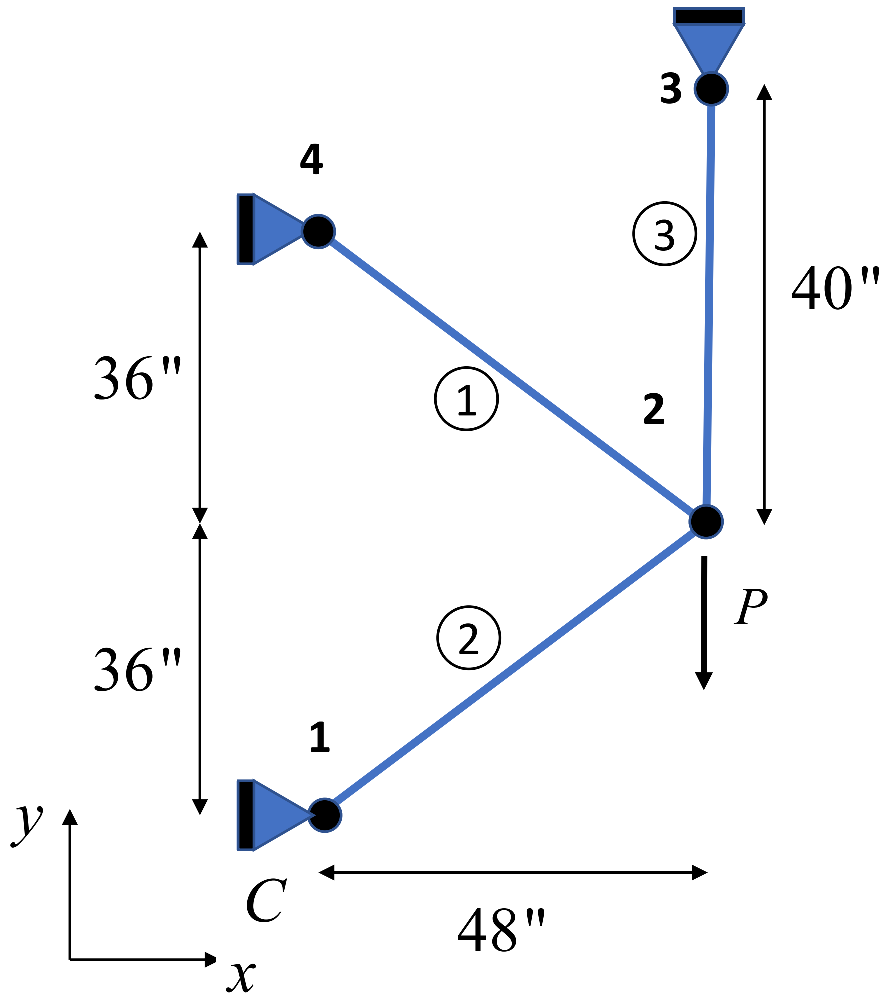
ASEN 3112 – Fall 2020

Note:

We will do Problems 1 and 2 in lecture on Tuesday, March 10.

Problem 1.

The truss is pinned at joints 1, 3, and 4. A load **P** is acting at node 2 downward in vertical direction.



$A_1 = 0.15\ in^2$

$A_2 = 0.25\ in^2$

$A_3 = 0.10\ in^2$

P = 3000lb

E = 30000ksi

Use the FEM method to answer the following questions:

a) Determine the size of the following matrices:
   1. The **global** nodal displacement vector, $\boldsymbol{u}$
   2. The **global** external nodal force vector , $\boldsymbol{f}$
   3. The **global** stiffness matrix , $\boldsymbol{K}$
   4. One **member**'s nodal displacement vector, $\boldsymbol{u}$
   5. One **member**'s external nodal force vector
   6. One **member**'s stiffness matrix

b) Write the **global** vectors of nodal displacements and external nodal forces as variables.

c) Use the boundary conditions to determine which components of the displacement vector and force vector are known and unknown. What are these known values?

d) Decompose the truss into bar elements. (Draw the $\bar{x}$ axis for each bar.)

e) For each bar, give (with values) the **member** stiffness equation in **global coordinates**.

a. Global matrix sizes are functions of $2J = 2(4) = 8$

$\underline{u} = 8 \times 1$, $\underline{f} = 8 \times 1$

$\underline{K} = 8 \times 8$

Local matrix sizes are <u>always</u> functions of 4

$\bar{\underline{u}} = 4 \times 1$, $\bar{\underline{f}} = 4 \times 1$

$\bar{\underline{K}} = 4 \times 4$

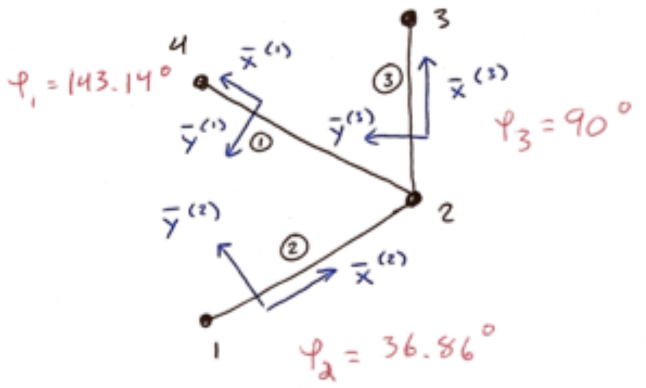
b.

$$\underline{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix}$$

c.

	Known	Unknown
u	$u_{x1}, u_{y1}, u_{x3}, u_{y3}, u_{x4}, u_{y4} = 0$	u_{x2}, u_{y2}
f	$f_{x2} = 0$, $f_{y2} = -P$	f_{x1}, f_{y1}, f_{x3}, f_{y3}, f_{x4}, f_{y4}

d. $\bar{x}$ goes from $i \rightarrow j$ where $i < j$



Solution 1.

e. In general, $\underline{f}^e = \underline{K}^e \underline{u}^e$

$$\underline{K}^e = \begin{bmatrix} [\hat{K}^e] & [-\hat{K}^e] \\ [-\hat{K}^e] & [\hat{K}^e] \end{bmatrix} \quad \text{where} \quad [\hat{K}^e] = \frac{E^e A^e}{L^e} \begin{bmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{bmatrix}$$

Bar 1: $[\hat{K}^{(1)}] = 7.5 \times 10^4 \begin{bmatrix} 0.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix}$

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 7.5 \times 10^4 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ f_{x4} \\ f_{y4} \end{bmatrix}$$

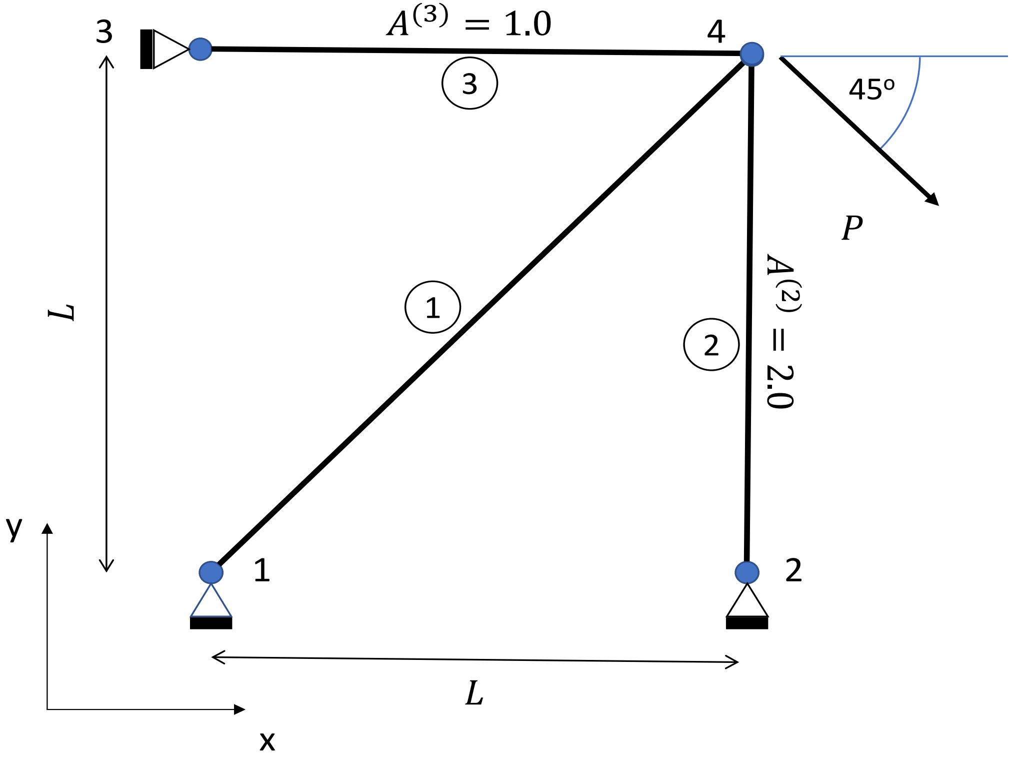
Bar 2. $[\hat{K}^{(2)}] = 1.25 \times 10^5 \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = 1.25 \times 10^5 \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

Bar 3 $[\hat{K}^{(3)}] = 7.5 \times 10^4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = 7.5 \times 10^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Problem 2.



Member stiffness equation for Bar 1:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x4}^{(1)} \\ f_{y4}^{(1)} \end{bmatrix} = 20 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x4}^{(1)} \\ u_{y4}^{(1)} \end{bmatrix}$$

The figure to the right shows a three-bar truss subject to a load $P = 5.0\sqrt{2}$ at Joint 4. Joints 1, 2, and 3 are pinned. Joint 4 is free to move in x- and y-direction. Bar 1 has a cross-sectional area $A^{(1)} = 1.0$ and a Young's modulus of $E = 282.84$ such that $(EA/L)^{(1)} = 40.0$. Bars 2 and 3 are made of the same material with a Young's modulus of $E = 100.0$ and have the same length with $L = 5.0$. Bar 2 has a cross-sectional area $A^{(2)} = 2.0$, and Bar 3 a cross-sectional area $A^{(3)} = 1.0$. All problem parameters are given in consistent units.

Use the FEM method to answer the following questions:

a) Use the boundary conditions to determine which components of the displacement vector and force vector are known and unknown. What are these known values?

b) Give the **member** stiffness equations for **Bar 2** and **Bar 3** in **global coordinates**.

c) On the next page, show how the **global** stiffness matrix is assembled from the $[\widehat{K}]$ matrices for each bar.

d) Show the reduced **global** stiffness equations and solve for the unknown displacements.

e) Compute the **axial force** in **Bar 2**. Specify whether the bar is in tension or compression.

f) Compute the **axial force** in **Bar 1**. Specify whether the bar is in tension or compression.

Solution 2.

a.

	Known	Unknown
u	u_{x1}, u_{y1}, u_{x2}, u_{y2}, $u_{x3}, u_{y3} = 0$	u_{x4}, u_{y4}
F	$f_{x4} = P\cos 45$, $f_{y4} = P\sin 45$	f_{x1}, f_{y1}, f_{x2}, f_{y2}, f_{x3}, f_{y3}

b. Bar 2 ($\varphi = 90°$)

$$\begin{bmatrix} u_{x2} \\ f_{y2} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 40 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Bar 3 ($\varphi = 0°$)

$$\begin{bmatrix} f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 20 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

c.

$$\underline{K} = \begin{array}{c} \left[\begin{array}{c|c|c|c} [\hat{K}^{(1)}] & [0] & [0] & [-\hat{K}^{(1)}] \\ \hline [0] & [\hat{K}^{(2)}] & [0] & [-\hat{K}^{(2)}] \\ \hline [0] & [0] & [\hat{K}^{(3)}] & [-\hat{K}^{(3)}] \\ \hline [-\hat{K}^{(1)}] & [-\hat{K}^{(2)}] & [-\hat{K}^{(3)}] & [\hat{K}^{(1)}] + [\hat{K}^{(2)}] + [\hat{K}^{(3)}] \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array}$$

d.

$$\begin{array}{r} \\ \\ \\ \\ \\ \\ \\ 5 = \\ -5 = \end{array} \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \text{cell} \\ & & (4,4) \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = 0 \\ = 0 \\ = 0 \\ \\ \\ \end{array}$$

$$\text{cell }(4,4) = [\hat{K}^{(1)}] + [\hat{K}^{(2)}] + [\hat{K}^{(3)}]$$

$$\begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 40 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{f} \quad \underset{\sim}{K} \quad \underset{\sim}{u}$$

$$\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 40 & 20 \\ 20 & 60 \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\rightarrow u_{x4} = \frac{1}{5}, \quad u_{y4} = \frac{-3}{20}$$

e. Axial force in Bar 2

By displacements (because we don't know reaction forces)

$$F^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}}\left(\bar{u}_{x4} - \bar{u}_{x2}\right) \qquad \varphi = 90^\circ$$

$$\bar{u}_{x4} = \cos\varphi\, u_{x4} + \sin\varphi\, u_{y4} \qquad \bar{u}_{x2} = \cos\varphi\, \underbrace{\cancel{u_{x2}}}_{=0} + \sin\varphi\, \underbrace{\cancel{u_{y2}}}_{=0}$$

$$\bar{u}_{x4} = \underbrace{\cancel{\cos 90}}_{=0}\left(\frac{1}{5}\right) + \sin 90\left(\frac{-3}{20}\right) \qquad \bar{u}_{x2} = 0$$

$$\bar{u}_{x4} = \frac{-3}{20}$$

$$F^{(2)} = \frac{100(2)}{5}\left(\frac{-3}{20}\right) = -6 \quad \text{(compression)}$$

f. Axial force in Bar 1

By displacements

$$F^{(1)} = \frac{E^{(1)}A^{(1)}}{L^{(1)}}\left(\bar{u}_{x4} - \bar{u}_{x1}\right) \qquad \varphi = 45^\circ$$

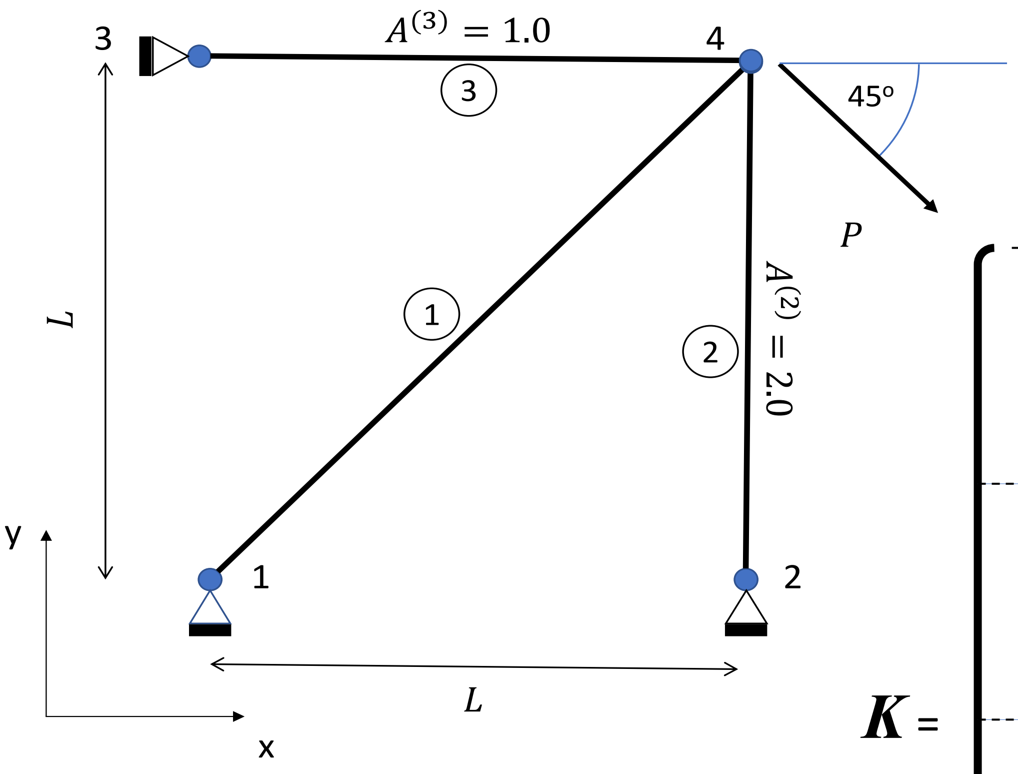
$$\bar{u}_{x4} = \cos\varphi\, u_{x4} + \sin\varphi\, u_{y4} \qquad \bar{u}_{x1} = \cos\varphi\, \underbrace{\cancel{u_{x1}}}_{=0} + \sin\varphi\, \underbrace{\cancel{u_{y1}}}_{=0}$$

$$\bar{u}_{x4} = \frac{1}{\sqrt{2}}\left(\frac{1}{5}\right) + \frac{1}{\sqrt{2}}\left(\frac{-3}{20}\right) \qquad \bar{u}_{x1} = 0$$

$$\bar{u}_{x4} = \frac{1}{20\sqrt{2}}$$

$$F^{(1)} = 40\left(\frac{1}{20\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \text{(tension)}$$

Problem 2.

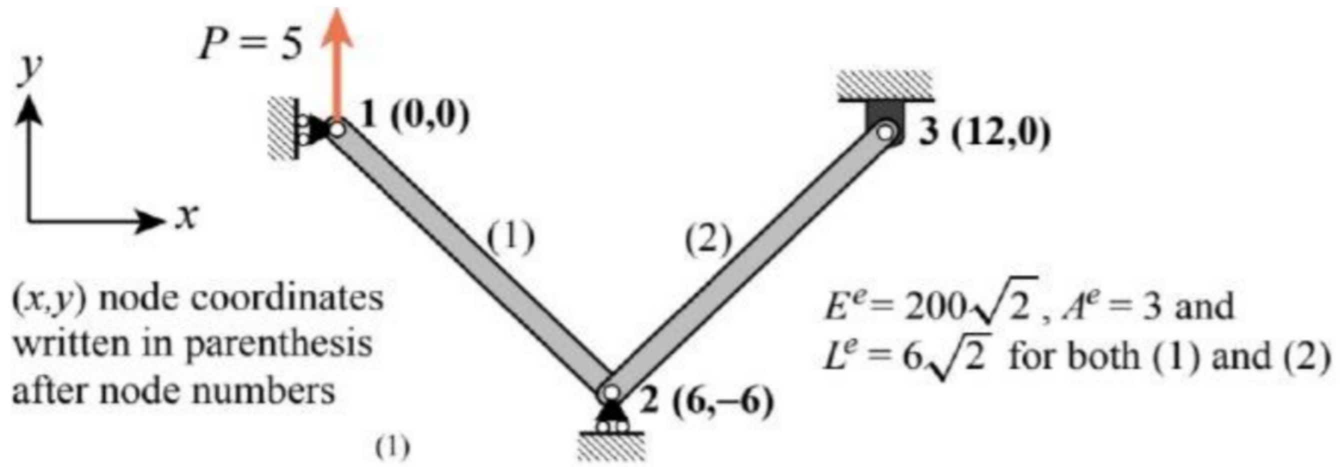


c) show how the **global** stiffness matrix is assembled from the $[\hat{K}]$ matrices for each bar.

$$\boldsymbol{K} = \left[\begin{array}{c|c|c|c} \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \end{array}\right]$$

Problem 3.

Figure shows a pin-jointed plane truss discretized with 2 elements and 3 nodes. Node 3 is fixed whereas 1 and 2 move over rollers as shown. The only nonzero applied load acts upward on node 1. Solve this problem by the Direct Stiffness Method. Start from the element stiffness equations given below in (1). These **already incorporate the $E_e A_e/L_e$ factor** in the stiffness matrices. The element stiffness equations **in global coordinates** are:



$$\begin{bmatrix} 50 & -50 & -50 & 50 \\ -50 & 50 & 50 & -50 \\ -50 & 50 & 50 & -50 \\ 50 & -50 & -50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} 50 & 50 & -50 & -50 \\ 50 & 50 & -50 & -50 \\ -50 & -50 & 50 & 50 \\ -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} \qquad (1)$$

(a). Assemble the master stiffness equations.

(b). Apply the given force and displacement BCs to get a reduced system of 2 equations and show it.

(c). Solve the reduced stiffness system for the unknown displacements and show the complete node displacement vector. *Skip recovery of node forces and reactions.*

(d). Recover the axial force $F_{(2)}$ in element (2) using the displacements you got in (c), noting sign.

Solution 3.

(a) Using any method (e.g., augment-element-stiffness-matrices-and-add), the assembled *master* stiffness equations are

$$\boxed{\begin{bmatrix} 50 & -50 & -50 & 50 & 0 & 0 \\ -50 & 50 & 50 & -50 & 0 & 0 \\ -50 & 50 & 100 & 0 & -50 & -50 \\ -50 & 50 & 0 & 100 & -50 & -50 \\ 0 & 0 & -50 & -50 & 50 & 50 \\ 0 & 0 & -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}}$$

(b) BCs: $u_{x1} = u_{y2} = u_{x3} = u_{y3} = 0$, $f_{y1} = +5$, $f_{x2} = 0$. Crossing out rows and columns 1, 4, 5 and 6 gives the *reduced* stiffness equation:

$$\boxed{\begin{bmatrix} 50 & 50 \\ 50 & 100 \end{bmatrix} \begin{bmatrix} u_{y1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}}$$

(c) Solving this linear system by any method yields

$$u_{x2} = -1/10 = -0.10, \quad u_{y1} = 1/5 = 0.20$$

Complete displacement solution:

$$\boxed{\mathbf{u} = [0 \quad 0.20 \quad -0.10 \quad 0 \quad 0 \quad 0]^T}$$

(written as a row vector to save space).

(d) Compute the internal (axial) force $F^{(2)}$ in member (2), which goes from node 2 to node 3. The orientation angle from x to $2 \to 3$ (+CCW from x) is $+45°$. We have $c = \cos 45° = 1/\sqrt{2}$ and $s = \sin 45° = 1/\sqrt{2}$. Local displacements of this member are recovered from the displacement transformation

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_{x2} = -1/10 \\ u_{y2} = 0 \\ u_{x3} = 0 \\ u_{y3} = 0 \end{bmatrix} = \begin{bmatrix} \bar{u}_{x2} = -1/(10\sqrt{2}) \\ * \\ \bar{u}_{x3} = 0 \\ * \end{bmatrix}$$

where the * mark local displacement entries (along $\bar{y}$) of no interest for this computation. The member elongation is $d^{(2)} = \bar{u}_{x3} - \bar{u}_{x2} = 0 - (-1/(10\sqrt{2}) = +1/(10\sqrt{2})$, whence Mechanics of Materials gives the axial force as

$$\boxed{F^{(2)} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} d^{(2)} = \frac{600\sqrt{2}}{6\sqrt{2}} \times \frac{1}{10\sqrt{2}} = \frac{10}{\sqrt{2}} = +7.07 \ \text{(T)}}$$