ASEN 3112

Spring 2020

Lecture 18

Whiteboard

March 19, 2020

Continue 1DoF, Damping, Free

Characteristic equation;

$$\underbrace{\lambda^2}_{a} + \underbrace{2\zeta\omega_n}_{b}\lambda + \underbrace{\omega_n^2}_{c} = 0$$

Quadratic equation: $a=1$, $b=2\zeta\omega_n$, $c=\omega_n^2$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\boxed{\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}}$$

Recall solution; $x(t) = Ae^{\lambda t}$

complex $A = A_{real} + i A_{imag}$

Consider 4 possible classes of problems:

[I] Undamped case: $\zeta = 0$, $\lambda = \pm\omega_n\sqrt{-1} = \pm i\omega_n$

$$x(t) = Ae^{\pm i\omega_n}$$

$$= \underbrace{A_1}\cos\omega_n t + \underbrace{A_2}\sin\omega_n t \quad \text{—— [1]}$$

real numbers

(like last lecture, except we are now using A_1, A_2 instead of B_1, B_2)

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A_1, A_2 obtained using the initial conditions.

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

$t=0$

See last lecture...

Now consider three different damping cases:

$\boxed{2}$ Underdamped $\zeta < 1$, $\lambda = -\zeta\omega_n \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$

damped frequency of vibration (damped natural frequency)

$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$

$$= e^{-\zeta\omega_n t}\left[A_1(\cos\omega_d t + i\sin\omega_d t) + A_2(\cos\omega_d t - i\sin\omega_d t)\right]$$

$$x(t) = e^{-\zeta\omega_n t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right]$$

where $\underbrace{B_1 = A_1 + A_2, \quad B_2 = (A_1 - A_2)i}_{\text{real}}$

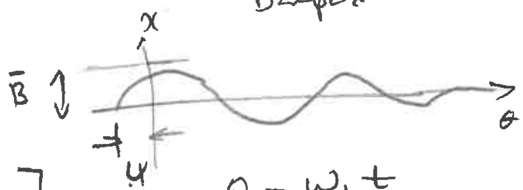
$$= e^{-\zeta\omega_n t}\,\bar{B}\sin(\omega_d t + \psi)$$

where $\bar{B} = \sqrt{B_1^2 + B_2^2}$, $\psi = \tan^{-1}\dfrac{B_1}{B_2}$

See lecture notes — for shape of signal

Undamped

Damped



$\theta = \omega_d t$

Solve B_1, B_2 using initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]} \quad [2]$$

$\frac{40}{3}$ [3] Overdamped $\xi > 1$, $\lambda = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

Note: $\xi > 1$ implies $c > c_c$ ie $c > \underbrace{2\sqrt{km}}_{c_c}$

$$x(t) = e^{-\xi\omega_n t}\left[A_1 e^{\overbrace{(\omega_n\sqrt{\xi^2-1})}^{\omega^*} t} + A_2 e^{-\overbrace{(\omega_n\sqrt{\xi^2-1})}^{\omega^*} t}\right]$$

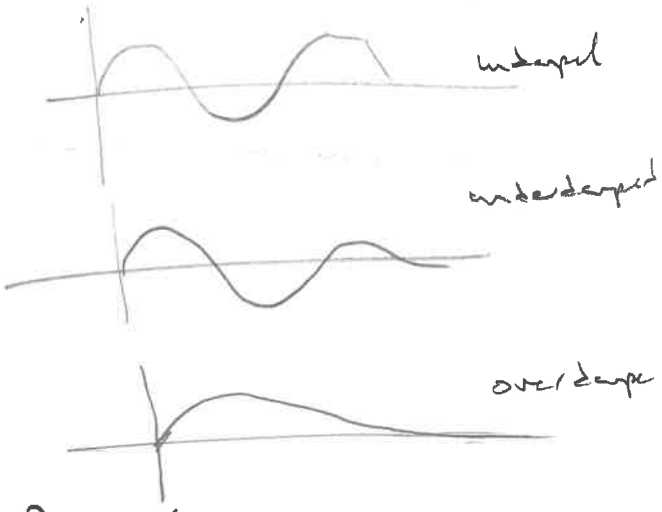
$$\omega^* = \omega_n\sqrt{\xi^2-1}$$

Consider initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$

$$\boxed{x(t) = e^{-\xi\omega_n t}\left[x_0 \cosh\omega^* t + \frac{v_0 + \xi\omega_n x_0}{\omega^*}\sinh\omega^* t\right]}$$

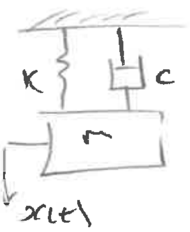
— [3]

See lecture notes for figures showing response



[4] Critically damped $\xi = 1$, $\lambda = -\xi\omega_n$

Note: $\xi = 1$ implies $c = c_c$ ie $c = 2\sqrt{KM}$



Assume solution $x(t) = (A + At)e^{-\xi\omega_n t}$

Note: As $t \to \infty$, $e^{-\xi\omega_n t}$ goes to zero, $x(t) \to 0$

Consider initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$:

$$\boxed{x(t) = \left[x_0 + (v_0 + \omega_n x_0)t\right]e^{-\omega_n t}}$$ — [4]

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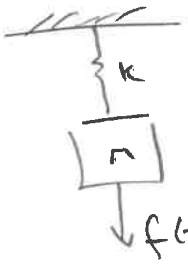
SDoF
- Undamped
- Forced

Steady-state harmonic excitation with amplitude F, frequency ω



$f(t) = F\sin\omega t$, ω: excitation frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

<u>EOM</u>

$$m\ddot{x} + kx = f(t)$$
$$m\ddot{x} + kx = F\sin\omega t$$
$$\ddot{x} + \frac{k}{m}x = \frac{F}{m}\sin\omega t$$
$$\ddot{x} + \omega_n^2 x + \frac{F}{m}\sin\omega t$$

Solution: $x(t) = x_p(t) + x_h(t)$

particular solution (steady-state)

homogeneous solution (transient)

Form of x_h is just like what we obtained for the free 1DOF case.

Assume $x_p(t) = X\sin\omega t$

real

$$\dot{x}_p(t) = \omega X\sin\omega t$$
$$\ddot{x}_p(t) = -\omega^2 X\sin\omega t$$

Substitute x_p into EOM (assuming EOM in x_p)

$$[-\omega^2 + \omega_n^2]X\cancel{\sin\omega t} = F\cancel{\sin\omega t}$$

$$X = \frac{F/m}{\omega_n^2 - \omega^2}$$

Note: if $\omega = \omega_n$, $X \to \infty$ (resonance!)

Define $\eta = \frac{\omega}{\omega_n}$

Divide by ω_n^2 (Note: $\frac{F}{m} \div \frac{k}{m} = \frac{F}{\cancel{m}} \times \frac{\cancel{m}}{k} = \frac{F}{k}$)

$$X = \frac{F/k}{1-\eta^2}$$

let $\delta_o = F/k$: static deflection

$$X = \frac{\delta_o}{1-\eta^2}, \quad \frac{X}{\delta_o} = \frac{1}{1-\eta^2} \text{ : Receptance}$$

$$\boxed{x_p(t) = \frac{\delta_o}{1-\eta^2} \sin \omega t}$$ S.S. Solution of 1DOF, undamped forced.

Total solution:

$$x(t) = x_p(t) + x_n(t)$$

in terms of B_1, B_2

To solve for B_1, B_2 at $t=0$

Need initial conditions to solve for B_1, B_2

considering the total solution $x(t)$

$$\dot{x}(t) = \dot{x}_p(t) + \underline{\dot{x}_n(t)}$$

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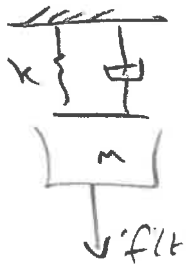
SDOF
- Damped
- Forced



$f(t) = F e^{i\omega t}$ (define in this manner ∵ in damped system, the response has a phase)

excitation frequency

EOM

$$m\ddot{x} + c\dot{x} + kx = f(t) = F e^{i\omega t}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m} = \frac{F}{m}e^{i\omega t}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}e^{i\omega t}$$

Assume $x_p(t) = X e^{i\omega t}$

X: complex.

ie $X = X_{real} + iX_{imag}$

$$\dot{x}_p(t) = +i\omega X e^{i\omega t}$$

$$\ddot{x}_p(t) = -\omega^2 X e^{i\omega t}$$

Plug into EOM:

$$\left[-\omega^2 + 2\zeta\omega_n i\omega + \omega_n^2\right] X = \frac{F}{m}$$

$$X = \frac{F/m}{\omega_n^2 - \omega^2 + 2\zeta\omega_n i\omega}$$

Divide by ω_n^2

$$X = \frac{F/k}{1-\eta^2 + i2\zeta\eta} = \frac{\delta_{st}}{1-\eta^2+i2\zeta\eta}$$

$$\frac{X}{\delta_{st}} = \frac{1}{1-\eta^2+i2\zeta\eta}$$

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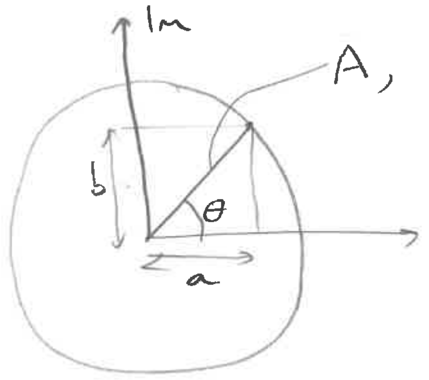
Side Note

Consider A: complex $\quad A = a + ib$

A, A has an amplitude $\bar{A}$



$$\bar{A} = \sqrt{a^2+b^2}\,, \quad \theta = \tan^{-1}\frac{b}{a}$$

Therefore:

$$X = \frac{\delta_{st}}{\sqrt{(1-\eta^2)^2+(2\xi\eta)^2}}\, e^{-i\phi}\,, \quad \phi = \tan^{-1}\frac{2\xi\eta}{1-\eta^2}$$

$$\boxed{x_p(t) = \frac{\delta_{st}}{\sqrt{(1-\eta^2)^2+(2\xi\eta)^2}}\, e^{i(\omega t-\phi)}}$$

cos + i sin

Physical steady state response.

$$x_p(t)\Big|_{phys} = \mathrm{Im}\left[x_p(t)\right]$$

imaginary part