ASEN 3112

Spring 2020

Lecture 10

Whiteboard

February 18, 2020

L10
1

Strain energy density

$$u_0\Big|_{xx} = \frac{1}{2}\sigma_{xx}\epsilon_{xx}$$

strain energy density considering deformation due to ~~to~~ due normal stress σ_{xx}

Recall stress tensor σ_{ij}

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

9 components

For moment equilibrium conditions in stress cubic:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx}=\tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx}=\tau_{xz} & \tau_{zy}=\tau_{yz} & \sigma_{zz} \end{bmatrix}$$

6 unique component

$\longrightarrow$ Recall $\underbrace{\gamma_{xy}}_{\text{engineering shear strain}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \underbrace{\epsilon_{xy} + \epsilon_{yx}}_{\text{true shear strain}} = 2\epsilon_{xy} = 2\epsilon_{yx}$

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

Also $\gamma_{yx} = \gamma_{xy}$, etc.

L10
2

~~Cons~~ strain energy density

$$U_0 = \frac{1}{2}\Big(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\Big)$$

strain energy in engineering component

Rod (bar)

$$U_{rod} = \frac{1}{2}\int \sigma_{xx}\,\epsilon_{xx}\,\underbrace{A\,dx}_{dV}$$

Apply Hooke's law: $\sigma_{xx} = E\,\epsilon_{xx}$

$$U_{rod} = \frac{1}{2}\int EA\,\epsilon_{xx}^2\,dx$$

Assume constant E, A over length

$$= \frac{1}{2} EAL\,\epsilon_{xx}^2$$

$$= \frac{1}{2}\frac{EA}{L}(\Delta L)^2$$

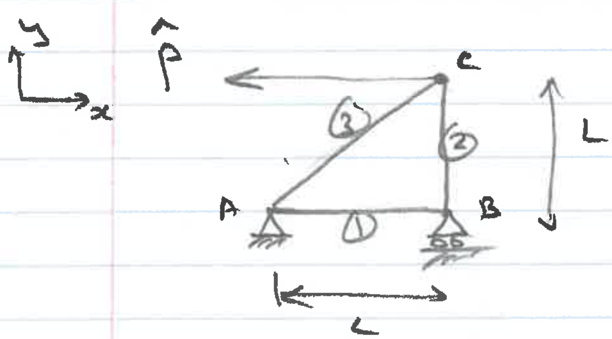
$$U_{rod} = \frac{1}{2}\int \frac{A}{E}\sigma_{xx}^2\,dx$$

$$= \frac{1}{2}\frac{AL}{E}\sigma_{xx}^2$$

$$= \frac{1}{2}\frac{AL}{E}\left(\frac{N}{A}\right)^2$$

$$= \frac{1}{2}\frac{L}{EA}N^2$$

Ex Consider a truss (made of bars)



E, A constant for all bars

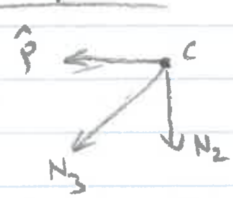
Principle of Conservation of Energy

$$W_e = U$$

$$U = \sum_{i=1}^{3} \frac{1}{2}\frac{N_i^2 L_i}{E_i A_i}$$

$$= \frac{1}{2}\frac{1}{EA}\sum_{i=1}^{3} N_i^2 L_i$$

Compute N_i



$$\sum F_x = 0 \Rightarrow -\hat{P} - \frac{\sqrt{2}}{2}N_3 = 0$$

$$\sum F_y = 0 \Rightarrow -N_2 - \frac{\sqrt{2}}{2}N_3 = 0$$

$$N_3 = -\sqrt{2}\,\hat{P},\quad N_2 = \hat{P}$$

L10

4

$$\sum F_x = 0 \Rightarrow -N_1 = 0$$

$$N_1 = 0$$

$$U = \frac{1}{2}\frac{1}{EA}\left(\underbrace{0}_{①} + \underbrace{\hat{P}^2 L}_{②} + \underbrace{2\hat{P}^2\sqrt{2}L}_{③}\right)$$

$$= \frac{\hat{P}^2 L}{2EA}\left(1+2\sqrt{2}\right)$$

$$\boxed{W_e = U}$$

$$-\cancel{\tfrac{1}{2}}\cancel{\hat{P}}\,\hat{u}_c = \frac{\hat{P}^{\cancel{2}}L}{\cancel{2}EA}\left(1+2\sqrt{2}\right)$$

$$\boxed{\hat{u}_c = -\frac{\hat{P}L}{EA}\left(1+2\sqrt{2}\right)}$$

$$\vec{P} = -\hat{P}\,\underline{I}$$

L10
5

Flexural beam

$$U_{beam} = \frac{1}{2}\int_L \sigma_{xx}\,\epsilon_{xx}\,A\,dx$$

Apply Hooke's law. $\sigma_{xx} = E\,\epsilon_{xx}$

$$U_{beam} = \frac{1}{2}\int_L \underline{EA\,\epsilon_{xx}^2}\,dx$$

$$U_{beam} = \frac{1}{2}\int \frac{A}{E}\,\sigma_{xx}^2\,dx$$

curvature

$$\epsilon_{xx} = -\frac{y}{\rho}, \quad \frac{1}{\rho} = \kappa$$

$$\sigma_{xx} = -\frac{My}{I_{zz}}$$

$$EA\,\epsilon_{xx}^2 = EA\frac{y^2}{\rho^2} = E\,\underbrace{Ay^2}_{I_{zz}}\,\kappa^2$$

Assume constant properties over cross section

$$\frac{A}{E}\sigma_{xx}^2 = \frac{A}{E}\frac{M^2y^2}{I_{zz}^2} = \frac{M^2 I_{zz}}{E\,I_{zz}^2} = \frac{M^2}{E\,I_{zz}}$$

$$I_{zz} = I$$

$$I_{zz} = \int y^2\,dA = Ay^2$$

$$I_{zz} = I$$

$$U_{beam} = \frac{1}{2}\int_L E\,I\,\kappa^2\,dx$$

$$U_{beam} = \frac{1}{2}\int_L \frac{M^2}{EI}\,dx \quad *$$

More general derivation of *

$$U_{beam} = \frac{1}{2}\int_V \frac{1}{E}\,\sigma_{xx}^2\,\underbrace{dy\,dz}_{dA}\,dx$$

Assume M, I, E constant over cross section

$$= \frac{1}{2}\int_L \frac{M^2}{E^2 I^2}\,y^2\,dy\,dz\,dx$$

$$= \frac{1}{2}\int_L \frac{M^2}{E^2I^2}\Big(\underbrace{\iint_A y^2\,dy\,dz}_{I_{zz} = I}\Big)dx = \frac{1}{2}\int \frac{M^2}{EI}\,dx$$

Also recall $M = EI\, v''$

$$U_{beam} = \frac{1}{2}\int EI\,(v'')^2\, dx$$

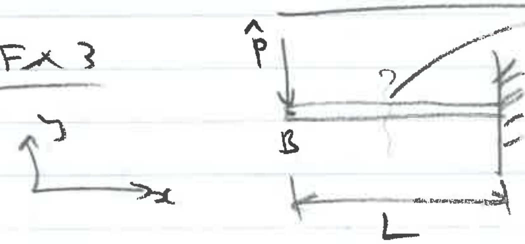
<u>Ex 2</u> Cantilever beam with multiple loading

units force

$$W_e = -\frac{1}{2}\hat{P}\,\hat{v}_B + \int \frac{1}{2}\hat{w}(x)\, v(x)\, dx$$
$$+ \frac{1}{2}\hat{M}_{ext}\,\hat{v}_B'$$

$\longrightarrow U = - - - -$

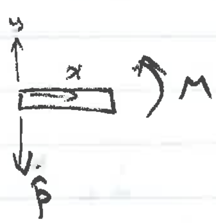
<u>Ex 3</u>



<u>Find $\hat{v}_B$</u>

$$W_e = U$$

$$-\frac{1}{2}\hat{P}\,\hat{v}_B = \frac{1}{2}\int_L \frac{M^2}{EI}\, dx$$



$$M(x) = -\hat{P}\, x$$

$$-\frac{1}{2}\hat{P}\,\hat{v}_B = \frac{1}{2}\int_L \frac{\hat{P}^2 x^2}{EI}\, dx = \frac{1}{2}\left(\frac{\hat{P}^2 L^3}{3EI}\right)$$

$$\hat{v}_B = -\frac{\hat{P}\, L^3}{3EI}$$

Clicker Question 1

Consider the cantilevered beam of length L shown below. An external moment (couple) of magnitude M is applied at the free end. The 2nd area moment of inertia, I, and the elastic modulus, E, are constant.
Let v(x) be the vertical displacement of the beam in y-direction.

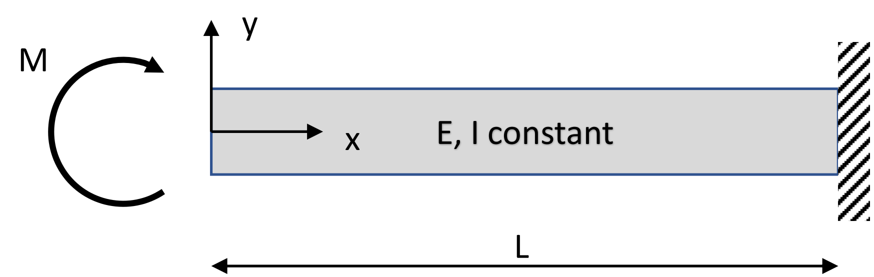
Give the expression for the magnitude of the **external work** $|\boldsymbol{W_e}|$

(a) $|W_e| = \frac{1}{2}\, v_{x=0}\, M$

(b) $|W_e| = \frac{1}{2}\, v'_{x=0}\, M$

(c) $|W_e| = \frac{1}{2}\, v'_{x=0}\, M\, L$

(d) none of the above



Clicker Question 2

Consider the cantilevered beam of length L shown below. An external moment (couple) of magnitude M_{ext} is applied at the free end. The 2nd area moment of inertia, I, and the elastic modulus, E, are constant.
Let v(x) be the vertical displacement of the beam in y-direction.

Give the expression for the magnitude of **internal work** $|\boldsymbol{W_i}|$

(a) $|W_i| = \frac{{M_{ext}}^2 L}{EI}$

(b) $|W_i| = \frac{1}{2}\frac{{M_{ext}}^2 L}{EI}$

(c) $|W_i| = \frac{1}{2} EI\, {M_{ext}}^2 L$

(d) none of the above