ASEN 3112 - Structures - Spring 2020
Homework 10

Due to **Gradescope** on Monday, April 27 at 11:59 PM MST
Make sure to follow the **homework guidelines** on the **last page** of this document!

Problem 10.1:

The column shown in Figure 1 is fabricated with two equal-length rigid struts and propped by a lateral extensional spring of stiffness k. It is pinned at both ends and compressed by load P at the top. Obtain the critical load by geometrically exact bifurcation analysis. To help the analysis process, Figure 1 depicts some of the key steps.

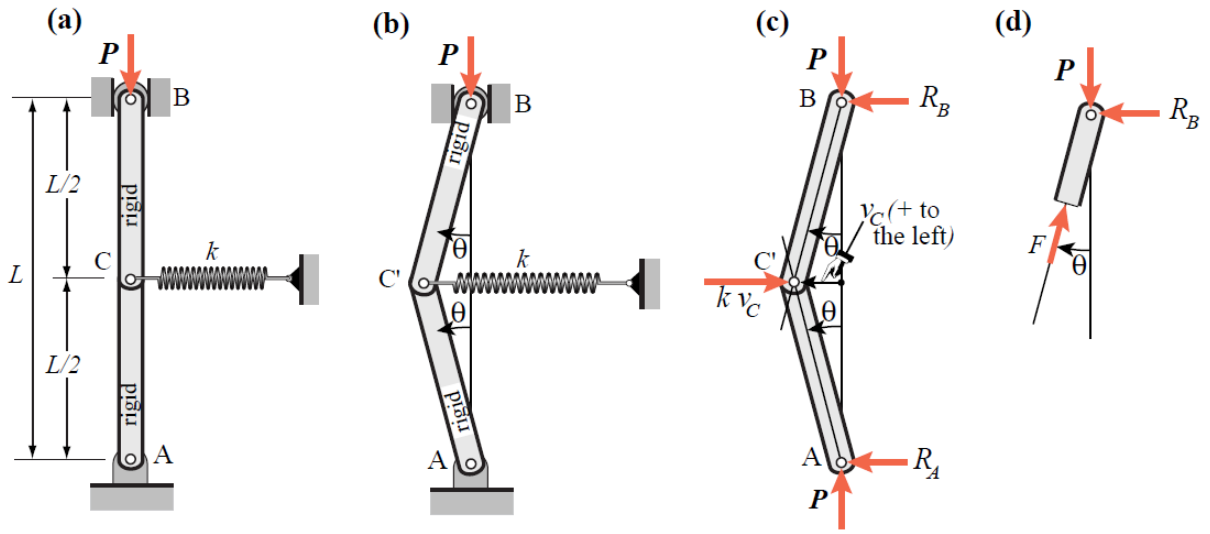


Figure 1: Buckling problem for for Exercise 10.1: (a) propped two-strut column; (b) deflected buckling shape and independent degree of freedom θ (tilt angle); (c) FBDs of whole column; (d) FBD of portion of strut BC. Deflections and tilt angles in (b) through (d) are *not* infinitesimal; thus geometrically exact analysis is required. Deflection v_C is taken positive to the left: this choice is arbitrary — results are independent of the sign convention since the column, being perfect, can buckle either way and its post-buckling behavior is symmetric.

First show that $F\cos\theta = P$ and $R_B = P\tan\theta$, in which F is the internal force in a strut, and R_B the horizontal reaction at B; cf. Figure 1(d). The spring elongates by $v_C = \frac{1}{2}L\sin\theta$. Taking moments with respect to hinge A (to get rid of the reactions there) you should get the exact equilibrium equation $L\sin\theta(P - \frac{1}{4}kL\cos\theta) = 0$. This has two solutions, one being $\theta = 0$. For the other, solve for P as function of θ. Find out at which value of P the two equilibrium paths intersect, that is, when $\theta = 0$ and P is equal to the expression you just found. That new value of P happens to be P_{cr}, as obtained by a geometrically exact bifurcation analysis. Does the load P rises or drops after buckling, that is, as θ increases?

To finish with a flourish, linearize the exact equilibrium equation for $\theta << 1$ (so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$) to obtain P_{cr}. Is it the same as that found by geometrically exact analysis?

Note: The analysis method covered in the **Pages 1-3** in "ASEN 3112 - Sp20 - Lecture 22 - Whiteboard.pdf" posted on Canvas is called "bifurcation analysis".

Hint:

This discrete system has only one degree of freedom. Consequently the buckling shape can only be as sketched in Figure 1(b), except that it may go either way. The tilt angle θ defined in Figure 1(c) is taken as the degree of freedom. Assuming θ is finite for an exact analysis, the spring elongates by $v_C = (L/2)\sin\theta$. The restoring spring force is $kv_C = k(L/2)\sin\theta$, which acts opposite to the assumed lateral deflection, as indicated in the whole-column FBD of Figure 1(c). From the FBD of link-BC shown in Figure 1(d), horizontal force equilibrium gives $F\cos\theta = P$ and $R_A = F\sin\theta = P\tan\theta$.

The next step is to balance moments with respect to the bottom hinge A.

Solution

Taking moments with respect to the bottom hinge A, positive CCW, gives the stability equation:

$$\begin{aligned} \sum M_A &= R(\tfrac{L}{2}\cos\theta + \tfrac{L}{2}\cos\theta) - \tfrac{kL}{2}\sin\theta\,\tfrac{L}{2}\cos\theta = P\tan\theta\, L\cos\theta - \tfrac{kL^2}{4}\sin\theta\cos\theta \\ &= L\sin\theta\left(P - \tfrac{kL}{4}\cos\theta\right) = 0. \end{aligned} \tag{1}$$

This exact equilibrium equation has two solutions: (I) $\theta = 0$ for any P, in which the column stays vertical, and (II) $P = \frac{1}{4}kL\cos\theta$, which pertains to the tilted equilibrium configuration. The two equilibrium paths intersect in the $\{P,\theta\}$ plane at the bifurcation point.

$$\boxed{\theta = 0, \quad P = P_{cr} = \frac{1}{4}kL.} \tag{2}$$

To linearize (1) for $\theta << 1$, one sets $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ to get

$$L\left(P - \frac{kL}{4}\right)\theta = 0. \tag{3}$$

For nontrivial solutions $\theta \neq 0$, the term in parentheses must vanish, whence

$$\boxed{P_{cr} = \frac{1}{4}kL.} \tag{4}$$

This is the same critical load given by the exact analysis. The exact analysis shows that after buckling the load P goes *down*, since the cosine function decreases away from zero.

Problem 10.2:

Two rigid-bar struts of equal length a are connected at the joints and at the bottom support by frictionless hinges as shown in Figure 2(a). The column is compressed by axial load P. The struts are maintained in vertically straight positions by torsional springs of the stiffnesses shown in the figure. Use linearized prebuckling (LPB) analysis (thus assuming very small displacements and rotations) throughout, do the following:

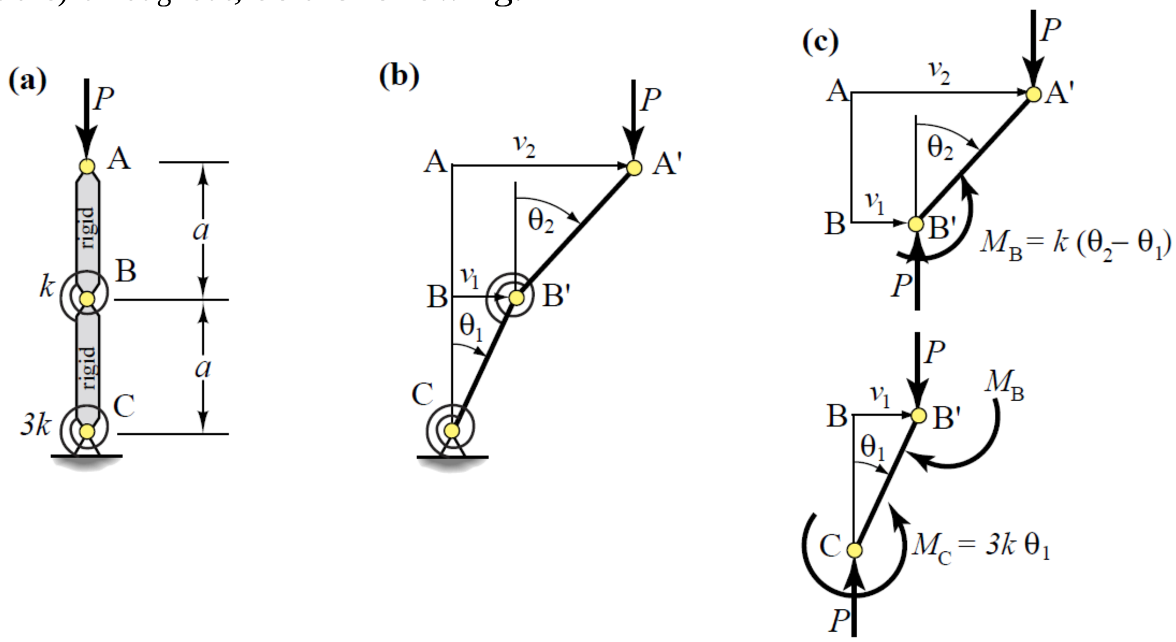


Figure 2: Column for Exercise 10.2: (a) configuration; (b) deflected shape and degrees of freedom, (c) FBDs of struts AB and BC. All deflections and tilt angles are actually very small (infinitesimal). Buckling shape is highly exaggerated for visibility.

(a) Derive the stability equation and extract the 2×2 stability matrix.
(b) Determine the two critical loads (eigenvalues of that matrix) in terms of a and k, and identify which one is the critical load.
(c) Compute the associated eigenvectors and use them to show the buckling shape sketches as separate diagrams.

Partial answer: $P_{cr} = \frac{1}{2}\left(5 - \sqrt{13}\right)\left(\frac{k}{a}\right) = 0.697\ k/a$

Note: The analysis method covered in **Pages 4-6** in "ASEN 3112-Sp20 - Lecture 22 - Whiteboard.pdf" posted on Canvas is called "linearized prebuckling (LPB) analysis".

Solution. Draw an arbitrary buckling shape as in Figure 2(b) (the deflections are highly exaggerated for readability; they are actually infinitesimally small). This shape is defined by the lateral deflections v_A and v_B of A and B, positive to the right. These in turn define the tilt angles θ_1 and θ_2 shown. These two angles are taken to be positive CW. The necessary geometric relations, assuming small deflections, are

$$v_A = v_B + a\theta_2 = a(\theta_1 + \theta_2), \quad v_B = a\theta_1. \tag{1}$$

The two tilt angles are independent and are taken as DOFs. The restoring moments at joints B and C induced by the torsional springs are

$$M_B = k(\theta_2 - \theta_1), \quad M_C = 3k\theta_1 \tag{2}$$

with M_B and M_C positive as shown. The FBD analysis of the complete structure sketched in Figure 2(b) shows that horizontal reactions are zero because M_B is self equilibrating (that is, hinge moments act in action-reaction pairs, similarly to internal forces).

We will set up the stability eigensystem selecting θ_1 and θ_2 as independent variables. Two equilibrium equations, obtained by analyzing the displaced links shown in Fig 2(c), are required to set up the stability eigensystem. The FBD of the displaced link AB gives $P(v_A - v_B) - M_B = 0$, or $Pa\theta_2 - k(\theta_2 - \theta_1) = (Pa - k)\theta_2 + k\theta_1 = 0$. The FBD of the displaced link CB gives $Pv_B - M_C + M_B = 0$ or $(Pa - 4k)\theta_1 + k\theta_2 = 0$. Collecting these two relations into a matrix system gives the stability eigensystem:

$$\begin{bmatrix} Pa-k & k \\ k & Pa-4k \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} P-\kappa & \kappa \\ \kappa & P-4\kappa \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{with} \quad \kappa = k/a. \tag{3}$$

Buckling loads are obtained by seeking nontrivial solutions of (3). Solving $\det \begin{bmatrix} P-\kappa & \kappa \\ \kappa & P-4\kappa \end{bmatrix} = P^2 - 5\kappa + 3\kappa^2 = 0$ gives two roots: $P_{1,2} = \frac{1}{2}(5 \mp \sqrt{13})\kappa = \frac{1}{2}(5 \mp \sqrt{13})\, k/a$. The critical load is the smallest one:

$$\boxed{P_{cr} = P_1 = \frac{5 - \sqrt{13}}{2}\frac{k}{a} = 0.6972\frac{k}{a}} \tag{4}$$

The corresponding eigenvectors (normalized so that the largest component is one) are

$$\begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0.3028 \end{bmatrix}, \quad \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}_2 = \begin{bmatrix} -0.3028 \\ 1 \end{bmatrix}$$

These can be transformed to lateral deflections using (1). Normalizing so that the largest component is ± 1 we get

$$\boxed{P_1 = P_{cr} = 0.6972\frac{k}{a} \Rightarrow \begin{bmatrix} v_A \\ v_B \end{bmatrix}_1 = \begin{bmatrix} 1. \\ 0.2324 \end{bmatrix}, \quad P_2 = 4.3028\frac{k}{a} \Rightarrow \begin{bmatrix} v_A \\ v_B \end{bmatrix}_2 = \begin{bmatrix} 0.6972 \\ 1. \end{bmatrix}} \tag{5}$$

These eigenvectors are sketched in Figure 3 below. They provide the buckling mode shapes.

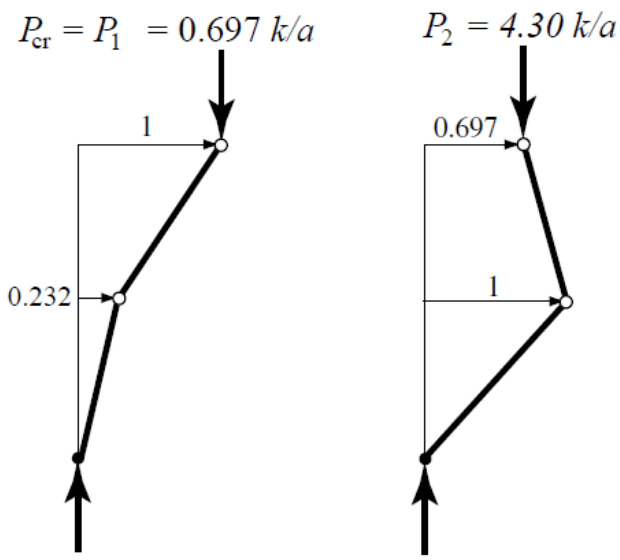


Figure 3: Normalized eigenvectors. These are the buckling mode shapes.

ASEN 3112 - Structures – Spring 2020 – Homework Submission Guidelines

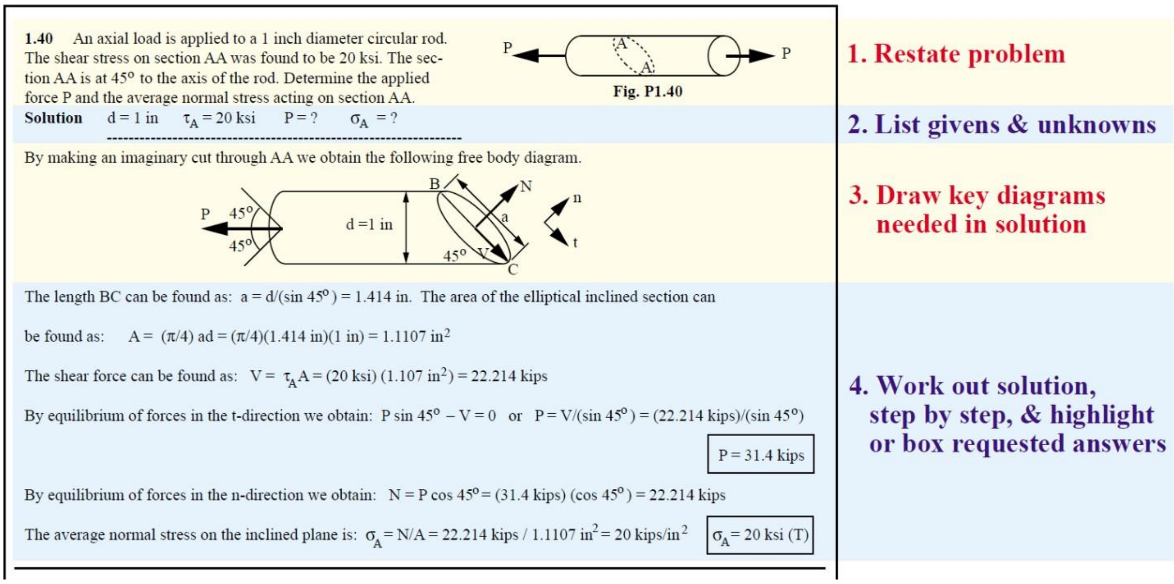
HW submission guidelines:

- **Write clearly your <u>name</u>, <u>student ID</u>, and <u>lab section ID</u> (011, 012, 013, or 014) on <u>each</u> sheet you turn in.**
- **Begin answering a new exercise on a new page to facilitate grading.**
- **Restate the questions so the grader is sure which exercise you solved.**
- **Do no fold your solution sheets.**

When writing out the homework, organize your solution to each assigned problem into four parts:

- **Restate the question.** Short hand is OK if the question is long or elaborated. Restating makes sure the grader knows you are answering the right problem, and will help you to organize the subsequent exposition as your subconscious gets going.
- **State the givens and unknowns to be found.** For givens always write down the physical units if those are stated in the problem.
- **Draw key diagrams at the start of the solution.** For example, Free Body Diagrams (FBD) are essential part of many problems in statics.
- **Write out the solution.** Be sure to **show work**. **Highlight** your answer(s) by a box, underline, arrow, or hi-lite marker. Also, do not forget to show the physical units. (This last item is also important in exams).

Example of Well Organized Homework Solution: Taken from the Solution Manual of M. Vable's *Mechanics of Materials*, Oxford, 1st ed.



1.40 An axial load is applied to a 1 inch diameter circular rod. The shear stress on section AA was found to be 20 ksi. The section AA is at 45° to the axis of the rod. Determine the applied force P and the average normal stress acting on section AA.

Fig. P1.40

Solution $d = 1$ in $\tau_A = 20$ ksi $P = ?$ $\sigma_A = ?$

By making an imaginary cut through AA we obtain the following free body diagram.

The length BC can be found as: $a = d/(\sin 45^\circ) = 1.414$ in. The area of the elliptical inclined section can be found as: $A = (\pi/4)\, ad = (\pi/4)(1.414 \text{ in})(1 \text{ in}) = 1.1107 \text{ in}^2$

The shear force can be found as: $V = \tau_A A = (20 \text{ ksi})\,(1.107 \text{ in}^2) = 22.214$ kips

By equilibrium of forces in the t-direction we obtain: $P \sin 45^\circ - V = 0$ or $P = V/(\sin 45^\circ) = (22.214 \text{ kips})/(\sin 45^\circ)$

$\boxed{P = 31.4 \text{ kips}}$

By equilibrium of forces in the n-direction we obtain: $N = P \cos 45^\circ = (31.4 \text{ kips})\,(\cos 45^\circ) = 22.214$ kips

The average normal stress on the inclined plane is: $\sigma_A = N/A = 22.214 \text{ kips} / 1.1107 \text{ in}^2 = 20 \text{ kips/in}^2$ $\boxed{\sigma_A = 20 \text{ ksi (T)}}$