ASEN 3112

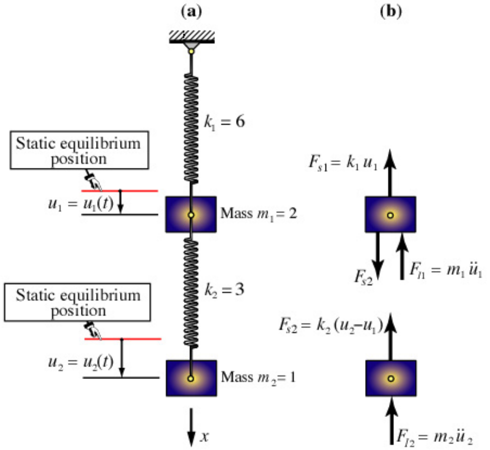
Spring 2020

Lecture 20

April 2, 2020

Modal Analysis of MDOF Unforced Undamped Systems

2-DOF, Unforced, Undamped Mass-Spring Example System



Matrix Equations of Motion of Example System

$$m_1 = 2, \quad m_2 = 1, \quad c_1 = c_2 = 0, \quad k_1 = 6, \quad k_2 = 3, \quad p_1 = p_2 = 0$$

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Natural Frequencies of Vibration

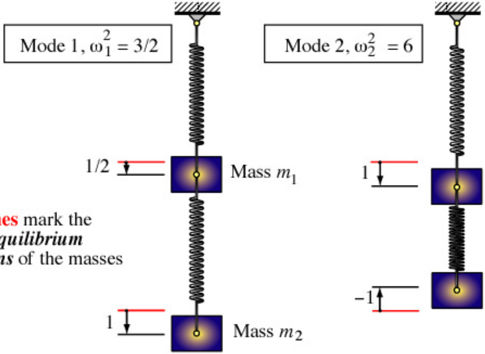
$$\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} = 18 - 15\omega^2 + 2\omega^4 = (3 - 2\omega^2)(6 - \omega^2) = 0$$

$$\boxed{\omega_1^2 = \frac{3}{2} = 1.5 \qquad \omega_2^2 = 6}$$

Vibration Mode Shape Pictures



Red lines mark the ***static equilibrium positions*** of the masses

Mode Shape Mass Orthonormalization

$$M_1 = \phi_1^T\,\mathbf{M}\,\phi_1 = [\,1/2 \quad 1\,]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 3/2$$

$$M_2 = \phi_2^T\,\mathbf{M}\,\phi_2 = [\,1 \quad -1\,]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3$$

$$K_1 = \phi_1^T\,\mathbf{K}\,\phi_1 = [\,1/2 \quad 1\,]\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 9/4$$

$$K_2 = \phi_2^T\,\mathbf{K}\,\phi_2 = [\,1 \quad -1\,]\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18$$

$$\omega_1^2 = \frac{K_1}{M_1} = \frac{9/4}{3/2} = 3/2 \qquad \omega_2^2 = \frac{K_2}{M_2} = \frac{18}{3} = 6$$

To orthonormalize, divide ϕ_i by the square root of M_i (i=1,2):

$$\phi_1 = \sqrt{\frac{2}{3}}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4088 \\ 0.8165 \end{bmatrix}, \quad \phi_2 = \sqrt{\frac{1}{3}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5773 \\ -0.5773 \end{bmatrix}$$

Mode Shape Mass Orthonormalization

$$M_1 = \phi_1^T \mathbf{M} \phi_1 = [\,1/2 \quad 1\,] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 3/2$$

$$M_2 = \phi_2^T \mathbf{M} \phi_2 = [\,1 \quad -1\,] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3$$

$$K_1 = \phi_1^T \mathbf{K} \phi_1 = [\,1/2 \quad 1\,] \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 9/4$$

$$K_2 = \phi_2^T \mathbf{K} \phi_2 = [\,1 \quad -1\,] \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18$$

$$\omega_1^2 = \frac{K_1}{M_1} = \frac{9/4}{3/2} = 3/2 \qquad \omega_2^2 = \frac{K_2}{M_2} = \frac{18}{3} = 6$$

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Superposition Through The Modal Matrix

$$\boxed{\mathbf{u} =} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \stackrel{\text{def}}{=} \phi_1\,\eta_1(t) + \phi_2\,\eta_2(t) = [\,\phi_1 \quad \phi_2\,] \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \boxed{= \boldsymbol{\Phi}\,\boldsymbol{\eta}}$$

$$\boxed{\boldsymbol{\Phi} = [\,\phi_1 \quad \phi_2\,] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0.4082 & 0.5773 \\ 0.8165 & -0.5773 \end{bmatrix}}$$

$$\boldsymbol{\Phi}^T\,\mathbf{M}\,\boldsymbol{\Phi} = \mathbf{M}_g = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\Phi}^T\,\mathbf{K}\,\boldsymbol{\Phi} = \mathbf{K}_g = \mathbf{diag}[\omega_i^2] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix}$$

Modal Equations of Motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\Phi}^T [\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u}] = \boldsymbol{\Phi}^T \mathbf{0} = \mathbf{0}$$

$$\mathbf{u}(t) = \boldsymbol{\Phi}\,\boldsymbol{\eta}(t) \quad \Rightarrow \quad \dot{\mathbf{u}}(t) = \boldsymbol{\Phi}\,\dot{\boldsymbol{\eta}}(t), \quad \ddot{\mathbf{u}}(t) = \boldsymbol{\Phi}\,\ddot{\boldsymbol{\eta}}(t)$$

$$\Downarrow$$

$$\boldsymbol{\Phi}^T \mathbf{M}\,\boldsymbol{\Phi}\,\ddot{\boldsymbol{\eta}} + \boldsymbol{\Phi}^T \mathbf{K}\,\boldsymbol{\Phi}\,\boldsymbol{\eta} = \boldsymbol{\Phi}^T \mathbf{0} = \mathbf{0}$$

$$\boxed{\ddot{\boldsymbol{\eta}} + \mathbf{K}_g\,\boldsymbol{\eta} = \mathbf{0}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1(t) \\ \ddot{\eta}_2(t) \end{bmatrix} + \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\ddot{\eta}_1(t) + (3/2)\,\eta_1(t) = 0, \quad \ddot{\eta}_2(t) + 6\,\eta_2(t) = 0}$$

Modal Initial Conditions

$$\boxed{\mathbf{u}(0) = \mathbf{u}_0 \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0}$$

$$\mathbf{u}(t) = \mathbf{\Phi}\,\boldsymbol{\eta}(t) \quad \Rightarrow \quad \mathbf{u}_0 = \mathbf{u}(0) = \mathbf{\Phi}\,\boldsymbol{\eta}(0)$$

$$\dot{\mathbf{u}}(t) = \mathbf{\Phi}\,\dot{\boldsymbol{\eta}}(t) \quad \Rightarrow \quad \mathbf{v}_0 = \dot{\mathbf{u}}(0) = \mathbf{\Phi}\,\dot{\boldsymbol{\eta}}(0)$$

$$\boldsymbol{\eta}_0 = \boldsymbol{\eta}(0) = \mathbf{\Phi}^{-1}\,\mathbf{u}_0 \qquad \dot{\boldsymbol{\eta}}_0 = \dot{\boldsymbol{\eta}}(0) = \mathbf{\Phi}^{-1}\,\mathbf{v}_0$$

To avoid inverting $\mathbf{\Phi}$ use this linear algebra trick: postmultiply both sides of $\mathbf{\Phi}^T\mathbf{M}\,\mathbf{\Phi} = \mathbf{I}$ by $\mathbf{\Phi}^{-1}$ to get $\mathbf{\Phi}^{-1} = \mathbf{\Phi}^T\,\mathbf{M}$, and replace

$$\boxed{\boldsymbol{\eta}_0 = \mathbf{\Phi}^T\mathbf{M}\,\mathbf{u}_0 \qquad \dot{\boldsymbol{\eta}}_0 = \mathbf{\Phi}^T\mathbf{M}\,\mathbf{v}_0}$$

Sample IC: Unit Velocity At Mass 1, Else Zero

$$\boxed{\mathbf{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\boldsymbol{\eta}_0 = \begin{bmatrix} \eta_{10} \\ \eta_{20} \end{bmatrix} = \mathbf{\Phi}^T \mathbf{M}\,\mathbf{u}_0 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{\eta}}_0 = \begin{bmatrix} \dot{\eta}_{10} \\ \dot{\eta}_{20} \end{bmatrix} = \mathbf{\Phi}^T \mathbf{M}\,\mathbf{v}_0 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} \end{bmatrix}$$

Solve Modal Equations And Combine Via Modal Matrix

$$\eta_1(t) = \frac{\dot\eta_{10}}{\omega_1}\,\sin\omega_1 t = \frac{2}{3\sqrt{3}}\,\sin(\sqrt{6}\,t) = \frac{2}{3}\,\sin(\sqrt{3/2}\,t) = 0.6667\,\sin(1.2247\,t),$$

$$\eta_2(t) = \frac{\dot\eta_{20}}{\omega_2}\,\sin\omega_2 t = \frac{2/\sqrt{3}}{\sqrt{6}}\,\sin(\sqrt{6}t) = \frac{2}{3\sqrt{2}}\,\sin(\sqrt{6}\,t) = 0.4714\,\sin(2.4495\,t).$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{2}{3}\,\sin(\sqrt{3/2}\,t) \\ \frac{2}{3\sqrt{2}}\,\sin(\sqrt{6}\,t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sqrt{\frac{2}{3}}\Big(\sin(\sqrt{3/2}t) + \sin(\sqrt{6}t)\Big) \\ \frac{1}{3}\sqrt{\frac{2}{3}}\Big(2\sin(\sqrt{3/2}t) - \sin(\sqrt{6}t)\Big) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2722\,\Big(\sin(1.2247t) + \sin(2.4495t)\Big) \\ 0.2722\,\Big(2\sin(1.2247t) - \sin(2.4495t)\Big) \end{bmatrix}$$

$$\begin{bmatrix} \dot u_1(t) \\ \dot u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\Big(\cos(\sqrt{3/2}t) + 2\cos(\sqrt{6}t)\Big) \\ \frac{2}{3}\Big(\cos(\sqrt{3/2}t) - \cos(\sqrt{6}t)\Big) \end{bmatrix} = \begin{bmatrix} 0.3333\,\Big(\cos(1.2247t) + 2\cos(2.4495t)\Big) \\ 0.6667\,\Big(\cos(1.2247t) - \cos(2.4495t)\Big) \end{bmatrix}$$

Response to Unit Initial Velocity on Mass 1