Recitation 10

ASEN 3112 – Spring 2020

Problem 1 Buckling of two-DOF lumped parameter model of a column

The 3-hinged column shown in Figure 2(a) consists of two rigid links (struts) AB and BC, both of length L. The column is pinned at support C, propped at A by an extensional spring of stiffness k, and rotationally stiffened at B by a torsional spring of stiffness $k_T = \beta k L^2$, in which $\beta \geq 0$ is a dimensionless parameter.

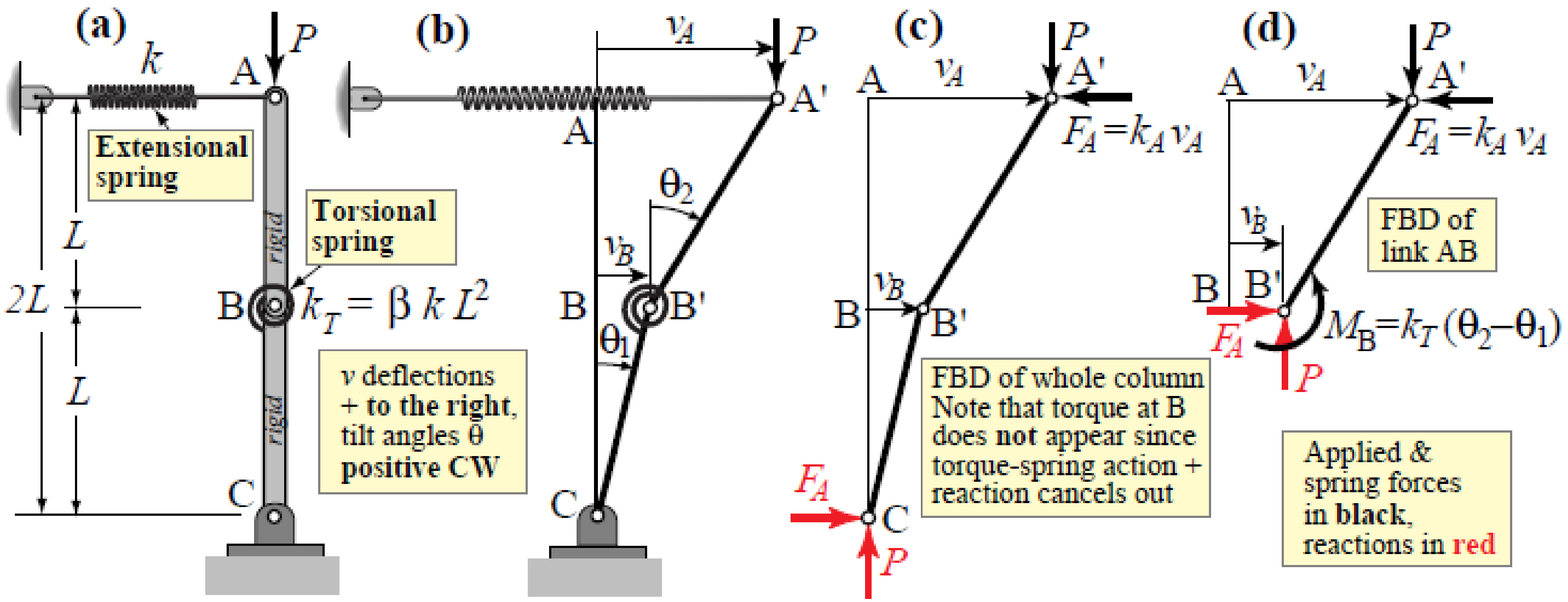


Figure 2: Three-hinged column for Problem 2.

Using the tilt angles θ_1 and θ_2 shown in Figure 2(b) as degrees of freedom (DOF) and linearized stability analysis ($|\theta_1| << 1$ and $|\theta_2| << 1$),

(a) Derive the linearized equilibrium equations in a deflected (tilted) configuration. Two equilibrium

(b) Place the foregoing equations in matrix form

$$\mathbf{A}\theta = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

where the entries of matrix $\mathbf{A}$ are functions of P, L, k and β.

(c) Find the determinant $\Delta = \det(\mathbf{A})$. This is a quadratic polynomial in P, called the *characteristic polynomial*. The *characteristic equation* is $\Delta = 0$. By solving for its P roots, find the two critical load values. Call them P_{cr1} and P_{cr2}. The smallest of the two is *the* critical load P_{cr}, but which one is the one?

(d) Find the buckling mode shapes.

Solution of Problem 1

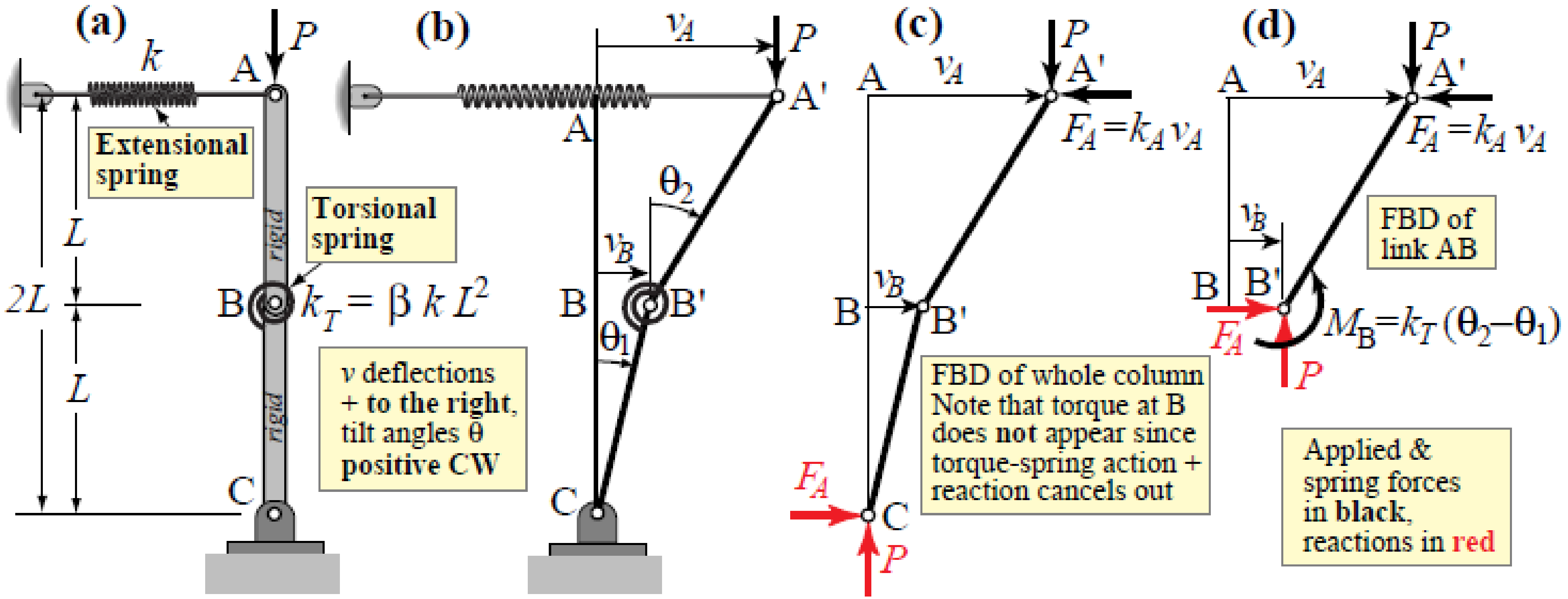


Figure 5: Three-hinged column for Problem 2.

Since the system has 2 DOFs, we need two equilibrium equations. To get them consider the moment equilibrium of: (1) deflected whole column ABC, and (2) link AB. Those are diagramed in Figures 5(c) and 5(d), respectively. Moments are taken positive CW.

$$\begin{aligned}
\text{FBD of ABC:} \sum M_C &= P\,v_A - k\,v_A(2L) = (P-2\,k\,L)\,v_A = (P-2\,k\,L)\,L(\theta_1+\theta_2) = 0. && (3)\\
\text{FBD of AB:} \sum M_{B'} &= P\,(v_A - v_B) - k\,v_A\,L - M_B && (4)\\
&= P\,L\theta_2 - k\,L^2(\theta_1+\theta_2) - \beta\,k\,L^2(\theta_2-\theta_1) && (5)\\
&= (\beta-1)\,k\,L^2\,\theta_1 + \left[P\,L - (\beta+1)\,k\,L^2\right]\theta_2 = 0. && (6)
\end{aligned}$$

In Equation 6, the lateral deflections v_A and v_B were eliminated in favor of the small tilt angles θ_1 and θ_2 using the linearized relations $v_B = L\theta_1$ and $v_A = v_B + L\theta_2 = L(\theta_1+\theta_2)$. Collecting and putting in matrix form:

$$\mathbf{A}\theta = \begin{bmatrix} (P-2\,k\,L)\,L & (P-2\,k\,L)\,L \\ (\beta-1)\,k\,L^2 & P\,L-(\beta+1)\,kL^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

This is the *stability eigenproblem*, or *buckling eigenproblem*, with load P playing the role of eigenvalue. Since the entries in the first row are the same, the determinant $\Delta = \det(\mathbf{A})$ is easily obtained as

$$\Delta = (P-2\,k\,L)\,L\left[-(\beta-1)\,k\,L^2 + P\,L - (\beta+1)\,k\,L^2\right] = (P-2k\,L)\,L(P\,L-2\beta\,k\,L^2) \tag{8}$$

This is the *characteristic polynomial*, which is quadratic in P. The factored form on the right of Equation 8 provides directly the two critical loads as roots of $\Delta = 0$:

$$P_{cr1} = 2k\,L, \qquad P_{cr2} = 2\beta\,k\,L. \tag{9}$$

Comparing these values gives the final answer

$$\boxed{P_{cr} = P_{cr1} = 2\,k\,L \quad \text{if} \quad \beta \geq 1, \quad \text{else} \quad P_{cr} = P_{cr2} = 2\beta\,k\,L.} \tag{10}$$

If $\beta = 1$, the two critical loads coalesce, whereas if $\beta = 0$ (no torsional spring at B) $P_{cr} = 0$. Eigenvectors can be obtained and are found to be:

$$\theta^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \; \theta^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{11}$$

Problem 2: Cantilever Beam Buckling

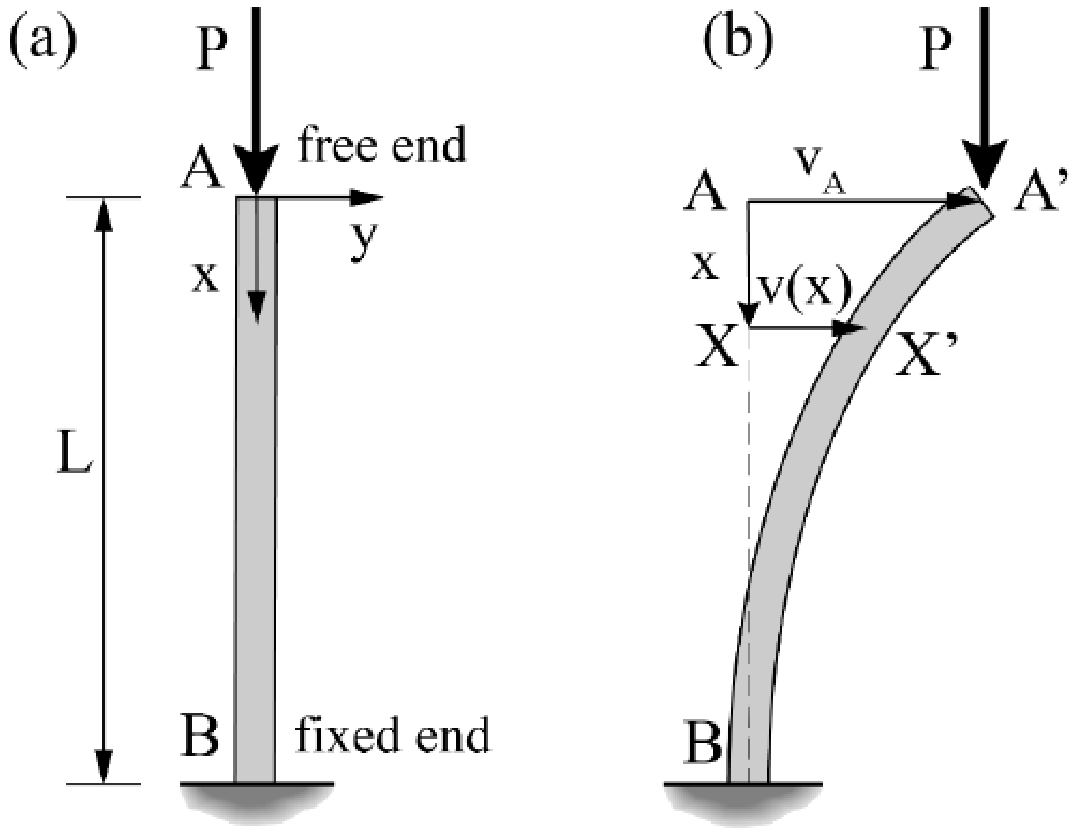


Figure 1. - Column structure for Question I: (a) configuration, (b) sketched buckling

The column of Figure 1(a) has length L, uniform Young's modulus E, and constant second moment of area I. It is fixed at B and free at A. A vertical load P is applied on the free end, causing the beam to buckle as P exceeds a critical value.

(a) Give a FBD of the *entire buckled* column, and calculate the reactions at Point B. Keep v_A as a parameter. Figure 1(b) is *not* a column FBD, only a buckling shape sketch.

(b) Give a FBD of the column cut at distance x from top, to derive the ODE for the lateral deflection $v(x)$. Hint: Take moments with respect to X'.

(c) Verify that the solution has the form: $v(x) = A\cos(\lambda x) + B\sin(\lambda x) + C$, and find the value for λ and C in terms of the parameters E, I, P, and v_A.

(d) Three kinematic boundary conditions are needed in this problem. One boundary condition is obvious from Figure 1(b): $v(0) = v_A$. What are the other two?

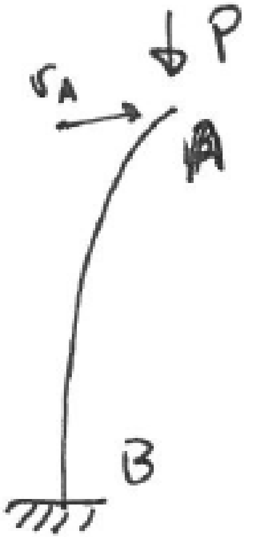
(e) Using the boundary conditions, give the characteristic equation for the buckling load. Solve this equation and calculate the critical load, P_{cr}. Hint: As a check, the effective length parameter for this boundary condition is $K = 2$. Note: even if you have the value in the crib-sheets, you still need to calculate P_{cr}.

(f) Write down the final solution for the deflection of the beam, using the results from pars (c), (d), and (e).

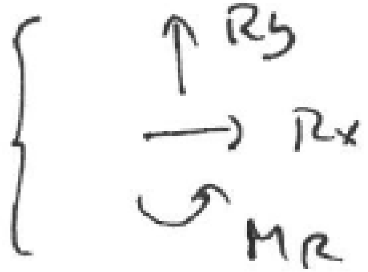
(g) Consider now that the beam has indeed buckled, and the displacement at Point A is v_A. Calculate the maximum stress due to bending as a function of the displacement v_A. Assume that the beam has a uniform circular cross section of radius r.

Solution of Problem 2

(a)



Reactions



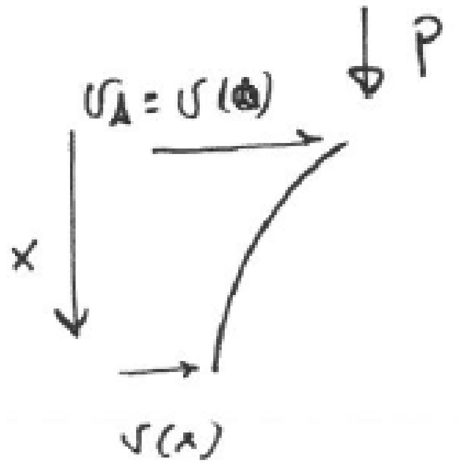
Equilibrium of forces:

$$\Sigma F_x = 0 \implies R_x = 0$$

$$\Sigma F_y = 0 \implies R_y = P$$

$$\Sigma M_o = 0 \implies M_R = P \cdot v_A$$

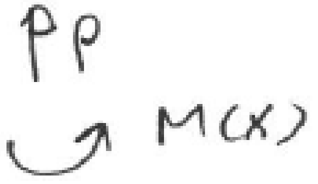
(b)



we only care about equilibrium of moments:

$$\Sigma M = 0 \Rightarrow \quad M(x) = P(v_A - v(x))$$

$$M(x) + P \cdot v(x) = P \cdot v_A$$



Now we use $M = EIv''$ to get

$$EIv'' + Pv = P \cdot v_A$$

$$v'' + \frac{P}{EI} v = v'' + \lambda^2 v = \frac{P}{EI} v_A$$

$$\boxed{v'' + \lambda^2 v = \lambda^2 v_A}$$

(c) we are told: $v(x) = A\cos\lambda x + B\sin\lambda x + C$

$$v'' = -A\lambda^2 \cos\lambda x - B\lambda^2 \sin\lambda x$$

So plug-in ODE:

$$v'' + \lambda^2 v = \lambda^2 v_A$$

$$-A\lambda^2\cos\lambda x - B\lambda^2 \sin\lambda x + A\lambda^2 \cos\lambda x + B\lambda^2 \sin\lambda x + \lambda^2 C = \lambda^2 v_A$$

The only condition is: $\lambda^2 C = \lambda^2 v_A$

$$\boxed{C = v_A}$$

Plus the fact that $\boxed{\lambda = \sqrt{\frac{P}{EI}}}$

(d) Three BC's: $v(0) = v_A$ ← Free BC

$$\left.\begin{aligned} v(L) &= 0 \\ v'(L) &= 0 \end{aligned}\right\} \text{Fixed BC}$$

(e) So we know

$$v = A\cos\lambda x + B\sin\lambda x + v_A$$

we start applying BC's

$$v(0) = v_A \Rightarrow A + v_A = v_A \Rightarrow \boxed{A = 0}$$

Then, we apply

$$v'(L) = 0 \Rightarrow \lambda B\cos\lambda L + 0 = 0$$

So either $B = 0$, which doesn't make sense, ~~or~~

(that would be a rigid body motion $v(x) = v_A$)

or we need

$$\cos \lambda L = 0$$

which requires $\lambda L = \frac{\pi}{2} + n\pi$

If we only care about the first critical load, then

$$\lambda L = \pi/2$$

$$\lambda = \frac{\pi}{2L}$$

$$\lambda^2 = \frac{P}{EI} = \frac{\pi^2}{(2L)^2}$$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(2L)^2}}$$

And, in fact,

we have $K = 2$ for

the effective length

finally, we apply

$$\sigma(L) = 0 \quad \Rightarrow \quad B \sin \lambda L + \sigma_A = 0$$

$$\lambda = \frac{\pi}{2L}, \text{ so} \quad B \sin \frac{\pi}{2} + \sigma_A = 0$$

$$\boxed{B = -\sigma_A}$$

(f) The final solution is:

$$\boxed{\sigma(x) = \sigma_A - \sigma_A \sin \frac{\pi x}{2L}}$$

(g) The maximum strain due to bending, for a circular beam, is:

$$\varepsilon_{max} = \kappa_{max}\, r = v''_{max}\, r$$

The stress is then:

$$\sigma_{max} = E\, v''_{max}\, r$$

Since $\quad v(x) = v_A - v_A \sin \frac{\pi x}{2L}$

$$v'' = +v_A \frac{\pi^2}{4L^2} \sin \frac{\pi x}{2L}$$

σ''_{max} takes place when $\sin \frac{\pi x}{2L} = 1$, so

$$\boxed{\sigma_{max} = E \frac{v_A \pi^2}{4L^2} r}$$