3/31/2020

Typos:

L18/2: $x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \dfrac{v_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$ — not ω_b

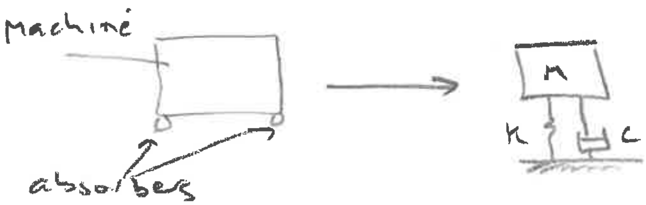
L18/3: Overdamped $c > 2\sqrt{km}$ — correct

critically damped $c = 2\sqrt{km}$ — correct

L18/6: $\dot{x}_p = i\omega X e^{i\omega t}$ — correct.

Experimental Identification of Damping: Logarithmic Decrement Approach

Consider a machine in a factory supported by damping absorbers.



Mass, m: weigh

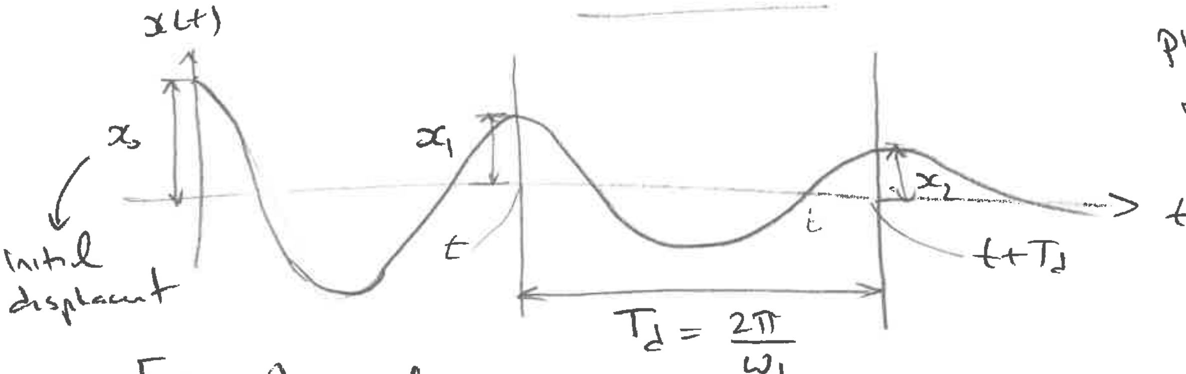
Stiffness, k: static load, measure deformation, calculate stiffness

How to quantify the damping c, ζ?

Procedure

Introduce initial displacement, and let vibrate freely, measure ~~output~~ response (i.e. motion) of machine as a function of time; extract damping ratio ζ.

(obtain c from ζ, k, m)



Plot of measured response of machine.

From observation: underdamped because it exhibits oscillations

L19
2

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\cancel{\omega_n t}}\cancel{\overline{X}}\sin(\cancel{\omega_d t + \psi})}{e^{-\zeta\omega_n(t+T_d)}\cancel{\overline{X}}\sin[\cancel{\omega_d(t+T_d)+\psi}]}$$

$$= \frac{1}{e^{-\zeta\omega_n T_d}} = e^{\zeta\omega_n T_d}$$

Recall $T_d = \frac{2\pi}{\omega_d}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (underdamped)

$$\frac{x_1}{x_2} = e^{\frac{\zeta\cancel{\omega_n}2\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}} = e^{\frac{\zeta 2\pi}{\sqrt{1-\zeta^2}}}$$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{\zeta 2\pi}{\sqrt{1-\zeta^2}} \approx \underbrace{2\pi\zeta \text{ for } \zeta \ll 1}$$

in practice, ζ is very small, eg $\zeta = 0.001$ or $\zeta = 0.01$

$$\therefore \zeta \approx \frac{\ln\left(\frac{x_1}{x_2}\right)}{2\pi}$$

$$\left[\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} \Rightarrow c = 2\zeta\sqrt{km}\right]$$

L19

3

Multi-Degree of Freedom (MDOF) System

— Damped
— Forced

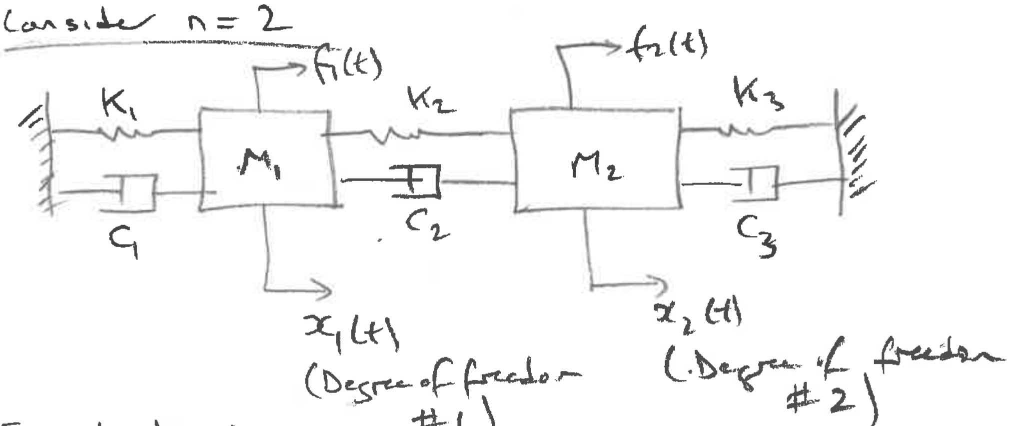
Steps

1. Derive Equations of Motion (EoM)
2. Remove "temporarily" damping and forcing
3. Obtain system "vibration characteristics"
4. Reinstate damping and/or forcing as needed
5. Use "vibration characteristics" from (3) to systematically solve for the vibration response of all degrees of freedom (Modal analysis)

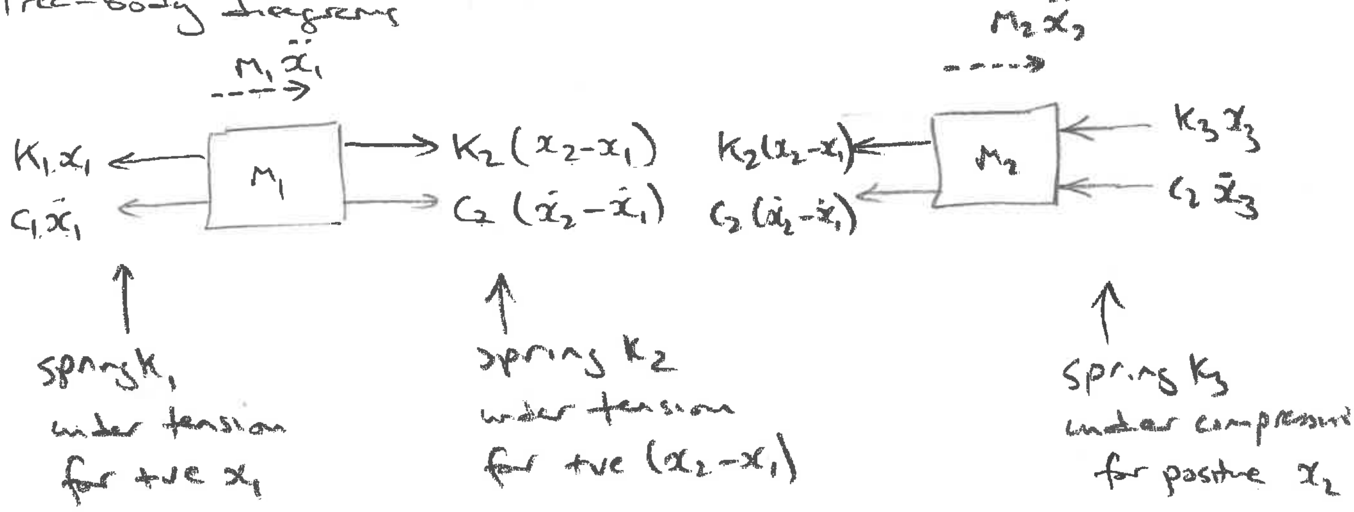
n: Number of degrees of freedom

Consider $n = 2$

Ex



(1) Free-body diagrams



$\frac{L19}{4}$

Mass#1 $\xrightarrow{+ve}$ $\sum F_1 = m_1 \ddot{x}_1$

$$-k_1 x_1 - c_1 \dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + f_1(t) = m_1 \ddot{x}_1$$

$$\Rightarrow m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = f_2 \quad \underline{\text{1st EOM}}$$

Mass#2 $\xleftarrow{+ve}$ $\sum F_2 = m_2 \ddot{x}_2$

$$k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + k_3 x_2 + c_3 \dot{x}_2 - f_2(t) = -m_2 \ddot{x}_2$$

$$\Rightarrow m_2 \ddot{x}_2 - c_2 \dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2 \quad \underline{\text{2nd EOM}}$$

In ~~matrix~~ ~~not~~ ~~form~~:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Using ~~matrix~~ matrix notation:

$$\underline{M}\,\ddot{\underset{\sim}{x}} + \underline{C}\,\dot{\underset{\sim}{x}} + \underline{K}\,\underset{\sim}{x} = \underset{\sim}{f}$$

where $\underset{\sim}{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$, $\underset{\sim}{f} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$

$$\underline{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}, \quad \underline{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

for $\underline{n = 3}$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}, \quad \underline{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

$\frac{4/9}{5}$ ② $\underline{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\underline{f} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

③ Obtain vibration characteristics of "basic system" ie/ undamped, unforced system

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{M}\,\underset{\sim}{\ddot{x}} + \underline{K}\,\underset{\sim}{x} = \underset{\sim}{0} \qquad (1)$$

zero vector

Assume $\underset{\sim}{x} = \underset{\sim}{X} e^{i\omega t}$ where $\underset{\sim}{X}$ is unknown; assume two masses can oscillate harmonically at the same frequency.

$$\underset{\sim}{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\underset{\sim}{\dot{x}} = i\omega \underset{\sim}{X} e^{i\omega t}$$

$$\underset{\sim}{\ddot{x}} = -\omega^2 \underset{\sim}{X} e^{i\omega t}$$

let $\lambda = \omega^2$

Plug into (1) (Plug $\underset{\sim}{x}, \underset{\sim}{\ddot{x}}, \lambda$ into (1)):

$$\left[-\omega^2 \underset{\sim}{X}\,\underline{M}\,\cancel{e^{i\omega t}} + \underset{\sim}{X}\,\underline{K}\,\cancel{e^{i\omega t}}\right] = 0$$

$$\left[\underline{K} - \lambda \underline{M}\right] \underset{\sim}{X} = \underset{\sim}{0} \longrightarrow \text{Eigenvalue problem (EVP)}$$

Can solve as an EVP, <u>or</u> can solve directly by deriving the corresponding "characteristic equation".

$$\underbrace{[\underline{K} - \lambda \underline{M}]}\ \underset{\sim}{X} = \underset{\sim}{0} \qquad \text{——————} \quad (2)$$

For non-trivial solution, this has to be uninvertible

$$\underbrace{\left| \underline{K} - \lambda \underline{M} \right| = 0}_{\text{determinant of } [\underline{K} - \lambda \underline{M}] = 0}$$

→ derive a characteristic equation

→ solve for roots of characteristic equation

→ "vibration characteristics"

[Alternatively, can obtain "vibration characteristics" by solving EVP]

Steps
- Remove $\underline{C}$, $\underline{f}$
- Assume $\underset{\sim}{x} = X e^{i\omega t}$
- Obtain [2]
- Setup characteristic equation
- Solve for $\underbrace{\lambda_1, \lambda_2}_{\text{roots}}$ (ie $\underbrace{\omega_1 = \sqrt{\lambda_1}, \omega_2 = \sqrt{\lambda_2}}_{\text{two natural frequencies}}$) - Eigenvalues
- Plug λ_1, λ_2 into (1), to get $\underset{\sim}{u_1}, \underset{\sim}{u_2}$ - Eigenvectors

Eigenvalues (natural frequencies) $n=2$ of them + Eigenvectors (mode shapes) $n=2$ of them = Eigensolutions $n=2$ sets

→ 2 scalars λ_1, λ_2 (ω_1, ω_2) → 2 vectors $\underset{\sim}{u_1}, \underset{\sim}{u_2}$ $(\lambda_i, \underset{\sim}{u_i})$ $i = 1, 2$

For $n=3$, we will get $\omega_1, \omega_2, \omega_3$; $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \underset{\sim}{u_3}$ - -