ASEN 3112

Spring 2020

Lecture 10

Whiteboard

February 25, 2020

Internal Virtual Work for Structural Models

The following expressions are often used for the **Virtual Displacement Method**:

- General 3D body:

$$\delta W_{ie} = \int\int\int_V \left(\sigma_{xx}\ \delta\varepsilon_{xx} + \sigma_{yy}\ \delta\varepsilon_{yy} + \sigma_{zz}\ \delta\varepsilon_{zz} + \tau_{xy}\ \delta\gamma_{xy} + \tau_{xy}\ \delta\gamma_{xy} + \tau_{xy}\ \delta\gamma_{xy} \right) dx\ dy\ dz$$

- Bar (constant E, A along bar; stress and strain constant along bar):

$$\delta W_{ie,bar} = \frac{E\ A}{L} \left(\Delta L \right) \delta \left(\Delta L \right)$$

- Beam (constant E constant over cross section):

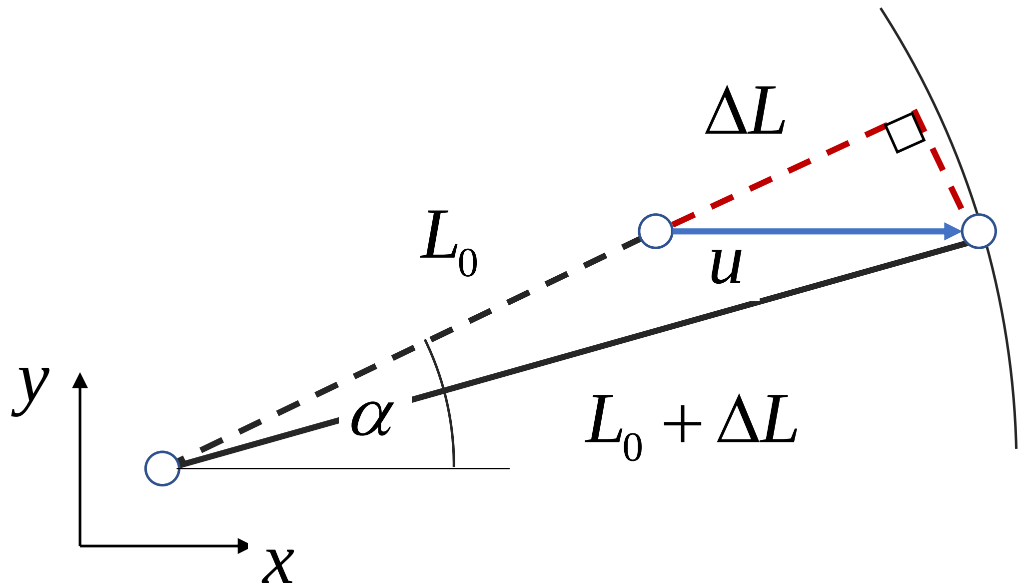
$$\delta W_{ie,beam} = \int_L E\ I\ \kappa\ \delta\kappa\ dx$$

- Shaft (constant G constant over cross section):

$$\delta W_{ie,shaft} = \int_L G\ J \left(\frac{d\phi}{dx} \right) \delta \left(\frac{d\phi}{dx} \right) dx$$

Kinematics of Bar

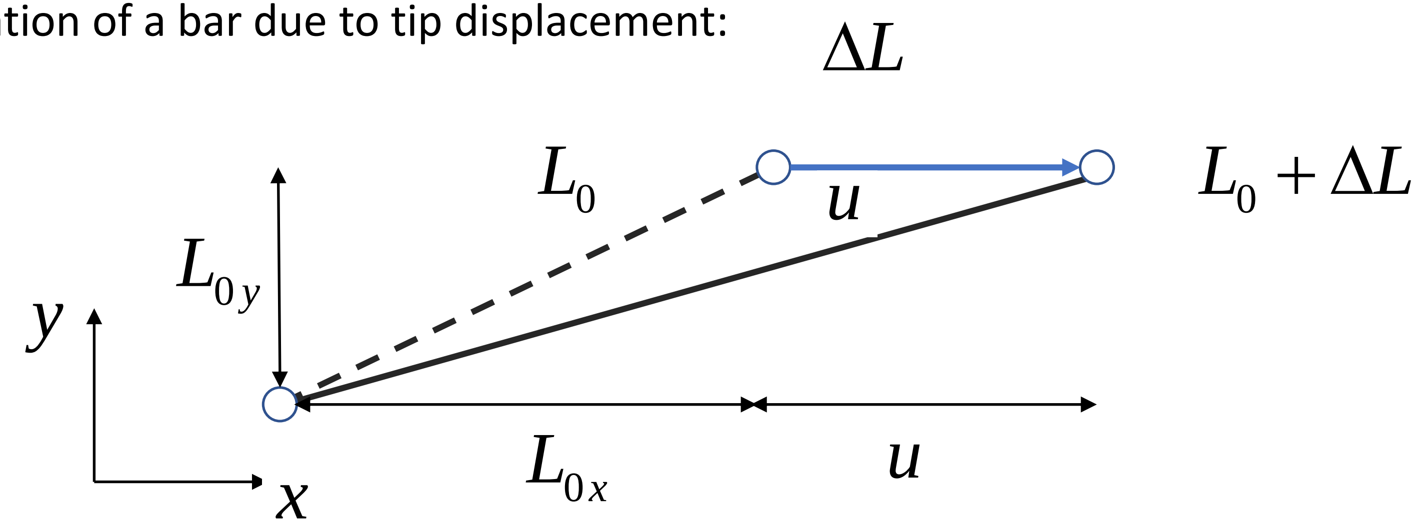
Elongation of a bar due to tip displacement:



$$\boxed{\Delta L = u\cos(\alpha)}$$

Kinematics of Bar

Elongation of a bar due to tip displacement:



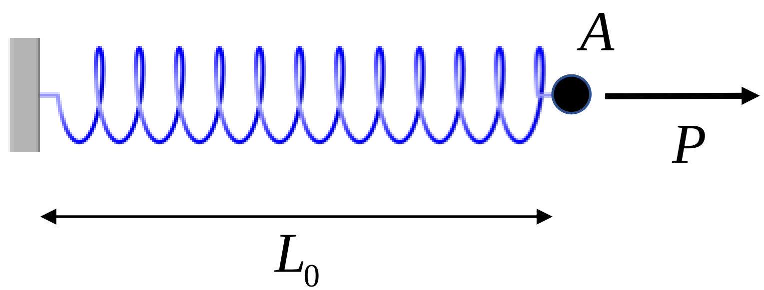
$$L_0^2 = L_{0x}^2 + L_{0y}^2 \qquad \left(L_0 + \Delta L\right)^2 = \left(L_{0x} + u\right)^2 + L_{0y}^2 \qquad \cos\left(\alpha\right) = \frac{L_{0x}}{L_0}$$

$$\frac{2\Delta L}{L_0} + \cancel{\left(\frac{\Delta L}{L_0}\right)^2} = \frac{2L_{0x}u}{L_0^2} + \cancel{\left(\frac{u}{L_0}\right)^2} \qquad \frac{2\Delta L}{L_0} = \frac{2u}{L_0}\cos\left(\alpha\right) \qquad \boxed{\Delta L = u\cos\left(\alpha\right)}$$

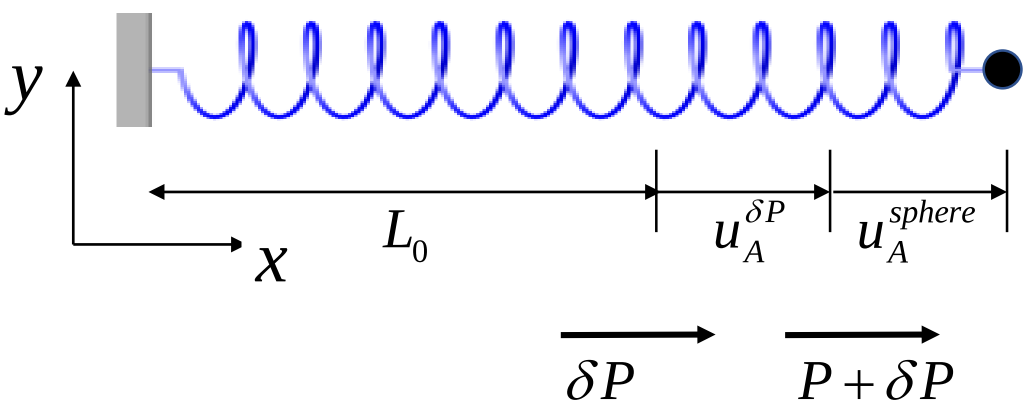
$$\left|\frac{\Delta L}{L_0}\right| \ll 1 \qquad \left|\frac{u}{L_0}\right| \ll 1$$

Infinitesimal stain assumption

Virtual Force Method (1)



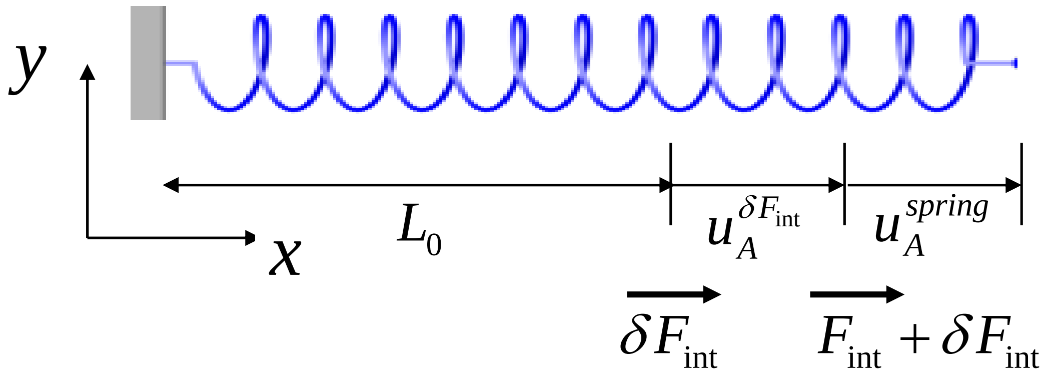
The external virtual work done by **virtual force $\boldsymbol{\delta P}$** on **real displacement $\boldsymbol{u_A^{sphere}}$**:

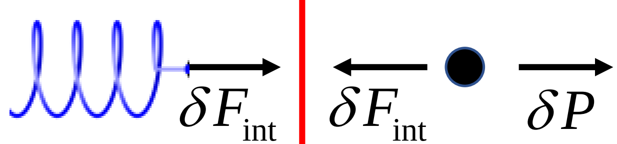


$$\boxed{\delta W_e^* = \delta P \, u_A^{sphere}}$$

Virtual Force Method (2)

The internal virtual work done by **virtual force δF_{int}** on **real displacement u_A^{spring}**:





$$\delta W_i^* = -\delta F_{int}\; u_A^{spring}$$

Principle of Virtual Work for Virtual Forces

Static equilibrium: $\delta P = \delta F_{\text{int}}$

Principle of Virtual Work:

$$\delta W^* = \delta W_e^* + \delta W_i^* = 0 \qquad \text{or} \qquad \boxed{\delta W_e^* = \delta W_{ie}^*}$$

$$\delta W = \left(u_A^{sphere} - u_A^{spring} \right) \delta P = 0$$

Since δP_A arbitrary:

$$P = F_{int}$$

The virtual work done by the virtual forces **that are in static equilibrium** on the real displacements vanishes if the displacements are compatible.

Internal Virtual Work for Structural Models

The following expressions are often used for the **Virtual Force Method**:

- General 3D body:

$$\delta W_{ie}^{*} = \int\int\int_V \left(\delta\sigma_{xx}\ \varepsilon_{xx} + \delta\sigma_{yy}\ \varepsilon_{yy} + \delta\sigma_{zz}\ \varepsilon_{zz} + \delta\tau_{xy}\ \gamma_{xy} + \delta\tau_{xy}\ \gamma_{xy} + \delta\tau_{xy}\ \gamma_{xy} \right) dx\ dy\ dz$$

- Bar (constant E, A along bar; stress and strain constant along bar):

$$\delta W_{ie,bar}^{*} = \frac{L\,N\,\delta N}{EA}$$

- Beam (constant E constant over cross section):

$$\delta W_{ie,beam}^{*} = \int_L \frac{M\ \delta M}{E\ I}\,dx$$

- Shaft (constant G constant over cross section):

$$\delta W_{ie,shaft}^{*} = \int_L \frac{T\ \delta T}{G\ J}\,dx$$

Unit Dummy-Load Method (1)

External virtual work:

- Virtual force δP*=1* :

$$\delta W_e^* = \bar{1}\, d \qquad d:\ \text{displacement in direction of dummy force}$$

- Virtual force δM*=1* :

$$\delta W_e^* = \bar{1}\, \phi \qquad \phi:\ \text{rotation angle about same axis as dummy moment}$$

Internal virtual work:

- Bar (constant E, A along bar; stress and strain constant along bar):

$$\delta W^*_{ie,bar} = \frac{L\ N\ \bar{n}}{E\ A} \qquad \bar{n}:\ \text{normal force in bar due to dummy load}$$

- Beam (constant E constant over cross section):

$$\delta W^*_{ie,beam} = \int_L \frac{M\ \bar{m}}{E\ I}\, dx \qquad \bar{m}:\ \text{internal bending moment due to dummy load}$$

Unit Dummy-Load Method (2)

Truss:

$$\bar{1}\ d = \sum_{i=1}^{N_b} \frac{L_i\ N_i\ \bar{n}_i}{E_i\ A_i}$$

N_i : normal force due to real load

$\bar{n}_i$: normal force due to dummy load in the direction of displacement d

Beam:

$$\bar{1}\ d = \int_L \frac{M\ \bar{m}}{E\ I}\ dx \qquad \text{or} \qquad \bar{1}\ \phi = \int_L \frac{M\ \bar{m}}{E\ I}\ dx$$

M : bending moment due to real load

$\bar{m}$: bending moment due to dummy force in the direction of displacement d

or dummy moment about the axis of rotation ϕ