ASEN 3112

Spring 2020

Lecture 22

Whiteboard

April 9, 2020

4/9/2020

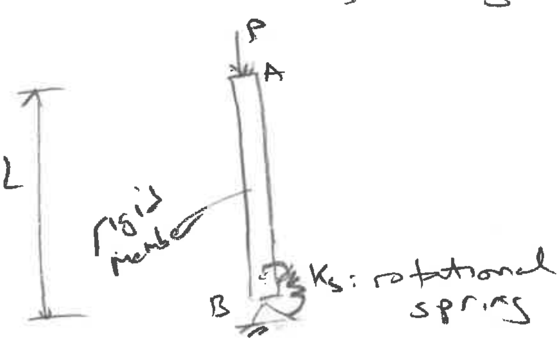
Stability of Structures

In structural mechanics, structures or structural systems under certain conditions — configuration, nature of loading, geometric properties, material properties — can experience sudden and excessive deformation under compressive-type loading

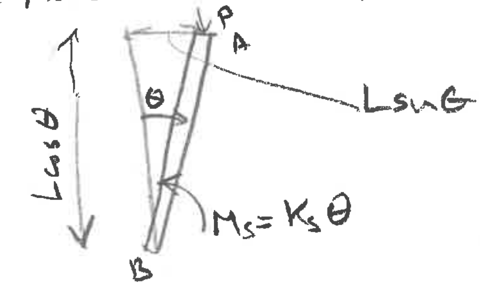
- commonly known as "buckling"
- once buckling occurs, the structure may be <u>stable</u> or <u>unstable</u>
- if post-buckling is unstable, the system may no longer bear load

Theory — Read textbook chapter & lecture notes

Consider following structural system:



1. Assume some "deformed" configuration



2. Moment equilibrium

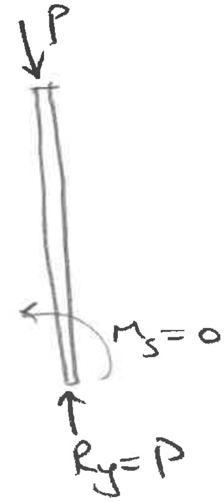
$$\Sigma M_B = 0 \quad \curvearrowleft +$$

$$M_s - PL\sin\theta = 0$$

$$\boxed{K_s\theta - PL\sin\theta = 0}$$

Two possible states.

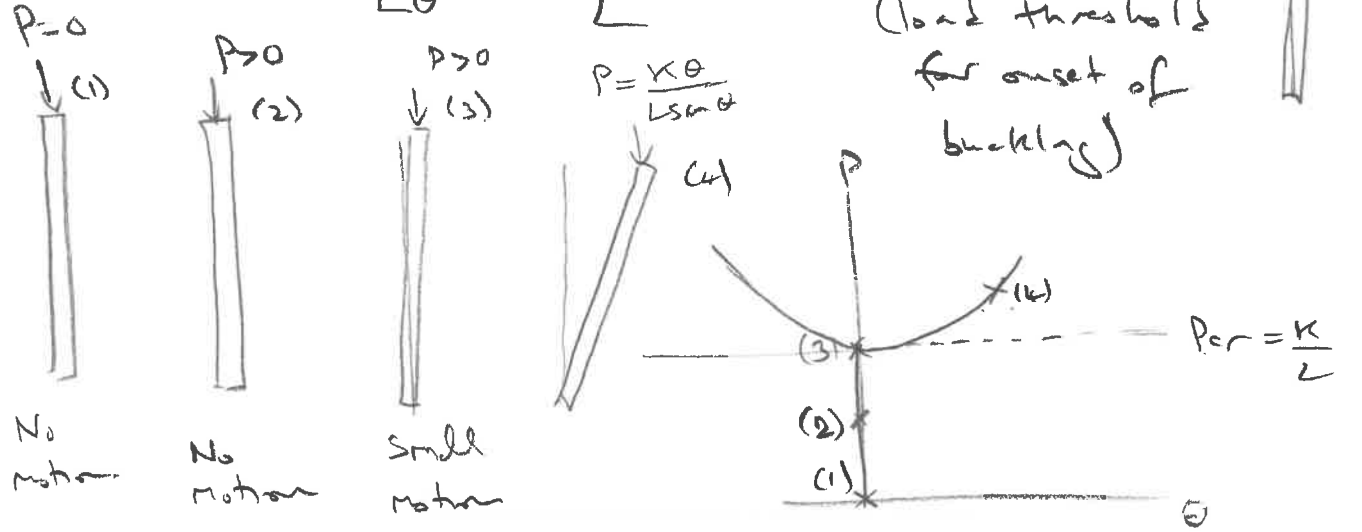
a) $\theta = 0 \Longrightarrow \sin\theta = 0$



b) $P = \dfrac{K\theta}{L\sin\theta}$

Take limit as $\theta \to 0$, $\sin\theta \to \theta$

$$P_{cr} \approx \frac{K\theta}{L\theta} = \frac{K}{L}$$ $\rightsquigarrow$ critical load (load threshold for onset of buckling)



P vs θ : θ grows faster than sinθ

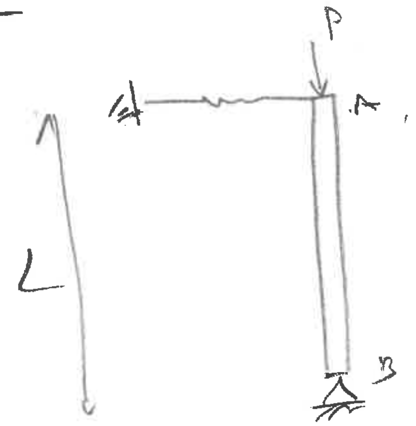
Post-buckling

$$\left.\frac{\partial P}{\partial\theta}\right|_{\theta=0} = \frac{\partial}{\partial\theta}\left(\frac{K\theta}{L\sin\theta}\right) > 0$$ $\rightsquigarrow$ stable post-buckling state

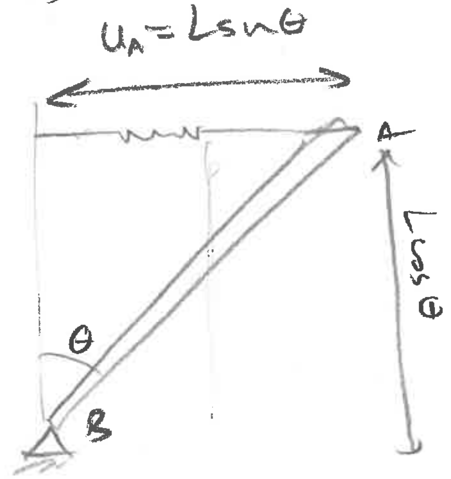
[Alternatively, look at energy $E = \underset{\text{potential}}{U} - \underset{\text{Work}}{W} = \frac{1}{2}K\theta^2 - P(L - L\cos\theta)$

$\frac{\partial E}{\partial\theta} = 0 \Rightarrow P = \frac{K\theta}{L\sin\theta}$; $\frac{\partial^2 E}{\partial\theta} < 0 \rightsquigarrow$ stable (if >0, unstable)]

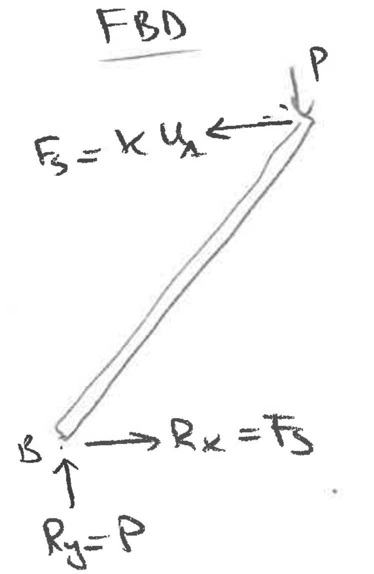
$\frac{L22}{3}$



1) Consider deformed config.



2) Draw FBD



3) $\sum M_B = 0$

$$PL\sin\theta - F_s L\cos\theta = 0$$

$$PL\sin\theta = F_s L\cos\theta$$

$$PL\sin\theta = k u_A L\cos\theta$$

$$PL\sin\theta = kL\sin\theta L\cos\theta$$

$$(P - kL\cos\theta)\, L\sin\theta = 0$$

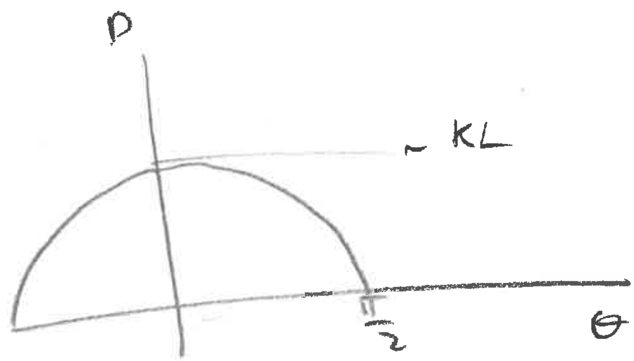
For nontrivial solution:

$$P - kL\cos\theta = 0$$

$$P_{cr} = kL\cos\theta$$

$$\theta \to 0,\ \cos\theta \to 1$$

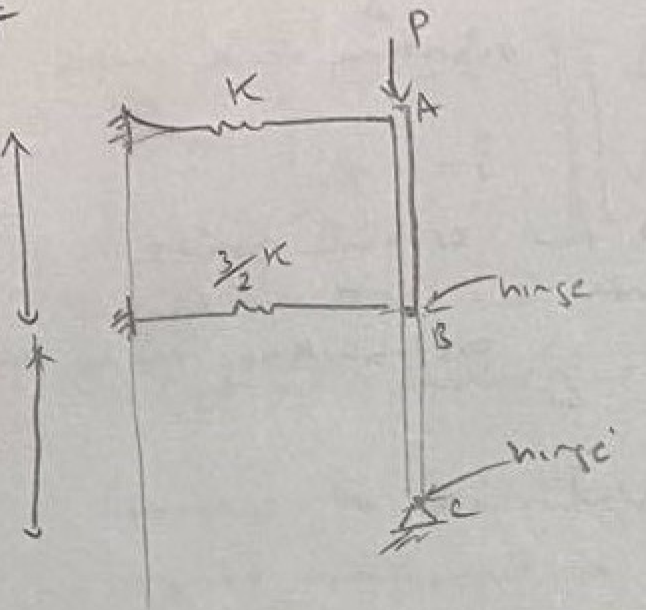
$$P_{cr} \approx kL$$

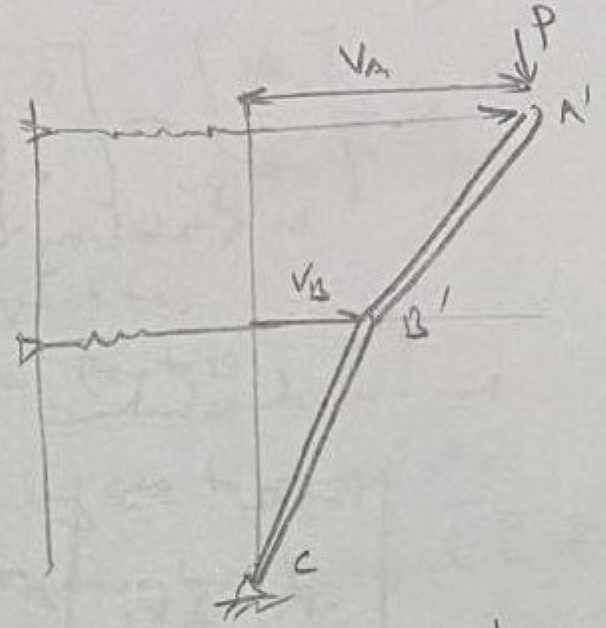


$$\left.\frac{\partial P}{\partial \theta}\right|_{\theta=0} = -kL\sin\theta < 0$$

$\Longrightarrow$ unstable

Now consider a system consisting of 2 degrees of freedom





V_A, V_B independent degrees of freedom

Consider 2 subsystem (full system being an option):

- strut AB & strut BC
- strut AB & column ABC ⟵ select this one
- strut BC & column ABC

Strut AB

$$\Sigma M_{B'} = 0$$

$$P(v_A - v_B) - F_A\left(\frac{L}{2}\right) = 0$$

$$\boxed{P(v_A - v_B) - K v_A \frac{L}{2} = 0}$$

Column ABC

$$\Sigma M_C = 0$$

$$\boxed{P v_A - KL v_A - 3KL \frac{v_B}{4}}$$

$$\underbrace{\begin{bmatrix} P - \frac{1}{2}KL & -P \\ P - KL & -\frac{3}{4}KL \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} v_A \\ v_B \end{bmatrix}}_{\underset{\sim}{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{0}}$$

matrix stability Equation

⟹ EVP for stability

stability matrix

$\frac{1 \cdot 22}{5}$

Note: In vibrations $\longrightarrow$ EVP for natural frequencies & vibration mode shapes

In stability $\longrightarrow$ EVP for critical loads & post-buckling mode shapes

Non-trivial solution

$$\begin{vmatrix} P-\frac{1}{2}KL & -P \\ -P & -\frac{3}{4}KL \end{vmatrix} = 0$$

$\Rightarrow$ characteristic equation:

$$P^2 - \frac{7}{4}KLP + \frac{3}{8}K^2L^2 = 0 \longrightarrow \text{quadratic equation.}$$

$$P_{cr_1} = \frac{1}{4}KL = 0.25\,KL, \quad P_{cr_2} = \frac{3}{2}KL = 1.5\,KL$$

The critical load is the smallest of two

$$P_{cr} = P_{cr_1} = \frac{1}{4}KL = \underline{0.25KL}$$

$\Rightarrow$ Eigenvalues give critical loads (lowest one is "the" critical load)

$\Rightarrow$ Eigenvectors give post buckling mode shapes

Post-Buckling mode shapes (interested in both)

Substitute $P_{cr_1} = \frac{1}{4}KL$ into $\underset{\sim}{A}\,\underset{\sim}{v} = \underset{\sim}{0}$

$$\frac{KL}{4}\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} V_{A_1} \\ V_{B_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} V_{A_1} \\ V_{B_1} \end{bmatrix} = C_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1st mode shape.

Substitute $P_{cr_2} = \frac{3}{2} kL$ into $\underset{\sim}{A}\,\underset{\sim}{v} = \underset{\sim}{0}$

$$\frac{kL}{4}\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_{A2} \\ v_{B2} \end{bmatrix} = c_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

Eigenvectors are not absolute quantities

c_1, c_2 are arbitrary, nonzero factors

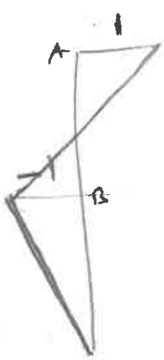
Common to normalize in a way as to make the largest component in each eigenvector to equal to 1.

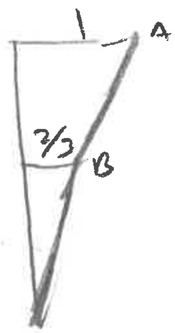
$c_1 = -1$, $c_2 = 2/3$

$$\underset{\sim}{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \underset{\sim}{v_2} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

for P_{cr_1} — 1st stability mode shape

for P_{cr_2} — 2nd stability mode shape





These are "discrete" buckling problems