# ASEN 3112:
# Spring 2020
# Exam 2

Date: March 17, 2020

You are expected to finish the exam by 2:15 pm. **We will then give you an additional half hour, until 2:45 pm, to submit your exam to Gradescope.** This half hour is not intended to you extra time to solve the exam problems; it is solely intended for you to upload your exam to Gradescope. So, we expect you to hold to the Honor Code and only spend 1 hour and 15 minutes solving the problems.

At the end of the exam, please write the following text somewhere and sign your name, indicating that you abided by the CU Boulder Honor Code:

*"On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this exam."*

**If you do not write and sign the honor code we are not obligated to grade your exam.**

Please label your work with the part of the problem you are working on and circle your final answer for each part.

1. This exam is **open-book** and **open-notes**. You are NOT allowed to collaborate with other students to solve the problem.

2. Solve all four problems.

3. Total time for the exam is 1 hour and 15 minutes. You then have an additional 30 minutes to scan and upload your exam to Gradescope. This 30 minutes is not to be used to continue working on solving the exam problems.

4. Do not redefine the problem; carefully read the problem statement and answer the questions that are asked.

5. Make sure that one can follow your analysis; describe briefly what you are doing.

6. You must show how you got to your solution. Simply showing the final results will lead to point reduction.

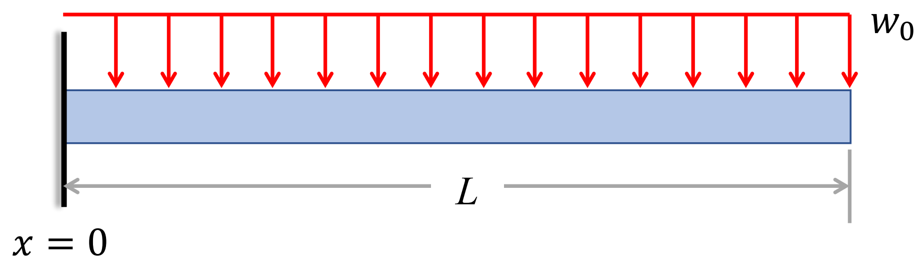
7. You must cross out all work you do not want graded. Any work that is not crossed out is fair game for grading.

8. Include units on final answers whenever applicable

9. Write your solution on any paper or electronic medium (e.g. on your iPad), but please make it clear where your work is for every problem.

**Question 1.** **20 points**

A beam of length $L$ is welded to a wall at $x = 0$. The beam is loaded by its own weight, which is modeled as a downward distributed load $w(x)$ with a constant magnitude of known value $w_0$. The beam has a constant flexural rigidity $EI$.




a) What are the units of flexural rigidity $EI$ in SI units?

b) Draw a global free-body diagram of the entire bar.

c) Calculate an equation for the bending moment throughout the entire beam $M(x)$ using any method. Your equation should be in terms of some or all of the following variables: $L$, $EI$, $w_0$, and $x$. **Please clearly write the method you are using to facilitate grading**.

d) Given your answer in part b, is the stress $\sigma$ on the **bottom** of the beam at $x = {}^{L}/_{2}$ tensile or compressive? Justify your answer. **Hint:** Recall the flexure formula from ASEN 2001, $\sigma = -\frac{My}{I}$.

e) Calculate the elastic strain energy $U$ of the beam.

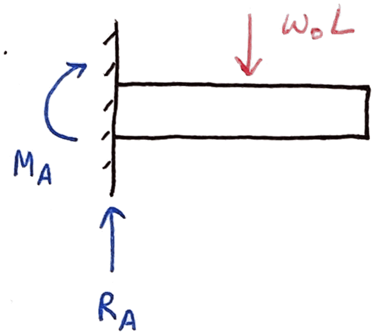
# Question 1

a. $E = [Pa] = [N/m^2]$

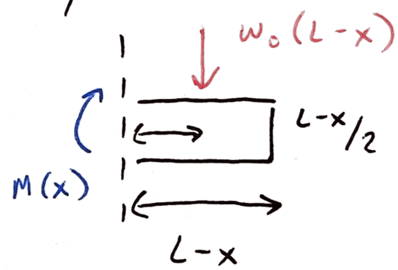
$I = [m^4]$

$EI = [N \cdot m^2]$

b.




c. By cut




$$\Sigma M_{cut} = 0 = -M(x) - \frac{w_0}{2}(L-x)^2$$

$$M(x) = \frac{-w_0}{2}(L-x)^2$$

$=$

By integration

$$w(x) = -w_0 = EIv''''(x)$$

$$-V(x) = -w_0 x + C_1 = EIv'''(x)$$

$$M(x) = \frac{-1}{2}w_0 x^2 + C_1 x + C_2 = EIv''(x)$$

BCs: $V(L) = M(L) = 0$

$$0 = -w_0 L + C_1 \qquad 0 = \frac{-1}{2}w_0 L^2 + w_0 L^2 + C_2$$

$$C_1 = w_0 L \qquad C_2 = \frac{-1}{2}w_0 L^2$$

$$M(x) = \frac{-1}{2}w_0 x^2 + w_0 L x - \frac{1}{2}w_0 L^2$$

d. $M\left(\frac{L}{2}\right) = \frac{-w_0}{2}\left(L - \frac{L}{2}\right)^2 = \frac{-w_0}{2}\left(\frac{L}{2}\right)^2$

$M\left(\frac{L}{2}\right) < 0$, so beam is bending like 

Stress on bottom is <u>compressive</u>.

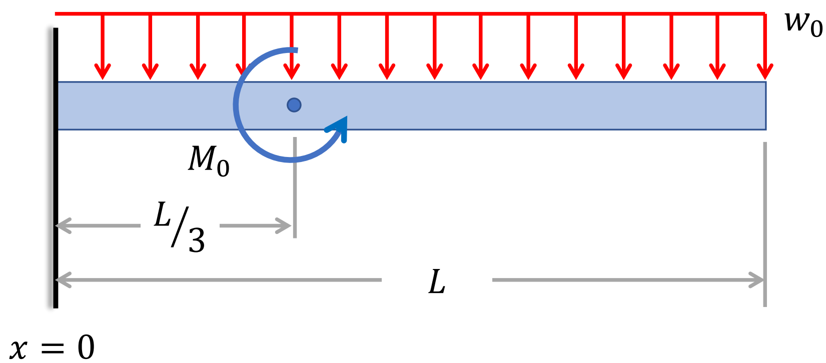
e. $$U = \int_0^L \frac{M^2}{2EI}\,dx = \frac{1}{2EI}\int_0^L \frac{w_0^2}{4}(L-x)^4 = \frac{w_0^2}{8EI}\int_0^L (L-x)^4$$

$$= \frac{w_0^2}{8EI}\left(\frac{-1}{5}(L-x)^5\right)_0^L = \frac{w_0^2}{8EI}\left[0 + \frac{1}{5}L^5\right] = \frac{w_0^2 L^5}{40EI}$$

**Question 2. 20 points**

This question features the same beam as in Question 1, with an external point moment added. The beam is of length $L$ and has a constant flexural rigidity $EI$. It is welded to a wall at $x = 0$. The beam is loaded by its own weight, which is modeled as a downward distributed load $w(x)$ with a constant magnitude of known value $w_0$.

Now, an external point moment with a known value $M_0$ in the counter-clockwise direction is applied to the beam at $x = {}^{L}/_{3}$.




a) Is this system statically determinate? Justify your answer.

b) If this system is statically determinate, describe a way it could be made statically indeterminate. If the system is statically indeterminate, describe a way it could be made statically determinate.

c) Write a fourth-order ordinary differential equation (which may be piecewise), that, when integrated, will define the deflection $v(x)$ over the length of the beam. If your equation is piecewise, be sure to define the x-limits of each piece. Your equation should be in terms of some or all of the following variables: $L$, $EI$, $w_0$, $M_0$, $v$ and its derivatives, and $x$. **Note: You should not do any integration for this part.**

d) Write all of the boundary conditions and matching conditions needed to fully solve for all the constants that arise when finding the deflection $v(x)$ by integrating the fourth-order ordinary differential equation you wrote in part c. Your equation should be in terms of some or all of the following variables: $L$, $EI$, $w_0$, $P_0$, $M_0$, $v$ and its derivatives, and $x$. **Note: You should not do any integration for this part.**

## Question 2

a. Yes, this beam is statically determinate. There are 3 unknown reactions at the wall $(R_x, R_y, M)$ & 3 eqns $(\Sigma F_x, \Sigma F_y, \Sigma M)$.

b. You could make this beam statically indeterminate by adding an additional (redundant) support.

c. $w(x) = -w_o$

$EIv''''(x) = -w_o$ But eqn is piecewise!

$$EIv''''(x) = \begin{cases} -w_o & 0 \le x \le \frac{L}{3} \\ -w_o & \frac{L}{3} \le x \le L \end{cases}$$

d. BCs: $v_1(0) = 0$ $\quad V_2(L) = 0 = -EIv_2'''(L)$

$v_1'(0) = 0$ $\quad M_2(L) = 0 = EIv_2''(L)$

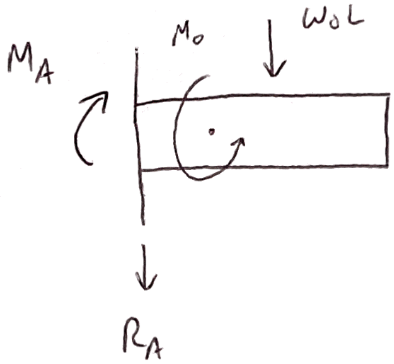
Also OK to say v_2'''(L) = 0 and v_2''(L) = 0

MCs: $v_1\left(\frac{L}{3}\right) = v_2\left(\frac{L}{3}\right)$

$v_1'\left(\frac{L}{3}\right) = v_2'\left(\frac{L}{3}\right)$

$V_1\left(\frac{L}{3}\right) = V_2\left(\frac{L}{3}\right)$ Also OK to say v_1'''(L) = v_2'''(L)

$M_2\left(\frac{L}{3}\right) - M_1\left(\frac{L}{3}\right) = -M_o$




V(x=0) and M(x=0)
boundary conditions

$$\Sigma F_y = 0 = -w_o L - R_A$$

$$R_A = -w_o L$$

$$V(0) = R_A = -w_o L = -EIv'''(x)$$

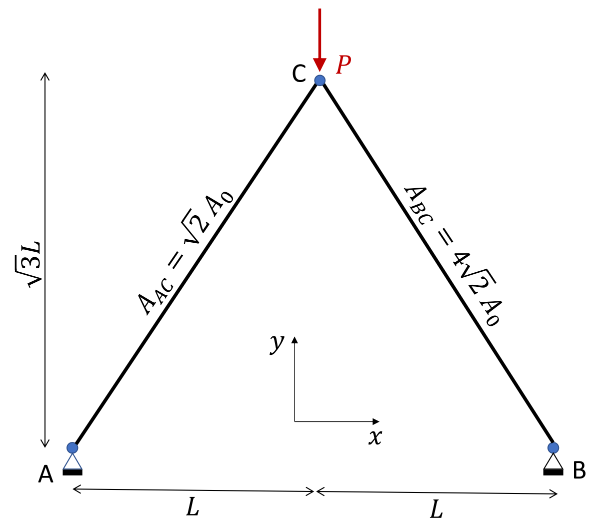
$$\Sigma M_A = 0 = -M_A + M_o - w_o L\left(\frac{L}{2}\right)$$

$$M_A = M_o - \frac{1}{2} w_o L^2$$

$$M(0) = M_A = M_o - \frac{1}{2} w_o L^2 = EIv''(x)$$

**Question 3. 30 points**

Consider the 2-bar truss shown in the figure. At joint C, a force of magnitude $P$ is acting downwards in the vertical direction, i.e., in negative y-direction. Joints A and B are pinned. The dimensions and geometric properties are given in the figure. Note the cross-sectional area of bar BC is four times that of bar AC. Both bars are made of an isotropic material with Young's modulus $E$. In your calculations, keep the length $L$, the load $P$, the cross-sectional area $A_0$, and the Young's modulus $E$ as symbols.




Assume a linear elastic response, infinitesimal strains, and small displacements and rotations.

a) Compute the forces in the bars.

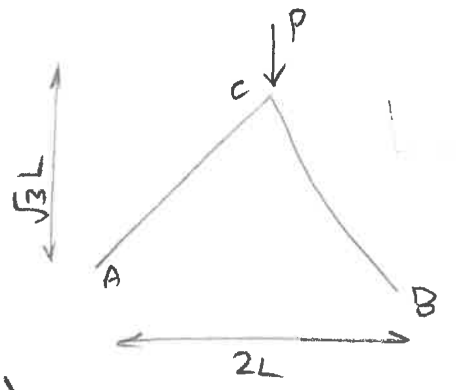
b) Compute the elastic strain energy stored in the truss.

c) Compute the displacement of joint C in the **vertical direction** by the **Conservation of Energy Principle**.

d) Compute the displacement of joint C in **horizontal direction** by the **Virtual Displacement Method**.
Hint: You may use the result from Part c to solve this problem. This will reduce the number of unknowns in Part d from two to one. Thus, you only need **one equation** to solve for the remaining unknown. If you have not solved Part c, assume that the displacement in $y$-direction at joint C is $v_C = -(5\sqrt{2}PL)/(12EA_0)$.

e) Using the results in Part d, compute the force in the bar BC and verify that this force equals the one computed in Part a.

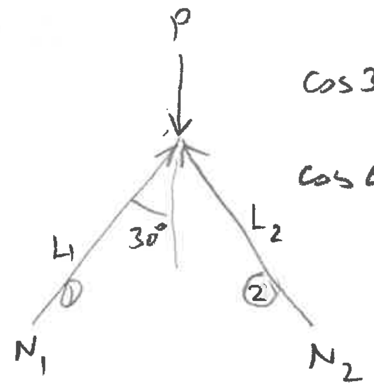
# Question 3




$A_{AC} = \sqrt{2} A_0$,

$A_{BC} = 4\sqrt{2} A_0$

$E_{AC} = E_{BC} = E$




$\cos 30° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

(a)

$\sum F_x = 0$

$\frac{1}{2} N_1 - \frac{1}{2} N_2 = 0 \implies N_1 = N_2$

$\sum F_y = 0$

$-P + \frac{\sqrt{3}}{2} N_1 + \frac{\sqrt{3}}{2} N_2 = 0$

$P = \frac{\sqrt{3}}{2} N_1 + \frac{\sqrt{3}}{2} N_1 = \sqrt{3} N_1$

$N_1 = \frac{P}{\sqrt{3}} = \frac{\sqrt{3}}{3} P$

$N_2 = \frac{\sqrt{3}}{3} P$

$L^2 + (\sqrt{3}L)^2 = L_1^2$

$L_1 = \sqrt{4} L$

$L_2 = \sqrt{4} L$

(b)

$$U = \frac{1}{2}\left(\frac{N_1^2 L_1^2}{E_1 A_1} + \frac{N_2 L_2}{E_2 A_2}\right)$$

$$= \frac{1}{2}\left(\frac{P^2}{3}\frac{\sqrt{4}L}{E\sqrt{2}A_0} + \frac{P^2}{3}\frac{\sqrt{4}L}{E4\sqrt{2}A_0}\right)$$

$$= \frac{1}{2}\left(\frac{P^2\sqrt{2}L}{3EA_0} + \frac{P^2\sqrt{2}L}{3E4A_0}\right)$$

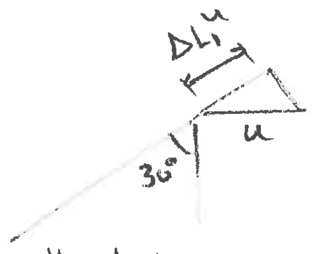
$$= \frac{P^2 L\sqrt{2}}{6EA_0}\left(1 + \frac{1}{4}\right)$$

$$\boxed{U = \frac{5}{24}\sqrt{2}\frac{P^2 L}{EA_0}}$$

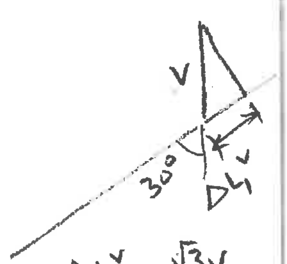
(c)

$$W_e = U \implies \frac{1}{2}\cancel{P}V_c = \frac{5}{24}\sqrt{2}\frac{P^{\cancel{2}}L}{EA_o}$$

$$\boxed{V_c = \frac{5\sqrt{2}}{12}\frac{PL}{EA_o}}$$




$\Delta L_1^u = \frac{1}{2}u_c$




$\Delta L_1^v = \frac{\sqrt{3}}{2}V_c$

(d)

$$\Delta L_1 = \frac{1}{2}u_c + \frac{\sqrt{3}}{2}V_c$$

$$\Delta L_2 = -\frac{1}{2}u_c + \frac{\sqrt{3}}{2}V_c$$

Considering virtual displacement $\delta u_c$

$$\delta \Delta L_1^u = \frac{1}{2}\delta u_c$$

$$\delta \Delta L_2^u = -\frac{1}{2}\delta u_c$$

$$\delta W_e^u = \delta W_{ie}^u$$

$$0 = \frac{E_1 A_1}{L_1}(\Delta L_1)(\delta L_1^u) + \frac{E_2 A_2}{L_2}(\Delta L_2)(\delta L_2^u)$$

$$0 = \frac{E\sqrt{2}A_o}{\sqrt{4}L}\left(\frac{1}{2}u_c + \frac{\sqrt{3}}{2}V_c\right)\left(\frac{1}{2}\delta u_c\right) + \frac{E4\sqrt{2}A_o}{\sqrt{4}L}\left(-\frac{1}{2}u_c + \frac{\sqrt{3}}{2}V_c\right)\left(-\frac{1}{2}\delta u_c\right)$$

$$0 = \frac{EA_o}{4\sqrt{2}L}\left(u_c + \sqrt{3}V_c\right)\delta u_c + \frac{E\cancel{4}A_o}{\cancel{4}\sqrt{2}L}\left(u_c - \sqrt{3}V_c\right)\delta u_c$$

$$0 = \frac{EA_o}{\sqrt{2}L}\left[\frac{1}{4}\left(u_c + \sqrt{3}V_c\right) + \left(u_c - \sqrt{3}V_c\right)\right]\delta u_c$$

$$0 = \frac{EA_o}{\sqrt{2}L}\left[\frac{5}{4}u_c + \frac{\sqrt{3}}{4}V_c - \sqrt{3}V_c\right]\delta u_c$$

$$0 = \frac{EA_o}{\sqrt{2}L}\left[\frac{5}{4}u_c - \frac{3\sqrt{3}}{4}V_c\right]\delta u_c \implies \frac{EA_o}{4\sqrt{2}L}\left[\underbrace{5u_c - 3\sqrt{3}V_c}_{=0}\right]\delta u_c$$

$$u_c = \frac{3\sqrt{3}}{5}V_c = \frac{3\sqrt{3}}{\cancel{5}}\left[\frac{\cancel{5}\sqrt{2}}{\cancel{12}_{4}}\frac{PL}{EA_o}\right] = \frac{\sqrt{6}}{4}\frac{PL}{EA_o} = u_c$$

$$\boxed{u_c = \frac{3}{2\sqrt{6}}\frac{PL}{EA_o}}$$

(d)

$$N_{BC} = EA_{BC} \frac{\Delta L_{BC}}{L_{BC}}$$

$$= EA_2 \frac{\Delta L_2}{L_2}$$

$$= \frac{E 4\sqrt{2} A_0}{\sqrt{4} L} \left( -\frac{1}{2} u_c + \frac{\sqrt{3}}{2} v_c \right)$$

$$= \frac{4 \cancel{E} \cancel{A_0}}{\sqrt{2} \cancel{L}} \left( -\frac{1}{2} \left[ \frac{3}{2\sqrt{6}} \frac{P\cancel{L}}{\cancel{EA_0}} \right] + \frac{\sqrt{3}}{2} \left[ \frac{5\sqrt{2}}{12} \frac{P\cancel{L}}{\cancel{EA_0}} \right] \right)$$

$$= \frac{\cancel{4}}{\cancel{4}\sqrt{2}} \left( -\frac{3}{\sqrt{6}} + \frac{5\sqrt{6}}{6} \right) P$$

$$= \frac{1}{\sqrt{2}} \left( -\frac{3}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} + \frac{5\sqrt{6}}{6} \right) P$$

$$= \frac{1}{\sqrt{2}} \left( -\frac{3\sqrt{6}}{6} + \frac{5\sqrt{6}}{6} \right) P$$

$$= \frac{1}{\sqrt{2}} \left( \frac{2\sqrt{6}}{6} \right) P$$

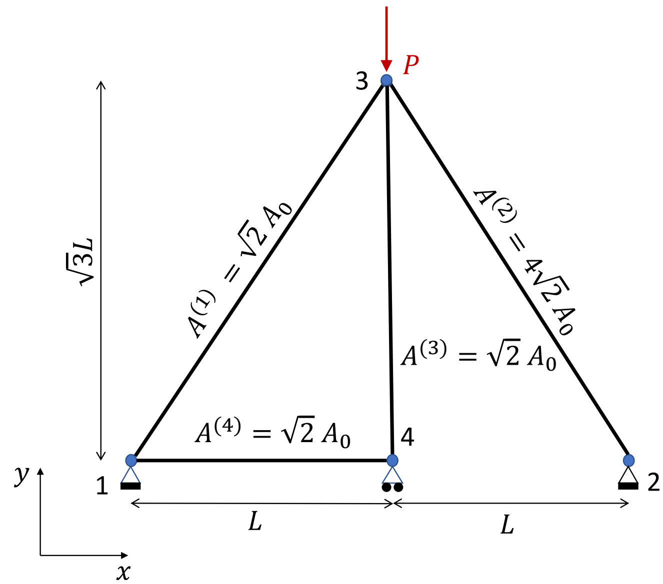
$$N_{BC} = \frac{\sqrt{3}}{3} P$$

From (a)

$$N_2 = \frac{\sqrt{3}}{3} P$$

**Question 4.** **30 points**

Consider the 4-bar truss shown in the figure to the right. It is the same truss as in Problem 3, with 1 additional node and 2 additional bars added. Note that the joints have been numbered 1-4, and the bars have also been numbered (1)-(4). At joint 3, a force of known magnitude *P* is acting downwards in the vertical direction, i.e., in negative y-direction. Joints 1 and 2 are pinned, and joint 4 is a roller joint that restricts motion in the y-direction but allows motion in the x-direction. The dimensions and geometric properties are given in the figure. <u>Note the cross-sectional area of bar (2) is four times that of any other bar.</u> All bars are made of an isotropic material with Young's modulus *E*. In your calculations, keep the length *L*, the load *P*, the cross-sectional area $A_0$, and the Young's modulus *E* as symbols.




Recall that the member stiffness matrix $\boldsymbol{K}$ is a $4 \times 4$ matrix, and it can also be considered to be a $2 \times 2$ matrix of $[\widehat{K}]$ matrices, where each $[\widehat{K}]$ matrix is itself $2 \times 2$. In other words,

$$\boldsymbol{K} = \begin{bmatrix} [\widehat{K}] & -[\widehat{K}] \\ -[\widehat{K}] & [\widehat{K}] \end{bmatrix} \text{ where } [\widehat{K}] = \begin{bmatrix} cos^2\varphi & sin\varphi cos\varphi \\ sin\varphi cos\varphi & sin^2\varphi \end{bmatrix}$$

**Question 4 continues on the next page**

**Question 4, Continued.**

a) What are the dimensions of the global $\boldsymbol{K}$, $\boldsymbol{u}$, and $\boldsymbol{F}$ matrices for the whole truss? What are the dimensions of the reduced global stiffness matrix ($\widetilde{\boldsymbol{K}}$) for the whole truss after you account for the boundary conditions?

b) Show schematically how the **four grey cells** in the global stiffness matrix $\boldsymbol{K}$ (for the whole truss) shown below would be assembled from $[\widehat{K}^{(1)}]$, $\widehat{K}^{(2)}]$, $[\widehat{K}^{(3)}]$, $[\widehat{K}^{(4)}]$, and $[0]$. As an example the top left cell is completed with $[\widehat{K}^{(1)}]+[\widehat{K}^{(4)}]$. You should not calculate the value of any $[\widehat{K}]$ matrices. You do not have to re-draw this matrix, but you must clearly indicate which cell you're writing the answer for (C1–C4).

$$\boldsymbol{K} = \begin{bmatrix} [\widehat{K}^{(1)}] + [\widehat{K}^{(4)}] & & \mathrm{C1} & \\ & \mathrm{C2} & & \\ & & \mathrm{C3} & \\ & \mathrm{C4} & & \end{bmatrix}$$

c) For this part only, assume that $L$ = 12 in., $A_0$ = 1 in$^2$, $P$ = 10,000 lb, and $E = 1 \times 10^7$ psi. The displacements of nodes 1-4 in global coordinate are calculated to be as follows (all units in inches):

| **Node** | **ux** | **uy** |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | $-4.13\times10^{-4}$ | $-3.98\times10^{-4}$ |
| 4 | 0 | 0 |

What is the stress (give a numerical answer) in bar (1)?
Is bar (1) in tension or compression? Justify your answer.

# Question 4

a. 4 joints

$\underline{u} = 2J \times 1 = 8 \times 1$

$\underline{F} = 2J \times 1 = 8 \times 1$

$\underline{K} = 2J \times 2J = 8 \times 8$

Unknown displacements

$\left.\begin{matrix} u_{x3}, u_{y3} \\ u_{x4} \end{matrix}\right\} 3$

Therefore, $\underline{\tilde{K}} = 3 \times 3$

b. $C_1 = (1,3)$ bar 1

$C_2 = (2,2)$ bar 2

$C_3 = (3,3)$ bars 1, 2, 3

$C_4 = (4,2)$ nodes not connected

$C_1 = -[\hat{K}^{(1)}]$

$C_2 = +[\hat{K}^{(2)}]$

$C_3 = [\hat{K}^{(1)}] + [\hat{K}^{(2)}] + [\hat{K}^{(3)}]$

$C_4 = [0]$

c. $\sigma = \frac{E_1}{L_1}(\bar{u}_{x3} - \underbrace{\cancel{\bar{u}_{x1}}}_{=0})$

$L_1 \neq L_g \quad L_1 = \sqrt{L^2 + (\sqrt{3}L)^2} = \sqrt{L^2 + 3L^2} = 2L = 2(12) = 24 \text{ in}$

$\varphi = \tan^{-1}\left(\frac{\sqrt{3}L}{L}\right) = 60°$

$E_1 = E = 1 \times 10^7 \text{ psi}$

$\bar{u}_{x3} = \cos\varphi \, u_{x3} + \sin\varphi \, u_{y3} = \cos 60(-4.13 \times 10^{-4}) + \sin 60(-3.98 \times 10^{-4})$

$\bar{u}_{x3} = -5.51 \times 10^{-4} \text{ in}$

$\sigma = \frac{1 \times 10^7}{24}(-5.51 \times 10^{-4}) = -229.66 \text{ psi}$

compression

4c. Force method

$$\underline{f}^{(1)} = \underline{K}^{(1)} \underline{u}^{(1)}$$

$$\underline{u}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -4.13\times10^{-4} \\ -3.98\times10^{-4} \end{bmatrix}$$

$$\hat{\underline{K}}^{(1)} = \frac{1\times10^{7}\left(\sqrt{2}A_0\right)}{24}\begin{bmatrix} 1/4 & 0.433 \\ 0.433 & 3/4 \end{bmatrix}$$

$$\underline{K}^{(1)} = 4.167\times10^{5} A^{(1)} \begin{bmatrix} 0.25 & 0.43 & -0.25 & -0.43 \\ 0.43 & 0.75 & -0.43 & -0.75 \\ -0.25 & -0.43 & 0.25 & 0.43 \\ -0.43 & -0.75 & 0.43 & 0.75 \end{bmatrix}$$

$$\underline{f}^{(1)} = \begin{bmatrix} 114.83 \\ 198.89 \\ -114.83 \\ -198.89 \end{bmatrix} \cdot A^{(1)}$$

$$\underline{T}^{(1)} = \begin{bmatrix} 0.5 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & 0.5 \end{bmatrix}$$

$$\bar{\underline{f}}^{(1)} = \underline{T}^{(1)} \underline{f}^{(1)}$$

$$\bar{\underline{f}}^{(1)} = \begin{bmatrix} 229.65 \\ 0 \\ -229.65 \\ 0 \end{bmatrix} \cdot A^{(1)}$$

$$\sigma^{(1)} = \frac{-\bar{f}_{x1}}{A^{(1)}} = -229.65 \text{ psi}$$

or

$$\sigma^{(1)} = \frac{+\bar{f}_{xj}}{A^{(1)}} = -229.65 \text{ psi}$$