ASEN 3112

Spring 2020

Lecture 4

January 23, 2020

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Stress - Strains Material Laws

1D Hooke's Law Including Thermal Effects

Stress To Strain:

$$\varepsilon = \frac{\sigma}{E} + \alpha\ \Delta T = \varepsilon^M + \varepsilon^T$$

expresses that total strain = mechanical strain + thermal strain: the strain superposition principle

Strain To Stress:

$$\sigma = E\,(\,\varepsilon - \alpha\ \Delta T\,)$$

3D Generalized Hooke's Law (1)

Stresses To Strains (Omitting Thermal Effects)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

For derivation using the strain superposition principle, as well as inclusion of thermal effects, see Lecture notes

3D Generalized Hooke's Law (2)

Strains To Stresses (Omitting Thermal Effects)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

in which

$$\hat{E} = \frac{E}{(1-2\nu)(1+\nu)}$$

This is derived by inverting the matrix of previous slide. For the inclusion of thermal effects, see Lecture notes

2D Plane Stress Specialization

Stresses

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Strains

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

2D Plane Stress Generalized Hooke's Law

Strains To Stresses (Omitting Thermal Effects)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

Stresses To Strains (Omitting Thermal Effects)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \tilde{E} & \tilde{E}\,\nu & 0 \\ \tilde{E}\,\nu & \tilde{E} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

in which

$$\tilde{E} = \frac{E}{1-\nu^2}$$

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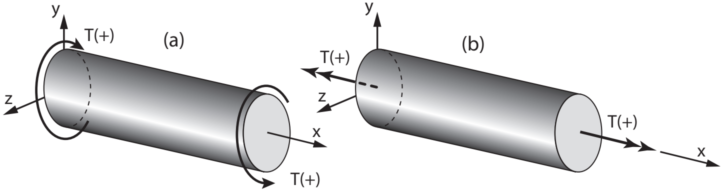
Torsion of Circular Sections

Notation and Terminology

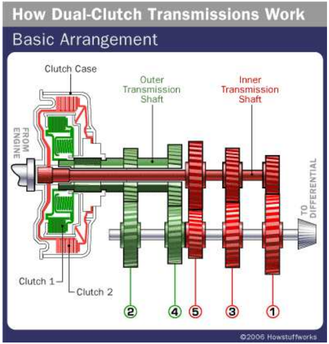
We consider a bar-like, straight, prismatic structural member of constant cross section subjected to applied torque T:

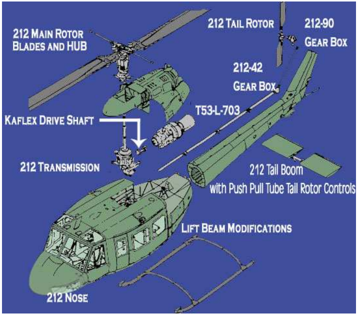


This kind of member, designed primarily to transmit torque, is called a shaft. Axis x is directed along the longitudinal shaft dimension, as shown. The applied torque is a moment about that axis. It is positive when it acts as shown, complying with the right-hand screw rule. Axes y and z lie on a cross section conventionally taken as origin.

The right figure shows another symbol (double-headed arrow) for torque. Both symbols (harpoon and double arrow) are used in this course.

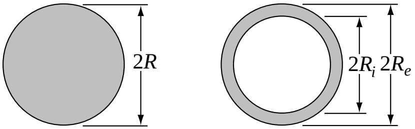
Shafts Are Used for Power Transmission





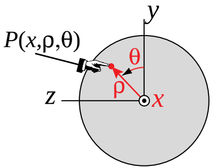
An important use of shafts is to transmit power between parallel planes, as in cars, aircraft engines or helicopters. See figures above (taken from Google Images)

Circular Cross Sections



In this lecture we restrict consideration to shafts of **circular** cross section, which may be either solid or hollow (also called annular & tubular)

Cylindrical Coordinate System



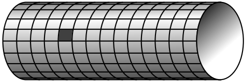
Point P referred to cylindrical coordinate system (looking at cross section from +x down)

In addition to the {x,y,z} Rectangular Cartesian Coordinate (RCC) system shown in the previous slide we shall also use frequently a cylindrical coordinate system $\{x,\rho,\theta\}$ as depicted above for a typical cross section

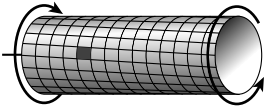
The position coordinates of an arbitrary shaft point P • are $\{x,y,z\}$ in RCC and $\{x,\rho,\theta\}$ in cylindrical coordinates. Here ρ is the radial coordinate (an unfortunate notational choice because of potential confusion with density, but it is that used in most textbooks) and θ the angular coordinate, taken positive CCW from y as shown. Note that $y = \rho \cos \theta$ and $z = \rho \sin \theta$.

Twisting A Shaft

To visualize what happens when a circular shaft is torqued, consider first the undeformed state under zero torque. A rectangular-like mesh is marked on the surface

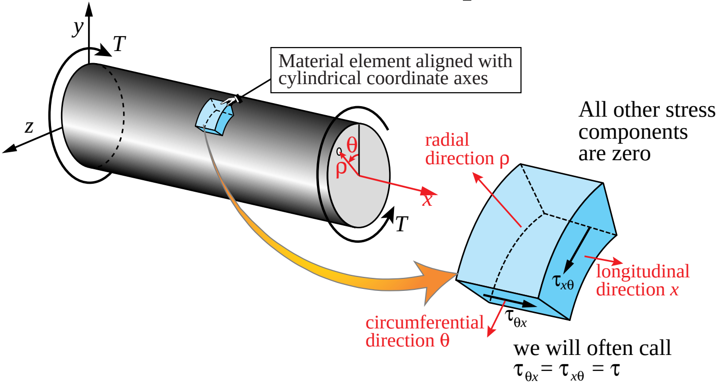


The rectangles have sides parallel to the longitudinal axis *x* and the circumferential direction θ. Then apply torques to the end. The rectangles shear into parallelograms



This figure displays the main effect: shear stresses and associated shear strains develop in the "cylinders" ρ = constant. Next slide goes into details

Stress & Strain Components



Cut out a material element aligned with the cylindrical coordinate axes as shown above. The only nonzero stress component is the shear stress $\tau_{x\theta} = \tau_{\theta x} = \tau$.

Stress and Strain Distribution

Consequently the stress and strain matrices have the form

$$\begin{bmatrix} 0 & \tau_{x\theta} & 0 \\ \tau_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Hooke's law} \Rightarrow \begin{bmatrix} 0 & \gamma_{x\theta} & 0 \\ \gamma_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stresses and strains are connected, point by point, by Hooke's law:

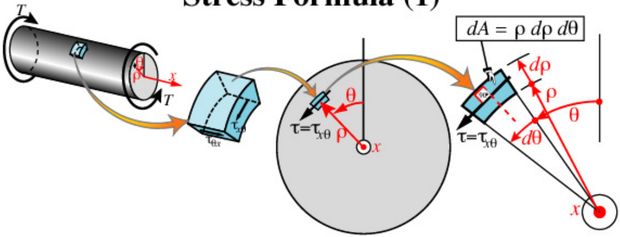
$$\tau = G\gamma \quad \text{or} \quad \gamma = \tau/G$$

Because of circular symmetry τ and γ do not depend on θ. They may depend on x if either the torque or the cross section redius change along the shaft length, but the dependence will not be made explicit. Both τ and γ depend **linearly** on ρ, and we scale that dependence as follows:

$$\tau = \frac{\rho}{R}\tau_{max}, \qquad \gamma = \frac{\rho}{R}\gamma_{max}$$

Here t_{max} and g_{max} are the maximum values, which occur at $\rho = R$, the radius of the shaft. (For an annular shaft, $R = R_e$, the exterior or outer radius.)

Stress Formula (1)



As shown above, cut out a material element aligned with the cylindrical coordinate axes. The only nonzero component on the faces of the material element is the shear stress $\tau_{x\theta} = \tau_{\theta x}$. It follows that the stress and strain matrices have the form

$$\begin{bmatrix} 0 & \tau_{x\theta} & 0 \\ \tau_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftarrow \text{Hooke's law} \Rightarrow \begin{bmatrix} 0 & \gamma_{x\theta} & 0 \\ \gamma_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in which the axes are reordered as (x,q,r) for convenience in identifying with plane stress and strain later. In the sequel we often call $\tau = \tau_{x\theta} = \tau_{\theta x}$ and $\gamma = \gamma_{x\theta} = \gamma_{\theta x}$ to reduce subscript clutter. Stresses and strains are connected, point by point, by Hooke's law

$$\tau = G\,\gamma, \qquad \gamma = \frac{\tau}{G}$$

Stress Formula (2)

Call T the internal torque acting on a cross section x = constant. The shear stress $\tau = \tau_{x\theta}$ acting on an elementary cross section area dA produces an elementary force $dF = \tau\, dA$ and an elementary moment $dM = dF\, \rho = (\tau\, dA)\, \rho$ about the x axis. Integration of the elementary moment over the cross section must balance the internal torque:

$$
\begin{aligned}
T &= \int_A (\text{shear stress} \times \text{elemental area}) \times \text{lever-arm} \\
&= \int_A (\tau\, dA)\rho = \int_A \frac{\rho}{R}\, \tau_{max}\, dA = \frac{\tau_{max}}{R} \int_A \rho^2\, dA \\
&= \frac{\tau_{max}}{R} \int_A \rho^2 (\rho\, d\rho\, d\theta) = \frac{\tau_{max}}{R} \int_A \rho^3\, d\rho\, d\theta = \frac{\tau_{max}\, J}{R},
\end{aligned}
$$

in which $J = \int_A \rho^3\, d\rho\, d\theta$ is the *polar moment of inertia* of the cross section.

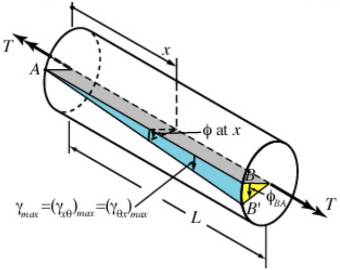
Solving for the maximum shear stress gives the **stress formula**

$$
\boxed{\tau_{max} = \frac{T\,R}{J}, \qquad \tau = \frac{\rho}{R}\, \tau_{max} = \frac{T\,\rho}{J}}
$$

For an solid circular cross section $J = \frac{\pi}{2} R^4$

For an annular cross section $J = \frac{\pi}{2} (R_e^4 - R_i^4)$

Twist Angle and Twist Rate (1)



$BB' \approx R\,\phi \approx \gamma_{max}\,x$ or $\phi = (x/R)\,\gamma_{max}$. The derivative $d\phi/dx$ is called the **twist rate.**

$$\frac{d\phi}{dx} = \frac{\gamma_{max}}{R} = \frac{\gamma}{\rho} \quad \text{so} \quad \gamma = \rho\,\frac{d\phi}{dx}$$

But

$$\gamma = \frac{\tau}{G} = \frac{T\rho}{GJ} = \rho\,\frac{d\phi}{dx}$$

$$\boxed{\frac{d\phi}{dx} = \frac{T}{GJ}}$$

This is the **twist rate formula.**

Twist Angle and Twist Rate (2)

To find the twist angle ϕ_{BA} from end A to end B, integrate along the length of the shaft,

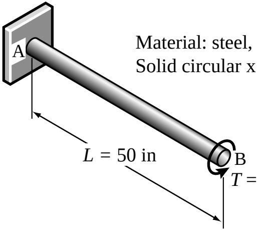
$$\boxed{\phi_{BA} = \int_0^L d\phi = \int_0^L \frac{d\phi}{dx}\,dx = \int_0^L \frac{T}{GJ}\,dx}$$

If T, G and J are constant along the shaft

$$\boxed{\phi_{BA} = \frac{T}{GJ}\int_0^L dx = \frac{TL}{GJ}}$$

This is the **twist angle** between ends.

Example



Material: steel, $G = 12 \times 10^6$ psi
Solid circular x section, $R = 1$ in

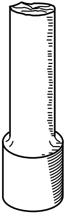
Find: maximum shear stress τ_{max}, maximum shear strain γ_{max} and twist angle ϕ_{BA}

$$\tau_{max} = \frac{T\,R}{J} = \frac{T\,R}{\frac{1}{2}\ldots R^4} = \frac{5000 \text{ lbs-in} \times 1 \text{ in}}{1.57 \times 1^4 \text{ in}^4} = 3183 \text{ psi}$$

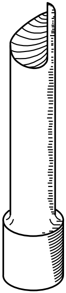
$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{3183 \text{ psi}}{12 \times 10^6 \text{ psi}} = 265\ \mu$$

$$\phi_{BA} = \frac{T\,L}{G\,J} = \frac{5000 \text{ lbs-in} \times 50 \text{ in}}{12 \times 10^6 \text{ psi} \times 1.57\ \ 1^4 \text{ in}} = 0.0133 \text{ rad} = 0.76^{-}$$

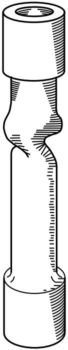
Torqued Shaft Failure Modes



Ductile material (mild steel).
Failure by shear stress.
Solid cross section



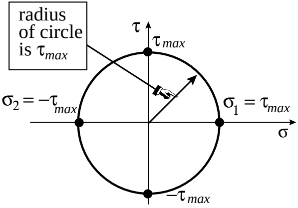
Brittle material (cast iron).
Failure by tensile normal stress. Solid cross section



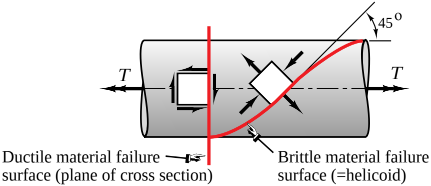
Wall buckling failure
Tubular cross section

Material Failure: Ductile vs. Brittle

Mohr's circle for (x,θ) plane at $\rho=R$ (outer shaft surface)



Material failure surfaces for two extreme cases: ductile vs. brittle



Photos of Specimen That Failed in Torsion

(a) Ductile failure by shear (mild steel)

(b) Brittle failure by tensile normal stress (cast iron)

(c) Brittle failure by tensile normal stress (chalk)