ASEN 3112

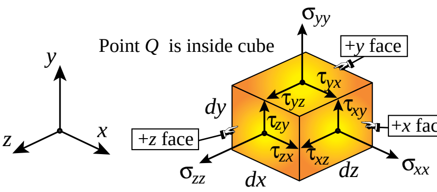
Spring 2020

Lecture 2

January 16, 2020

Visualization on "Stress Cube"

The foregoing nine stress components may be conveniently visualized on a ***"stress cube"*** as follows. Cut an ***infinitesimal cube*** *dx dy dz* around *Q* with sides parallel to the RCC axes {*x*,*y*,*z*}. Draw the components on the ***positive cube faces*** (defined below) as



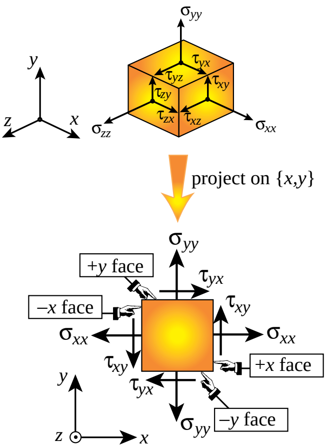
Note that stresses are ***forces per unit area***, **not** forces, although they look like forces in the picture.

Strictly speaking, this is a "cube" only if $dx=dy=dz$, else it should be called a parallelepiped; but that is difficult to pronounce.

The three positive cube faces are those with exterior (outward) normals aligned with +*x*, +*y* and +*z*, respectively. Positive (+) values for stress components on those faces are as shown. More on sign conventions later.

What Happens On The Negative Faces?

The stress cube has three ***positive*** (+) ***faces***. The three opposite ones are ***negative*** (–) ***faces***. Outward normals at – faces point along $-x$, $-y$ and $-z$. What do stresses on those faces look like? To maintain static equilibrium, stress components must be ***reversed***.

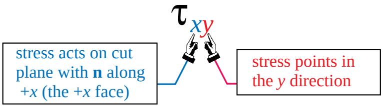


For example, a positive σ_{xx} points along $+x$ on the $+x$ face, but along $-x$ on the $-x$ face. A positive τ_{xy} points along $+y$ on the $+x$ face but along $-y$ on the $-x$ face.

To visualize the reversal, it is convenient to project the stress cube onto the $\{x,y\}$ plane by looking at it from the $+z$ direction. The resulting 2D figure clearly displays the "reversal recipe" given above.

Notational & Sign Conventions

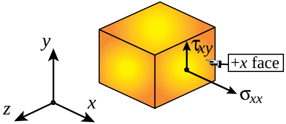
Shear stress components have two indices. The first one identifies the cut plane on which it acts, through the normal to that plane. The second index identifies component direction. For example:



Stress sign conventions are as follows.

For a ***normal stress***: positive (negative) if it produces tension (compression) in the material.

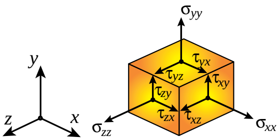
For a ***shear stress***: positive if, when acting on the + face identified by the first index, it points in the + direction identified by the second index. Example: τ_{xy} is + if on the +x face it points in the +y direction; see figure. The sign of a shear stress has no physical meaning; it is entirely conventional.



Both σ_{xx} and τ_{xy} are + as drawn above

Matrix Representation of Stress

The nine components of stress referred to the $\{x,y,z\}$ axes may be arranged as a 3 x 3 matrix, which is configured as



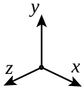
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

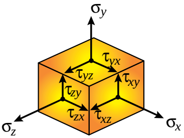
Note that normal stresses are placed in the ***diagonal*** of this square matrix.

We will call this a 3D ***stress matrix,*** although in more advanced courses this is the representation of a second-order tensor.

Shear Stress Reciprocity

From moment equilibrium conditions on the stress cube it can be shown that





$$\boxed{\tau_{xy} = \tau_{yx}\,, \quad \tau_{xz} = \tau_{zx}\,, \quad \tau_{yz} = \tau_{zy}}$$

in ***magnitude***. In other words: switching shear stress indices does not change its value. Note, however, that stresses point in different directions: τ_{xy}, say, points along y whereas τ_{yx} points along x.

This property is known as ***shear stress reciprocity***. It follows that the stress matrix is ***symmetric:***

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx}=\tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx}=\tau_{xz} & \tau_{zy}=\tau_{yz} & \sigma_{zz} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy} & \tau_{yz} \\ \textit{symm} & & \sigma_{zz} \end{bmatrix}$$

Consequently the 3D stress state depends on only ***six (6) independent parameters***: three normal stresses and three shear stresses.

Simplifications: 2D and 1D Stress States

For certain structural configurations such as thin plates, all stress components with a z subscript may be considered negligible, and set to zero. The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

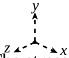
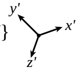
This 2D simplification is called ***plane stress state***. Since $\tau_{xy} = \tau_{yx}$, this state is characterized by ***three*** independent stress components: σ_{xx}, σ_{yy} and $\tau_{xy} = \tau_{yx}$.

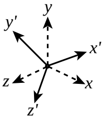
A further simplification occurs in structures such as bars or beams, in which all stress components except σ_{xx} may be considered negligible and set to zero, whence the stress matrix reduces to

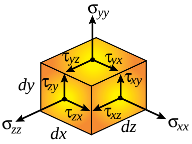
$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

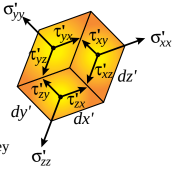
This is called a ***one-dimensional stress state***. There is only ***one*** independent stress component left: σ_{xx}

Changing Coordinate Axes (1)

Suppose we change axes $\{x,y,z\}$  to another set $\{x',y',z'\}$  that also forms a RCC system. The stress cube centered at Q is rotated to realign with $\{x',y',z'\}$ as pictured below.







Note that Q is *inside both cubes*; they are drawn offset for clarity

The stress components change accordingly, as shown in matrix form:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \xrightarrow{\text{becomes}} \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix}$$

Can the primed components be expressed in terms of the original ones?

Changing Coordinate Axes (2)

The answer is ***yes***. All primed stress components can be expressed in terms of the unprimed ones and of the direction cosines of $\{x',y',z'\}$ with respect to $\{x,y,z\}$. This operation is called a ***stress transformation.***

For a general 3D state this operation is quite complicated because there are three direction cosines. In this introductory course we will cover only transformations for the 2D ***plane stress*** state. These are simpler because changing axes in 2D depends on only one direction cosine or, equivalently, the rotation angle about the z axis.

Why do we bother to look at stress transformations? Well, material failure may depend on the ***maximum normal tensile stress*** (for brittle materials) or the ***maximum absolute shear stress*** (for ductile materials). To find those we generally have to look at ***parametric rotations*** of the coordinate system, as in the skew-cut bar example studied later. Once such dangerous stress maxima are found for a given structure, the engineer can determine ***strength safety factors***.

4
Strains

From Statics To Kinematics

Lectures 1-3 dealt with topics in Statics:

Applied Forces $\Rightarrow$ Internal Forces $\Rightarrow$ Stresses

This lecture takes us into Kinematics:

Stresses $\Rightarrow$ Strains $\Rightarrow$ Displacements $\Rightarrow$ Size & Shape Changes

Specifically, we cover strains and their connections to displacements

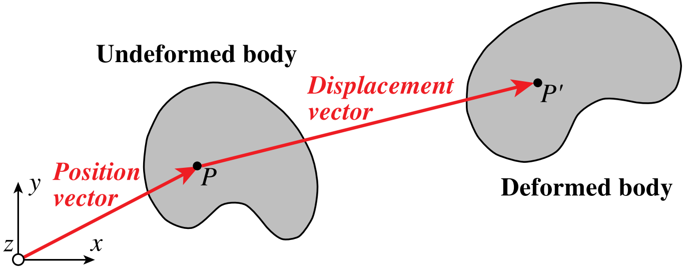
What is Strain (in Mechanics)?

A measure of deformation of a flexible body

All real materials deform under the action of stresses as well as temperature changes.

The connection between strains and stresses (and temperature changes) is done through constitutive equations that encode material properties. These are covered in Lecture 5.

Strains and Displacements



The displacement of a body particle (mathematically a point P in a continuum model of the body) is a vector that defines its motion, joining the initial position P to the final position P'. Displacements of all particles form a vector field.

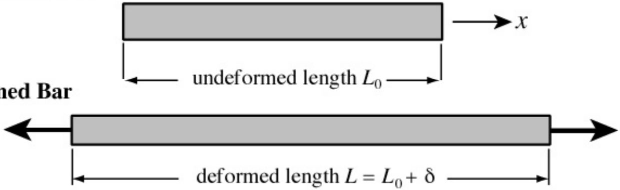
Point strains are connected to partial derivatives of displacements with respect to the position coordinates $\{x,y,z\}$

These are called the strain-displacement equations

Average Normal Strain in 1D

(a) Undeformed Bar

(b) Deformed Bar



$$\epsilon_{av} \stackrel{\text{def}}{=} \frac{L - L_0}{L_{ref}} = \frac{\delta}{L_{ref}}$$

Two common choices for the reference length are

$L_{ref} = L_0$, **the initial gage length: Lagrangian strain**

$L_{ref} = L$, **the final gage length: Eulerian strain**

Strains Flavors (1)

Average vs. Point. Average strain is that taken over a finite portion of the body, for example using a strain gage or rosette. **Point strain** is obtained by a limit process in which the dimension(s) of the gaged portion is made to approach zero.

Normal vs. Shear. Normal strain (a.k.a. **extensional or dimensional strain)** measures changes in length along **one** specific direction. **Shear strain** measures changes in angles with respect to **two** specific directions.

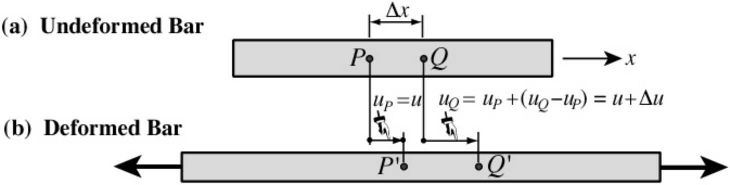
Mechanical vs. Thermal. Mechanical strain is produced by stresses. **Thermal strain** is produced by temperature changes.

Strains Flavors (2)

Finite vs. Infinitesimal. **Finite strains** are obtained using exact measures of the changes in dimensions or angles.
Infinitesimal strains (a.k.a. linearized or small strains) are obtained by linearizing a finite strain measure with respect to displacement gradients.

Strain Measures. For **finite strains,** several mathematical measures are in use, often identified with a person name in front. For example Lagrangian strains, Eulerian strains, Hencky strains, Almansi strains, Murnaghan strains, Biot strains, etc.
They have one common feature: as strains get small, meaning $<<1$, all measures coalesce into the infinitesimal (linearized) version. A brief discussion of Lagrangian versus Eulerian in provided when defining 1D normal strains later.

Point Normal Strain in 1D



Take the limit as the distance between P and Q goes to zero:

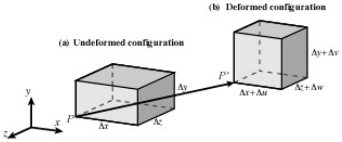
$$\varepsilon_P \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{(u + \Delta u) - u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \qquad (*)$$

In 3D (*) generalizes to

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

with involves three changes: (1) point label P is dropped as point becomes "generic", (2) subscript xx is appended to identify normal component, and (3) the ordinary derivative becomes a partial derivative.

Normal Strains in 3D (2)



Average normal strains:

$$\epsilon_{xx,av} \stackrel{\text{def}}{=} \frac{u+\Delta u-u}{\Delta x} = \frac{\Delta u}{\Delta x} \qquad \epsilon_{yy,av} \stackrel{\text{def}}{=} \frac{v+\Delta v-v}{\Delta y} = \frac{\Delta v}{\Delta y}$$

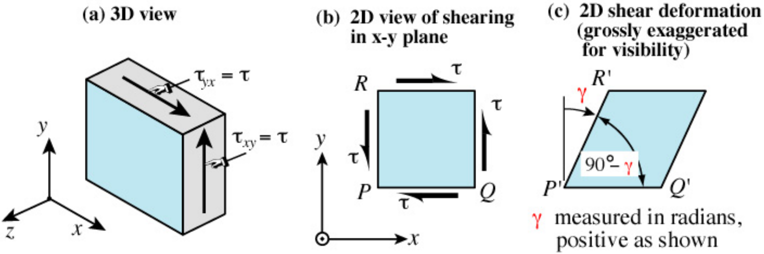
$$\epsilon_{zz,av} \stackrel{\text{def}}{=} \frac{w+\Delta w-w}{\Delta z} = \frac{\Delta w}{\Delta z}$$

Point strains at P:

$$\epsilon_{xx} \stackrel{\text{def}}{=} \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} \stackrel{\text{def}}{=} \lim_{\Delta y\to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} \stackrel{\text{def}}{=} \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z}$$

Average Shear Strain in x,y Plane

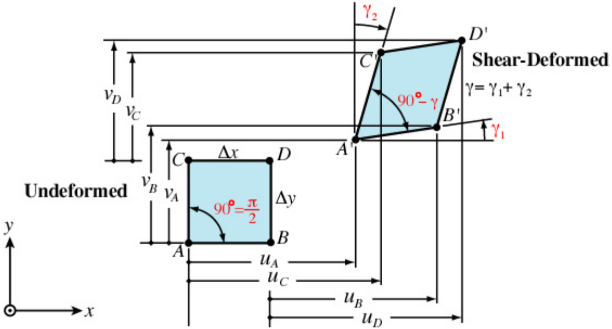


The average shear strain is

$$\gamma_{xy,av} \stackrel{\text{def}}{=} \gamma.$$

Positive if original right angle **decreases** by γ, as shown

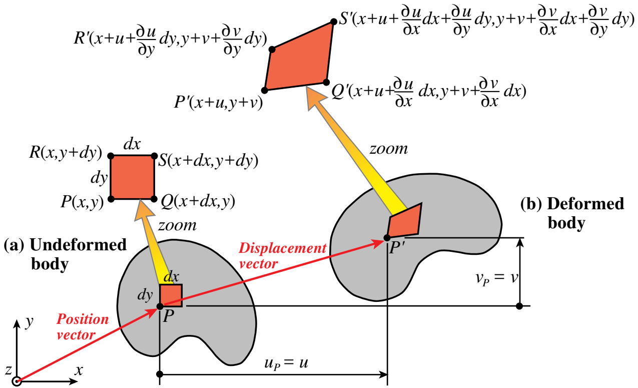
Average Shear Strain in x,y Plane in Terms of Corner Displacements



$$\gamma_{xy,av} = \gamma = \gamma_1 + \gamma_2 \approx \frac{\Delta v_{BA}}{\Delta x} + \frac{\Delta u_{CA}}{\Delta y} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}.$$

See Notes for derivation details

Arbitrary Body in 3D - see Notes



3D (Point) Strain-Displacement Equations

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

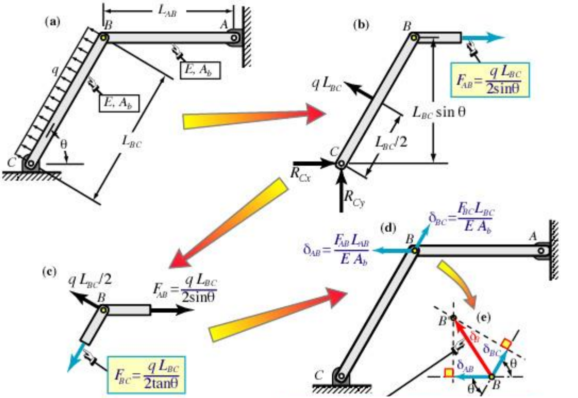
Strain matrix:

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

in which the shear strain components verify reciprocity:

$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{yz} = \gamma_{zy} \qquad \gamma_{zx} = \gamma_{xz}$$

Displacement Calculations for Truss



$$\delta_B = \frac{q\,L_{BC}}{2E\,A_b \sin\theta}\sqrt{\frac{(L_{BC}+L_{AB})^2\cos^2\theta}{\sin^2\theta} + L_{AB}^2}$$

See Section 4.5 of Notes for the worked out example