ASEN 3112 Structures – Spring 2020: Homework 3

Solutions

Exercise 3.1. As Dr. Johnson showed in class, brittle and ductile cylindrical shafts fail differently when an external torque is applied. Ductile materials (like most metals, which fail due to shear stresses) fail at a plane oriented perpendicular to the longitudinal axis, while brittle materials (like chalk or cast iron, which fail due to normal stresses) fail at a plane oriented 45° to the longitudinal axis. Draw Mohr's circle and use it to explain these two failure modes for the two materials.

Solution: See pages 7-9 in Ch. 7 of the textbook for a complete explanation of this phenomena.

Exercise 3.2 A solid circular shaft of 40 mm diameter is to be replaced by a hollow circular tube made of the same material. The outside diameter of the hollow tube is limited to 70 mm. Find: (a) the thickness of the hollow tube if both designs are to work at the same maximum shear stress, (b) the ratio of weights between the two shafts, commenting on which one is more weight efficient. (Use the exact theory for circular shafts presented in 7). Hint: solve a quartic equation.

Solution: Let T be the given torque, while R_s and R_t are the external radii of the solid and tube shafts, respectively. (*Note: the exact theory for torsion of circular shafts is used because the hollow tube that solves the problem is not necessarily thin since its thickness is unknown; that way one covers all bases*.) The corresponding polar moment of inertias are $J_s = \pi R_s^4/2$ and $J_t = \pi[R_t^4-(R_t-t)^4]/2$, where t is the internal tube thickness. Equating the maximum shear stress in both shafts gives the condition

$$\tau_{max} = \frac{TR_s}{J_s} = J_s = \frac{TR_s}{\frac{1}{2}\pi R_s^4} = \frac{TR_t}{J_t} = \frac{TR_t}{\frac{1}{2}\pi[R_t^4-(R_t-t)^4]}$$

Cancelling out T and $\frac{1}{2}\pi$, inserting $R_s = 40/2 = 20$ mm and $R_t = 70/2 = 35$ mm, and rationalizing yields the equation

$$20\times[35^4-(35-t)^4] = 35\times 20^4$$

Solving gives $(35-t)^4$, which has 4 roots for $35-t$, two real. The real root giving positive thickness is

t=1.76 mm Ans

The weight ratio of solid shaft to tube is the ratio of their cross section areas:

$$\frac{W_s}{W_t} = \frac{A_s}{A_t} = \frac{\pi R_s^2}{\pi[R_t^2-(R_t-t)^2]} = \frac{20^2}{35^2-(35-1.76)^2} = 3.33$$

Hence the tube weighs 3.33 times less than the solid shaft.
The cross sections of the original (solid) and replacement (hollow) shaft are shown below to scale in Figure 1. (Drawing these figures is not required in the HW solution; they are shown here to illustrate the fact that the hollow tube uses less material.) Since the hollow shaft is indeed thin wall, the CTW approximate theory could have been used, and the answer would be very close. However, that possibility was not known *a priori*.

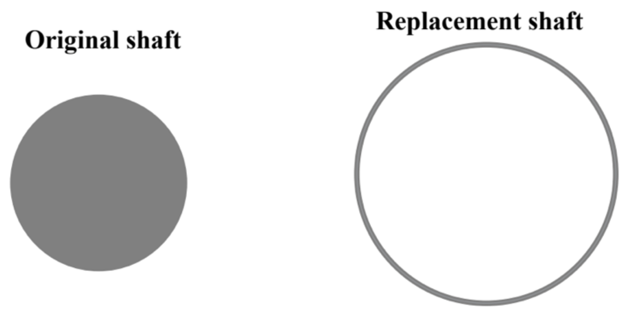

Figure 1: Comparison of both designs for Exercise 3.1.

Exercise 3.3

The solid 80-mm-diameter steel (G = 90 GPa) line shaft shown in Figure 2(a) is driven by a 50-kW motor at 3 Hz. The power is transmitted to gears located at B, D and E, which take up power as shown. Converting power to torque [1] gives the N-m values listed in Figure 2(b). (For the applied torque signs, note that the longitudinal x axis is directed from A to E.) (a) Find the maximum shear stresses in sections BC, CD and DE of the shaft. (Draw first a torque diagram that plots T along the shaft from A through E. In between gears T is constant.) (b) Determine the total angle of twist in degrees between A and E. Note: signs of internal torques are important for this computation.

Solution. (a) The drive power acting on the bearings at B, C, D and E given in kW at a driving frequency of $f = 3$ Hz, is converted to torque in Nm using $P = 2\pi f T$. The results of the conversion are shown in (b) of the figure below, already provided in the HW assignment. The sense adopted for the driven torque is arbitrary because the only important thing for max stress is magnitude. However if the motor-applied torque at C, which is $50kW \times \dfrac{159Nm}{(kW/Hz) \times 3Hz} = 2652$ Nm, is taken as positive,

[1] $P = 2\pi f T$, where P is the power and f is the rotational frequency in Hz. In SI units, 1 watt (W) = 1 Nm/sec.

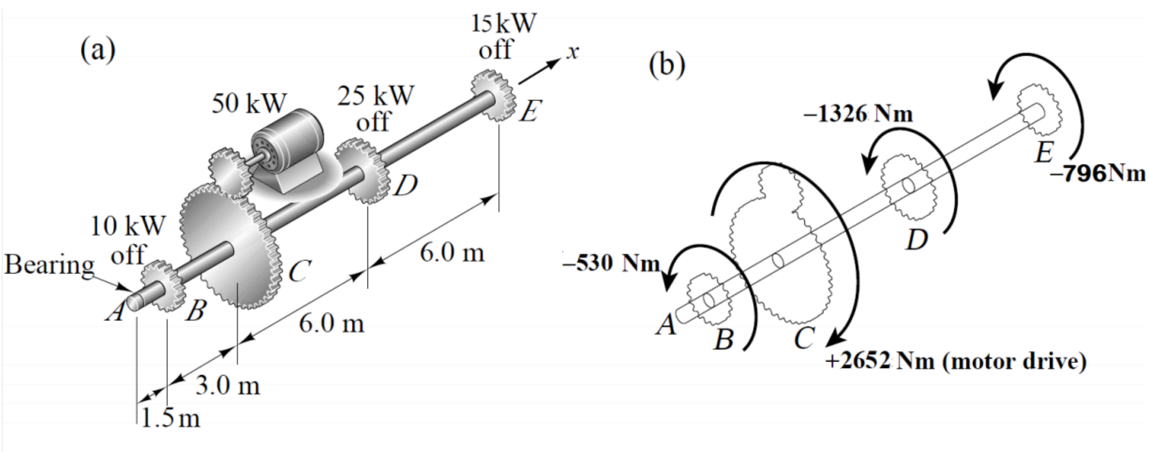

Figure 2: Power transmission shaft for Exercise 3.2.

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a) The polar moment of inertia of the shaft is $J = \frac{\pi r^4}{2} = 4.0212 \times 10^{-6}\ m^4$. The maximum shear stresses in the shaft segments are (ignoring the signs) :

$$\tau_{AB} = 0\,, since\ T_{AB} = 0$$

$$\tau_{BC} = \frac{T_{BC}\, r}{J} = 5.272 \times 10^6 \frac{N}{m^2} = 5.272\ MPa$$

0

$$\tau_{CD} = \frac{T_{CD}\, r}{J} = 21.108 \times 10^6 \frac{N}{m^2} = 21.108\ MPa$$

0

$$\tau_{DE} = \frac{T_{DE}\, r}{J} = 7.918 \times 10^6 \frac{N}{m^2} = 7.918\ MPa$$

Hence, the maximum shear stress occurs on shaft CD.

b) Since the internal torque is constant over each one, the angle of twist between A and E is the sum of twist angles of the individual segments.

$$\phi_{AB} = \frac{T_{AB} L_{AB}}{GJ} = 0$$

$$\phi_{BC} = \frac{T_{BC}L_{BC}}{GJ} = -0.0044 \ radians$$

$$\phi_{CD} = \frac{T_{CD}L_{CD}}{GJ} = 0.0352 \ radians$$

$$\phi_{DE} = \frac{T_{DE}L_{DE}}{GJ} = 0.0132 \ radians$$

Total twist angle is : $0 - 0.0044 + 0.0352 + 0.0132 = 0.0440\, radians = 2.52\, degrees$

Note on sign conventions. The exercise was taken from the Mechanics of Materials textbook by E. P. Popov. To agree with Vable's sign convention for torque, the x axis should be directed from A to E. If so the twist angle is -8° instead of 8°. Credit is given if the answer differs only in sign; since the only important value is the magnitude.

Exercise 3-4

The shaft pictured in Figure 3.3(A) has an L-shaped cross section, and both flanges have length D. It is subjected to a torque T applied at both ends. The numerical values for the properties are:

- Shaft length L = 50 in
- Flange length D = 5.5 in
- Applied torque T = 5500 lb-in
- Shear modulus $G = 5.2 \text{x} 10^6$ psi
- Maximum allowed shear stress τmax = 10000 psi
- Material density $\rho = 0.1$ lb/in^3

Note that D is a midline dimension. Two different cross sections are considered: in the cross section in Figure 3.3(B) both flanges have the same thickness t_1, and in the cross section in Figure 3.3(C) the flanges have thickness t_2 and $2t_2$. Use Open Thin Wall theory, assuming $\alpha = \beta = 1/3$.

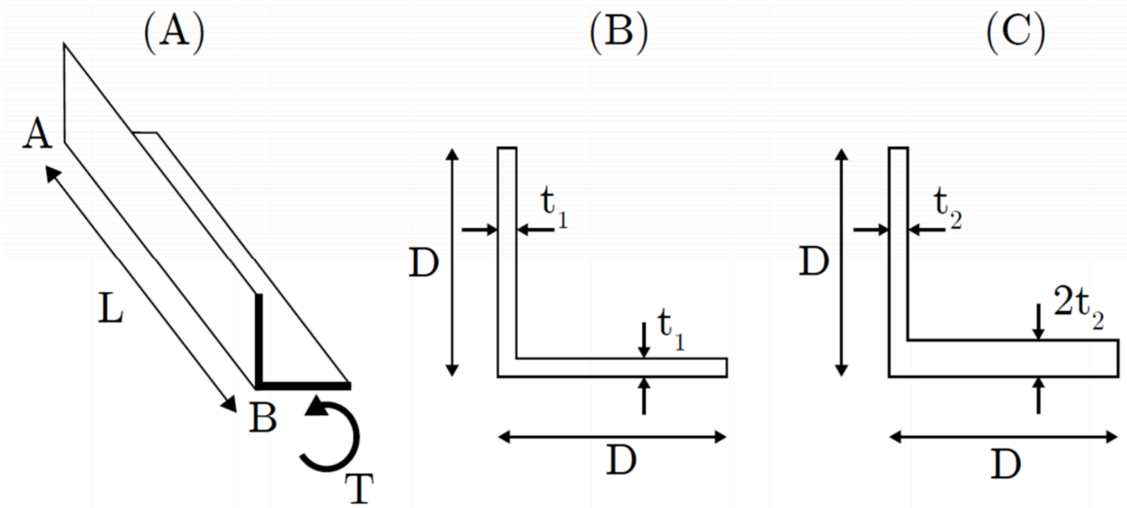

Figure 3.3: Shaft under torque T for Exercise 3.3.

(a) Find thicknesses t_1 and t_2 so that the maximum stress at any point of both cross sections is $\tau = \tau max = 10000$ psi.
(b) Find the twist angle, φ_{BA} over the shaft length in radians and degrees, for both cases.
(c) Calculate the total mass of both shafts. Given the results from (b), and assuming that the goal is to have the less twist possible, which one is the more efficient design?

Solution:

Given :

- Shaft length $L = 50$ in
- Flange length $D = 5.5$ in
- Applied torque $T = 5500$ lb-in
- Shear modulus $G = 5.2\text{x}10^6$ psi
- Maximum allowed shear stress $\tau max = 10000$ psi
- Material density $\rho = 0.1$ lb/in^3

Unknowns :

$t_1, t_2, M_1, M_2, \phi_1, \phi_2$

Case 1 :

a) From OTW theory, we know that

$$J_\propto = \frac{bt_1^3}{3}$$

Maximum shear stress formula is:

$$\tau_{max} = \frac{Tt_1}{J_\propto}$$

From figure 3.3 for case1 it can be seen that,

$J_\propto = \frac{1}{3}(Dt_1^3 + Dt_1^3) = \frac{2}{3}(Dt_1^3)$

Since $\propto = \beta = \frac{1}{3}$

We have

$$\frac{2}{3}(Dt_1^3) = \frac{Tt_1}{\tau_{max}}$$

by substituting the given values appropriately, we get t_1 = 0.3873 in

b) To calculate angle, $\phi_1 = \frac{TL}{GJ_\propto} x \frac{180}{\pi}$ = 14.2247°

c) Mass = $\rho LDt_1 \ x \ 2$ = 21.3014 lb

Case 2:

Similarly for the L-section (C), thickness
can be found using the relation

$$\tau_{max} = \frac{Tt_2}{J_\propto}$$

For section with thickness and 2 we have:

$$J_1 = \frac{1}{3}(Dt_2^3)$$

$$J_2 = \frac{1}{3}(8Dt_2^3)$$

Now we can calculate torques in the sections using the relation below:

$$T_1 = \frac{T(J_1)}{J_1 + J_2} = \frac{1}{9}T = 611.1 \ lb - in$$

$$T_2 = \frac{T(J_2)}{J_1 + J_2} = \frac{8}{9}T = 4888.9 \ lb - in$$

We can use these values of T_1 and T_2 to evaluate thickness for the two cases using the relations:

$\frac{1}{3}(Dt_2^3) = \frac{T_1 t_2}{\tau_{max}}$ for the vertical bar with thickness t_2

$\frac{1}{3}(8Dt_2^3) = \frac{T_2 2t_2}{\tau_{max}}$ for the horizontal bar with thickness $2t_2$

By substituting the known values, we get two values of t:

t_2= 0.1826 for the vertical bar

t_2 = 0.2582 for the horizontal bar

We consider the maximum of these two values as t_2 which is = 0.2582 in

We can check that the maximum values of τ_{max} is the one we want using the relation

$$\tau_{max} = \frac{Tt}{J_\propto}$$

When thickness is t_2 with $J_\propto = J_1$ and T = T_1 it
can be found that τ_{max} = 5000 psi

When thickness is $2t_2$ with $J_\propto = J_2$ and T = T_2
can be found that τ_{max} = 10000 psi

b). To calculate angle, $\phi_2 = \frac{TL}{GJ_\propto}\ x \frac{180}{\pi}$ = 10.67°

c). Mass = $\rho L(Dt_2 + Dt_2\ x\ 2)$ = 21.3014 lb

Case2 design has a lesser twist for the same mass and is considered as a more efficient design.

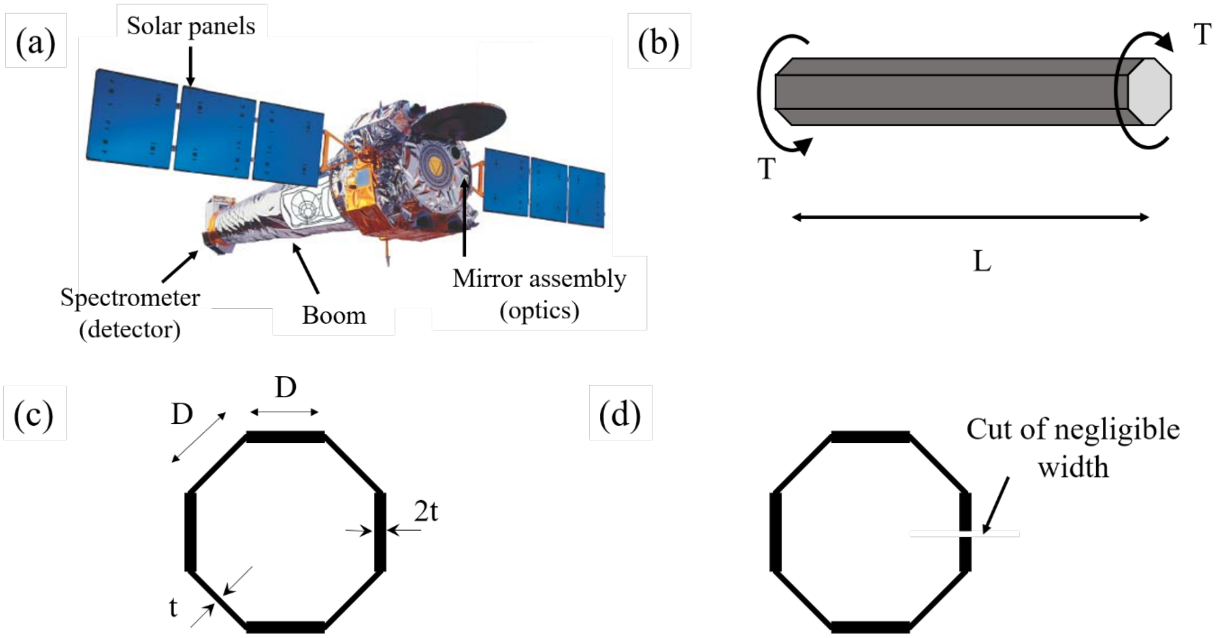

Figure 4.- Structure for Question 3-4: (a) Image of spacecraft, (b) schematic of boom with load,
(c) cross section before cut and (d) cross section after cut.

Exercise 3-5. The optics and detectors of an X-ray space telescope (Fig. 4a) are connected by a boom of length L = 10.5 m, that can be modeled as a <u>thin-walled regular octagon</u> with eight sides of length D = 0.5m (assumed to be at the midline), see Fig. 4b. The walls have two different thicknesses: four of the walls have thickness t, while the other four have thickness $2t$, see Fig. 4c. The material is a carbon fiber composite that can be modeled as an isotropic linearly elastic solid with E = 100 GPa, G = 20.1 GPa. Use the usual approximations for thin-walled structures and assume $\alpha=\beta=1/3$ when necessary for Open Thin Wall Theory.

During attitude maneuvers, it is expected that the boom in Fig. 4c will experience the equivalent to a torsional load of magnitude T = 2000 Nm. Using the appropriate TW theory:

1. Find the minimum value of t that will guarantee BOTH of the following design requirements:
   - The alignment between optics and detector requires a total rotation of $|\Delta\phi_{max}| < 0.055$ degrees
   - Maximum shear stress in the material is $|\tau_{max}| < 0.1$ GPa.
2. Was it reasonable to use the thin-walled approximation? If the answer is no, there is no need to redo calculations.

During the mission, due to impact with space debris, the section tears along its complete length, as sketched in Figure 4d. The width of the cut is negligible. Using the appropriate TW theory:

3. Find the maximum torque T so that the total rotation remains $|\Delta\phi_{max}| < 0.055$ degrees.
4. **** Find the maximum torque T so that the material does not fail, $|\tau_{max}| < 0.1$ GPa.

Solution 3-5:

The first part of the problem deals with a closed wall section, so we used CTW theory. We know that we will need the area enclosed by the section. An octagon can be split in one square of dimensions $D \times D$, four rectangles of dimensions $D \times D/\sqrt{2}$, and four triangles of area $1/2 \times D\sqrt{2} \times D/\sqrt{2}$, see Figure 1.

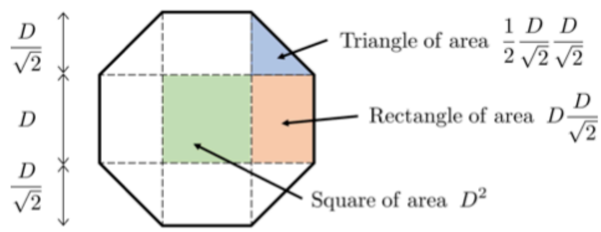

Figure 1: Simply supported beam with uniform load over right halfspan.

The enclosed area can then be calculated as:

$$A_E = D^2 + 4\frac{D^2}{\sqrt{2}} + 4\frac{D^2}{4} = 2D^2\left(1+\sqrt{2}\right) \tag{1}$$

which gives $A_E = 1.2071$ m^2.

To calculate the twist angle, we need the torsional rigidity:

$$J = \frac{4\,A_E^2}{\oint \frac{ds}{t}} = \frac{4\,A_E^2}{4\frac{D}{t} + 4\frac{D}{2t}} = \frac{2\,t\,A_E^2}{3\,D} \tag{2}$$

The total rotation is:

$$\Delta\phi = \frac{d\phi}{dx}L = \frac{T}{GJ}L < \Delta\phi_{max} \tag{3}$$

Putting everything together we get:

$$\frac{3\,T\,L\,D}{G\,2\,t\,A_E^2} < \Delta\phi_{max} \tag{4}$$

which yields the condition:

$$t > \frac{3\,T\,L\,D}{2\,G\,A_E^2\,\Delta\phi_{max}} \tag{5}$$

Substituting numerical values we get $t >$ 560.21 µm

Now, for the condition on maximum shear stress, we know that:

$$\tau_{max} = \frac{T}{2t_{min}A_E} \tag{6}$$

where $t_min = t$. Applying $\tau_{max} < 0.1$ GPa, we get the condition:

$$t > \frac{T}{2A_E\tau_{max}} \tag{7}$$

which yields $t >$ 8.2843 µm

The condition on maximum angle is clearly more restrictive, and so $\boxed{t = \text{560.21 µm}}$

The ratio D/t >892.5 and so the thin wall approximation is perfectly valid.

For the second part of the problem, the close section is now open, so we use OTW theory. We are told to assume $\alpha = \beta = 1/3$. As such, we can simplify our notation and use $J = J_\alpha = J_\beta$. Since we have two different thicknesses, we can divide our section into the region with thickness t and that with thickness $2t$, both of them with width $4D$ (the position of the cut is irrelevant). The respective torsional rigidities are:

$$J_t = \frac{1}{3}4\,D\,t^3 \qquad\qquad J_{2t} = \frac{1}{3}4\,D\,(2\,t)^3 \tag{8}$$

where the two divisions consider the whole length with each thickness, $4\,D$ each.

The total rotation is still given by Equation 3, and so we obtain a condition on the maximum torque T:

$$T_{max,\phi} = \frac{\Delta\phi_{max}\, G\, (J_t + J_{2t})}{L} \tag{9}$$

which gives $\boxed{T_{max,\phi} = 0.0019 \text{ Nm}}$

To calculate the shear stress we need to find out how much of the applied torque T will go into each region:

$$T_t = \frac{J_t}{J_t + J_{2t}} T = \frac{1}{9} T \qquad T_{2t} = \frac{J_{2t}}{J_1 + J_{2t}} T = \frac{8}{9} T \tag{10}$$

We know that the region with highest shear stress will be the thickest one, but we can calculate the maximum torque allowed by both of them. The expressions for the maximum shear stresses are:

$$\tau_{max,t} = \frac{T_t\, t}{J_t} = \frac{T\, t}{9\, J_t} \qquad \tau_{max,2t} = \frac{T_{2t}\, 2t}{J_{2t}} = \frac{16\, T\, t}{9\, J_{2t}} \tag{11}$$

and so we obtain the following conditions for T:

$$T_{max,\tau,t} = \frac{\tau_{max}\, 9\, J_t}{t} = 188.3031 \text{ Nm} \tag{12}$$

$$T_{max,\tau,2t} = \frac{\tau_{max}\, 9\, J_{2t}}{16\, t} = 94.1515 \text{ Nm} \tag{13}$$

and so the more restrictive is $\boxed{T_{max,\tau} = 94.1515 \text{ Nm}}$

The new open section is clearly worse, particularly regarding the maximum rotation. The difference is particularly large, due to the high slenderness of the walls: the torsional rigidity of the closed section scales with t, see Equation 2, while for the open section it scales with t^3, see Equation 8. As such, the torsional rigidity of the closed section is six times order of magnitude higher than that of the open section, $J/(J_t + J_{2t}) = 1.0317\text{E}6$.