ASEN 3112

Spring 2020

Lecture 11

February 20, 2020

Last Week ….

Conservation of Energy Principle:

$W_e = U$ $\quad W_e$: external work $\quad U$: elastic strain energy

Trusses and Bars (constant properties, tip loads only):

$$W_e = \frac{1}{2}\,\hat{u}\,\hat{P}$$

$$U_{bar} = \frac{1}{2}\frac{E\ A\ \Delta L^2}{L} \quad \text{or} \quad U_{bar} = \frac{1}{2}\frac{N^2\ L}{E\ A}$$

Beams:

$$W_e = \frac{1}{2}\,\hat{v}\,\hat{P} \quad \text{or} \quad W_e = \frac{1}{2}\,\hat{v}'\,\hat{M}_{ext}$$

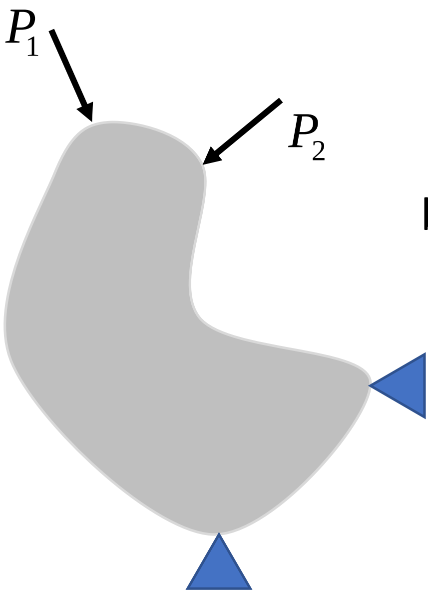
$$U_{beam} = \frac{1}{2}\int_L E\ I_{zz}\ \kappa^2\ dx \quad \text{or} \quad U_{beam} = \frac{1}{2}\int_L \frac{M^2}{E\ I}\,dx$$

Shafts:

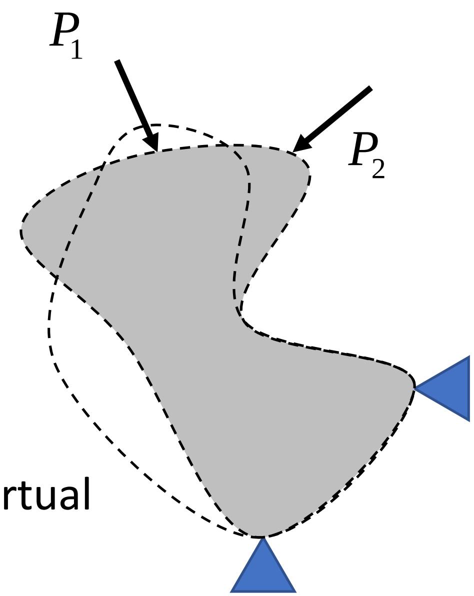
$$W_e = \frac{1}{2}\,\hat{\theta}\,\hat{T}_{ext}$$

$$U_{shaft} = \frac{1}{2}\int_L \frac{T^2}{G\ J}\,dx$$

Virtual Work Methods



body in equilibrium



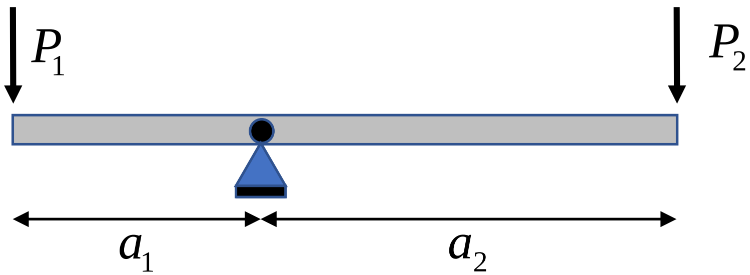
body due to virtual forces or virtual prescribed displacement

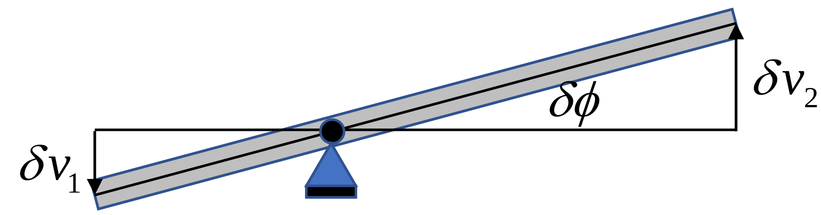
External virtual work + Internal virtual work = 0

$$\delta W = \delta W_e + \delta W_i = 0$$

Virtual Displacement Method

Rigid lever





$$\delta v_1 = a_1\ \delta\phi \qquad \delta v_2 = a_2\ \delta\phi$$

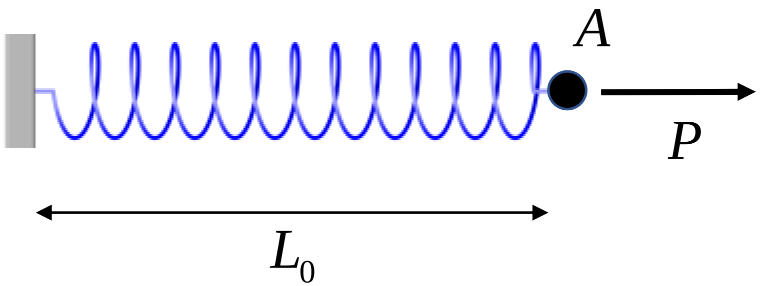
The external virtual work done by **real forces** on **virtual displacements**:

$$\delta W_e = \delta v_1\ P_1 + (-\delta v_2)\ P_2$$

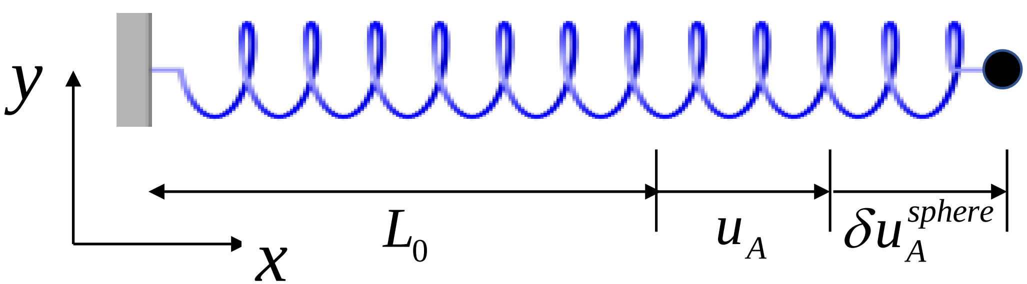
In the absence of internal virtual work (lever is rigid):

$$\delta W_e = \left(a_1\ P_1 - a_2\ P_2\right)\delta\phi = 0$$

External and Internal Virtual Work (1)



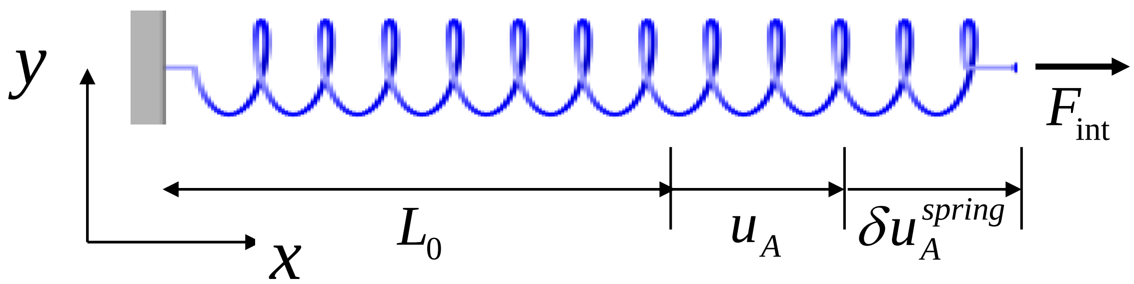
The external virtual work done by **real force *P*** on **virtual displacement $\boldsymbol{\delta u_A^{sphere}}$**:

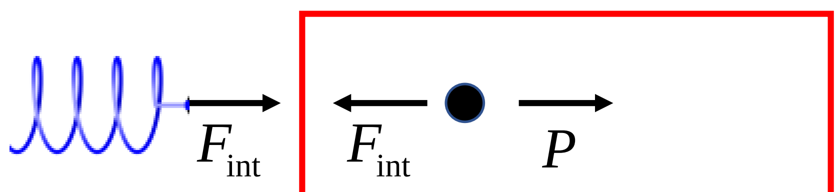


$$\boxed{\delta W_e = P\ \delta u_A^{sphere}}$$

External and Internal Virtual Work (2)

The internal virtual work done by **real force $\boldsymbol{F_{int}}$** on **virtual displacement $\boldsymbol{\delta u_A^{spring}}$**:





$$\delta W_i = -F_{int}\ \delta u_A^{spring}$$

Principle of Virtual Work for Virtual Displacements

Kinematic compatibility: $\delta u_A^{sphere} = \delta u_A^{spring} = \delta u_A$

Principle of Virtual Work for Virtual Displacements:

$$\delta W = P\ \delta u_A - F_{int}\ \delta u_A = 0$$

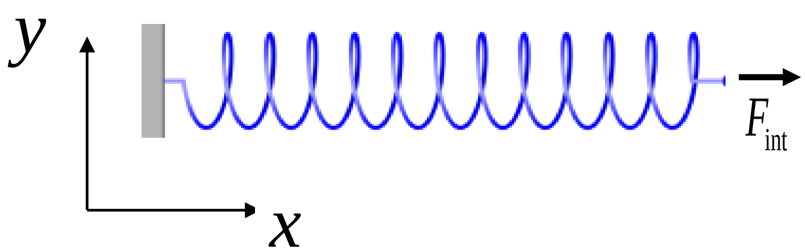
$$\delta W = \left(P - F_{int}\right)\delta u_A = 0$$

Since δu_A arbitrary:

$$P = F_{int}$$

The virtual work done by the real forces on *compatible* virtual displacements vanishes for a structure in static equilibrium.

More on Internal Virtual Displacements



$$F_{int} = k\, u_A$$

The internal virtual work:

$$\delta W_i = -k\, u_A\, \delta u_A$$

Alternative measure of internal work (to avoid negative sign):

$$\delta W_{ie} = -\delta W_i \qquad \left(= k\, u_A\, \delta u_A \right)$$

External virtual work $\boldsymbol{\delta W}$ = Internal virtual work $\boldsymbol{\delta W_{ie}}$

$$\delta W_e = \delta W_{ie}$$

Internal Virtual Work for Structural Models

The following expressions are often used for the Virtual Displacement Method:

- General 3D body:

$$\delta W_{ie} = \int\int\int_V \left(\sigma_{xx}\ \delta\varepsilon_{xx} + \sigma_{yy}\ \delta\varepsilon_{yy} + \sigma_{zz}\ \delta\varepsilon_{zz} + \tau_{xy}\ \delta\gamma_{xy} + \tau_{xy}\ \delta\gamma_{xy} + \tau_{xy}\ \delta\gamma_{xy} \right) dx\ dy\ dz$$

- Bar (constant E, A along bar; stress and strain constant along bar):

$$\delta W_{ie,bar} = \frac{E\ A}{L} \left(\Delta L \right) \delta \left(\Delta L \right)$$

- Beam (constant E constant over cross section):

$$\delta W_{ie,beam} = \int_L E\ I\ \kappa\ \delta\kappa\ dx$$

- Shaft (constant G constant over cross section):

$$\delta W_{ie,shaft} = \int_L G\ J \left(\frac{d\phi}{dx} \right) \delta \left(\frac{d\phi}{dx} \right) dx$$