ASEN 3112

Spring 2020

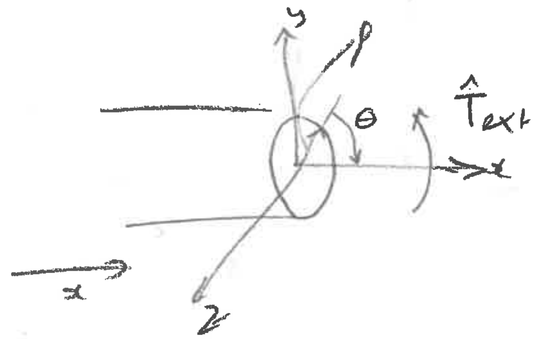
Lecture 11

Whiteboard

February 20, 2020

L11

1 Shafts



$$U_{shaft} = \frac{1}{2}\int_V \tau_{x\theta}\,\gamma_{x\theta}\,\underbrace{dx\,dy\,dz}_{dV}$$

Hookes law $\tau = G\gamma$

$$U_{shaft} = \frac{1}{2}\int_V \frac{\tau_{x\theta}^2}{G}\,dV \qquad \tau_{x\theta} = \frac{T\rho}{J}$$

$$= \frac{1}{2}\int_V \frac{T^2\rho^2}{J^2 G}\,\underbrace{dy\,dz}_{dA}\,dx$$

Assume T, J, G constant over cross section.

$$= \frac{1}{2}\int_V \frac{T^2}{J^2 G}\,\underbrace{\int \rho^2\,dy\,dz}_{J}\,dx$$

$$= \frac{1}{2}\int \frac{T^2}{JG}\,dx$$

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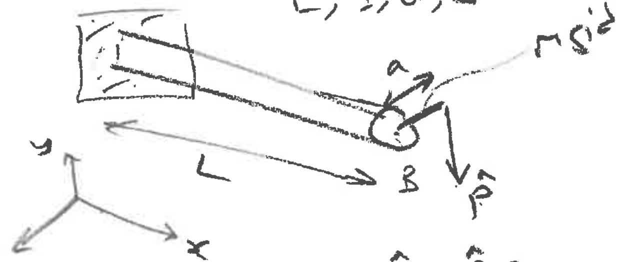
beam/shaft

E, I, G, J



find $\hat{v}_B$ using Principle of Conservation of Energy

$\hat{T} = \hat{P} a$

$$W_e = U$$

$$-\cancel{\tfrac{1}{2}}\cancel{\hat{P}}\,\hat{v}_B = \cancel{\tfrac{1}{2}}\frac{\hat{P}^{\cancel{2}} L^3}{3EI} + \cancel{\tfrac{1}{2}}\frac{\overbrace{\hat{P}^{\cancel{2}} a^2}^{\hat{T}^2}}{GJ} L$$

$$\hat{v}_B = -\left(\frac{\hat{P} L^3}{3EI} + \frac{\hat{P} a^2 L}{GJ}\right) = -\hat{P} L \left(\frac{L^2}{3EI} + \frac{a^2}{GJ}\right)$$

Limitations of Principle of Conservation of Energy ($W_e = U$)

- Cannot provide solution when there are more than one external load.
- Cannot provide solution (eg displacement) at a point in body that is not experiencing an external load.

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Summary

Conservation of Energy

External Work = elastic strain energy

$$W_e = U$$

Rod (bar)

$$W_e = \frac{1}{2}\hat{u}_A \hat{P} \qquad U_{rod} = \frac{1}{2}\frac{EA(\Delta L)^2}{L}$$

or

$$U_{rod} = \frac{1}{2}\frac{N^2 L}{EA}$$

Beam*

$$W_e = \frac{1}{2}\hat{v}\hat{P} \qquad U_{beam} = \frac{1}{2}\int_L EI\, \chi^2 dx$$

or

$$W_e = \frac{1}{2}\hat{v}'\hat{M}_{ext} \qquad U_{beam} = \frac{1}{2}\int_L \frac{M^2}{EI} dx$$

$$\left(\hat{v}' = \frac{d\hat{v}}{dx} = \hat{\theta}\right)$$

* We will consider only strain energy due to bending unless $L \leq 5a$ (a: width or diameter of cross section)
Strain energy due to shear is very small ($< 8\%$ for $L > 5a$), usually neglected in engineering analysis

Optional reading: see Example on Canvas under "Additional Handouts"

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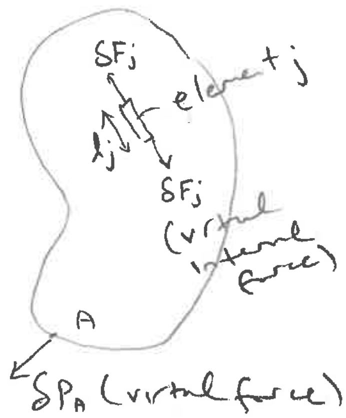
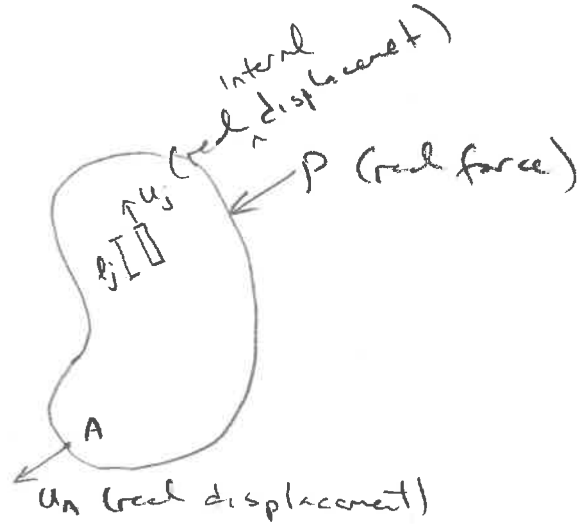
Principle of Virtual Work

- Developed by John Bernoulli 1717
- Also based on conservation of energy
- Used to solve for displacement or slope at a point in deformable body.
- Unlike previous methods, we can solve for u_A <u>even</u> if there is no external load acting on A (similarly, for θ_A even if there is not external moment acting at A)

<u>Underlying Concept</u>





- δP_A and δF_j can be related by equations of equilibrium
- Now, considering <u>only</u> the conservation of <u>virtual energy</u>:

$$\delta W_e = \delta W_{ie}$$

$$\underbrace{\delta P_A}_{\text{virtual}} \underbrace{u_A}_{\text{real}} = \sum_j \underbrace{\delta F_j}_{\text{virtual}} \underbrace{u_j}_{\text{real}}$$

δW_e = External virtual work

δW_{ie}: Internal virtual work done on all elements of the body

Note 1: δP_A is arbitrary —

Note 2: $\delta W_e = \delta P_A u_A$, not like $W_e = \frac{1}{2} P_A u_A$

(δP_A is virtually applied to final state) (P_A gradually applied)

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Alternate form:

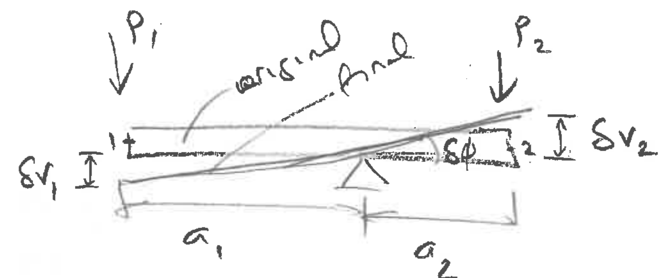
$$\delta W = \delta W_e + \delta W_i = 0$$

i.e. $\delta W_{ie} = -\delta W_i$ — example later

Virtual work formulation
- virtual forces/moments (previous demo)
- or
- virtual displacement/rotation

Case 1 Rigid-body ($\delta W_{ie} = 0$)

$+\downarrow$



No bending!

$$\delta v_1 = a_1\,\delta\phi$$
$$\delta v_2 = a_2\,\delta\phi$$

$$\delta W_e^{①} = \underbrace{P_1}_{\text{real force}}\ \underbrace{\delta v_1}_{\text{virtual displacement}} \qquad \delta W_e^{②} = -\underbrace{P_2}_{\text{real force}}\ \underbrace{\delta v_2}_{\text{virtual displacement}}$$

$$\delta W_e = \cancel{\delta W_{ie}} = 0 \text{ (rigid)}$$

$$\delta W_e = \delta W_e^{①} + \delta W_e^{②} = 0$$

$$\underbrace{(P_1 a_1 - P_2 a_2)}_{=0}\ \delta\phi = 0$$

$\delta\phi$ — arbitrary

$$\boxed{\frac{P_1}{P_2} = \frac{a_2}{a_1}}$$

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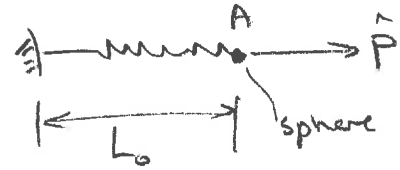
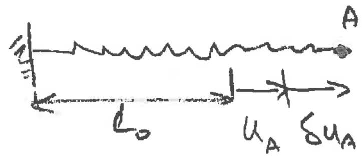
Case 2 : Spring

Undeformed



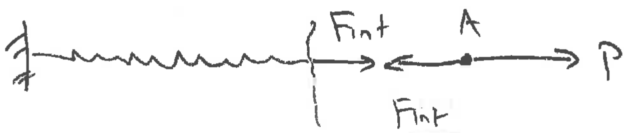
Deformed



Here, so simple that the real displacement (u_A) and virtual displacement (δu_A) are aligned.

$$\delta W_e = \hat{P}\,\delta u_A^{sphere}$$

Break at A:



real internal force

$$\delta W_i = -F_{int}\,\delta u_A^{spring}$$

Kinematic compatibility: $\delta u_A^{sphere} = \delta u_A^{spring} = \delta u_A$

$$\underbrace{\delta W}_{\text{total virtual work}} = \delta W_e + \delta W_i = 0$$

virtual work done by real forces on <u>compatible</u> virtual displacements for a deformable structure in static equilibrium.

$$= P\,\delta u_A - F_{int}\,\delta u_A = 0$$

$$= \underbrace{(P - F_{int})}_{=0}\,\delta u_A = 0$$

arbitrary

$$\boxed{P = F_{int}}$$

Clicker Question 1

Consider the cantilevered beam of length L shown below. An external moment (couple) of magnitude M_{ext} is applied at the free end. The 2nd area moment of inertia I and the elastic modulus E are constant. Let v(x) be the vertical displacement of the beam in y-direction.

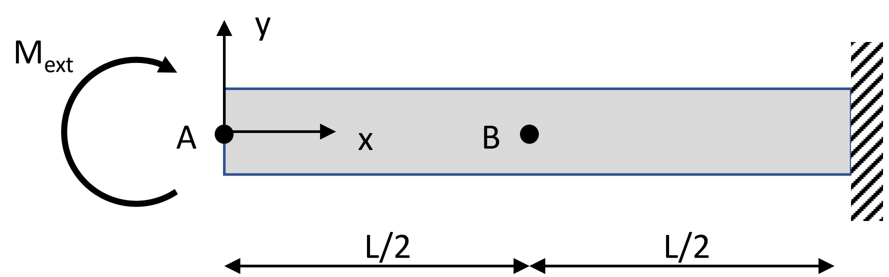
Can you compute the **displacements** at points A and/or B of the beam using the **Conservation of Energy principle**?

(a) Only at Point A

(b) Only at Point B

(c) Neither at Point A nor B

(d) Both at Point A and B



Clicker Question 2

What does the word “virtual” stand for in the Virtual Displacement Method.

(a) Virtual = imaginary; method uses imaginary displacements

(b) Virtual = almost; method only computes approximate displacements

(c) Virtual = computer based; method is strictly based on a computational procedure

Clicker Question 3

The external virtual work δW_e done by the external force P on the virtual displacements δu is:

(a) $\frac{1}{2} P \, \delta u$

(b) $P \, \delta u$

(c) 0

(d) none of the above