ASEN 3112

Spring 2020

Lecture 21

Whiteboard

April 7, 2020

4/7/2020

Recall 2DOF–Undamped, Free

$$\underline{M}\ddot{\underset{\sim}{x}} + \underline{K}\underset{\sim}{x} = \underset{\sim}{0} \; ; \quad \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \; ; \quad \dot{\underset{\sim}{x}}(0) = \underset{\sim}{v}_0$$

Set EVP

$$\left[\underline{K} - \lambda \underline{M}\right]\underset{\sim}{U} = \underset{\sim}{0}$$

Obtain eigensolutions:

eg $n = 2$

$$\underbrace{\left(\omega_{n_1}, \{\underset{\sim}{U}_1\}\right)}_{\text{1st Mode}}, \quad \underbrace{\left(\omega_{n_2}, \{\underset{\sim}{U}_2\}\right)}_{\text{2nd mode}}$$

Normalize eigenvectors with respect to mass matrix

$$\underset{\sim}{V}_1 = \alpha_1 \underset{\sim}{U}_1 \; , \quad \alpha_1 = \frac{1}{\sqrt{\underset{\sim}{U}_1^T \underline{M}\, \underset{\sim}{U}_1}}$$

$$\underset{\sim}{V}_2 = \alpha_2 \underset{\sim}{U}_2 \; , \quad \alpha_2 = \frac{1}{\sqrt{\underset{\sim}{U}_2^T \underline{M}\, \underset{\sim}{U}_2}}$$

Normalized <u>modal</u> matrix:

$$\underline{V} = \left[\underset{\sim}{V}_1 \,\middle|\, \underset{\sim}{V}_2\right] = \begin{bmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{bmatrix}$$

Test orthonormal property of $\underline{V}$:

$$\underline{V}^T \underline{M}\, \underline{V} = \underline{I} \; ; \qquad \underline{V}^T \underline{K}\, \underline{V} = \begin{bmatrix} \omega_{n_1}^2 & 0 \\ 0 & \omega_{n_2}^2 \end{bmatrix}$$

identity matrix

$\underline{I}$: diagonal (orthogonal)
diagonal elements are 1 (normalization)

$\begin{bmatrix} \omega_{n_1}^2 & 0 \\ 0 & \omega_{n_2}^2 \end{bmatrix}$: diagonal (orthogonal)
diagonal elements are eigenvalues (normalization)

Apply Modal Analysis on EOM:

$$\underset{\sim}{x}(t) = \beta_1(t)\underset{\sim}{V}_1 + \beta_2(t)\underset{\sim}{V}_2 = \underline{V}\,\underset{\sim}{\beta}(t)$$

modal matrix

$$\underset{\sim}{\beta}(t) = \begin{Bmatrix} \beta_1(t) \\ \beta_2(t) \end{Bmatrix}$$ — modal coordinates
principal "
generalized "

Notes: Think of $\beta_1(t)$, $\beta_2(t)$ as the "participation factors" for the contribution of normalized eigenvector $\underset{\sim}{V}_1$, and $\underset{\sim}{V}_2$ to the response $\underset{\sim}{x}(t)$.

$$\underline{V}^T\underline{M}\,\underline{V}\,\ddot{\underset{\sim}{\beta}} + \underline{V}^T\underline{K}\,\underline{V}\,\underset{\sim}{\beta} = 0$$

$$\left.\begin{aligned} \ddot{\beta}_1 + \omega_{n_1}^2\beta_1 &= 0 \\ \ddot{\beta}_2 + \omega_{n_2}^2\beta_2 &= 0 \end{aligned}\right] \quad \text{Uncoupled EOM in modal space} \qquad (*)$$

Treat (*) as 1DOF systems as we did in Lecture 17

i.e. $\ddot{\beta}_i + \omega_{n_i}^2\beta_i = 0$ is analogous to $\ddot{x}_i + \omega_{n_i}^2 x_i = 0$

But need to transform also the IC's to modal space:

$$\underset{\sim}{\beta}_0 = \underline{V}^T\underline{M}\,\underset{\sim}{x}_0 \quad , \quad \dot{\underset{\sim}{\beta}}_0 = \underline{V}^T\underline{M}\,\underset{\sim}{v}_0$$

modal space — original physical space — modal space — original physical space

Once solve for $\beta_1(t)$, $\beta_2(t)$ (as shown in Lecture 17), we transform back to $\underset{\sim}{x}$-space.

$$\underset{\sim}{x}(t) = \underline{V}\,\underset{\sim}{\beta}(t) \quad \text{where} \quad \underset{\sim}{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

Solving for "response" of MDOF System
- Undamped
- Forced

$\rightarrow$ by Modal analysis (Expansion theorem)

$$\underline{M}\ddot{\underset{\sim}{x}} + \underline{K}\underset{\sim}{x} = \underset{\sim}{f} \quad , \quad \underset{\sim}{f} = \begin{Bmatrix} F_1 e^{i\omega t} \\ F_2 e^{i\omega t} \end{Bmatrix} = \underset{\sim}{F} e^{i\omega t}$$

F_1, F_2 real

ω: excitation frequency

(assume single excitation freq)

LHS of EOM: identical to free vibrations case

RHS of EOM: Force to be transformed to modal space

Interested in particular (steady-state) solution

$$\underset{\sim}{x}_p = X_p e^{i\omega t} \quad , \quad \lambda = \omega^2$$

$$\left[\underline{K} - \lambda \underline{M}\right] \underset{\sim}{X}_p = \underset{\sim}{F} \quad \text{where} \quad \underset{\sim}{F} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Set up EVP

$$\left[\underline{K} - \lambda \underline{M}\right] \underset{\sim}{U} = 0$$

temporarily set $\underset{\sim}{F} = 0$ in order to solve for eigensolution

Solve for eigensolution:

$$(\lambda_1, \underset{\sim}{U}_1), \quad (\lambda_2, \underset{\sim}{U}_2)$$

Normalize eigenvectors w.r.t mass matrix : $\underline{U} \rightarrow \underline{V}$

Modal transformation: $\underset{\sim}{x}_p(t) = \underline{V}\,\underset{\sim}{\beta}$

particular solution for MDOF problem

$$\underline{M}\,\underline{V}\,\ddot{\underset{\sim}{\beta}} + \underline{K}\,\underline{V}\,\underset{\sim}{\beta} = \underset{\sim}{f}$$

$$\underline{V}^T\underline{M}\,\underline{V}\,\ddot{\underset{\sim}{\beta}} + \underline{V}^T\underline{K}\,\underline{V}\,\underset{\sim}{\beta} = \underline{V}^T\underset{\sim}{f} = \underset{\sim}{\gamma}$$

↑ modal force vector

$$\ddot{\underset{\sim}{\beta}} + \begin{bmatrix} \omega_{n_1}^2 & 0 \\ 0 & \omega_{n_2}^2 \end{bmatrix} \underset{\sim}{\beta} = \underset{\sim}{\gamma}$$

$$\left[\begin{aligned} \ddot{\beta}_1 + \omega_{n_1}^2 \beta_1 &= \gamma_1 \\ \ddot{\beta}_2 + \omega_{n_2}^2 \beta_2 &= \gamma_2 \end{aligned}\right] \quad (**)$$

$$\underset{\sim}{\gamma} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} = \begin{bmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} V_{11}f_1 + V_{21}f_2 \\ V_{12}f_1 + V_{22}f_2 \end{bmatrix}$$

γ_1 : measure of how much forcing distribution across the 2 masses "conforms" with the 1st mode shape

γ_2 : measure of how much forcing distribution across the two masses "conforms" with the 2nd mode shape

Forcing distribution is the relative values of F_1 and F_2.

Solve $(**)$ for $\beta_1(t)$, $\beta_2(t)$

Transform back into modal space

$$\underset{\sim}{x}_p(t) = \underline{V}\,\underset{\sim}{\beta}(t)$$

Notes

(A) — If we need to obtain total solution

$$\underset{\sim}{x}_t = \underset{\sim}{x}_h + \underset{\sim}{x}_p$$

due to non-zero initial conditions (transient solution)

due to ongoing harmonic forcing (steady-state solution)

— Solve for $\underset{\sim}{x}_h$ expression and add to $\underset{\sim}{x}_p$.

Then apply IC's to obtain coefficients in $\underset{\sim}{x}_h$

(B) — If for the forced problem, we have more than one excitation frequency:

$$\underset{\sim}{f} = \begin{Bmatrix} F_1 e^{i\omega_1 t} \\ F_2 e^{i\omega_2 t} \end{Bmatrix}$$

ω_1, ω_2 two different excitation frequencies

Remedy:

Solve problem considering $\underset{\sim}{f}_A = \begin{Bmatrix} F_1 e^{i\omega_1 t} \\ 0 \end{Bmatrix}$

" " " $\underset{\sim}{f}_B = \begin{Bmatrix} 0 \\ F_2 e^{i\omega_2 t} \end{Bmatrix}$

get $\underset{\sim}{x}_{p_A}(t)$, $\underset{\sim}{x}_{p_B}(t)$

Then $\underset{\sim}{x}_p(t) = \underset{\sim}{x}_{p_A}(t) + \underset{\sim}{x}_{p_B}(t)$

i.e. solve by linear superposition.

Continue with numerical example:

Recall:

$$\underline{M} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \underline{k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solved EVP:

$$\left(\frac{1}{3}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right), \quad \left(1, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\right)$$

$$\underline{U} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Normalize $\underline{U}$ w.r.t $\underline{M}$:

$$\underset{\sim}{V_i} = \alpha_i \underset{\sim}{U_i} \Rightarrow \alpha_i = \frac{1}{\sqrt{\underset{\sim}{U_i}^T \underline{M} \underset{\sim}{U_i}}} \qquad i = 1, 2$$

$\underline{i=1}$

$$\alpha_1 = \frac{1}{\sqrt{(1 \;\; 1)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} = \frac{1}{\sqrt{(3 \;\; 3)\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} = \frac{1}{\sqrt{6}}$$

$i=2$

$$\alpha_2 = \frac{1}{\sqrt{(1 \; -1)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} = \frac{1}{\sqrt{(1 \; -1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} = \frac{1}{\sqrt{2}}$$

$$\underset{\sim}{V_1} = \frac{1}{\sqrt{6}} \underset{\sim}{U_1} = \frac{1}{\sqrt{6}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \end{Bmatrix}$$

$$\underset{\sim}{V_2} = \frac{1}{\sqrt{2}} \underset{\sim}{U_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$\frac{L21}{7}$ check:

$$\underline{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underline{V}^T \underline{M} \underline{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I} \qquad \underline{\underline{\text{ok}}}$$

$$\underline{V}^T \underline{K} \underline{V} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \underline{\underline{\text{ok}}}$$

Suppose in this numerical example, we have a RHS

$$\underset{\sim}{f} = \underset{\sim}{F} \sin \omega t \,, \quad \underset{\sim}{F} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix} N$$

$$\omega = 1\,Hz = 2\pi \text{ rad}/s$$

Note: OK to use $\sin \omega t$ (instead of $e^{i\omega t}$) because no damping.

Solve steady-state response

Transform EOM.

$$\ddot{\beta}_1 + \underbrace{\tfrac{1}{3}}_{\lambda_1} \beta_1 = \gamma_1 \,, \quad \ddot{\beta}_2 + \underbrace{1}_{\lambda_2} \beta_2 = \gamma_2$$

$$\underset{\sim}{\gamma} = \underline{V}^T \underset{\sim}{f} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} 0 \\ 5 \end{Bmatrix} \sin \omega t$$

$$= \begin{Bmatrix} \frac{5}{\sqrt{6}} \\ -\frac{5}{\sqrt{2}} \end{Bmatrix} \sin \omega t = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix}$$

$$\ddot{\beta}_1 + \tfrac{1}{3} \beta_1 = 5/\sqrt{6} \sin \omega t \,, \quad \ddot{\beta}_2 + \beta_2 = \frac{-5}{\sqrt{2}} \sin \omega t \qquad (***)$$

where $\omega = 2\pi$ rad/s

Recall SDOF – undamped/force (Lect 17/18)

$$\left.\begin{aligned} \beta_i(t) &= B_i \sin \omega t \\ \dot{\beta}_i(t) &= \omega B_i \cos \omega t \\ \ddot{\beta}_i(t) &= -\omega^2 B_i \sin \omega t \end{aligned}\right] \quad i = 1, 2$$

$\frac{L21}{9}$ Plug into (***):

$$\left[\omega_{n_i}^2 - \omega^2\right] B_i \cancel{\sin}\,\omega t = \gamma_i \cancel{\sin}\,\omega t$$

$$B_i = \frac{\gamma_i}{\omega_{n_i}^2 - \omega^2}$$

Define $\eta_i = \dfrac{\omega}{\omega_{n_i}}$

Divide by $\omega_{n_i}^2$

$$B_i = \frac{\gamma_i / \omega_{n_i}^2}{1 - \eta_i^2}$$

$$\beta_i = \frac{\gamma_i / \omega_{n_i}^2}{1 - \eta_i^2} \sin \omega t \quad , \quad i = 1, 2$$

Plug in the numbers:

$$\beta_1(t) = \frac{\frac{5}{\sqrt{6}} / \frac{1}{3}}{1 - \left(\frac{2\pi}{\sqrt{1/3}}\right)^2} \sin 2\pi t = \frac{15/\sqrt{6}}{1 - 12\pi^2} \sin 2\pi t$$

$$\beta_2(t) = \frac{-5/\sqrt{2}}{1 - (2\pi)^2} \sin 2\pi t = \frac{-5/\sqrt{2}}{1 - 4\pi^2} \sin 2\pi t$$

$$\underline{x}_p(t) = \underline{V}\,\underline{\beta}$$

$$x_p(t) = \begin{Bmatrix} x_{p1}(t) \\ x_{p2}(t) \end{Bmatrix} = \begin{bmatrix} 1/\sqrt{6} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} \beta_1(t) \\ \beta_2(t) \end{Bmatrix} = \begin{bmatrix} 1/\sqrt{6}\, \beta_1(t) + \frac{1}{\sqrt{2}} \beta_2(t) \\ \frac{1}{\sqrt{6}} \beta_1(t) - \frac{1}{\sqrt{2}} \beta_2(t) \end{bmatrix}$$

$\frac{L21}{10}$

$$x_1(t) = \frac{15/6}{1-12\pi^2}\sin 2\pi t - \frac{5/2}{1-4\pi^2}\sin 2\pi t = \left[\left(\frac{15/6}{1-12\pi^2}\right) - \left(\frac{5/2}{1-4\pi^2}\right)\right]\sin 2\pi t$$

$$x_2(t) = \frac{15/6}{1-12\pi^2}\sin 2\pi t + \frac{5/2}{1-4\pi^2}\sin 2\pi t = \left[\left(\frac{15/6}{1-12\pi^2}\right) + \left(\frac{5/2}{1-4\pi^2}\right)\right]\sin 2\pi t$$