ASEN 3112

Spring 2020

Lecture 23

April 16, 2020

Stability of Structures: Continuous Models

Objective

This Lecture covers continuous models for structural stability. Focus is on axially loaded columns with various end support conditions. Columns are treated with Linearized PreBuckling (LPB) analysis assumptions to set up the equilibrium equation in a perturbed shape. Formulas for beam deflection (Part 3 of course) reused.

Equilibrium equation is a linear second order ODE - may be homogeneous or not.

Seeking nontrivial ODE solutions leads to a trascendental eigenproblem, from which critical loads and buckling mode shapes can be obtained. For prismatic columns with standard BC, the stability analysis can be entirely done by hand.

Three Example Problems

Three column buckling problems are worked out in this Lecture:

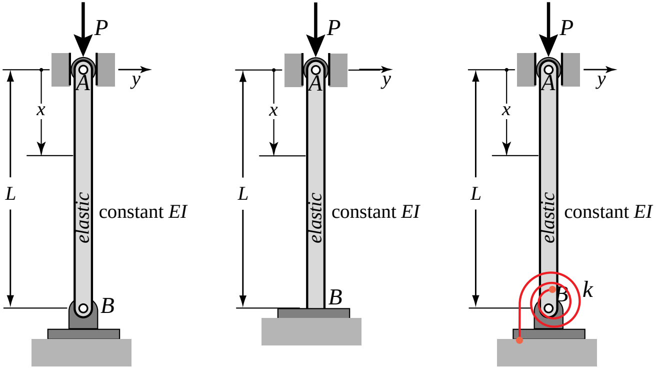
1. **The Euler column: a prismatic, elastic, axially loaded column pinned (same as SS) at both ends**

2. **The pinned-fixed column: a prismatic, elastic, axially loaded column pinned at one end and fixed at the other**

3. **The pinned-elastically-restrained column: a prismatic, elastic, axially loaded column SS at both ends but connected to a torsional spring at the other**

These 3 BC cases were tested Friday, and are used in Exercise 10.6 of HW#10

The Three BC Cases Treated in This Lecture



Pinned-pinned
(a.k.a Euler column)

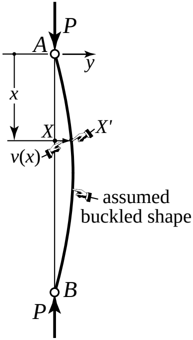
Pinned-fixed

Pinned-restrained

Pinned-Pinned Column: Assumed Buckling Shape

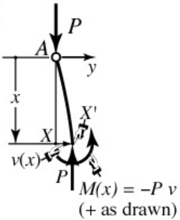
Figure (b) of Slide 4, reproduced on the right for convenience, is a FBD done in an **admissible perturbed configuration** for the Euler column.

It shows reactions at end B, which reduce to just a vertical force balancing *P*. There are no end moment reactions because the column is hinged at both A and B. Furthermore, taking moments with respect to A and B shows that both horizontal reactions (along *y*) vanish.



Pinned-Pinned Column: FBD at Variable x

Next, do the FBD of a segment AX, with X located at distance x from A, as shown in (c) of last slide, reproduced on the right for convenience. The displaced X is labeled X', and the distance from X to X' is the **lateral column deflection** $v(x)$. At this cross section we will have a bending moment $M_z(x)$, which is positive as drawn (reason: a positive $M_z(x)$ compresses the "beam top surface", which by convention lies on the $+y$ side.) Taking moments with respect to X' for convenience, equilibrium requires



$$\boxed{M_z(x) + P\, v(x) = 0}$$

But according to beam deflection theory (developed in Part 3 of the course)

$$\boxed{M_z(x) = E\, I_{zz}\, v''(x) = E\, I\; v''(x)}$$

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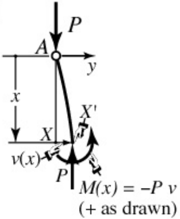

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Pinned-Pinned Column: ODE Derivation

Here $I = I_{zz}$ denotes the *minimum* moment of inertia of the column cross-section. This defines the z direction. The y axis is taken normal to x and z forming a right-handed RCC frame. The column will buckle in the y direction; that is, the deflected shape will be in the x-y plane. Replacing $M_z(x) = E\,I\,v''(x)$ gives

$$\boxed{E\,I\;v''(x) + P\,v(x) = 0}$$

This is the ODE that governs stability analysis. It is a **linear, homogeneous, second-order** ODE in the deflection $v(x)$. For convenience in the reduction to standard (canonical) form, introduce

$$\lambda \overset{\text{def}}{=} +\sqrt{\frac{P}{E\,I}}$$

Replacing P by $E\,I\,\lambda^2$ and dividing through by $E\,I$ yields the canonical ODE form

$$\boxed{v''(x) + \lambda^2\,v(x) = 0}$$

Pinned-Pinned Column: General ODE Solution

The **general solution** of the foregoing ODE is

$$\boxed{v(x) = A \cos \lambda x + B \sin \lambda x}$$

Coefficients A and B are determined from the kinematic BC of simple support at both ends: $v(0)$=0 and $v(L)$ =0. The first one requires $A = 0$, whence the general solution reduces to

$$\boxed{v(x) = B \sin \lambda x}$$

This is called a *characteristic solution*. Two cases can be distinguished:

$$\boxed{\begin{cases} B = 0 & \Rightarrow \; v(x) = 0\text{: the column remains straight for any load} \\ B \neq 0 & \Rightarrow \; v(x) \neq 0\text{: the column buckles with shape } \sin \lambda x \end{cases}}$$

These are called the **trivial** and **nontrivial** solutions, respectively, in the literature. Since the case $B = 0$ or $v(x) = 0$ is that of the reference (undeformed) state at 0 load, it represents the **primary equilibrium path**.

Pinned-Pinned Column: Critical Loads

The critical loads are the values of P at which **nontrivial solutions are possible**. These are determined by applying the second end condition $v(L) = 0$. Since B is nonzero (else we would get a trivial solution) we must have

$$\boxed{\sin \lambda L = 0}$$

This **characteristic equation** is actually a **trascendental eigenproblem** in λ. Its solutions are $\lambda L = n\,\pi$, in which $n = 1,2, \ldots$ is a positive integer. Square both sides for convenience: $\lambda^2 L = n^2\pi^2$, replace λ^2 by $P/(E\,I)$, solve for P, and tag those loads as critical:

$$\boxed{P_{cr,n} = \frac{n^2\,\pi^2\,E\,I}{L^2}, \qquad n = 1, 2, \ldots}$$

As can be observed, there is an **infinite number of critical loads**. These are the nontrivial solutions of the characteristic equation.

Pinned-Pinned Column: Critical Load

The **lowest** critical load, denoted simply by P_{cr} , corresponds to $n = 1$:

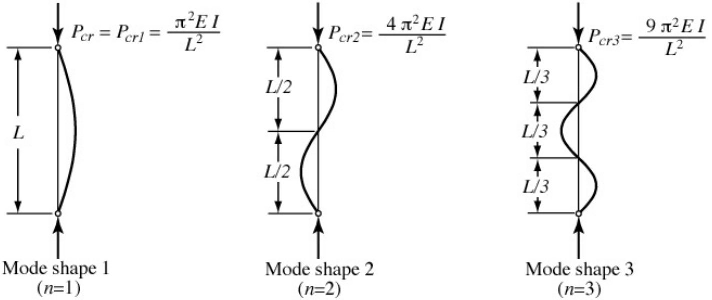
$$\boxed{P_{cr} = P_{cr,1} = \frac{\pi^2 EI}{L^2}}$$

This is called the **Euler critical load**. The **buckling mode shapes** associated with the set of critical loads $P_{cr,n}$ are given by

$$\boxed{v_{cr,n}(x) = B \sin \frac{n\,\pi\, x}{L}}$$

If $n = 1$ the mode shape is a **half sine wave**, similar to the one pictured as assumed mode shape. If $n = 2$, we get a complete sine wave, if $n = 3$ a one-and-a-half sine wave, etc. Those buckling shapes are drawn for $n = 1, 2, 3$ in the next slide.

Pinned-Pinned Column: Buckling Mode Shapes for n = 1, 2 ,3



Pinned-Pinned Column: Determinant Form of Stability Equation

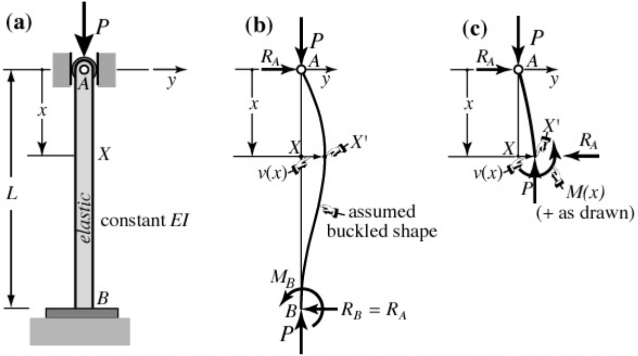
The boundary condition equations $\boxed{A = 0}$ and $\boxed{B \ \sin \lambda L = 0}$ can be recast in matrix form as

$$\boxed{\begin{bmatrix} 1 & 0 \\ 0 & \sin \lambda L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

For a nontrivial solution in which A and B are not simultaneouly zero the matrix determinant, which is simply $\sin \lambda L$, must be zero. Thus we recover the characteristic equation $\boxed{\sin \lambda L = 0.}$

For this particular problem this detour is a waste of time. But for more complicated cases it can shortcut error prone hand derivations especially with the help of a computer algebra system (CAS), which can directly compute and simplify the determinant.

Pinned-Fixed Column: Assumed Buckling Shape



In the configuration shown in (a) the column is pinned at A and fixed at B. The FBD of a kinematically admissible buckling shape is illustrated in (b). The most notable difference with respect to the pinned-pinned column case is the presence of **additional reaction forces**: the lateral reactions R_A and R_B, and the fixed end moment M_B . These are positive as pictured above.

Pinned-Fixed Column: ODE and its Solution

Equilibrium of x forces in (b) of the previous slide is satisfied by the vertical reaction $R_B = P$ at B. Equilibrium of y forces gives $R_A = R_B$. Equilibrium of moments taken with respect to either A or B yields $M_B = R_A L$, or $R_A = M_B/L$. For now M_B will be kept as an unknown until the ODE is solved. Next consider the FBD of the portion AX shown in (c) of the previous slide Moment equilibrium with respect to X' gives the non-homogeneous ODE

$$\boxed{E\,I\,v''(x) + P\,v(x) = R_A x = M_B\,x/L}$$

The LHS is the same as that of the pinned-pinned column, but the RHS is no longer zero. Dividing through by $E\,I$ and setting $\lambda^2 = P/(EI)$ gives the canonical form

$$\boxed{v''(x) + \lambda^2\,v(x) = M_B\,x/(E\,I\,L) = \lambda^2\,M_B\,x/(P\,L)}$$

The general solution of this ODE is the sum of the homogeneous solution $v_H(x) = A\cos\lambda L + B\sin\lambda L$ and the particular solution $v_P(x) = \lambda^2 M_B x/(\lambda^2\,P\,L) = M_B x/(P\,L)$:

$$\boxed{v(x) = A\cos\lambda\,x + B\sin\lambda\,x + M_B x/(P\,L)}$$

Pinned-Fixed Column: Critical Load

The three **kinematic BC** are $v_A = v(0) = 0$, $v_B = v(L) = 0$, and $v'_B = v'(L) = 0$. These provide 3 equations. One is trivial: $\boxed{A = 0}$, but the other two are not

$$\boxed{A \cos \lambda L + B \sin \lambda L + \frac{M_B}{P} = 0, \quad -A\,\lambda \sin \lambda L + B\,\lambda \cos \lambda L - \frac{M_B}{P\,L} = 0}$$

Soving these equations simultaneously gives the **characteristic equation**

$$\boxed{\tan \lambda L = \lambda L}$$

The smallest root of this trascendental equation, to 4 places, is $\lambda L = 4.493$. The corresponding **critical load** is

$$\boxed{P_{cr} = \frac{20.19\,E I}{L^2}}$$

Since $20.19 \sim 2.05\,\pi^2$, this critical load is approximately **twice that of the pinned-pinned Euler column**. Thus fixing one end has substantially increased the critical load.

Pinned-Fixed Column: Determinant Form of Stability Equation

The three boundary condition equations of the previous slide can be recast in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ \cos\lambda L & \sin\lambda L & 1/P \\ -\lambda\cos\lambda L & \lambda\sin\lambda L & 1/(PL) \end{bmatrix} \begin{bmatrix} A \\ B \\ M_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution in which A, B and M_B are not all simultaneously zero the matrix determinant must vanish. Expanding gives

$$(\sin\lambda L - \lambda L\cos\lambda L) / (P L) = 0$$

Since P and L are nonzero, the factor $1 / (P L)$ may be removed. That leaves $\sin\lambda L = \lambda L\cos\lambda L$. Dividing through by $\cos\lambda L$ we recover the characteristic equation

$$\tan\lambda L = \lambda L$$

which was found in the previous slide by an elimnation method