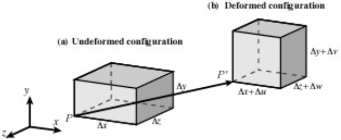
ASEN 3112

Spring 2020

Lecture 3

January 21, 2020

Normal Strains in 3D (2)



Average normal strains:

$$\epsilon_{xx,av} \stackrel{\text{def}}{=} \frac{u + \Delta u - u}{\Delta x} = \frac{\Delta u}{\Delta x} \qquad \epsilon_{yy,av} \stackrel{\text{def}}{=} \frac{v + \Delta v - v}{\Delta y} = \frac{\Delta v}{\Delta y}$$

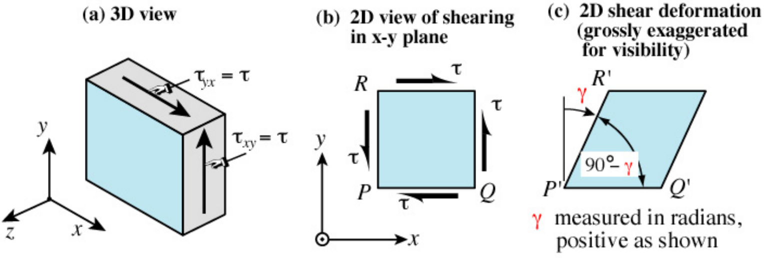
$$\epsilon_{zz,av} \stackrel{\text{def}}{=} \frac{w + \Delta w - w}{\Delta z} = \frac{\Delta w}{\Delta z}$$

Point strains at P:

$$\epsilon_{xx} \stackrel{\text{def}}{=} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} \stackrel{\text{def}}{=} \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} \stackrel{\text{def}}{=} \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z}$$

Average Shear Strain in x,y Plane

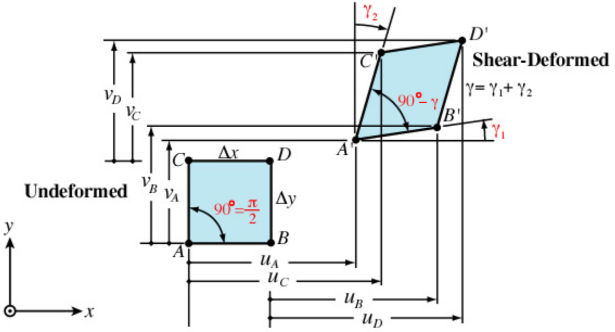


The average shear strain is

$$\gamma_{xy,av} \stackrel{\text{def}}{=} \gamma.$$

Positive if original right angle **decreases** by γ, as shown

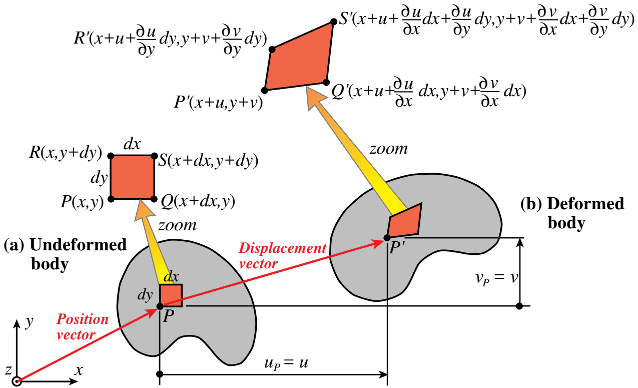
Average Shear Strain in x,y Plane in Terms of Corner Displacements



$$\gamma_{xy,av} = \gamma = \gamma_1 + \gamma_2 \approx \frac{\Delta v_{BA}}{\Delta x} + \frac{\Delta u_{CA}}{\Delta y} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}.$$

See Notes for derivation details

Arbitrary Body in 3D - see Notes



3D (Point) Strain-Displacement Equations

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

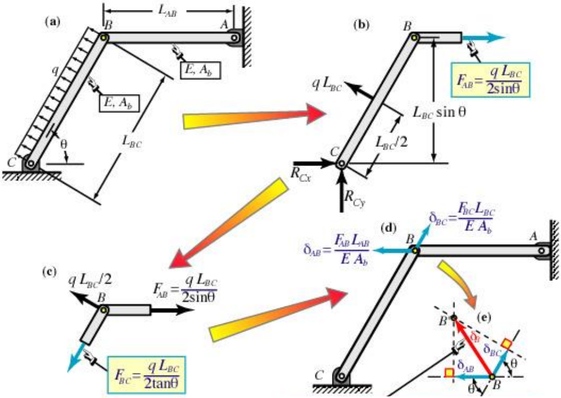
Strain matrix:

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

in which the shear strain components verify reciprocity:

$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{yz} = \gamma_{zy} \qquad \gamma_{zx} = \gamma_{xz}$$

Displacement Calculations for Truss



$$\delta_B = \frac{q\,L_{BC}}{2E\,A_b\sin\theta}\sqrt{\frac{(L_{BC}+L_{AB})^2\cos^2\theta}{\sin^2\theta}+L_{AB}^2}$$

See Section 4.5 of Notes for the worked out example

Displacement Vector Composition Differs From That of Force Vectors

(a) force composition

(b) displacement composition (assuming small deformations)

The physical significance of the diagram on the right will be illustrated with a problem in Recitation #1.

5

Stress - Strains Material Laws

Strains and Stresses are Connected by Material Properties of the Body (Structure)

Recall the connections displayed in previous lecture:

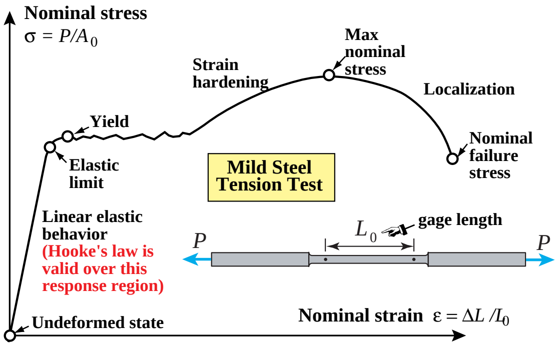
internal forces $\Rightarrow$ stresses $\xrightarrow{\text{MP}}$ strains $\Rightarrow$ displacements $\Rightarrow$ size&shape changes

displacements $\Rightarrow$ strains $\xrightarrow{\text{MP}}$ stresses $\Rightarrow$ internal forces

The linkage between stresses and strains is done through material properties, as shown by symbol MP over red arrow Those are mathematically expressed as constitutive equations

Historically the first C.E. was Hooke's elasticity law, stated in 1660 as "ut tensio sic vis" Since then recast in terms of stresses and strains, which are more modern concepts.

The Tension Test Revisited: Response Regions for Mild Steel



Material Properties For A Linearly Elastic Isotropic Body

E **Elastic modulus, a.k.a. Young's modulus**
Physical dimension: stress=force/area (e.g. ksi)

ν **Poisson's ratio**
Physical dimension: dimensionless (just a number)

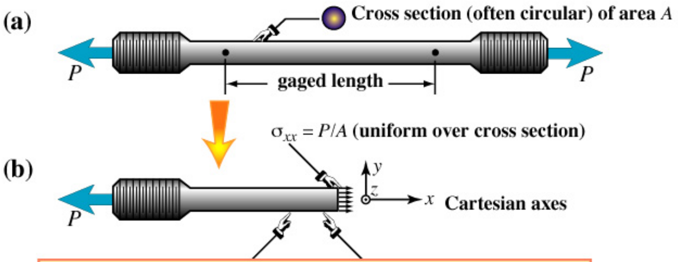
G **Shear modulus, a.k.a. modulus of rigidity**
Physical dimension: stress=force/area (e.g. MPa)

α **Coefficient of thermal expansion**
Physical dimension: 1/degree (e.g., 1/°C)

E, ν **and** G **are not independent. They are linked by**

$$E = 2G\,(1+\nu), \qquad G = E/(2(1+\nu)), \qquad \nu = E/(2G) - 1$$

State of Stress and Strain In Tension Test



Stress state $\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **Strain state** $\begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$

at all points in the gaged region

For isotropic material, $\varepsilon_{yy} = \varepsilon_{zz}$ **is called the lateral strain**

Defining Elastic Modulus and Poisson's Ratio

Isotropic material properties E and ν are obtained from the linear elastic response region of the uniaxial tension test (last slide). For simplicity call

$$\sigma = \sigma_{xx} = \text{axial stress}, \quad \epsilon = \epsilon_{xx} = \text{axial strain}, \quad \epsilon_{yy} = \epsilon_{zz} = \text{lateral strain}$$

The elastic modulus E is defined as the ratio of axial stress to axial strain:

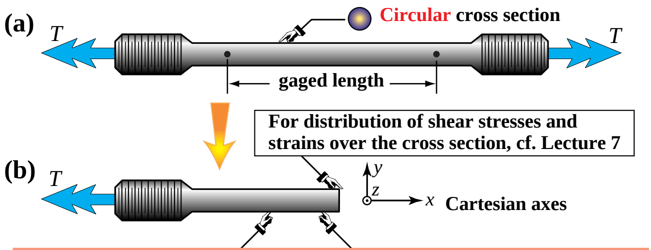
$$E \overset{\text{def}}{=} \frac{\sigma}{\epsilon} \quad \text{whence} \quad \sigma = E\,\epsilon, \quad \epsilon = \frac{\sigma}{E}$$

Poisson's ratio ν is defined as the ratio of lateral strain to axial strain:

$$\nu \overset{\text{def}}{=} \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\text{lateral strain}}{\text{axial strain}}$$

The $-$ sign in the definition of ν is introduced so that it comes out positive. For structural materials ν lies in the range [0,1/2). For most metals (and their alloys) ν is in the range 0.25 to 0.35. For concrete and ceramics, $\nu \approx 0.10$. For cork $\nu \approx 0$. For rubber $\nu \approx 0.5$ to 3 places. A material for which $\nu = 0.5$ is called incompressible. If ν is very close to 0.5, it is called nearly incompressible.

State of Stress and Strain In Torsion Test



Stress state $\begin{bmatrix} 0 & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **Strain state** $\begin{bmatrix} 0 & \gamma_{xy} & 0 \\ \gamma_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

at all points in the gaged region. Both the shear stress $\tau_{yx} = \tau_{xy}$ as well as the shear strain $\gamma_{xy} = \gamma_{yx}$ vary linearly as per distance from the cross section center (Lecture 7). They attain maximum values on the specimen surface. For simplicity, call those values $\tau = \tau_{xy}^{max}$ and $\gamma = \gamma_{xy}^{max}$

Defining Shear Modulus Of An Isotropic Linearly Elastic Material

Isotropic material property G (the shear modulus, also called modulus of rigidity) is obtained from the linear elastic response region of the torsion test of a circular cross section specimen (last slide).
For simplicity call

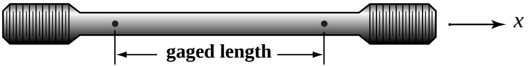
$$\tau = \tau_{xy}^{max} = \text{ max shear stress on specimen surface over gauged region}$$
$$\gamma = \gamma_{xy}^{max} = \text{ max shear strain on specimen surface over gauged region}$$

The shear modulus G is defined as the ratio of the foregoing shear stress and strain:

$$G \overset{\text{def}}{=} \frac{\tau}{\gamma} \quad \text{whence} \quad \tau = G\,\gamma, \quad \gamma = \frac{\tau}{G}$$

Defining The Coefficient of Thermal Expansion Of An Isotropic Material

Take a standard uniaxial test specimen:



At the reference temperature T_0 (usually the room temperature) the gaged length is L_0. Heat the unloaded specimen by ΔT while allowing it to expand freely in all directions. The gaged length changes to $L = L_0 + \Delta L$. The coefficient of thermal expansion is defined as

$$\alpha \overset{\text{def}}{=} \frac{\Delta L}{L_0 \Delta T} \quad \textbf{whence} \quad \Delta L = \alpha L_0 \Delta T$$

The ratio $\epsilon^T = \epsilon^T_{xx} = \Delta L / L_0 = \alpha \Delta T$ is called the thermal strain in the axial (x) direction. For an isotropic material, the material expands equally in all directions: $\epsilon^T_{xx} = \epsilon^T_{yy} = \epsilon^T_{zz}$, whereas the thermal shear strains are zero.

1D Hooke's Law Including Thermal Effects

Stress To Strain:

$$\varepsilon = \frac{\sigma}{E} + \alpha\ \Delta T = \varepsilon^M + \varepsilon^T$$

expresses that total strain = mechanical strain + thermal strain: the strain superposition principle

Strain To Stress:

$$\sigma = E\,(\,\varepsilon - \alpha\ \Delta T\,)$$

3D Generalized Hooke's Law (1)

Stresses To Strains (Omitting Thermal Effects)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

For derivation using the strain superposition principle, as well as inclusion of thermal effects, see Lecture notes

3D Generalized Hooke's Law (2)

Strains To Stresses (Omitting Thermal Effects)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

in which

$$\hat{E} = \frac{E}{(1-2\nu)(1+\nu)}$$

This is derived by inverting the matrix of previous slide. For the inclusion of thermal effects, see Lecture notes

2D Plane Stress Specialization

Stresses

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Strains

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

2D Plane Stress Generalized Hooke's Law

Strains To Stresses (Omitting Thermal Effects)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

Stresses To Strains (Omitting Thermal Effects)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \tilde{E} & \tilde{E}\,\nu & 0 \\ \tilde{E}\,\nu & \tilde{E} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

in which

$$\tilde{E} = \frac{E}{1 - \nu^2}$$