ASEN 3112

Spring 2020

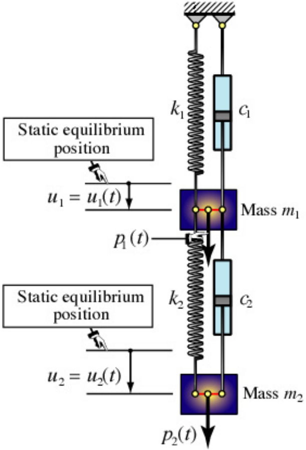
Lecture 19

March 31, 2020

MDOF Dynamic Systems

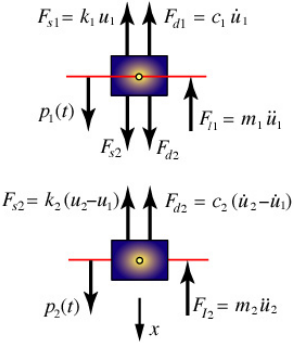
A Two-DOF Mass-Spring-Dashpot Dynamic System

Consider the lumped-parameter, mass-spring-dashpot dynamic system shown in the Figure. It has two point masses m_1 and m_2, which are connected by a spring-dashpot pair with constants k_2 and c_2, respectively. Mass m_1 is linked to ground by another spring-dashpot pair with constants k_1 and c_1, respectively. The system has **two degrees of freedom** (DOF). These are the displacements $u_1(t)$ and $u_2(t)$ of the two masses measured from their static equilibrium positions. Known dynamic forces $p_1(t)$ and $p_2(t)$ act on the masses. Our goal is to formulate the **equations of motion** (EOM) and to place them in matrix form.



Dynamic Free-Body-Diagrams to Derive EOM

The EOM are derived from the dynamic Free Body Diagrams (DFBD) shown in the figure. Isolate the masses and apply the forces acting on them. Spring and dashpot forces are denoted by F_{si} and F_{di} for the ith spring and ith dashpot, respectively. The inertia force acting on the ith mass is F_{Ii}.



Positive force conventions are those shown in the Figure. Summing forces along x we get the two force equilibrium equations

DFBD #1: $\sum F_x$ at mass 1 : $\quad -F_{I1} - F_{s1} - F_{d1} + F_{s2} + F_{d2} + p_1 = 0$

DFBD #2: $\sum F_x$ at mass 2 : $\quad -F_{I2} - F_{s2} - F_{d2} + p_2 = 0$

EOM in Scalar Form

Replace now the expression of motion-dependent forces in terms of the displacement DOF, their velocities and accelerations:

$$F_{s1} = k_1\, u_1 \qquad F_{s2} = k_2\,(u_2 - u_1) \qquad F_{d1} = c_1\,\dot u_1$$
$$F_{d2} = c_2\,(\dot u_2 - \dot u_1) \qquad F_{I1} = m_1\,\ddot u_1 \qquad F_{I2} = m_2\,\ddot u_2$$

Note that for the spring and dashpot that connect masses 1 and 2, **relative** displacements and velocities must be used. Replacing now into the force equilibrium equations furnished by the two DFBD gives

$$-m_1\,\ddot u_1 - k_1\,u_1 - c_1\,\dot u_1 + k_2\,(u_2 - u_1) + c_2\,(\dot u_2 - \dot u_1) + p_1 = 0$$
$$-m_2\,\ddot u_2 - k_2\,(u_2 - u_1) - c_2\,(\dot u_2 - \dot u_1) + p_2 = 0$$

Finally, collect all terms that depend on the DOF and their time derivatives in the LHS, while moving everything else (here, the given applied forces) to the RHS. Sign convention: $m\,\ddot u$ terms on the LHS must be positive.

$$\boxed{\begin{aligned} m_1\,\ddot u_1 + c_1\,\dot u_1 + c_2\,\dot u_1 - c_2\,\dot u_2 + k_1\,u_1 + k_2\,u_1 - k_2\,u_2 &= p_1 \\ m_2\,\ddot u_2 - c_2\,\dot u_1 + c_2\,\dot u_2 - k_2\,u_1 + k_2\,u_2 &= p_2 \end{aligned}}$$

Numerical Example

For the two-DOF example of Slide 2, assume the numerical values

$$m_1 = 2 \quad m_2 = 1 \quad c_1 = 0.1 \quad c_2 = 0.3$$
$$k_1 = 6 \quad k_2 = 3 \quad p_1 = 2\,\sin 3t \quad p_2 = 5\,\cos 2t$$

Then the matrix EOM are

$$\boxed{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 0.4 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2\,\sin 3t \\ 5\,\cos 2t \end{bmatrix}}$$

The known matrices and vectors (the givens for the problem) are

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0.4 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} 2\,\sin 3t \\ 5\,\cos 2t \end{bmatrix}$$

Vibration Eigenproblem: Assumptions on Damping and Forces

Suppose that the example two-DOF system is **undamped**:
$c_1 = c_2 = 0$, whence $\mathbf{C} = \mathbf{0}$, the null 2 x 2 matrix. It is also **unforced**:
$p_1 = p_2 = 0$, whence $\mathbf{p} = \mathbf{0}$, the null 2 x 1 vector.
The matrix EOM reduces to

$$\boxed{\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{0}}$$

This is the MDOF generalization of the free, undamped SDOF oscillator covered in Lecture 17. Next, assume that this unforced and undamped dynamic system is undergoing in-phase **harmonic motions** of circular frequency ω.

Vibration Eigenproblem: Assumptions on Damping and Forces

Suppose that the example two-DOF system is **undamped**:
$c_1 = c_2 = 0$, whence $\mathbf{C} = \mathbf{0}$, the null 2 x 2 matrix. It is also **unforced**:
$p_1 = p_2 = 0$, whence $\mathbf{p} = \mathbf{0}$, the null 2 x 1 vector.
The matrix EOM reduces to

$$\boxed{\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{0}}$$

This is the MDOF generalization of the free, undamped SDOF oscillator covered in Lecture 17. Next, assume that this unforced and undamped dynamic system is undergoing in-phase **harmonic motions** of circular frequency ω.

Vibration Eigenproblem: Assumed Harmonic Motion

The assumption of harmonic motion can be mathematically stated using either trigonometric functions, or complex exponentials:

$$\boxed{\mathbf{u}(t) = \mathbf{U}\cos(\omega t - \alpha) \quad \text{or} \quad \mathbf{u}(t) = \mathbf{U}\, e^{i\omega t}}$$

in which $\mathbf{U}$ is a nonzero 2-vector that collects amplitudes of the motions of the point masses as entries, and (in the first form) α is a phase shift angle. Both expressions lead to identical results.
For the ensuing derivations we select the first form. The corresponding velocities and accelerations are

$$\dot{\mathbf{u}}(t) = -\,\omega\,\mathbf{U}\sin(\omega t - \alpha) \qquad \ddot{\mathbf{u}}(t) = -\,\omega^2\mathbf{U}\cos(\omega t - \alpha)$$

Vibration Eigenproblem: Characteristic Equation

Substitute the accelerations into the matrix EOM, and extract $\mathbf{U}\cos(\omega\, t - \alpha)$ as common post-multiplier factor:

$$\boxed{\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{U}\,\cos(\omega t - \alpha) = \mathbf{0}}$$

If this vector expression is to be identically zero for any time t, the product of the first two terms (matrix times vector) must vanish. Thus

$$\boxed{\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{U} = \mathbf{D}(\omega)\,\mathbf{U} = \mathbf{0}}$$

This $\mathbf{D}$ is called the **dynamic stiffness matrix**.

The foregoing equation states the **free vibrations eigenproblem** for an undamped MDOF system. For nontrivial solutions $\mathbf{U} \neq \mathbf{0}$ the determinant of the dynamic stiffness matrix must be zero, whence

$$\boxed{C(\omega^2) = \det\mathbf{D}(\omega) = \det\left[\mathbf{K} - \omega^2\mathbf{M}\right] = 0}$$

This is called the **characteristic equation** of the dynamic system.

Vibration Eigenproblem: Natural Frequencies

For a two-DOF system $C(\omega)$ is a **quadratic polynomial** in ω^2, which will yield two roots: ω_1^2 and ω_2^2. We will later see that under certain conditions on $\mathbf{K}$ and $\mathbf{M}$, which are satisfied here, those roots are **real** and **nonnegative**. Consequently the positive square roots

$$\omega_1 = +\sqrt{\omega_1^2} \qquad \omega_2 = +\sqrt{\omega_2^2}$$

are also real and nonnegative. Those are called the **undamped natural circular frequencies**. (Qualifiers "undamped" and "circular" are often omitted unless a distinction needs to be made.) As usual they are measured in radians per second, or rad/sec. They can be converted to frequencies in cycles per second, or Hertz (Hz) by scaling through $1/(2\pi)$, and to natural periods in seconds by taking the reciprocals:

$$f_i = \frac{\omega_i}{2\pi}, \qquad T_i = \frac{1}{f_i} = \frac{2\pi}{\omega_i}, \quad i = 1, 2$$

Eigenvectors As Vibration Modes

How about the $\mathbf{U}_1$ vector? Insert $\omega^2 = \omega_1^2$ in the eigenproblem to get

$$\boxed{(\mathbf{K} - \omega_1^2\mathbf{M})\ \mathbf{U}_1 = \mathbf{0}}$$

and solve this homogeneous system for a nonzero $\mathbf{U}_1$ (This operation is always possible because the coefficient matrix is singular by definition.) Repeat with $\omega^2 = \omega_2^2$, and call the solution vector $\mathbf{U}_2$. Vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are mathematically known as **eigenvectors.** Since they can be scaled by arbitrary nonzero factors, we must **normalize** them to make them unique. Normalization criteria often used in structural dynamics are discussed in a later slide.

For this application (i.e., structural dynamics), these eigenvectors are called **undamped free-vibrations natural modes**, a mouthful often abbreviated to **vibration modes** or simply **modes**, on account of the physical interpretation discussed later. The visualization of an eigenvector as a motion pattern is called a **vibration shape** or **mode shape**.

Generalization to Arbitrary Number of DOF

Thanks to matrix notation, the generalization of the foregoing derivations to an **undamped**, **unforced, Multiple-Degree-Of-Freedom** (MDOF) dynamical system with n degrees of freedom is immediate.

In this general case, matrices **K** and **M** are both $n \times n$. The characteristic polynomial is of order n in ω^2. This polynomial provides n roots labeled ω_i^2, $i = 1, 2, \ldots n$, arranged in **ascending order**. Their square roots give the natural frequencies of the system.

The associated vibration modes, which are called $\mathbf{U}_i$ **before** normalization, are the **eigenvectors** of the vibration eigenproblem. Normalization schemes are studied later.

Linkage to Linear Algebra Material

To facilitate linkage to the subject of algebraic eigenproblems covered in Linear Algebra sophomore courses, as well as to capabilities of matrix oriented codes such as *Matlab*, we effect some notational changes. First, rewrite the vibration eigenproblem as

$$\boxed{\mathbf{K}\,\mathbf{U}_i = \omega_i^2\,\mathbf{M}\,\mathbf{U}_i \qquad i = 1, 2, \ldots n}$$

where i is a natural frequency index that ranges from 1 through n. Second, rename the symbols as follows: **K** and **M** become **A** and **B**, respectively, ω_i^2 becomes λ_i, and $\mathbf{U}_i$ becomes $\mathbf{x}_i$. These changes allow us to rewrite the vibration eigenproblem as

$$\boxed{\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{B}\,\mathbf{x}_i \qquad i = 1, 2, \ldots n}$$

This is the **canonical form** of the **generalized algebraic eigenproblem** that appears in Linear Algebra textbooks. If both **A** and **B** are **real symmetric**, this is called the **generalized symmetric algebraic eigenproblem**.

Properties of the Generalized Symmetric Algebraic Eigenproblem

If, in addition, **B** is positive definite (PD) it is shown in Lineaar Algebra textbooks that

(i) All eigenvalues λ_i are **real**
(ii) All eigenvectors $\mathbf{x}_i$ have **real entries**

Property (i) can be further strengthened if **A** enjoys additional properties:

(iii) If **A** is nonnegative definite (NND) and **B** is PD all eigenvalues are **nonnegative real**
(iv) If both **A** and **B** are PD, all eigenvalues are **positive real**

The nonnegativity property is obviously important in vibration analysis since frequencies are obtained by taking the positive square root of the eigenvalues. If a squared frequency is negative, its square roots are imaginary. The possibility of negative squared frequencies arises in the study of dynamic stability and active control systems, but those topics are beyond the scope of the course.

The Standard Algebraic Eigenproblem

If matrix **B** is the identity matrix, the generalized algebraic eigenproblem reduces to the **standard algebraic eigenproblem**

$$\boxed{\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i \qquad i = 1, 2, \ldots n}$$

If **A** is real symmetric this is called the **standard symmetric algebraic eigenproblem**, or **symmetric eigenproblem** for short. Since the identity is obviously symmetric and positive definite, the aforementioned properties (i) through (iv) also hold for the standard version.

Some higher level programming languages such as *Matlab*, *Mathematica* and *Maple*, can solve eigenproblems directly through their built-in libraries. Of these 3 commercial products, *Matlab* has the more extensive capabilities, being able to process generalized eigenproblem forms directly. On the other hand, *Mathematica* only accepts the standard eigenproblem as built-in function. Similar constraints may affect hand calculators with matrix processing abilities. To get around those limitations, there are transformation methods that can convert the generalized eigenproblem into the standard one.

Reduction to the Standard Eigenproblem (cont'd)

Two very simple schemes to reduce the generalized eigenproblem to standard form are

(I) If **B** is nonsingular, premultiply both sides by its inverse.

(II) If **A** is nonsingular, premultiply both sides by its inverse.

For further details, see Remark 19.1 in Lecture 19.

The matrices produced by these schemes will not generally be symmetric, even if **A** and **B** are. There are fancier reduction methods that preserve symmetry, but those noted above are sufficient for this course.

Using the Characteristic Polynomial To Get Natural Frequencies

For a small number of DOF, say 4 or less, it is often expedient to solve for the eigenvalues of the vibration eigenproblem directly by finding the **roots of the characteristic polynomial**

$$\boxed{C(\lambda) = \det(\mathbf{D}) = 0}$$

in which $\lambda = \omega^2$ and $\mathbf{D} = \mathbf{K} - \omega^2\,\mathbf{M} = \mathbf{K} - \lambda\,\mathbf{M}$ is the dynamic stiffness matrix. This procedure will be followed in subsequent Lectures for two-DOF examples. It is particularly useful with hand calculators that can compute polynomial roots but cannot handle eigenproblems directly. Eigenvectors may then be obtained as illustrated in the next Lecture.

Eigenvector Uniqueness

We go back to the mass-stiffness notation for the vibration eigenproblem, which is reproduced below for convenience:

$$\boxed{\mathbf{K}\,\mathbf{U}_i = \omega_i^2\,\mathbf{M}\,\mathbf{U}_i}$$

For specificity we will assume that

(I) The mass matrix is **positive definite** (PD) while the stiffness matrix is **nonnegative definite** (NND). As a consequence, all eigenvalues $\lambda_i = \omega_i^2$ are **nonnegative real**, and so are their positive square roots ω_i.

(II) All frequencies are **distinct**:

$$\omega_i \neq \omega_j \quad \text{if} \quad i \neq j$$

Under these assumptions, the theory behind the algebraic eigenproblem says that for each ω_i there is **one and only one eigenvector** $\mathbf{U}_i$, which is **unique** except for an **arbitrary nonzero scale factor**.

What Happens If There Are Multiple Frequencies?

Life gets more complicated. See Remark 19.3 in Lecture Notes.

Normal Modes and Their Orthogonality Properties

An eigenvector $\mathbf{U}$ scaled as per some normalization condition (for example, unit Euclidean length) is called a **normal mode**. We will write

$$\mathbf{U}_i = c_i\,\phi_i \qquad i = 1, 2, \ldots n$$

Here ϕ_i is the notation for the ith normal mode. This is obtained from $\mathbf{U}$ on dividing through some scaling factor c_i, chosen as per some normalization criterion. (Those are discussed in a later slide).

Normal modes enjoy the following **orthogonality properties** with respect to the mass and stiffness matrices:

$$\boxed{\phi_i^T\,\mathbf{M}\,\phi_j = 0 \qquad \phi_i^T\,\mathbf{K}\,\phi_j = 0 \qquad i \neq j}$$

Generalized Mass and Stiffness

If $i = j$, the foregoing quadratic forms provide two important quantities called the **generalized mass** M_i, and the **generalized stiffness** K_i :

$$\boxed{M_i = \phi_i^T\,\mathbf{M}\,\phi_i > 0 \qquad K_i = \phi_i^T\,\mathbf{K}\,\phi_i \geq 0 \qquad i = 1, 2, \ldots n}$$

These are also called the **modal mass** and the **modal stiffness**, respectively, in the structural dynamics literature.

The squared natural frequencies appear as the ratio of generalized stiffness to the corresponding generalized masses:

$$\boxed{\omega_i^2 = \frac{K_i}{M_i} \qquad i = 1, 2, \ldots n}$$

This formula represents the generalization of the SDOF expression for the natural frequency: $\omega_n^2 = k/m$, to MDOF.

Mass-Orthonormalized Eigenvectors

The foregoing definition of generalized mass provides one criterion for **eigenvector normalization**, which is of particular importance in modal response analysis. If the scale factors are chosen so that

$$\boxed{M_i = \phi_i^T \, \mathbf{M} \, \phi_i = 1 \qquad i = 1, 2, \ldots n}$$

the normalized eigenvectors (a.k.a. normal modes) are said to be **mass-orthogonal**. If this is done, the generalized stiffnesses become the squared frequencies:

$$\boxed{\omega_i^2 = K_i = \phi_i \, \mathbf{K} \, \phi_i \qquad i = 1, 2, \ldots n}$$

Eigenvector Normalization Criteria

Three eigenvector normalization criteria are in common use in structural dynamics. They are summarized here for convenience. The *i*th **unnormalized** eigenvector is generically denoted by $\mathbf{U}_i$, whereas a **normalized** one is called ϕ_i. We will assume that eigenvectors are **real**, and that each entry has the same physical unit. Furthermore, for (III) we assume that $\mathbf{M}$ is positive definite (PD).

(I) **Unit Largest Entry**. Search for the largest entry of $\mathbf{U}_i$ in absolute value, and divide by it. (This value must be nonzero, else $\mathbf{U}_i$ would be null.)

(II) **Unit Length**. Compute the Euclidean length of the eigenvector, and divide by it. The normalized eigenvector satisfies $\mathbf{U}_i^T\,\mathbf{U}_i = 1$.

(III) **Unit Generalized Mass**. Get the generalized mass $M_i = \mathbf{U}_i^T\mathbf{M}\,\mathbf{U}_i$, which must be positive under the assumption that $\mathbf{M}$ is PD, and divide $\mathbf{U}_i$ by the positive square root of M_i. The normalized eigenvector satisfies $\phi_i^T\mathbf{M}\,\phi_i = 1$. The resulting eigenvectors are said to be **mass-orthogonal**.

If the mass matrix is the identity matrix, the last two methods coalesce. For just showing mode shapes pictorially, the simplest normalization method (I) works fine. For the modal analysis presented in following lectures, (III) offers computational and organizational advantages, so it is preferred for that task.

Frequency Computation Example

Consider the two-DOF system introduced in an earlier slide. As numeric values take $m_1 = 2$, $m_2 = 1$, $c_1 = c_2 = 0$, $k_1 = 6$, $k_2 = 3$, $p_1 = p_2 = 0$. The mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}$$

whereas the damping matrix $\mathbf{C}$ and dynamic force vector $\mathbf{p}$ vanish. The free vibrations eigenproblem is

$$\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \text{ or } \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The characteristic polynomial equation is

$$\det \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} = 18 - 15\omega^2 + 2\omega^4 = (3 - 2\omega^2)(6 - \omega^2) = 0$$

The roots of this equation give the two undamped natural squared frequencies

$$\omega_1^2 = \frac{3}{2} = 1.5 \qquad \omega_2^2 = 6$$

The undamped natural frequencies (to 4 places) are $\omega_1 = 1.225$ and $\omega_2 = 2.449$. If the physical sets of units is consistent, these are expressed in radians per second. To convert to cycles per second, divide by 2π: $f_1 = 0.1949$ Hz and $f_2 = 0.3898$ Hz. The calculation of eigenvectors is carried out in the next Lecture.