ASEN 3112

Spring 2020

Lecture 24

April 21, 2020

Elastically restrained pinned-pinned problem
(See details in Chapter 28, Section 28.4)

Problem description and FBDs

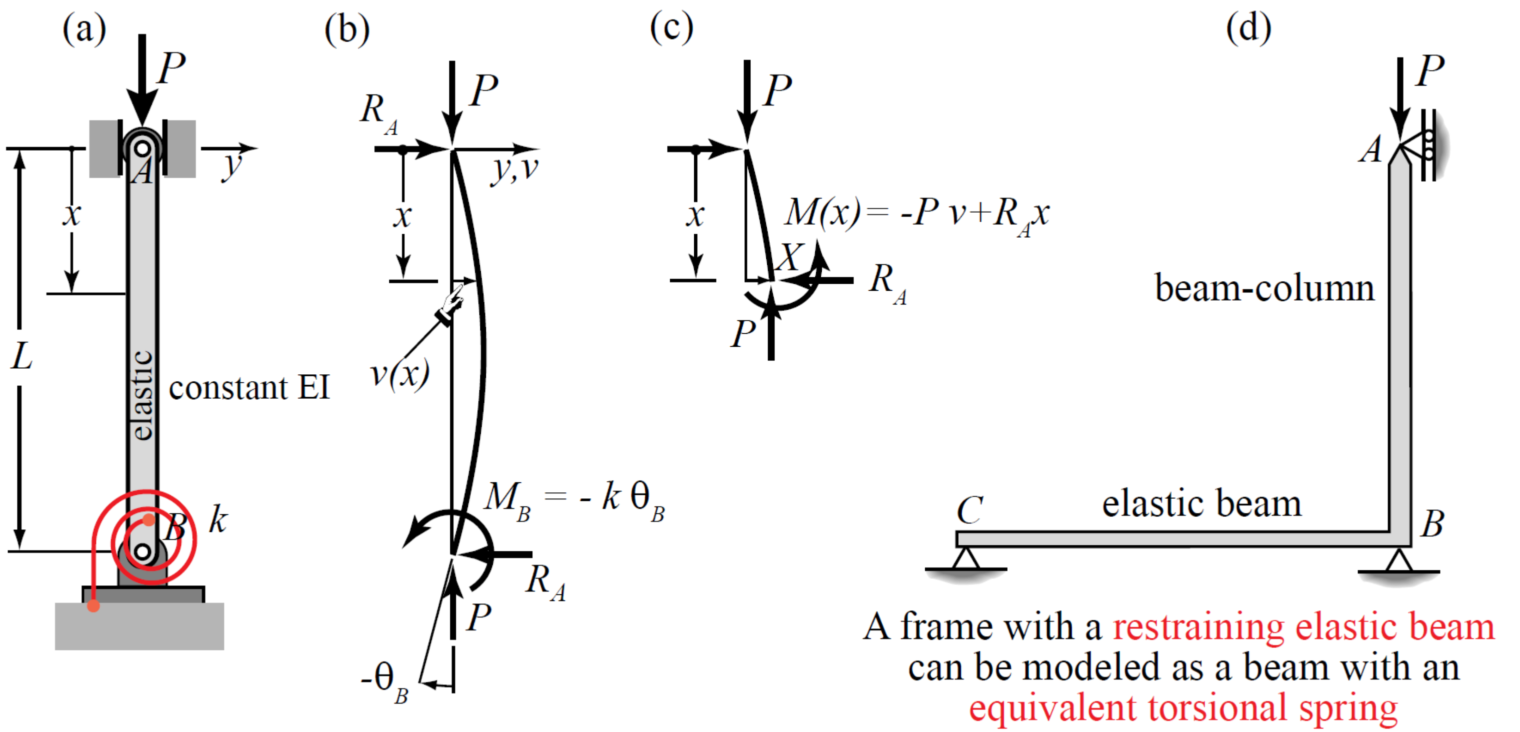

Figure 28.5: Column with elastic restraint: a torsional spring, at end B.

Elastically restrained pinned-pinned problem
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Column AB has length L and constant flexural rigidity EI, in which $I \equiv I_{zz}$ is the minimum moment of inertia of the cross section that controls buckling. (For a rectangular cross section of width b and thickness $t < b$, $I = I_{zz} = b\,t^3/12$.) The column is simply supported (pinned) at both ends A and B, and axially loaded by P. The rotation at B is further restrained by a torsional spring with stiffness k. Select x as shown and consider the buckling shape $v(x)$ sketched in Figure A.1(b). The end rotation at B is $\theta_B = v'(L)$, positive CCW. The spring at B applies a restoring end moment $M_B = -k\,\theta_B$. For convenience in obtaining dimensionless equations we define

$$k = \beta \frac{EI}{L} \tag{28.22}$$

where β is a numerical coefficient. If $\beta = 0$ the problem reduces to that of the classical Euler column, whereas if $\beta \to \infty$ we obtain the case of a column simply supported at A and fixed (clamped) at B.

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Critical load analysis

The FBD of the complete column is shown in Figure 28.5(b). Taking moments with respect to B one finds that

$$R_A = M_B/L = -k\,\theta_B/L = -\beta EI\,\theta_B/L^2. \tag{28.23}$$

Now cut the column at section X at distance x as sketched in Figure 28.5(c). Moment equilibrium with respect to X yields the second order differential equation

$$EI\,v'' + P\,v = R_A\,x = -k\,\theta_B\frac{x}{L} = -\beta\frac{EI\,\theta_B x}{L^2}, \tag{28.24}$$

which results from equating the bending moment $M(x) = EIv''$ to $-Pv+R_Ax$. Dividing through by EI and calling $\lambda^2 = P/EI$ produces the canonical form

$$v'' + \lambda^2\,v = -\beta\frac{\theta_B\,x}{L^2}. \tag{28.25}$$

The general solution of (28.25) is the sum of the homogeneous and particular solutions:

$$v(x) = C_1\sin\lambda x + C_2\cos\lambda x - \beta\frac{\theta_B\,x}{\lambda^2 L^2}. \tag{28.26}$$

Since $v(0) = 0$, $C_2 = 0$. The rotation is $\theta(x) = v'(x) = C_1\lambda\cos\lambda x - \beta\theta_B/(\lambda^2L^2)$. Evaluating this at $x = L$ yields $\theta_B = \theta(L) = C_1\lambda\cos\lambda L - \beta\theta_B/(\lambda^2L^2)$, from which we can solve for the end rotation:

$$\theta_B = C_1\frac{\lambda^3L^2}{\beta+\lambda^2L^2}\cos\lambda L. \tag{28.27}$$

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Inserting this into the solution (28.28) with $C_2 = 0$ gives $v(x) = C_1 \sin \lambda x - C_1 \beta \lambda x \cos \lambda L/(\beta + \lambda^2 L^2)$. Applying now the second boundary condition: $v(L) = 0$, furnishes the stability equation

$$v(L) = C_1 \left(\sin \lambda L - \frac{\beta \lambda L}{\beta + \lambda^2 L^2} \cos \lambda L \right) = 0. \tag{28.28}$$

For buckling to occur, $C_1 \neq 0$, and the expression in parentheses in (28.29) must vanish. Calling $\alpha = \lambda L$, which is also a dimensionless variable, we obtain the trascendental equation

$$\boxed{\tan \alpha = \frac{\alpha\, \beta}{\alpha^2 + \beta}} \tag{28.29}$$

For a given $\beta \geq 0$ we seek the solution $\alpha_{cr} > 0$ of (28.29) closest to zero. If $\beta \neq 0$ there is no closed form solution and it is better to proceed numerically, using for example a Newton solver. It is easily shown that for $\beta = [0, \infty]$, $\pi \leq \alpha_{cr} < 4.5$, so $\alpha \approx 4$ is a good start point for a Newton iteration. Here is a table for selected values of β:

β	0	1	3	10	100	1000	10000	∞
α_{cr}	π	3.4056	3.7264	4.1323	4.4494	4.4889	4.4930	4.4934
L_e/L	1.0000	0.9224	0.8431	0.7602	0.7061	0.7000	0.6992	0.6991

The critical load P_{cr} and the effective length L_e (tabulated above) are

$$\boxed{P_{cr} = \frac{\alpha_{cr}^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}, \qquad L_e = \frac{\pi}{\alpha_{cr}} L}$$