Homework -1

Pankaj Kumar

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Q1/2 (i) $y - a^x = 0$, $a > 0$

$\therefore y = a^x$

$= f(x) = a^x$

but we know that condition number is represented by $\left| \frac{x f'(x)}{f(x)} \right|$

$= \left| \frac{x \times f'(a^x)}{a^x} \right| = \left| \frac{x \times \ln a \times \cancel{a^x}}{\cancel{a^x}} \right|$

$= |x \ln a|$ for $a > 0$.

(ii) $x - y + 1 = 0$

$y = x+1$ $\therefore$ $f(x) = x+1$

$\therefore$ condition number $= \left| \frac{x \times f'(x+1)}{x+1} \right|$

$= \left| \frac{x \times (1)}{x+1} \right| = \left| \frac{x}{x+1} \right|$

$= \left| \frac{x+1-1}{x+1} \right| = \left| \frac{x+1}{x+1} - \frac{1}{x+1} \right|$

$= \left| 1 - \frac{1}{x+1} \right|$

Q2 To prove :- any vector form is Lipschitz continous, which satisfies the inequality.

$$|\,\|x\|-\|y\|\,| < \|x-y\| \quad , x,y \in \mathbb{R}^n$$

Proof :- Here, let any $f(x) = \|x\|$ and $f(y) = \|y\|$ provided that $x,y \in \mathbb{R}^n$

Now we know that for Lipschitz conitnuity we have

$$|f(x)-f(y)| \leq |x-y|$$

$\therefore$ we need to prove it first.

Now.

$$f(x) = \|x-y+y\|$$

$$\leq \|x-y\| + \|y\|$$

(we know that $\|x-y+y\| \leq \|x-y\|+\|y\|$)

$$\therefore f(y) = \|y-x+x\|$$

$$\leq \|y-x\| + \|x\|$$

$$= \|-(x-y)\| + \|x\|$$

(Similar as above)

$$= \|x-y\| + \|x\|$$

($\|-a\| = \|a\|$)

Thus,

$$f(x) \leq \|x-y\| + \|y\|$$

(from above)

$$\|x\| \leq \|x-y\| + \|y\|$$

$$\|x\| - \|y\| \leq \|x-y\| \quad \text{--- (I)}$$

Now

$f(y) \leq ||x-y|| + ||x||$

$= ||y|| - ||x|| \leq ||x-y||$ (Similar as in $f(x)$)

$= ||x|| - ||y|| \geq -||x-y||$ (inequality changes) — II

∴ from I and II

$$\boxed{\big|\, ||x|| - ||y|| \,\big| \leq ||x-y||}$$

Q4 Since from the Question, we already have

Given: $A^2 = 0$, for $A \in R^{n \times n}$ and $0 \in R^{n \times n}$

to prove: A is singular matrix.

proof: let's say A is an non-singular matrix.

thus,

$AX = 0$ for some $X \in R^n$

Now. solution of eqn is

$X = 0$ (Since A is non-singular as we already supposed that)

and $A^2 = \vec{0}$ $|A^2| = 0$

Now for som $y \in R^n$, such that $y \neq 0$

and. $A^2 y = 0$ (since $A^2 = 0$)

$2 \quad A(AY) = 0 \qquad (A^2 = A \cdot A)$

$\therefore$ it's either $AY = 0$ or $A = 0$

but we already know that $A \neq 0$ (as we supposed that)

Now

$AY = 0$ but again $Y \neq 0$

$\therefore AY = 0$ has sol^n which creates the contradiction between the assumption that A is singular.

$\therefore$ By contraction, we can say that A is a singular Matrix

Hence Proved.

Q5 to prove: for any singular matrices

$A, B \in R^{n \times n}$ we have to prove that

$$k(AB) \leq k(A) * k(B)$$

proof: Since we already know that for any matrix A, condition number is given by

$$k(A) = ||A|| \cdot ||A^{-1}||$$

similarly for matrix B.

$$k(B) = ||B|| \cdot ||B^{-1}||$$

$\therefore$ $k(AB) = ||AB|| \cdot ||(AB)^{-1}||$

$= ||AB|| \, ||B^{-1}A^{-1}||$ $\quad$ ($(PQ)^{-1} = Q^{-1}P^{-1}$ property of matrix.)

also, we have $||AB|| \leq ||A|| \, ||B||$ $\quad$ (Multiplicative property)

Thus, $k(AB) = ||AB|| \cdot ||B^{-1}A^{-1}||$

$\leq (||A|| \, ||B||) \cdot (||B^{-1}|| \cdot ||A^{-1}||)$

$= (||A|| \, ||A^{-1}||) \cdot (||B|| \cdot ||B^{-1}||)$

$= k(A) \cdot k(B)$ $\quad$ (from above)

$$\boxed{k(AB) \leq k(A) \cdot k(B)}$$

~~Solu~~ 3(a) $x \in R^n$, $y \in R^m$

$$x \otimes y := \begin{bmatrix} xy_1 \\ \vdots \\ xy_m \end{bmatrix}_{mn}$$

$\|x \otimes y\|_p$ in term of $\|x\|_p$ and $\|y\|_p$ for $p = 1, 2, \ldots$

for $p = 1$

Now let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ for $x \in R^n$, $y \in R^m$

Since tensor product is already defined

$$\therefore X \otimes Y = \begin{bmatrix} xy_1 \\ xy_2 \\ \vdots \\ xy_m \end{bmatrix}_{mn}$$

$$= \begin{bmatrix} y_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \vdots \\ y_m \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \end{bmatrix}$$ (Just simplification)

$$= \begin{bmatrix} x_1 y_1 \\ x_2 y_1 \\ \vdots \\ x_n y_1 \\ \vdots \\ x_n y_m \end{bmatrix}$$

Now, we have

$$\|X\| = \sum_{i=1}^{n} |x_i|$$

$$= |x_1| + |x_2| + \cdots + |x_n|$$

$\therefore$ Similarly

$$\|Y\| = \sum_{i=1}^{n} |y_i| = |y_1| + |y_2| + \cdots + |y_n|$$

Now the tensor product

$$\|X \otimes Y\| = \sum_{i=1}^{mn} |(X \otimes Y)_i|$$

$$= \left(|x_1 y_1| + |x_2 y_1| + |x_3 y_1| + \cdots (x_n y_1) \right)$$

$$+ \cdots + \left(x_1 y_m + x_2 y_m + \cdots (x_n y_m) \right)$$

$$= \left(|x_1| \cdot |y_1| + |x_2 y_1| + \cdots |x_n||y_1| \right) +$$

$$\cdots + |x_1| \cdot |y_m + \cdots |x_n||y_m| \Big)$$

$$= |y_1| \left(|x_1| + \cdots |x_n| \right) + |y_2| \left(|x_1 + \cdots |x_n| \right)$$

$$\cdots + |y_m| \left(|x_1| + \cdots + |x_n| \right)$$

$=$ taking common $(|x_1| + \cdots |x_n|)$

2) $(|x_1|+|x_2|+ \cdots |x_n|)(|y_1|+|y_2|+ \cdots +|y_m|)$

$$= \sum_{i=1}^{n} |x_i| * \sum_{j=1}^{m} |y_j|$$

$$= \|X\| * \|Y\|$$

$$= \boxed{\|X \otimes Y\| = \|X\| \cdot \|Y\|}$$

for $p=2$

$$\|X\|_2 = \left(\sum_{i=1}^{n} (|x_i|^2)\right)^{1/2}.$$

∴ on Squaring both side.

$$\|X\|_2^2 = \sum_{i=1}^{n} |x_i|^2$$

∴ $\|Y\|_2^2 = \sum_{i=1}^{m} |y_i|^2$

Now, similarly tensor product is given by

$$\|X \otimes Y\|_2^2 = \sum_{i=1}^{m \times n} (X \otimes Y)_i^2$$

on Expanding.

$$= [(x_1y_1)^2 + (x_2y_1)^2 + \cdots (x_ny_1)^2]$$

$$+ \cdots + [(x_1y_m)^2 + (x_2y_m)^2 + \cdots + (x_ny_m)^2]$$

$$= y_1^2 (x_1^2+x_2^2+ \cdots +x_n^2) + \cdots + y_m (x_1^2 + \cdots + x_n^2)$$

$$= (x_1^2+x_2^2+ \cdots x_n^2)(y_1 + \cdots y_m^2)$$

$$\|X \otimes Y\|_2^2 = \|X\|_2^2 \cdot \|Y\|_2^2$$

as $\sum_{i=1}^{n} X_i^2 = (x_1^2 + \cdots x_n^2) = \|X\|_2$

$$\sum_{i=1}^{m} Y_i^2 = (y_1^2 + \cdots y_m^2) = \|Y\|_2$$

$$\therefore \boxed{\|X \otimes Y\|_2 = \|X\|_2 \cdot \|Y\|_2}$$

for $p = \infty$, Since

$$\|X\|_\infty = \max_{1 \le i \le n} (x_i)$$

$$\|Y\|_\infty = \max_{1 \le i \le n} (y_i)$$

Now here for the tensor product,

$$\|X \otimes Y\|_\infty = \max_{i \le i \le m \times n} (|(X \otimes Y)|_i)$$

∴ we can written as

$$= \max\left((|y_1| * \max(|x_1|, |x_2| \ldots |x_n|) \right],$$

$$\ldots (|y_m| * \max(|x_1|, |x_2| \ldots |x_n|)]$$

Now

$|y_i|$ and $|x_i| \geq 0$ (as it stated)

$$\therefore \max_{j \in [1,m]} \left(\max_{i \in [1,n]} (|x_i| |y_j|) \right)$$

$$= \max_{i \in [1,n]} (|x_i|) \cdot \max_{j \in [1,m]} (|y_i|)$$

ii

$$\|x \otimes y\|_\infty = \max\left(\max_{1 \le i \le n} (|x_i|) \ \max(|y_1| \ldots |y_m|) \right)$$

$$= \max_{i \in [1,n]} (|x_i|) \cdot \max_{j \in [1,m]} (|y_j|)$$

$$\boxed{= \|x \otimes y\|_\infty = \|x\|_\infty \cdot \|y\|_\infty}$$

3(b) $A \in R^{m \times n}$, $B \in R^{k \times l}$

tensor product $\in R^{mk \times nl}$

$$A \otimes B = \begin{bmatrix} AB_{11} & \cdots & AB_{1l} \\ AB_{k1} & \cdots & AB_{kl} \end{bmatrix}$$

$\|A \times B\|_p$ for $p = 1, 2, \infty$

Firstly for $p = 1$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{j=1}^{m} |a_{ij}|$$

$$\therefore \|B\|_1 = \max_{1 \le j \le l} \sum_{i=1}^{k} |b_{ij}|$$

Now for the tensor product

$A \otimes B =$

$$= \|A \otimes B\|_1 = \max_{1 \le i \le nl} \sum_{i=1}^{mk} |A \otimes B|_{ij}$$

Now, we already know that

for any $A \in R^n$ and $d \in R$.

$$\|\alpha A\|_p = |\alpha| \, \|A\|_p$$

$$\therefore \quad \|A \otimes B\|_1 = \max_{1 \le j \le l} \|A\| \sum_{i=1}^{k} |b_{ij}|$$

$$= \|A\|_1 \cdot \|B\|_1$$

Hence proved

~~Similarly for p=2~~

~~$\|A\|_2 =$~~

For $p = \infty$

$$\|A\|_\infty = \max_{1 \le j \le m} \sum_{i=1}^{n} |a_{ij}|$$

$$\& \quad \|B\|_\infty = \max_{1 \le j \le k} \sum_{i=1}^{l} |b_{ij}|$$

$$\|A \otimes B\|_\infty = \max_{1 \le i \le mk} \sum_{i=1}^{nl} \|A \otimes B\|_{ij}$$

$$= \max_{1 \le j \le k} \Big(\sum_{i=1}^{l} |b_{ij}| \cdot \|A\| \Big)_\infty$$

$$= \|A\|_\infty \cdot \|B\|_\infty$$

Q6 Codes of Q6 with output is present in the ipynb fileas the output is very large to paste as a snippet.

a)

Underflow

The output is obtained because it computes the recurrence as in

a(n+1) = 2^(-a(n)) where a(1)=1

But we know that in floating-point there are upper bounds and thus underflow, the value after the 0 that python can output is there in the ipynb file.

6b)

Machine Epsilon

The code computes the minimum gap between the two consecutive no for which it follows The definition of machine epsilon i,e |fx| + e >|fx|

6c)

Overflow

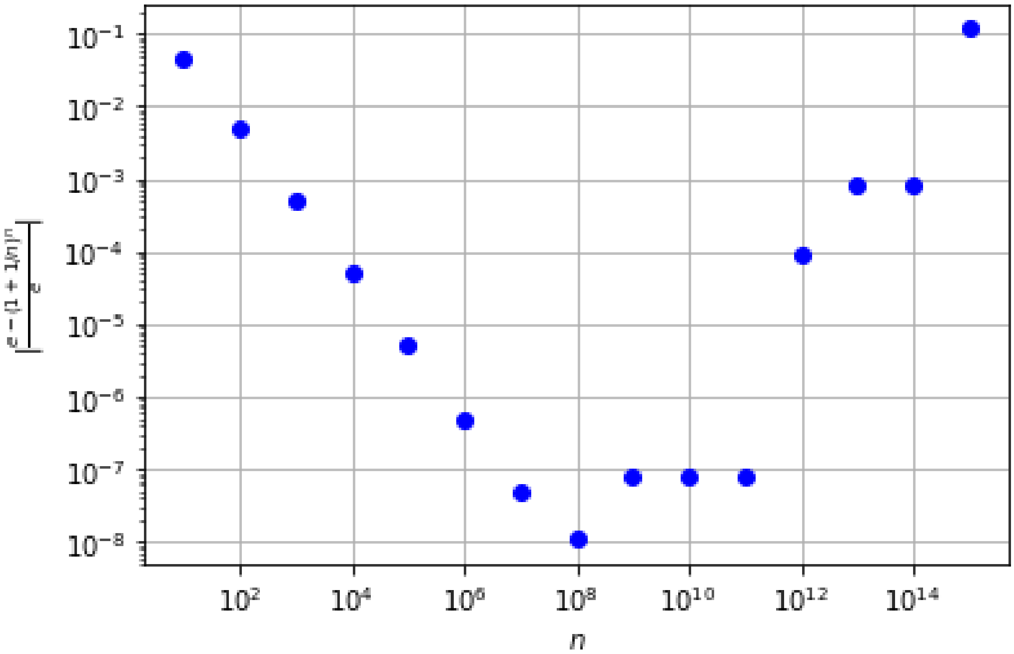
The output is obtained because it computes the recurrence as in

a(n+1)=2^a(n) where a(1)=1

But we know in floating-point there is upper bound and thus overflow as the value of 2^n (just before inf) that python can output is there in ipynb file.

Q7 a)

```
import matplotlib.pyplot as p
def limit_e(n):
  return (1+1/n)**n
reler=[]
N = [10**m for m in range(1, 16)]
for n in N:
  y=limit_e(n)
  t=abs((y-np.e)/np.e)
  reler.append(t)
p.figure()
p.plot(N, reler, 'bo')
p.xscale('log');p.yscale('log');p.grid(True)
p.ylabel("$|\\frac{e - (1 + 1/n)^n}{e}|$");p.xlabel("$n$")
```



At m=8 we compute the most accurate value of e using the formula but our answer doesn't match with the theoretical computation because the formula is reached to more accuracy as the n increases and because of the round-off of floating-point numbers.
We are computing the power of 10^k for a very small float no thus we saw the increment in the relative error.

Q7 b)

```python
import numpy as np
import scipy as sp
import scipy.special as sc
curr = 1.
e_val = 1.
esmp=1.1102230246251565e-16 # I have taken it from question 6b
def error(curr,e_val,esmp):
  while True:
    a = 1/sc.factorial(curr, exact=True)
    e_val += a
    if (a<=esmp):
      break
    curr += 1
  reler = abs((np.e-e_val)/np.e)
  return reler
r=error(curr,e_val,esmp)
print("\nThe relative error is %1.15f" % r)
```

We need to calculate the stopping condition of the Taylor Series of e
Firstly, we have to compute the term for which if we add that number to some summation we get the same summation. Thus we keep on adding terms until we compute that term provided that it should not be less than machine epsilon.
As we know that number that is less than machine epsilon does not make summation different after adding on it.