Wavelets and Their Applications

Pankaj Patil

University of Toronto

pankaj.patil@mail.utoronto.ca

August 21, 2021

Overview

Fourier Series Pitfalls

- Fourier coefficients, and hence the Fourier series approximation, depend on all values of the function
- For example, if you change a function f by a small amount, it is possible that every Fourier coefficient changes. This will then have an effect on the partial sums $S_n f(\theta)$ for all values of θ
- For a badly behaved function, such as a nondifferentiable or discontinuous one, the coefficients decrease slowly.
- One can show that the Fourier coefficients of a function go rapidly to 0 only when the functions has several continuous derivatives. Thus, we may need many terms to get a close approximation, even at a point relatively far away from the discontinuity.

Fourier Series Pitfalls (Continued..)

- The partial sums $S_n f(\theta)$ do not always converge to $f(\theta)$ when f is merely continuous.
- According to Gibbs's phenomenon, $S_n f(\theta)$ will always exhibit bad behaviour near discontinuities, no matter how large n is.
- While we can get better approximations by using $\sigma f(\theta)$ instead of $S_n f(\theta)$, this will not resolve such problems as slowly decreasing Fourier coefficients.

$\Longrightarrow$

We need series expansions with better local properties.

$\Longrightarrow$

Solution is Wavelets.

Wavelets

Definition (15.1.1)

A wavelet is a function

$$\psi \in L^2(\mathbb{R})$$

such that the set

$$\{\psi_{kj}(x) = 2^{k/2}\psi(2^k x - j) : k, j \in \mathbb{Z}\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$

- ψ is called the mother wavelet.
- called dyadic wavelet to stress that dilations are taken to be powers of 2
- the wavelet basis has two parameters, whereas the Fourier basis for $L^2(\mathbb{R})$ has only one, given by dilation alone

Haar System for $L^2(\mathbb{R})$

Start with description for $L^2([0,1])$, which will lead us to $L^2(\mathbb{R})$

For $a < b$, let $\chi_{[a,b)}$ denote the characteristic function of $[a,b)$.

$$\phi = \chi_{[0,1)} \text{ and } \psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$$

Then define,

$$\{\psi_{kj}(x) = 2^{k/2}\psi(2^k x - j)\} \quad \text{for all} \quad k, j \in \mathbb{Z}$$

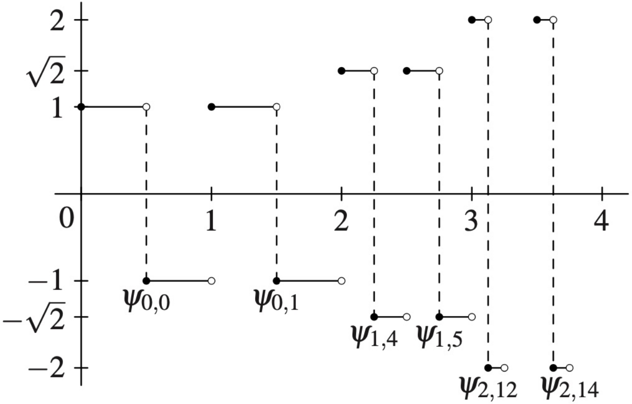
We use only those functions that are supported on $[0,1)$, namely $0 \leq j < 2k$ for each $k \geq 0$

The Haar system is the family

$$\{\phi, \psi_{kj} : k, j \geq 0 \text{ and } 0 \leq j < 2^k\}$$

Haar System for $L^2(\mathbb{R})$ (Continued..)

Some elements of Haar System



Haar System for $L^2(\mathbb{R})$ (Continued..)

Lemma (15.2.1)

The Haar system is orthonormal.

Proof.
Each of these functions has norm 1.

$$\int_0^1 \psi_{kj}^2(x)dx = \int_0^1 (2^{k/2})^2 \psi^2(2^k x - j)dx = \int_{-j}^{2^k - j} \psi_{kj}^2(t)dt = 1$$

Now ψ_{kj} and $\psi_{kj'}$ for $j \neq j'$ have disjoint supports and thus are orthogonal.

$$\int_0^1 \psi_{kj}(x)dx = 0 \text{ for all } k, j$$

$\Longrightarrow$ these functions are pairwise orthogonal. $\square$

Haar System for $L^2(\mathbb{R})$ (Continued..)

Haar Coefficients

Consider inner product expansion with respect to orthonormal basis and define

$$H_n f = \langle f, \phi \rangle \phi(x) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \langle f, \psi_{kj} \rangle \psi_{kj}(x)$$

The Haar Coefficients are the inner products $\langle f, \psi_{kj} \rangle$ used in this expansion.

Haar System for $L^2(\mathbb{R})$ (Continued..)

For more explicit description of $H_n f$ we have following lemma

Lemma (15.2.2)

Let $f \in L^2(0,1)$. Then $H_n f$ is the unique function that is constant on each dyadic interval of length 2^{-n} in $[0,1]$ and satisfies

$$H_n f(x) = 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} H_n f(t)dt = 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} f(t)dt$$

for $x \in [j2^{-n}, (j+1)2^{-n})$, $0 \leq j < 2$. Moreover $\|H_n f\|_2 \leq \|f\|_2$

Proof. $\{\phi, \psi_{00}\}$ span the functions that are constant on $[0, 1/2)$ and $[1/2, 1)$. By induction, it follows that

$$\boldsymbol{M_n} := \{\phi, \psi_{\boldsymbol{j}}\boldsymbol{k}\}$$

is a subspace of all the functions that are constant on the dyadic intervals.

Haar System for $L^2(\mathbb{R})$ (Continued..)

We note that M_n is also spanned by the characteristic functions over dyadic intervals.
Now, $H_n f$ is also contained in this span, hence must be constant on these dyadic intervals. Thus, $H_n f$ is unique function which satisfies

$$\langle H_f, \phi \rangle = \langle f, \phi \rangle \text{ and } \langle H_f, \psi_{kj} \rangle = \langle f, \psi_{kj} \rangle$$

As noted above, the basis of M_s can be replaced by characteristic functions, hence $H_n f$ is unique function in M_n such that

$$H_n f(x) = 2^n \langle H_n f, \chi_{n,j} \rangle = 2^n \langle f, \chi_{n,j} \rangle$$

which is what we wanted.
Now H_n is orthogonal projection of $L^2(0,1)$ onto M_n. By projection theorem it follows that

$$\|H_n f\|_2 \leq \|f\|_2$$

$\square$

Haar System for $L^2(\mathbb{R})$ (Continued..)

Theorem (15.2.3)

Let $f \in L^2(0,1)$. Then $H_n f$ converges to f in the L^2 norm. Consequently, the Haar system is an orthonormal basis for $L^2(0,1)$. Moreover, if f is continuous on $[0,1]$, then $H_n f$ converges uniformly to f.

Proof.

- Last Part: $\boldsymbol{f}$ is continuous on compact set $[0,1] \implies$ it is uniformly continuous on [0,1]. The modulus of continuity is given by

$$\boldsymbol{w}(\boldsymbol{f};\boldsymbol{\delta}) = \boldsymbol{sup}\{|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| : |\boldsymbol{x} - \boldsymbol{y}| \leq \boldsymbol{\delta}\}$$

. Uniform coninuity implies that

$$\lim_{\boldsymbol{n}\to\infty} \omega(\boldsymbol{f};2^{-\boldsymbol{n}}) = 0$$

Now let $\boldsymbol{x} \in [\boldsymbol{j}2^{-\boldsymbol{n}}, (\boldsymbol{j}+1)2^{-\boldsymbol{n}})$ (called dyadic interval of length $2^{-\boldsymbol{n}}$)

Haar System for $L^2(\mathbb{R})$ (Continued..)

$$
\begin{aligned}
|H_n f(x) - f(x)| &= |2^n \int_{j2^{-n}}^{(j+1)2^{-n}} f(t)dt - 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} f(x)dt| \\
&\leq 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} |f(t) - f(x)|dt \\
&\leq 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} \omega(f; 2^{-n})dt \\
&= \omega(f; 2^{-n})
\end{aligned}
$$

Hence $\|H_n f - f\|_\infty \leq \omega(f; 2^{-n})$ tends to 0. Therefore, $H_n f$ converges uniformly to f on $[0, 1]$.

Haar System for $L^2(\mathbb{R})$ (Continued..)

Now

$$\begin{aligned} \|H_n f - f\|_2 &\leq \left(\int_0^1 \|H_n f - f\|_\infty dt \right)^{1/2} \\ &= \|H_n f - f\|_\infty \end{aligned}$$

So we have convergence in $L^2(0,1)$ norm as well.

- Let $f \in L^2(0,1)$, and fix $\epsilon > 0$. Since f is the L^2 lmit of a seqeence of continuous functions, we can find a continuous function g such that $\|f - g\|_2 < \epsilon$. Choose n large enough so that $\|H_n g - g\|_2 < \epsilon$, then

$$\begin{aligned} \|H_n f - f\|_2 &\leq \|H_n f - H_n g\|_2 + \|H_n g - g\|_2 + \|g - f\|_2 \\ &\leq \|H_n(f - g)\|_2 + \epsilon + \epsilon \\ &\leq \|f - g\|_2 + 2\epsilon \\ &< 3\epsilon \end{aligned}$$

So $H_n f$ converges to f in L^2.

Haar System for $L^2(\mathbb{R})$ (Continued..)

Since the orthogonal expansion of f in the Haar system sums to f in the L^2 norm, we deduce that this orthonormal set spans all of $L^2(0,1)$ and thus is a basis.

$\square$

Haar Wavelets

Definition (15.2.4)

The Haar wavelet is the function

$$\psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$$

The Haar wavelet basis is the family

$$\{\psi_{kj} : k, j \in \mathbb{Z}\}$$

Haar Wavelets (Continued..)

Theorem (15.2.5)

The Haar wavelet basis spans all of $L^2(\mathbb{R})$.

Proof.

- Enough to show that any continuous function of bounded support is spanned by the Haar wavelet basis.
- Each such function is the finite sum of (piece- wise) continuous functions supported on an interval $[\boldsymbol{m}, \boldsymbol{m} + 1)$.
- But our basis is invariant under integer translations $\implies$ it is enough to show that a function on $[0, 1)$ is spanned by the Haar wavelet basis.
- But Theorem 15.2.3 shows that the functions $\psi_{\boldsymbol{kj}}$ supported on $[0, 1)$ together with ϕ span $\boldsymbol{L}^2(0, 1) \implies$ it is enough to approximate ϕ alone.

Haar Wavelets (Continued..)

Consider the functions $\phi_{-k,0} = 2^{-k/2}\chi_{[0,2^{k-1})} - \chi_{[2^{k-1},2^k)}$ for $k \geq 1$. An easy computation shows that

$$h_N := \sum_{k=1}^{N} 2^{-k/2}\phi_{-k,0} = (1 - 2^{-N})\chi_{[0,1]} - 2^{-N}\chi_{[1,2^N)}$$

Thus $\|\phi - h_N\|_2 = \|2^{-N}\chi_{[0,2^N)}\|_2 = 2^{-N/2}$. Hence ϕ is in the span of rhe wavelet basis. Therefore, the Haar wavelet basis spans all of $L^2(\mathbb{R})$. $\square$

Multiresolution Analysis

Objective

Develop a general framework to contruct other wavelet system.

Define:

$$\phi = \chi_{[0,1)}$$

$$\phi_{kj}(x) = 2^{k/2}\phi(2^k x - j) \text{ for all } k, j \in \mathbb{Z}$$

- The system as is, is not orthonormal.
- But for each k, the family $\{\phi_{kj} : j \in \mathbb{Z}\}$ is orthonormal.

Define:

$$V_k = span\{\phi_{kj} : j \in \mathbb{Z}\}$$

$$\implies V_k \subset V_{k+1}$$

Multiresolution Analysis (Continued..)

Some important properties of this decomosition are,

Lemma (15.3.1)

Let $\phi = \chi_{[0,1)}$, and with V_k defined as before, we have

- orthogonality*: $\{\phi(x-j) : j \in \mathbb{Z}\}$ is orthonormal basis for V_0.*
- nesting*: $V_k \subset V_{k+1}$ for all $k \in \mathbb{Z}$.*
- scaling*: $f(x) \in V_k$ if and only if $f(2x) \in V_{k+1}$.*
- density*: $\overline{\bigcup_{k \in \mathbb{Z}} V_k} = L^2(\mathbb{R})$.*
- separation*: $\cap_{k \in \mathbb{Z}} V_k = 0$.*

Multiresolution Analysis (Continued..)

Definition (Multiresolution)

A multiresolution of $L^2(\mathbb{R})$ with scaling function $\phi$is the sequence of subspaces

$$V_k = span\{\phi_{kj} : j \in \mathbb{Z}\}$$

provided that the sequence satisfies the five properties described in the preceding lemma.

The function ϕ is sometimes called a father wavelet.

Orthogonality

$$\langle \phi_{kj}, \phi_{kl} \rangle = \int_{-\infty}^{\infty} 2^k \phi(2^k x - j)\phi(2^k x - l)dx = \int_{-\infty}^{\infty} \phi(t-j)\phi(t-l)dt = \delta_{jl}$$

Multiresolution Analysis (Continued..)

Once we have a nested sequence Vk with these properties, we can decompose $L^2(\mathbb{R})$ into a direct sum of subspaces. Set

$$W_k = \{f \in V_{k+1} : f \perp V_k\}$$

This is the orthogonal complement of V_k in V_{k+1}. We write $V_{k+1} = V_k \bigoplus W_k$, that is, a sum of orthogonal subspaces.
So each vector $f \in V_{k+1}$ can be written uniquely as $f = g + h$ with $g \in V_k$ and $h \in W_k$

Multiresolution Analysis (Continued..)

Theorem (15.4.2)

Let ϕ be the scaling function generating a multiresolution $\{V_k\}$ of $L^2(\mathbb{R})$ with scaling relation $\phi(x) = \sum_{j=-\infty}^{\infty} a_j \phi(2x - j)$. Define

$$\psi(x) = \sum_{j=-\infty}^{\infty} (-1)^j a_{j-1} \phi(2x - j)$$

Then ψ is a wavelet that generates the wavelet basis $\{\psi_{kj} : k, j \in \mathbb{Z}\}$ such that $W_k = span\{\psi_{kj} : j \in \mathbb{Z}\}$ for each $k \in \mathbb{Z}$

Multiresolution Analysis (Continued..)

Proof.

- Mainly consists of long computations
- The orthonormality of $\{\phi(\boldsymbol{x}-\boldsymbol{j}) : \boldsymbol{j} \in \mathbb{Z}\}$ yields conditions on $\boldsymbol{a_j}$.
- $\{\psi(\boldsymbol{x}-\boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}\}$ is orthonormal and is orthogonal to $\phi(\boldsymbol{x}-\boldsymbol{j})$'s.
- Then we show that $\boldsymbol{V}_1$ is spanned by $\boldsymbol{V}_0$ and then $\psi(\boldsymbol{x}-\boldsymbol{k})$'s.

Define $\boldsymbol{\delta}_{0\boldsymbol{n}} = 1$ if $\boldsymbol{n} = 0$ and 0 otherwise. We have

$$\begin{aligned}
\boldsymbol{\delta}_{0\boldsymbol{n}} &= \langle \phi(\boldsymbol{x}), \phi(\boldsymbol{x}-\boldsymbol{n}) \rangle = \left\langle \sum_{\boldsymbol{i}} \boldsymbol{a_i} \phi(2\boldsymbol{x}-\boldsymbol{i}), \sum_{\boldsymbol{j}} \boldsymbol{a_j} \phi(2\boldsymbol{x}-2\boldsymbol{n}-\boldsymbol{i}) \right\rangle \\
&= \sum_{\boldsymbol{i}} \sum_{\boldsymbol{j}} \boldsymbol{a_i} \boldsymbol{a_j} \langle \phi(2\boldsymbol{x}-\boldsymbol{i}), \phi(2\boldsymbol{x}-2\boldsymbol{n}-\boldsymbol{j}) \rangle \\
&= \frac{1}{2} \sum_{\boldsymbol{j}} \boldsymbol{a}_{\boldsymbol{j}+2\boldsymbol{n}} \boldsymbol{a_j}.
\end{aligned}$$

Multiresolution Analysis (Continued..)

- Orthonormality of $\{\phi(x-j) : j \in \mathbb{Z}\}$

$$\begin{aligned}
\langle \psi(x), \psi(x-n) \rangle &= \left\langle \sum_i (-1)^i a_{1-i} \phi(2x-i), \sum_j (-1)^j a_{1-j} \phi(2x-2n-j) \right\rangle \\
&= \sum_i \sum_j (-1)^{i+j} a_{1-i} a_{1-j} \langle \phi(2x-i), \phi(2x-2n-j) \rangle \\
&= \frac{1}{2} \sum_j (-1)^{2j+2n} a_{1-j-2n} a_{1-j} \\
&= \frac{1}{2} \sum_i a_{i+2n} a_i = \delta_{0n}.
\end{aligned}$$

$\implies \{\phi(x-j) : j \in \mathbb{Z}\}$ is orthonormal.

Multiresolution Analysis (Continued..)

- Orthogonality of ψ's and ϕ's.

$$
\begin{aligned}
\langle \psi(x-m), \phi(x-n) \rangle \\
&= \left\langle \sum_i (-1)^i a_{1-i}\phi(2x-2m-i), \sum_j a_j \phi(2x-2n-j) \right\rangle \\
&= \sum_i \sum_j (-1)^i a_{1-i} a_j \langle \phi(2x-2m-i), \phi(2x-2n-j) \rangle \\
&= \frac{1}{2} \sum_j (-1)^{j+2m-2n} a_{1-j+2m-2n} a_j = \frac{1}{2} \sum_j (-1)^j a_{p-j} a_j \\
&= \frac{1}{2} \sum_i (-1)^{p-i} a_{p-i} a_i \\
&= 0
\end{aligned}
$$

$\implies$ $\{\phi(x-j) : j \in \mathbb{Z}$ is orthogonal to $\{\psi(x-k) : j \in \mathbb{Z}$.

Multiresolution Analysis (Continued..)

- A rather lengthy computation gives us the third part of the proof, which is that $\{\psi(x-j) : j \in \mathbb{Z}\}$ spans $L^2(\mathbb{R})$.
- Hence it follows that $\{\psi(x-j) : j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$

$\implies \psi$ is s wavelet.

$\square$

Daubechies Wavelets

- Use Multiresolution Analysis to design continuous wavelet.
- Haar wavelets do a good job of approximating functions that are locally constant.
- Can approximate continuous functions if we use a wavelet that also satisfies
$$\int x\psi(x)dx = 0$$
It is called first moment of ψ.

Theorem (15.5.1)

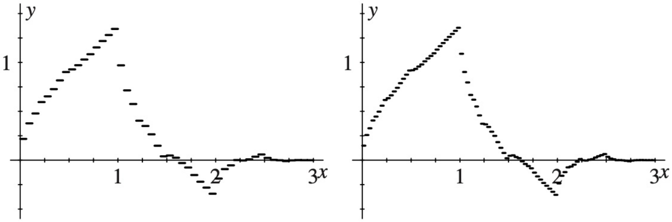
There is a continuous function ϕ of compact support in $L^2(\mathbb{R})$ that generates a multiresolution of $L^2(\mathbb{R})$ such that the associated wavelet ψ is continuous, has compact support, and satisfies

$$\int \psi(x)dx = \int x\psi(x)dx = 0$$

.

Daubechies Wavelets (Continued..)

- This is called Daubechies scaling function.
- It has all the good properties we wanted.
- It has support on $[0, 3]$.
- The graph of ϕ_4 and ϕ_5



Function Approximation using Haar Wavelets

$f(x) = x^3$ approximation using Haar Wavelets

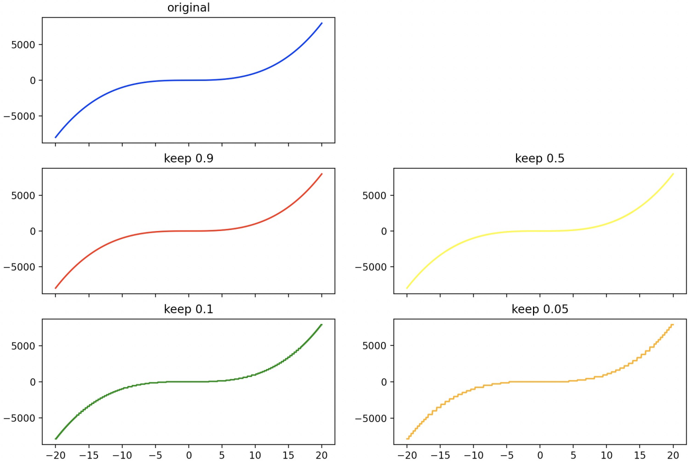
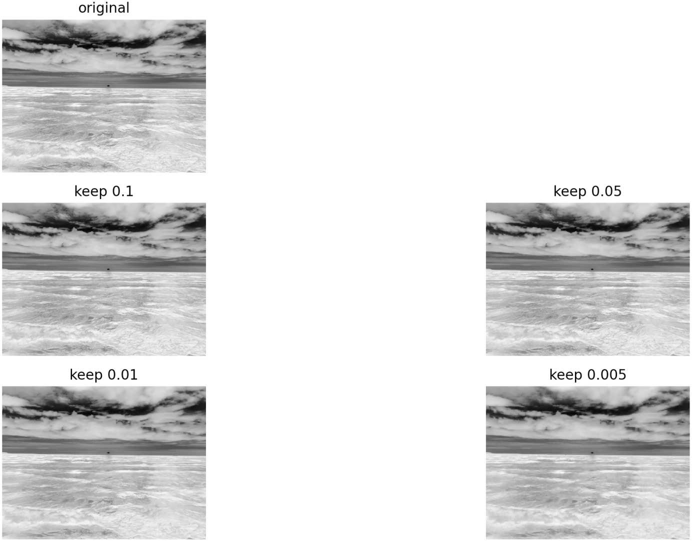


Image Compression using Daubechies Wavelets

- Signals and Images have lot of redundant data because of correlation within the data.
- If signal is continuous, then we can store only the value at certain time and then only increments to it, so that the original signal can be reconstructed.
- This reduces storage requirements and is an example of data compression.
- In case of images, the same principle follows. The wavelet expansion for images can generate many coefficients that are insignificant and can be discarded without losing must in terms of quality of the reconstructed image.
- Caution: more coefficients discarded more is the error.

Image Compression using Daubechies Wavelets



Thank You.