Basic Mathematics for Portfolio Management

“If you stop at general math, you’re only going to make general math money.”
- Snoop Dogg

Statistics

- Variables *x, y, z*
- Constants *a, b*
- Observations $\{x_n, y_n, z_n \mid n=1,\ldots N\}$
- Mean

$$\bar{x} = E\{x\} = \text{mean or expected value of } x$$

$$E\{a \cdot x + b \cdot y\} = a \cdot \bar{x} + b \cdot \bar{y}$$

$$\bar{x} = \frac{1}{N} \cdot \sum_{n=1}^{N} x_n$$

Variance

$$\sigma_x^2 = Var\left\{x\right\} = E\left\{\left(x - \overline{x}\right)^2\right\}$$

$$Var\left\{a \cdot x\right\} = a^2 \cdot \sigma_x^2$$

$$\sigma_x^2 = \frac{1}{N-1} \cdot \sum_{n=1}^{N} \left(x_n - \overline{x}\right)^2$$

Covariance

$$\sigma_{xy} = Cov\{x, y\} = E\{(x - \overline{x}) \cdot (y - \overline{y})\}$$

$$\sigma_{xy} = \sigma_{yx}$$

$$Cov\{x, x\} = \sigma_x^2$$

$$Cov\{a \cdot x + b \cdot y, z\} = a \cdot \sigma_{xz} + b \cdot \sigma_{yz}$$

$$Var\{a \cdot x + b \cdot y\} = Cov\{a \cdot x + b \cdot y, a \cdot x + b \cdot y\}$$

$$= a^2 \cdot \sigma_x^2 + b^2 \cdot \sigma_y^2 + 2ab \cdot \sigma_{xy}$$

$$\sigma_{xy} = \frac{1}{N-1} \cdot \sum_{n=1}^{N} (x_n - \overline{x}) \cdot (y_n - \overline{y})$$

Correlation

- Closely related to covariance:

$$\rho_{xy} \equiv \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$

- Note that correlation is dimensionless, and that:

$$-1 \leq \rho_{xy} \leq 1$$

  – This facilitates our intuition for the magnitude of a correlation.

A bit more on Correlation

- What is $Corr\left\{a \cdot x, y\right\}$?

$$Corr\left\{a \cdot x, y\right\} = \frac{Cov\left\{a \cdot x, y\right\}}{\sqrt{Var\left\{a \cdot x\right\}, Var\left\{y\right\}}} = \frac{a \cdot \sigma_{xy}}{a \cdot \sigma_x \cdot \sigma_y}$$

$$= \rho_{xy}$$

Regression

$$y_n = a + b \cdot x_n + \varepsilon_n = \hat{y}_n + \varepsilon_n$$

- This is an example of a linear model.
- The model is unbiased if:

$$\sum_{n=1}^{N} \varepsilon_n = 0$$

- How do we choose the best {*a*,*b*}?

Regression

- The best estimates for {*a*,*b*} should minimize the sum of squared errors:

$$ESS = \sum_{n=1}^{N} \varepsilon_n^2 = \sum_{n=1}^{N} \left(y_n - \hat{y}_n \right)^2$$

- This corresponds to minimizing:

$$\sum_{n=1}^{N} \hat{y}_n^2 - 2 \cdot y_n \cdot \hat{y}_n$$

- Which leads to two equations:

$$\sum_{n=1}^{N} \left[\hat{y}_n - y_n \right] \cdot \left(\frac{\partial \hat{y}_n}{\partial a} \right) = 0$$

$$\sum_{n=1}^{N} \left[\hat{y}_n - y_n \right] \cdot \left(\frac{\partial \hat{y}_n}{\partial b} \right) = 0$$

Regression

- The first equation leads to an unbiased model.

$$\left(\frac{\partial \hat{y}_n}{\partial a}\right) = 1 \Rightarrow \sum_{n=1}^{N}\left[\hat{y}_n - y_n\right] = 0$$

- We can use this to solve for *a*:

$$y_n = a + b \cdot x_n + \varepsilon_n$$

$$\overline{y} = a + b \cdot \overline{x}$$

$$a = \overline{y} - b \cdot \overline{x}$$

Regression

- The second equation relates the coefficient *b* to the sample covariance of *x* and *y*:

$$\left(\frac{\partial \hat{y}_n}{\partial b}\right) = x_n \Rightarrow \sum_{n=1}^{N} \left[\hat{y}_n - y_n\right] \cdot x_n = 0$$

$$\Rightarrow b = \frac{Cov\{x, y\}}{Var\{x\}}$$

- Putting this together:

$$y_n = \bar{y} + \left(\frac{Cov\{x, y\}}{Var\{x\}}\right) \cdot \left[x_n - \bar{x}\right] + \varepsilon_n$$

- The model $\hat{y}$ is the best linear unbiased estimate (BLUE), given only the sample data.
- Before some examples, I need to introduce portfolio management notation.

Portfolio Management Definitions and Notation

t ——— $t + \Delta t$

- Total rate of return:

new price dividend old price

$$trr \equiv \frac{p(t+\Delta t) + d(t+\Delta t) - p(t)}{p(t)}$$

price and dividend notation are standard but we almost never focus on total rate of return.

- Risk-free return: i_F

(Usually T-Bill return over period Δt)

- Excess return:

excess returns are a standard focus of investment theory

$$r \equiv trr - i_F$$

$$r_n = \text{stock} - n \text{ excess return}$$

Portfolios and Holdings

- A portfolio is a set of holdings.
- Holdings are value-weighted fractions:
  - Example: portfolio consists of 100 shares each of A and B. The price of A is \$50 and the price of B is \$25.

$$h_A = \frac{100 \cdot 50}{100 \cdot 50 + 100 \cdot 25} = 0.67$$

$$h_B = \frac{100 \cdot 25}{100 \cdot 50 + 100 \cdot 25} = 0.33$$

- Fully invested portfolios have holdings that sum to 1.
- Long-only portfolios have all holdings greater than or equal to 0.
- A benchmark is a particular portfolio, one outside the control of the portfolio manager. Examples include the S&P 500, the Russell 2000, and the MSCI EAFE.

Returns and Portfolios

- Portfolio returns: $r_P = \mathbf{h}_P^T \cdot \mathbf{r}$
- Residual returns (defined via regression)

$$r_P = \beta_P \cdot r_B + \theta_P = \alpha_P + \beta_P \cdot r_B + \varepsilon_P$$

- Active returns (defined via subtraction)

$$r_P = r_B + \delta_P$$

- Active portfolio holdings (defined via subtraction)

$$\mathbf{h}_{PA} \equiv \mathbf{h}_P - \mathbf{h}_B$$

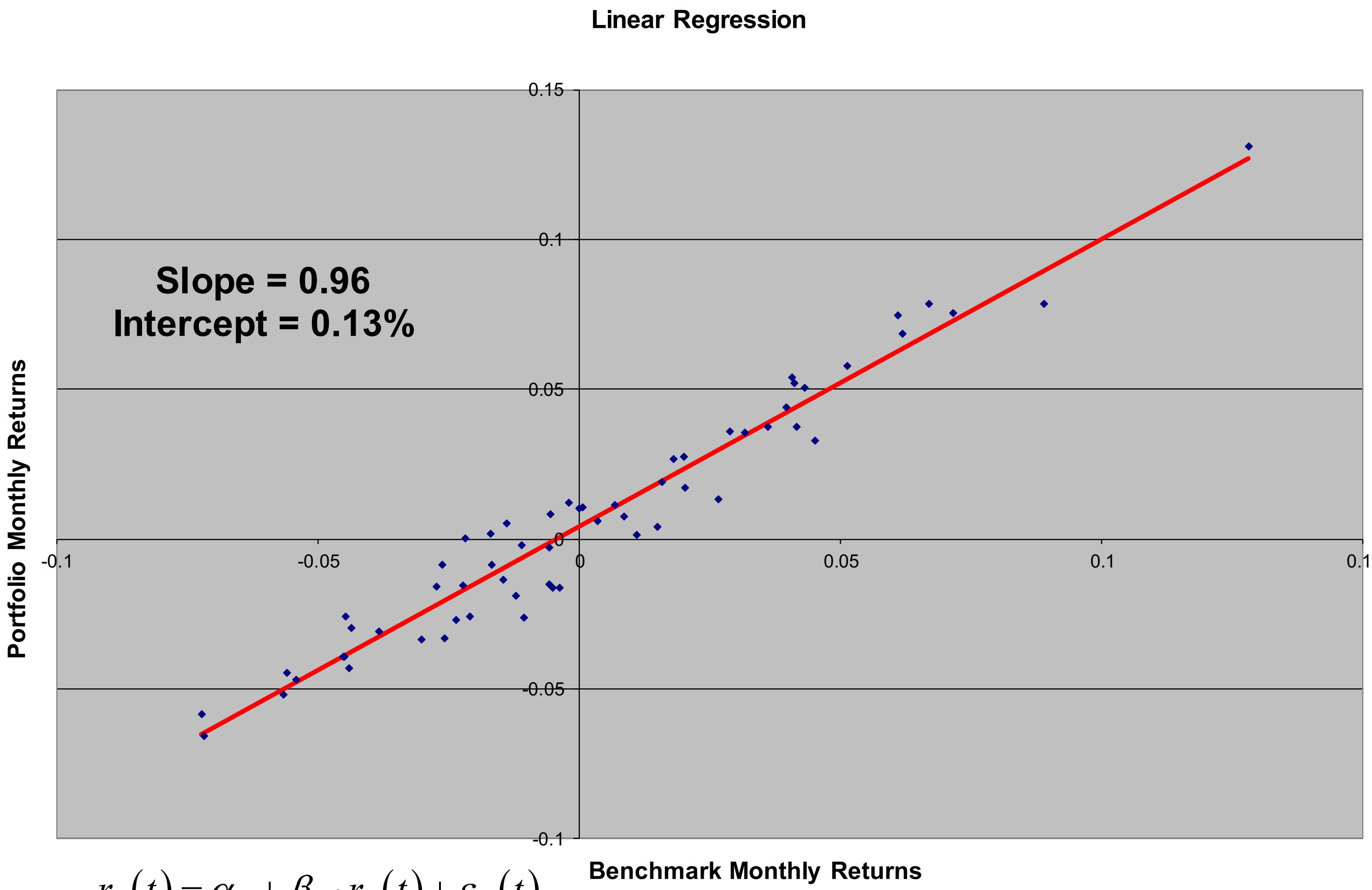
Returns and Portfolios

Returns	**Variable**	**Mean**	**Variance**
Excess (above risk-free)	r	f	σ^2
Residual	θ	α	ω^2
Active	δ	$\alpha + (\beta - 1) \cdot f_B$	$\omega^2 + (\beta - 1)^2 \cdot \sigma_B^2$

Portfolios:	$\mathbf{h}_P$	=	portfolio holdings
	$\mathbf{h}_B$	=	benchmark holdings
	$\mathbf{h}_{PA}$	=	active holdings
Covariance Matrix:	$\mathbf{V}$	=	covariance matrix (Cov$\{\mathbf{r},\mathbf{r}\}$)

- Now to some portfolio management examples of regression.

Investment Example: Time-series Regression

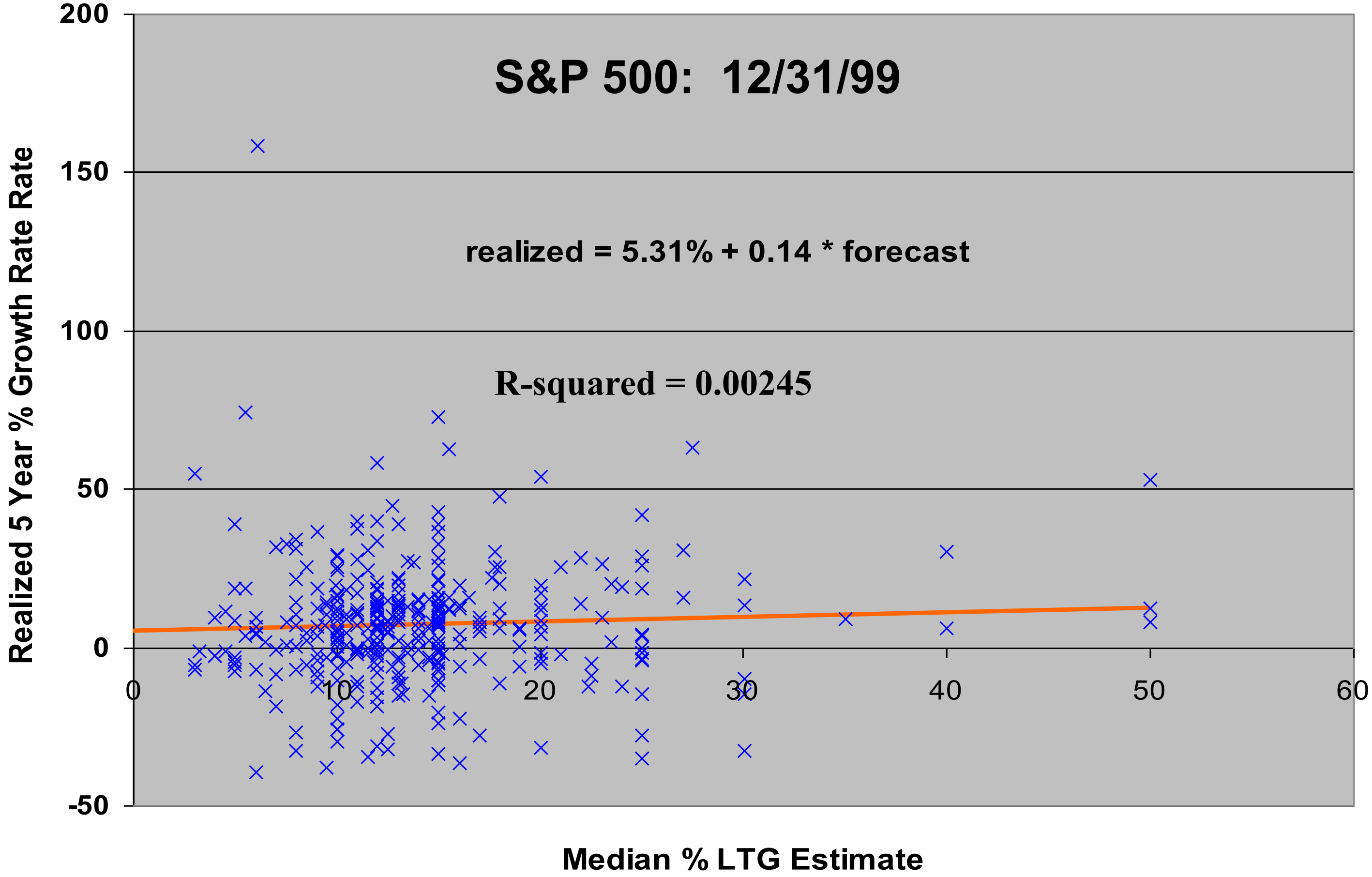


$$r_P(t) = \alpha_P + \beta_P \cdot r_B(t) + \varepsilon_P(t)$$

$$E\{r_P(t)\} = \alpha_P + \beta_P \cdot E\{r_B(t)\}$$

$$E\{r_P(t) \mid r_B(t)\} = \alpha_P + \beta_P \cdot r_B(t)$$

Investment Example: Cross-sectional Regression



BLUE of y conditional on x

- More generally, the BLUE has the form:

$$E\{y \mid x\} = E\{y\} + Cov\{x, y\} \cdot Var^{-1}\{x\} \cdot \left(x - E\{x\}\right)$$

- You can see how this directly relates to our prior regression result.
- We will apply this result more generally than just in the regression context. For example, our estimates of variances and covariances may improve upon sample estimates.

Matrices: Transpose and Inverse

- Transpose
  - Switch rows and columns

$$A^T_{mn} \equiv A_{nm}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

- Inverse (Assume A, B are square (*N*x*N*) matrices)

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y}$$

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{y}$$

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$$

$$\left(\mathbf{A} \cdot \mathbf{B}\right)^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

Example: 2x2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\left(a_{11} \cdot a_{22} - a_{12} \cdot a_{21}\right)} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Verify that $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Using Linear Algebra

- Portfolio Variance

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & & \ddots & \vdots \\ \sigma_{N1} & & & \sigma_N^2 \end{pmatrix}$$

$$\sigma_P^2 = Var\left\{h_1 \cdot r_1 + h_2 \cdot r_2 + \ldots + h_N \cdot r_N\right\}$$

$$= h_1^2 \cdot \sigma_1^2 + \ldots + h_N^2 \cdot \sigma_N^2 + 2h_1 h_2 \cdot \sigma_{12} + 2h_1 h_3 \cdot \sigma_{13} + \ldots$$

$$= \mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h}$$

Multivariate Regression

- The multivariate linear model is:

$$\mathbf{y} = \mathbf{X} \cdot \mathbf{b} + \boldsymbol{\varepsilon}$$

$$\boldsymbol{\Omega} = Cov\{\boldsymbol{\varepsilon}\}$$

- We can include an intercept as a column of **X**.
- To generalize, we minimize the weighted sum of squared errors:

$$\boldsymbol{\varepsilon}^T \cdot \boldsymbol{\Omega}^{-1} \cdot \boldsymbol{\varepsilon}$$

- Our resulting estimates are:

$$\mathbf{b} = \left[\mathbf{X}^T \cdot \boldsymbol{\Omega}^{-1} \cdot \mathbf{X} \right]^{-1} \cdot \mathbf{X}^T \cdot \boldsymbol{\Omega}^{-1} \cdot \mathbf{y}$$

Example: Simple 2-factor model

- Previously we saw a “beta regression” model:

$$r_P(t) = \alpha_P + \beta_P \cdot r_B(t) + \varepsilon_P(t)$$

- Now consider a different type of model:

$$r_n(t) = b_{Market}(t) + X_{Size,n} \cdot b_{Size}(t) + u_n(t)$$

where we have only two factors—a market factor and a measure of size—to model cross-sectional asset returns

What is X? What is b?

$$\begin{array}{cc} & \begin{array}{cc}\text{Market exposure} & \text{Size exposure}\end{array} \\ \mathbf{X} = & \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \end{array}$$

$$\mathbf{b} = \left[\mathbf{X}^T \cdot \mathbf{X}\right]^{-1} \cdot \mathbf{X}^T \cdot \mathbf{r}$$

(ignore GLS weighting)

All *N* stocks are in the market (all Market exposures = 1). *N*/2 stocks are large (size exposure = 1) and *N*/2 are small (size exposure = -1).

This is an odd definition of “the market.”

Basic Utility Function

- Mean/Variance

$$U = f_P - \lambda \cdot \sigma_P^2$$

$$= \mathbf{h}^T \cdot \mathbf{f} - \lambda \cdot \mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h}$$

Portfolio Optimization

- Choose portfolio **h** to maximize *U*.
- What does that mean? We must take the derivative of *U* with respect to each of the *N* elements of **h**. That leads to *N* equations in *N* unknowns.

$$\frac{\partial U}{\partial h_n} = 0$$

$$\frac{\partial U}{\partial \mathbf{h}^T} = 0$$

More Hints

- Hint 2: Try out 2x2 or 3x3 examples.
- Hint 3: To start, try working with individual elements. For example, I will use shorthand like:

$$\frac{\partial}{\partial \mathbf{h}^T}$$

  – This says to take derivatives with respect to each element of **h**. This should lead to *N* separate equations.

  – Example:

$$\frac{\partial}{\partial \mathbf{h}^T}\left(\mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h}\right) = 2\mathbf{V} \cdot \mathbf{h}$$

Portfolio Construction with Constraints

- We usually need to include constraints in our portfolio optimization.
  - For example, the full investment constraint:

$$\sum_{n=1}^{N} h_n = 1$$

  - There is nothing in the mathematical optimization problem to guarantee the holdings sum to 1 unless we explicitly add it.
- We can add linear constraints using Lagrange multiplier:

$$U: \quad \left[\mathbf{h}^T \cdot \mathbf{f} - \lambda \cdot \mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h}\right] \Rightarrow \left[\mathbf{h}^T \cdot \mathbf{f} - \lambda \cdot \mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h} + c \cdot \left(\mathbf{h}^T \cdot \mathbf{e} - 1\right)\right]$$

  - We have added a new variable, *c*. Now when we optimize *U*, we take derivatives with respect to **h** and *c*. Setting derivative with respect to *c* to zero leads to the constraint.