MFE 230G
Key Formulas: Math for Portfolio Management, Framework, Risk

Math

Variance and Covariance:

Start with random variables x and y:

$$\sigma_x^2 = Var\{x\}, \sigma_y^2 = Var\{y\}$$
$$Var\{a \cdot x + b \cdot y\} = a^2 \cdot \sigma_x^2 + b^2 \cdot \sigma_y^2 + 2ab \cdot Cov\{x, y\} \tag{1}$$

Regression:

$$y_n = a + b \cdot x_n + \varepsilon_n$$
$$b = \frac{Cov\{x, y\}}{Var\{x\}} \tag{2}$$

$$E\{\varepsilon_n\} = 0 \tag{3}$$

Best Linear Unbiased Estimator:

$$E\{y \mid x\} = E\{y\} + Cov\{x, y\} \cdot Var^{-1}\{x\} \cdot \left[x - E\{x\}\right] \tag{4}$$

Framework:

Definitions:

Returns	**Variable**	**Mean**	**Variance**
Excess (above risk-free)	r	f	σ^2
Residual	θ	α	ω^2
Active	δ	$\alpha + (\beta - 1) \cdot f_B$	$\omega^2 + (\beta - 1)^2 \cdot \sigma_B^2$

Portfolios:	$\mathbf{h}_P$	=	portfolio holdings
	$\mathbf{h}_B$	=	benchmark holdings
	$\mathbf{h}_{PA}$	=	active holdings
Covariance Matrix:	$\mathbf{V}$	=	covariance matrix (Cov{**r**,**r**})

Active Management Utility (for institutional equity managers with $\beta_P = 1$):

$$U_P = \alpha_P - \lambda \cdot \omega_P^2 \tag{5}$$

Information Ratios:

$$IR_P = \frac{\alpha_P}{\omega_P} \tag{6}$$

- By definition, we annualize the Information Ratio. So, for example, if we start with the ratio of return to risk from monthly data, we must multiple the result by the square root of 12. We need to multiply by different amounts if we start with daily or quarterly returns.
- Typical median Information Ratios are zero. Typical top quartile Information Ratios are about 0.5 for long-only portfolios. Typical top decile Information Ratios are about 1.0 for long-only portfolios.
- The Information Ratio is a measure of consistency, where we define consistency as the fraction of months of positive residual return. As the Information Ratio increases, that fraction increases.

Optimizing the active management utility function leads to:

$$\omega^* = \frac{IR}{2\lambda} \tag{7}$$

$$U_P^* = \frac{(IR)^2}{4\lambda} \tag{8}$$

$$\begin{aligned} \mathbf{h}_{PA} &= \left(\frac{1}{2\lambda}\right) \cdot \mathbf{V}^{-1} \cdot \boldsymbol{\alpha} \\ \boldsymbol{\alpha} &= 2\lambda \cdot \mathbf{V} \cdot \mathbf{h}_{PA} \end{aligned} \tag{9}$$

$$h_{PA}(n) = \frac{\alpha_n}{2\lambda\omega_n^2} \quad \text{if active returns uncorrelated} \tag{10}$$

- Equation (7) tells us that we take more active risk as the Information Ratio increases, and/or the risk aversion decreases. It also helps us understand risk aversion, by relating it to the Information Ratio and the optimal active risk level. According to Equation (7), and assuming we measure risk in percent, risk aversions of 0.05 are low, 0.10 are moderate, and 0.15 are high.
- Equation (8) tells us that every investor—independent of their level of risk aversion—want to find the manager with the highest Information Ratio.
- Equation (10) tells us that our active holdings are proportional to alpha and inversely proportional to active variance, assuming active returns uncorrelated.

The Fundamental Law of Active Management:

$$IR = IC \cdot \sqrt{BR} \tag{11}$$

- Deriving this requires several assumptions:
  - No constraints on portfolio construction.
  - Zero transactions costs.
  - Optimal implementation of ideas.
  - Information coefficients significantly less than one.
  - Normally distributed signals.

  - We will show below how to drop the first three assumptions (or more accurately how to account for deviations from them).
- Understanding Skill (*IC*):
  - The Information Coefficient (*IC*) is the correlation of forecasts (α, or equivalently *z*-score or information *g*) with realizations (θ).
  - If residual returns are normally distributed, and the forecast alpha is a linear combination of information and noise, we can show that the fraction of alphas, *fr*, which correctly forecast the sign of the residual return is:

$$fr \approx \frac{1}{2} + \frac{IC}{\pi} \tag{12}$$

  - Typical *IC*'s are small. The average *IC* is zero. A good *IC* is 0.05. A great *IC* is 0.10. We tend to see higher *IC*s forecasting markets than forecasting individual stock returns.
- Understanding Breadth (*BR*):
  - Independent bets per year. The "per year" part comes from our definition of *IR* as an annualized quantity.
  - For a process in equilibrium, covering *N* stocks, with information decaying at $e^{-\gamma t}$ (i.e. $\alpha_n(t) = e^{-\gamma\Delta t} \cdot \alpha_n(t - \Delta t) + \tilde{s}_n(t) \cdot \sqrt{\Delta t}$):

$$BR = \gamma \cdot N \tag{13}$$

- Additivity:
  - What if your breadth is divided among bets of different skill? So you have BR_1 bets of skill IC_1, BR_2 bets of skill IC_2? Assuming you optimally allocate among these bets, then:

$$IR^2 = IC_1^2 \cdot BR_1 + IC_2^2 \cdot BR_2 \tag{14}$$

    More generally, this is how Information Ratios add.

Adjustment due to implementation efficiency (Transfer Coefficient)

$$IR = IC \cdot \sqrt{BR} \cdot TC \tag{15}$$

The transfer coefficient is the correlation between the actual (implemented) portfolio and the ideal portfolio you would build in the absence of constraints and costs. With this formulation, we can drop the assumptions of no constraints, zero costs, and optimal implementation of ideas. The transfer coefficient captures the impact of all three.

Risk Modeling

Problems with historical risk modeling:

- Data requirements. Need the number of periods of history to exceed the number of stocks.
- Handling of new assets.

- Handling of assets whose characteristics (and variances and covariances) have changed over time.

Single Factor Model (Sharpe's Market Model):

$$\mathbf{r} = \boldsymbol{\beta} \cdot r_B + \boldsymbol{\theta} \tag{16}$$

$$Cov\{\theta_n, \theta_m\} = 0 \text{ if } n \neq m \tag{17}$$

$$\mathbf{V} = \boldsymbol{\beta} \cdot \boldsymbol{\beta}^T \cdot \sigma_B^2 + \boldsymbol{\Delta} \tag{18}$$

- Note that the residual covariance matrix is diagonal.

Factor Models:

$$\begin{aligned} \mathbf{r} &= \mathbf{X} \cdot \mathbf{b} + \mathbf{u} \\ \mathbf{V} &= \mathbf{X} \cdot \mathbf{F} \cdot \mathbf{X}^T + \boldsymbol{\Delta} \end{aligned} \tag{19}$$

- Each month, we separate returns into common factor components, and a specific component.
- Our covariance matrix consists of a common factor piece, and a specific piece. We assume specific returns are uncorrelated. We choose common factors to completely capture the common components of returns.
- There are three main approaches to determining factors:
  - Fundamental factors: we specify exposures and estimate monthly or daily factor returns.
  - Macroeconomic factors: we specify changes in macreconomic variables, and estimate individual stock exposures to those factors via time-series regressions for each stock.
  - Statistical factors: We jointly estimate factors and exposures, under the assumption that exposures do not change over our estimation period.
- Given a covariance matrix, we can estimate betas more accurately than by simply running historical regressions.

Marginal Contributions to Risk

The marginal contribution to residual risk measures the change in residual risk given a small change in position:

$$\mathbf{MCRR} = \frac{\partial \omega_P}{\partial \mathbf{h}_{PA}^T} = \frac{\mathbf{V} \cdot \mathbf{h}_{PA}}{\omega_P} \tag{20}$$

At optimality this marginal contribution is directly related to alphas:

$$\boldsymbol{\alpha} = 2\lambda \cdot \omega_P \cdot \mathbf{MCRR} \tag{21}$$

This allows us to back out implied alphas from a given portfolio P.

Testing Risk Forecasts:

Given a set of observed returns $\{r(t)\}$, and risk forecasts $\{\sigma(t)\}$, we can calculate the standardized outcomes:

$$x(t) = \frac{r(t)}{\sigma(t)}. \tag{22}$$

To test the accuracy of these forecasts after T observations, calculate the bias:

$$bias = StDev\left\{x(t) \middle| t = 1, \ldots T\right\} \tag{23}$$

If the bias>1, we have underestimated risk. If the bias<1, we have overestimated risk. We can estimate the statistical significance of the difference from 1 using the standard error of a sample standard deviation (assuming normal distributions):

$$SE\{bias\} = \frac{bias}{\sqrt{2T}} \tag{24}$$