Suggested Solutions for the MFE 230I Sample Final

Question 1

1a

The Macaulay duration of the 5% coupon (c), 2 year (T) bond with yield to maturity (y) of 4% and face value (F) of \$100 is given by

$$\left(\frac{\sum_t^T \frac{c}{2} \times F \times t/(1+y/2)^{2t} + F \times t/(1+y/2)^{2t}}{\sum_t^T \frac{c}{2} \times F/(1+y/2)^{2t} + F/(1+y/2)^{2t}} \right) = 1.93.$$

The Modified duration is just the Macaulay duration divided by $(1 + y/2)$, which equals 1.89.

1b

The convexity of the bond is given by the following formula:

$$\left(\frac{\sum_t^T \frac{c}{2} \times F \times t \times (t+1/2)/(1+y/2)^{2t} + F \times t \times (t+1/2)/(1+y/2)^{2t}}{\sum_t^T \frac{c}{2} \times F/(1+y/2)^{2t} + F/(1+y/2)^{2t}} \right) \times \frac{1}{(1+y/2)^2} = 4.578.$$

1c

Remember that

$$\frac{\Delta P}{P} \approx \frac{-D_{mac}}{1+y/2} \Delta y + \frac{1}{2} C \, \Delta y^2.$$

Given $\Delta y = 0.25\%$ and using the values calculated in part a and part b, we have $\frac{\Delta P}{P} = -0.00471$ or 0.471%.

1d

With the three zero coupon bonds in our portfolio, we can match the value, the duration and the convexity of the two year bond we would like hedged. Suppose $A = 0.5$ yr ZCB, $B = 1$ yr ZCB and $C = 1.5$ yr ZCB.

For hedging,

$$\begin{aligned} n_A P_A + n_B P_B + n_C P_C &= P_{2yr} \\ n_A D_A + n_B D_B + n_C D_C &= D_{2yr} \\ n_A C_A + n_B C_B + n_C C_C &= C_{2yr} \end{aligned}$$

where P is price, D is dollar duration and C is dollar convexity. Solving this system of three equations and three unknowns gives us $n_a = 0.99, n_b = -2.93$ and $n_c = 3.04$.

1e

Similar to the inverse-floater example in class, we can deduce that the following relationship must hold:

$$50D(InvFloater) + 100D(Floater) = 150PV(Fixed; c = 20\%/3).$$

The PV of a 20%/3 coupon bond is 157.6155 and its Macaulay duration is 1.9. The duration of the floater is 6 months or 0.5. Now following the example in class, we solve the equation

$$1.9 = \left(\frac{100}{157.62} \times 0.5\right) + \left(\frac{157.62 - 100}{157.62} \times D_{mac}(IF)\right)$$

Solving we obtain $D_{mac}(IF) = 4.35$.

Question 2

2a

The value of the 1 year bond is \$95.198, so we can solve the following equation for X:

$$0.9518 = \frac{0.42 \times \frac{1}{1+X/2} + (1 - 0.42) \times \frac{1}{1+0.04/2}}{1 + 0.05/2}.$$

This returns $X = 6.31\%$. Once we have X, we may solve for Y by matching the value of the 18 month bond. You should get approximately 4.84% for Y. The value of Z is calculated straighforwardly from the 6 month bond as $1/(1 + 0.05/2) = 97.56$.

The final interest rate tree looks like this:

5.00%	6.31%	7.00%
	4.00%	4.84%
		3.00%

2b

While there are many ways to solve this, one can think of a callable bond as a non-callable bond minus a call option. This method is very similar to the one you implemented in homework 4. The non-callable component is simply a coupon bond evaluated along each interest rate path and has a value of 99.9062, calculated as follows:

99.90617	98.68102	99.03382	102.50
	100.7893	100.0784	102.50
		100.9852	102.50

The call option is an American option with expected payout equal to $\mathbf{max}\left(NonCallable - 100, 0\right)$, where 100 is the par value of the bond, discounted along the interest rate path. The value of the call option is 0.4544, calculated from the following tree:

0.454421	0.018998	0
	0.789313	0.078392
		0.985222

The value of the callable bond is the difference between these values, 99.4518:

99.45175	98.66202	99.03382
	100	100
		100

Note that you can construct the callable bond tree in one step, replacing each value with \$100 whenever you calculate a value greater than \$100.

2c

Note that the Asian option is a path dependent option and the average interest rate depends on the path you followed to maturity. Let us index the final period payoffs as uu, ud, du and dd; where uu indicates that the interest rate followed the up node in both periods, ud indicates that the interest rate followed the up node in the first period and then the down node in the next period and so on.

State	Probability	Average r (%)	Payout, \$1M $\times$ $\max(\bar{r} - K, 0)$
uu	$0.42 \times 0.75 = 0.315$	$(7 + 6.31 + 5)/3 = 6.10$	11037.09
ud	$0.42 \times (1 - 0.75) = 0.105$	$(4.84 + 6.31 + 5)/3 = 5.38$	3835.17
du	0.435	4.61	0
du	0.145	3.10	0

Note that the average interest rate need not be above the strike of 5% in all states (it is not when the states of *du* and *dd* occur). The probability of each state and the corresponding payout is calculated in table . Since this is a futures contract, the current futures price is simply the expected value of the final payouts *without discounting*. Thus the current futures price for this contract is given by $0.315 \times 11037.09 + 0.105 \times 3835.17 = 3879.38$.

2d

To calculate the position in the futures contract we calculate the hedge ratio between the callable bond and the futures. For this we need the Deltas of each:

$$\Delta_{CallBond} = \frac{C_u - C_d}{r_u - r_d} = \frac{98.66 - 100}{0.0631 - 0.04} = -57.89$$

$$\Delta_{Option} = \frac{O_u - O_d}{r_u - r_d} = \frac{9236.61 - 0}{0.0631 - 0.04} = 399658.2$$

To hedge the callable bond with the option, we need to hold $\frac{\Delta_{CallBond}}{\Delta_{Option}} = -0.00014$ Asian futures option (a short position).

Question 3

3a

There are closed-form solutions for zero-coupon bonds in the Vasicek model of interest rates (see lecture notes on continuous-time spot rate models). The forward rate is easy to compute from zero-coupon bond prices:

$$\frac{1}{\left(1 + \frac{f_2(0,2,5)}{2}\right)^6} = \frac{P(r,5)}{P(r,2)} = 0.755/0.898 = 0.84.$$

So $f_2(0,2,5) = 2 \times \left(0.84^{-\frac{1}{6}} - 1\right) = 5.864\%$. Continuously compounded, we get $f(0,2,5) = 5.780\%$.

3b

The futures prices is simply the expected value of a three-year bond two years in the future.

$$\begin{aligned} E_0[P(r,T=3)|t=2] &= E_0\left[e^{A(T=3)-B(T=3)r(2)}\right] \\ &= e^{A(3)} E_0\left[e^{-B(3)r(2)}\right] \\ &= e^{A(3)} \exp\left\{-B(3)E[r(2)] + \frac{1}{2}B(3)^2 Var[r(2)]\right\} \end{aligned}$$

The mean and variance of $r(2)$ are given in the lecture notes. Calculations give us a futures price of 0.84 and an implied rate of 5.789% (continuously compounded) or 5.874% (semiannual).

Question 4

4a

This statement is false. A callable bond is a combination of a non-callable bond minus a call option. The value of the non-callable bond falls as the interest rate rises and the value of the call option increases as the volatility increases. Thus the value of the callable bond falls with rising interest rates and volatilities.

4b

This statement is true. As the interest rates increase, the prepayments to principal may become more likely. This will decrease future cash flows to the Interest Only securities.

4c

This statement is false. More volatility has an ambiguous effect. All else equal, more volatility makes it more valuable to keep the call alive, which increases the expected life. However more volatility also makes it more likely that the call boundary will be reached sooner.

4d

This statement is true.

4e

This statement is true.

Question 5

The Strike price is straightforwardly calculated as $K = F(0, 1.25, 1.5) = Z(0, 1.5)/Z(0, 1.25) = 0.993$. **Hint: The next step is to decompose the call option to the floorlet to use Black's Formula.** The method is similar to the one employed in class.

Call option payoff at $t = 1.25$ is

$$N \times Max\left[\frac{1}{1 + \Delta\, r(1.25, 1.5)} - K, 0\right].$$

This has the same PV as a payout at $t = 1.5$ of

$$N \times (1 + \Delta\, r(1.25, 1.5)) \times Max\left[\frac{1}{1 + \Delta\, r(1.25, 1.5)} - K, 0\right] = N \times Max[1 - K(1 + \Delta\, r(1.25, 1.5)), 0]$$
$$= N \times Max[1 - K - K\Delta\, r(1.25, 1.5), 0]$$
$$= NK\Delta \times Max\left[\frac{1-K}{K\Delta} - r(1.25, 1.5), 0\right].$$

This is the payoff of a floorlet with notional $NK = 99.28$ and strike rate $r_K = \frac{1-K}{K\Delta} = 0.0291$. We can now use Black's formula for a floorlet. First calculate the forward rate:

$$f_4(0, 1.25, 1.5) = 4 \times (1/(F(0, 1.25, 1.5)) - 1) = 0.0291.$$

Now following Veronesi section 21.4.4, equation 21.39, we calculate the cumulative volatility,

$$\sigma_f \sqrt{1.25} = \sqrt{\Delta \sum_{i=1}^{5} S_i^2} = 0.4055.$$

Formulas for d_1 and d_2 are given in Veronesi Section 20.1, and result in

$$d_1 = 0.2028,$$
$$d_2 = -0.2028.$$

Now using Black's formula for a floorlet, we obtain an option value of \$0.1117.

Question 6

The question is pretty open-ended. Good answers would cover some of these topics:

First, BDT is a term structure model, while HJM is a framework of term structure models, which allows more flexible volatility assumptions.

The difference in the calibration process. Model inputs are different. Observability of the main factors are different too: Short rate for BDT model, and forward rates for HJM model.

HJM model is more general than short rate models.

Using HJM to price financial instruments usually involves Monte Carlo simulation. BDT is tree based.