MFE 230I: Fixed Income
Topic 2: Duration, Convexity and Interest Rate Risk

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Reading assignments for this topic

- Required: Veronesi, Chapters 3 and 4.

Outline

Duration

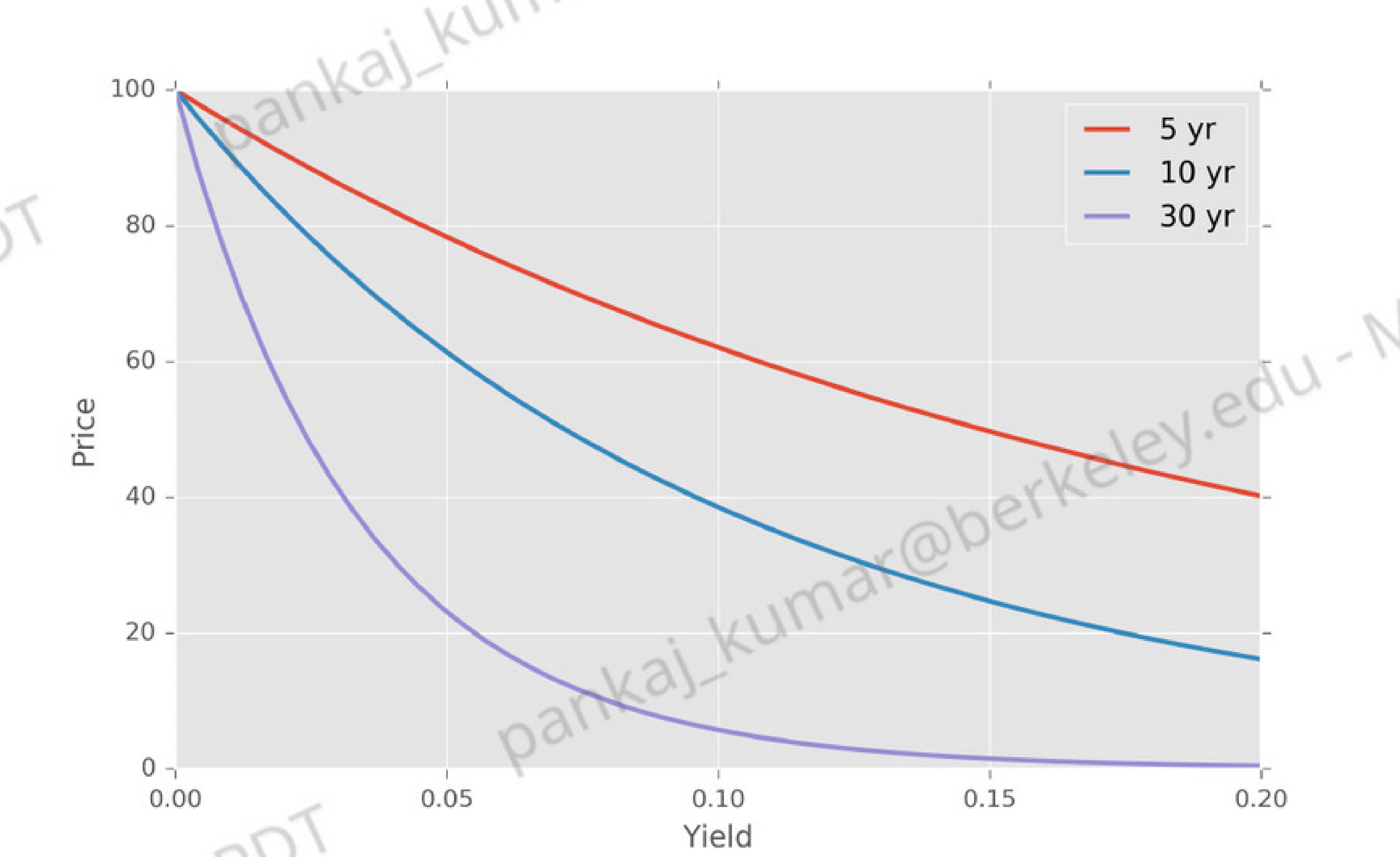
The Interest Sensitivity of Derivatives

Effective Duration

Convexity

Cash Flow Variance

Price vs. yield (zero coupon bonds)



Macaulay duration

- Macaulay duration is a weighted average of the time to each payment, the weights proportional to the PV of each payment.

$$D_{\text{mac}} = \frac{1}{P}\sum_{i=1}^{k} t_i \times \text{PV}(C_i)$$
$$= \begin{cases} \frac{1}{P}\sum_{i=1}^{k} t_i \times \frac{C_i}{(1+y)^{t_i}} & \text{(annual compounding)} \\ \frac{1}{P}\sum_{i=1}^{k} t_i \times \frac{C_i}{(1+y/n)^{nt_i}} & \text{(n times per year)} \\ \frac{1}{P}\sum_{i=1}^{k} t_i \times C_i e^{-yt_i} & \text{(continuous compounding)} \end{cases}$$

Macaulay duration and interest rate sensitivity

- Bond price (annual compounding): $P(y) = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$.
- Differentiating with respect to y and rearranging, we obtain

$$\frac{dP}{dy} = -\frac{D_{\text{mac}}}{1+y} \times P.$$

- Approximating $\frac{dP}{dy}$ with $\frac{\Delta P}{\Delta y}$, we obtain

$$\frac{\Delta P}{P} \approx \begin{cases} \frac{-D_{\text{mac}}}{1+y} \times \Delta y & \text{(annual compounding)} \\ \frac{-D_{\text{mac}}}{1+y/n} \times \Delta y & \text{(n times per year)} \\ -D_{\text{mac}} \times \Delta y & \text{(continuous compounding)} \end{cases}$$

- D_{mac} measures a bond's interest rate sensitivity.

Example: Calculating Macaulay duration (annual compounding)

- 5 year bond, 5% annual coupon, yield = 10%:

t	CF	PV(C)	t * PV(C)
1	5	4.5455	4.5455
2	5	4.1322	8.2645
3	5	3.7566	11.2697
4	5	3.4151	13.6603
5	105	65.1967	325.9837
Total		81.0461	363.7236

- Macaulay duration = 363.72 / 81.05 = 4.49.

Macaulay duration of a zero coupon bond

- Consider a zero coupon bond paying \$1 at time T. Its Macaulay duration is

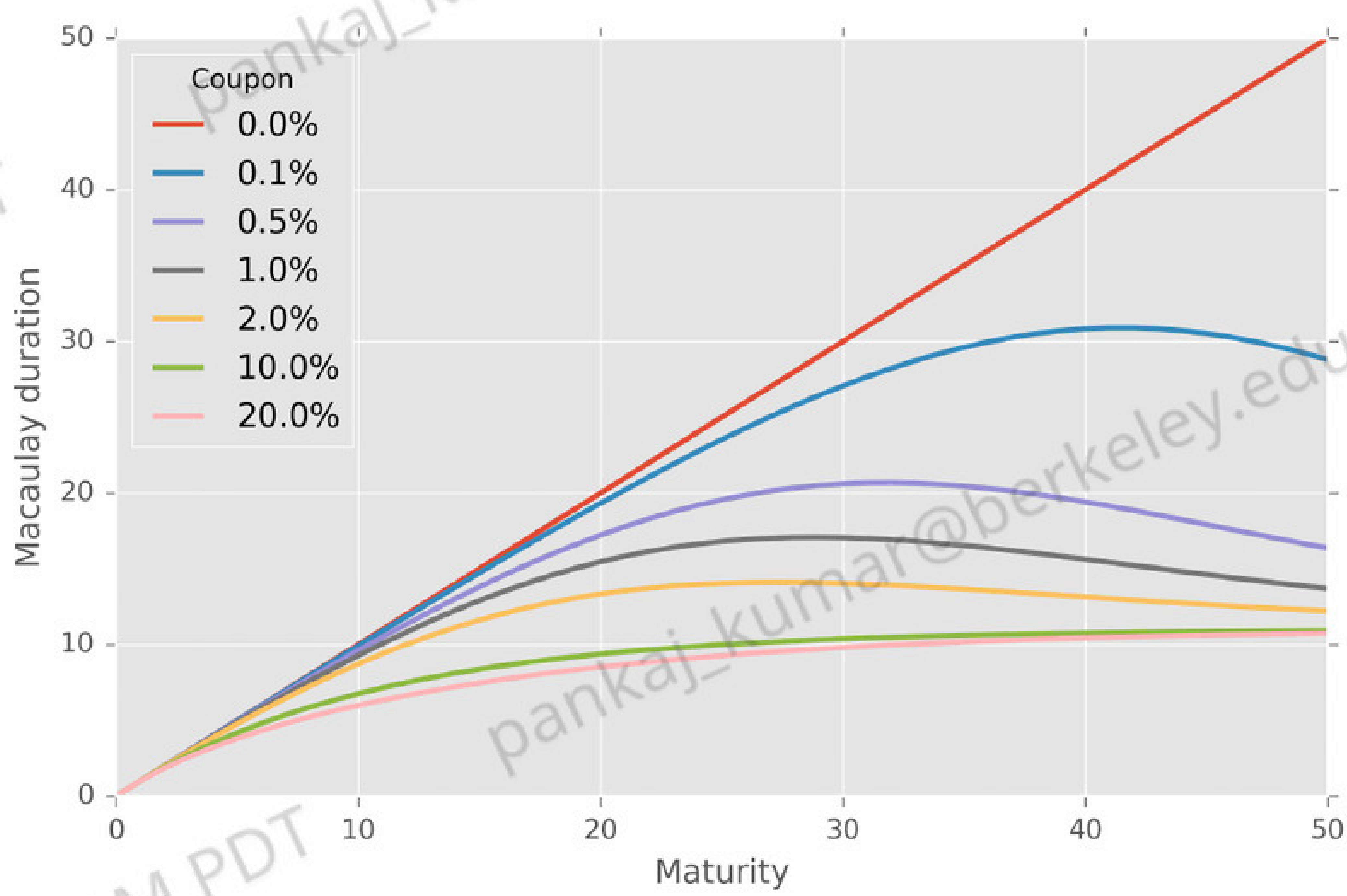
$$D_{\text{mac}} = \frac{T/(1+y)^T}{1/(1+y)^T} = T.$$

- Macaulay duration equals maturity for a zero coupon bond.

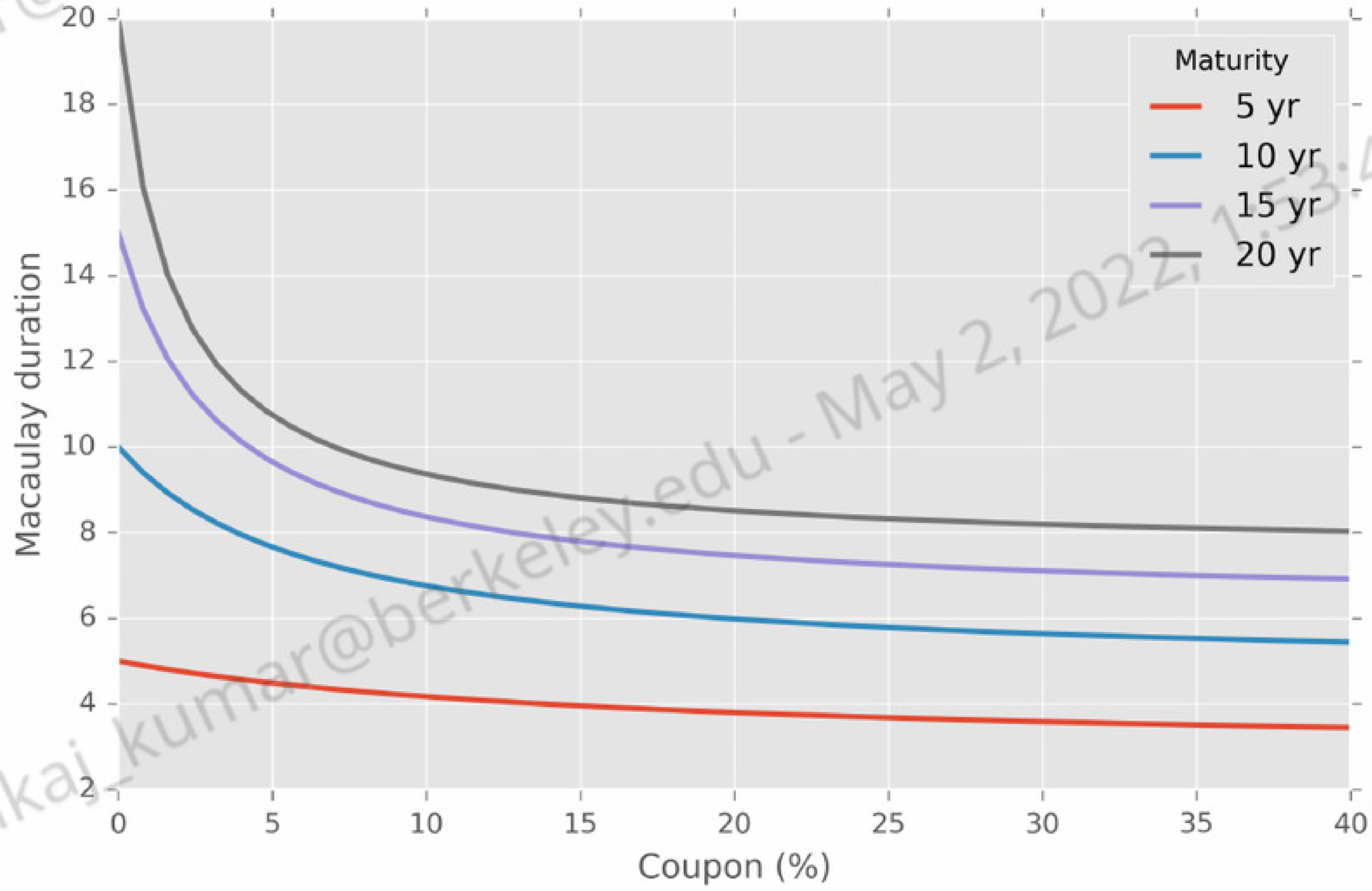
Example

- We saw previously that at a yield of 10%, a 5 yr. 5% annual coupon bond has a value of $81.05 and a Macaulay duration of 4.49 years.
- Assume the bond's yield increases from 10% to 10.01%.
  - Use duration to approximate the bond's % and $ change in value.
  - Calculate the new value exactly using the new yield.

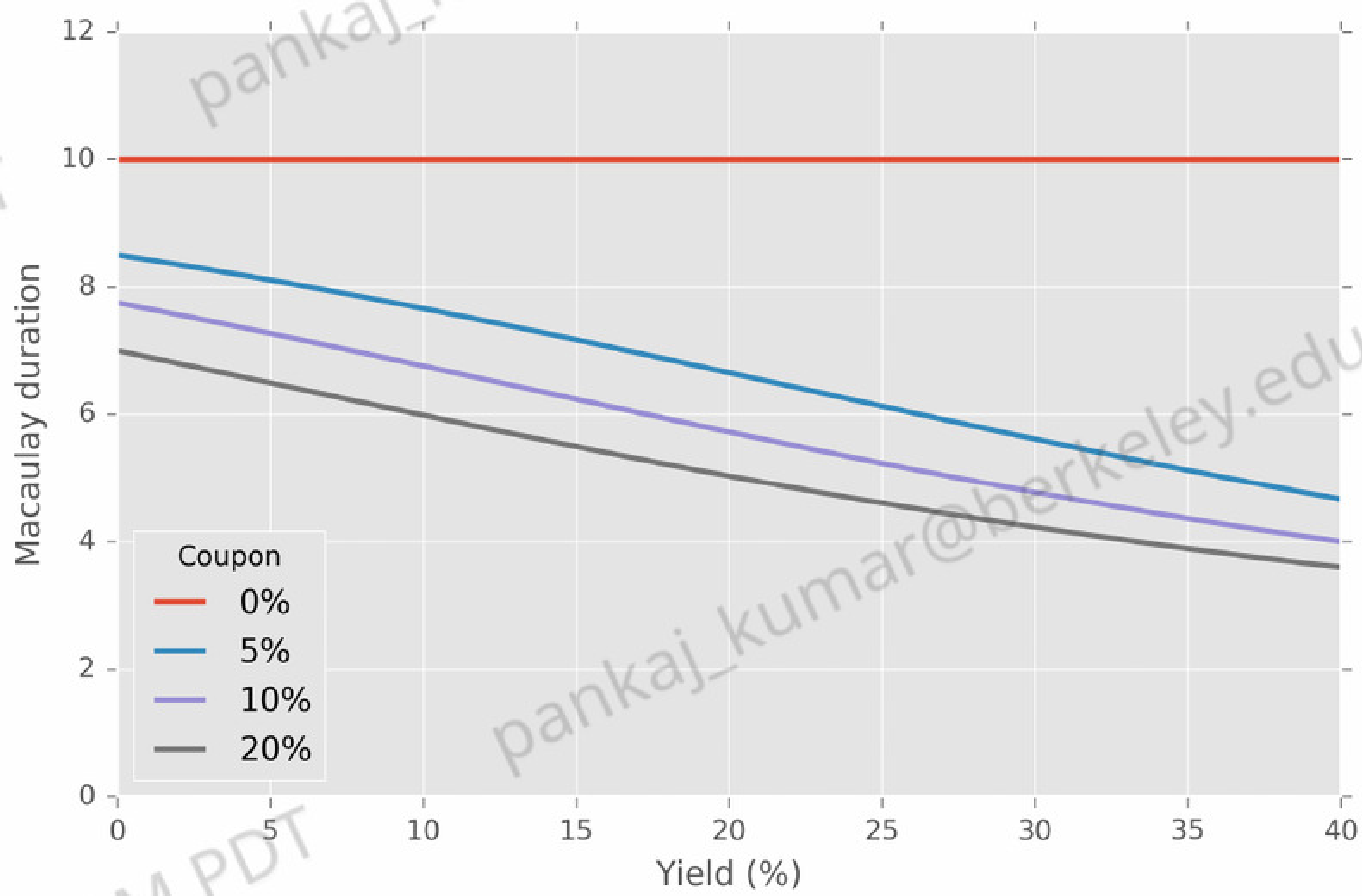
Macaulay duration vs. maturity (y = 10%)



Macaulay duration vs. coupon (y = 10%)



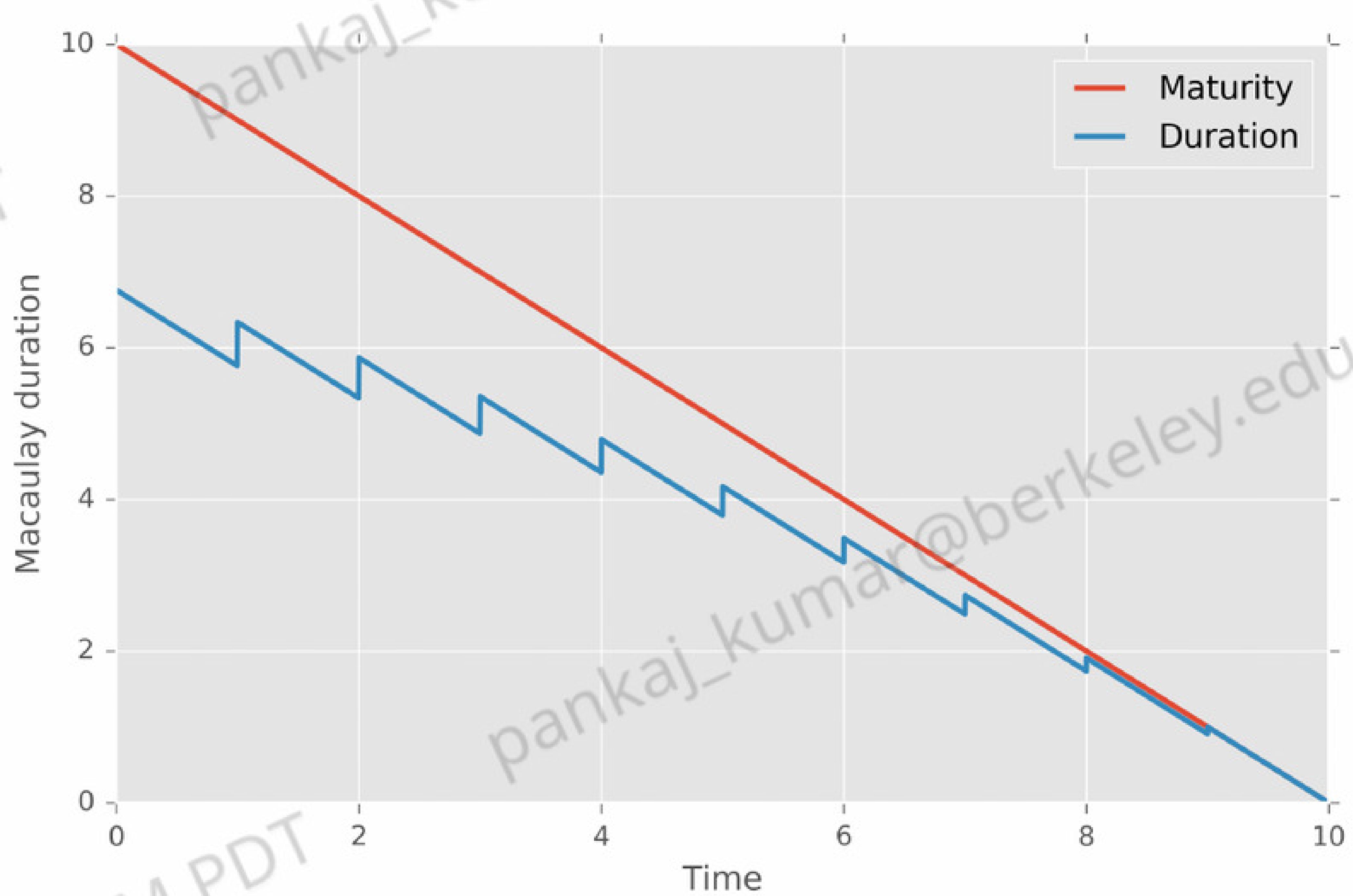
Macaulay duration vs. yield (T = 10)



Macaulay duration and accrued interest

- Remember that Macaulay duration is a weighted average of times to each coupon payment, the weights summing to 1.
- Between coupon payments, every day that passes, each payment gets 1 day closer, so the weighted average also decreases by exactly 1 day.
- What happens when a coupon payment is made?

Macaulay duration and accrued interest (10-year 10% bond)



Modified duration

- Rather than calculating $D_{\text{mac}}/(1+y)$ every time, we define modified duration by

$$D_{\text{mod}} = \begin{cases} \dfrac{D_{\text{mac}}}{(1+y)} & \text{(annual compounding)} \\ \dfrac{D_{\text{mac}}}{(1+y/n)} & \text{(n times per year)} \\ D_{\text{mac}} & \text{(continuous compounding)} \end{cases}$$

- Modified duration tells us the percentage change in price for each 1% increase in the yield,

$$\frac{\Delta P}{P} \approx -D_{\text{mod}} \times \Delta y$$

Dollar duration and DV01

- Dollar duration is defined by

$$\begin{aligned} D_{\$} &= D_{\text{mod}} \times \frac{P}{100} \\ &\approx -\frac{1}{100} \times \frac{\Delta P}{\Delta y} \end{aligned}$$

  - The dollar change in price for a 1% change in rates.

- Dollar Value of 1 Basis Point (DV01) is the dollar price change of a security for a one basis point change in yield.
  - DV01 = Dollar Duration / 100.
  - Also known as PVBP (price value of a basis point)

Example

- 5 year bond, 5% annual coupon, yield = 10%.
- We calculated its Macaulay duration earlier: 4.487
  - Modified duration = 4.487 / 1.1 = 4.08
  - Dollar duration = 4.08 x 81.046 / 100 = $3.306
  - DV01 = 3.306 / 100 = $0.03306

Duration of a portfolio

- Let P be the price of a portfolio, P_i the value of security i, and n_i the number of security i in the portfolio. From the definitions,

$$D_{\text{mac}} = \sum_i \frac{n_i P_i}{P} \times D^i_{\text{mac}} \qquad D_{\text{mod}} = \sum_i \frac{n_i P_i}{P} \times D^i_{\text{mod}}$$

$$D_{\$} = \sum_i n_i \times D^i_{\$} \qquad \text{DV01} = \sum_i n_i \times \text{DV01}^i$$

- Macaulay (modified) duration: multiply the assets' durations by their fraction in the portfolio.
- Dollar duration (DV01): multiply the assets' durations by their number in the portfolio.

Example

- Two bonds have prices of $80 and $90, and Macaulay durations of 4.5 and 7.6 years respectively. Their yields are 6.0%.
  - What are the securities' modified durations?
  - What are their dollar durations?

Example

- You form a portfolio by buying 4 of the first bond, and 6 of the second bond.
  - What is the portfolio's modified duration?
  - What is its dollar duration?

Outline

Duration of other securities

- So far we can calculate the duration (interest rate sensitivity) of bonds making risk-free fixed coupon payments, e.g., Treasury bonds
- We need to be able to measure the interest rate risk of other securities too.
  - We may own securities other than T-Bonds.
  - We may want to use other securities to hedge.

Defaultable bonds

- Even if the payment amounts are fixed, many bonds are subject to default risk.
  - We might not get the payments.
- How does the duration of a risky bond compare with that of an otherwise identical riskless bond?

Defaultable bonds

- Consider the 5 year, 5% coupon bond we looked at earlier.
- Default-free, its Macaulay duration was 4.49.
- Suppose there's a 10% chance each year that the bond will default completely (no recovery).
  - Suppose the event of default is uncorrelated with the state of the economy overall.
- What is the bond's Macaulay duration?

Defaultable bonds

t	Promised CF	Prob. of receipt	Expected CF	PV(C)	t * PV(C)
1	5	90%	4.500	4.0909	4.0909
2	5	81%	4.050	3.3471	6.6942
3	5	73%	3.645	2.7385	8.2156
4	5	66%	3.281	2.2406	8.9625
5	105	59%	62.001	38.4980	192.4901
Total				50.9152	220.4534

- $D_{\text{mac}} = 220.45/50.91 = 4.33$.
- Smaller than for default-free bond.

Interest rate derivatives

- A derivative security is one whose payoffs (and hence value) depend on the payoffs/value of another security. Examples include
  - Futures contracts
  - Forward Contracts
  - Options
- We shall examine many of these during the course.

Options in fixed income securities

- Embedded options
  - Callable (puttable) bonds
  - Caps, floors, collars
  - Mortgage-backed securities

- Bond and bond futures options

- Swaptions…

Example: Callable bond

- A callable bond may be redeemed by the issuer before its scheduled maturity.
  - The call provision gives the borrower the right to buy back the bond at a predetermined price (the call price).
  - Usually there is an early period of call protection. After this period, the option behaves like an American call.

Callable bonds

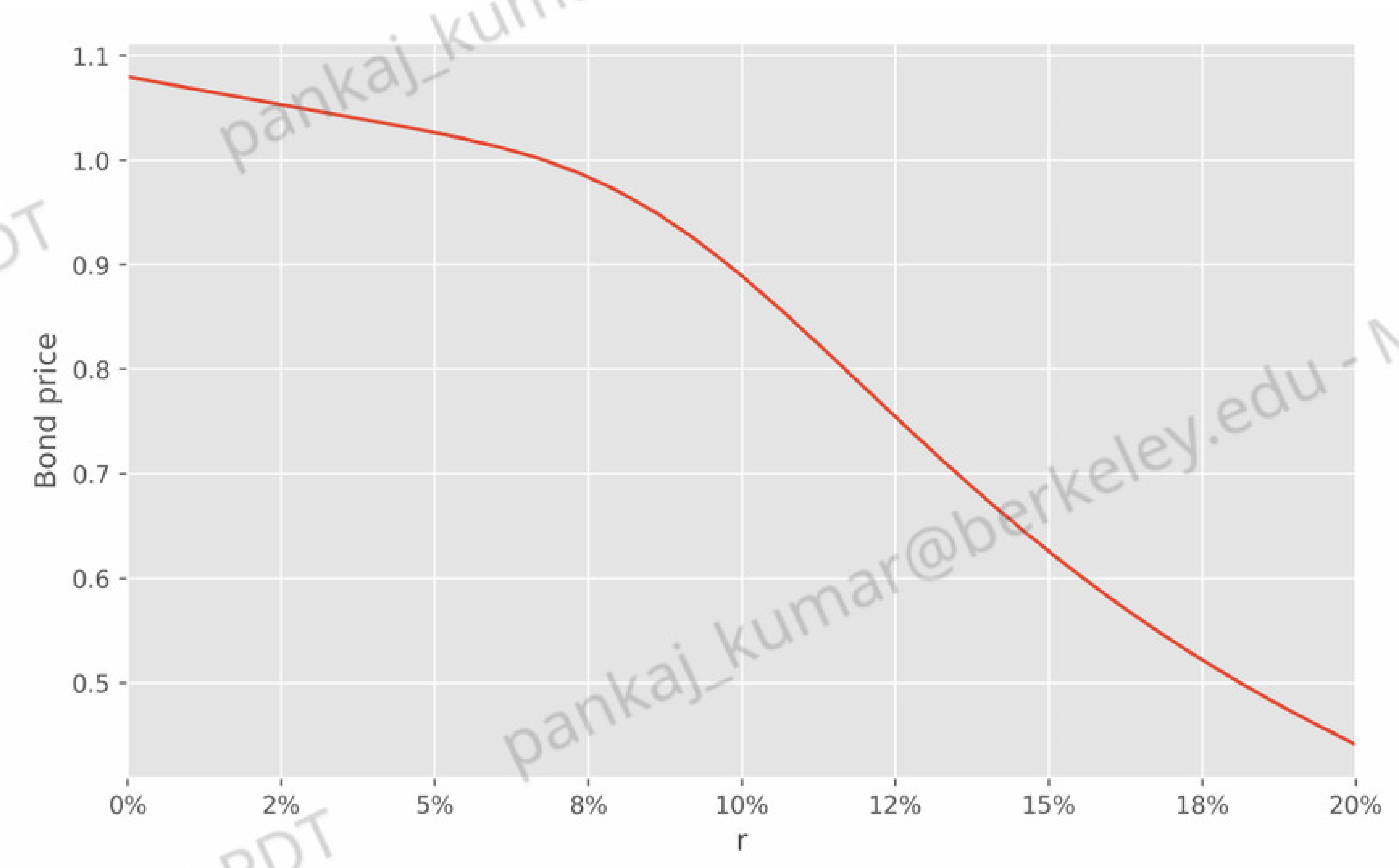
- The buyer of a callable bond is
  - Long a noncallable bond with the same coupon and maturity.
  - Short a call option on this bond.
- Thus

$$P_{cb} = P_{nb} - C,$$

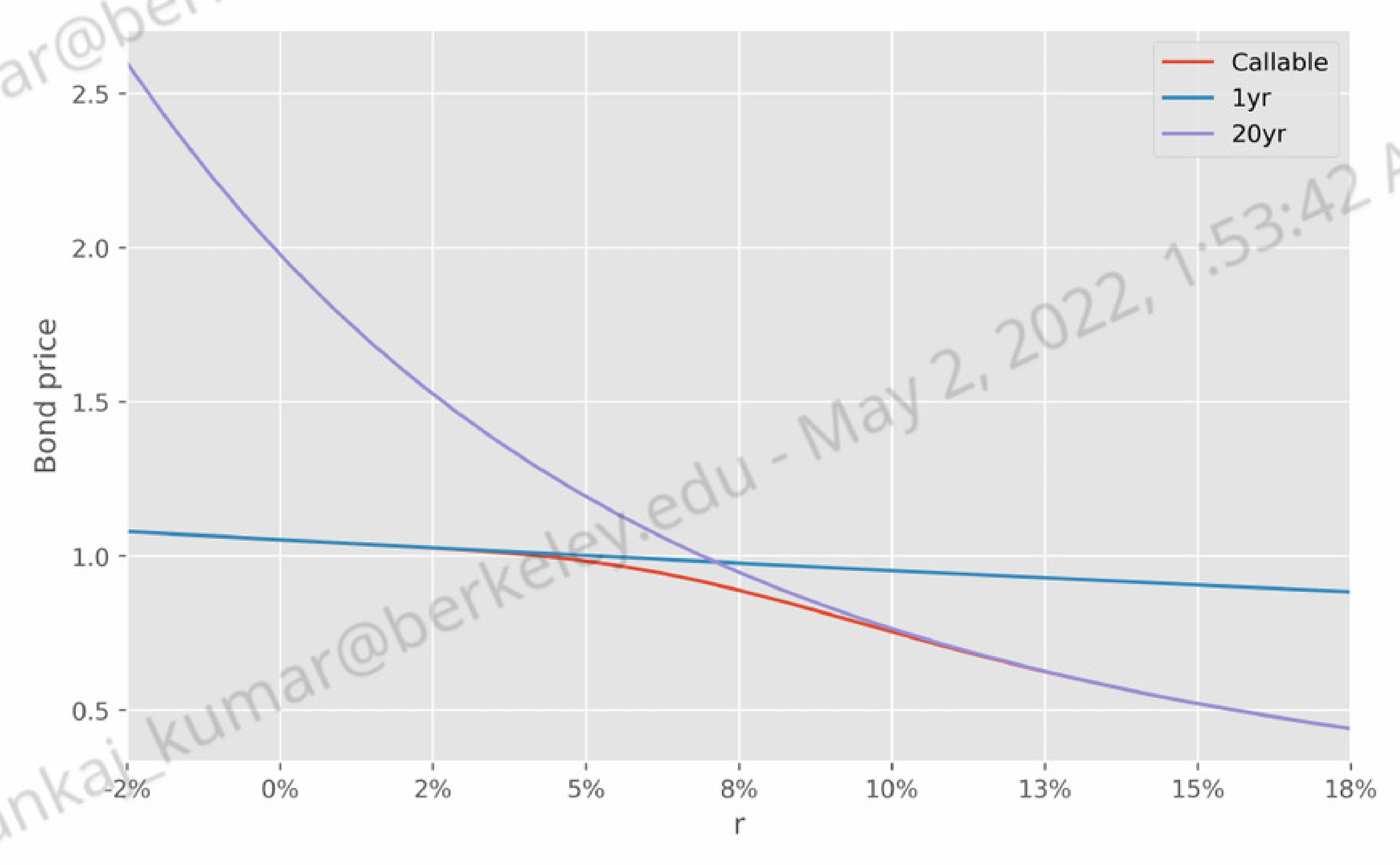
where P_{cb} is the callable price, P_{nb} the non-callable price, and C the option price.

- How does the yield to maturity of a callable bond compare with that of a non-callable bond?

Price of a callable bond



Non-callable vs. callable bonds



Outline

Duration

The Interest Sensitivity of Derivatives

Effective Duration

Convexity

Cash Flow Variance

Duration and interest rate derivatives

- Our original definitions of duration (Macaulay, modified etc.) cannot be used for interest rate derivatives.
  - The cash flows from the security vary depending on the level of interest rates.
- However, it still makes sense to ask about the interest rate sensitivity of such securities.
- Effective duration is a generalized measure of interest rate sensitivity.

Effective duration

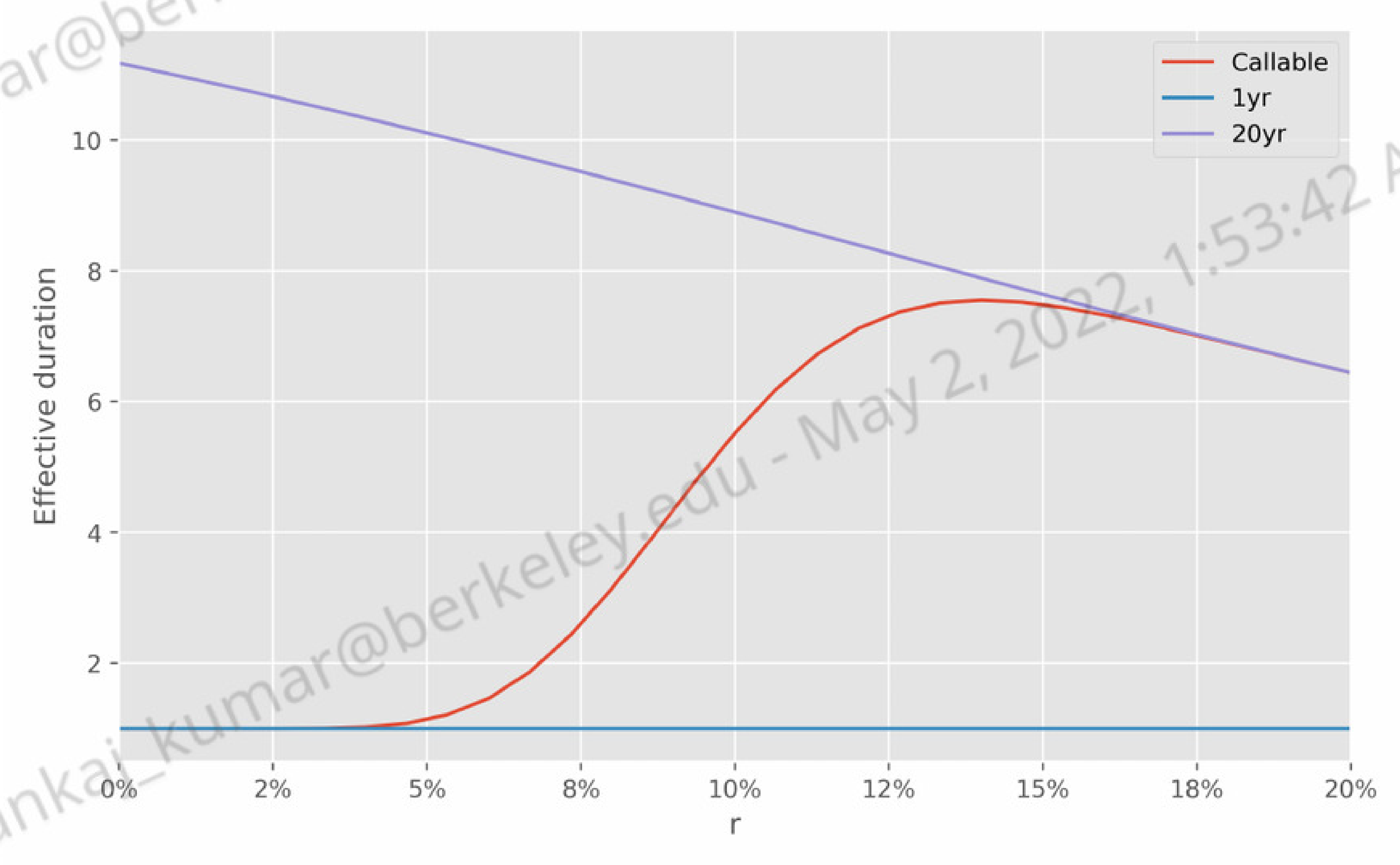
- Remember the interpretation of modified duration as a sensitivity to changes in interest rates:

$$D_{\text{mod}} = -\frac{1}{P}\frac{dP}{dy}.$$

- Our earlier calculation of modified duration doesn't make sense for derivatives.
  - The cash flows from the security vary depending on the level of interest rates.
- However, its interpretation as a sensitivity still makes sense.
- Define effective duration by

$$D_{\text{eff}} = -\frac{1}{P}\frac{dP}{dy}.$$

Comparison of effective duration



Floating-rate note

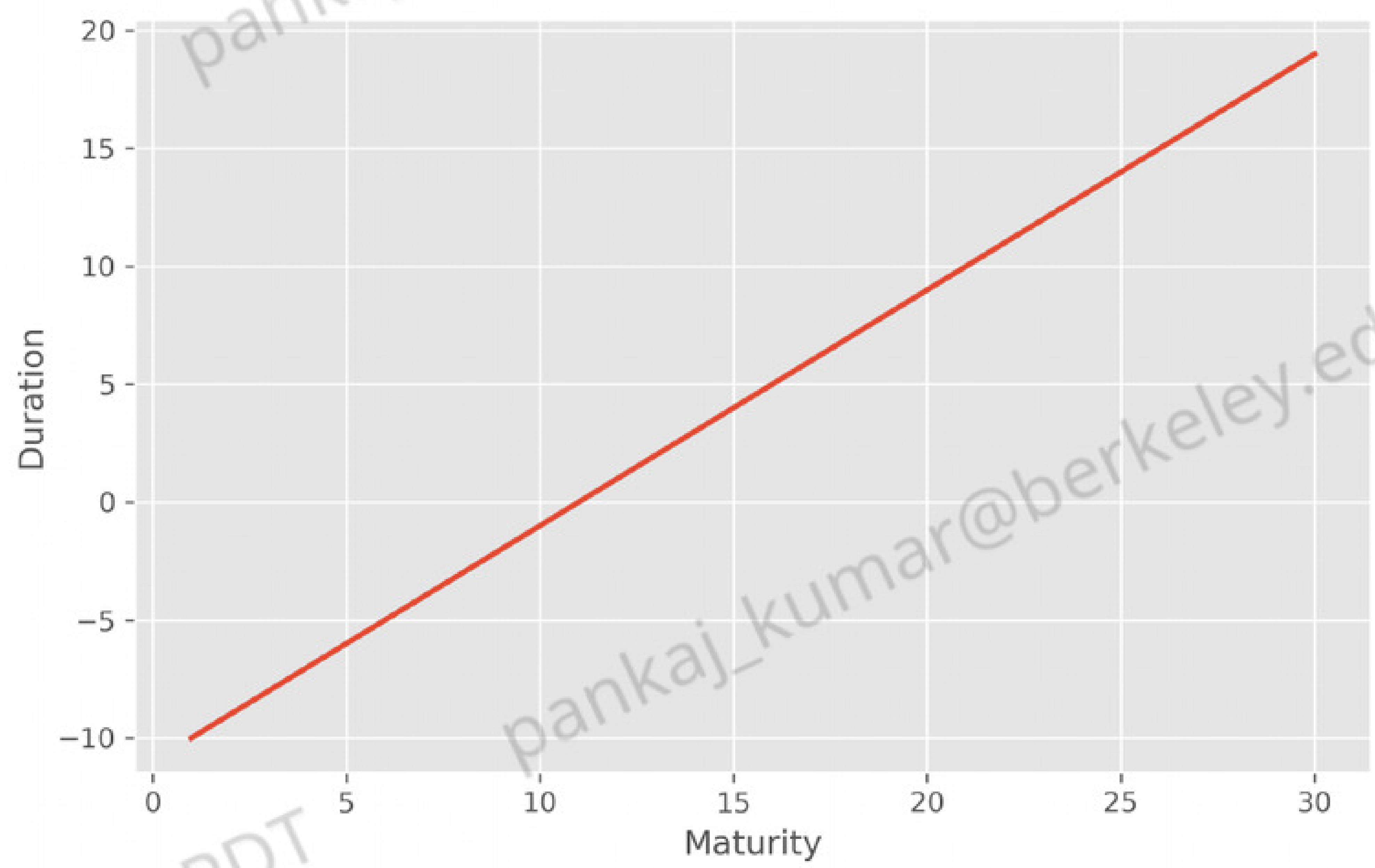
- Assuming $\Delta = 6m$ and the coupon just reset to 4%, the price of the bond resets to par every 6 months.
- Thus the FRN is equivalent to a zero-coupon bond paying \$100 (value after reset) + \$2 (coupon) = \$102 in 6 months time.
- Its Macaulay duration therefore equals 6 months.
  - More generally, D_{mac} = time until next reset date.

Duration of a single floating payment

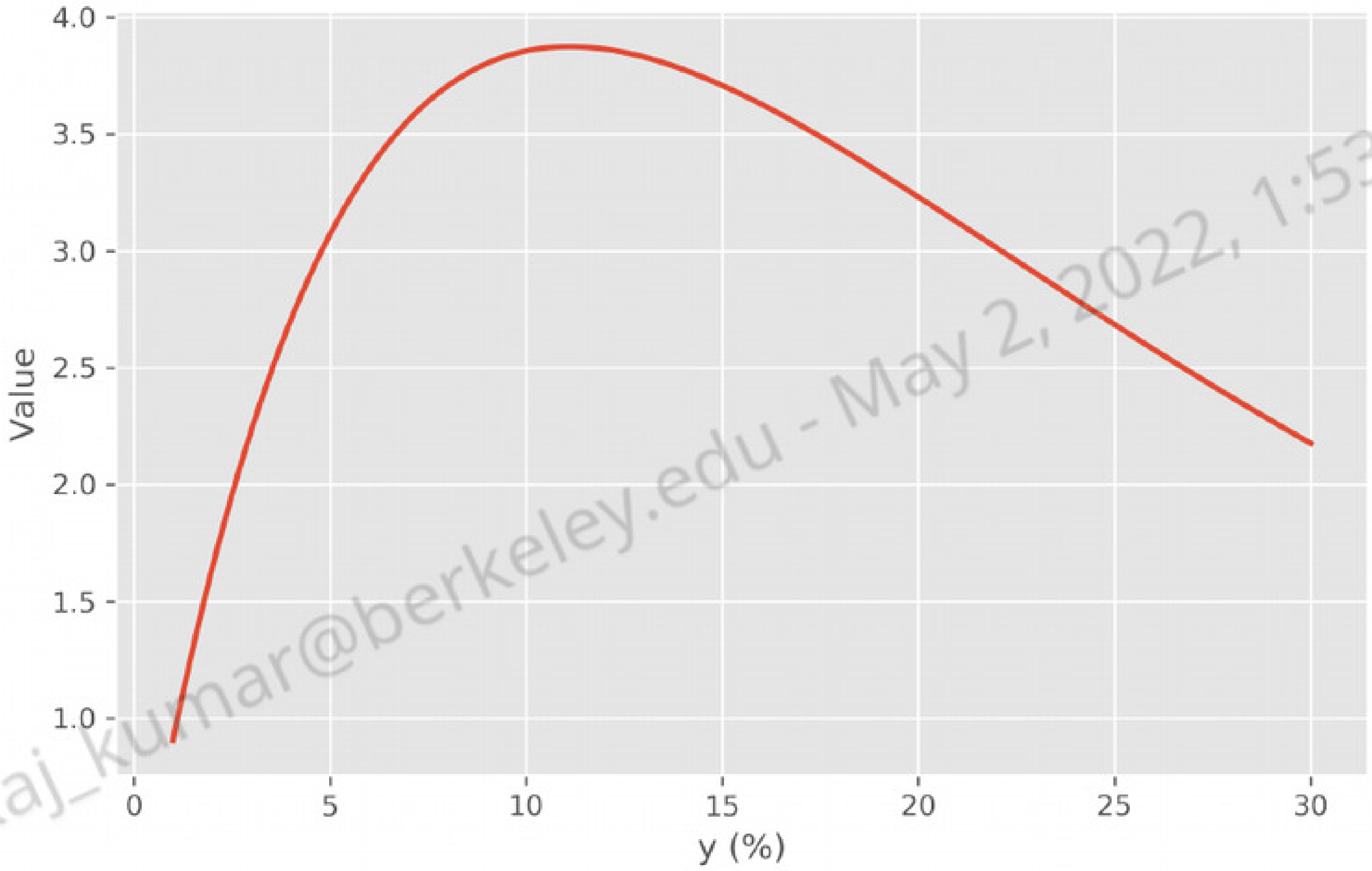
- We saw earlier that we can replicate a single floating payment as follows:
  1) Buy a zero-coupon bond with face value N and maturity $(T - \Delta)$.
  2) Sell a zero-coupon bond with face value N and maturity T.
  3) Invest N at date $T - \Delta$ for one period at rate $r_n(T - \Delta, T)$.
- The duration of a single floating payment is thus the same as that of a portfolio that is
  - Long a zero-coupon bond with maturity $T - \Delta$.
  - Short a zero-coupon bond with maturity T.
- Thus (after some calculation),

$$D_{\text{mac}} = T - \left(\frac{1 + y\Delta}{y}\right).$$

Macaulay duration of a single floating payment ($y = 10\%$)



Value of a single floating payment ($T = 10$ years)



Inverse floaters

- The coupon payments on an inverse floater are negatively related to interest rates.
- For example, suppose the (semiannually compounded) coupon rate on an inverse floater is $10\% -$ the 6m rate.
- We can calculate its price/duration using the relationship:

$$\$50 \text{ FV Floater} + \$50 \text{ FV Inverse Floater} = \$100 \text{ FV of a } 5\% \text{ Coupon Bond}$$

- Floater + inverse floater = fixed-coupon bond

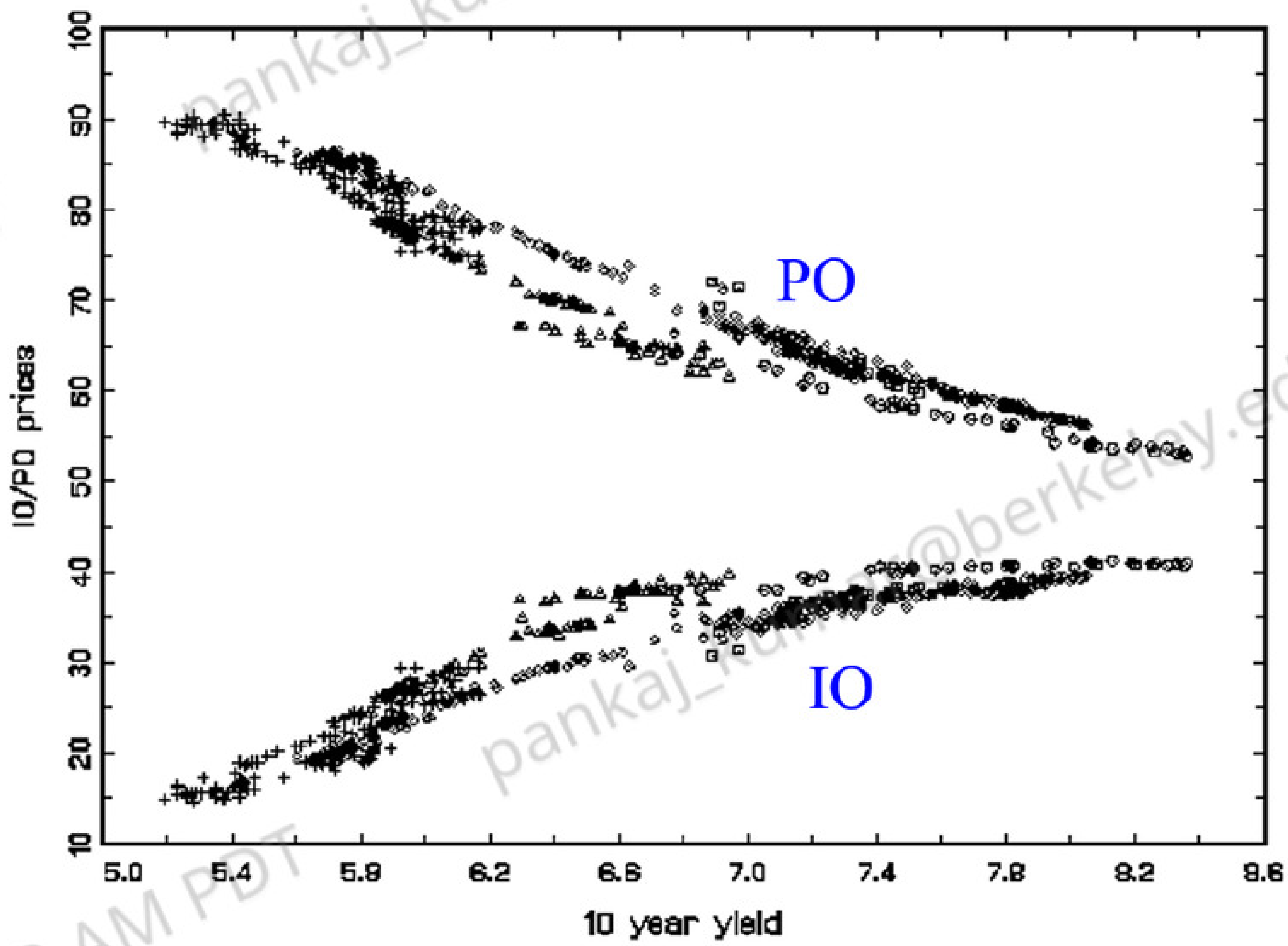
Inverse floaters: Example

- With maturity 2 years, current interest rate 4%,
  - 2 year 5% bond has PV = \$101.90.
  - 2 year 5% bond has $D_{\text{mac}} = 1.929$.
- So PV(\$50 Inverse Floater) = $\$101.90 - \$50 = \$51.90$.
- For duration, use portfolio formula,

$$1.929 = \left(\frac{50}{101.90} \times 0.5\right) + \left(\frac{51.90}{101.90} \times D_{\text{mac}}(\text{IF})\right).$$

- So $D_{\text{mac}}(\text{IF}) = 3.305$.

Example: IO/PO stripped mortgage-backed securities



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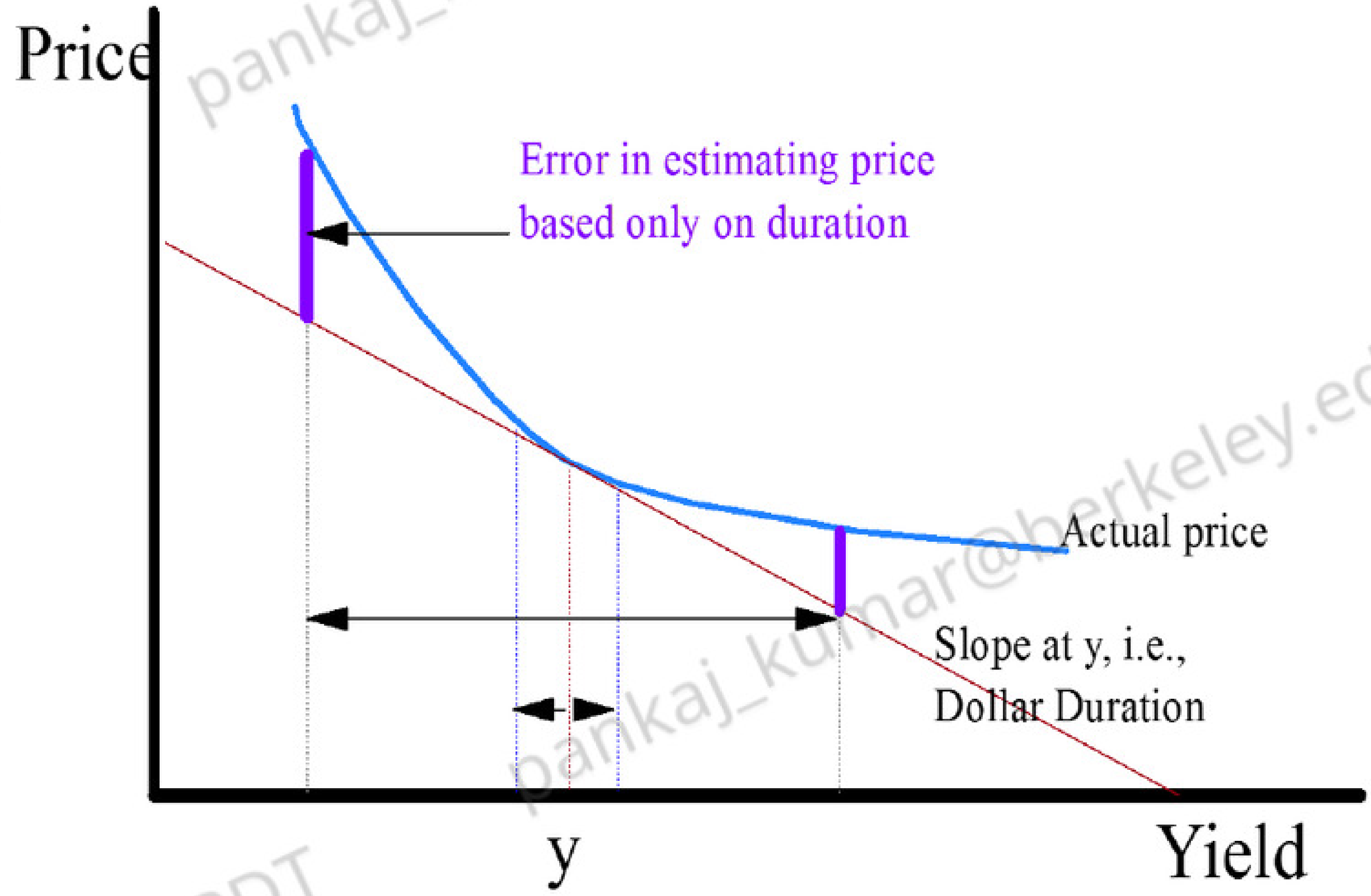
What’s wrong with duration?



Convexity

- Convexity is defined as

$$
\begin{aligned}
C &= \frac{1}{P}\frac{d^2P}{dy^2} \\
&= \begin{cases}
\dfrac{1}{(1+y)^2} \times \dfrac{1}{P}\displaystyle\sum_{i=1}^{k} t_i(t_i+1)\frac{C_i}{(1+y)^{t_i}} & \text{(annual)} \\
\dfrac{1}{(1+y/n)^2} \times \dfrac{1}{P}\displaystyle\sum_{i=1}^{k} t_i\left(t_i+\frac{1}{n}\right)\frac{C_i}{(1+y/n)^{nt_i}} & \text{(n times per year)} \\
\dfrac{1}{P}\displaystyle\sum_{i=1}^{k} t_i^2 \times C_i e^{-yt_i} & \text{(continuous)}
\end{cases}
\end{aligned}
$$

Convexity of a zero coupon bond

- The convexity of a zero-coupon bond is given by

$$C(\text{zero}) = \frac{T(T+1/n)}{(1+y/n)^2}$$

- Approximately equal to D_{mod}^2.

Calculating convexity

- What is the convexity of a 5 yr. 5% annual coupon bond, with a yield of 10%?

t	CF	PV(C)	t * PV(C)	t(t+1)PV
1	5	4.5455	4.5455	9.0909
2	5	4.1322	8.2645	24.7934
3	5	3.7566	11.2697	45.0789
4	5	3.4151	13.6603	68.3013
5	105	65.1967	325.9837	1955.9022
Total		81.0461	363.7236	2103.1667

- $C = 2103.2/81.05/1.1^2 = 21.4465.$
- As with dollar duration, define dollar convexity by $C_\$ = C \times P/100$.
- In this case, $C_\$ = 21.4465 \times 81.0461/100 = 17.382$.

Estimating price change using duration and convexity

- The % change in price is approximately given by

$$\boxed{\frac{\Delta P}{P} \approx -D_{\text{mod}} \times \Delta y + \frac{1}{2} C\, \Delta y^2.}$$

- Similar to our previous duration equation, but with an extra term involving convexity and Δy^2.
- This is a second-order Taylor series approximation.
  - Using convexity is equivalent to approximating the true graph with a parabola.
  - Using convexity as well as duration allows us to improve our estimates when price movements are larger.

Example

- Consider a 5 year, 5% (annual) coupon bond, with yield to maturity 10%.
- We have already calculated its D_{mod} (4.08) and convexity (21.447).
- Suppose interest rates decrease from 10% to 8%.
  - What is the % increase in the bond's price using duration alone?
  - What is the % increase using duration + convexity?
  - What is the true % increase?

Example: Hedging with duration and convexity

- There are three bonds: A, B, and C.
  - A is a 1 year zero coupon bond.
  - B is a 3 year 5% (annual) coupon bond.
  - C is a 10 year zero coupon bond.
- The annual interest rate is 10%.
- You own 1 of bond B that you want to hedge.
  - If you used bond A, what position would you need?
  - If you used bond C, what position would you need?

Hedging using duration only

- We want to short enough of bond A to make portfolio's modified duration zero.
  - Will we want to short more or less than one of bond A?
- We can think of this in two (equivalent) ways:
  - Modified duration of portfolio must equal zero.
  - Dollar duration of position in bond A must equal $-$ Dollar duration of position in bond B.

Calculating duration

- First, calculate the duration of bond B:

T	Disc. fac.	CF	PV(C)	t * PV(C)
1	0.909091	5	4.545455	4.545455
2	0.826446	5	4.132231	8.264463
3	0.751315	105	78.88805	236.6642
TOTAL			87.56574	249.4741

- $P(B) = \$87.57$.
- $D_{\text{mod}}(B) = 249.47/87.57/1.1 = 2.5899$.
- $D_{\$}(B) = 2.5899 \times 87.57/100 = 2.2679$.

Hedging with bond A

- To hedge, we need to find a position in bond A with the same dollar duration as bond B.
- Consider one Bond A:
  - $P(A) = 100/1.1 = \$90.91.$
  - $D_{\text{mod}}(A) = 1/1.1 = 0.9091.$
  - $D_{\$}(A) = 0.9091 \times 90.91/100 = 0.8264.$
- So to hedge B we need to short 2.2679 / 0.8264 = 2.744 of bond A.

Hedging with bond C

- Consider one Bond C:
  - $P(C) = 100/1.1^{10} = \$38.554.$
  - $D_{\text{mod}}(C) = 10/1.1 = 9.091.$
  - $D_{\$}(C) = 9.091 \times 38.554/100 = 3.505.$
- So to hedge B we need to short 2.2679 / 3.505 = 0.647 of bond C.
- Note that we do not need as much of bond C, as it is much more interest sensitive.

Hedging with multiple instruments

- What position in both bonds A and C would allow you to hedge "better"?

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Cash Flow Variance

Cash flow variance

- How does the pattern of cash flows affect a bond's convexity for a given duration?
- Define cash flow variance, a measure of dispersion, by

$$\mathrm{CFV} = \sum_{t=1}^{T} \frac{C_t}{P(1+y)^t} \left(t - D_{\mathrm{mac}}\right)^2 .$$

- Note: For a zero coupon bond, CFV = 0. For any other bond, CFV is greater than zero.
- What is the CFV for our 5 year, 5% coupon bond, with yield = 10%?

Cash flow variance: Example

T	Disc. fac.	CF	PV(C)	t * PV(C)	t(t+1)PV	(t-D)^2*PV
1	0.909091	5	4.545455	4.545455	9.090909	55.296295
2	0.826446	5	4.132231	8.264463	24.79339	25.576281
3	0.751315	5	3.756574	11.26972	45.07889	8.316059
4	0.683013	5	3.415067	13.66027	68.30135	0.8128196
5	0.620921	105	65.19674	325.9837	1955.902	17.100113
TOTAL			81.04607	363.7236	2103.167	107.10157

- Cash flow variance = 107.10 / 81.05 = 1.321

CFV versus duration and convexity

- Define weights $w_i = \mathrm{PV}(C_i)/P$ and assume annual compounding:

$$\begin{aligned}
\mathrm{CFV} &= \sum_{i=1}^{N} w_i (t_i - D_{\mathrm{mac}})^2 \\
&= \sum_{i=1}^{N} w_i t_i^2 - 2D_{\mathrm{mac}} \sum_{i=1}^{N} w_i t_i + D_{\mathrm{mac}}^2 \sum_{i=1}^{N} w_i \\
&= \sum_{i=1}^{N} w_i t_i (t_i + 1) - \sum_{i=1}^{N} w_i t_i - 2D_{\mathrm{mac}} \sum_{i=1}^{N} w_i t_i + D_{\mathrm{mac}}^2 \sum_{i=1}^{N} w_i \\
&= C(1+y)^2 - (1+y)D_{\mathrm{mod}} - 2(1+y)^2 D_{\mathrm{mod}}^2 + (1+y)^2 D_{\mathrm{mod}}^2 \\
&= C(1+y)^2 - (1+y)D_{\mathrm{mod}} - (1+y)^2 D_{\mathrm{mod}}^2
\end{aligned}$$

- Thus

$$\boxed{C = \frac{\mathrm{CFV}}{(1+y)^2} + D_{\mathrm{mod}}^2 + \frac{D_{\mathrm{mod}}}{1+y}.}$$

Convexity, duration and cash flow variance

- Repeating for different compounding frequencies, we get

$$C = \begin{cases} \dfrac{\mathrm{CFV}}{(1+y)^2} + D_{\mathrm{mod}}^2 + \dfrac{D_{\mathrm{mod}}}{1+y} & \text{(annual)} \\[2ex] \dfrac{\mathrm{CFV}}{(1+y/n)^2} + D_{\mathrm{mod}}^2 + \dfrac{D_{\mathrm{mod}}}{n+y} & \text{(n times per year)} \\[2ex] \mathrm{CFV} + D_{\mathrm{mod}}^2 & \text{(continuous)} \end{cases}$$

- For our 5% coupon bond, this becomes

$$\begin{aligned} C &= 1.321/1.21 + 4.08^2 + 4.08/1.1 \\ &= 21.4465, \end{aligned}$$

the same as we obtained earlier via direct calculation.

- For a given duration, the higher a coupon bond's CFV, the larger its convexity.
  - A zero-coupon bond has the lowest convexity for a given duration.

Barbells versus bullets

- We can construct a portfolio of a long-term and a short-term bond (a barbell) that has the same market value and duration as an intermediate-term bond (a bullet).
- The barbell will have more convexity.

Barbell versus bullet: example

- Duration and convexity data for \$100 par value of 10, 20 and 30-year zeros.
  - All interest rates semiannually compounded.

Maturity	Rate	Price	Dollar Duration	Modified Duration	Dollar Convexity	Convexity
10	6.00%	55.3676	5.375493	9.70874	54.7987	98.9726
20	6.50%	27.8226	5.389364	19.37046	107.0043	384.5951
30	6.40%	15.1084	4.391974	29.06977	129.8015	859.1356

- \$100,000 par value of the 20 year bond has market value \$27,822, modified duration 19.37.
- Consider a portfolio consisting of
  - \$25,174 par value of the 10-year zero (value \$13,938)
  - \$91,898 par value of the 30-year zero (value \$ 13,884)
- Market value = 13,938 + 13,884 = \$27,822.
- $D_{\text{mod}} = (13{,}938/27{,}822 \times 9.71) + (13{,}884/27{,}822 \times 29.07) = 19.37.$

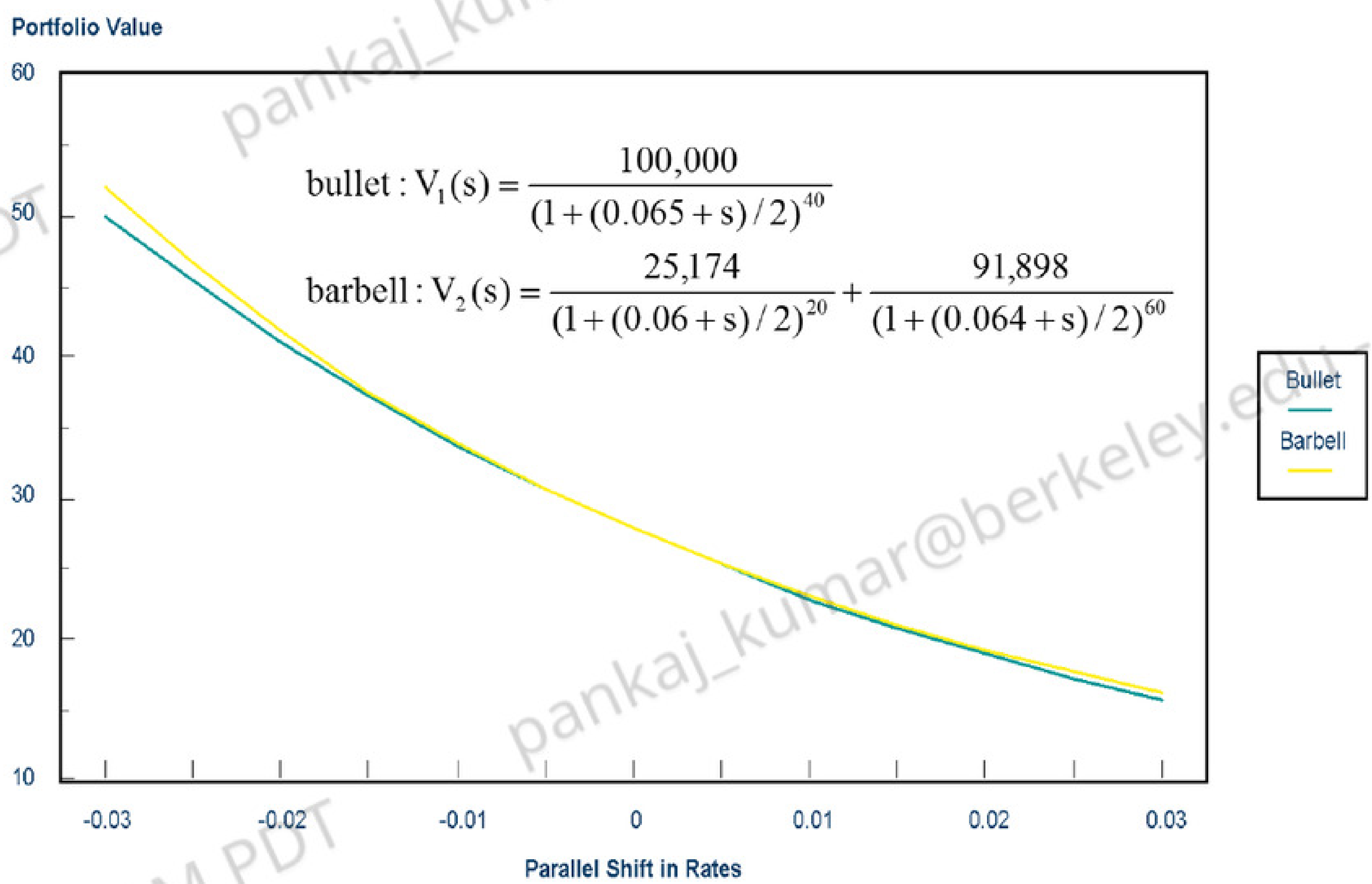
Convexity of barbell

- The barbell has the same market value and modified duration as the bullet.

$$C = (13{,}938/27{,}822 \times 98.9726) + (13{,}884/27{,}822 \times 859.1356)$$
$$= 478.32.$$

- Its convexity is higher than that of the bullet (384.60)

Value of barbell and bullet



Does the barbell always outperform the bullet?

- If there is an immediate parallel shift in interest rates, either up or down, then the barbell will outperform the bullet.

- (If the shift is not parallel, anything could happen).

- If rates stay exactly the same, then as time passes the bullet will outperform the barbell:
  - the bullet will return 6.5%
  - the barbell will return about 6.2%, the market value-weighted average of the 6% and 6.4% on the 10- and 30-year zeros.

Convexity and the shape of the yield curve

- It *cannot* be the case that the yield curve is flat and always makes parallel shifts.
  - Barbell outperforms bullet if rates move in either direction.
  - If rates stay constant, barbell and bullet perform the same.
  - So we can make an arbitrage profit by going long a barbell and short a bullet.

- To prevent arbitrage, yield on the bullet needs to be higher than weighted average of barbell yields.
  - This is why yield curve tends to curve downwards at long maturities.

References I

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