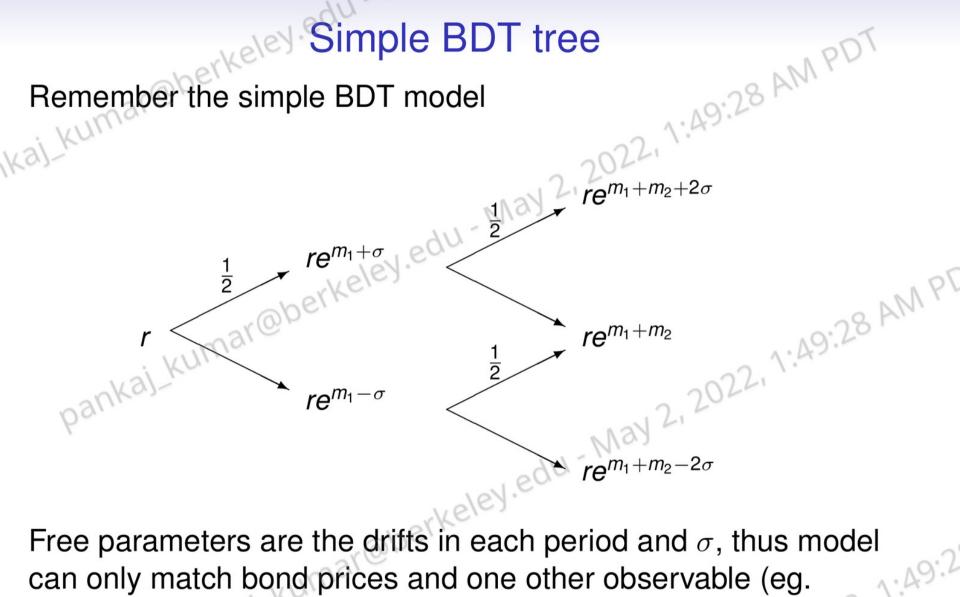
pankaj_kumar@berkeley.edu - Wita (1994) MFE2301 Section 5 pankaj_kumar@berkeley.edu -

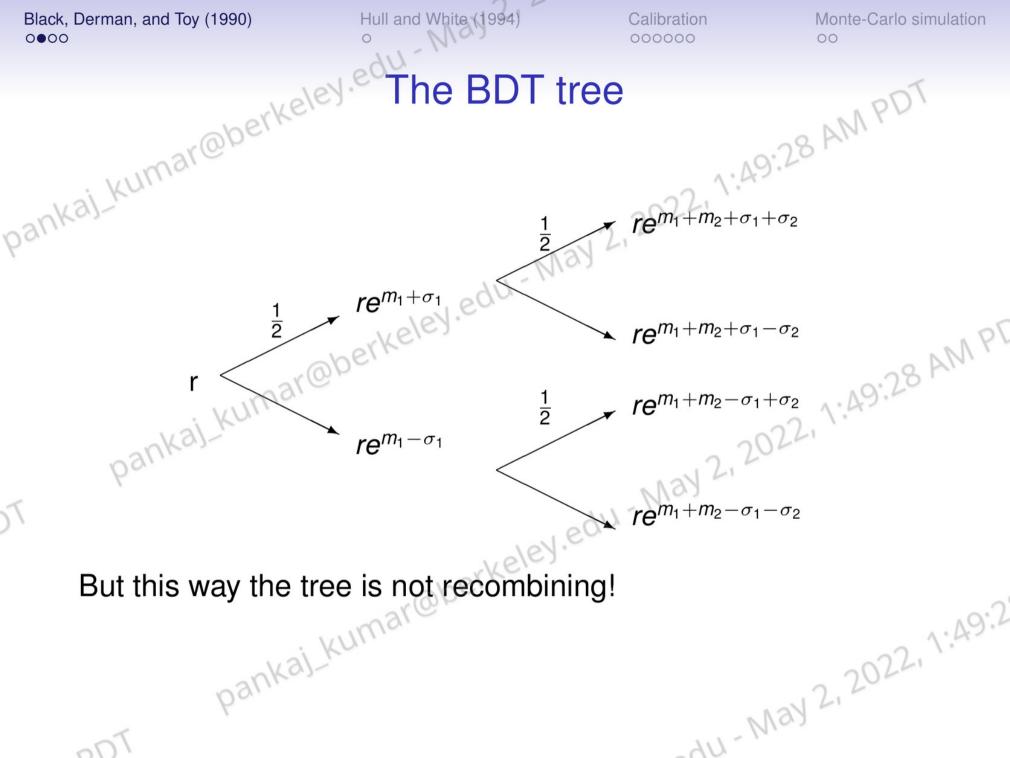
^{-1 -}May 2, 2022, 1:49:28 AM Pr -1 -May 2, 2022, 1:49:28 AM Pr ¹Based on notes by David Echeverry, Jiakai Chen, Tamás Bátyi and Christoph Kröner. Errors are mine.

Simple BDT tree

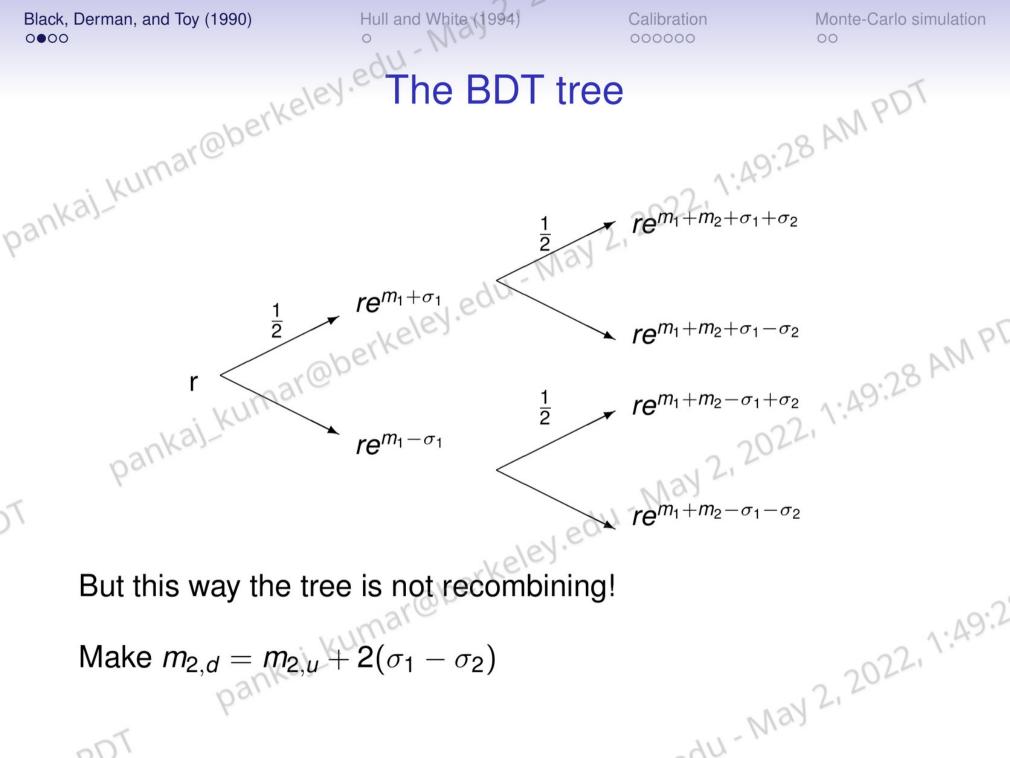


Free parameters are the drifts in each period and σ , thus model can only match bond prices and one other observable (eg. volatility of short term interest rate or some cap or option price) May 2 \Rightarrow we can make σ time varying too!

MAN 2022. 1:49:2



MAN 2022. 1:49:2



Make
$$m_{2,d} = m_{2,u} + 2(\sigma_1 - \sigma_2)$$

Calibration Constructing the BDT tree 2022. 1:49:28 AM PDT Keeping track of all different values of *m* is inconvenient. What can make it easier?

pankaj_kumar@berkeley.edu - May 2, 2022, 1:49:28 AM Pr May 7.2022, 1:49:2

Calibration 000000

Constructing the BDT tree Keeping track of all different values of m is inconvenient. What can make it easier?

Observe that ratio of every two node above each other is the same $e^{-2\sigma_t}$!

- each period need to keep track of only the lowest rate r_{0,t} and σ_t
- calibration essentially choosing these two values instead of pankaj_kumar@berkeley. *m* and σ



Continuous time limit of BDT

As $\Delta t \Rightarrow$ 0 the continuous time BDT model takes the form

$$d \log r_t = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \log r_t\right) dt + \sigma(t) dW$$

Thus $\log r$ is normally distributed $\to r$ is lognormal, and the variance only depends on $\sigma(t)$ In this limiting case the existence of mean reversion depends on the form of $\sigma(t)$, what is an outcome of the $\sigma(t)$ don't have control over the existence and size of mean pankaj_kumar@ reversion)



The Hull-White tree

The Hull and White (1994) tree is a discrete time generalization of the Vasiček (1977) model when $dr_t = \kappa(\mu - r_t)dt + \sigma dW$ of the Vasiček (1977) model, where

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW$$

- trinomial approximation
- transition probabilities chosen to match expected value V.edu - May and variance

Question: what happens with probabilities if T gets large? pankaj_kumar@ May 7.2022, 1:49:2

The Hull-White tree

The Hull and White (1994) tree is a discrete time generalization of the Vacionk (1977) models. $dr_t = \kappa(\mu - r_t)dt + \sigma dW$ of the Vasiček (1977) model, where

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW$$

- trinomial approximation
- transition probabilities chosen to match expected value redu-May and variance

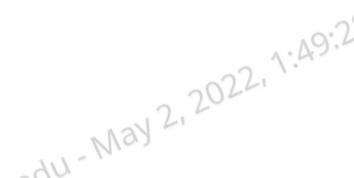
Question: what happens with probabilities if T gets large? May 7.2022, 1:49:2

For large T need to change branching!

Calibration

22.1:49:28 AM PDT Calibration is choosing the model's free parameters to match some observable data, for example ZCB prices. A tree with T periods needs pricing of T bonds

- this would lead to T bond-price trees, each of them with $O(T^2)$ calculations \rightarrow total of $O(T^3)$ calculations
- however for calibration we only need the t=0 prices, not the whole price tree!
- if we compute the state-price tree, that is sufficient to obtain all t = 0 ZCB prices using only one tree \rightarrow leading to only $O(T^2)$ calculations pankaj_kumar



May 2022. 1:49:2

Example – interest rate tree

2022, 1:49:28 AM PDT Suppose the tree for the one year rate after calibration is same as used in previous section

	10V.e	C			
year	6/0	1	2	3	2,2022,1:49:28 AM PT
	5%	4%	3%	2%	19:28
vai Kulli		6%	5%	4%	2211:4
pankaj_kumar@ T			7%	6%	2,202
7				8%	

How does the tree of state-prices look like? pankaj_kumar@ber

May 2022, 1:49:2

pankaj kumar@berkeley Example – $\pi_0(u)$

For one year state prices:

$$0.4762 \text{ pankal kumar@berkeley.edu}$$

$$0.4762 = 0.5 \times \frac{1}{1.05}$$

$$0$$

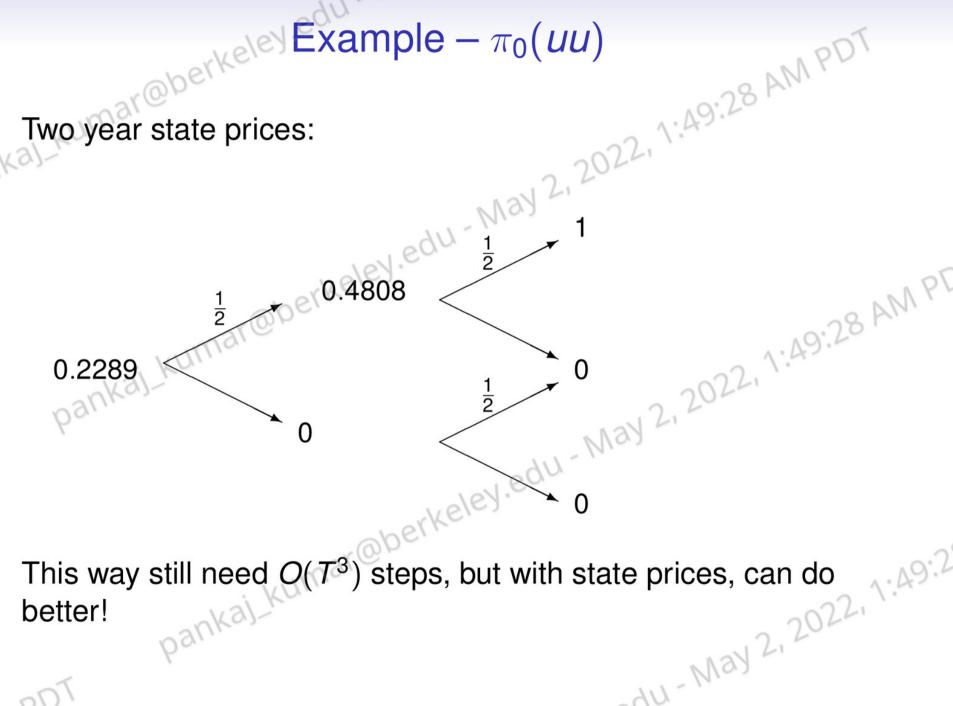
$$0.4762 = 0.5 \times \frac{1}{1.05}$$

$$0$$

$$0.4762 = 0.5 \times \frac{1}{1.05}$$

Example – $\pi_0(uu)$

Two year state prices:



Black, Derman, and			Hull and Whit	te (1994)	Calibration 0000●0		onte-Carlo simulation o
	or\	Ξxε	ample -	state-p	orice tre	е	MPDT
pankaj_kum	ar@per					1:49:28 F	ZIA,
ankal-k	year	0	1	2	3322	4	
			0.4762	0.2289	0.1111	0.0545	_
			0.4762	0.4536	0.3271	0.2117	
			rkeley	0.2246	0.3209	0.3087	1 P
		v @	perio		0.1050	0.2000	:49:28 AM P
	· KUM	al.				0.0486	.49.2
-30	Kal-1		0.4762 berkeley			2024	
ba					May	21	
T				. 0	du		
				rkeley.			
			ar@b	SI.			0:5
		, \	Mua,				1.49.
	nank	(9)-			du-May	2	2022, 1:49:
700	Par					May	-1
207					4	11	

Black, Derman, and			Hull and Whi	0.)	Calibration 0000●0	00	onte-Carlo simulation
	- 1	Ξха	ample -	-state-	orice tre	ee	1 PDT
pankaj_kum	ar@per.					4 0.0545	W.
-ankaj-ko	year	0	1	2	3)22	4	
ba	$\pi_0(\omega)$	1	0.4762	0.2289	0.1111	0.0545	-
			0.4762	0.4536	0.3271	0.2117	
			keley	0.2246	0.3209	0.3087	PT
		v (0)	perk		0.1050	0.2000	28 AM
	. KUM	ar				0.0486	:49:28 AM PT
bauk	(al-K			0 2220	0.4	152622	
pank	0.3	271	$= 0.5 \times$	$\frac{0.2269}{1.03}$ +	$-0.5 \times \frac{0.4}{1}$	05	
)				1.00	du	.03	
				erkeley.			
			agr@b				.0.2
		; V	KUMICA				7:49.
	nank	(9)-				. 2	2022
4	P					May	2022, 1:49:2
DOT					- 4	U.	

$$0.3271 = 0.5 \times \frac{0.2289}{1.03} + 0.5 \times \frac{0.4536}{1.05}$$

Black, Derman, and		Hull and White (1994) o			Calibration 0000●0	01	onte-Carlo simulation
	E	Exa	ample -	state-p	orice tre	е	1 PDT
pankaj_kum	ar@ber.					1:49:28 F	ZIVI .
-ankaj-ko	year	0	1	2	30221	4	
ba	$\pi_0(\omega)$	1	0.4762	0.2289	0.1111	0.0545	_
			0.4762	0.4536	0.3271	0.2117	
			berkeley	0.2246	0.3209	0.3087	70.
		v (0)	perk		0.1050	0.2000	28 AM
	. vum	are				0.0486	:49:28 AM PE
bauk	kaj_kum	271	$= 0.5 \times$	0.2289	$0.5 \times \frac{0.4}{}$	536	

$$0.0486$$

$$0.3271 = 0.5 \times \frac{0.2289}{1.03} + 0.5 \times \frac{0.4536}{1.05}$$

$$0.11111 = 0.5 \times 0 + 0.5 \times \frac{0.2289}{1.03}$$

$$0.1111 = 0.5 \times 0 + 0.5 \times \frac{0.2289}{1.03}$$
aj Kumar (2.2022) 1.49.2

Black, Derman,	and Toy (1990)		Hull and White	(1994)	Calibration ○○○○○●	Mont 00	e-Carlo simulation
	. 0\	KeEX	ample	– ZCB	prices		1 PDT
. W	war@pe				^	.49:28 An	1
Black, Derman,	year	0	1	2 _{nay}	2,2022	.49:28 AN	
	$\pi_0(\omega)$	1	0.4762	0.2289	0.1111	0.0545	
	100 14		0.4762	0.4536	0.3271	0.2117	70.
		do	ELK	0.2246	0.3209	0.3087	19:28 AM Pr
	nkaj-kur	nare			0.1050	_ \ .	19:10
03				90.71	86.42	0.0486 82.34	
1	$\angle(0,t)$	100	35.24	90.71	00.42	02.04	
) \			95.24 mar@be	weley.ed	,U		
			ogr@be	111			.0.2
		kaj-Ki	ILICA				2022, 1:49:2
	ball	IN				12V21	20-
\sim T					1.1	Mico	

M2117 2022. 1:49:2

May 2022, 1:49:2

Black, Derman, ar	nd Toy (1990)	Hull and White (1994) Example – ZCB 0 1 2			Calibration 00000●	Mon	te-Carlo simulation
	. 0	KeEX	ample	- ZCB	prices		1 PDT
, VUY	uar@be				_ ^	.49:28 Al	1,
ankaj-ka					20221		
bar	year	0	1	2,00	2, 3	4	
	$\pi_0(\omega)$	1	0.4762	0.2289	0.1111	0.0545	-
			0.4762	0.4536	0.3271	0.2117	70.
		do	ELK	0.2246	0.3209	0.3087	19:28 AM PE
	kaj-kur	Jake			0.1050	0.2000	49:10
- 1	Kaj-Ka					0.0486	
bai	Z(0,t)	100	95.24	90.71	86.42	82.34	-

 $82.34 = 100 \times (0.0545 + 0.2117 + 0.3087 + 0.2 + 0.0486)$ pankaj kumaraberkel 7 + 0.3087 + 0.2 + 0.0486

Monte-Carlo simulation If payoffs are path dependent, the valuation tree won't recombine anymore

- using the tree (or listing all paths) is $O(2^T)$ computations
- can use only a sample

Monte-Carlo simulation is same as with stock trees

- simulate n series of up and down movements ($n \times T$ computations)
- calculate payoffs and discount back through those paths
- compute average and standard deviation May 7.2022, 1:49:2 pankaj_kumar(

May 2022. 1:49:2

pankaj_kumar@berkeleyVariance reduction

12.2022, 1:49:28 AM PDT Can get faster convergence using antithetic variates: for every path simulated, generate another by 'flipping' each up and • if $q=\frac{1}{2}$, both paths drawn from the correct distribution • with most assets the values a^{-1} down movement

- correlated (one with high, one with low interest rates)
- same number of paths lead to more precise results pankaj_kumar@berkeley.e