MFE230I Section 3

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June 18, 2021

Definition of duration

- D_{Mac}, D_{mod}, $D_\$$ and $DV01$ (Dollar Value of 1 Basis Point)
- Assuming annual coupon and compounding, the yield on a bond solves the following equation

$$P = \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{1}{(1+y)^T}$$

- Macaulay duration: The average time of payment weighted by PV of each payment, then normalized by price

$$D_{Mac} \equiv \frac{1}{P}\left[\sum_{t=1}^{T} t \times \frac{c}{(1+y)^t} + T \times \frac{1}{(1+y)^T}\right]$$

$$= -\frac{1+y}{P}\frac{dP}{dy} = -\frac{dP/P}{d(1+y)/(1+y)}$$

Macaulay duration

By the formula of power series:

$$P(T) = \frac{c}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \left[\frac{1}{(1+y)^T}\right]$$

$$D_{Mac} = \frac{(1+\frac{1}{y})c[(1+y)^T - 1] + T(y-c)}{c[(1+y)^T - 1] + y}$$

Modified Duration & Dollar Duration

$$D_{\text{mod}} = \begin{cases} \frac{D_{\text{mac}}}{(1+y)} & \text{(annual compounding)} \\ \frac{D_{\text{mac}}}{(1+y/n)} & (n \text{ times per year}) \\ D_{\text{mac}} & \text{(continuous compounding)} \end{cases}$$

$$D_{\$} = D_{\text{mod}} \times \frac{P}{100} \approx -\frac{1}{100} \times \frac{\Delta P}{\Delta y}$$

$$D_{\text{eff}} = -\frac{1}{P}\frac{dP}{dy}$$

Convexity

Interest rate sensitivity of duration.

$$C = \frac{1}{P}\frac{d^2P}{dy^2}$$

$$\frac{\Delta P}{P} \approx -D_{\mathrm{mod}} \times \Delta + \frac{1}{2}C\Delta y^2$$

Applications: immunization and hedging

Immunization - insurance

- match interest rate risk of assets and liabilities

Two ways of thinking:

- Make the portfolio duration zero.
- Match the dollar duration.

Hedging Example

Sample Midterm Q2

- Semi-annual coupon bond: 2 years, 5% coupon, 4% flat yield curve.
- Hedging by zero-coupon bonds with maturities 0.5, 1 and 1.5 years.
- How to design your position?

- See excel.

Calculate duration by forming portfolio

Sample Midterm Q5(3)

- 2-year 6% annual coupon bond. Discount rate 8%
- What is the dollar duration of a newly issued forward contract to buy the bond in 5a in one year? [To clarify, when you buy the bond, it will have 1yr remaining]

Calculate duration by forming portfolio

Sample Midterm Q5(3)

- 2-year 6% annual coupon bond. Discount rate 8%
- What is the dollar duration of a newly issued forward contract to buy the bond in 5a in one year? [To clarify, when you buy the bond, it will have 1yr remaining]
- forward contract $==$ long a 2-year zero coupon bond with face value $N(1+6\%)$ and short 1-year zero coupon bond to make $CF_0 = 0$
- $ZCP_2 = 90.88$, $D_{\$2} = 2/1.08 \times 90.88/100 = 1.68$,
  $D_{\$1} = 1/1.08 \times -90.88/100 = -0.84$,
  $D_{\$} = 1.68 - 0.84 = 0.84$

Ho and Lee Binomial Model

Ho and Lee propose a binomial model for interest rate dynamics where

- interest rate volatility is constant over time (measured in basis points),
- the drift of the short rate evolves over time (but not across states), and
- the transition probability under the risk-neutral measure is constant $\frac{1}{2}$ for all times and states.

Binomial Ho-Lee tree

$$\begin{cases} \text{initial one year rate: } r \\ \text{drift: } m_t \\ \text{volatility: } \sigma \end{cases}$$

$$\begin{tikzcd}
 & & r+(m_1+m_2)+2\sigma \\
 & r+m_1+\sigma \arrow[ur, "\frac{1}{2}"] \arrow[dr] & \\
r \arrow[ur, "\frac{1}{2}"] \arrow[dr] & & r+(m_1+m_2) \\
 & r+m_1-\sigma \arrow[ur, "\frac{1}{2}"] \arrow[dr] & \\
 & & r+(m_1+m_2)-2\sigma
\end{tikzcd}$$

Calibration exercise for Ho-Lee

Calibration is choosing the model's free parameters, typically in a way to match observed prices

- ZCB prices:

Maturity	Price
1 year	95
2 years	90
3 years	85
4 years	80

- volatility of the (annually compounded) one-year rate is 1.5% per year
- See excel.

Pricing with interest rate trees

After calibration of the model, pricing is similar to pricing derivatives on stocks

- start from payoff and work backwards,
- only difference is that discounting is always with the appropriate interest rate value from the tree

For the following exercises suppose the calibrated interest rate tree is

year	0	1	2	3
r	5%	4%	3%	2%
		6%	5%	4%
			7%	6%
				8%

Zero coupon bonds

Construct the tree for a 4 year ZCB.

year	0	1	2	3
Z(t,4)	82.34	88.93	94.27	98.04
		83.99	90.71	96.15
			87.35	94.34
				92.59

Zero coupon bonds

Construct the tree for a 4 year ZCB.

year	0	1	2	3
Z(t,4)	82.34	88.93	94.27	98.04
		83.99	90.71	96.15
			87.35	94.34
				92.59

$$98.04 = \frac{100}{1.02}$$

$$94.27 = \frac{1}{1.03}\frac{1}{2}(98.04 + 96.15)$$

Options & Callable bonds

What is the price of the 4 year zero coupon bond if the issuer has the right to buy it back 2 years from now for 92? Is the price bigger or smaller?

- a long zero coupon bond and a short two year call option on that bond
- price of callable bond = price of ZCB - value of call option

What is the price of a 2-year American put option on this bond?

What is the price of a 2 year American put option but instead of trading the 4 year bond when exercising, it gives a bond with 2 years of maturity left independently of the exercise date.

Swap and Swaption

Sample Midterm Q6