MFE230I Section 5

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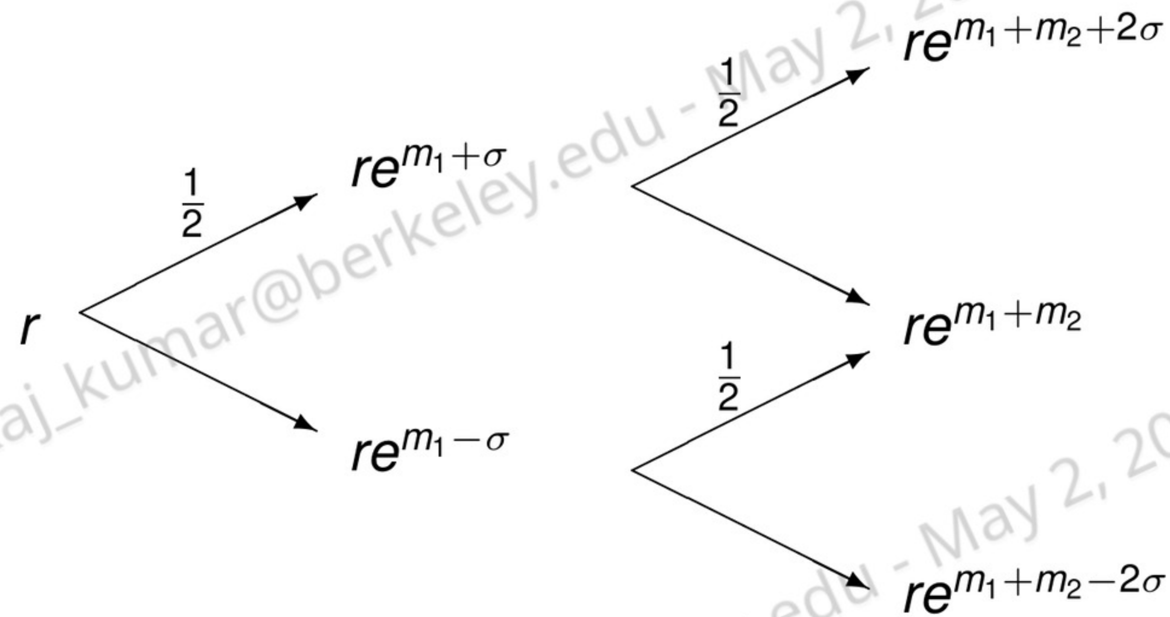
[1] Based on notes by David Echeverry, Jiakai Chen, Tamás Bátyi and Christoph Kröner. Errors are mine.

Simple BDT tree

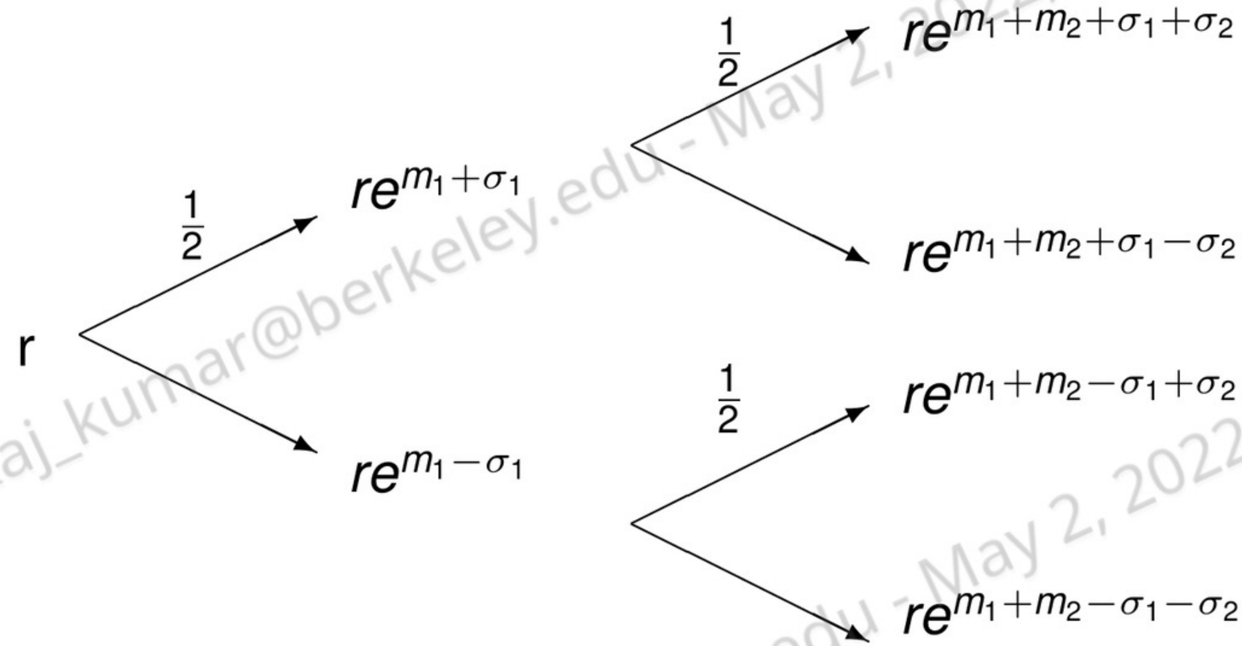
Remember the simple BDT model



Free parameters are the drifts in each period and σ, thus model can only match bond prices and one other observable (eg. volatility of short term interest rate or some cap or option price)

$\Rightarrow$ we can make σ time varying too!

The BDT tree



But this way the tree is not recombining!

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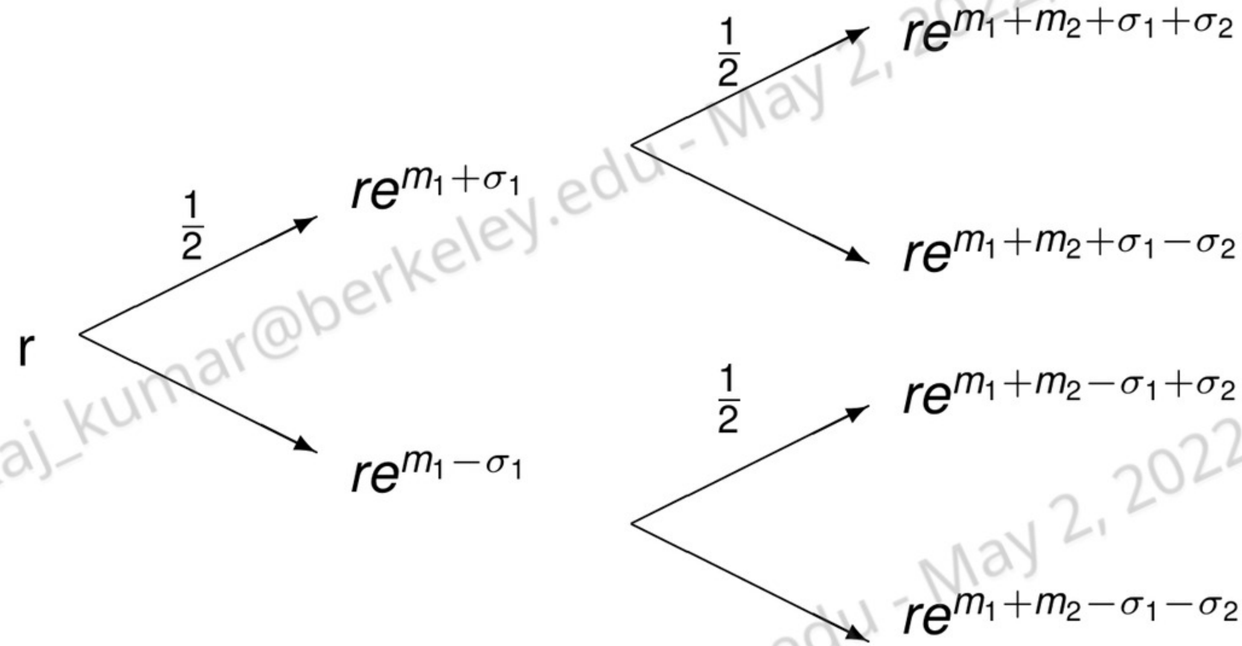

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Make $m_{2,d} = m_{2,u} + 2(\sigma_1 - \sigma_2)$

Constructing the BDT tree

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Constructing the BDT tree

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Observe that ratio of every two node above each other is the same $e^{-2\sigma_t}$!

- each period need to keep track of only the lowest rate $r_{0,t}$ and σ_t
- calibration essentially choosing these two values instead of m and σ

Continuous time limit of BDT

As $\Delta t \Rightarrow 0$ the continuous time BDT model takes the form

$$d \log r_t = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \log r_t \right) dt + \sigma(t) dW$$

Thus $\log r$ is normally distributed $\rightarrow$ r is lognormal, and the variance only depends on $\sigma(t)$

In this limiting case the existence of mean reversion depends on the form of $\sigma(t)$, what is an outcome of the calibration (we don't have control over the existence and size of mean reversion)

The Hull-White tree

The Hull and White (1994) tree is a discrete time generalization of the Vasiček (1977) model, where

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW$$

- trinomial approximation
- transition probabilities chosen to match expected value and variance

Question: what happens with probabilities if T gets large?

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Question: what happens with probabilities if T gets large?

For large T need to change branching!

Calibration

Calibration is choosing the model's free parameters to match some observable data, for example ZCB prices. A tree with T periods needs pricing of T bonds

- this would lead to T bond-price trees, each of them with $O(T^2)$ calculations $\rightarrow$ total of $O(T^3)$ calculations
- however for calibration we only need the $t = 0$ prices, not the whole price tree!
- if we compute the state-price tree, that is sufficient to obtain all $t = 0$ ZCB prices using only one tree $\rightarrow$ leading to only $O(T^2)$ calculations

Example – interest rate tree

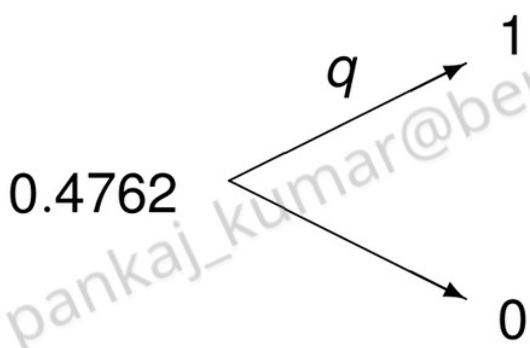
Suppose the tree for the one year rate after calibration is same as used in previous section

year	0	1	2	3
r	5%	4%	3%	2%
		6%	5%	4%
			7%	6%
				8%

How does the tree of state-prices look like?

Example – $\pi_0(u)$

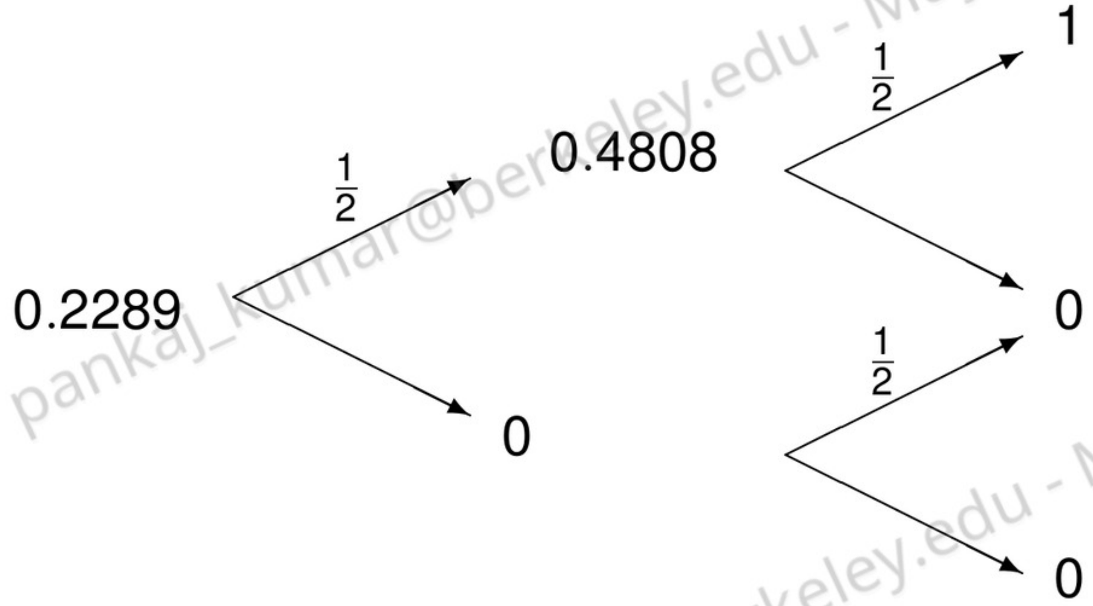
For one year state prices:



$$0.4762 = 0.5 \times \frac{1}{1.05}$$

Example – $\pi_0(uu)$

Two year state prices:



This way still need $O(T^3)$ steps, but with state prices, can do better!

Example – state-price tree

year	0	1	2	3	4
$\pi_0(\omega)$	1	0.4762	0.2289	0.1111	0.0545
		0.4762	0.4536	0.3271	0.2117
			0.2246	0.3209	0.3087
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$$82.34 = 100 \times (0.0545 + 0.2117 + 0.3087 + 0.2 + 0.0486)$$

Monte-Carlo simulation

If payoffs are path dependent, the valuation tree won't recombine anymore

- using the tree (or listing all paths) is $O(2^T)$ computations
- can use only a sample

Monte-Carlo simulation is same as with stock trees

- simulate n series of up and down movements ($n \times T$ computations)
- calculate payoffs and discount back through those paths
- compute average and standard deviation

Variance reduction

Can get faster convergence using antithetic variates: for every path simulated, generate another by 'flipping' each up and down movement

- if $q = \frac{1}{2}$, both paths drawn from the correct distribution
- with most assets the values on two path are negatively correlated (one with high, one with low interest rates)
- same number of paths lead to more precise results