MFE 230I: Problem Set 4

Continuous-time spot rate models and HJM

Due: Monday, July 19, 2021, by 9:30 a.m.

1. **Two-factor Vasicek model:** Consider the *two-factor* Vasicek model, with risk-neutral dynamics

$$
\begin{aligned}
r &= x + y, \\
dx &= \kappa_1 (\mu_1 - x)\, dt + \sigma_1\, dW_1, \\
dy &= \kappa_2 (\mu_2 - y)\, dt + \sigma_2\, dW_2, \\
dW_1\, dW_2 &= \rho\, dt.
\end{aligned}
$$

Explain why it is impossible to estimate unique values for all of the parameters κ_1, κ_2, μ_1, μ_2, σ_1, σ_2 and ρ.

Consider $x' = x + 0.01$ and $y' = y - 0.01$, so $r = x' + y'$. The dynamics of x' are given by

$$
\begin{aligned}
dx' &= dx \\
&= \kappa_1(\mu_1 - x)\, dt + \sigma_1\, dW_1 \\
&= \kappa_1(\mu_1 + 0.01 - x')\, dt + \sigma_1\, dW_1 \\
&= \kappa_1(\mu_1' - x')\, dt + \sigma_1\, dW_1
\end{aligned}
$$

where $\mu_1' = \mu_1 + 0.01$. Similarly,

$$dy' = \kappa_2(\mu_2' - y')\, dt + \sigma_2\, dW_2,$$

with $\mu_2' = \mu_2 - 0.01$.

We have two different models that generate exactly the same observed data, but have different values for μ_1 and μ_2. It is therefore impossible to estimate *unique* values for all parameters.

2. **Continuous-time Hull-White model:** The t-year discount factor (for $0 \le t \le 30$) is

$$Z(t) = e^{at+bt^2+ct^3},$$

where $a = -.04$, $b = -.001$, $c = .0001$. Assuming $\kappa = 0.15$ and $\sigma = 0.015$, fit the continuous-time Hull and White model,

$$dr = (\theta_t - \kappa r)\, dt + \sigma\, dZ^Q,$$

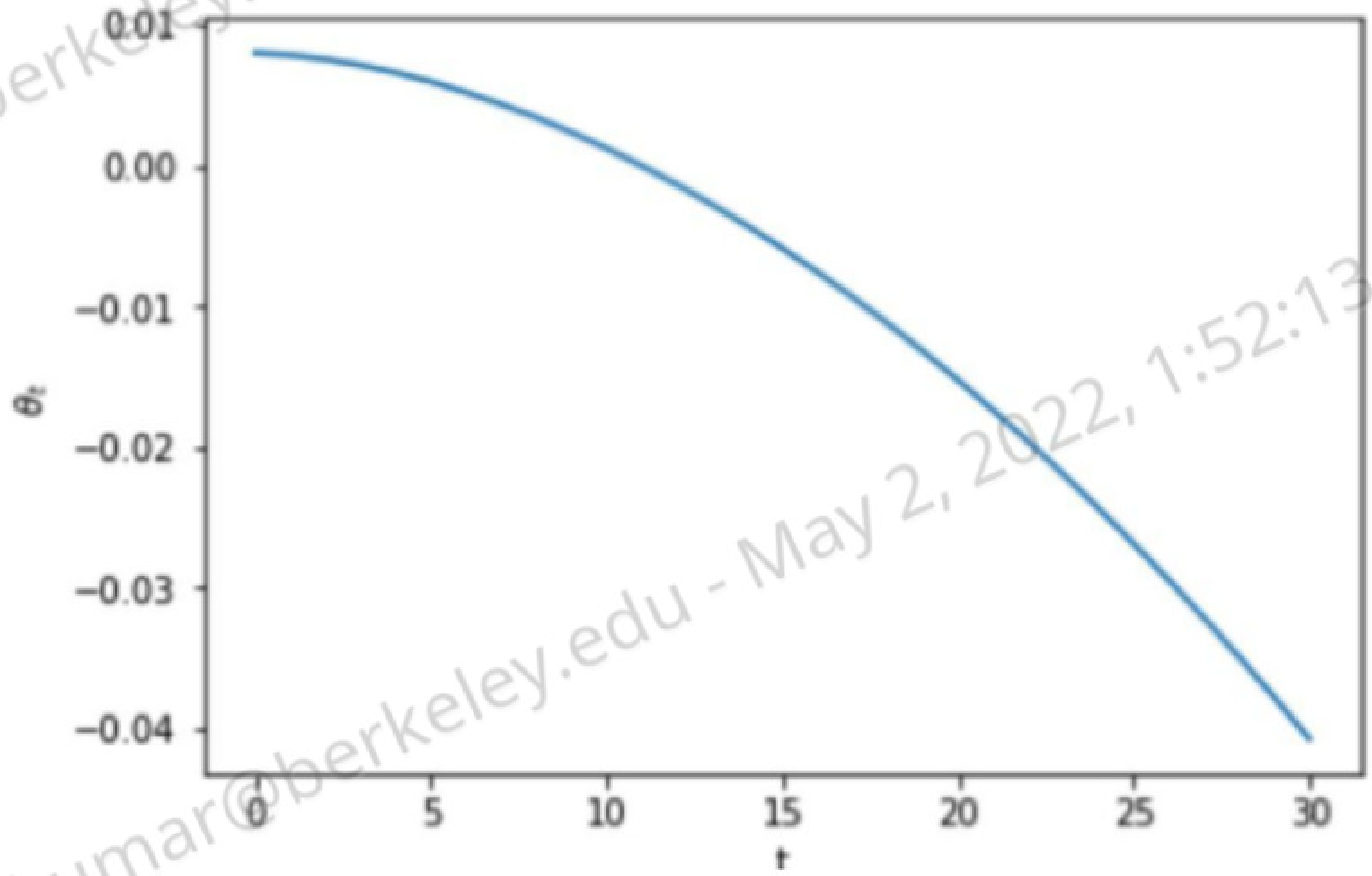
to the discount function above.

   (a) Plot θ_t against t for values of t between 0 and 30 years. **Use this estimated model for the rest of the question.**

   Using results from Veronesi,

$$\theta_t = \frac{\partial f(0,t)}{\partial t} + \kappa f(0,t) + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}) = -2b - 6ct - \kappa\left(a + 2bt + 3ct^2\right) + \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

   Here is a plot of θ_t against t:



(b) A security pays, 10 years from today, an amount equal to \$10 million times the difference between the highest and lowest 2-year rates (compounded semi-annually) observed between today and 10 years from today, i.e.,

$$\text{Payout}_{10} = \$10{,}000{,}000 \times \left(\max_{0 \le t \le 10} r_2(t, t+2) - \min_{0 \le t \le 10} r_2(t, t+2) \right).$$

i. Value the security using Monte Carlo simulation + antithetic variates. To do this, generate 10,000 total paths (5,000 independent paths plus 5,000 antithetic paths) of interest rates at monthly time intervals.

ii. What is the standard error of your estimated price?

iii. Assuming the Hull/White model you are using is the correct model for the evolution of interest rates, do you think the price you estimated in (a) is more likely to over- or understate the true value of the security? Explain why.

In this problem for <u>each</u> short rate r_t along <u>each</u> simulated path

- Calculate continuously compounded two-year rates using the formula:[1]

$$r(t, t+2) = -\frac{A(t; t+2)}{2} + \frac{(1 - e^{-\kappa 2})}{2\kappa} r_t,$$

  then convert to a semiannually compounded APR, $r_2(t, t+2)$.
- Along each path, choose the highest and the lowest value of $r_2(t, t+2)$, and use these two rates to calculate the corresponding payout at year 10.
- To use antithetic variates, for each path (5,000 in total) construct another path using the negative of each random number.

(a) For each pair of paths, take the average of the two discounted year-10 payoffs, then average across all pairs of paths. You should obtain (at least close to)

$$P = \$363{,}827.76.$$

[1]See Veronesi p. 606.

(b) Compute the sample standard deviation for each pairwise average value from (a), then divide by $\sqrt{N}$:

$$S.E. = \frac{\sigma^{average}}{\sqrt{5{,}000}} = 1{,}064.785.$$

(c) We only sample over each month instead of continuously, so the spread between minimum and maximum from our samples must be less than the true spread, and thus we underestimate the value of this security.

(c) Value a 2 year American *put* option with strike price \$100, written on a 5-year par bond that was just issued earlier today,[2] using the *implicit* finite-difference method to solve the pricing p.d.e.,

$$\frac{1}{2}\sigma^2 P_{rr} + (\theta(t) - \kappa r)\, P_r + P_t - rP = 0. \tag{1}$$

Report the price of the option for today's value of r, and also plot a graph of the option price against r, for values of r between 0 and 20%.

Here are guidelines for the implicit method.

$$P_t \approx \frac{P_{i,j+1} - P_{i,j}}{\Delta t}$$
$$P_r \approx \frac{P_{i+1,j+1} - P_{i-1,j+1}}{2\Delta r}$$
$$P_{rr} \approx \frac{P_{i+1,j+1} - P_{i,j+1} + P_{i-1,j+1}}{\Delta r^2}.$$

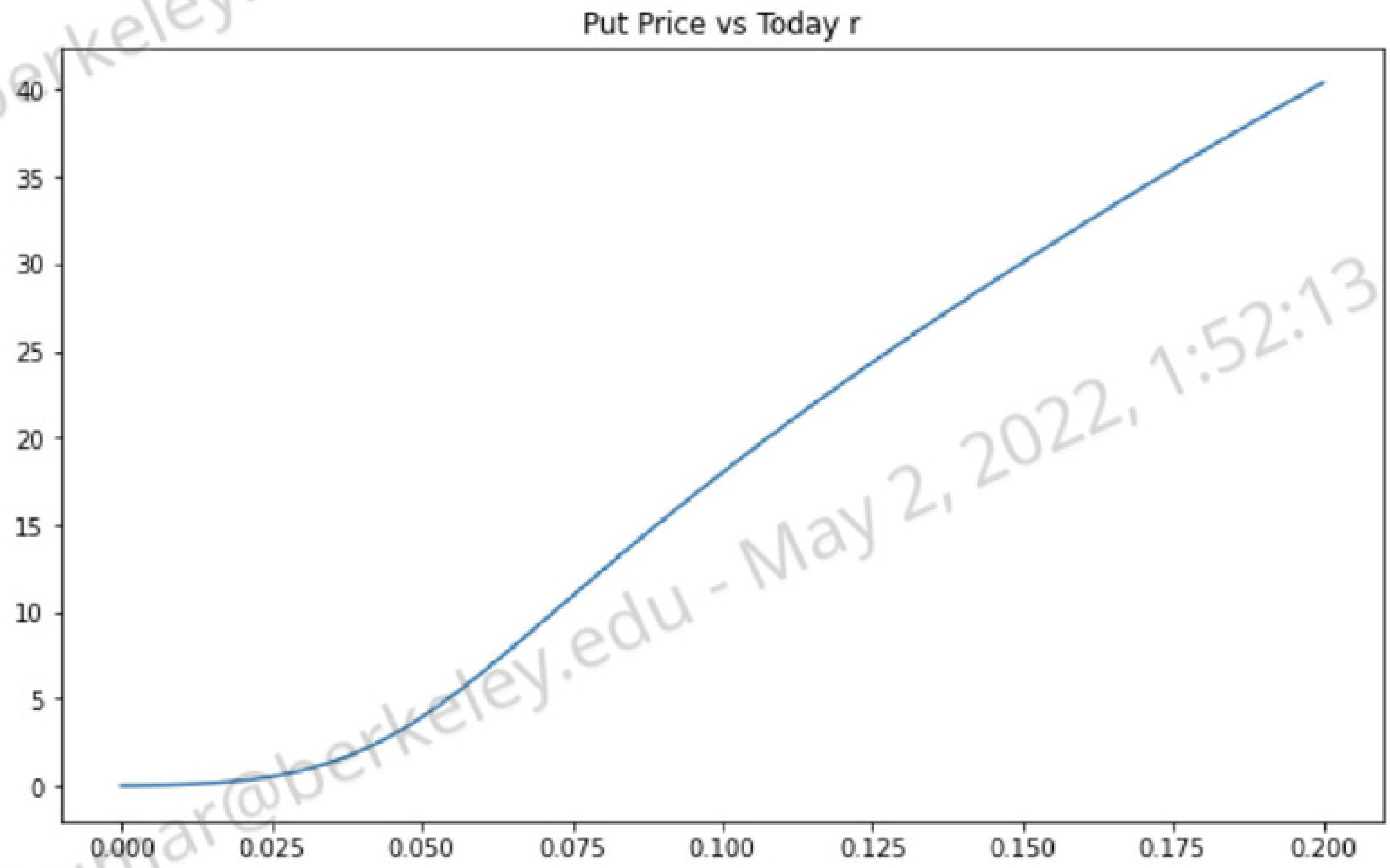
Plug-in the above approximations into the pricing p.d.e. to get:

$$\frac{1}{2}\sigma^2 \left[\frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta_r^2}\right] + (\theta(t) - \kappa r)\left[\frac{P_{i+1,j} - P_{i-1,j}}{2\Delta_r}\right] + \left[\frac{P_{i,j+1} - P_{i,j}}{\Delta_t}\right] - rP_{i,j} = 0$$

$$P_{i,j+1} = \Delta_t\left(-\frac{\sigma^2}{2\Delta_r^2} + \frac{\theta(t) - \kappa r}{2\Delta_r}\right) P_{i-1,j} + \Delta_t\left(r + \frac{\sigma^2}{\Delta_r^2} + \frac{1}{\Delta_t}\right) P_{i,j} - \Delta_t\left(\frac{\sigma^2}{2\Delta_r^2} + \frac{\theta(t) - \kappa r}{2\Delta_r}\right) P_{i+1,j}$$

The price of the option for today's value of r (= 4%) is about 2.24.
The plot of option price against r

[2] To clarify, the underlying bond remains fixed throughout the life of the option. Its coupon rate was set so that it trades at par today, but it will not in general trade at par in the future. The bond makes annual coupon payments and matures 5 years from today, so that when the option expires the bond will only have *three* years remaining. Finally, assume that the strike price is quoted as a *clean* price. In other words, when you exercise, you receive \$100 + Accrued Interest in exchange for the bond.



(d) Using Monte Carlo simulation, calculate the futures price for a contract expiring in 5 years, where the short must deliver at maturity a 4-year bond with coupon rate 4% (i.e., the bond matures 9 years from today). Assume annual coupon payments.

Based on the values for simulated short rates at $T = 5$ and the corresponding closed-form solutions for ZCBs, calculated prices of the 4-year bond with coupon rate of 4% paid semiannually at $T = 5$. The futures price is the risk-neutral expected value (no discounting) of the bond prices. Do not forget about using antithetic variates. The futures price is about \$99.6.

(e) If you were going to use a finite-difference method instead of Monte Carlo simulation to solve for the futures price, F, in 2d, what is the p.d.e. you would need to solve?

Equation (1) is equivalent to saying that the risk-neutral expected return on asset P over the next instant equals r, i.e.,

$$E\left(dP\right) = rP\,dt.$$

For a futures price, F, since its value today equals the risk-neutral expectation of its value tomorrow, the expected change over the next instant equals zero, i.e.,

$$E\left(dF\right) = 0.$$

Write the futures price as some function of r and t, $F(r,t)$, use Ito's Lemma to determine the drift and diffusion of F, and set the drift equal to zero, to obtain

$$\frac{1}{2}\sigma^2 F_{rr} + \left(\theta(t) - \kappa r\right) F_r + F_t = 0.$$

[This is the same as Equation (1) but without the $-rP$ term.]

3. **Discrete-time HJM:** You are given the following semi-annually compounded zero-coupon bond yields:

Maturity	Yield
6 months	4.70%
1 year	5.00%
1.5 years	5.50%
2 years	6.00%

As in class (where our discussion was based on Heath, Jarrow, and Morton, 1990), define $f(t,T)$ to be the continuously compounded forward rate, quoted at date t, for a forward loan between dates T and $T+\Delta$, where $\Delta = 1/2$. Assume that the one-period-ahead variance of $f(t,T)$ is given by

$$\mathrm{var}_{t-\Delta}\left(f(t,T)\right) = \sigma^2(T-t)\,\Delta,$$

where the function $\sigma(x)$ takes on the following values:

x	$\sigma(x)$
0.0	1.50%
0.5	2.10%
1.0	2.50%
1.5	2.00%

(a) As in class, construct a binomial tree (risk-neutral prob. of up/down jumps = 0.5) showing in detail the evolution of forward rates over the next 18 months.

HJM tree for f(0, 0)

$t=0$
0.0465

HJM tree for f(0, 6m)

$t=0$	$t=6m$
0.0523	0.0629
	0.0417

HJM tree for f(0, 12m)

$t=0$	$t=6m$	$t=12m$
0.0640	0.0790	0.0896
		0.0684
	0.0493	0.0599
		0.0387

HJM tree for f(0, 18m)

$t=0$	$t=6m$	$t=12m$	$t=18m$
0.0736	0.0917	0.1067	0.1173
			0.0960
		0.0770	0.0876
			0.0664
	0.0563	0.0713	0.0819
			0.0607
		0.0416	0.0522
			0.0310

(b) Using the HJM tree you just constructed, calculate the value of a one-year American put option on a two-year zero coupon bond, face value = \$100, strike price = \$93.30.

Prob	Z(0m,24m)	Z(6m,24m)	Z(12m,24m)	Z(18m,24m)	V(0m,12m)	V(6m,12m)	V(12m,12m)
0.125	88.848	88.975	90.652	94.303	4.451	4.324	2.647
0.125				95.309			
0.125			92.989	95.714			0.31
0.125				96.735			
0.125		92.897	93.649	95.985		0.402	0
0.125				97.009			
0.125			96.063	97.421			0
0.125				98.46			

4. **Continuous-time HJM:** Let $f(t,T)$ be the continuously compounded *instantaneous* forward rate quoted at date t for an "instantaneous" loan at date T. Assume the risk-neutral dynamics of the forward rates are given by

$$df(t,T) = m(t,T)\,dt + \sigma_1(t,T)\,dW_1 + \sigma_2(t,T)\,dW_2,$$

where dW_1 and dW_2 are independent and

$$\sigma_1(t,T) = 0.01,$$
$$\sigma_2(t,T) = 0.01e^{-0.1(T-t)}.$$

To prevent arbitrage, what must the drift function, $m(t,T)$, be?

The drift for forward rates implied by non-arbitrage implied by the HJM model is:

$$\begin{aligned} m(t,T) &= \sum_{k=1}^{2} \sigma_k(t,T) \int_t^T \sigma_k(t,\tau)d\tau \\ &= 0.01 \int_t^T 0.01d\tau + 0.01e^{-0.1(T-t)} \int_t^T 0.01e^{-0.1(T-\tau)}d\tau \\ &= 0.01^2(T-t) + 0.01^2 e^{-0.1(T-t)}\left[\frac{1-e^{-0.1(T-t)}}{0.01}\right] \\ &= 0.01^2\left[T-t+10(e^{-0.1(T-t)} - e^{-0.2(T-t)})\right] \end{aligned}$$

References

Heath, D., R. A. Jarrow, and A. Morton, 1990, Bond pricing and the term structure of interest rates: A discrete time approximation, *Journal of Financial and Quantitative Analysis* 25, 419–440.

Heath, D., R. A. Jarrow, and A. Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology, *Econometrica* 60, 77–105.