Name: ______________________

MFE 230I Final Exam, 2021

Monday, July 26, 9:30 a.m.–12:30 p.m.

Important Instructions — read these carefully before starting the exam:

- You agree to follow the **UC Berkeley Honor Code**, which states that "As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others." In particular, in taking this exam you are making the following commitments:
  - I alone am taking this exam.
  - I will neither receive assistance from anyone while taking the exam, nor will I provide assistance to anyone while the exam is still in progress.
  - Other than with the instructor and GSIs, I will not have any verbal, written, or electronic communication with anyone else while I or others are taking the exam.
- You may use any of the course materials in your possession (lecture notes, text book, problem sets, sample exams) to help you during the exam. You may *not* use any external internet sources or do any searches for additional information during the exam.
- Show all your work in detail to earn as much credit as possible. Just writing the solution, with no explanation, will earn few points. If you cannot answer one part of a question, you can still get full marks for the remaining parts, so don't leave anything blank.
- **Answers should be written in the space provided for each question.**
- **Do not write anything on the back side of any page.** Instead, use the (front side of the) blank pages at the end if you need more space, making it clear that you've done so.
- Points (out of **150**) are roughly proportional to the suggested number of minutes for each question. This exam contains **3** questions and **22** pages in total.
- **If taking the exam remotely,** you must be logged on to Zoom while taking the exam, with both your microphone and webcam turned on.
- **If taking the exam remotely or using a tablet/laptop to write your solutions,** you have **5 minutes** (i.e., until **12:35 p.m.**) to upload your solutions as a single PDF file to **bCourses**.

Note: You may find Ito's Lemma useful in solving this exam:

One dimension: If $dx = \mu(x)\,dt + \sigma(x)\,dW$, and $f \equiv f(x,t)$, then

$$df = \left[\frac{\partial f}{\partial t} + \mu(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2 f}{\partial x^2}\right] dt + \sigma(x)\frac{\partial f}{\partial x}\,dW.$$

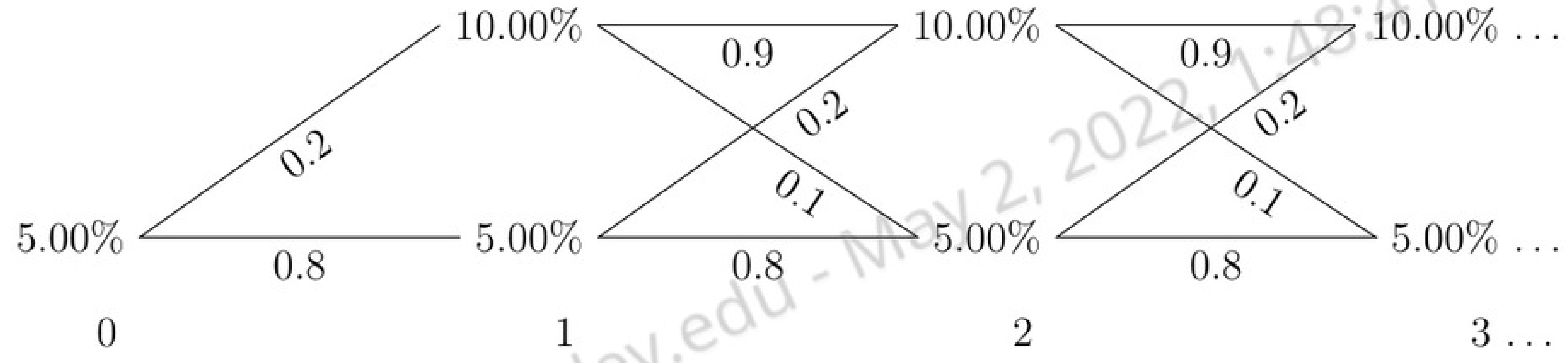
Two dimensions: If $dx = \mu(x)\,dt + \sigma(x)\,dW_1$, $dy = m(y)\,dt + s(y)\,dW_2$, where $dW_1\,dW_2 = \rho\,dt$, and $f \equiv f(x,y,t)$, then

$$df = \left[\frac{\partial f}{\partial t} + \mu(x)\frac{\partial f}{\partial x} + m(y)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma(x)^2\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}s(y)^2\frac{\partial^2 f}{\partial y^2} + \rho\sigma(x)s(y)\frac{\partial^2 f}{\partial x\,\partial y}\right] dt + \sigma(x)\frac{\partial f}{\partial x}\,dW_1 + s(y)\frac{\partial f}{\partial y}\,dW_2.$$

Unless otherwise specified in a question, assume all payments are annual, interest rates are compounded annually, and all bonds have a face value of $100.

[Questions begin on next page]

1. (**50 points.** Interest-rate trees)
The following tree shows possible values of the one-year (annually compounded) riskless interest rate, r, over the next 3 years. r is currently 5%. In all future years, r can take on one of two possible values, 5% and 10%, with (risk-neutral) jump probabilities as shown in the tree.



The tree continues indefinitely to the right, with interest-rate movements between any future years t and $t+1$ governed by the same probabilities as between years 1 and 2.

(a) What is the value of a 3-year zero-coupon bond today?

(b) What is the value of a one-year European call option on the bond in 1a, with strike price \$85?

(c) What is the forward price today for delivery in one year of a zero-coupon bond maturing 3 years from today?

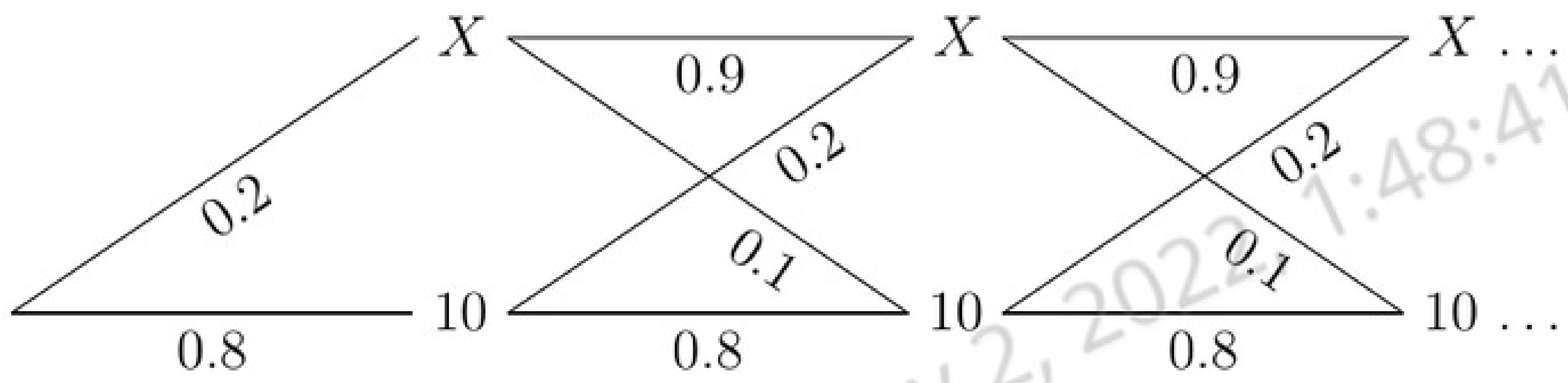
(d) What is the *futures* price today for delivery in one year of a zero-coupon bond maturing 3 years from today, assuming marking to market occurs once per year?

(e) If you own one of the call options in 1b, what position in the futures contract in 1d do you need to hedge yourself against interest rate movements over the next year?

(f) What is the value today of a security that makes a one-time payment in year 4 equal to \$1,000 $\times$ the average value of r_t observed up to and including year 3? I.e., four years from today it pays out the amount

$$\$1{,}000 \times \frac{(r_0 + r_1 + r_2 + r_3)}{4}.$$

(g) A security pays out \$10 each year that the interest rate is 5%, and \$X each year that the interest rate is 10%, starting one year from today and going on forever. I.e., the cash flows look like this:



You look at the valuation tree for this asset, and notice that its price (calculated immediately *after* each year's payment has just been made) is the same in *every* future node of the tree (in particular, the value is the same regardless of whether the interest rate is 5% or 10%). Determine X and the value of the asset today.

(h) What is the value today of a perpetuity paying \$1 every *two* years, with the first payment two years from today?

2. (**50 points.** Ho and Lee)
Assume the instantaneous (continuously compounded) riskless rate r_t follows the risk-neutral process

$$dr = \mu\, dt + \sigma\, dW_t^Q,$$

and define $M_T = \int_t^T r_s\, ds$.

(a) Derive the distribution of M_T under Q, conditional on $r_t = r$.

(b) We can write the price of a zero-coupon maturing at date T as

$$Z(t,T) = E_t^Q\left[e^{-M_T}\right].$$

Calculate $Z(t,T)$ by explicitly evaluating this expectation, using the result you obtained in 2a.

(c) Derive the *joint* distribution of $(r_T, M_T)'$ under Q, conditional on $r_t = r$.

(d) Consider a European call option with expiry T_O on a zero-coupon bond with maturity $T_B > T_O$. We can write the value of this security at date $t \leq T_O$ in the form

$$C(r,t) = E_t^Q \left[e^{-\int_t^{T_O} r_s \, ds} \max\left(Z(T_O, T_B) - K, 0\right) \right].$$

Calculate $C(r,T)$ by explicitly evaluating this expectation, using the result you obtained in 2c. You may find the following result useful:

Theorem 2 from Lien (1985): If $(x,y)'$ are bivariate lognormal random variables with

$$\begin{pmatrix} \log x \\ \log y \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 t \end{pmatrix} \right],$$

then for any $A > 0$ with $B = \log A - \mu_1$,

$$E(y \mid x \geq A) = \frac{d\,\Phi\left(\frac{\sigma_{12} - B}{\sigma_1}\right)}{\Phi\left(\frac{-B}{\sigma_1}\right)}, \qquad \text{where}$$

$$d = e^{\mu_2 + \frac{\sigma_2^2}{2}}.$$

For the rest of this question, assume $\mu = 0$.

(e) What is the value of a security with a payoff 3 years from today of $\$1{,}000 \times Z(3,5)^3$ (where $Z(3,5)$ is, as usual, the value 3 years from today of a zero-coupon bond with face value \$1 and maturity date 5 years from today)?

(f) What is the value of a security that makes a continuous payout at rate r_t between today and date T (i.e., in each small time interval $[\tau, \tau + \delta t]$, the security makes a payment of approximately $r_\tau \, \delta t$), and then in addition makes a payment of \$1 at date T?

(g) What is the value of a perpetuity, i.e., a security that makes a continuous payout at a rate 1 (i.e., in each small time interval $[\tau, \tau + \delta t]$, the security makes a payment of δt)?

(h) You want to price a security with a single payoff $g(r_T, T)$ at date T using Monte Carlo simulation (where g is some function that is easy to calculate, but complicated enough to prevent pricing the asset in closed form). Describe in detail a pricing algorithm for this asset that is as efficient as possible (i.e., you want to obtain the largest possible number of digits of precision from running your algorithm for any given, fixed, amount of time).

(i) How would your answer to 2h change, if at all, if the payoff were a function of the entire path of r, not just its final value?

3. (**50 points.** Annuities and adjustable-rate mortgages)

(a) You take out a T-year loan in the amount of B_0. The loan requires you to make *continuous* payments at a constant rate x per unit time. I.e., in any small period of length δt between dates 0 and T, you pay $x\,\delta t$. If the loan's quoted interest rate is c (continuously compounded), derive an expression for the value of x.

(b) What is the remaining balance on the loan at date $t \in [0, T]$, B_t?

Now let us value a simplified **adjustable-rate mortgage (ARM)** with initial principal amount B_0 and T years to expiration.

Riskless interest rate, r_t: Assume the continuously compounded, instantaneous risk-free rate follows the risk-neutral process

$$dr = \sigma \, dW^Q.$$

Mortgage rate, c_t: At every date $t \in [0,T]$, the interest rate on the mortgage is set equal to the then-current riskless rate, r_t, subject to a *cap* of 6% and a *floor* of 2%. In other words the mortgage rate at time t is set to

$$c_t = c(r_t) \equiv \begin{cases} 2\% & \text{if } r_t \leq 2\%, \\ r_t & \text{if } 2\% < r_t \leq 6\%, \\ 6\% & \text{if } r_t > 6\%. \end{cases}$$

Mortgage payment, x_t: The mortgage makes *continuous* payments, as in 3a. The payment rate for the next instant at each date t, x_t, is set equal to the payment rate you would have calculated in 3a for a (fixed-rate) loan in the amount B_t with continuously compounded interest rate c_t and remaining time to maturity $T - t$.

Prepayment: Assume no early prepayment is allowed on the mortgage, i.e., the borrower will continue to make the scheduled payments until the loan is fully paid off at date T.

(c) Derive the stochastic process followed by the loan balance, B_t. I.e., determine the values of C and D (each of which might be functions of any of the other variables in this problem) in the expression

$$dB_t = C \, dt + D \, dW.$$

(d) Write the value of the ARM at date t as $V(r, B, t)$. Using the fact that the risk-neutral expected return on the security over the next instant must equal $r_t\, dt$, use Ito's Lemma (see page 1) to derive a partial differential equation (p.d.e.) that must be satisfied by $V(r, B, t)$.

(e) Consider two different ARMs that are identical except that one has a current total loan balance B_t and one has a current loan balance $B_t/2$. What can you say about the relationship between the prices on these two ARMs?

(f) Using the result from 3e combined with the p.d.e. from 3d, derive a simpler p.d.e. that can be used to price the ARM more easily. [The simpler the equation, the better. For example, a p.d.e. in one variable plus time is better than a p.d.e. in two variables plus time. An o.d.e. is better still.]

(g) Describe how you would solve this equation numerically.

References

Lien, Da-Hsiang Donald, 1985, Moments of truncated bivariate log-normal distributions, *Economics Letters* 19, 243–247.