Suggested Solutions for MFE 230I Sample Midterm

1. (a) The price of a 2-year bond is given by its present value:

$$\frac{3}{(1+0.06)} + \frac{100+3}{(1+0.06)^2} = 94.5.$$

(b) To compute the 2-year spot rate, we need to know the price of a 2-year zero-coupon bond. Consider the following replicating portfolio:

$$5 \times 3\%\text{-coupon bond} - 3 \times 5\%\text{-coupon bond} = 2 \times \text{zero-coupon bond}$$

The price of the 5% coupon bond is

$$\frac{5}{(1+0.062)} + \frac{100+5}{(1+0.062)^2} = 97.8.$$

It follows from the principle of no-arbitrage that the price of a 2-year coupon bond must be

$$5 \times 94.5 - 3 \times 97.8 = 2 \times \text{Price of zero-coupon bond}$$

So its price is 89.54. The 2-year spot rate, r, is therefore the solution to the equation

$$\frac{1}{(1+r)^2} = 89.54$$

The answer is 5.68%.

(c) True. Recall from class that

$$\frac{\partial r(t,T)}{\partial T} = \frac{1}{T-t}(f(t,T) - r(t,T)).$$

When the forward rate is increasing, it is always higher than the corresponding spot rate, so this expression is positive, and hence the spot rate is also increasing.

A more intuitive, discrete-time argument (which doesn't rely on continuous compounding) is merely to note that the spot rate is (some kind of) average of the forward rates. If the forward curve is increasing, then the spot rate r_{t+1} is an average of all of the forward rates used to calculate r_t, plus the additional forward rate f_{t+1}, which is the highest so far. Thus r_{t+1} must be higher than r_t. This could be proved formally, e.g., using induction.

(d) False. The instantaneous forward rate, $f(t, T_1)$ is given (see class notes) by

$$f(t,T_1) = r(t,T_1) + (T_1 - t)r'(T_1)$$

where r is the spot rate. Take the derivative of both sides with respect to T_1:

$$f'(t,T_1) = 2r'(t,T_1) + (T_1 - t)r''(T_1).$$

Without any restriction on the sign or magnitude of r'', we can draw no conclusion. Alternatively, see the counter example in lecture 1. If $f > r$, r is increasing even if f is decreasing.

(e) Use the formula from part (d) to get

$$\begin{aligned} f(t,T) &= r(t,T) + (T-t)r'(T) \\ &= r(t,T) + (T-t) \times 0.3\% \end{aligned}$$

Substituting T = t+3, the solution is $4\% + 3 \times 0.3\% + 3 \times 0.3\% = 5.8\%$.

(f) We solve for the spot rate recursively. We can derive the par rate from the bond price equation with annual coupon rate C, frequency of compounding k, and maturity T. The price (present value) of this bond is

$$P = \frac{C}{k} Z\left(\frac{1}{k}\right) + \frac{C}{k} Z\left(\frac{2}{k}\right) + \ldots + \frac{C}{k} Z\left(T - \frac{1}{k}\right) + \left(\frac{C}{k} + P\right) Z(T)$$

The par rate is the coupon rate C that solves the equation when the bond price P = 1. Given three par yields and $k = 2$, we can progressively solve for $Z(\frac{i}{k})$ for $i = 1, 2, 3$ and therefore the spot rate. The solutions are

$$\begin{aligned} r(0.5) &= 0.05 \\ r(1) &= 0.0602 \\ r(1.5) &= 0.0705 \end{aligned}$$

(g) False. Even when the counterparties are riskless, the floating rate is LIBOR, which is not default-free. The fixed rate has to be higher than the risk-free rate to compensate for the fact that the floating rate is higher than the risk-free rate.

2. (a) The Macaulay duration of the 5% coupon (c), 2 year (T) bond with yield to maturity (y) of 4% and face value (F) of \$100 is given by

$$\left(\frac{\sum_t^T \frac{c}{2} \times F \times t/(1+y/2)^{2t} + F \times t/(1+y/2)^{2t}}{\sum_t^T \frac{c}{2} \times F/(1+y/2)^{2t} + F/(1+y/2)^{2t}} \right) = 1.93.$$

The Modified duration is just the Macaulay duration divided by $(1 + y/2)$, which equals 1.89.

(b) The convexity of the bond is given by the following formula:

$$\left(\frac{\sum_t^T \frac{c}{2} \times F \times t \times (t+1/2)/(1+y/2)^{2t} + F \times t \times (t+1/2)/(1+y/2)^{2t}}{\sum_t^T \frac{c}{2} \times F/(1+y/2)^{2t} + F/(1+y/2)^{2t}} \right) \times \frac{1}{(1+y/2)^2} = 4.578.$$

(c) Remember that

$$\frac{\Delta P}{P} \approx \frac{-D_{mac}}{1+y/2} \Delta y + \frac{1}{2} C \, \Delta y^2.$$

Given $\Delta y = 0.25\%$ and using the values calculated in part a and part b, we have $\frac{\Delta P}{P} = -0.00471$ or 0.471%.

(d) With the three zero-coupon bonds in our portfolio, we can match the value, the duration and the convexity of the two-year bond we would like hedged. Suppose $A = 0.5$ yr ZCB, $B = 1$ yr ZCB and $C = 1.5$ yr ZCB.

For hedging,

$$\begin{aligned} n_A P_A + n_B P_B + n_C P_C &= P_{2yr} \\ n_A D_A + n_B D_B + n_C D_C &= D_{2yr} \\ n_A C_A + n_B C_B + n_C C_C &= C_{2yr} \end{aligned}$$

where P is price, D is dollar duration and C is dollar convexity. Solving this system of three equations and three unknowns gives us $n_a = 0.99, n_b = -2.93$ and $n_c = 3.04$.

(e) Similar to the inverse-floater example in class, we can deduce that the following relationship must hold:

$$50\,D(\text{Inverse Floater}) + 100\,D(\text{Floater}) = 150\,PV(\text{Fixed}; c = 20\%/3).$$

The PV of a 20%/3 coupon bond is 157.6155 and its Macaulay duration is 1.9. The duration of the floater is 6 months or 0.5. Now following the example in class, we solve the equation

$$1.9 = \left(\frac{100}{157.62} \times 0.5\right) + \left(\frac{157.62 - 100}{157.62} \times D_{mac}(IF)\right)$$

Solving, we obtain $D_{mac}(IF) = 4.35$.

3. (a) The value of the 1 year bond is \$95.198, so we can solve the following equation for X:

$$0.9518 = \frac{0.42 \times \frac{1}{1+X/2} + (1 - 0.42) \times \frac{1}{1+0.04/2}}{1 + 0.05/2}.$$

This returns $X = 6.31\%$. Once we have X, we may solve for Y by matching the value of the 18 month bond. You should get approximately 4.84% for Y. The value of Z is calculated straighforwardly from the 6 month bond as $1/(1 + 0.05/2) = 97.56$.

The final interest rate tree looks like this:

5.00%	6.31%	7.00%
	4.00%	4.84%
		3.00%

(b) While there are many ways to solve this, one can think of a callable bond as a non-callable bond minus a call option. The non-callable component is simply a coupon bond evaluated along each interest rate path and has a value of 99.9062, calculated as follows:

99.90617	98.68102	99.03382	102.50
	100.7893	100.0784	102.50
		100.9852	102.50

The call option is an American option with expected payout equal to $\mathbf{max}\,(NonCallable - 100, 0)$, where 100 is the par value of the bond, discounted along the interest rate path. The value of the call option is 0.4544, calculated from the following tree:

0.454421	0.018998	0
	0.789313	0.078392
		0.985222

The value of the callable bond is the difference between these values, 99.4518:

99.45175	98.66202	99.03382
	100	100
		100

Note that you can construct the callable bond tree in one step, replacing each value with \$100 whenever you calculate a value greater than \$100.

(c) Note that the Asian option is path-dependent, and the average interest rate depends on the path followed to maturity. Let us index the final period payoffs as uu, ud, du and dd, where uu indicates that the interest rate followed the up node in both periods, ud indicates that the interest rate followed the up node in the first period and then the down node in the next period and so on.

State	Probability	Average r (%)	Payout, \$1M $\times \max(\bar{r} - K, 0)$
uu	$0.42 \times 0.75 = 0.315$	$(7 + 6.31 + 5)/3 = 6.10$	11037.09
ud	$0.42 \times (1 - 0.75) = 0.105$	$(4.84 + 6.31 + 5)/3 = 5.38$	3835.17
du	0.435	4.61	0
dd	0.145	3.10	0

Note that the average interest rate need not be above the strike of 5% in all states (it is not when the states of du and dd occur). The probability of each state and the corresponding payout is calculated in table 3. Since this is a futures contract, the current futures price is simply the expected value of the final payouts *without discounting*. Thus the current futures price for this contract is given by $0.315 \times 11037.09 + 0.105 \times 3835.17 = 3879.38$.

(d) To calculate the position in the futures contract we calculate the hedge ratio between the callable bond and the futures. For this we need the Deltas of each:

$$\Delta_{CallBond} = \frac{C_u - C_d}{r_u - r_d} = \frac{98.66 - 100}{0.0631 - 0.04} = -57.89$$

$$\Delta_{Option} = \frac{O_u - O_d}{r_u - r_d} = \frac{9236.61 - 0}{0.0631 - 0.04} = 399658.2$$

To hedge the callable bond with the option, we need to hold $-\frac{\Delta_{CallBond}}{\Delta_{Option}} = 0.00014$ Asian futures option (a long position).

4. (a) This statement is false. A callable bond is a combination of a non-callable bond minus a call option. The value of the non-callable bond falls as the interest rate rises and the value of the call option increases as the volatility increases. Thus the value of the callable bond falls with rising interest rates and volatilities.

(b) This statement is true. As interest rates decrease, prepayments become more likely. This will decrease future cash flows to the Interest Only securities. At low interest rates, this can cause the duration of the IO to be negative. At high rates, however, prepayment is less of an issue, and a decrease in interest rates causes the value of the security to increase, as with most bonds.

(c) This statement is false. More volatility has an ambiguous effect. All else equal, more volatility makes it more valuable to keep the call alive, which increases the expected life. However more volatility also makes it more likely that the call boundary will be reached sooner.

(d) This statement is true. We saw in class that duration for long-maturity bonds can be smaller than duration for shorter-maturity bonds, as long as they are coupon bonds with coupon rates below the

YTM. The higher duration, the more prices move when rates move, and thus the higher the price volatility.

(e) This statement is false, at least for the models we have seen in class. A simple argument is that the duration of a ZCB is always equal to maturity, so higher maturity implies a higher duration and hence a higher volatility. However, the full answer is a bit more subtle, since the volatility of the short vs. long interest rates may be different. As a simple counterexample, consider a simple binomial model where the one-period rate is 5% today, may be 2% or 8% next year, and in the following year will always be (10% − the rate in period 1) (i.e., if the rate in period 1 is 2%, the rate in period 2 will be 8%, and vice versa). Then a three-period zero-coupon bond today will be worth

$$\frac{100}{1.02}1.08 = \$90.777$$

in *either* state of the world next period, i.e., its volatility over the next period is zero. By contrast, a *two*-year bond will be worth either $100/1.02 = \$98.039$ or $100/1.08 = \$92.593$, so its volatility is strictly greater than that of the three-year bond.

5. (a) The Macaulay duration of the 6% coupon (c), 2 year (T) bond with yield to maturity (y) of 8% and face value (F) of \$100 is given by

$$\left(\frac{\sum_t^T c \times F \times t/(1+y)^t + F \times t/(1+y)^t}{\sum_t^T c \times F/(1+y)^t + F/(1+y)^t}\right) = 1.94$$

(b) The convexity of the bond is given by the following formula:

$$\left(\frac{\sum_t^T c \times F \times t \times (t+1)/(1+y)^t + F \times t \times (t+1)/(1+y)^t}{\sum_t^T c \times F/(1+y)^t + F/(1+y)^t}\right) \times \frac{1}{(1+y)^2} = 4.95$$

(c) Assume the bond is to be bought after the coupon is paid at T = 1. The forward contract is equivalent to

- Buying one bond today. Price = \$90.88, dollar duration = $2/1.08 \times 90.88/100 = 1.68$.
- Borrowing \$90.88 for one year, i.e., shorting a one-year zero coupon bond. Price = -\$90.88, dollar duration = $1/1.08 \times -90.88/100 = -.84$.

Overall dollar duration of forward contract is $1.68 - 0.84 = 0.84$.

(d) PO has a higher effective duration. When the interest rate declines, the prices of both securities increase due to the discount effect. For the PO, however, there's an additional effect from prepayments. As rates fall, prepayments go up so the cash flows to the PO occur earlier, raising price further.

(e) The price of this annuity is

$$\begin{aligned} P &= \frac{1}{1+y}\sum_{t=1}^{T}\left(\frac{1+g}{1+y}\right)^t \\ &= \frac{1}{1+y}\frac{1-\left(\frac{1+g}{1+y}\right)^T}{1-\frac{1+g}{1+y}} \\ &= \frac{1-\left(\frac{1+g}{1+y}\right)^T}{y-g} \end{aligned}$$

Take the derivative with respect to y and multiply by $(1+y)$:

$$(1+y)\frac{dP}{dy} = \frac{1}{1+y}\sum_{i=0}^{T-1}(i+1)\left(\frac{1+g}{1+y}\right)^i$$
$$= \frac{(1+y)\frac{1-\left(\frac{1+g}{1+y}\right)^T}{y-g} - T\left(\frac{1+g}{1+y}\right)^T}{y-g}$$

The Macaulay duration is therefore

$$\frac{1+y}{P}\frac{dP}{dy} = \frac{1+y}{y-g} - T\frac{\left(\frac{1+g}{1+y}\right)^T}{1-\left(\frac{1+g}{1+y}\right)^T}$$

(f) You could answer this question by directly taking the limit of the expression you just calculated, but this turns out not to be too much fun. A simpler way to answer this question is to note that when $g = y$, the present value of each cash flow from the annuity is the same. Thus the Macaulay duration is a simple (unweighted) average of the time to each payment,

$$\frac{1}{T}\sum_{i=1}^{T} i = \frac{1}{T}\times\frac{T(T+1)}{2} = \frac{T+1}{2}.$$

6. (a) Note that

$$\text{Puttable Bond} = \text{Vanilla Bond} + \text{Put Option}$$

The price of the bond is given by

101.16	98.68	98.56	103.00
	102.69	100.49	103.00
		101.98	103.00
			103.00

and that of the put option by

0.64	1.32	1.44	0.00
	0.00	0.00	0.00
		0.00	0.00
			0.00

So the price of the puttable bond is 101.80.

(b) Assume that payment is made in arrears. First calculate the cash flows to the fixed receiver:

0.50%	-0.90%
	1.50%

Now value them from the perspective of the fixed receiver by discounting back, adding in cash flows as you go:

0.79%	-0.87%
	1.48%

The price is 0.79.

(c) At T = 1, the party paying fixed will cancel the swap when the interest rate is lower than the fixed rate. At T = 0, the party paying fixed will cancel the swap if the cash flow from canceling (i.e., 0) is greater than the value of waiting (i.e., paying the T = 0 interest rate differential + the discounted expected value of the swap in the next period.) To the fixed payer, the value of the cancelable swap is given by

-0.065	0.87
	0.00

The value of waiting is -0.065 (it is 0.065 to the fixed receiver, but he doesn't control the option). The party paying fixed will cancel the swap.

(d) The value of the cancelable swap is positive (1.21) to the party receiving fixed, so she will not cancel it.

1.21	0.00
	1.48