MFE 230I: Fixed Income
Topic 1: Introduction to Fixed Income

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U.C. Berkeley

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Outline

MFE 230I: Fixed Income

- Instructor: Richard Stanton
  - Office Hours: Monday, 3:00–4:00.
- GSIs:
  - Can Huang (can_huang@haas.berkeley.edu).
  - Yunbo Liu (liuyb@berkeley.edu).
  - Office hours: Thursday, 2:00–3:00.
- Discussion sections: Friday, 2:00–4:00 p.m.
- Check MFE calendar for details.

About me

- Professor of Finance and Real Estate; Kingsford Capital Management Chair in Business.
  - BA/MA in mathematics from Cambridge University.
  - Ph.D. in Finance from Stanford.
- Consult on fixed income and mortgage pricing.
- Director of Montgomery Street Income Securities, a fixed-income closed-end fund.
- Hedge fund collapses in 1998 caused by my taking a job at Salomon fixed-income arbitrage group in New York!
- Teach undergraduate, MBA, MFE and PhD students.
  - Three-time winner of Cheit teaching award at Haas.
- Research interests: Mortgage and lease markets; interest rates and term structure modeling; financial institutions and risk management; employee stock options.

Course Organization

- Instruction this summer will be in two phases:
  1. Phase 1, through June 15: 50% classroom capacity. Attend based on pod no.
  2. Phase 2, June 16 onward: 100% classroom capacity. All welcome to attend in person.

- When attending in person, show green status badge and complete symptom screen.
  - Detailed instructions in May 21 email from MFE Office.

- When attending via Zoom,
  - Treat Zoom class meetings as if attending class on campus.
  - You must have and use a camera and microphone for all class meetings.
  - Ensure you have a fast, stable internet connection in a quiet location.
  - Do not join the session from a cell phone or in the car.
  - Only log on from one device.
  - Use a headset rather than desktop speakers.

Course Organization

- Lecture: I will share a screen showing the lecture notes. Annotated notes will be posted after each topic.

- GSI discussion sections: Let the GSIs know at least a day beforehand (either via Slack channel `#mfe230i` or by emailing mfe230i@berkeley.edu) if you want them to cover a certain topic or answer specific questions.

- Office hours: Office hours will be organized in 20 minute sessions for groups of 5 students; you will need to sign up on a Google sheet beforehand.

Communication

- Outside class: Post questions/comments to the Slack channel `#mfe230i`.
  - Conversation can be seen and responded to by the entire class, the GSIs, and me.
- Private communication: Email sent to mfe230i@berkeley.edu will only be seen by the GSIs and me.

Course texts, etc.

- Required textbook (will also be used in 230M)
  - Pietro Veronesi, *Fixed Income Securities,* Wiley, 2010.
- Recommended optional texts:
  - John C. Hull, *Options, Futures and Other Derivatives*, 10th edition, Prentice Hall, 2017.
  - Bruce Tuckman and Angel Serrat, *Fixed Income Securities*, 3rd edition, Wiley, 2011.
  - Jessica James and Nick Webber, *Interest Rate Modelling*, Wiley, 2000.
  - Andrew J. G. Cairns, *Interest Rate Models: An Introduction*, Princeton University Press, 2004.
  - Brigo and Mercurio, *Interest Rate Models*, 2nd ed., Springer, 2006.
  - Andersen and Piterbarg, *Interest Rate Modeling*, Atlantic Financial Press, 2010.
- Lecture notes:
  - Available a few days prior to class on `https://berkeley.app.box.com`.
- Readings:
  - Available on Box.com.
- We'll be using and building on material from 230A, 230E and 230Q.

Course objectives

1. Understand state-of-the-art models and techniques for pricing and hedging fixed-income securities.
   - Basic concepts: yield, duration, convexity.
   - Numerical and analytical techniques to describe Treasury and corporate-bond data, e.g., "curve fitting," principal components.
   - Modern valuation models for fixed income instruments: e.g., Ho and Lee; Hull and White; Black-Derman-Toy; Vasicek; Cox, Ingersoll and Ross; Heath, Jarrow and Morton; LIBOR market model; SABR.

2. Build a toolbox of practical code implementing these models.
   - Fundamental knowledge on day 1 for any position you will take.
   - Also crucial for later MFE courses, especially 230H (Risk Management), 230M (Asset-Backed Securities).
   - Homeworks are a core part of the course.

Course requirements

Weekly Problem Sets

- Group work is encouraged — max. 4 students per group.
- Hand in one solution (write all the names on the solution).
- Do not split class assignments between group members!
- Lots of work, but this is the guts of the course.
- Problem sets will require programming in Python.

Exams

- Midterm: Details TBA.
- Final: Monday, July 26 from 9:30 a.m.–12:30 p.m.

Grade breakdown

- 30% Homework
- 30% Midterm exam
- 40% Final exam
- Note: Attendance and class participation may affect grades in borderline cases (e.g., to discriminate between an A− and a B+).

(Approximate) course outline

1. Introduction to fixed income
2. Duration and convexity
3. Tree-based (spot rate) models of interest rates
4. Calibrating interest-rate trees and Monte Carlo simulation.
5. Continuous-time spot rate models
6. Monte Carlo simulation and continuous-time models.
7. Numerical methods for pricing interest-rate derivatives
8. Heath, Jarrow and Morton and modeling forward rates
9. LIBOR market model (BGM, LMM)
10. Credit risk

Reading assignments for this topic

- Required: Veronesi, Chapters 1, 2 and 5.
- Additional resources:
  - Nelson and Siegel, “Parsimonious Modeling of Yield Curves,” Journal of Business 60, 473–489, 1987.
  - Svensson, “Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994,”, NBER Working paper 4871, 1994.
  - McCulloch, “The Tax-Adjusted Yield Curve,” Journal of Finance 30, 811–830, 1975.
  - Bloomberg, “Building the Bloomberg Interest Rate Curve: Definitions and Methodology,” 2012.

Outline

Introduction

Overview of Fixed-Income Markets

“Bond Math”

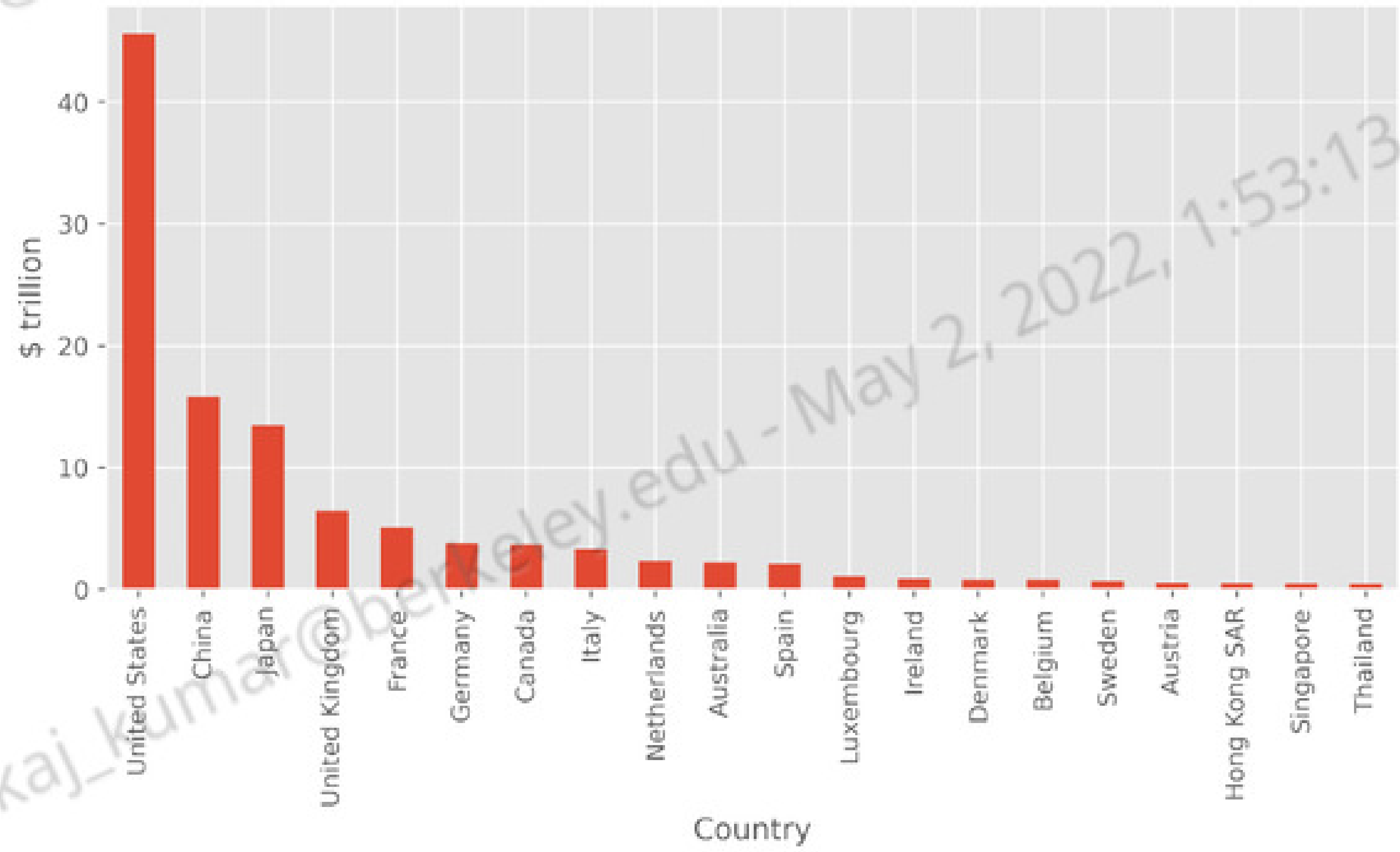
Day Count Conventions

Fitting the Yield Curve

Basic Fixed-Income Securities

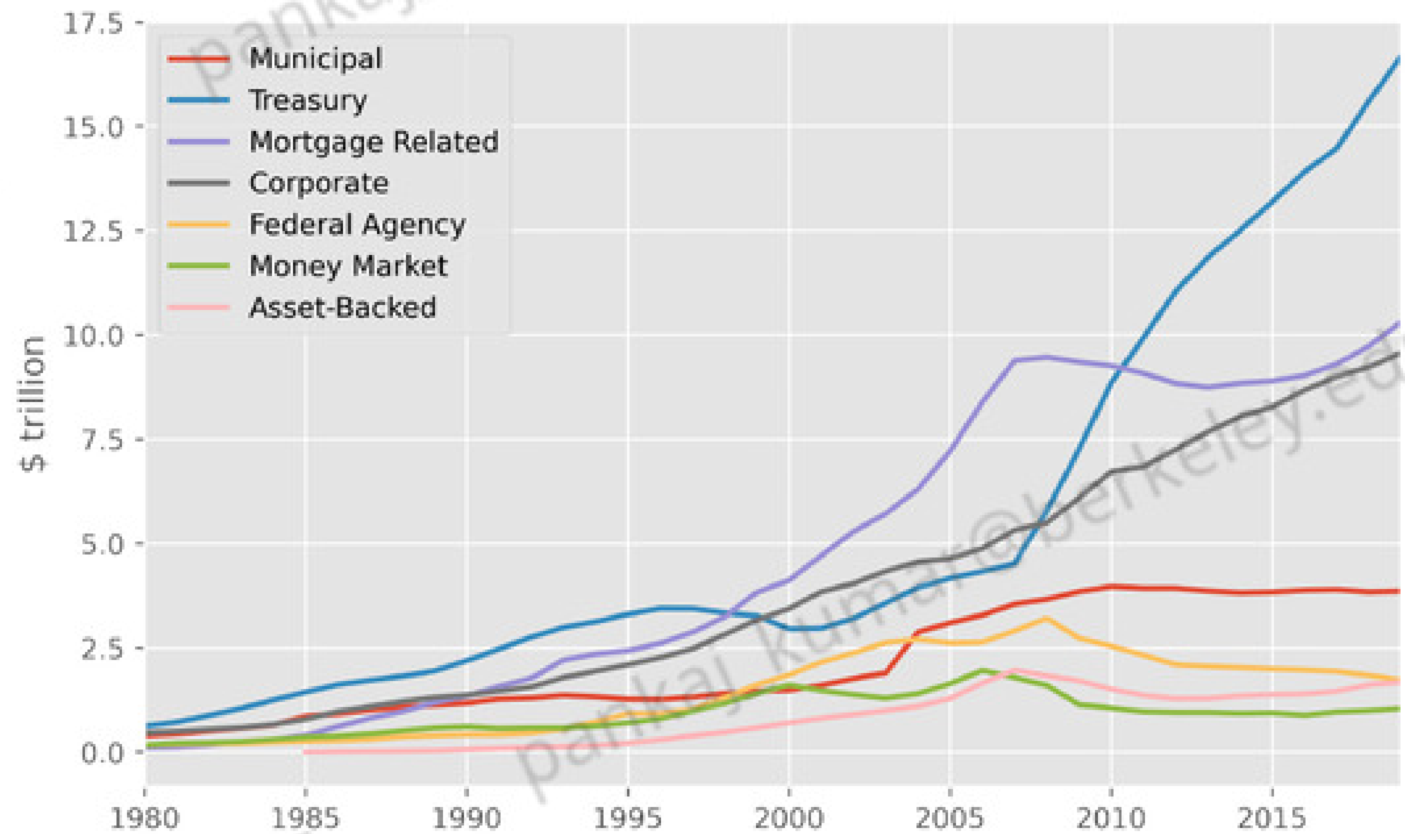
What riskless rate to use?

Total debt by country, 2020



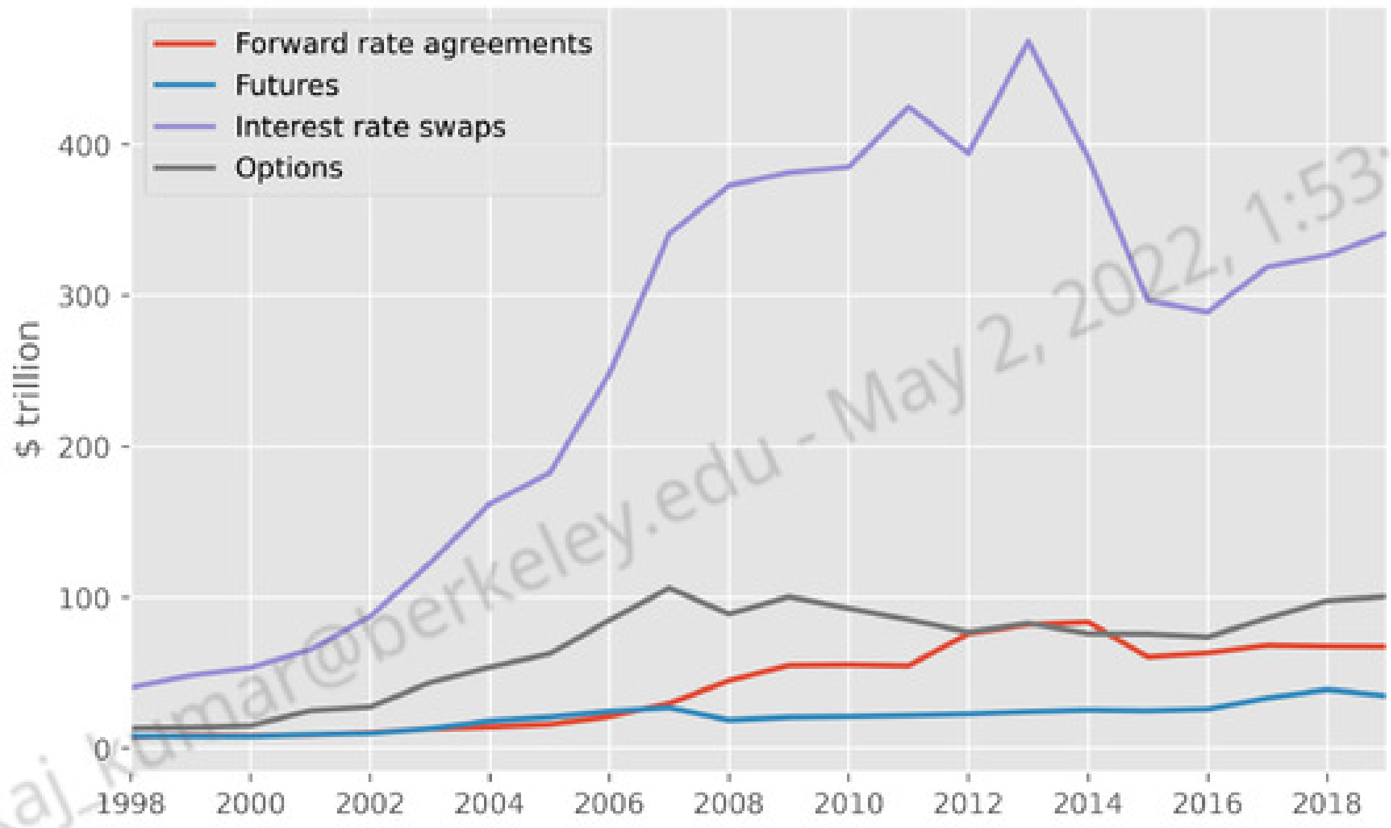
Data source: BIS

US debt markets, 1980–2019



Data source: SIFMA

Interest rate derivative markets 1998–2019 (notional)



Data source: BIS

Outline

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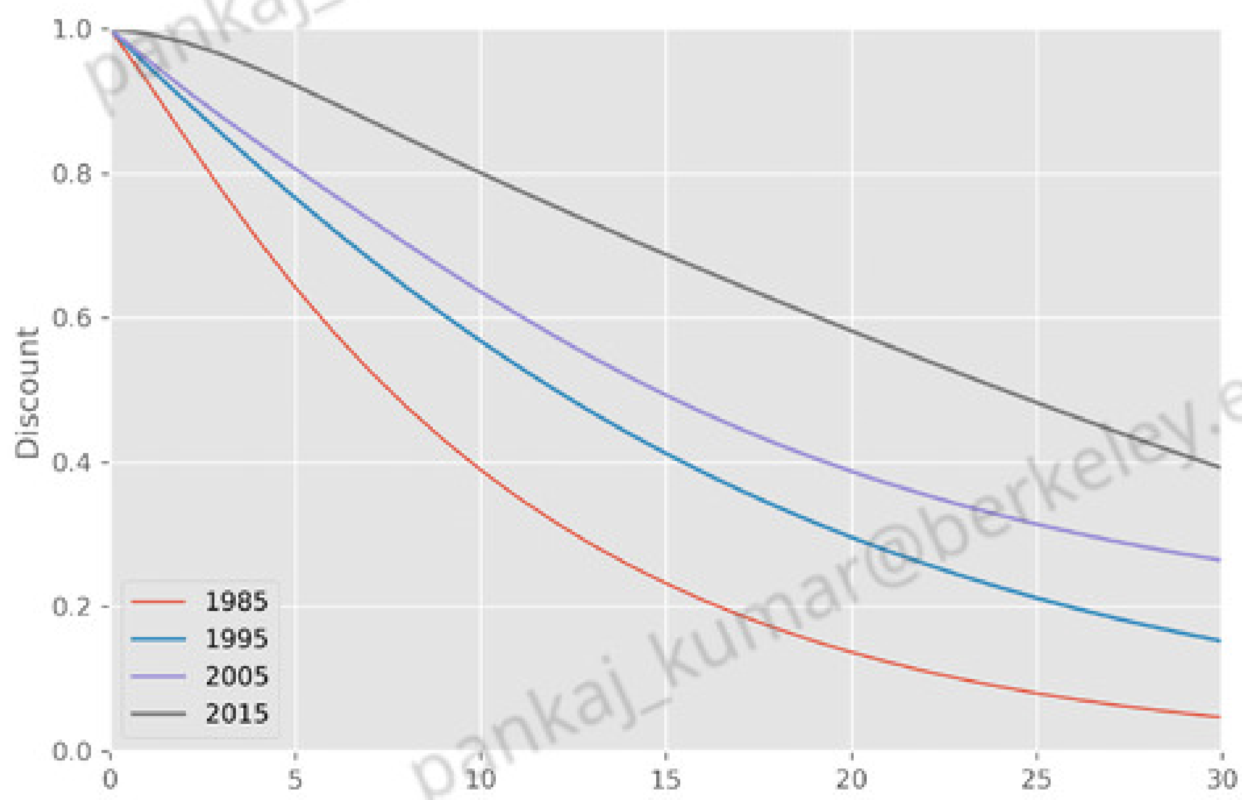
Basic Fixed-Income Securities

What riskless rate to use?

Present value and interest rates

- $\$X$ invested today (date t) at an (annualized) interest rate r grows to $\$X(1+r)^{T-t}$ in year T.
  - $X(1+r)^{T-t}$ is the future value of $\$X$ in year T.
- To have $\$C$ in year T, we must invest $C/(1+r)^{T-t}$ today.
  - $C/(1+r)^{T-t}$ is the present value (PV) of $\$C$ in year T.
- $1/(1+r)^{T-t}$ is called the year T discount factor, $Z(t,T)$.
  - PV of \$1 received at date T.
  - The price of a zero coupon bond paying \$1 at date T.
- We call $Z(t,T)$ the discount function (a function of T).

The discount function



Data source: Federal Reserve

Review problems

PV	Rate	Time	Future Value
\$100	5.25%	1.0 year(s)	________
\$600	8.00%	5.0 year(s)	________

FV	Rate	Time	Present Value
\$500	7.30%	4.0 year(s)	________

	Rate	Time	Discount Factor
	6.25%	3.0 year(s)	________

Compounding: EAR vs. APR

- Suppose we earn 3% interest every 6 months.
  - In one year, \$100 grows to $100 \times 1.03^2 = \$106.09$.
- We can quote this interest rate in several ways:
  - $r_{6m} = 3\%$, the (non-annualized) 6m interest rate.
  - 6.09% is the Effective Annual Rate (EAR).
  - 6.00% (3% x 2) is the Annual Percentage Rate (APR)
    - Semi-annual compounding.
- Converting between EAR and APR:

$$1 + \text{EAR} = (1 + \text{APR}/2)^2.$$

Compounding: EAR vs. APR

- If we compound k times per year, EAR and APR are related by

$$1 + \text{EAR} = \left(1 + \frac{\text{APR}}{k}\right)^k.$$

- As k goes to infinity, we get continuous compounding,

$$1 + \text{EAR} = \lim_{k\to\infty}\left(1 + \frac{r_{\text{cts}}}{k}\right)^k = e^{r_{\text{cts}}}.$$

- We can calculate PV using either APR or EAR:

$$\text{PV} = \frac{C}{(1+\text{EAR})^t} = \frac{C}{\left(1 + \frac{\text{APR}}{k}\right)^{kt}}.$$

  - It doesn't matter as long as we're consistent.

Compounding conventions

- Different markets use different compounding conventions, e.g.,

k	frequency	name	example
1	annual	EAR	Eurobonds
2	semi-annual	Bond-equivalent yield (BEY)	U.S. bonds
12	monthly	monthly	mortgages and credit cards

Example: APR = 8%

Frequency	EAR
Annual	8.000%
Semi-annual	______
Quarterly	8.243%
Monthly	8.300%
Daily	8.328%
Hourly	8.329%
Continuous	______

$$1 + \text{EAR} = \left(1 + \frac{\text{APR}}{k}\right)^k$$

$$1 + \text{EAR} = e^{r_{\text{cts}}}$$

Example: EAR = 11%

Frequency	APR
Annual	11.000%
Semi-annual	10.713%
Quarterly	______
Monthly	10.482%
Daily	10.437%
Hourly	10.436%
Continuous	______

$$\text{APR} = k\left((1 + \text{EAR})^{\frac{1}{k}} - 1\right)$$

$$r_{\text{cts}} = \ln(1 + \text{EAR})$$

Loans

- Loan balance = PV of payments, discounted at loan rate.

- E.g., you take out a four-year loan for $8,239.05.
  - 48 equal monthly payments.
  - Interest rate is 14.2% (APR, compounded monthly)
- How much do you pay each month?

Loans

- What's the remaining loan balance after you've made 3 payments?

Interest rates over different periods

- Interest rates usually differ over time and over different horizons.
- We call the interest rate for a t-year investment the t-year spot rate.
  - Also called the zero-coupon yield for year t.
- When we need to be precise, we can write $r_k(t, t+n)$ for the n-year spot rate at date t, compounded k times per year.
  - $r(t, t+n)$ for the continuously compounded rate.
- PV of riskless cash flows at years $t+1, t+2, t+3, \dots$ is

$$\begin{aligned} \text{PV} &= \frac{C_{t+1}}{\left(1+\frac{r_k(t,t+1)}{k}\right)^k} + \frac{C_{t+2}}{\left(1+\frac{r_k(t,t+2)}{k}\right)^{2k}} + \frac{C_{t+3}}{\left(1+\frac{r_k(t,t+3)}{k}\right)^{3k}} + \dots \\ &= C_{t+1}e^{-r(t,t+1)} + C_{t+2}e^{-2r(t,t+2)} + C_{t+3}e^{-3r(t,t+3)} + \dots \end{aligned}$$

Discount function vs. spot rates

- Discount function from spot rates:

$$\begin{aligned} Z(t, T) &= \frac{1}{\left(1+\frac{r_k(t,T)}{k}\right)^{k(T-t)}} \\ &= e^{-r(t,T)(T-t)}. \end{aligned}$$

- Spot rates from discount function:

$$\begin{aligned} r_k(t, T) &= k\left(\frac{1}{Z(t, T)^{\frac{1}{k(T-t)}}} - 1\right) \\ r(t, T) &= -\frac{1}{T-t}\ln\left[Z(t, T)\right] \end{aligned}$$

Yield to maturity (YTM)

- A bond's yield to maturity is the single rate that discounts the payments back to the current price.
- I.e., it is the value y that solves:

$$P = \frac{c}{1+y} + \frac{c}{(1+y)^2} + \ldots + \frac{1+c}{(1+y)^T}$$

for annual-coupon-paying bonds (with y compounded annually), or

$$P = \frac{c/2}{1+y/2} + \frac{c/2}{(1+y/2)^2} + \ldots + \frac{1+c/2}{(1+y/2)^{2T}}$$

for semiannual bonds (with y compounded semiannually).

Yield to maturity (YTM)

- What is the yield to maturity of a zero-coupon bond?
- What is the yield to maturity of a par bond (i.e., $P = 100$)?
- What is the yield to maturity of a coupon bond if the term structure is flat (i.e., $r(t, \tau) = r$ for all $\tau \geq t$)?

- In all other cases, YTM is a (complicated, nonlinear) weighted average of the spot rates corresponding to the different cash flows paid by a bond.
  - The higher a specific cash flow, the more weight on that spot rate.
  - Needs to be solved for numerically (for more than 4 payments).

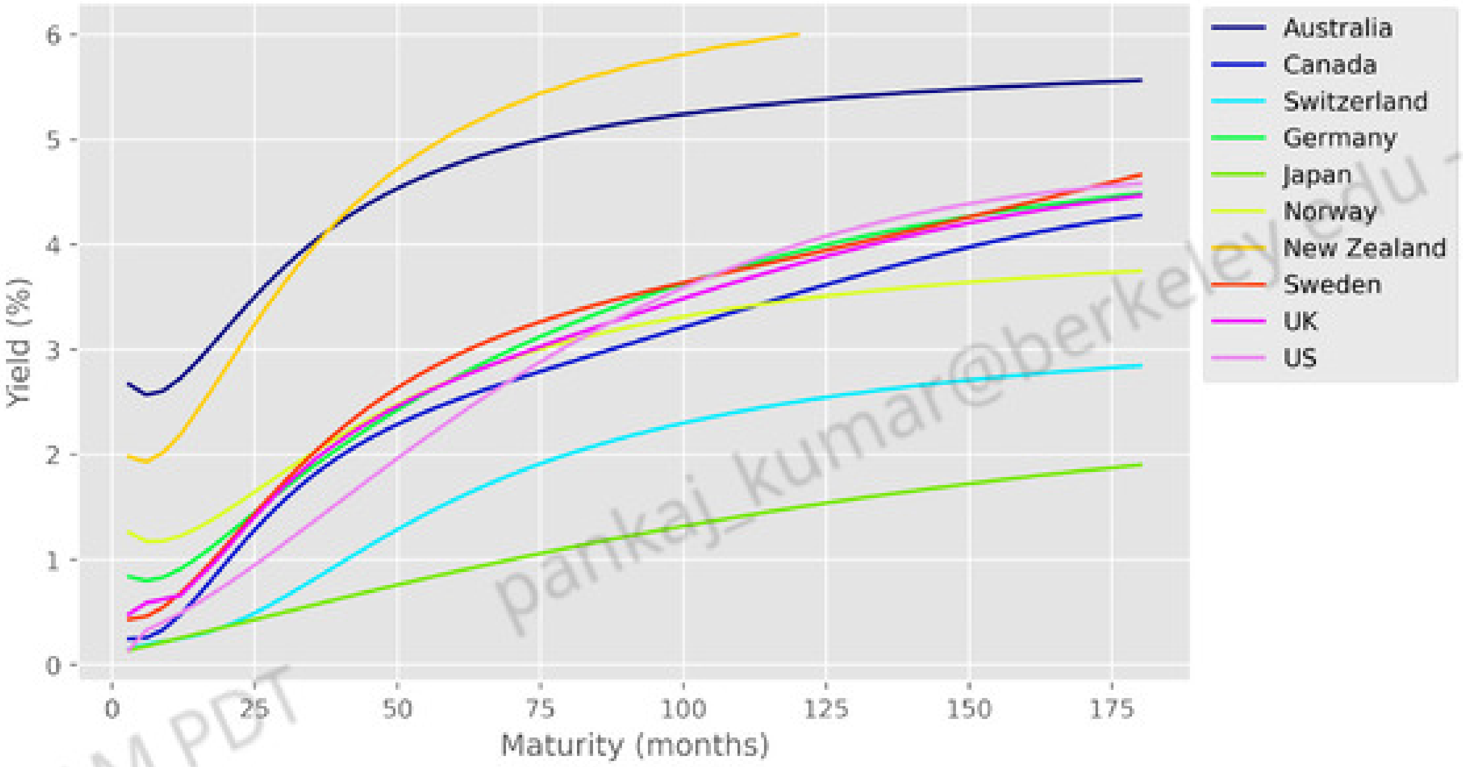
Example: Calculating yield to maturity

- Suppose we have 2-year, 6% coupon Eurobond (with annual payments).
  - Assume $r_1 = 1.3411\%$, $r_2 = 1.8757\%$.
- What is the bond's price?

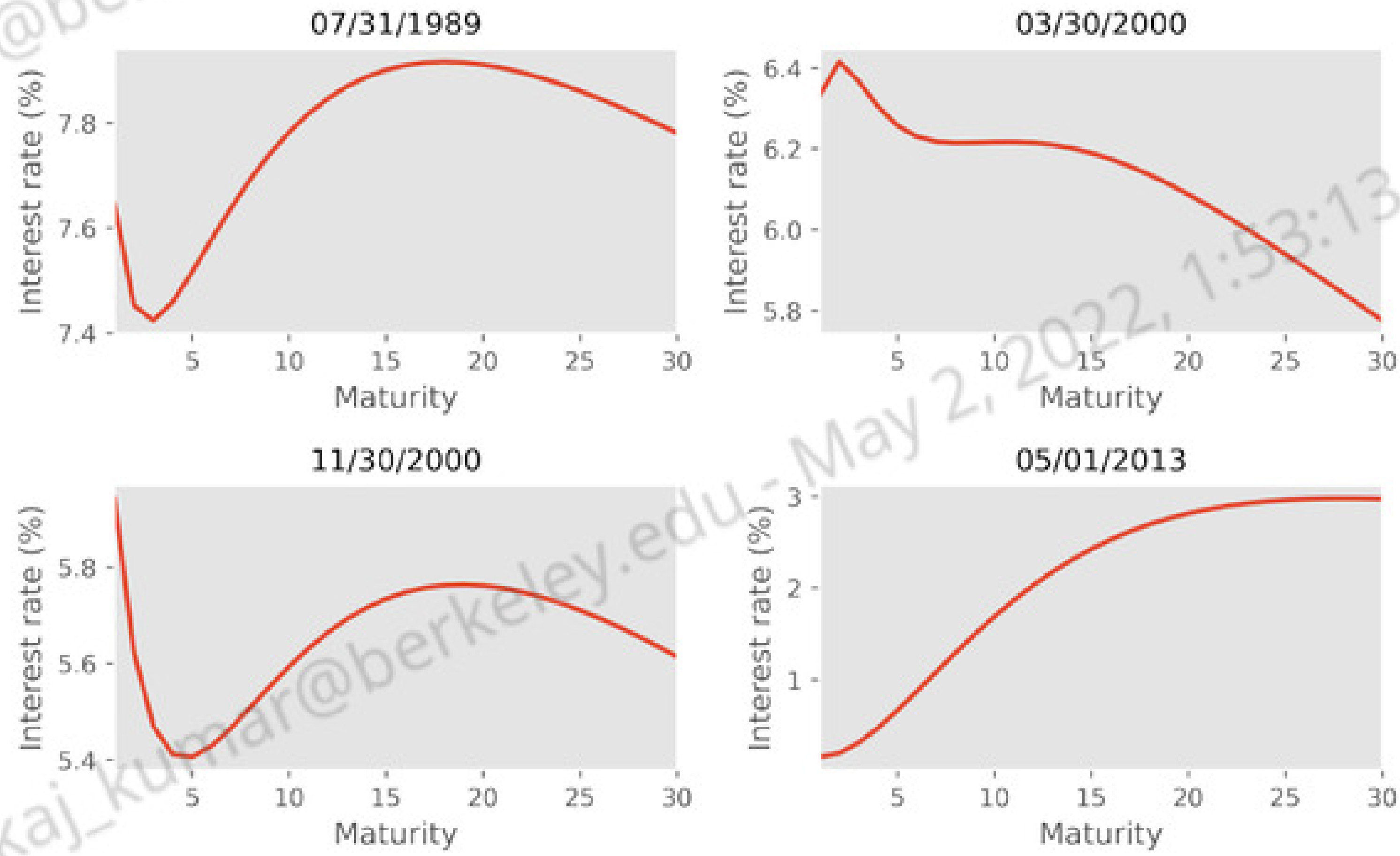
- What is its yield to maturity?

The Yield Curve

- A graph of interest rates (usually zero-coupon yields) plotted against maturity is called the yield curve or the term structure of interest rates.
- Here are some examples from May 2009 (data from Wright, 2011):



U.S. yield curves



Data source: Federal Reserve

Forward rates

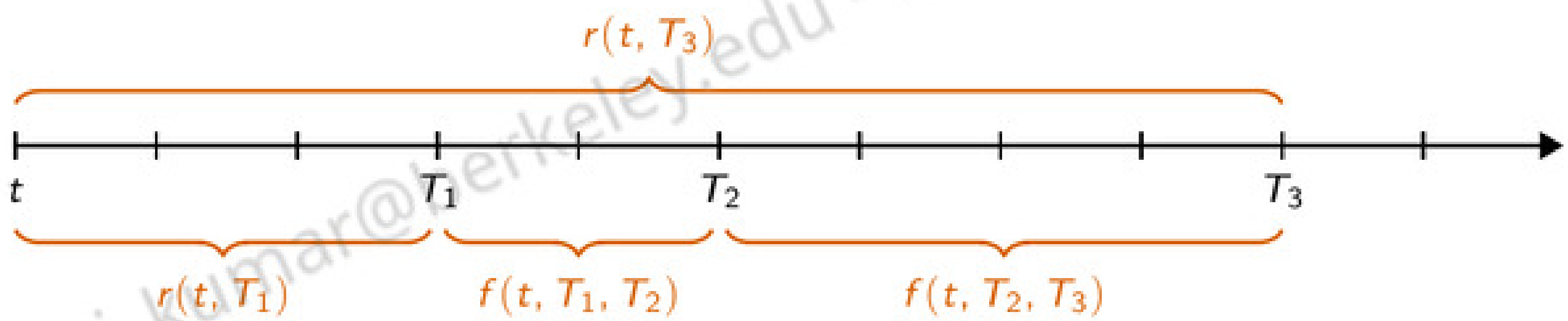
- Spot rate: today's rate for a loan starting today.
- Forward rate: today's rate for a loan starting in the future.
- As with spot rates, forward rates may apply over various periods, and may be quoted as EARs, APRs, or continuously compounded.
- Notation:
  - $f_k(t, T_1, T_2)$ = forward rate at date t for a loan between T_1 and T_2, compounded k times per year.
  - $f(t, T_1, T_2)$ = continuously compounded forward rate.

Forward rates versus spot rates

- Investing at the spot rate between t and T is the same as simultaneously agreeing to invest at
  - the spot rate between t and T_1, and
  - the forward rate between T_1 and T_2, and

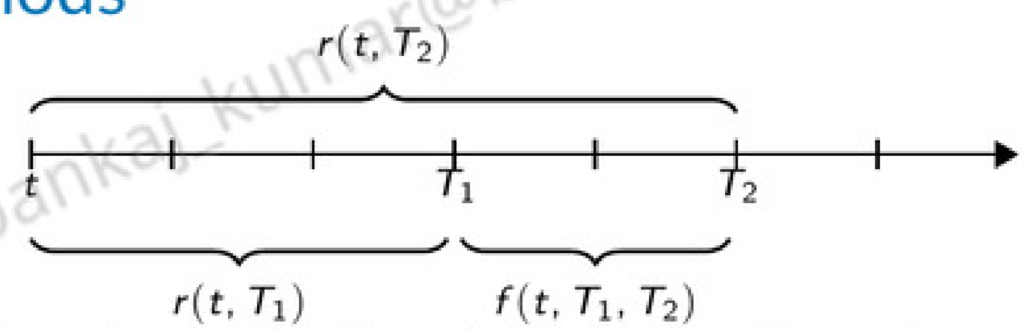
    $\vdots$
  - the forward rate between T_{n-1} and T,

  where $t = T_0 < T_1 < T_2 < \ldots < T_{n-1} < T_n = T$.



$$e^{r(t,T_3)(T_3-t)} = e^{r(t,T_1)(T_1-t)} \times e^{f(t,T_1,T_2)(T_2-T_1)} \times e^{f(t,T_2,T_3)(T_3-T_2)}.$$

Example: Two periods



$$e^{r(t,T_2)(T_2-t)} = e^{r(t,T_1)(T_1-t)} \times e^{f(t,T_1,T_2)(T_2-T_1)}. \tag{1}$$

$$\left(1 + \frac{r_k(t, T_2)}{k}\right)^{k(T_2-t)} = \left(1 + \frac{r_k(t, T_1)}{k}\right)^{k(T_1-t)} \left(1 + \frac{f_k(t, T_1, T_2)}{k}\right)^{k(T_2-T_1)} \tag{2}$$

- Rearranging (1) we get

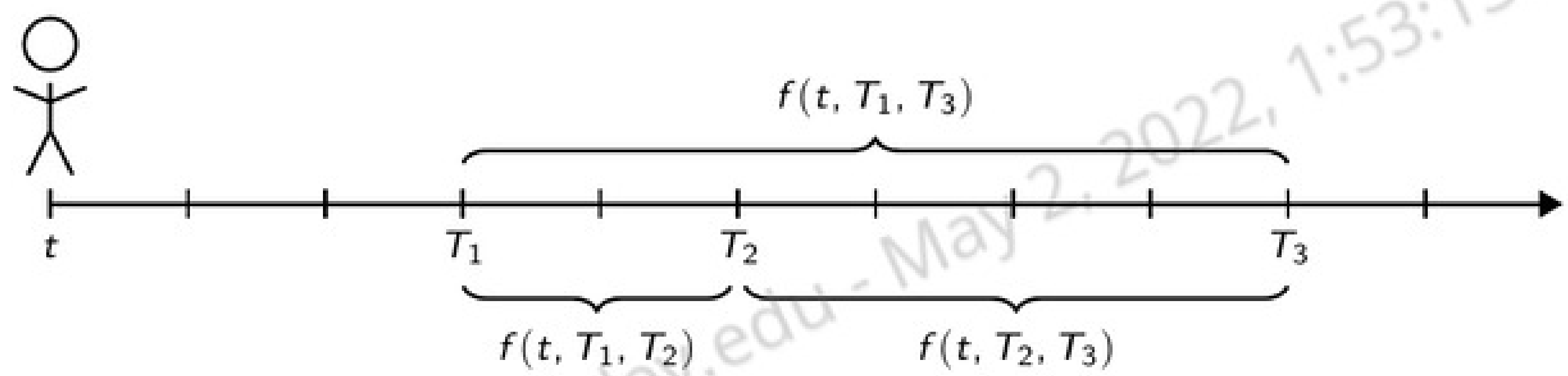
$$r(t, T_2) = \left(\frac{T_1 - t}{T_2 - t}\right) r(t, T_1) + \left(\frac{T_2 - T_1}{T_2 - t}\right) f(t, T_1, T_2).$$

$$e^{f(t,T_1,T_2)(T_2-T_1)} = \frac{Z(t, T_1)}{Z(t, T_2)}.$$

- $r(t, T_2)$ is a weighted average of $r(t, T_1)$ and $f(t, T_1, T_2)$.

Forward rates versus spot rates

- Similarly, investing at the forward rate between (future) dates T_1 and T_n is the same as rolling over successive forward rates between T_1 and $T_2, \ldots, T_{n-1}$ and T_n.



$$e^{f(t,T_1,T_3)(T_3-T_1)} = e^{f(t,T_1,T_2)(T_2-T_1)} \times e^{f(t,T_2,T_3)(T_3-T_2)}.$$

Instantaneous forward rates

- Rearranging (1) again, we get

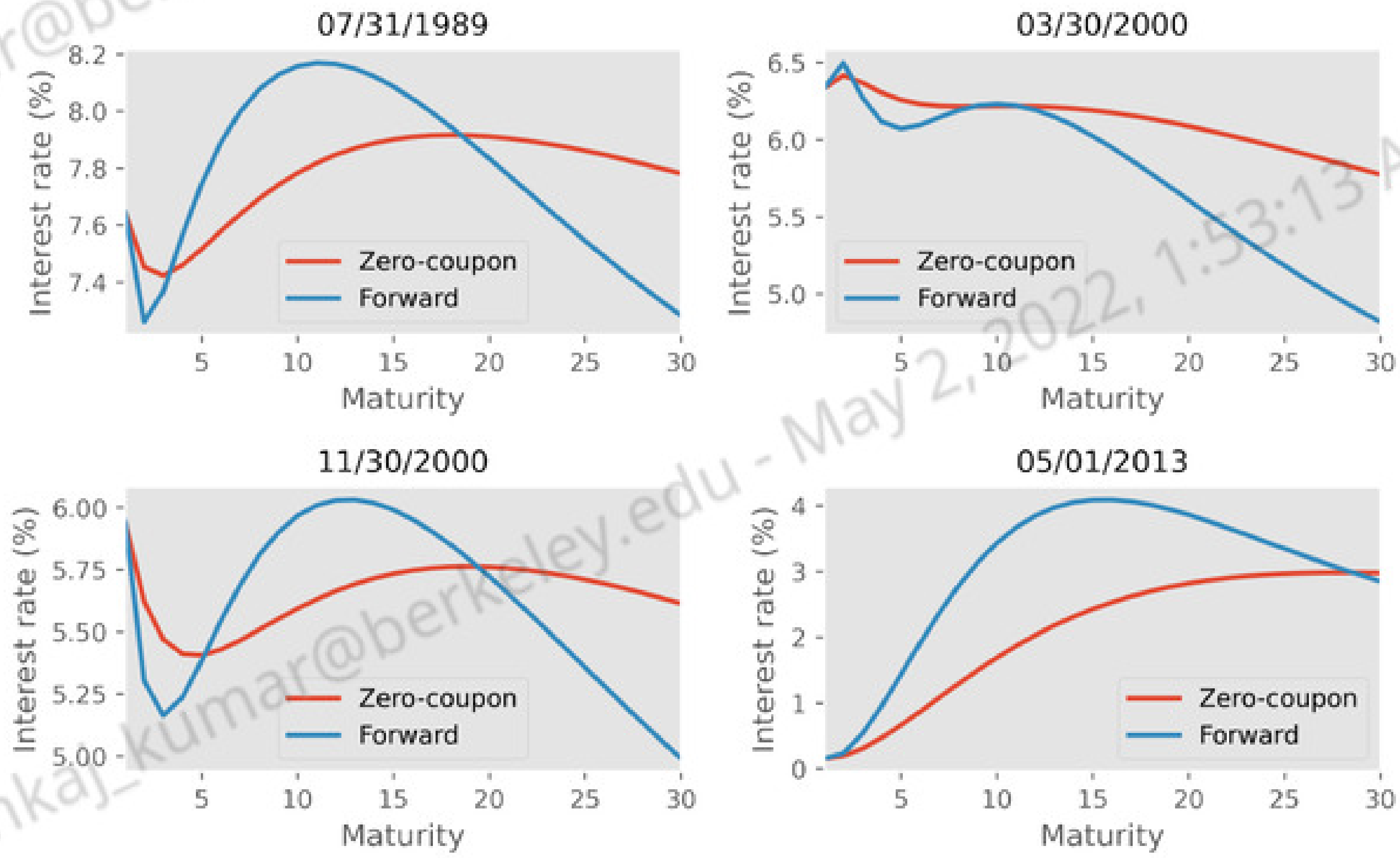
$$\begin{aligned} f(t, T_1, T_2) &= \frac{(T_2 - t)r(t, T_2) - (T_1 - t)r(t, T_1)}{T_2 - T_1}, \\ &= r(t, T_1) + (T_2 - t)\left(\frac{r(t, T_2) - r(t, T_1)}{T_2 - T_1}\right). \end{aligned}$$

- As T_2 converges to T_1, we get the instantaneous forward rate, $f(t, T_1)$, given by

$$\begin{aligned} f(t, T_1) &= \lim_{T_2 \to T_1} f(t, T_1, T_2) \\ &= \left.\frac{\partial\left[(T - t)r(t, T)\right]}{\partial T}\right|_{T=T_1} \\ &= r(t, T_1) + (T_1 - t)\left.\frac{\partial r(t, T)}{\partial T}\right|_{T=T_1} \end{aligned}$$

- Forward curve is above (below) spot curve if $\frac{\partial r(t,T)}{\partial T} > 0$ (< 0).

Zero-coupon yield curve vs. forward curve



Data source: Federal Reserve

Forward rates and bond prices

- Split interval $[t, T]$ into n pieces of length $\Delta = (T - t)/n$. From above,

$$r(t, T) = \frac{1}{T - t} \sum_{i=0}^{n-1} f(t, t + i\Delta, t + (i+1)\Delta)\Delta;$$

$$Z(t, T) = e^{-r(t,T)(T-t)} = e^{-\sum_{i=0}^{n-1} f(t, t+i\Delta, t+(i+1)\Delta)\Delta}.$$

- As Δ goes to zero, the sum becomes an integral:

$$r(t, T) = \frac{1}{T - t} \int_t^T f(t, \tau)\, d\tau,$$

$$Z(t, T) = e^{-\int_t^T f(t,\tau)\, d\tau}.$$

- Similarly for discrete-horizon forward rates:

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, \tau)\, d\tau.$$

Outline

Day Count Conventions

- Interest calculations are easy (enough) over whole periods.
- Things get messier over partial periods.
- Basic calculation:

$$\text{Interest amount} = \text{Principal} \times \text{interest rate} \times \frac{\text{days in interest period}}{\text{days in year}}$$

- Securities differ in how they calculate the numbers of days.
- Note: Interest rates quoted on a given trade date usually apply as of the settlement date, normally two business days after the trade date (one day for Treasury securities).

Actual/Actual

- Used for: Treasury notes and bonds.
- Number of days in period = actual calendar days between dates.
- Days in year = 365 (with various conventions for handling leap years).

30/360

- Used for: U.S. corporate bonds, municipal bonds, Eurobonds.
- Each month is treated as having 30 days, each year as having 360 days.
- Interest factor between date 1 and date 2 is calculated as

$$\text{Interest} = \text{Principal} \times \text{rate} \times \frac{360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)}{360}$$

- Dates may be adjusted for the end of a month. For details, see `https://en.wikipedia.org/wiki/Day_count_convention` or `http://strata.opengamma.io/day_counts/`

30/360 End-of-Month Conversions: US corporate bonds

- If investment pays at End of Month (EOM) and (date 1 is the last day of Feb.) and (date 2 is the last day of Feb.), then change D_2 to 30.
- If investment is EOM and (date 1 is the last day of Feb.), then change D_1 to 30.
- If $D_2 = 31$ and $D_1 = 30$ or 31, change D_2 to 30.
- If $D_1 = 31$, change D_1 to 30.

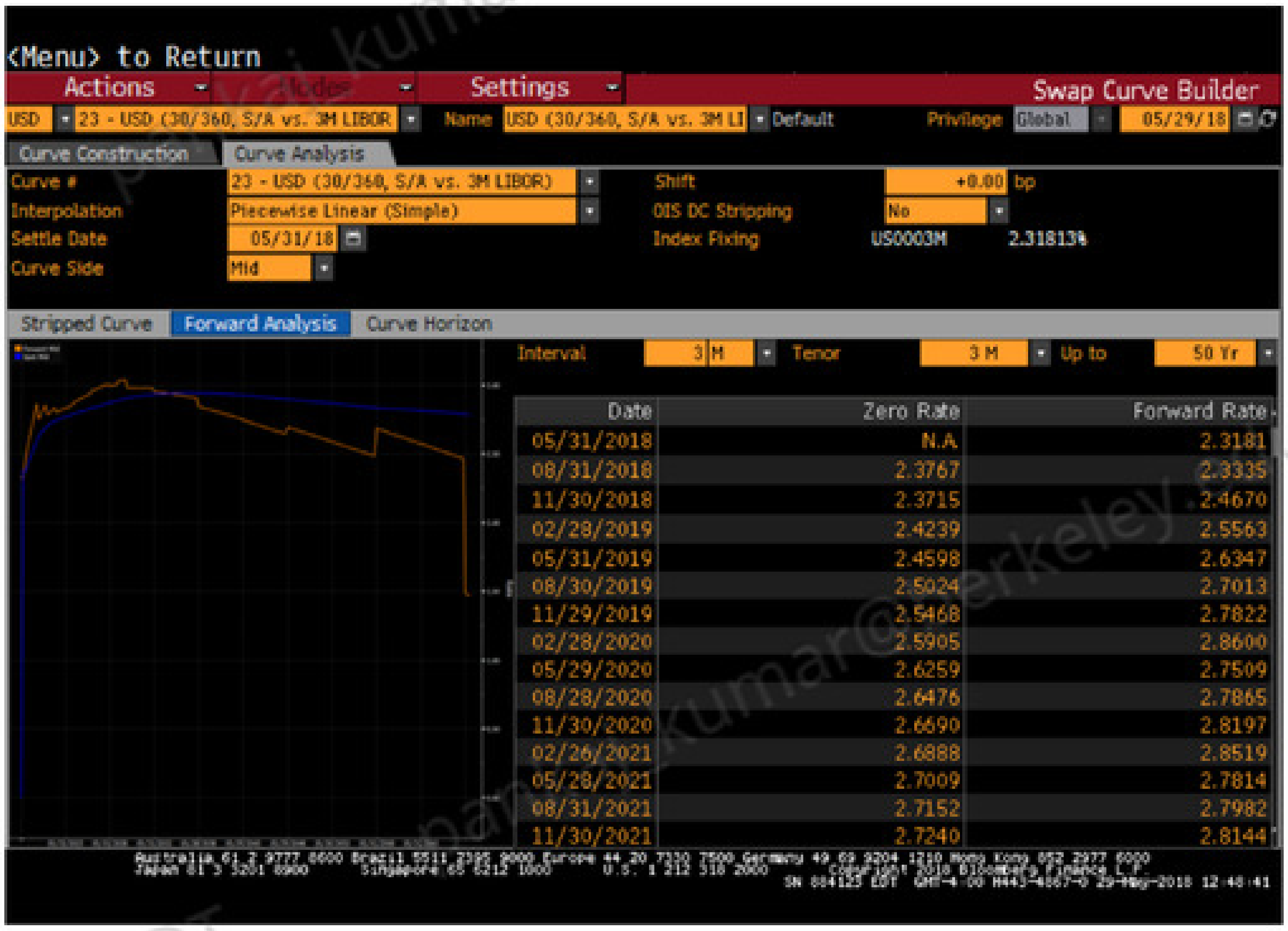
30/360 End-of-Month Conversion: ISDA Rules (e.g., Eurobonds)

- If D_1 is last day of month, change D_1 to 30.
- If D_2 is last day of month (unless date 2 is the maturity date and $M_2 =$ Feb.), change D_2 to 30.

Actual/360

- Used for: U.S. money markets, repo markets (Treasury bills).
- Number of days in period = actual calendar days between dates.
- Days in year = 360.

Example: Bloomberg "23" LIBOR curve, May 29, 2018



Source: Bloomberg

The numbers

Date	Zero Rate	Forward Rate
05/31/2018	0	2.31813
08/31/2018	2.376663016	2.333460953
11/30/2018	2.371504787	2.467024806
02/28/2019	2.423937532	2.556335148
05/31/2019	2.45980416	2.634724361
08/30/2019	2.502399439	2.70132263
11/29/2019	2.54683672	2.78219816
02/28/2020	2.590473685	2.859980339
05/29/2020	2.625908881	2.750872729

Note:

- Numbers exported to Excel are more precise than on screen.
- Zero rates are compounded semiannually and use 30I/360 day-count.
- Forward rates compounded quarterly and use Actual/360 day-count.

Example: Discount factor for 8/31/18

- First, calculate using forward rates:
  - Forward rates compounded quarterly and use Actual/360 day-count.
  - Interest calculations use simple interest:

$$\text{Interest} = \text{quoted rate} \times \frac{\text{(actual) days in interest period}}{360}$$

- (Actual) days between 5/31/18 and 8/31/18 = 92.
  - 30 for June, 31 each for July and August.
- Discount factor:

$$DF_{8/31/18} = \frac{1}{1 + \left(.0231813 \times \frac{92}{360}\right)} = 0.994111.$$

- Discount factor is fundamental quantity, from which we calculate spot/forward rates for a given compounding frequency, day-count convention.

Example: Discount factor for 8/31/18

- Now repeat calculation using zero rates:
  - Zero rates compounded semiannually and use 30I/360 day-count.
  - Interest calculations use compound interest:

$$1 + \text{interest} = \left(1 + \frac{\text{quoted rate}}{2}\right)^{2\times \frac{\text{(30I/360) days in interest period}}{360}}$$

- Days between 5/31/18 and 8/31/18 using 30I/360 convention:
  - Start with $(D_1, M_1, Y_1) = (31, 5, 2018)$ and $(D_2, M_2, Y_2) = (31, 8, 2018)$.
  - D_1 is last day of month, so change D_1 to 30.
  - D_2 is also the last day of month, so change D_2 to 30.
  - Day count = $360(2018 - 2018) + 30(8 - 5) + (30 - 30) = 90$.
- Discount factor:

$$DF_{8/31/18} = \frac{1}{\left(1 + \frac{.02376663}{2}\right)^{2\times\frac{90}{360}}} = 0.994111.$$

Example: Discount factor for 11/30/18

- Similar to the above calculations, we have
  - (Actual) days between 5/31/18 and 8/31/18 = 92.
  - (Actual) days between 8/31/18 and 11/30/18 = 91.
  - (30I/360) days between 5/31/18 and 11/30/18 = 180.

- So, using forward rates:

$$\begin{aligned} DF_{11/30/18} &= \frac{1}{1 + \left(.0231813 \times \frac{92}{360}\right)} \times \frac{1}{1 + \left(.02333461 \times \frac{91}{360}\right)} \\ &= 0.988281. \end{aligned}$$

- Or, using spot rates:

$$\begin{aligned} DF_{11/30/18} &= \frac{1}{\left(1 + \frac{.02371505}{2}\right)^{2\times\frac{180}{360}}} \\ &= 0.988281. \end{aligned}$$

Example: Discount factor for 2/28/19

Outline

Introduction

Overview of Fixed-Income Markets

"Bond Math"

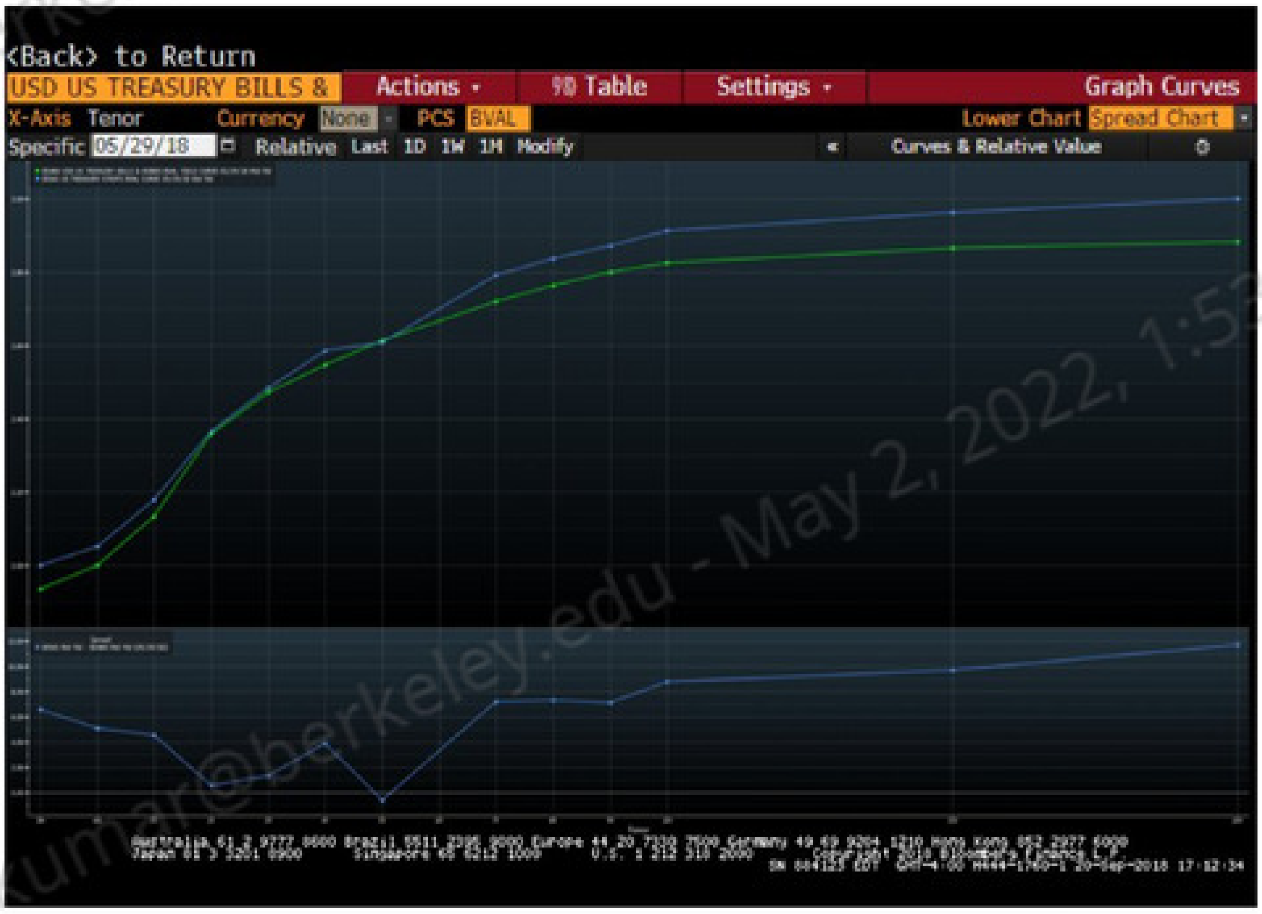
Day Count Conventions

Fitting the Yield Curve

Basic Fixed-Income Securities

What riskless rate to use?

Treasury STRIP vs. coupon-bond yields, May 29, 2018



- Why is the zero curve higher than the coupon-bond curve?

What yield curve to estimate?

- To be useful, we need to have a yield curve that is not affected by the coupon rates of outstanding bonds.
- One obvious choice: the zero-coupon yield curve.

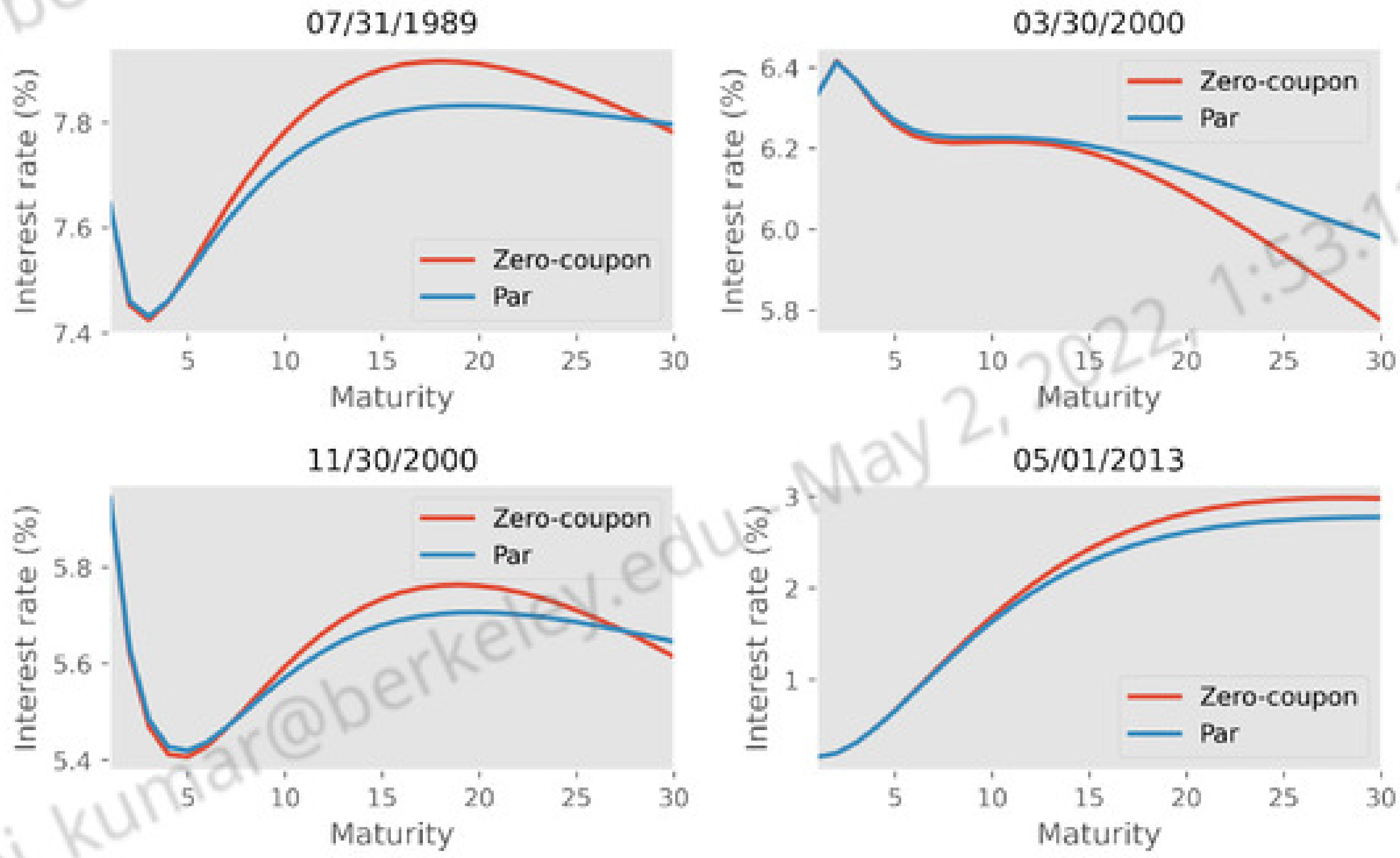
- Market participants often use the par yield curve instead.
  - Yields (coupons) on newly-issued (par) coupon bonds.
- Implicit definition for par yield, r_p (n_c = # coupons):

$$100 = 100\frac{r_p(t,T)}{2}\sum_{i=1}^{n_c} e^{-(T-t-(i-1)/2)\, r(t,T-(i-1)/2)} + 100\, e^{-(T-t)r(t,T)}.$$

- Solve for the par bond yield:

$$r_p(t,T) = \frac{2\left(1 - e^{-(T-t)r(t,T)}\right)}{\sum_{i=1}^{n_c} e^{-(T-t-(i-1)/2)\, r(t,T-(i-1)/2)}}.$$

Par yield curve vs. zero curve



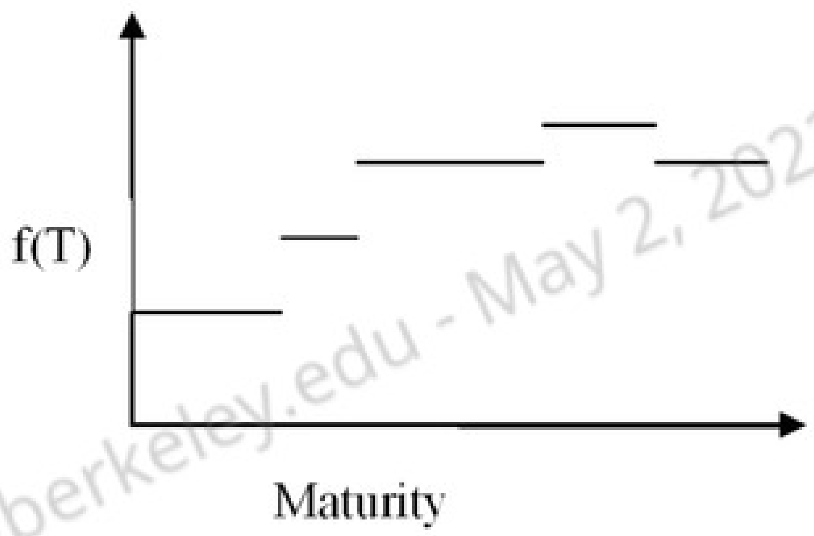
Data source: Federal Reserve

Fitting the yield curve

- We need to interpolate because there are gaps in the data: periods when no bonds mature.
- Interpolation can be done using any of three related functions:
  - $r(t, T)$ — spot rate
  - $Z(t, T)$ — price of bond paying \$1 at T
  - $f(t, T)$ — instantaneous forward rate.
- Two general approaches
  - Choose (not necessarily smooth) functions to fit (interpolate) observed data exactly.
  - Choose smooth functions that do not fit exactly.
- Either way, we need to make some functional-form assumptions.
- Gürkaynak, Sack, and Wright (2007) is a good reference and data source.

Example: $f(t, T)$ piecewise constant

- Goal: Exactly fit N bond prices, maturities τ_i, $i = 1, \ldots, N$.
- Method: Assume instantaneous forward rates follow a step function.
  - Steps occur at each bond maturity point.



- Proceed one bond at a time, starting with the shortest maturity.
- Process is called bootstrapping.
- See Fama and Bliss (1987).

Bootstrapping example

Mat. (months)	Coupon rate	Cts. comp. yield	BEY
5	5%	5%	5.063%
8	4%	5.5%	5.576%
10	6%	5.3%	5.371%

- Bond prices:

$$P_1 = \frac{102.5}{e^{0.05(5/12)}} = \$100.3867.$$

$$P_2 = \frac{2}{e^{0.055(2/12)}} + \frac{102}{e^{0.055(8/12)}} = \$100.3095.$$

$$P_3 = ?$$

Bootstrapping example (2)

- Calculation of forward rate f_1 (forward rate from 0 to 5 months)
  - First bond matures in five months; no information about shorter-maturity yields
  - Its continuously-compounded yield of 5% is the average of forward rates from zero to five months. Step function assumes that forward rates are flat over this range, so
- $f_1 = 0.05$.

Bootstrapping example (3)

- Calculation of f_2 (forward rate from 5 to 8 months)
  - Coupon at two months is discounted at average of forward rates from zero to two months; we know these rates
  - Face + principal at eight months is discounted at average of forward rates from zero to eight months; we know first five months of these rates
  - Forward rate from month 5 to month 8 is whatever is necessary to produce correct price for second bond

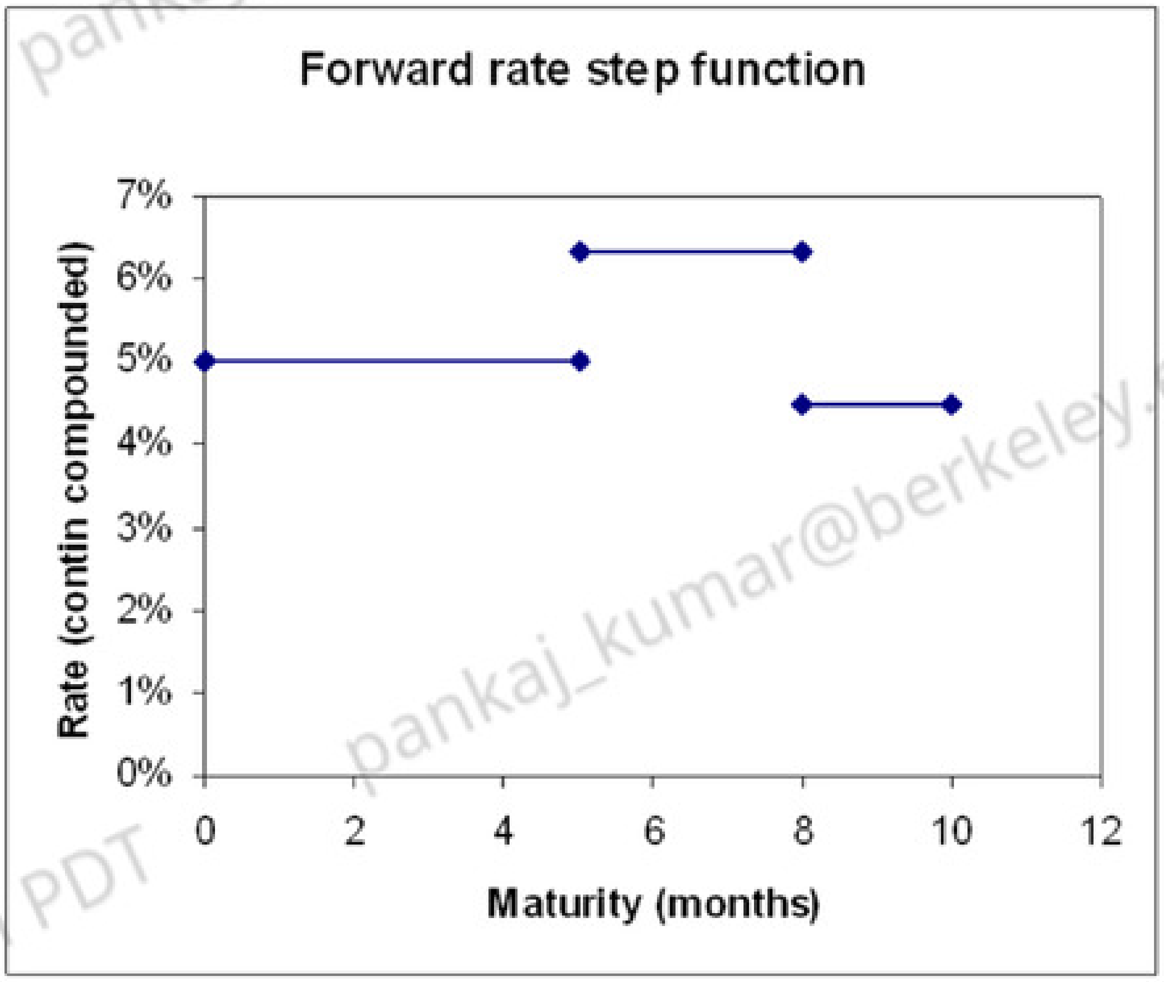
$$100.3095 = \frac{2}{e^{0.05(2/12)}} + \frac{102}{e^{0.05(5/12)+f_2(3/12)}}.$$
$$f_2 = 0.0634.$$

Bootstrapping example (4)

- Calculation of f_3 (forward rate from 8 to 10 months):

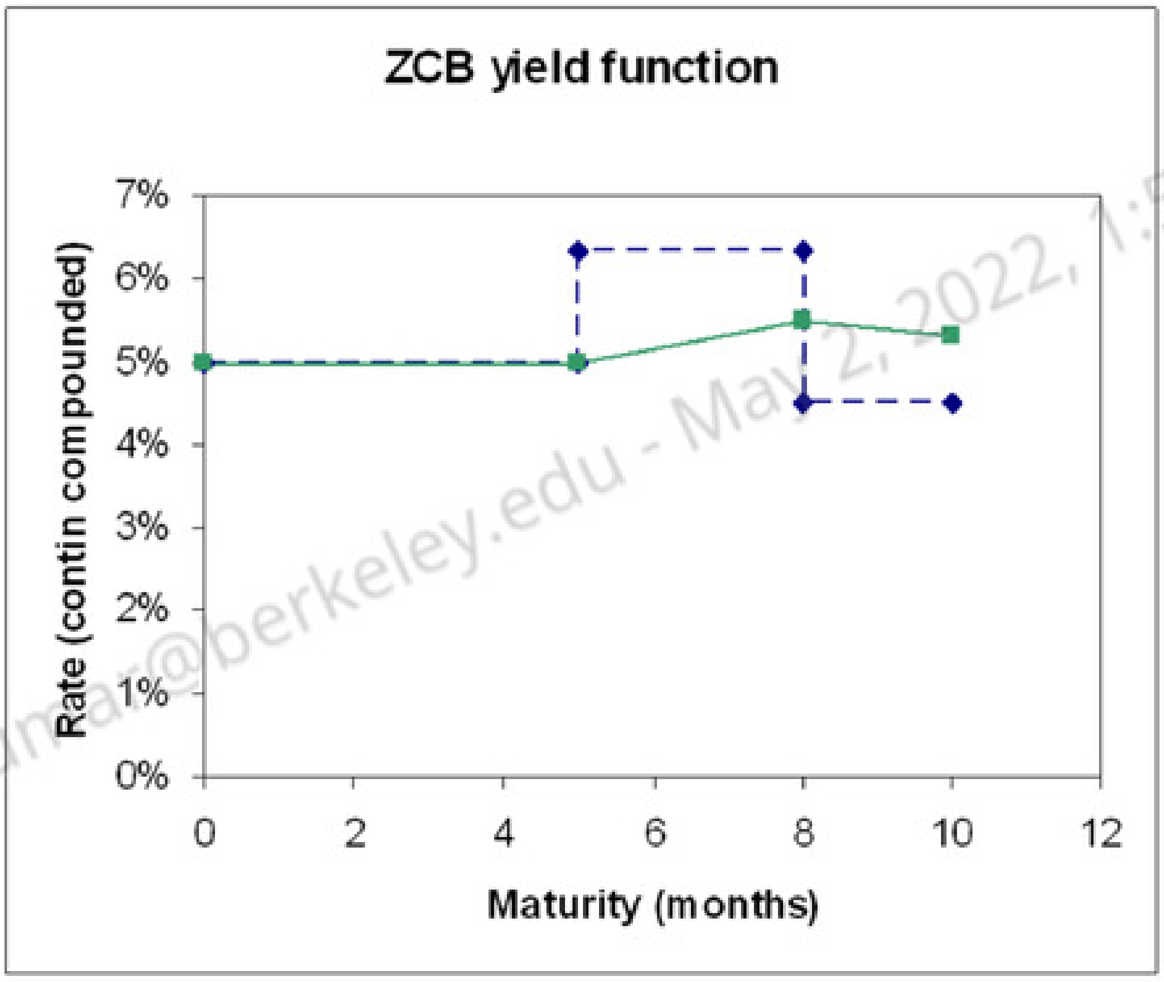
Bootstrapping example (5)

- Implied forward curve



Bootstrapping example (6)

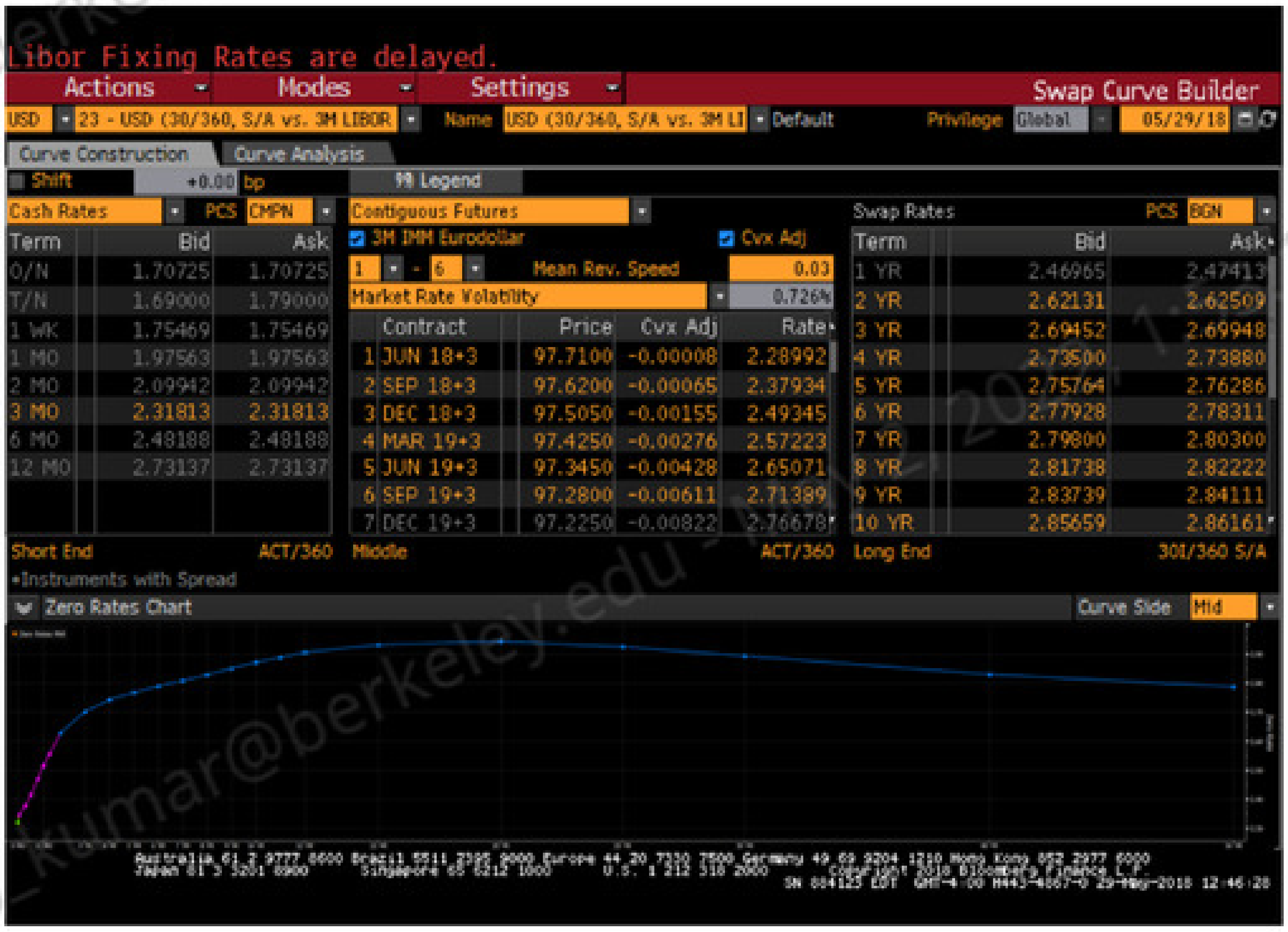
- Implied zero-coupon yield curve



Example: Bloomberg yield curve

- A standard yield curve provided by Bloomberg is the USD “23” LIBOR curve (found via **ICVS <GO>** on the terminal).
- This curve is based on the following rates:
  - the 3-month USD LIBOR deposit rate.
  - the first six Eurodollar futures that mature after the settlement date.
  - USD spot starting swaps with maturities of 2, …, 12, 15, 20, 25 and 30-years.
- Curve is fitted to match these rates perfectly, with interpolation for intermediate dates.
- See Bloomberg (2012) for details.

Input data for May 29, 2018



Source: Bloomberg

Fitting the Bloomberg USD “23” LIBOR curve

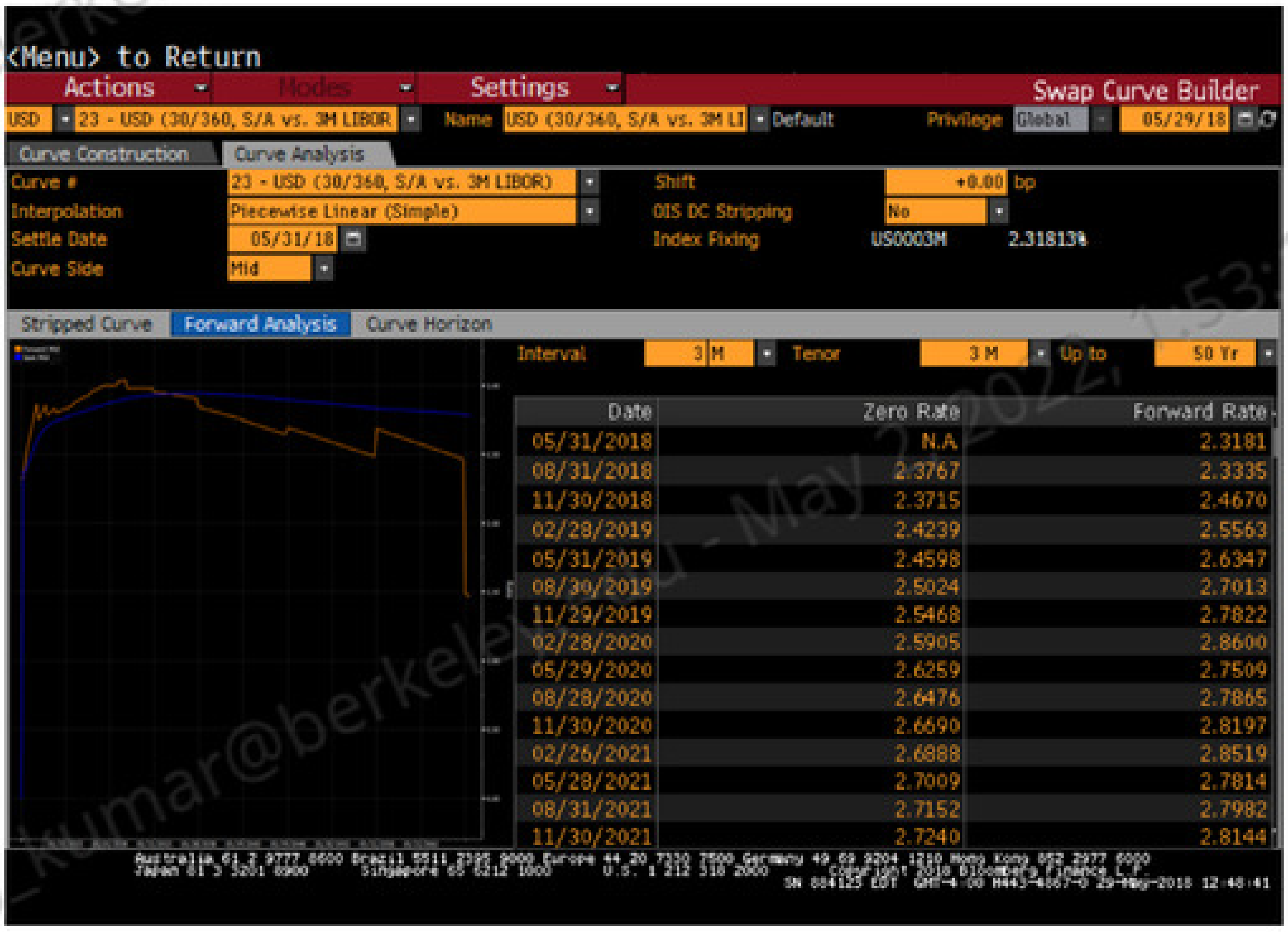
- Bloomberg has various fitting methods (see Bloomberg, 2012).
- Method 1 is “piecewise linear simple-compounded zero rate,” $r_s(t)$, defined via

$$Z(D_0, D) = \frac{1}{1 + r_s(\tau) \times \tau},$$

  where
  - D_0 is present date.
  - D is future date.
  - $Z(D_0, D)$ is discount factor for date D.
  - $\tau = \tau(D_0, D)$ is the time between D_0 and D in years, calculated using the relevant day-count convention(s).

- Another example of bootstrapping, working from shortest to longer maturities one at a time.

The fitted yield curve: method 1



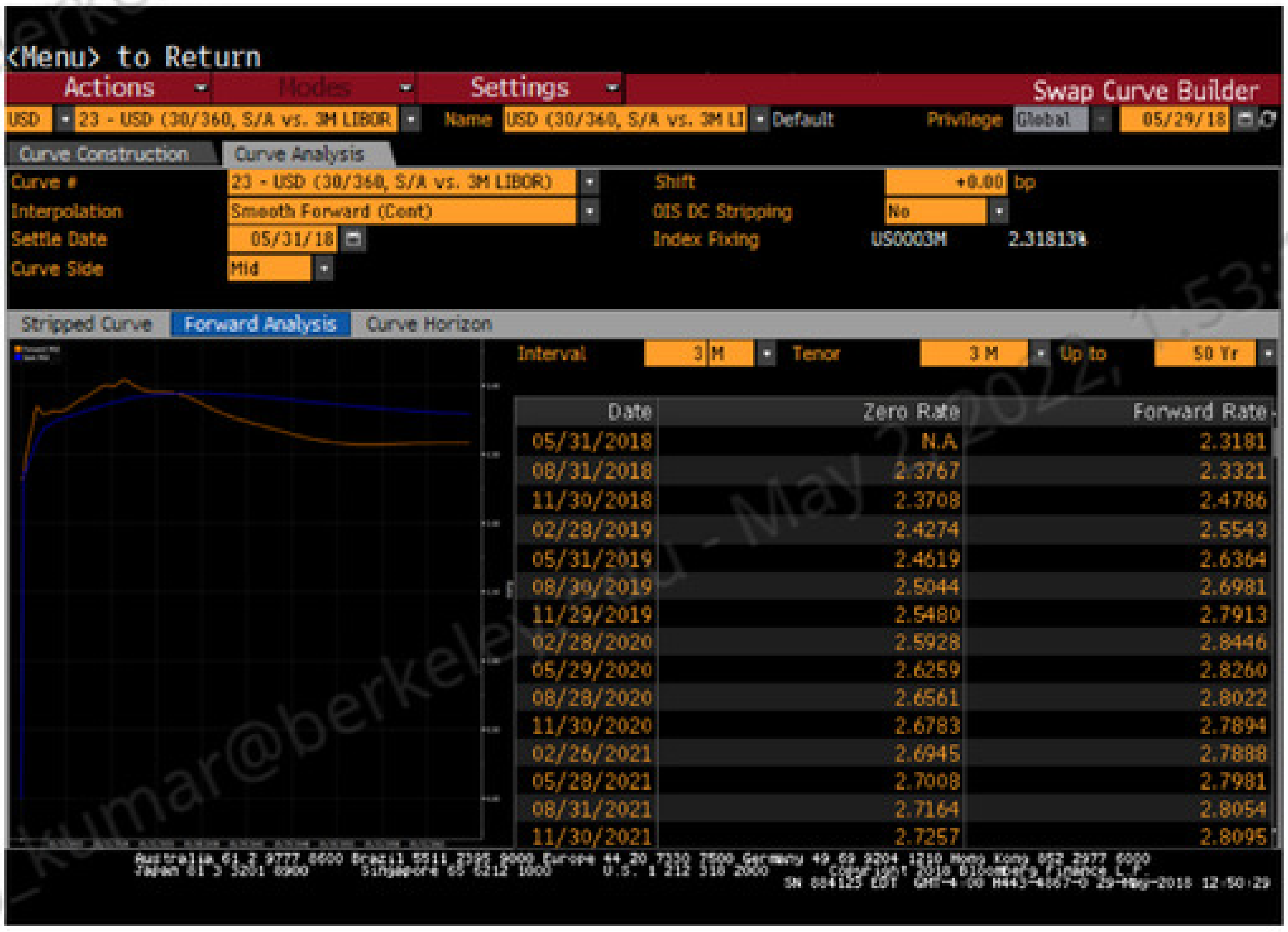
Date	Zero Rate	Forward Rate
05/31/2018	N.A	2.3181
08/31/2018	2.3767	2.3335
11/30/2018	2.3715	2.4670
02/28/2019	2.4239	2.5563
05/31/2019	2.4598	2.6347
08/30/2019	2.5024	2.7013
11/29/2019	2.5468	2.7822
02/28/2020	2.5905	2.8600
05/29/2020	2.6259	2.7509
08/28/2020	2.6476	2.7865
11/30/2020	2.6690	2.8197
02/26/2021	2.6888	2.8519
05/28/2021	2.7009	2.7814
08/31/2021	2.7152	2.7982
11/30/2021	2.7240	2.8144

Source: Bloomberg

Fitting the Bloomberg USD "23" LIBOR curve: method 2

- Bloomberg's second interpolation method (#2) is "smooth forward."
- Continuously compounded forward rates assumed piecewise quadratic between input dates.
- Neighboring pieces of the forward curve are connected so that curve *and its first derivative* are continuous at joins.
- This is an example of a spline, a piecewise polynomial where (some) derivatives match at joins.
  - If polynomial is of order n, derivatives of orders $0, 1, \ldots, n-1$ must match at joins.
  - Derivative of order n will in general have jumps.
- How does this compare with method 1? Is this another bootstrapping method?

Results: Method 2 (smooth forward/piecewise quadratic)



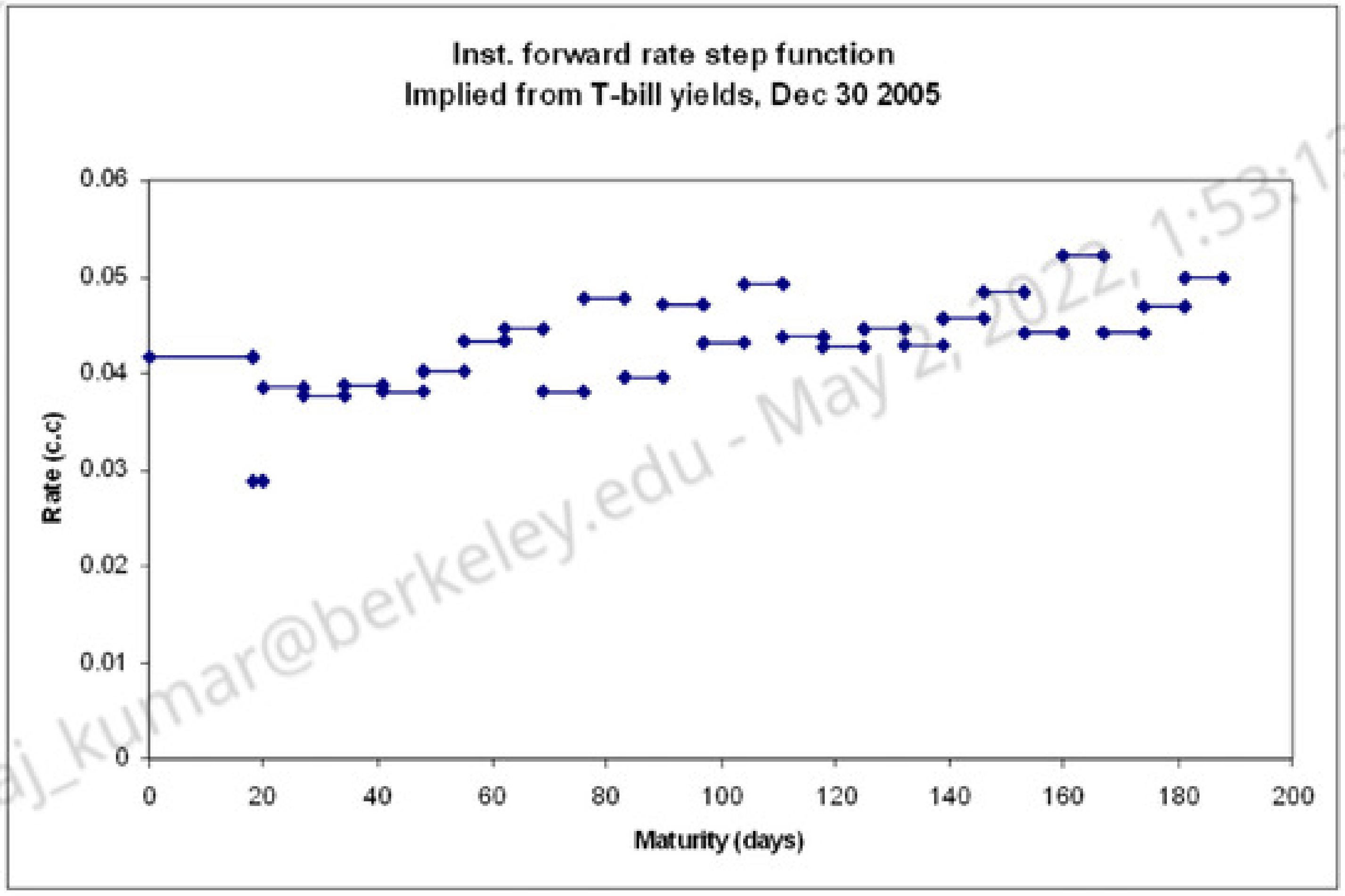
Source: Bloomberg

Does it always make sense to fit the input data exactly?

- The Bloomberg term structures fit the input rates exactly and are quite smooth.
- But they only use a relatively small number of input rates.
- What if we wanted to use lots more data?

Bootstrapped (piecewise-constant) forward rates

Treasury rates, Dec. 30, 2005



From bootstrapping to smoothing

- For many purposes, we prefer a curve that is smoother, even if it doesn't fit the data exactly.
  - Deemphasizes noise in bond prices, so may be more accurate than bootstrapped rates.
- Instead of a curve that fits the data exactly, we fit a smooth, low-dimensional curve as well as we can (not exactly).

Example: Polynomial in maturity

- We can parametrize $r(t, T)$, $f(t, T)$ or $Z(t, T)$. E.g., write

$$\hat{r}(t, t+T) = \sum_{j=0}^{J} b_j T^j.$$

  - A polynomial of order J.
- Given a coupon bond, fitted price is

$$\hat{P}(t, t+T; c) = \frac{c}{2}\left(\sum_{i=1}^{n_c} e^{-(T-(i-1)/2)\,\hat{r}(t,t+T-(i-1)/2)}\right) + e^{-T\hat{r}(t,t+T)}.$$

- Pricing errors for N coupon bonds with coupons c_i and prices P_i.

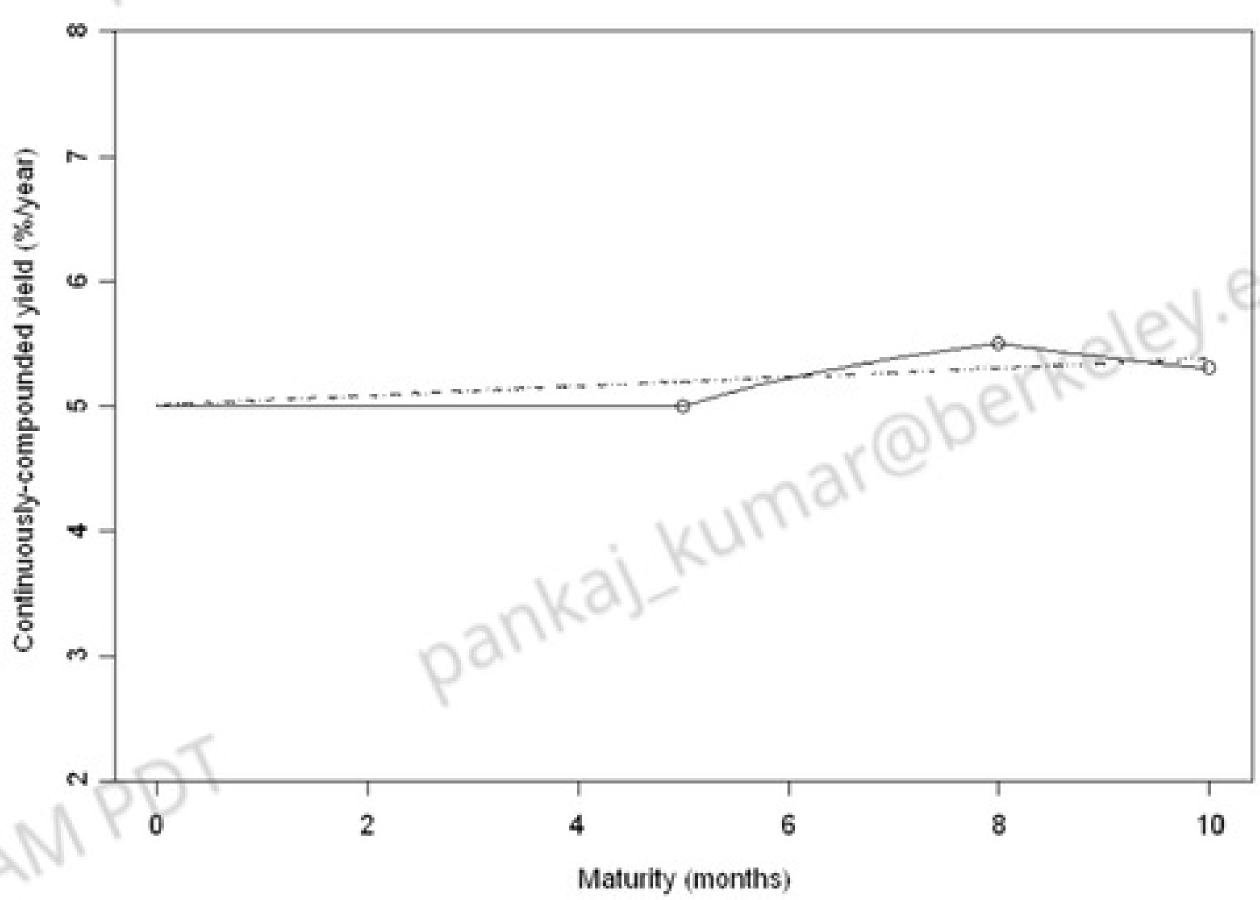
$$e_i = P(t, t+\tau_i; c_i) - \hat{P}(t, t+\tau_i; c_i).$$

- Estimation:

$$\min_{b_1,\ldots,b_J} \sum_{i=1}^{N} e_i^2.$$

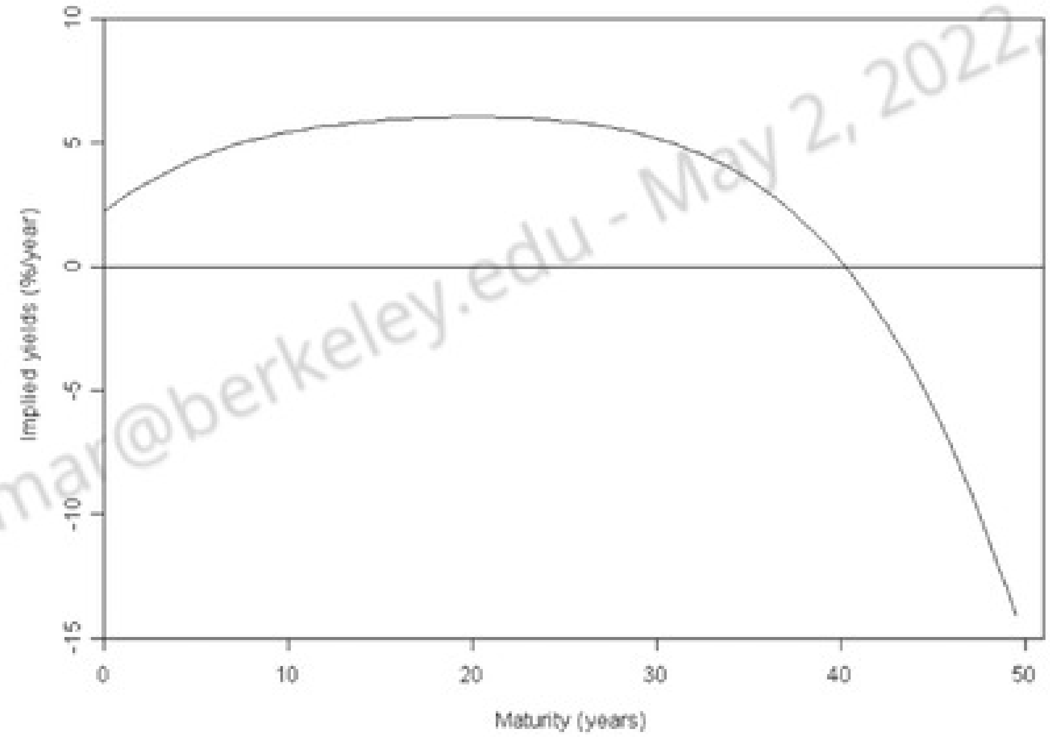
Polynomial example

- Same 3 bonds used in earlier bootstrapping example.
- Pick $J = 1$ (so not all bond prices can be fit)



Pros and cons of using polynomials

- Easy to implement
- Polynomials can uniformly approximate any continuous function on a finite interval (Weierstrass Theorem).
- Does a poor job of fitting local variations in yields
- Extrapolation outside the data works poorly



The Nelson-Siegel method

- Four parameters that can be chosen to produce the basic term structure shapes that we see in reality.
- Specification of continuously-compounded instantaneous forward rates:

$$f(t, t+T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \beta_2 \left(\frac{T}{\tau_1}\right) e^{-T/\tau_1}.$$

- Integration gives zero-coupon bond yields:

$$\begin{aligned} r(t, t+T) &= \frac{1}{T}\int_0^T f(t, t+\tau)\, d\tau \\ &= \beta_0 + \beta_1 \left(\frac{1-e^{-T/\tau_1}}{T/\tau_1}\right) + \beta_2 \left(\frac{1-e^{-T/\tau_1}}{T/\tau_1} - e^{-T/\tau_1}\right). \end{aligned}$$

- Parameters can be estimated using nonlinear least squares, just like the polynomial example.

Variants of Nelson-Siegel

- A five-parameter version of Nelson-Siegel (p. 475 in Nelson and Siegel, 1987) has instantaneous forward rates

$$f(t, t+T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \beta_2 \left(\frac{T}{\tau_2}\right) e^{-T/\tau_2}.$$

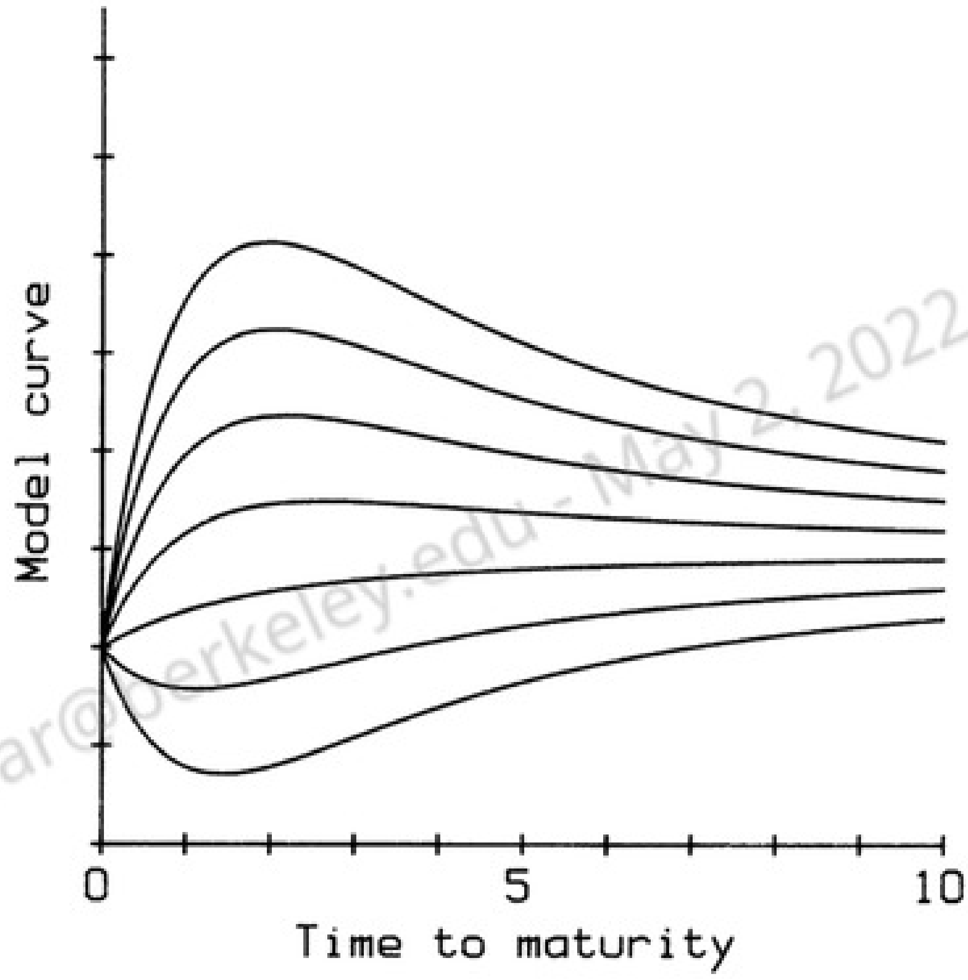
- Integration gives zero-coupon bond yields:

$$\begin{aligned} r(t, t+T) &= \frac{1}{T}\int_0^T f(t, t+\tau)\, d\tau \\ &= \beta_0 + \beta_1 \left(\frac{1-e^{-T/\tau_1}}{T/\tau_1}\right) + \beta_2 \left(\frac{1-e^{-T/\tau_2}}{T/\tau_2} - e^{-T/\tau_2}\right). \end{aligned}$$

Role of parameters

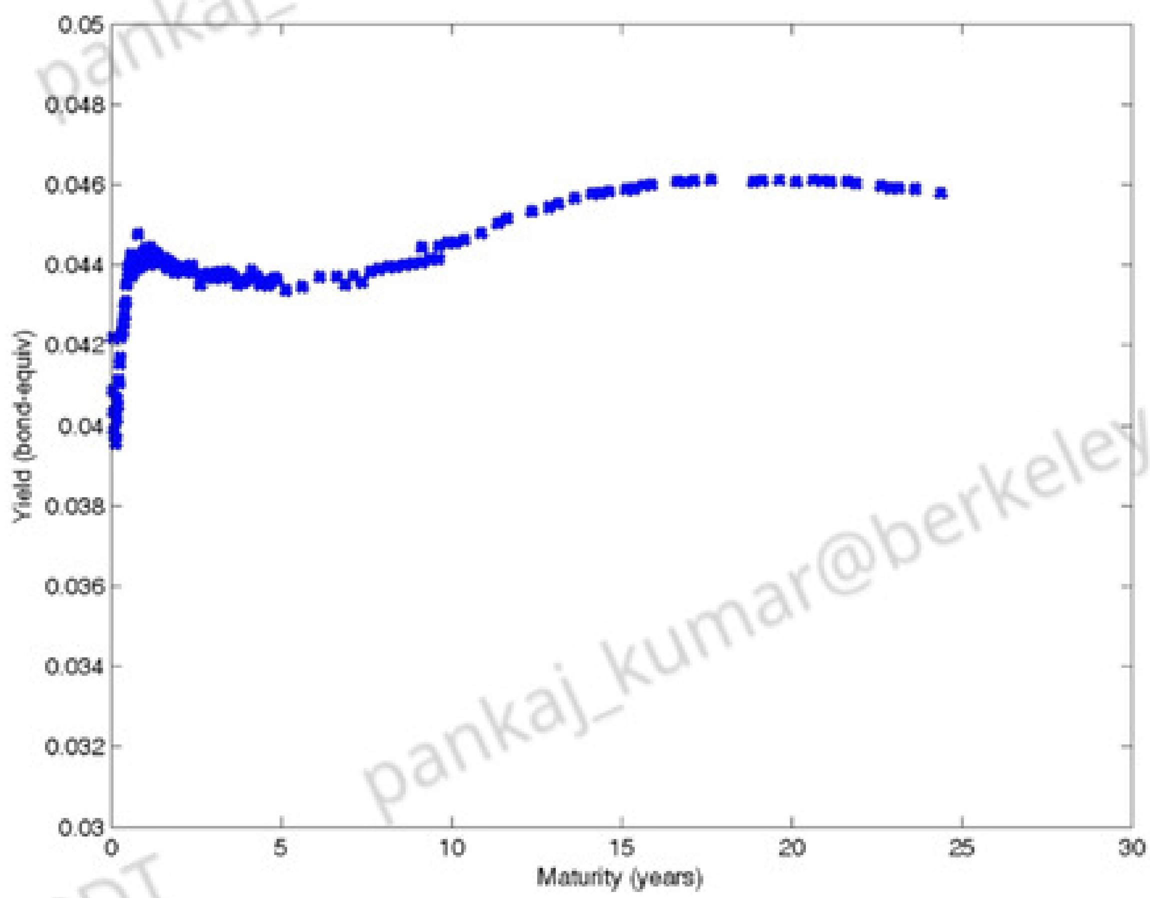
- Long end of term structure asymptotes to β_0.
- Short end begins at $\beta_0 + \beta_1$.
- Term structure has a "hump", with location determined by τ_1 (or τ_2).
- Size of hump determined by β_2.

Nelson-Siegel term structure shapes



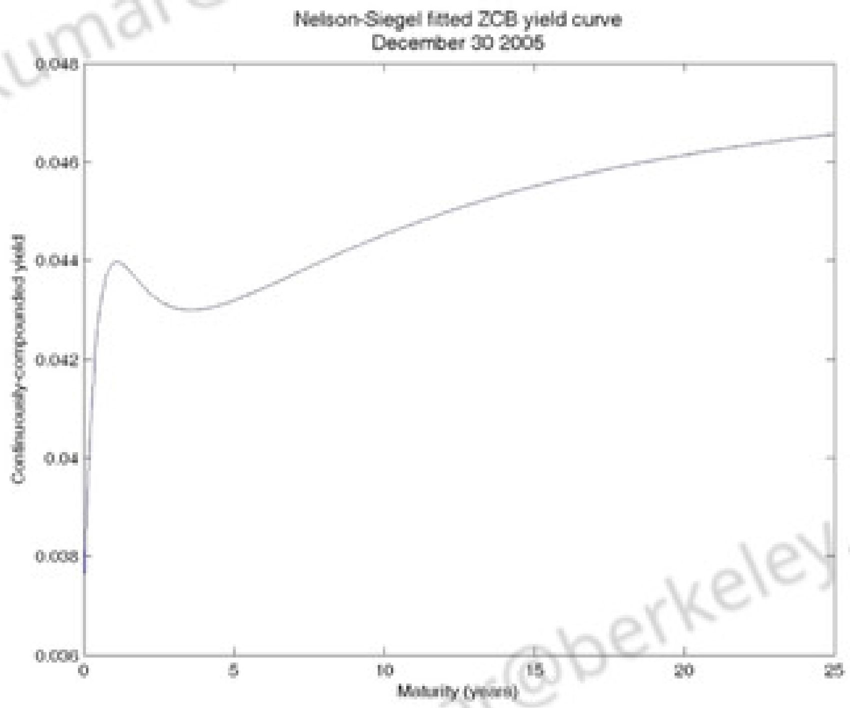
83

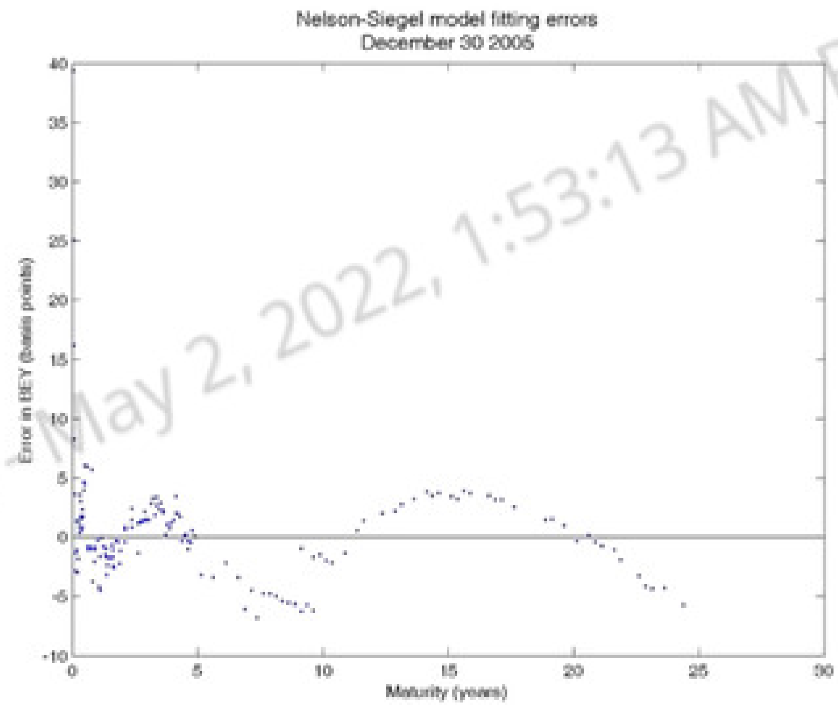
Raw data: Treasury yields, Dec. 30, 2005



84

Nelson-Siegel: results





- Error = actual minus model-implied zero-coupon bond yield.

Svensson (1994)

- Add an additional term (two parameters) to the Nelson-Siegel model, allowing for a second hump in the term structure:

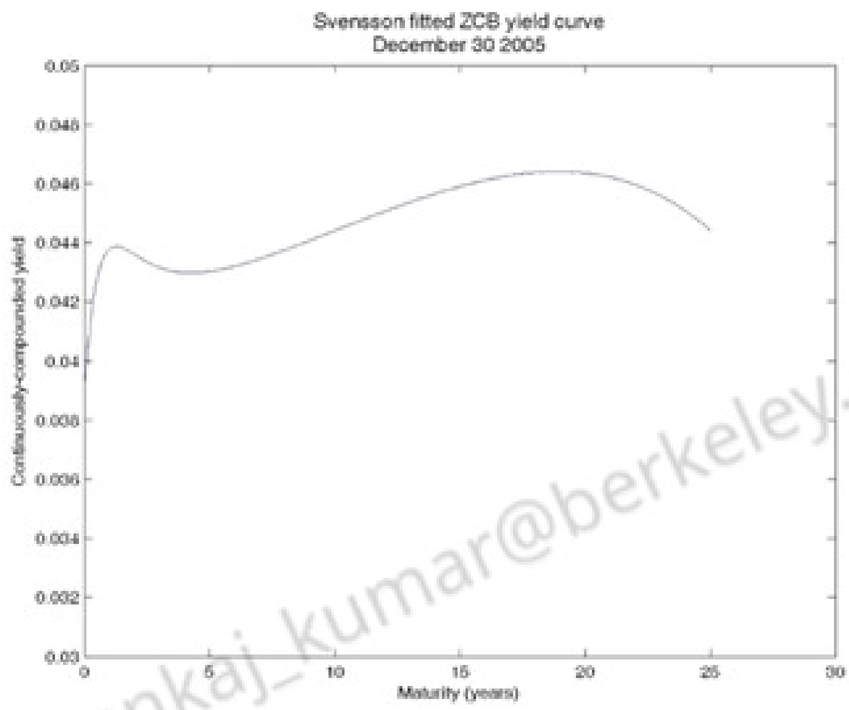
$$f(t, t+T) = \beta_0 + \beta_1 e^{-T/\tau_1} + \beta_2 \left(\frac{T}{\tau_1}\right) e^{-T/\tau_1} + \beta_3 \left(\frac{T}{\tau_2}\right) e^{-T/\tau_2}.$$

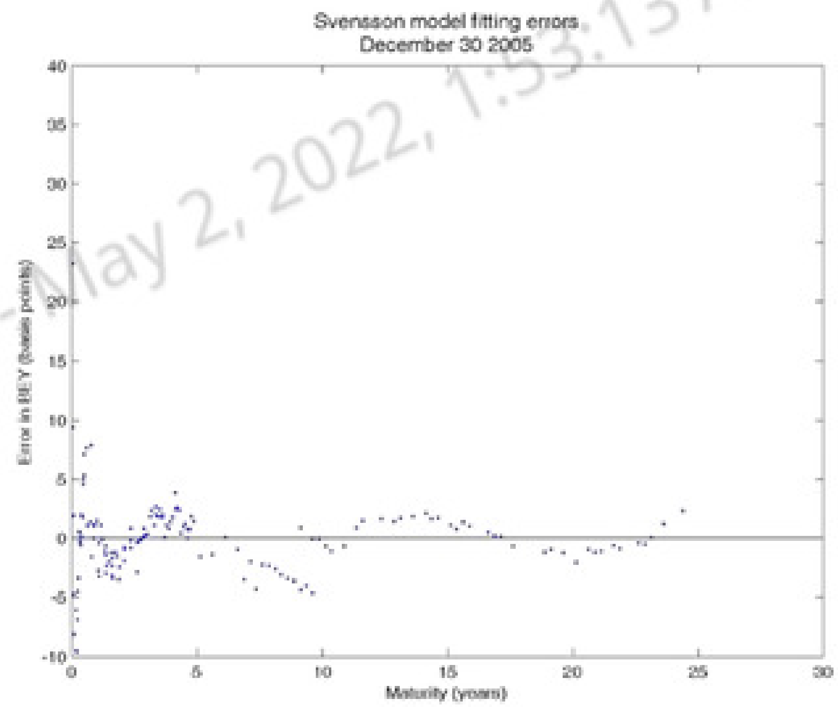
- Continuously-compounded zero-coupon-bond yield curve:

$$r(t, t+T) = \beta_0 + \beta_1 \left(\frac{1 - e^{-T/\tau_1}}{T/\tau_1}\right) + \beta_2 \left(\frac{1 - e^{-T/\tau_1}}{T/\tau_1} - e^{-T/\tau_1}\right) + \beta_3 \left(\frac{1 - e^{-T/\tau_2}}{T/\tau_2} - e^{-T/\tau_2}\right).$$

Svensson: results

- The same nonlinear least squares procedure is used to estimate the model's parameters for Treasury data of Dec 30, 2005



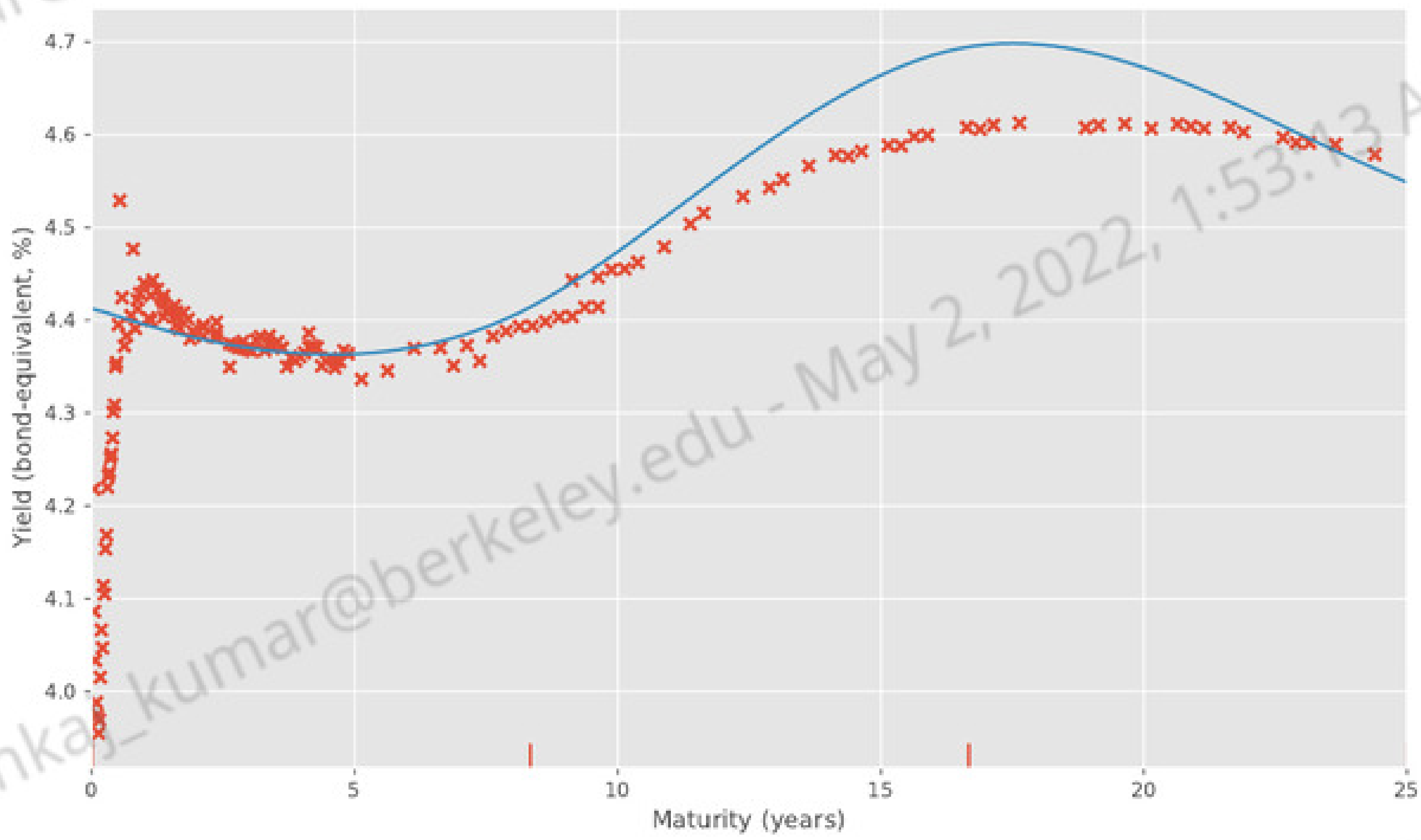


Cubic splines: McCulloch (1975)

- Yield curve is assumed piecewise cubic, with continuous first and second derivatives at a set of knot points.
- Similar to Bloomberg interpolation method 2, except that
  - Cubic instead of quadratic spline.
  - Joins (knot points) are not set equal to every bond maturity, but are selected as part of the fitting process.
  - Resulting curve does not fit all bond yields exactly, but is fit using least squares.

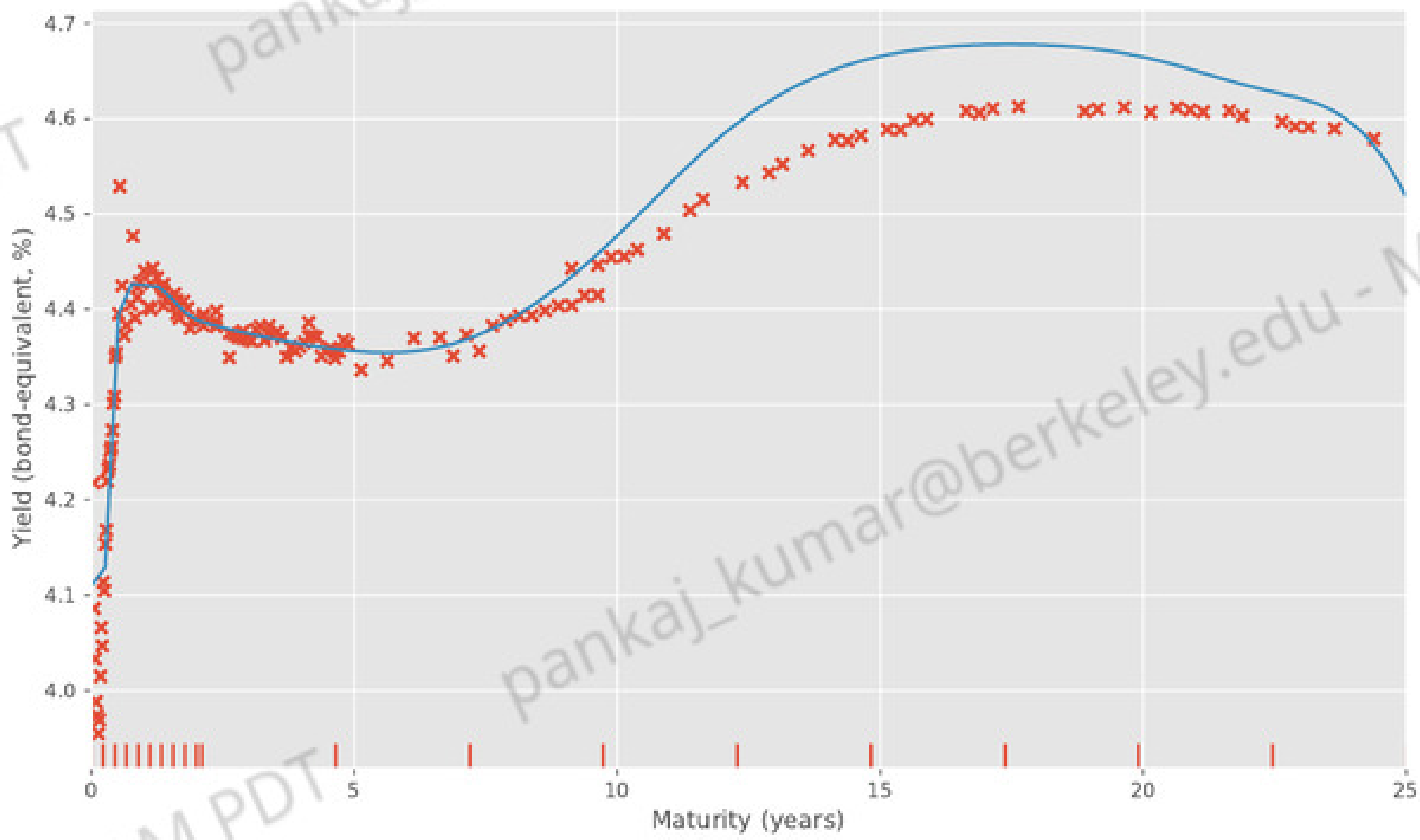
Cubic spline estimation: example

- $J = 5$; 4 equally spaced knot points in $[0, 25]$.



Cubic spline estimation: example

- $J = 21$; 20 non-equally spaced knot points in $[0, 25]$.



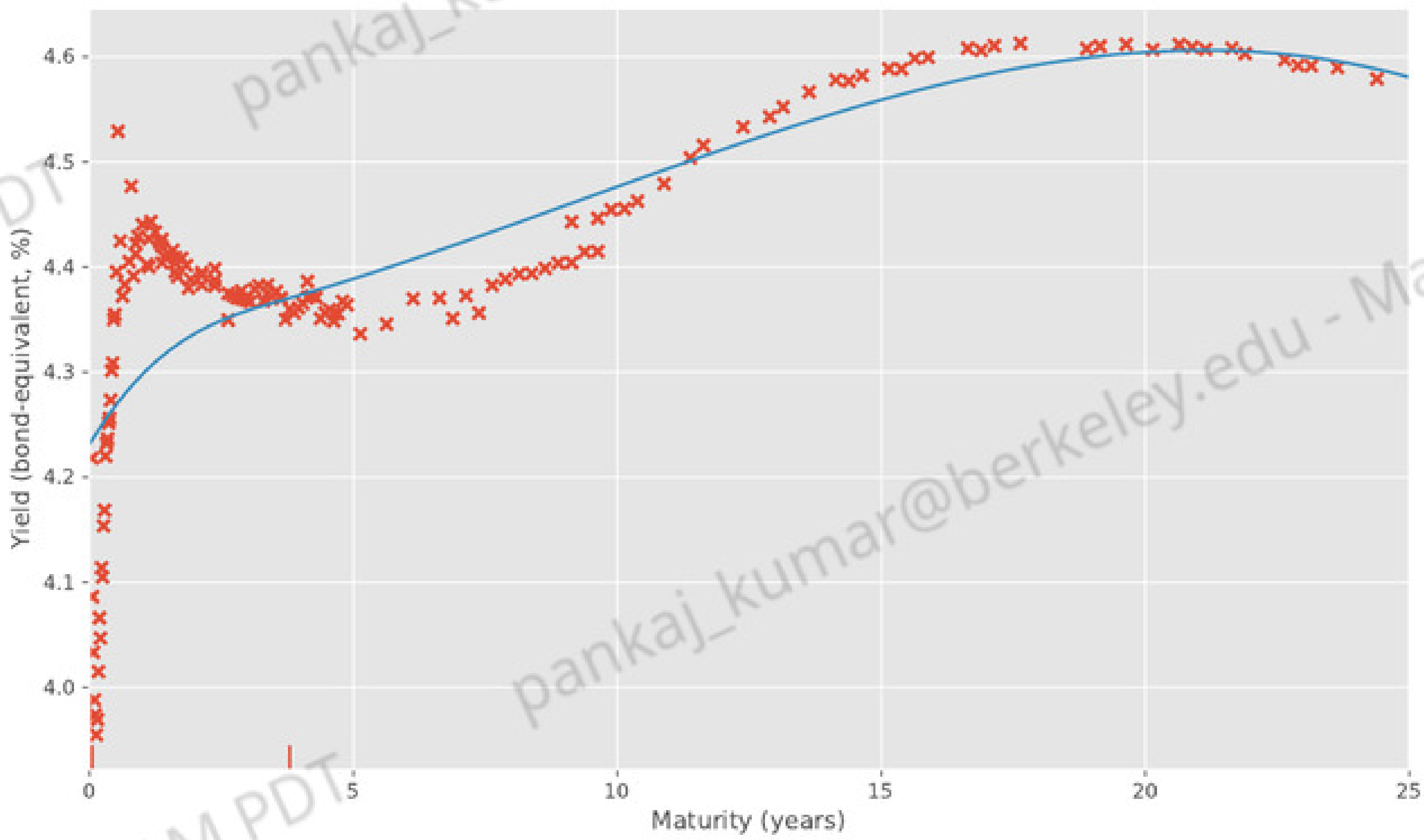
Smoothing Splines: Hastie, Tibshirani, and Friedman (2009)

- A smoothing spline is the *natural* cubic spline $\hat{f}$ that minimizes

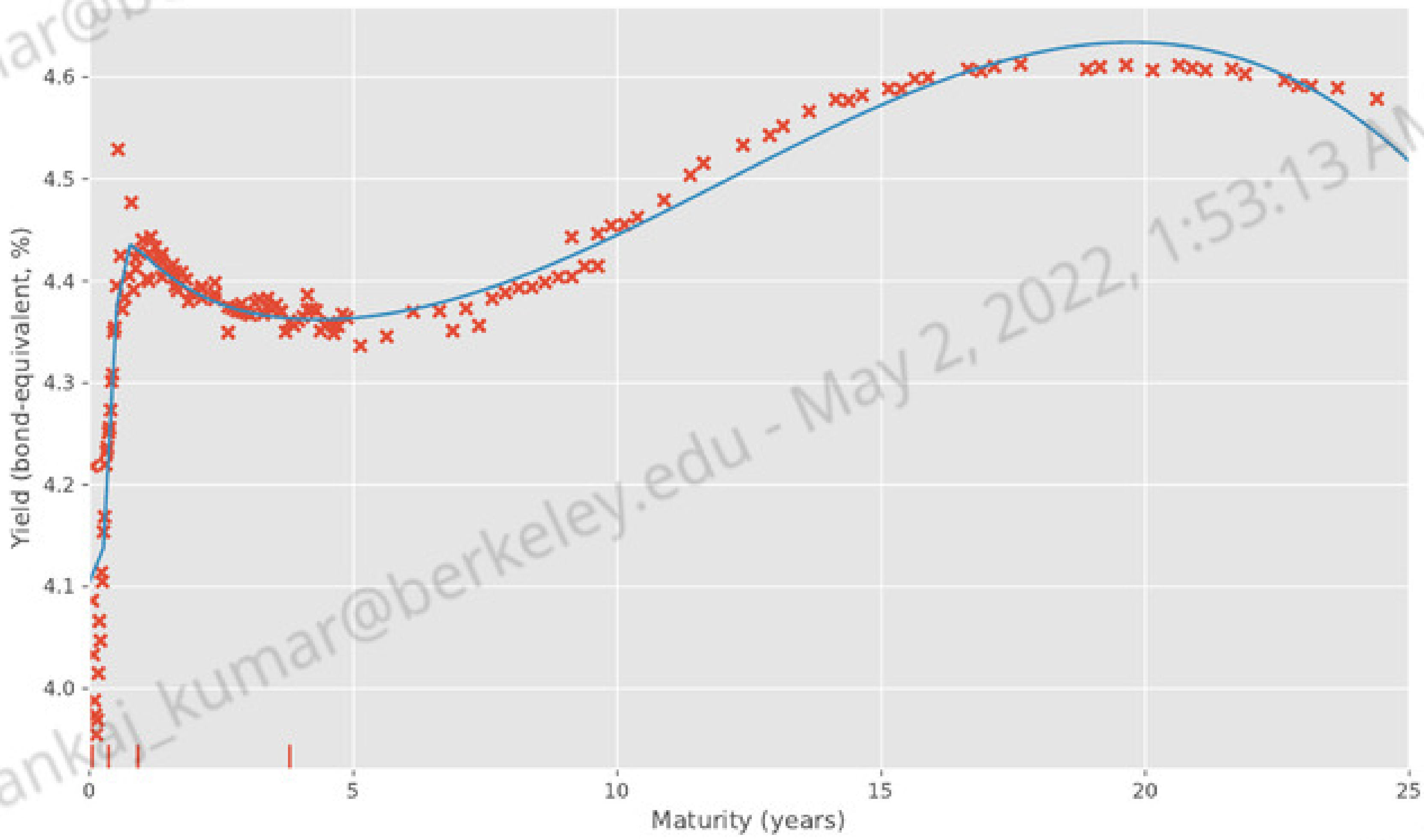
$$\sum_{i=1}^{n} \left[y_i - \hat{f}(x_i)\right]^2 + \lambda \int \hat{f}''(x)^2 \, dx. \tag{3}$$

- It penalizes for not closely fitting the data, but also for being too "rough," with smoothing parameter λ.
  - As $\lambda \to 0$, we get no smoothing, just interpolation.
  - As $\lambda \to \infty$, smoothness becomes most important, and estimate converges to OLS.
- E.g., Python function `scipy.interpolate.UnivariateSpline()`.
- Knots are (automatically) selected at a subset of the x_i values.
- Note: a smoothing spline is the solution to minimization (3), even when we allow $\hat{f}$ to be *any* function.
- For application to term-structure fitting, see Adams and van Deventer (1994); van Deventer and Imai (1997); van Deventer, Imai, and Mesler (2013).

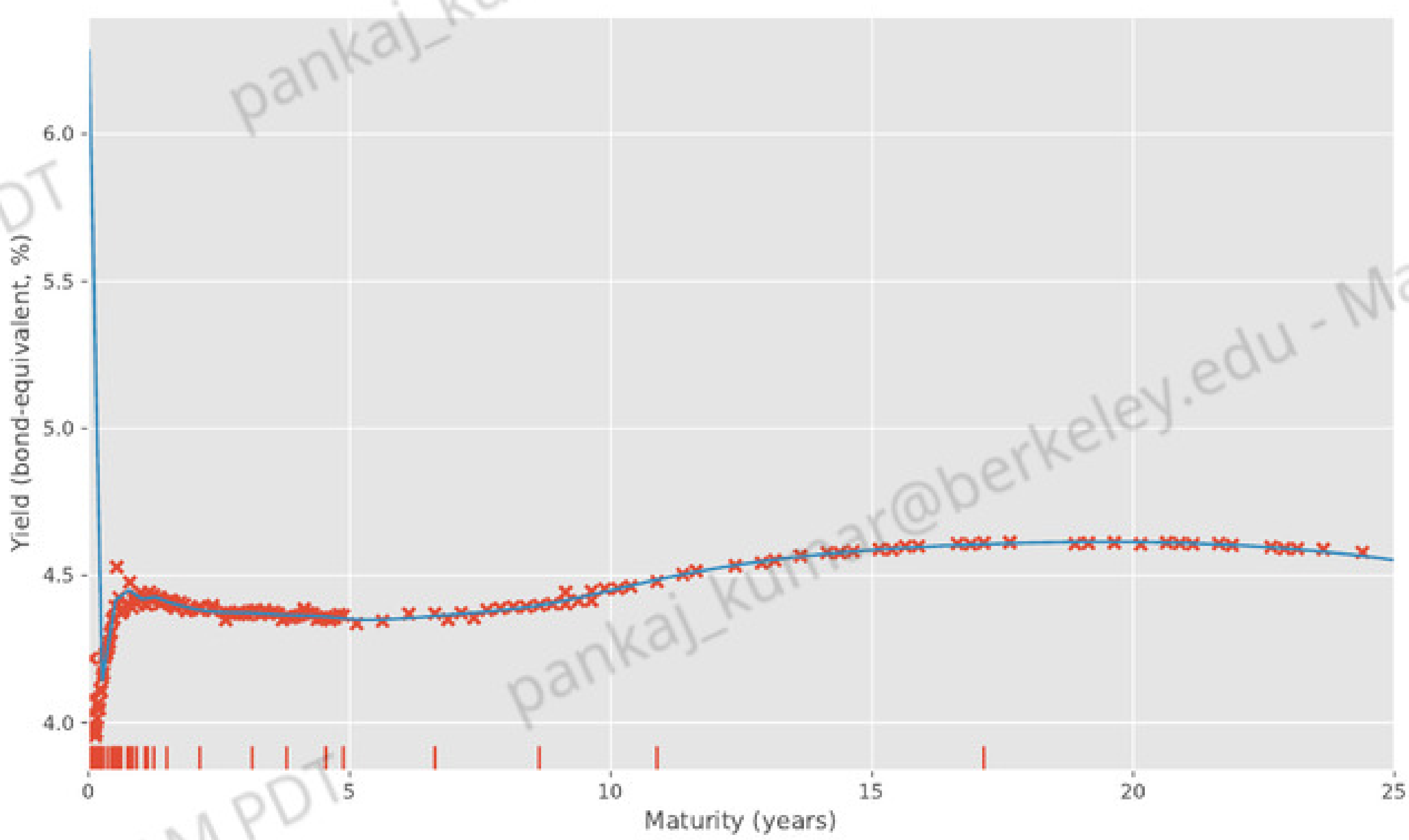
Smoothing spline example: $\lambda = 0.0001$ (3 knots)



Smoothing spline example: $\lambda = 0.00001$ (5 knots)



Smoothing spline example: $\lambda = 0.000001$ (34 knots)



Pros and cons of splines

- Easy to implement
  - Function definition is a straightforward algorithm
  - Included in standard software packages
- Captures local curvature (with enough knot points)
- Can be fine-tuned by adjusting knot points/smoothing
- Bad extrapolation properties (cubic term blows up for out-of-sample maturities)
- The US Treasury uses cubic splines to estimate the yield curve daily
  - `http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yieldmethod.html`

Techniques used by Central Banks (BIS, 2005)

Central bank	Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
Belgium	Svensson or Nelson-Siegel	Weighted prices	Treasury certificates: > few days Bonds: > one year	No	Couple of days to 16 years
Canada	Merrill Lynch Exponential Spline	Weighted prices	Bills: 1 to 12 months Bonds: > 12 months	Effectively by excluding bonds	3 months to 30 years
Finland	Nelson-Siegel	Weighted prices	≥ 1 day	No	1 to 12 years
France	Svensson or Nelson-Siegel	Weighted prices	Treasury bills: all Treasury Notes: : ≥ 1 month Bonds: : ≥ 1 year	No	Up to 10 years
Germany	Svensson	Yields	> 3 months	No	1 to 10 years

Techniques used by Central Banks (BIS, 2005)

Central bank	Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
Italy	Nelson-Siegel	Weighted prices	Money market rates: O/N and Libor rates from 1 to 12 months Bonds: > 1 year	No	Up to 30 years Up to 10 years (before February 2002)
Japan	Smoothing splines	Prices	≥ 1 day	Effectively by price adjustments for bills	1 to 10 years
Norway	Svensson	Yields	Money market rates: > 30 days Bonds: > 2 years	No	Up to 10 years
Spain	Svensson Nelson-Siegel (before 1995)	Weighted prices Prices	≥ 1 day ≥ 1 day	Yes No	Up to 10 years Up to 10 years
Sweden	Smoothing splines and Svensson	Yields	≥ 1 day	No	Up to 10 years
Switzerland	Svensson	Yields	Money market rates: ≥ 1 day Bonds: ≥ 1 year	No	1 to 30 years

Techniques used by Central Banks (BIS, 2005)

Central bank	Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
United Kingdom[1]	VRP (government nominal)	Yields	1 week (GC repo yield)	No	Up to around 30 years
	VRP (government real/implied inflation)	Yields	1.4 years	No	Up to around 30 years
	VRP (bank liability curve)	Yields	1 week	No	Up to around 30 years
United States	Smoothing splines (two curves)	Bills: weighted prices Bonds: prices	– ≥ 30 days	No No	Up to 1 year 1 to 10 years

[1] The United Kingdom used the Svensson method between January 1982 and April 1998.

Outline

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Basic Fixed-Income Securities

What riskless rate to use?

Basic fixed-income securities 1

- **Zero coupon bond**: Pays a fixed principal amount at a specified maturity date.
- **Coupon bond**: Like a zero, but also pays periodic fixed coupon payments.
- **STRIPS** (Separate Trading of Registered Interest and Principal of Securities): Artificial zero coupon bonds constructed by stripping off separate interest and principal payments from a coupon bond.
- **Floating rate note**: Like a coupon bond, but the coupon rate varies with some interest rate such as LIBOR (London Interbank Offer Rate)

Basic fixed-income securities 2

- **Forward Contract/Forward Rate Agreement (FRA)**: Agreement to lend/borrow a fixed amount at a fixed interest rate some time in the future, or buy/sell a security at a fixed price.

- **Swap:** Agreement to exchange (or swap) future cashflows based on two different interest rates, etc. E.g., one side pays fixed rate, one pays floating.

- **Futures contract**: Similar to a forward contract, but contracts are standardized, trade on an exchange, and are marked to market daily.

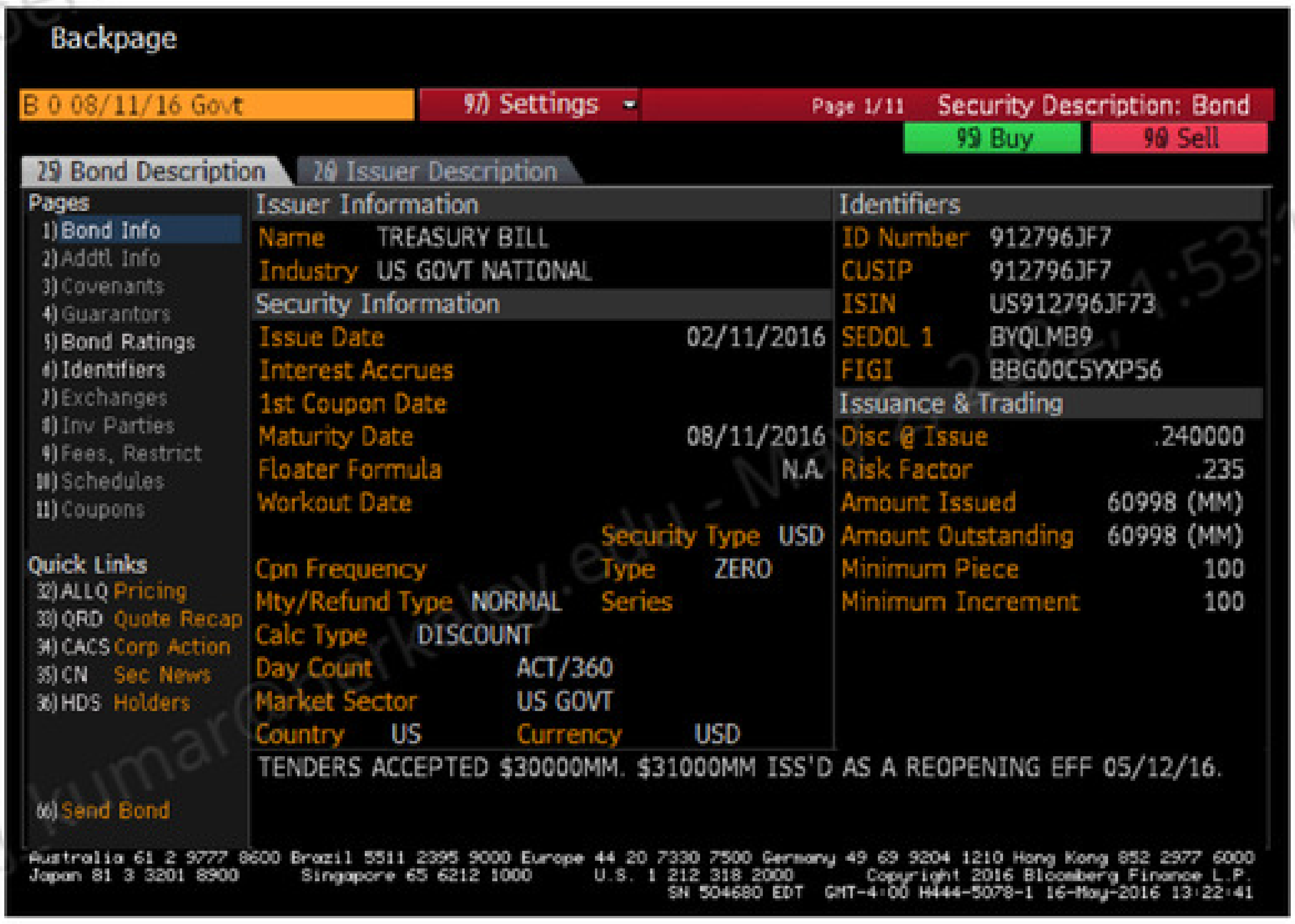
- **Option:** The right to buy or sell some asset at a fixed price.
  - Bond options
  - Swaptions
  - Embedded options

Treasury bills

- Treasury Bills are issued by the US Treasury, with 3 months, 6 months and 1 year to maturity
  - 3 and 6 month Bills usually auctioned weekly.
  - 1 year Bills usually auctioned monthly.

- Treasury Bills are zero-coupon bonds.
  - You pay for them today, and receive the face value, usually \$10,000, at maturity.
  - Bond prices are quoted relative to a \$100 face value
    - I.e., the quoted price is for 1/100 of a T-Bill

Bloomberg screen for a U.S. Treasury bill, May 16, 2016



Treasury bills: Discount

- T-bills are quoted as a discount.
- For a bill maturing in n days, the discount, d, is defined as:

$$d = \frac{360}{n} \times \frac{100 - P}{100}, \qquad \text{so}$$

$$P = 100 \times \left(1 - \frac{nd}{360}\right).$$

- E.g., the bill maturing on Aug. 11, 2016 has 86 days to maturity from May 17, 2016 (the settlement date, one day after trade date).
- Quoted (ask) discount is 0.260%, so its price is

$$P = 100 \times \left(1 - \frac{86 \times 0.00260}{360}\right) = \$99.93789.$$

- This is the price per \$100 FV, so for 1 bill (FV \$10,000) you'd pay

$$99.93789 \times 10{,}000/100 = \$9{,}993.79.$$

Treasury bills: Effective annual rate

- The non-annualized interest rate over the next 86 days is

$$r_{86\text{ day}} = 100/99.93789 - 1 = 0.0621\%.$$

- So the EAR is given by:

$$\text{EAR} = (1 + r_{86\text{ day}})^{365/86} - 1 = 0.2640\%.$$

- Slightly higher than the quoted discount rate.

Treasury bills: Quoted yield

- The ask yield is defined as:

$$y = \frac{365}{n} \times \frac{100 - P^{\text{ask}}}{P^{\text{ask}}}.$$

- For our bill,

$$y = \frac{365}{86} \times \frac{100 - 99.93789}{99.93789}$$
$$= .2638\%.$$

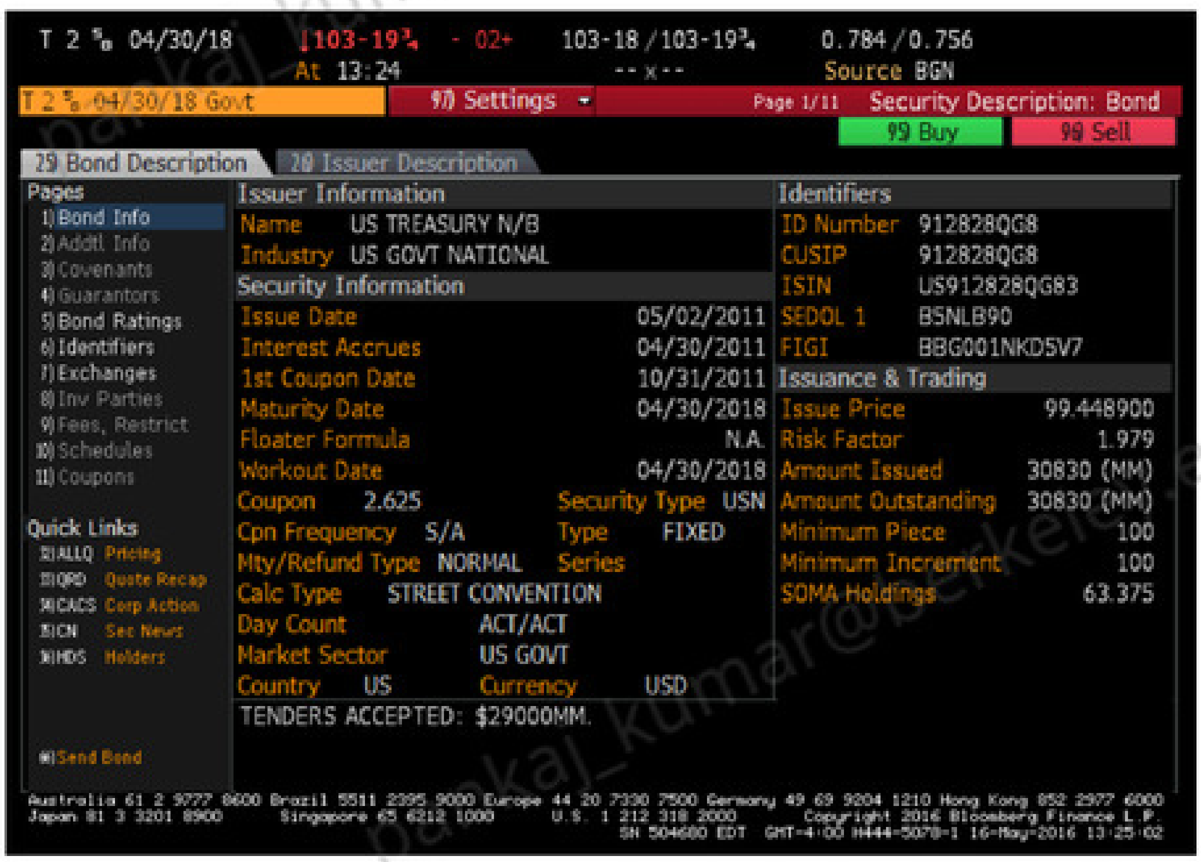
- Larger than the discount (but slightly smaller than the EAR).
- Quoted as an APR, compounded every n days.

Treasury notes and bonds

- Treasury Notes and Treasury Bonds are coupon paying bonds issued by the US government. The only difference is maturity:
  - Notes have more than 1, and up to 10, yrs. to maturity.
  - Bonds have more than 10, and up to 30, yrs. to maturity

- Coupon payments are made every 6 months.

Bloomberg screen for a U.S. Treasury note, May 16, 2016



- Prices are quoted in 1/32nds of a dollar.
- Ask price = $103 + \frac{19\ 3/4}{32} = 103.6171875$.

Accrued interest

- Quoted ("clean") prices are not the actual prices you pay.
  - You have to add accrued interest to get the "dirty price"
- If t_0 is the date of the last coupon payment, t_1 the date of the next payment, and t is settlement date (one business day after trade date).

$$\text{Accrued interest} = \text{Principal} \times \frac{\text{coupon rate}}{2} \times \frac{\text{Actual days between } t_0 \text{ and } t}{\text{Actual days between } t_0 \text{ and } t_1}.$$

- For our note, last coupon was April 30, 2016.
  - 17 days ago as of May 17, 2016 (settlement date, one day after trade date).
- Next coupon is October 31, 2016.
  - 184 days after April 30.
- Accrued interest = (17 / 184) x (2.625 / 2) = $0.12126.
- Price you'd actually pay = $103.6172 + $0.12126
  = $103.73845.

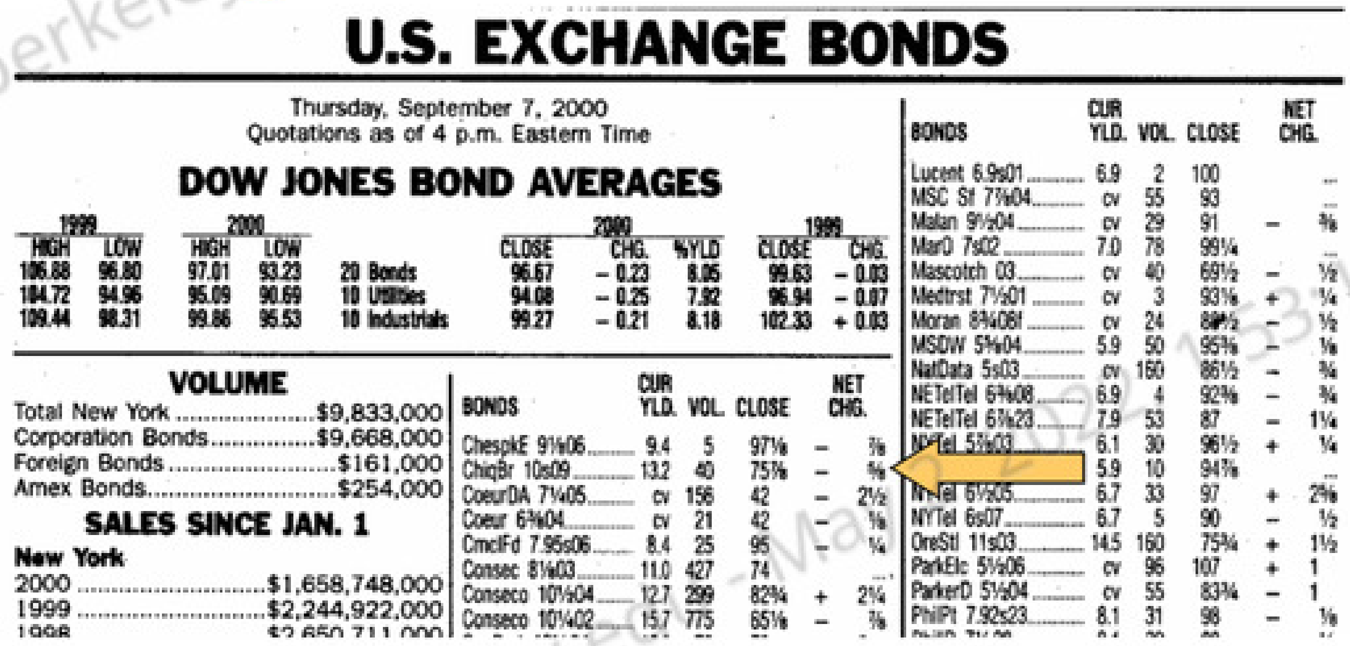
Calculating the yield

- Each coupon payment = 2.625/2 = $1.3125
- Next coupon payment is 167 days from settlement date.
  - 167/184 = 0.9076 compounding periods.
  - Then every 6 months until April 2018 (4 payments in all).
- Yield satisfies

$$103.73845 = \frac{1.3125}{\left(1+\frac{y}{2}\right)^{0.9076}} + \frac{1.3125}{\left(1+\frac{y}{2}\right)^{1.9076}} + \frac{1.3125}{\left(1+\frac{y}{2}\right)^{2.9076}} + \frac{101.3125}{\left(1+\frac{y}{2}\right)^{3.9076}}.$$

- Solution: $y = 0.756\%$.

Corporate bond listings



U.S. EXCHANGE BONDS

Thursday, September 7, 2000
Quotations as of 4 p.m. Eastern Time

DOW JONES BOND AVERAGES

1999 HIGH	1999 LOW	2000 HIGH	2000 LOW		2000 CLOSE	2000 CHG.	2000 %YLD	1999 CLOSE	1999 CHG.
106.88	96.80	97.01	93.23	20 Bonds	96.67	− 0.23	8.05	99.63	− 0.03
104.72	94.96	95.09	90.69	10 Utilities	94.08	− 0.25	7.92	96.94	− 0.07
109.44	98.31	99.86	95.53	10 Industrials	99.27	− 0.21	8.18	102.33	+ 0.03

VOLUME

Total New York $9,833,000
Corporation Bonds...............$9,668,000
Foreign Bonds $161,000
Amex Bonds...........................$254,000

SALES SINCE JAN. 1

New York

2000 ....$1,658,748,000
1999 ....$2,244,922,000

BONDS	CUR YLD.	VOL.	CLOSE	NET CHG.
ChespkE 9⅛06	9.4	5	97⅛	− ⅞
ChiqBr 10s09	13.2	40	75⅞	− ⅝
CoeurDA 7¼05	cv	156	42	− 2½
Coeur 6⅜04	cv	21	42	− ⅛
CmclFd 7.95s06	8.4	25	95	− ¼
Consec 8⅛03	11.0	427	74	...
Conseco 10½04	12.7	299	82¾	+ 2¼
Conseco 10¼02	15.7	775	65⅛	− ⅞

BONDS	CUR YLD.	VOL.	CLOSE	NET CHG.
Lucent 6.9s01	6.9	2	100	...
MSC Sf 7⅞04	cv	55	93	...
Malan 9½04	cv	29	91	− ⅝
MarO 7s02	7.0	78	99¼	...
Mascotch 03	cv	40	69½	− ½
Medtrst 7½01	cv	3	93⅝	+ ¼
Moran 8¾08f	cv	24	89½	− ½
MSDW 5⅝04	5.9	50	95⅝	− ⅛
NatData 5s03	cv	160	86½	− ¾
NETelTel 6⅜08	6.9	4	92⅝	− ¾
NETelTel 6⅞23	7.9	53	87	− 1¼
NYTel 5⅞03	6.1	30	96½	+ ¼
[illegible]	5.9	10	94⅞	...
NYTel 6½05	6.7	33	97	+ 2⅝
NYTel 6s07	6.7	5	90	− ½
OreStl 11s03	14.5	160	75¾	+ 1½
ParkElc 5½06	cv	96	107	+ 1
ParkerD 5½04	cv	55	83¾	− 1
PhilPt 7.92s23	8.1	31	98	− ⅛

- Current yield = coupon rate × par value/market price.
  - 10% × 100 / 75.875 = 13.2%.
  - For discount bond, YTM > current yield > coupon rate.
  - For premium bond, YTM < current yield < coupon rate.
- Accrued interest for corporate/municipal bonds assumes 30 days per month, 360 days per year.

Floating-rate notes

- A floating-rate note (FRN) has a coupon indexed to a benchmark interest rate.
  - E.g., Treasury rates, LIBOR, prime rate, municipal and mortgage interest rates.
- Examples of FRNs
  - Corporate (especially financial institutions)
  - Adjustable-rate mortgages (ARMs)
  - Governments (inflation-indexed notes)
- Other terms used for floating-rate notes include
  - Floaters and inverse floaters
  - Variable-rate notes (VRNs)
  - Adjustable-rate notes
- FRN usually refers to an instrument whose coupon is based on a short-term rate (3-month T-bill, 6-month LIBOR)
- VRNs based on longer-term rates (1yr T-bill, 5yr T-bond)

Floating-rate notes

- An FRN makes payments at intervals Δ, based on the prior period's value of the underlying interest rate.

- If face value = N,

$$\text{Coupon}_t = N\Delta \times r_n(t-\Delta, t),$$

  where $n = 1/\Delta$.

- For example, suppose $\Delta = 1/2$ and face value is \$100.
  - If current 6-month rate is 4%, the next coupon payment will be 0.5 x 4% x 100 = \$2.
  - If 6-month rate moves to 5% 6 months from now, the coupon payment in one year will be 0.5 x 5% x 100 = \$2.50.

Floating-rate notes

- Ex-coupon value at date $T-\Delta$, one period before maturity, is

$$P^{\text{Fl}}(T-\Delta) = \frac{N\left(1+\Delta r_n(T-\Delta, T)\right)}{1+\Delta r_n(T-\Delta, T)} = N.$$

- Similarly, value at date $T-2\Delta$, two periods before maturity, is

$$P^{\text{Fl}}(T-2\Delta) = \frac{N\left(1+\Delta r_n(T-2\Delta, T-\Delta)\right)}{1+\Delta r_n(T-2\Delta, T-\Delta)} = N.$$

- Continuing backwards, $P^{\text{Fl}}(t) = N$ at *every* reset date.

Valuing a single floating payment

- What is today's value of floating payment $N\Delta r_n(T-\Delta, T)$ at date T, where $n = 1/\Delta$?
- We can replicate this payment as follows:
  1) Buy a zero-coupon bond with face value N and maturity $(T-\Delta)$.
  2) Sell a zero-coupon bond with face value N and maturity T.
  3) Invest N at date $T-\Delta$ for one period at rate $r_n(T-\Delta, T)$.
- NPV of 3) is 0 at any time prior to $T-\Delta$, so

$$\begin{aligned}\text{Cost today} &= \frac{N}{(1+\Delta r_n(t, T-\Delta))^{nT-1}} - \frac{N}{(1+\Delta r_n(t, T))^{nT}} \\ &= \frac{N\Delta f_n(t, T-\Delta, T)}{(1+\Delta r_n(t, T))^{nT}}.\end{aligned}$$

- Calculate payoff as if today's forward rate were realized, then discount at riskless rate.

Forward Rate Agreement (FRA)

- Forward Rate Agreement (FRA): One counterparty pays fixed rate c on notional N during future period T_1 to $T_2 = T_1 + \Delta$; the other pays according to future market LIBOR, $r_n(T_1, T_2)$, where $n = 1/\Delta$.
  - A 1 x 4 FRA is for a 3-month loan between months 1 and 4 from today (underlying rate = 3m LIBOR).
  - A 12 x 18 FRA is for a 6-month loan between months 12 and 18 from today (underlying rate = 6m LIBOR).
- Net payment to fixed lender at T_2 is

$$N\Delta\left[c - r_n(T_1, T_2)\right].$$

- Note: In practice, the FRA is usually settled at T_1, with cash flow

$$\frac{N\Delta\left[c - r_n(T_1, T_2)\right]}{1 + r_n(T_1, T_2)\Delta}.$$

Example

- Suppose a 12 x 18 FRA with principal amount $1M has a contract interest rate of 5%, and 6m LIBOR at settlement (month 12) = 6%.
- In principle:
  - In month 12, fixed lender borrows at 6% for 6 months, investing proceeds at 5%, receiving cash flow in month 18 of

$$1M \times (5\% - 6\%)/2 = -\$5{,}000$$

- In practice:
  - In month 12, fixed lender pays fixed borrower

$$\frac{5{,}000}{1 + .06/2} = \$4{,}854.37$$

FRA valuation

- Net payment to fixed lender at T_2 is

$$N\Delta \left[c - r_n(T_1, T_2)\right].$$

- From earlier result about PV of floating payments, PV is

$$\frac{N\Delta \left(c - f_n(t, T_1, T_2)\right)}{(1 + r_n(t, T_2)\Delta)^{n(T_2 - t)}}.$$

- I.e., again calculate payoff as if today's forward rates were realized, then discount at riskless rate.

Example

- 12 x 18 FRA with principal amount \$1M and contract interest rate 5%.
- 1-year spot rate, $r_2(0,1) = 4.2\%$.
- Forward LIBOR between 12m and 18m, $f_2(0,1,1.5) = 4.0\%$.
- So 18-month spot rate, $r_2(0,1.5)$, is

$$\left(\left[\left(1+\frac{.042}{2}\right)^2\left(1+\frac{.04}{2}\right)\right]^{1/3} - 1\right) \times 2 = 4.1333\%.$$

- What is the PV of the fixed lender's position?
  1. "Payoff" at month 18 is
  $$\$1M \times (5\% - 4\%)/2 = \$5{,}000.$$
  2. So PV today is
  $$\text{PV} = \frac{5{,}000}{\left(1+\frac{.041333}{2}\right)^3} = \$4{,}702.39.$$

Swaps

- Interest rate swap: A contract which commits two counterparties to exchange, over an agreed period, two streams of interest payments.
  - Each is calculated using a different interest rate index, but
  - Each is applied to a common notional principal amount.
- Plain-vanilla or generic swap: fixed-for-floating swap, constant notional principal, constant fixed rate, floating 6-month interest rate, semi-annual payments.
- The swap rate is the quoted fixed rate.
- Terminology: "Fixed payer" vs. "fixed receiver."

Swap cash flows

- If the notional amount of the swap is N and the fixed swap rate is c, every six months until maturity the fixed receiver receives $Nc/2$ and pays the 6-month rate set 6 months earlier.
- I.e, if the swap maturity is T, the time-t cash flow to the fixed receiver is

$$N\left(c - r_2(t-0.5,t)\right)/2$$

  for $t = 0.5, 1, 1.5, \ldots, T$.
- No principal is exchanged.
- Note: Equivalent to a portfolio of FRAs with common fixed rate.

Swap = bond − floater

- Consider again the cash flows of the plain vanilla swap with fixed rate c, notional amount N and maturity T:

$$N\left(c - r_2(t-0.5,t)\right)/2$$

  for $t = 0.5, 1, 1.5, \ldots, T$.
- Same as the cash flows from a portfolio containing:
  - a long position in a T-year fixed-rate note with par amount N and coupon rate c; and
  - a short position in a T-year floating-rate note with par amount N.
- The difference between the coupons equals the swap payment.
  - The difference between their principal payments is zero.
- swap(c, T) = fixed-rate note(c, T) − floating-rate note

Swap valuation

- A fixed/floating interest rate swap is equivalent to a portfolio of FRAs with the same fixed rate.
- The value of the swap is thus the sum of the values of each of the FRAs.
- I.e., to value a plain vanilla interest rate swap,
  1. Calculate forward rates for each of the LIBOR rates that will affect the swap's cash flows.
  2. Calculate the swap cash flows assuming LIBOR at each future date will equal the appropriate forward rate today.
  3. Discount these cash flows back to the present.

Swap rate = par yield

- The fixed rate in the swap is called the swap rate.
  - Swap rate set to make the swap initially worth zero.
- Recall: swap(c, T) = fixed-rate note$(c, T) -$ floater
  - The value of the floater is par.
  - To make the swap worth zero, the swap rate must make the fixed rate bond worth par as well.

- The swap rate, c, must be the par yield for maturity T.

Swap rate = par yield

- Thus, if payments occur (and rates are compounded) n times per year,

$$c = n \times \left(\frac{1 - Z(0, T)}{\sum_{i=1}^{n_c} Z(0, T_i)} \right).$$

The swap curve

- The swap curve at time t is the set of swap rates for every maturity, $c(t, T_i)$.
- The discount function, $Z(t, T_i)$, can be calculated iteratively from the swap curve:

$$Z(t, T_1) = \frac{1}{1 + \frac{c(t,T_1)}{n}}$$

$$Z(t, T_2) = \frac{1 - \frac{c(t,T_2)}{n} Z(t, T_1)}{1 + \frac{c(t,T_2)}{n}}$$

$$Z(t, T_i) = \frac{1 - \frac{c(t,T_i)}{n} \sum_{j=1}^{i-1} Z(t, T_j)}{1 + \frac{c(t,T_i)}{n}}.$$

Swap-based products

- Forward swap: The swap begins at some specified future date with the terms set in advance (mutually binding).
- Amortizing swap: Notional amount of swap (and thus the size of the coupon payments) changes over time according to a schedule.
- Zero swap: There is no exchange of payments until maturity. Then a fixed amount of accumulated interest is exchanged for interest that has accumulated at the floating interest rate.
- Swaption: An option on a swap, usually with strike price 0, i.e., the right to enter into a swap with specified terms at some future date.
- Put(t)able swap: The fixed interest receiver has the right to cancel the swap before maturity (the premium for the cancelation option is paid up front)
- Callable swap: The fixed interest payer has right to cancel the swap before maturity (premium paid up front)

Repurchase agreement (repo)

- Repurchase agreement (repo): Agreement to sell securities to another party and buy them back at a later date for a higher price.
  - Very similar to a secured (collateralized) loan.
    - Implied interest rate is called the repo rate.
    - There are also usually margin requirements.
  - Reverse repo = same transaction from seller's viewpoint
    - ≈ secured lending.
- Main source of financing for Treasury dealers.
- Most repos are short maturity (e.g., overnight), but may be for 30 days or longer.
- For details, see Acharya and Öncü (2011).
- Will repo rates be higher or lower than Treasury rates? Unsecured lending rates?

Types of repo contract

- In most repo contracts, lender is willing to accept general (e.g., government + high-grade corporate bonds) collateral for the loan.
  - In this case, rate is called the general-collateral rate.
- Sometimes lender specifies a particular bond (e.g., for delivery on a separate transaction).
  - Rate on such a transaction is called a special-repo rate.
  - Dealers often need to short on-the-run Treasuries to hedge other securities.
  - If there is a lot of demand for the security, repo rate will be lower than general collateral rate.
    - Lower repo rate is an additional benefit of owning the security, separate from coupon and principal payments.

Outline

Introduction

Overview of Fixed-Income Markets

"Bond Math"

Day Count Conventions

Fitting the Yield Curve

Basic Fixed-Income Securities

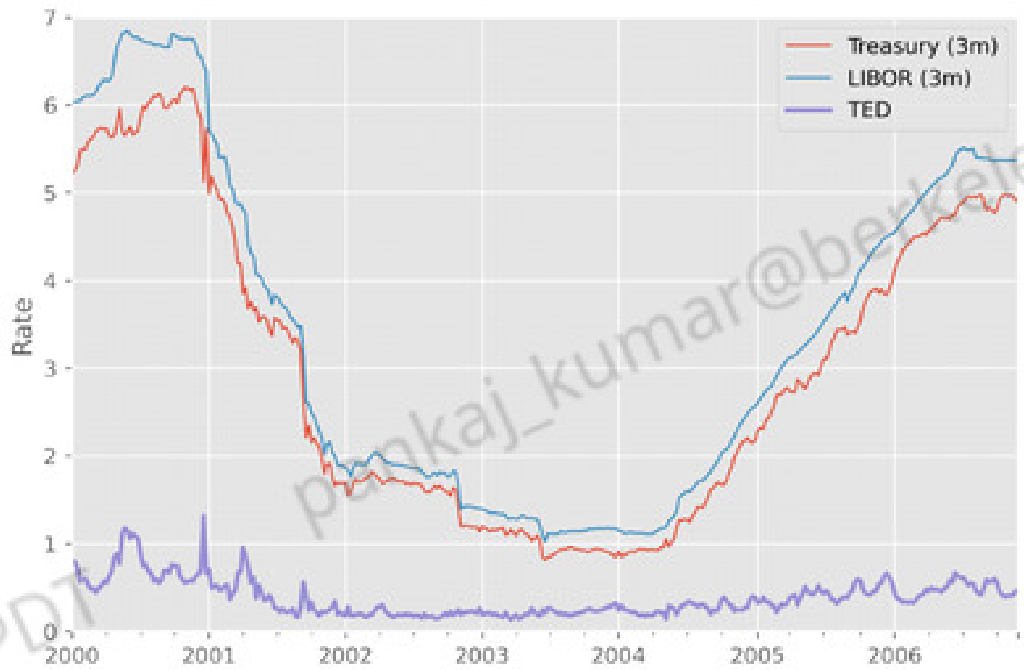
What riskless rate to use?

What riskless rate to use?

- At first sight, it seems sensible to use Treasury bonds to determine riskless interest rates.
- But Treasury rates may be artificially low:
  - Treasuries are not taxed at the state level.
  - Institutions must purchase Treasuries for various regulatory purposes.
  - Treasuries have favorable capital treatment.
  - Treasuries often provide below-market special repo rates.
- Prior to the 2007 crash, it was standard practice to use LIBOR (London Interbank Offered Rate) as the short-term riskless rate.
  - Short-term, uncollateralized borrowing rate for AA-rated financial institutions.
  - Swaps usually 3-month LIBOR vs. semi-annual fixed payments.
  - Fixed swap rates used as a proxy for longer-maturity riskless rates.
- $\geq$ \$350 trillion in derivatives/other financial products tied to LIBOR.

LIBOR versus Treasury rates: the TED spread

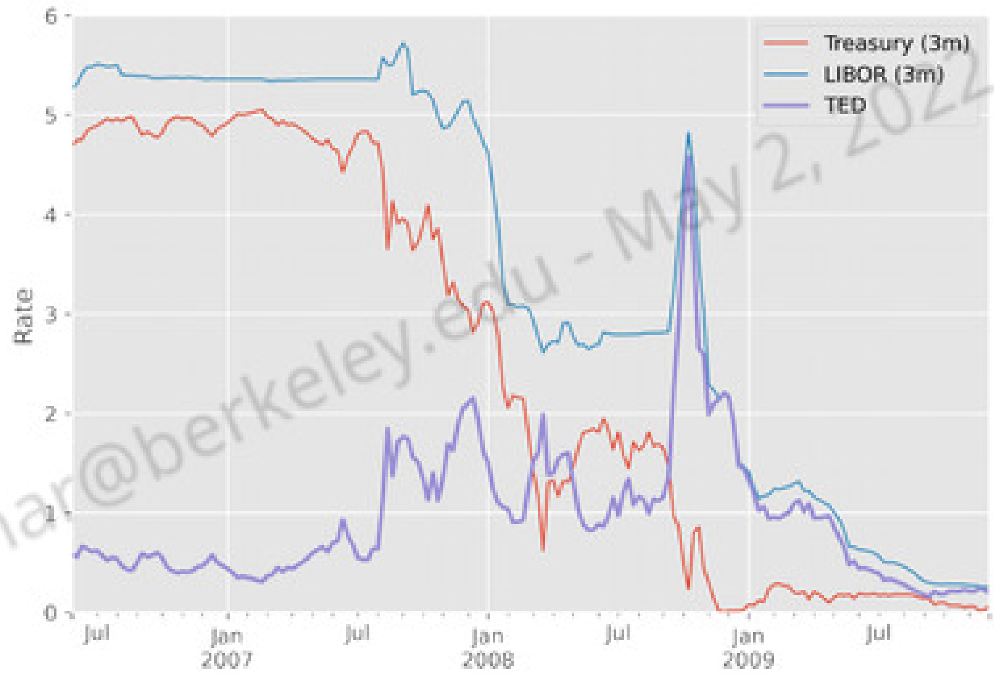
- The borrowing rates of AA-rated banks reflects (some) default risk, so are higher than riskless (Treasury) short-term rates.
- (3-month LIBOR $-$ 3-month T-Bill rate) is called the TED spread.
- Pre-crisis, TED spread usually below 0.5%.



Data source: Datastream

LIBOR during the crisis

- During the crisis, the credit risk over 3 months of even AA-rated financial institutions became significant.
- TED spread $>$ 1% throughout crisis, and briefly above 4%.



Data source: Datastream

- Could no longer view LIBOR as being riskless.

The LIBOR scandal

- Another problem: LIBOR based on reported rates, not transactions.
- New York Times article, "Behind the LIBOR Scandal," July 10, 2012 describes how Barclays manipulated the rates they submitted.
  - From 2005–2007, this was to help profits of other traders.
  - During crisis, rates set low to give false impression of creditworthiness.
  - `http://www.nytimes.com/interactive/2012/07/10/business/dealbook/behind-the-libor-scandal.html`
- E.g., Sept 13, 2006 message from trader to submitter:

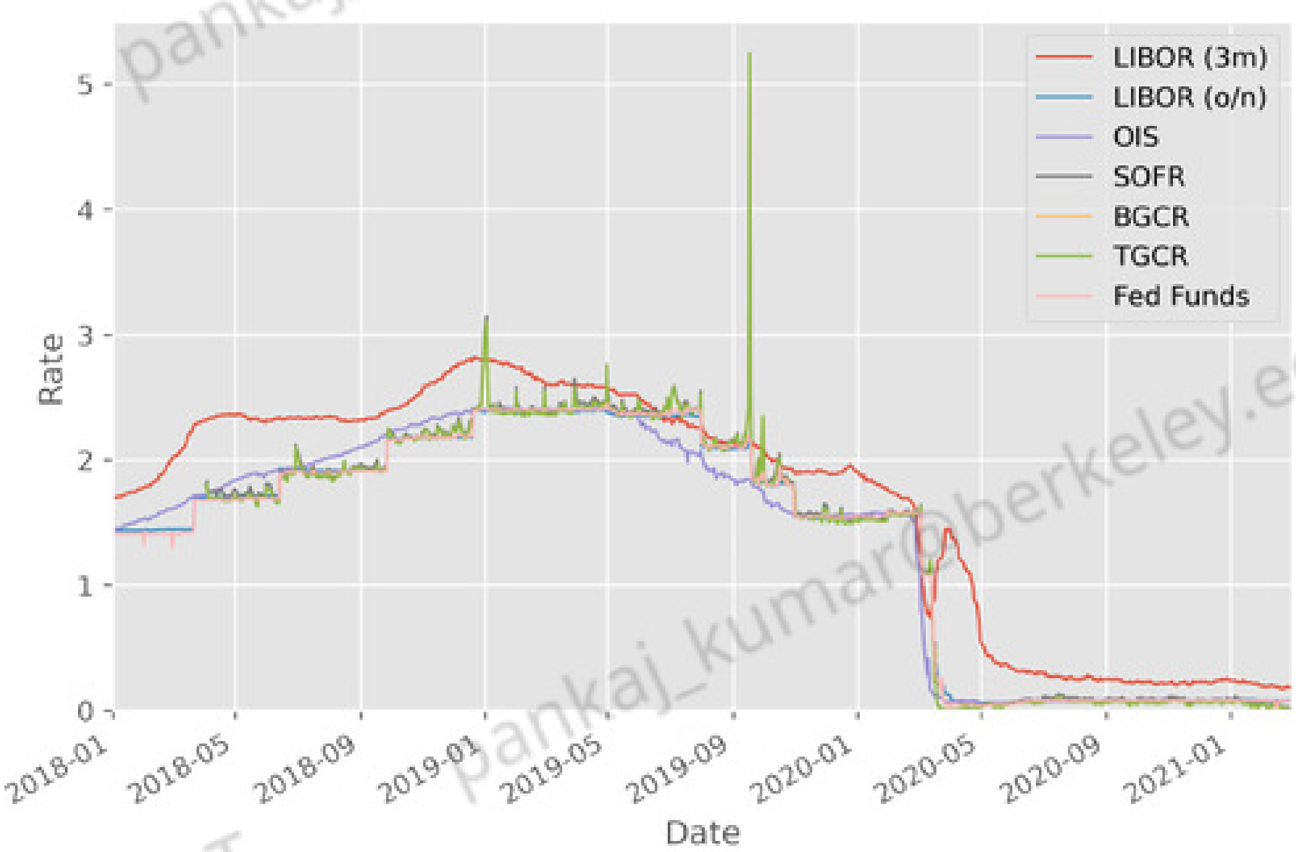
  "Hi Guys, We got a big position in 3m libor for the next 3 days. Can we please keep the libor fixing at 5.39 for the next few days. It would really help. ***We do not want it to fix any higher than that.*** *Tks a lot."*

- Billions in fines imposed on banks including Deutsche Bank, Royal Bank of Scotland, Lloyds and Citigroup.

Replacing LIBOR

- The Fed set up an Alternative Reference Rate Committee in 2014 to identify possible replacements for LIBOR.
- June 2017: recommended use of the Secured Overnight Financing Rate (SOFR).
  - Aggregate Treasury repo rate based on large volume of transactions.
  - Published daily by FRBNY starting April 3, 2018.
  - See A User's Guide to SOFR, The Alternative Reference Rates Committee, April 2019.
- Institutions starting to issue SOFR-based bonds:
  - Fannie Mae: $6 billion in July 2018.
  - World Bank: $1 billion in August 2018.
  - Credit Suisse: $100 million in August 2018.
- CME launched SOFR-based futures contracts in May 2018, swaps in October 2018.
  - vs. fixed, LIBOR, EFFR.
- March 5, 2021: UK FCA announced final fixing dates for LIBOR.
  - Dec. 31, 2012: All sterling, euro, Swiss franc, yen; 1-week, 2-month dollar.
  - June 30, 2023: All remaining dollar (overnight, 1, 3, 6, 12 months)

Short rates, 2018–21



Data source: Bloomberg

Overnight Indexed Swaps (OIS)

- SOFR-fixed swaps are Overnight Indexed Swaps (OIS).
  - Floating payments made every 3 months, but tied to an *overnight* rate.
  - Swaps with maturity ≤ 1 year make one payment at maturity.
- OIS also exist based on other overnight rates, e.g.,
  - Effective Federal Funds Rate (EFFR) in U.S.
  - EONIA (Euro OverNight Index Average) in Europe.
- Since the crisis, it has been common to use EFFR-OIS rates to derive riskless yield curve.

Overnight Indexed Swaps (OIS)

- Floating LIBOR payment at date T_i equals interest earned between T_{i-1} and T_i at date-T_{i-1} LIBOR (known at T_{i-1}):

$$CF^{\text{LIBOR}}_{\text{Floating}}\left(T_i\right) = N\Delta r_n\left(T_{i-1}, T_i\right).$$

- OIS floating payment at date T_i equals interest earned between T_{i-1} and T_i by rolling over overnight rate (*not* known at T_{i-1}):

$$CF^{\text{OIS}}_{\text{Floating}}\left(T_i\right) = N\left(\prod_{j=0}^{n_i-1}\left(1+\frac{d_j r_{t_j}}{360}\right)-1\right).$$

Bootstrapping riskless term structure from OIS rates

- Floating leg is worth par at reset dates (by similar argument to FRN).
- Fixed leg is equivalent to a bond with face value (and initial value) N:

$$V_0^{\text{Fixed}} = N \times c \times \Delta \times \sum_{i=1}^{n} Z^{\text{OIS}}(0, T_i) + N \times Z^{\text{OIS}}(0, T_n) = N.$$

- Rearrranging, we obtain the OIS rate c:

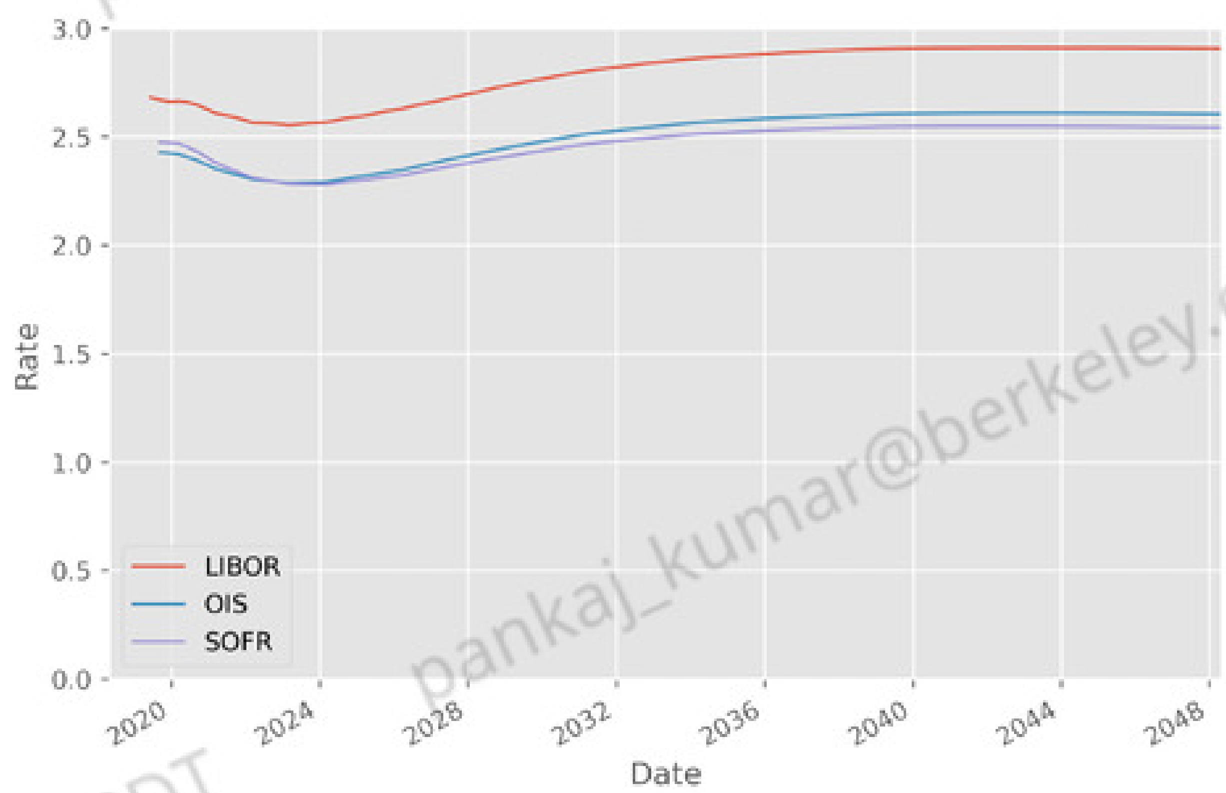
$$\boxed{c = \frac{1}{\Delta} \frac{1 - Z^{\text{OIS}}(0, T_n)}{\sum_{i=1}^{n} Z^{\text{OIS}}(0, T_i)}.}$$

- Hence we can bootstrap the OIS zero-coupon curve using

$$\boxed{Z^{\text{OIS}}(0, T_i) = \frac{1 - c(T_i) \times \Delta \times \sum_{j=1}^{i-1} Z^{\text{OIS}}(0, T_j)}{1 + c(T_i)\,\Delta}.}$$

LIBOR, EFFR-OIS and SOFR-OIS yield curves, Feb. 28, 2019

(Bloomberg curves 23, 42, 490)



Data source: Bloomberg

Dual-curve valuation

- LIBOR is still the main rate used today for swaps, other derivatives, and many mortgages and loans.
- Valuation now requires models for (at least) *two* curves:
  - LIBOR (to determine the cash flows) *and*
  - OIS rates (to determine the discount rate).
  - (SOFR rates).
- We'll talk about dual-curve valuation later in the course.

References I

Acharya, V. V., and T. S. Öncü, 2011, The repurchase agreement (repo) market, in V. V. Acharya, T. F. Cooley, M. P. Richardson, and I. Walter, eds., *Regulating Wall Street: The Dodd-Frank Act and the new Architecture of Global Finance*, chapter 11, 319–350 (John Wiley, New York).

Adams, K. J., and D. R. van Deventer, 1994, Fitting yield curves and forward curves with maximum smoothness, *Journal of Fixed Income* 4, 52–62.

Alternative Reference Rates Committee, 2019, A user's guide to SOFR, Technical report.

Bank for International Settlements, 2005, Zero-coupon yield curves: Technical documentation.

Bloomberg, 2012, Building the Bloomberg interest rate curve: Definitions and methodology, Technical report.

Fama, E. F., and R. R. Bliss, 1987, The information in long-maturity forward rates, *American Economic Review* 77, 680–692.

Gürkaynak, R. S., B. Sack, and J. H. Wright, 2007, The U.S. Treasury yield curve: 1961 to the present, *Journal of Monetary Economics* 54, 2291–2304.

References II

Hastie, T., R. Tibshirani, and J. Friedman, 2009, *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, second edition (Springer, New York).

James, J., and N. Webber, 2000, *Interest Rate Modelling: Financial Engineering* (Wiley, New York).

McCulloch, J. H., 1971, Measuring the term structure of interest rates, *Journal of Business* 44, 19–31.

McCulloch, J. H., 1975, The tax-adjusted yield curve, *Journal of Finance* 30, 811–830.

Nelson, C. R., and A. F. Siegel, 1987, Parsimonious modeling of yield curves, *Journal of Business* 60, 473–489.

Svensson, L. E. O., 1994, Estimating and interpreting forward rates: Sweden 1992–4, Working Paper 4871, NBER.

van Deventer, D. R., and K. Imai, 1997, *Financial Risk Analytics: A Term Structure Model Approach for Banking, Insurance, and Investment Management* (Irwin Professional Publishing, Chicago).

References III

van Deventer, D. R., K. Imai, and M. Mesler, 2013, Yield curve smoothing, in *Advanced Financial Risk Management: Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management*, second edition, chapter 5 (John Wiley).

Veronesi, P., 2010, *Fixed Income Securities* (Wiley, Hoboken, NJ).

Wright, J., 2011, Term premia and inflation uncertainty: Empirical evidence from an international panel, *American Economic Review* 101, 1514–1534.