MFE 230I: Problem Set 3

Due: Monday, July 12, 2021

Note: Unless otherwise specified in a question, assume all payments are annual, interest rates are compounded annually, and all bonds have a face value of \$100.

1. **Black, Derman and Toy:**

   (a) You are given the following zero-coupon bond yields (all annually compounded) and (proportional) yield volatilities:[1]

Maturity	Yield	Volatility
1 year	5.00%	15.0%
2 years	5.50%	16.0%
3 years	5.70%	17.0%
4 years	5.90%	18.0%
5 years	6.00%	19.0%
6 years	6.10%	20.0%

   As in class, calibrate a Black, Derman, and Toy (1990) tree (the full model, *not* just the simple BDT model), with annual time steps, to the given zero curve and volatilities of zero yields out to 6 years. Your tree should show the evolution of the annually compounded 1-year rate. Use this tree to answer the remaining parts of this question.

   As described in class, match current bond prices and the proportional volatility of the spot rates for different maturities sequentially to find m and σ at each time step in the tree. The following table shows the resulting tree:

```
[[0.05       0.06904723 0.08370018 0.11020841 0.14007713 0.19436303]
 [0.         0.05115144 0.05956077 0.07539734 0.09167574 0.12168862]
 [0.         0.         0.04238325 0.0515819  0.05999867 0.07618794]
 [0.         0.         0.         0.03528894 0.0392671  0.04770046]
 [0.         0.         0.         0.         0.02569899 0.02986475]
 [0.         0.         0.         0.         0.         0.018698  ]]
```

   (b) **Callable bond:** What is the price today of a 6-year bond with face value \$100 and coupon rate 5%, callable at par starting in 2 years?

   First, we calculate the price of a non-callable coupon bond. At maturity $T = 6$ years, the value of the bond will be par$= 100$. We can then discount the price backward, according to the formula

$$P_{i,j} = \frac{0.5 \cdot (P_{i,j+1} + P_{i+1,j+1}) + 2}{1 + r_{i,j}}$$

   The value of the coupon bond is $P_{0,0} = 94.803$.

[1] As in class, define the proportional volatility of (say) the 3-year rate as the standard deviation of its log over the next period, i.e.,

$$\sigma_3 = \frac{\log\left(\frac{\text{3-year rate in up-state next period}}{\text{3-year rate in down-state next period}}\right)}{2\sqrt{\Delta t}},$$

where Δt is the time step in the tree.

The value of the call option can then be determined at each node as

$$C_{i,j} = \frac{0.5 \cdot (C_{i,j+1} + C_{i+1,j+1})}{1 + r_{i,j}}$$

if the time $j < 2$ (before 2 years) and

$$C_{i,j} = \max\left(\frac{0.5 \cdot (C_{i,j+1} + C_{i+1,j+1})}{1 + r_{i,j}}, P_{i,j} - 100\right)$$

if $j \geq 2$ (starting at year 2). Computing an options price tree in the usual way, we get a price today for the call option of $C_{0,0} = 0.9054$. The value of the callable bond is then

$$\begin{aligned} P_{callable} &= P_{noncallable} - P_{CallOption} \\ &= 94.803 - 1.66 \\ &= 93.897 \end{aligned}$$

(c) **Forward price:** What is the current forward price for delivery in two years of a zero-coupon bond with face value \$100, maturing 6 years from today?

First compute the prices of a 6-year ZCB and 2-year ZCB using the rates tree, then compute the forward price today of a 2-year forward contract on a 6-year bond as

$$\begin{aligned} P_{fwd} &= \frac{P(0,6)}{P(0,2)} \times 100 \\ &= \frac{70.098}{89.845} \times 100 \\ &= 78.0212 \end{aligned}$$

(d) **Futures price:** What is the current *futures* price for delivery in two years of a zero-coupon bond with face value \$100, maturing 6 years from today? Assume the futures contract is marked to market every 12 months. Why is the price you obtained here different from that you obtained in 1c?

To price the futures contract, we build a price tree for a 6-year ZCB and extract the portion of the tree up to and including year two. We then iterate backwards from the year one price **without** discounting to get a futures price of 77.989, which is slightly less than the forward price from above. This is because futures are marked to market every 6 months.

(e) **Hedging 1:** If you own the callable bond from 1b, what position in the forward contract from 1c do you need to hedge yourself against interest rate movements over the next 12 months?

$$\begin{aligned} \Delta_{\text{callable}} &= \frac{P_u - P_d}{r_u - r_d} = -451.857 \\ \Delta_{\text{6-yr bond}} &= -461.382 \\ \Delta_{\text{2-yr bond}} &= -88.989 \\ \Delta_{\text{fwd}} &= \Delta_{\text{6-yr bond}} - 0.7802 \times \Delta_{\text{2-yr bond}} \\ &= -391.952. \end{aligned}$$

Hence, in order to hedge the position in one callable bond, we need to short

$$451.857/391.952 = 1.153 \text{ forward contracts.}$$

(f) **Hedging 2:** If you own the callable bond from 1b, what position in the *futures* contract from 1d do you need to hedge yourself against interest rate movements over the next 12 months?

$$\Delta_{future} = \frac{F_u - F_d}{r_u - r_d} = -415.507.$$

Hence, in order to hedge the position in one callable bond, we need to short

$$451.857/415.507 = 1.087 \text{ futures contracts.}$$

2. **Hull and White:** You are given the following zero-coupon bond yields (all annually compounded):

Maturity	Yield
1 year	5.00%
2 years	5.50%
3 years	5.70%
4 years	5.90%
5 years	6.00%
6 years	6.10%

(a) As in class, calibrate a Hull and White (1993, 1994) tree, with annual time steps, to these bond yields. Your tree should show the evolution of the *continuously compounded* one-year rate, whose annualized volatility is 1%. The mean-reversion parameter is 0.1. Use this tree to answer the rest of this question.

Calibrating the interest rate model as in class, we get the following tree:

	period 0	period 1	period 2	period 3	period 4	period 5
node 0	0.000000	0.000000	0.092366	0.096311	0.095567	0.097680
node 1	0.000000	0.074826	0.075877	0.079822	0.079078	0.081191
node 2	0.048790	0.058337	0.059387	0.063332	0.062588	0.064701
node 3	0.000000	0.041847	0.042898	0.046843	0.046099	0.048212
node 4	0.000000	0.000000	0.026408	0.030353	0.029609	0.031722

(b) **(Simplified) mortgage-backed security (MBS):** This security receives the cash flows from a set of newly issued underlying mortgages with total principal \$1,000,000. The mortgages all make one payments per year and have 6 years to expiration (so they make a total of 6 payments, 1, 2, ..., 6 years from today). Their coupon rate is a fixed 5.5% (compounded annually).

Prepayment: In the absence of any unscheduled prepayment, the borrowers would make 6 equal payments. However, borrowers have the right to prepay their loans early, starting one year from today, by paying back the remaining principal on the loan (plus any interest due). Any such unscheduled prepayment (i.e., prepayment in excess of scheduled amortization) changes the cash flows received by investors, so it needs to be taken into account in valuing the security. Assume:

- In any future period where interest rates (the continuously compounded rates shown in your tree) are greater than or equal to today's value, mortgages will prepay at a rate of 3% of their remaining balance every year (e.g., because some homeowners move).[2]
- Prepayment will accelerate to a total of 5 percent per year whenever the one-year rate is 50 b.p. below its current level (e.g., as homeowners refinance).
- A one-year rate that is 100 b.p. lower than today leads to a prepayment rate of 8% every year, and a 200 b.p. lower rate leads to a prepayment rate of 17% every year (the maximum possible prepayment rate).
- For other interest rates, use linear interpolation to calculate the prepayment rate.

Value the MBS using the Hull-White interest rate tree you just calibrated in three ways:

i. Using Monte Carlo simulation: Generate 1,000 independent interest rate paths for the valuation, and calculate the standard error of your price estimate.

ii. Using Monte Carlo, as above, but this time using 500 independent paths, and 500 *antithetic* paths, explaining in detail how you generate your antithetic paths and, again, calculating the standard error of your price estimate.

iii. Exactly, using a *recombining* tree.

i. With 1,000 independent paths, price is \$988,529.1954 with a SE of \$718.3137.

ii. With 500 independent paths and 500 antithetic paths, price is \$989,814.9123 with a SE of \$19.9727.

iii. To create a recombining tree, note that we can use the asset value per dollar of remaining principal, similar to the in-class example with index-amortizing swaps. This leads to the following (recombining) tree of values per dollar:

	0	1	2	3	4	5
0	0.000000	0.000000	0.986918	0.998973	1.015624	1.033497
1	0.000000	1.007259	1.012941	1.019714	1.030413	1.041419
2	0.989812	1.039329	1.040012	1.041156	1.045588	1.049473
3	0.000000	1.071240	1.067232	1.062920	1.060991	1.057654
4	0.000000	0.000000	1.088023	1.080221	1.073472	1.064698

[2] This 3% prepayment is *in addition to* regularly scheduled payment of principal due to amortization of the loan. I.e., treat it as though 3% of the remaining loans in the pool pay off in full immediately after making whatever scheduled interest and principal payments are due that period.

Multiplying the price per dollar by \$1,000,000, we obtain a price of \$989,811.8549.

3. **Vasicek:** Interest rates are described by the Vasicek (1977) model, with risk-neutral dynamics for the continuously compounded instantaneous rate given by

$$dr = \kappa\,(\mu - r)\;dt + \sigma\,dW.$$

Parameter values: $\kappa = 0.5$, $\mu = 0.06$, $\sigma = 0.01$. Current instantaneous rate: $r = .04$.

(a) Calculate the forward rate for a 3-year loan beginning in 2 years.

(b) What is the futures price for a 3-year zero-coupon bond to be delivered in 2 years? What is the implied interest rate (i.e., the yield on a 3-year zero-coupon bond with price equal to the value you just calculated)? Explain why this rate is different from that in 3a.

(c) Value a 4-year European call option with strike price \$100 on a bond with coupon rate 6% (calculated and paid annually) maturing in 10 years (assume the option exercise date is immediately after the year-4 coupon payment). What position do you need to take in the underlying bond to hedge away the interest-rate risk of the option over the next instant?

(a) Let $F_{fw}(t,T_1,T_2)$ be the forward price of a ZCB, and $F_{fut}(t,T_1,T_2)$ the futures price of the same asset. The the corresponding forward and futures rates are calculated from

$$F_i(t,T_1,T_2) = e^{-3f_i(t,T_1,T_2)}.$$

The difference arise from the difference between the futures and forward prices.
The forward price is calculated as

$$F_{fw}(t,T_1,T_2) = \frac{Z(t,T_2)}{Z(t,T_1)},$$

where in the Vasiček model we have closed form solution for $Z(0,T)$ as function of the current short rate, thus easy to compute

$$F_{fw}(0,2,5) = \frac{Z(0,5)}{Z(0,2)} = .8452$$

This leads to an implied forward rate $f_{fw}(0,2,5) = 5.606\%$.

(b) To calculate the futures price we'll use that the futures price of any asset is a martingale under the risk-neutral measure, moreover $F_{fut}(T_1,T_1,T_2) = Z(T_1,T_2)$ thus

$$F_{fut}(t,T_1,T_2) = E^Q[F_{fut}(T_1,T_1,T_2)] = E^Q[Z(T_1,T_2)].$$

Using the closed form solution for the ZCB price in the Vasiček model we get

$$\begin{aligned}F_{fut}(t,T_1,T_2) &= E^Q[Z(T_1,T_2)] = \\ &= E^Q\left[e^{A(T_2-T_1)-B(T_2-T_1)r_{T_1}}\right].\end{aligned}$$

Given that r_{T_1} is conditionally normal, $Z(T_1,T_2)$ is conditionally lognormal, thus its expected value can be written as

$$
\begin{aligned}
F_{fut}(t,T_1,T_2) &= E^Q\left[e^{A(T_2-T_1)-B(T_2-T_1)r_{T_1}}\right] \\
&= e^{A(T_2-T_1)-B(T_2-T_1)E^Q[r_{T_1}]+\frac{B^2(T_2-T_1)}{2}Var^Q(r_{T_1})},
\end{aligned}
$$

where we have closed form solutions for both $E^Q[r_{T_1}]$ and $Var^Q(r_{T_1})$. This leads to $F_{fut}(0,2,5) = 0.8451$ which implies a futures rate of $f_{fut}(0,2,5) = 5.609\%$. This is different from part a) due to convexity adjustment.

(c) Short .0487 coupon bonds to hedge.

References

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Hull, John, and Alan White, 1994, Numerical procedures for implementing term structure models I: Single-factor models, *Journal of Derivatives* 2, 7–16.

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