MFE 230I: Problem Set 4
Continuous-time spot rate models and HJM

Due: Monday, July 19, 2021, by 9:30 a.m.

1. **Two-factor Vasicek model:** Consider the *two-factor* Vasicek model, with risk-neutral dynamics

$$\begin{aligned} r &= x + y, \\ dx &= \kappa_1 (\mu_1 - x)\, dt + \sigma_1\, dW_1, \\ dy &= \kappa_2 (\mu_2 - y)\, dt + \sigma_2\, dW_2, \\ dW_1\, dW_2 &= \rho\, dt. \end{aligned}$$

Explain why it is impossible to estimate unique values for all of the parameters κ_1, κ_2, μ_1, μ_2, σ_1, σ_2 and ρ.

2. **Continuous-time Hull-White model:** The t-year discount factor (for $0 \le t \le 30$) is

$$Z(t) = e^{at+bt^2+ct^3},$$

where $a = -.04$, $b = -.001$, $c = .0001$. Assuming $\kappa = 0.15$ and $\sigma = 0.015$, fit the continuous-time Hull and White model,

$$dr = (\theta_t - \kappa r)\, dt + \sigma\, dZ^Q,$$

to the discount function above.

(a) Plot θ_t against t for values of t between 0 and 30 years. **Use this estimated model for the rest of the question.**

(b) A security pays, 10 years from today, an amount equal to \$10 million times the difference between the highest and lowest 2-year rates (compounded semi-annually) observed between today and 10 years from today, i.e.,

$$\text{Payout}_{10} = \$10{,}000{,}000 \times \left(\max_{0 \le t \le 10} r_2(t, t+2) - \min_{0 \le t \le 10} r_2(t, t+2) \right).$$

i. Value the security using Monte Carlo simulation + antithetic variates. To do this, generate 10,000 total paths (5,000 independent paths plus 5,000 antithetic paths) of interest rates at monthly time intervals.

ii. What is the standard error of your estimated price?

iii. Assuming the Hull/White model you are using is the correct model for the evolution of interest rates, do you think the price you estimated in (a) is more likely to over- or understate the true value of the security? Explain why.

(c) Value a 2 year American *put* option with strike price \$100, written on a 5-year par bond that was just issued earlier today,[1] using the *implicit* finite-difference method to solve the pricing p.d.e.,

$$\frac{1}{2}\sigma^2 P_{rr} + (\theta(t) - \kappa r)\, P_r + P_t - rP = 0. \qquad (1)$$

[1] To clarify, the underlying bond remains fixed throughout the life of the option. Its coupon rate was set so that it trades at par today, but it will not in general trade at par in the future. The bond makes annual coupon payments and matures 5 years from today, so that when the option expires the bond will only have *three* years remaining. Finally, assume that the strike price is quoted as a *clean* price. In other words, when you exercise, you receive \$100 + Accrued Interest in exchange for the bond.

Report the price of the option for today's value of r, and also plot a graph of the option price against r, for values of r between 0 and 20%.

(d) Using Monte Carlo simulation, calculate the futures price for a contract expiring in 5 years, where the short must deliver at maturity a 4-year bond with coupon rate 4% (i.e., the bond matures 9 years from today). Assume annual coupon payments.

(e) If you were going to use a finite-difference method instead of Monte Carlo simulation to solve for the futures price, F, in 2d, what is the p.d.e. you would need to solve?

3. **Discrete-time HJM:** You are given the following semi-annually compounded zero-coupon bond yields:

Maturity	**Yield**
6 months	4.70%
1 year	5.00%
1.5 years	5.50%
2 years	6.00%

As in class (where our discussion was based on Heath, Jarrow, and Morton, 1990), define $f(t,T)$ to be the continuously compounded forward rate, quoted at date t, for a forward loan between dates T and $T+\Delta$, where $\Delta = 1/2$. Assume that the one-period-ahead variance of $f(t,T)$ is given by

$$\mathrm{var}_{t-\Delta}\left(f(t,T)\right) = \sigma^2(T-t)\,\Delta,$$

where the function $\sigma(x)$ takes on the following values:

x	$\sigma(x)$
0.0	1.50%
0.5	2.10%
1.0	2.50%
1.5	2.00%

(a) As in class, construct a binomial tree (risk-neutral prob. of up/down jumps = 0.5) showing in detail the evolution of forward rates over the next 18 months.

(b) Using the HJM tree you just constructed, calculate the value of a one-year American put option on a two-year zero coupon bond, face value = \$100, strike price = \$93.30.

4. **Continuous-time HJM:** Let $f(t,T)$ be the continuously compounded *instantaneous* forward rate quoted at date t for an "instantaneous" loan at date T. Assume the risk-neutral dynamics of the forward rates are given by

$$df(t,T) = m(t,T)\,dt + \sigma_1(t,T)\,dW_1 + \sigma_2(t,T)\,dW_2,$$

where dW_1 and dW_2 are independent and

$$\sigma_1(t,T) = 0.01,$$
$$\sigma_2(t,T) = 0.01e^{-0.1(T-t)}.$$

To prevent arbitrage, what must the drift function, $m(t,T)$, be?

References

Heath, D., R. A. Jarrow, and A. Morton, 1990, Bond pricing and the term structure of interest rates: A discrete time approximation, *Journal of Financial and Quantitative Analysis* 25, 419–440.

Heath, D., R. A. Jarrow, and A. Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology, *Econometrica* 60, 77–105.