MFE230I Section 8

Yunbo Liu [1]

July 23, 2021

[1] Based on notes by Paulo Manoel, all errors are mine.

Girsanov's Theorem and Change of Measure

Girsanov's theorem (simplified version): Consider the process

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW^P$$

Let's say that you want to find a measure M (equivalent to P) such that $X(t)$ follows a diffusion process with a different drift. Let $\theta(t)$ denote any "well behaved" adapted process. Let:

$$W^M(t) = W^P(t) + \int_0^t \theta(s)ds$$

Then:

- There is a measure M, equivalent to P, such that $W^M(t)$ is a Brownian motion under M.
- Under M, X will follow the process

$$dX(t) = \Big\{\mu(t, X(t)) - \theta(t, X(t))\sigma(t, X(t))\Big\}dt + \sigma(t, X(t))dW^M$$

Exercise

Consider a model with a single state variable (e.g., interest rates). Let V(t) and N(t) denote the price of two arbitrary securities following

$$\frac{dV}{V} = \mu_V dt + \sigma_V dW$$

$$\frac{dN}{N} = \mu_N dt + \sigma_N dW$$

Find a measure Q^N that makes the relative price $R(t) \equiv V(t)/N(t)$ a martingale. Conclude that

$$\frac{V(t)}{N(t)} = E_t^{Q^N}\left[\frac{V(T)}{N(T)}\right] \text{ for any } T > t$$

Basic application of the Ito's Lemma leads to

$$\frac{dR}{R} = (\mu_V - \mu_N + \sigma_N^2 - \sigma_N\sigma_V)dt + (\sigma_V - \sigma_N)dW$$

By no arbitrage, $\mu_i = r + \lambda\sigma_i$ we get

$$\frac{dR}{R} = (\sigma_V - \sigma_N)(\lambda - \sigma_N)dt + (\sigma_V - \sigma_N)dW$$

If we apply the Girasnov theorem using $\theta = \lambda - \sigma_N$ we get

$$\frac{dR}{R} = (\sigma_V - \sigma_N)dW^M$$

T-Forward Measure

Consider the particular case of a security with terminal value V(T) and numeraire N(t)=Z(t,T).
There is a measure Q^T such that:

$$\frac{V_t}{Z(t,T)} = E^{Q_T}\left[\frac{V(T)}{Z(T,T)}\right] = E_t^{Q_T}[V_T]$$

Therefore:

$$V_t = Z(t,T)E_t^{Q_T}[V_T]$$

MC Simulation with T-Forward Measure

The diffusion process of state variable X under the T-forward measure will be

$$dX = (\mu_X + \sigma_X \sigma_{Z(T)})dt + \sigma_X dW^{QT}$$

If we can solve this equation and find the terminal distribution of $X(T)$, prices can be simply simulated as

$$V_t = Z(t, T) \frac{1}{N} \sum_{i=1}^{N} \hat{V}_i(X(T), T)$$

LIBOR Market Model Question

Assuming the LIBOR market model holds, you are given

- Three-year discount factor, Z(0,3)=0.90.
- Forward LIBOR, $f_4(0, 2.75, 3.0) = 3.0\%$
- Implied volatility from three-year caplet, $\sigma_f^{Fwd}(3) = 28.0\%$

- (a) Calculate the value of a three-year caplet, payments in arrears (i.e., payment at date 3 is based, as usual, on observed 3-month LIBOR at date 2.75), notional = \$100 million, strike rate = 3.0%.

The value of the caplet is given by:

$$10^8 Z(0,3)\Delta\left[f_4(0,2.75,3)N(d_1) - r_K N(d_2)\right]$$

where the constants d_1 and d_2 are given by:

$$\begin{aligned} d_1 &= \frac{1}{\sigma_f\sqrt{2.75}}\log\left(\frac{f_4(0,2.75,3)}{r_K}\right) + \frac{1}{2}\sigma_f\sqrt{2.75} \\ &= \frac{1}{2}\times 28\% \times \sqrt{2.75} \\ &= 0.232 \\ d_2 &= d_1 - \sigma_f\sqrt{2.75} \\ &= 0.232 - 28\% \times \sqrt{2.75} \\ &= -0.232 \end{aligned}$$

The value of the caplet is:

$$\begin{aligned} V &= 10^8 \times 0.90 \times 1/4 \times 3\% \times [N(0.232) - N(-0.232)] \\ &= 123,922.68 \end{aligned}$$

- (b) Calculate the value of a security paying at year 3 the amount

$$N max(r_n(2.75, 3)^3 - K, 0)$$

where $N = \$100$ million and $K = 0.03^3$

Remember that, if x is lognormal, so is x^3. In particular, if $\log(x) \sim N\left(\mu, \sigma^2\right)$, then $\log\left(x^3\right) \sim N\left(3\mu, 9\sigma^2\right)$, and therefore $E\left(x^3\right) = e^{3\sigma^2} E(x)^3$. This implies that we can apply the formula used to price caplets, as long as we replace the values of the means and of the variances appropriately. Let $g(0, 2.75, 3)$ denote the expected value of $r_n(2.75, 3)^3$, and let σ_T^2 denote the variance of $\log\left[r_n(2.75, 3)^3\right]$.

Then:

$$
\begin{aligned}
g(0, 2.75, 3) &= f_4(0, 2.75, 3)^3 e^{3\times\sigma_f^2\times 2.75} \\
&= 3\%^3 \times \exp\left\{3 \times 28\%^2 \times 2.75\right\} \\
&= 3.72\%^3 \\
\sigma_T^2 &= 9 \times 28\%^2 \times 2.75 \\
&= 1.940 \\
d_1 &= \frac{1}{\sigma_T}\log\left(\frac{g(0, 2.75, 3)}{K}\right) + \frac{1}{2}\sigma_T \\
&= \frac{1}{1.393}\log\left(\frac{3.72\%^3}{3\%^3}\right) + \frac{1}{2}1.393 \\
&= 1.161 \\
d_2 &= d_1 - \sigma_T \\
&= 1.161 - 1.393 \\
&= -0.232
\end{aligned}
$$

$= 3.72\%^3$

The value of the security is:

$$
\begin{aligned}
V &= 10^8 \times 0.90 \times [g(0, 2.75, 3)N(d_1) - KN(d_2)] \\
&= 10^8 \times 0.90 \times \left[3.72\%^3 \times 0.877 - 3\%^3 \times 0.408\right] \\
&= 3,077.91
\end{aligned}
$$