MFE 230I: Problem Set 2

Due: Monday, June 21, 2021

Note: For all questions in this problem set, unless specified otherwise, assume annual payments, annually compounded interest rates, and that the face value of all bonds is \$100.

1. Zero-coupon bond yields of all maturities are equal to 9%. Bond A is a 3-year 6% coupon bond. Calculate

   (a) The Macaulay duration of bond A.
   (b) The modified duration of bond A.
   (c) The dollar duration of bond A.
   (d) The convexity of bond A.
   (e) The cash-flow variance of bond A.

2. Bond B is a one-year zero-coupon bond. With the same assumptions as in Q1, if you own one bond A, what position in bond B do you need to hedge yourself against interest rate movements? Explain intuitively why this number is greater or less than one (whichever you find it to be)?

3. Bond C is a six-year zero-coupon bond. With the same assumptions as in Q1, what position in both bonds B *and* C would allow you to hedge your position in bond A over as wide a range of interest rate movements as possible?

4. Assume that interest rates are continuously compounded; that cash flows on annuities/perpetuities and coupon payments on bonds are made continuously, starting today;[1] and that the yield to maturity on bonds of all maturities is $y > 0$.

   Calculate a *closed-form* expression (i.e., one that does not involve sums or integrals over lots of cash flows) for the Macaulay duration of

   (a) A *perpetuity* making a continuous payment at a rate of \$1 per year forever, starting today. Call this duration D_P.
   (b) An *annuity* making a continuous payment at a rate of \$1 per year, starting today and ending at date T. Call this duration $D_A(T)$.
   (c) A *coupon bond* with continuous coupon rate c and maturity T years. Call this duration $D(T)$.

5. With the same assumptions as in Q4,

   (a) Calculate the limit of $D_A(T)$, as the yield y goes to zero.
   (b) What numerical problems do you encounter as you try to calculate your expression for $D_A(T)$ from Q4 for very small values of y, and why?
   (c) Explain how to calculate $D_A(T)$ in a way that gives results accurate to at least 10 significant figures for all values of y with $0 \leq y \leq 0.001$ and all values of T with $0 \leq T \leq 10$.

[1] I.e., for a bond with coupon rate c, the total cash flow between t and $t + \delta t$ for t prior to maturity T is $c\,\delta t$. Similarly for annuities/perpetuities.

6. [**Extra credit**] Prove that

   (a) For par and premium bonds ($c \geq y > 0$):
      i. $D(T)$ is always below D_P.
      ii. $D(T)$ is always increasing in T.
      iii. $D(T)$ approaches a limit of D_P as $T \to \infty$.

   (b) For discount bonds ($0 < c < y$):
      i. There is a unique $T_{max} > 0$ such that $D(T)$ is increasing in T for $T < T_{max}$ and decreasing in T for $T > T_{max}$.
      ii. $D(T)$ hits a maximum greater than D_P at $T = T_{max}$.
      iii. $D(T)$ approaches a limit of D_P as $T \to \infty$.

7. **Ho and Lee:**

   (a) You are given the following zero-coupon bond yields (all annually compounded):

Maturity	Yield
1 year	5.00%
2 years	5.50%
3 years	5.70%
4 years	5.90%
5 years	6.00%
6 years	6.10%

   Assuming the volatility of the (annually compounded) one-year rate is 1.5% per year, calibrate a Ho and Lee (1986) tree (as we did in class) to these bond prices, where the interest rate in the tree is the annually compounded one-year rate. Print the tree, and use it to price the following securities and to calculate their spot rate durations.

   (b) A *non-prepayable* 6-year mortgage. Assume this is a self-amortizing loan, requiring equal payments at the end of each of the next six years, with a quoted (annual) interest rate of 5.5% and an initial balance of $100.

   (c) A *prepayable* 6 year mortgage. This is the same mortgage as above, except that the loan may be prepaid at any time. When a borrower prepays a mortgage, the borrower pays back the remaining principal on the loan, and does not make any of the remaining scheduled payments. Assume that the borrower refinances optimally, incurring no prepayment penalties or other transaction costs upon refinancing.

   (d) A principal only (PO) security based on the prepayable mortgage above. The holder of this security receives any *principal* payments made by the mortgage holder(s) (either scheduled or unscheduled), but none of the interest.

   (e) An interest only (IO) security based on the prepayable mortgage above. The holder of this security receives any *interest* payments made by the mortgage holder(s), but none of the principal.

References

Ho, Thomas S. Y., and Sang Bin Lee, 1986, Term structure movements and pricing interest rate contingent claims, *Journal of Finance* 41, 1011–1029.