MFE230I Section 4

Can Huang

June 25, 2021

Forward and futures contracts

Main difference is that futures are exchange traded while forwards are bilateral agreements

	Future	Forward
Flexibility	standardized	more flexible
Trading	more liquid	unique
Settlement	marked-to-market	end of contract

- When interest rates correlated with the underlying asset price, then forward and futures prices are different!
- $f(t) = \frac{Z(t,T_2)}{Z(t,T_1)}$
- $F(t) = E^Q[F(T_1)]$

What is the initial forward and futures price for a contract on the previous 4 year bond, that expires after 2 years?

Suppose the calibrated interest rate tree is

year	0	1	2	3
r	5%	4%	3%	2%
		6%	5%	4%
			7%	6%
				8%

What is the initial forward and futures price for a contract on the previous 4 year bond, that expires after 2 years?
Suppose the calibrated interest rate tree is

year	0	1	2	3
r	5%	4%	3%	2%
		6%	5%	4%
			7%	6%
				8%

P_4

$$f = \frac{Z(0,4)}{Z(0,2)} \times 100 = 90.78$$

P_2

year	0	1	2
F	90.76	92.49	94.27
		89.03	90.71
			87.35

$$94.27 = Z(2,4)$$

$$92.49 = 0.5\,(94.27 + 90.71)$$

Hedging using Forward and Future

Class example: Use forward or future to hedge the interest risk of 5-year zero-coupon bond.

1. Calculate forward delta by replication.
2. Calculate Future delta by tree.

Hedging using Forward and Future

Class example: Use forward or future to hedge the interest risk of 5-year zero-coupon bond.

1. Calculate forward delta by replication.
2. Calculate Future delta by tree.
3. We can also calculate forward delta by tree.

$t=1$

$$P_2(t), \quad P_5(t)$$

$$P_2(0) \begin{cases} P_2^u(1) \\ P_2^l(1) \end{cases} \qquad P_5(0) \begin{cases} P_5^u(1) \\ P_5^d(1) \end{cases}$$

$$\Delta \quad \pi(2) = P_5(2) - f$$

$$PV_0(\pi(2)) = PV_0(P_5(2)) - PV_0(f)$$

$$= P_5(0) - f \cdot Z(0,2) = 0$$

$$\Rightarrow f = \frac{P_5(0)}{Z(0,2)} = \frac{P_5(0)}{P_2(0)} \times 100$$

$$\underline{PV_1^u(\pi(2))} = PV_1^u(P_5(2)) - PV_1^u(f)$$

$$= P_5^u(1) - f \cdot Z^u(1,2)$$

$$\underline{= P_5^u(1) - f \cdot P_2^u(1)/100}$$

$$PV_1^d(\pi(2)) = P_5^d(1) - f \cdot P_2^d(1)/100$$

$$\Delta = \frac{PV_1^u - PV^d}{\Delta y}$$

Quality option

What is the futures price for a contract where the short side can deliver either the 4 or the 3 year bond (with 2 or 1 year remaining)? Computation similar as before:

- first construct futures prices at expiration (year 2),
- then take expectation under risk-neutral measure.

Conversion factors 6%

- for 3 year bond (1 year remaining at expiration): $\frac{1+c}{1+y}$
  $C_3 = \frac{1}{1.06} = 0.94$
- for 4 year bond (2 year remaining at expiration): $\frac{c}{1+y} + \frac{c+1}{(1+y)^2}$
  $C_4 = \frac{1}{1.06^2} = 0.89$

Pricing tree with conversion factors

3 year bond:

year	0	1	2
P	100.97	101.93	102.91
		100.01	100.95
			99.07

4 year bond:

year	0	1	2
P	92.52	99.92	105.92
		94.37	101.92
			98.15

Futures price with quality option

Short party can choose which bond to deliver, thus in each state she pays

$$\min(Z(2,3) - FC_3, Z(2,4) - FC_4)$$

. This leads to

year	0	1	2
F	100.74	101.93	102.91
		99.55	100.95
			98.15

$$102.91 = \min\left(\frac{Z(2,3)}{C_3}, \frac{Z(2,4)}{C_4}\right) = \min\left(\frac{97.09}{0.94}, \frac{94.27}{0.89}\right)$$

$$101.93 = \frac{1}{2}(102.91 + 100.95)$$

r

(a) $100 r_1(t-1,t)$; $t=1, 3, \ldots$

$\boxed{\text{Price}}$

$$P = \frac{100r}{1+r} + \frac{100r}{(1+r)^3} + \cdots$$

$$P_{even} = \frac{100r}{(1+r)^2} + \frac{100r}{(1+r)^4} + \cdots$$

$$P_{even} = \frac{P}{1+r}$$

$$P + P_{even} = 100$$

$$P + \frac{P}{1+r} = 100$$

$$\boxed{P = 100 \times \frac{1+r}{2+r}}$$

$\boxed{\text{Duration}}$

$$D_{mac} = -\frac{1+r}{P} \times \frac{dP}{dr} \qquad r, y$$

$$P = \frac{100r}{1+y} + \left(100\frac{1+y}{2+y} - \frac{100y}{1+r}\right)$$

$$D_{mac} = -\frac{1+y}{P} \times \frac{dP}{dy}\Big|_{y=r}$$

(b) P, D (P, D) P, D

(a) + (b) = Perpetuity. \$200, 5%.

$$P_p = \sum_{i=1}^{10} \frac{10}{(1+r)^{2i-1}} \qquad \checkmark \qquad \Rightarrow P_b = P_p - P_a$$

$$D_{mac,p} = \sum_{odd} \frac{10t}{(1+r)^t} \Big/ P$$

$$= -\frac{1+y}{P} \frac{dP}{dy}$$

$$\Rightarrow \frac{P_a}{P_p} \times D_a + \frac{P_b}{P_p} \times D_b = D_p$$

(c & d)