Name: ______________________

MFE 230I Midterm Exam, Summer 2021

Wednesday, June 23, 9:30 a.m.–12:30 p.m.

Important Instructions — read these carefully before starting the exam:

- **You must be logged on to Zoom while taking the exam, with both your microphone and webcam turned on.**
- You agree to follow the **UC Berkeley Honor Code**, which states that "As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others." In particular, in taking this exam you are making the following commitments:
  - I alone am taking this exam.
  - I will neither receive assistance from anyone while taking the exam, nor will I provide assistance to anyone while the exam is still in progress.
  - Other than with the instructor and GSIs, I will not have any verbal, written, or electronic communication with anyone else while I or others are taking the exam.
- **Your exam solutions must be submitted as a single PDF file to Gradescope (either directly or via bCourses, just like the problem sets) before 12:30 p.m.**
- Show all your work in detail to earn as much credit as possible. Just writing the solution, with no explanation, will earn few points. If you cannot answer one part of a question, you can still get full marks for the remaining parts, so don't leave anything blank.
- Points (out of **120**) are roughly proportional to the suggested number of minutes for each question. There are **4** questions and **9** pages in total.

[Unless otherwise specified in a question, assume all payments are annual, interest rates are compounded annually, and all bonds have a face value of $100.]

1. (**28 points, 7 each.** Short-answer questions)

   (a) The one-, two- and three-year spot rates, $r_1(t,t+1)$, $r_1(t,t+2)$ and $r_1(t,t+3)$, are 3%, 4% and 5%, respectively. What are the one-, two-, and three-year par yields?

$$100 = \frac{100(1+y_1)}{1+3\%}, \qquad \text{so}$$
$$y_1 = 3\%$$

$$100 = \frac{100 \times y_2}{1+3\%} + \frac{100(1+y_2)}{(1+4\%)^2}, \qquad \text{so}$$
$$y_2 = 3.9803\%$$

$$100 = \frac{100 \times y_3}{1+3\%} + \frac{100 \times y_3}{(1+4\%)^2} + \frac{100(1+y_3)}{(1+5\%)^3}, \qquad \text{so}$$
$$y_3 = 4.9347\%$$

(b) The continuously compounded one-year spot rate $r(t, t+1)$ is 7%; the semiannually compounded forward rate between years 1 and 2, $f_2(t, t+1, t+2)$, is 9%; and the year-3 par yield (annually compounded) is 10%. What is the three-year discount factor?

The price of a 3-year bond with coupon rate 10% is \$100:

$$100 = \frac{10}{e^{0.07}} + \frac{10}{e^{0.07}(1.045)^2} + 110 \times DF(t, t+3), \quad \text{so}$$
$$DF(t, t+3) = 0.7467.$$

(c) For a settlement date of June 14, 2021, the discount factor for September 14, 2021 is 0.991. What is the zero rate for September 14, 2021 that you see quoted on your Bloomberg screen? [Report your answer as a percentage with 5 decimal places (e.g., 5.12345%) and assume (as in the example in class) that quoted zero rates are semiannually compounded using the 30I/360 day-count convention and that interest is calculated using compound interest.]

Day count $= 360(2021 - 2021) + 30(9 - 6) + (14 - 14) = 90$.

$$DF = \frac{1}{(1 + r/2)^{\frac{2\times 90}{360}}}, \qquad \text{so}$$
$$r = 3.64919\%.$$

(d) Given the same information as in 1c, what is the forward rate for the period between June 14 and Sept. 14, 2021 that you see quoted on your Bloomberg screen? [Report your answer as a percentage with 5 decimal places and assume that interest is calculated from the forward rates within each 3-month period using simple interest with the Actual/360 day-count convention.]

Day count $= 30 + 31 + 31 = 92$.

$$DF = \frac{1}{\left(1 + f \times \frac{92}{360}\right)}, \qquad \text{so}$$
$$f = 3.55372\%.$$

2. (**25 points.** Duration/convexity)

The yield curve is currently flat at 7% for all maturities.

(a) What are the price and modified duration of a 3-year, 6% coupon bond?

The price is given by

$$P = \frac{6}{1.07} + \frac{6}{(1.07)^2} + \frac{106}{(1.07)^3}$$
$$= 97.37568.$$

The modified duration is

$$\begin{aligned}
D_{\text{mod}} &= \frac{\left(1 \times \frac{6}{1.07}\right) + \left(2 \times \frac{6}{(1.07)^2}\right) + \left(3 \times \frac{106}{(1.07)^3}\right)}{97.37568 \times 1.07000} \\
&= 2.64580.
\end{aligned}$$

(b) What is the convexity of the bond in part 2a?

$$\begin{aligned}
C &= \frac{1}{(1+y)^2} \times \frac{1}{P} \times \sum t_i(t_i+1)PV_i \\
&= 9.69624.
\end{aligned}$$

(c) Use your results from parts 2a and 2b to calculate the approximate price of the bond if yields move to 9%.

$$\begin{aligned}
\frac{\Delta P}{P} &= -D_{\text{mod}}\,\Delta y + \frac{1}{2} C\,\Delta y^2 \\
&= -2.64580 \times 2\% + \frac{1}{2} \times 9.69624 \times 2\%^2 \\
&= -5.098\%, \qquad \text{so} \\
P_{new} &= 92.4118.
\end{aligned}$$

(d) What position in a 1-year zero-coupon bond is needed to hedge the bond in part 2a against movements in interest rates?

The dollar duration of the coupon bond is

$$D_{\$} = D_{\text{mod}} \times \frac{P}{100} = 2.57637.$$

The dollar duration of the 1-year bond is 0.87344, so we need to short

$$\frac{2.57637}{0.87344} = 2.94968$$

of the one year bond.

(e) Calculate the price, Macaulay duration, and convexity of a just-issued 2-year inverse floater, whose coupon payment at date t is given by

$$\text{Coupon}_t = 100 \times [15\% - r_1(t-1,t)].$$

As in class,
\$100 FV Floater + \$100 FV Inverse Floater = \$200 FV of a 7.5% coupon bond.

$$\begin{aligned}
P_{bond} &= \frac{15}{1.07} + \frac{215}{1.07^2} = 201.8080182 \\
P_F &= 100 \\
P_{IF} &= 201.8080182 - 100 = 101.8080182.
\end{aligned}$$

The Macaulay duration of the coupon bond is a weighted average of the Macaulay durations of the floater and the inverse floater:

$$\begin{aligned}
D_{Bond} &= \frac{1}{201.808}\left(\frac{1\times 15}{1.07}+\frac{2\times 215}{1.07^2}\right) = 1.930534516 \\
D_F &= 1 \\
D_{Bond} &= \frac{P_F}{P_{bond}}\times D_F + \frac{P_{IF}}{P_{bond}}\times D_{IF}, \qquad \text{so} \\
D_{IF} &= 2.844543583.
\end{aligned}$$

Similarly for convexity:

$$\begin{aligned}
C_{Bond} &= \frac{1}{201.808\times 1.07^2}\left(\frac{1\times 2\times 15}{1.07}+\frac{2\times 3\times 215}{1.07^2}\right) = 4.99794 \\
C_F &= \frac{1\times 2}{1.07^2} = 1.747 \\
C_{Bond} &= \frac{P_F}{P_{bond}}\times C_F + \frac{P_{IF}}{P_{bond}}\times C_{IF}, \qquad \text{so} \\
C_{IF} &= 8.1912.
\end{aligned}$$

3. (**37 points.** More fun with floaters)

The yield curve is currently flat, with interest rate r (annually compounded) for all maturities.

(a) A just-issued floating rate bond with *infinite* maturity has one unusual feature: it skips all the *even*-numbered payments. Its payments are given by

$$\text{Payment}_t = \begin{cases} 100\times r_1(t-1,t) & \text{for } t = 1,3,5,7,\dots, \\ 0 & \text{for } t = 2,4,6,8,\dots. \end{cases}$$

Calculate closed-form expressions (no summation signs or $+\ldots+$) for the price and Macaulay duration of this bond today.

Price

We know that the PV today of a single floating payment to be made in the future is the discounted value of the future payment calculated using today's *forward rate* as though it were the realized future spot rate. Here, all forward and spot rates today equal r, so

$$\begin{aligned}
P &= 100r\times\left[\frac{1}{(1+r)}+\frac{1}{(1+r)^3}+\frac{1}{(1+r)^5}+\ldots\right] \\
&= (1+r)\times \text{PV of perpetuity making fixed payment \$100}r\text{ at dates }2,4,6,\ldots \\
&= (1+r)\times\frac{100r}{r(2+r)} \qquad (1)\\
&= 100\times\frac{(1+r)}{(2+r)}, \qquad (2)
\end{aligned}$$

where Equation (1) follows from the previous line because the two-year interest rate is

$$(1+r)^2-1 = r(2+r).$$

Duration
It is tempting, **but wrong**, to calculate the duration as follows:

$$\begin{aligned} D_{\text{mac}} &= -\frac{(1+r)}{P} \times \frac{dP}{dr} \\ &= -\frac{(1+r)(2+r)}{100(1+r)} \times \frac{100}{(2+r)^2} \\ &= -\frac{1}{(2+r)}. \end{aligned}$$

Unfortunately, Equation (2) is not valid for interest rates other than r, because the date-1 payment remains $100r$ even if rates move. Splitting the first payment from the rest, the value of the security, valid for arbitrary interest rate y, is

$$P(y) = \frac{100r}{1+y} + \left[100 \times \frac{(1+y)}{(2+y)} - \frac{100y}{1+y}\right]. \tag{3}$$

This is equivalent to Equation (2) **for** $y = r$**, but not otherwise.** Now,

$$\begin{aligned} D_{\text{mac}} &= -\frac{(1+y)}{P} \times \left.\frac{dP}{dy}\right|_{y=r} \\ &= 1 + \frac{1}{1+r} - \frac{1}{2+r} \\ &= 1 + \frac{1}{(1+r)(2+r)}. \end{aligned}$$

(b) A just-issued *inverse floater* with *infinite* maturity has the same feature as above: it skips all the *even*-numbered payments. Its payments are given by

$$\text{Payment}_t = \begin{cases} 100 \times [10\% - r_1(t-1,t)] & \text{for } t = 1,3,5,7,\ldots, \\ 0 & \text{for } t = 2,4,6,8,\ldots. \end{cases}$$

Calculate closed-form expressions (no summation signs or $+\ldots+$) for the price and Macaulay duration of this bond today.

Holding a floater in (a) and a inverse floater in (b) is equivalent to holding a perpetuity with \$200 FV that pays 5% in odd-numbered periods. The price of this perpetuity is

$$\begin{aligned} P_p &= \sum_{i=1}^{\infty} \frac{10}{(1+r)^{2i-1}} \\ &= \frac{10(1+r)}{r(2+r)}. \end{aligned}$$

Therefore, the price of the inverse floater is

$$\begin{aligned} P_i =& P_p - P_f \\ =& \frac{10(1+r)}{r(2+r)} - \frac{100(1+r)}{(2+r)} \\ =& \frac{10(1+r)(1-10r)}{r(2+r)}. \end{aligned}$$

The Macaulay duration of the perpetuity is

$$D_p = \frac{1}{P}\sum_{i=1}^{\infty} \frac{(2i-1)}{(1+r)^{2i-1}}$$
$$= 1 + \frac{2}{r(r+2)}.$$

Using the formula for the duration of a portfolio, we have

$$D_p = \frac{P_f}{P_p} \times D_f + \frac{P_i}{P_p} \times D_i, \text{ so}$$
$$D_i = 1 - \frac{1}{r+2} + \frac{1}{r(1-10r)} - \frac{10r}{(1+r)(1-10r)}.$$

(c) Answer 3a for a just-issued floater with finite (odd) maturity T, whose payments are

$$\text{Payment}_t = \begin{cases} 100 \times r_1(t-1,t) & \text{for } t = 1,3,5,\ldots,T-2, \\ 100 \times [1 + r_1(t-1,t)] & \text{for } t = T, \\ 0 & \text{for } t = 2,4,6,\ldots,T-1. \end{cases}$$

Calculate closed-form expressions (no summation signs or $+\ldots+$) for the price and Macaulay duration of this bond today, valid for any *odd* value of T (i.e., $T = 1,3,5,\ldots$).

Similar to above, the price of a finite floater is

$$P = \sum_{i=1}^{\frac{T+1}{2}} \frac{100r}{(1+r)^{2i-1}} + \frac{100}{(1+r)^T}$$
$$= \frac{100(1+r)}{2+r}\left(1 + \frac{1}{(1+r)^T}\right)$$

To calculate the Macaulay duration, separate the first payment from the rest as in 3a:

$$P(y) = \frac{100r}{1+y} + \left[100 \times \frac{(1+y)}{(2+y)}\left(1 + \frac{1}{(1+y)^T}\right) - \frac{100y}{1+y}\right].$$

Therefore, the Macaulay duration is

$$D_{\text{mac}} = -\frac{(1+y)}{P} \times \left.\frac{dP}{dy}\right|_{y=r}$$
$$= -\frac{1}{r+2} + \frac{(r+2)(1+r)^{T-1} + T}{1 + (1+r)^T}.$$

(d) Answer 3b for a just-issued *inverse floater* with (odd) maturity T, whose payments are

$$\text{Payment}_t = \begin{cases} 100 \times [10\% - r_1(t-1,t)] & \text{for } t = 1,3,5,\ldots,T-2, \\ 100 \times [1 + 10\% - r_1(t-1,t)] & \text{for } t = T, \\ 0 & \text{for } t = 2,4,6,\ldots,T-1. \end{cases}$$

Calculate closed-form expressions (no summation signs or $+\ldots+$) for the price and Macaulay duration of this bond today, valid for any *odd* value of T (i.e., $T = 1, 3, 5, \ldots$).

Similar to 3b, holding a floater in c and an inverse floater in d is equivalent to holding a \$200 5% coupon bond. The price of the coupon bond is

$$P_c = \sum_{i=1}^{\frac{T+1}{2}} \frac{10}{(1+r)^{2i-1}} + \frac{200}{(1+r)^T}$$
$$= \frac{10(r+1)}{r(r+2)} \left(1 - \frac{1 - 20r(r+2)}{(1+r)^{T+1}}\right).$$

Therefore the price of the inverse floater is

$$P_i = P_c - P_f$$
$$= \frac{10(r+1)}{r(r+2)} \left(1 - \frac{1 - 20r(r+2)}{(1+r)^{T+1}}\right) - \frac{100(1+r)}{2+r} \left(1 + \frac{1}{(1+r)^T}\right).$$

The duration of the coupon bond is

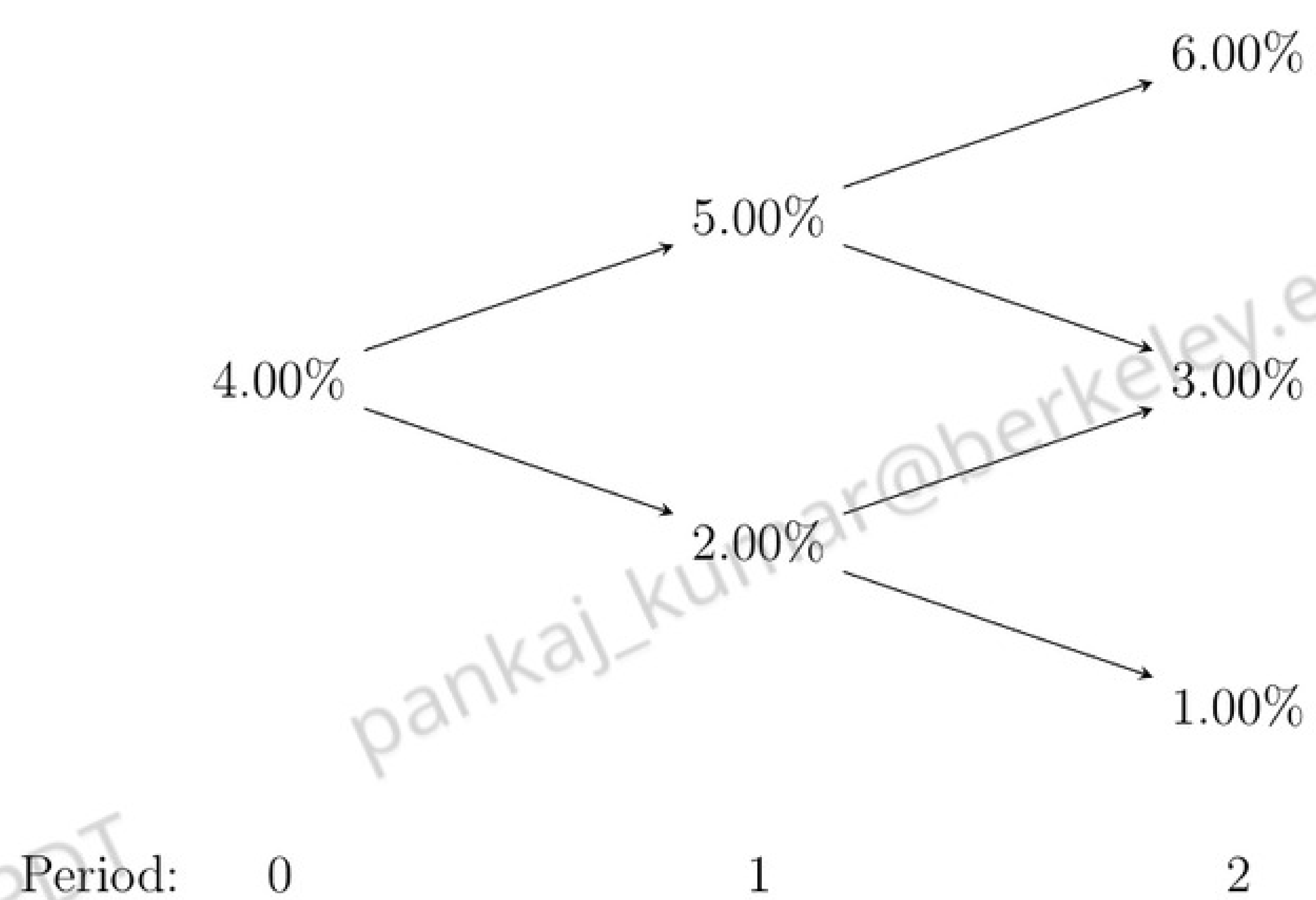
$$D_c = -\frac{(1+y)}{P_c} \times \left.\frac{dP_c}{dy}\right|_{y=r}$$
$$= \frac{(r^2+2r+2)(1+r)^{T+1} - 2(1+r)^2 - Tr(r+2) + 20Tr^2(r+2)^2}{r(r+2)\left[(1+r)^{T+1} - 1 + 20r(r+2)\right]}$$

Using the same relationship between floater and coupon bond,

$$D_c = \frac{P_f}{P_c} \times D_f + \frac{P_i}{P_c} \times D_i,$$

we can solve for the duration of this bond.

4. (**30 points.** Trees) The following tree shows possible movements in the 1-year interest rate (compounded annually) over the next two years. The risk-neutral probability of an upward shift in interest rates is 0.5 in each period:



(a) What are today's one-, two-, and three-year *swap rates*, $c(0,1)$, $c(0,2)$, and $c(0,3)$?

First calculate the bond price for each maturity:

2-year bond

92.9217841	95.2381
	98.03922

3-year bond

90.03934248	91.15571	94.339623
	96.12612	97.087379
		99.009901

Then the spot rate and swap rates can be calculated as follows:

z

t=1	2	3
96.15384615	92.92178	90.039342

c

t=1	2	3
4.00000%	3.7436%	3.56866%

(b) Construct a tree showing the value to the *fixed payer* in each time and state of a newly issued 3-year fixed-for-floating swap with swap rate equal to the value you calculated in 4a.

The swap tree is as follows (note that the value equals 0 in period 0).

t=1	2	3
0	2.192528	2.2937193
	−3.05521	−0.552095
		−2.543225

(c) You possess a 2-year European *chooser option*. This allows you, 2 years from today, to receive the payoff of either a 2-year European call option or a 2-year European put option (or neither, if you prefer). The underlying asset in each case is (today) a 3-year bond with coupon rate 3%, and the strike price is \$100. If you choose to exercise the option, you must do so immediately *after* the bond makes its year-2 coupon payment. What is the price of this option today?

First we price the bond using the binomial tree (note that the coupon payments are in arrears). Then the chooser option value is always positive when the bond price deviates from the strike price, \$100:

Bond price

t=0	1	2	3
98.41279166	96.74753	97.169811	103
	101.9511	100	103
		101.9802	103
			103

European chooser

t=0	1	2
1.114612596	1.347709	2.8301887
	0.970685	0
		1.980198

(d) Answer 4c again, this time assuming that the option is *American*. Thus, you may choose to exercise the option either today or in 1 year or in 2 years. As above, whatever year you choose to exercise the option, you must do so immediately *after* a coupon payment.

For an American option, compare each discounted value with value of exercising immediately, giving the following tree (red means early exercise is optimal):

t=0	1	2
2.501705894	3.252471	2.8301887
	1.951077	0
		1.980198