MFE 230I: Problem Set 2

Due: Monday, June 21, 2021

Note: For all questions in this problem set, unless specified otherwise, assume annual payments, annually compounded interest rates, and that the face value of all bonds is \$100.

1. Zero-coupon bond yields of all maturities are equal to 9%. Bond A is a 3-year 6% coupon bond. Calculate

   (a) The Macaulay duration of bond A.
   (b) The modified duration of bond A.
   (c) The dollar duration of bond A.
   (d) The convexity of bond A.
   (e) The cash-flow variance of bond A.

   Using the definitions from the notes, we obtain

   (a) $\mathrm{D}_{\mathrm{Mac}}(\mathrm{A}) = 2.82621$.
   (b) $\mathrm{D}_{\mathrm{Mod}}(\mathrm{A}) = 2.59285$.
   (c) $\mathrm{D}_{\$}(\mathrm{A}) = 2.39595$.
   (d) $\mathrm{C}(\mathrm{A}) = 9.32278$.
   (e) $\mathrm{CFV}(\mathrm{A}) = 0.26273$.

2. Bond B is a one-year zero-coupon bond. With the same assumptions as in Q1, if you own one bond A, what position in bond B do you need to hedge yourself against interest rate movements? Explain intuitively why this number is greater or less than one (whichever you find it to be)?

   Dollar duration of bond B is 0.84168, so short $2.39595/0.84168 = 2.84663$ of bond B. Answer is greater than 1 because each bond B is less interest-sensitive than bond A.

3. Bond C is a six-year zero-coupon bond. With the same assumptions as in Q1, what position in both bonds B *and* C would allow you to hedge your position in bond A over as wide a range of interest rate movements as possible?

   Set overall dollar duration and dollar convexity to zero. Suppose the portfolio to be shorted contains b units of bond B, and c units of bond C. Then

   $$0.84168b + 3.28221c = 2.39595 \qquad \text{(Dollar duration)}$$
   $$1.54437b + 21.07838c = 8.61482 \qquad \text{(Dollar convexity)}$$

   Solution: short 1.75400 of bond B and 0.28019 of bond C.

4. Assume that interest rates are continuously compounded; that cash flows on annuities/perpetuities and coupon payments on bonds are made continuously, starting today;[1] and that the yield to maturity on bonds of all maturities is $y > 0$.

   Calculate a *closed-form* expression (i.e., one that does not involve sums or integrals over lots of cash flows) for the Macaulay duration of

   (a) A *perpetuity* making a continuous payment at a rate of \$1 per year forever, starting today. Call this duration D_P.

   Value of perpetuity is

   $$P = \int_0^\infty e^{-yt}\,dt = \frac{1}{y}.$$

   With continuous compounding,

   $$\begin{aligned} D_P &= -\frac{1}{P} \times \frac{dP}{dy} \\ &= \frac{1}{y}. \end{aligned}$$

   (b) An *annuity* making a continuous payment at a rate of \$1 per year, starting today and ending at date T. Call this duration $D_A(T)$.

   Value of annuity equals value of perpetuity minus a perpetuity starting at date T,

   $$A = \frac{1}{y}\left[1 - e^{-yT}\right].$$

   So

   $$\begin{aligned} D_A(T) &= -\frac{1}{P} \times \frac{dP}{dy} \\ &= \frac{1}{y} - \frac{T}{e^{yT} - 1}. \end{aligned} \tag{1}$$

   (c) A *coupon bond* with continuous coupon rate c and maturity T years. Call this duration $D(T)$.

   A coupon bond is equivalent to a portfolio containing an annuity paying c per year up to date T plus a zero-coupon bond paying \$1 at date T. Its Macaulay duration is thus a weighted average of D_A and T, which simplifies to

   $$D(T) = \frac{1}{y} + \frac{T(y - c) - 1}{c\left[e^{yT} - 1\right] + y}. \tag{2}$$

[1]I.e., for a bond with coupon rate c, the total cash flow between t and $t + \delta t$ for t prior to maturity T is $c\,\delta t$. Similarly for annuities/perpetuities.

5. With the same assumptions as in Q4,

(a) Calculate the limit of $D_A(T)$, as the yield y goes to zero.

Discounting disappears as $y \to 0$, so $D_A(T)$ converges to an equally weighted average of the time to each cash flow, $T/2$. This can also be computed by taking the limit of Equation (1) as $y \to 0$, but this requires the application of l'Hôpital's rule twice.

(b) What numerical problems do you encounter as you try to calculate your expression for $D_A(T)$ from Q4 for very small values of y, and why?

Calculating Equation (1) directly for small y involves subtracting one very large number from another of very similar magnitude, resulting in significant loss of precision (see Goldberg, 1991).

(c) Explain how to calculate $D_A(T)$ in a way that gives results accurate to at least 10 significant figures for all values of y with $0 \leq y \leq 0.001$ and all values of T with $0 \leq T \leq 10$.

Rewrite Equation (1) as

$$D_A = \frac{e^{yT} - (1 + yT)}{y\left(e^{yT} - 1\right)}.$$

Direct calculation of both the numerator and denominator still involves subtracting two numbers of almost the same magnitude, again leading to loss of precision. However, taking Taylor series approximations to the exponentials in both numerator and denominator, and dividing both top and bottom by y^2T, we obtain

$$D_A = \frac{T\left(\frac{1}{2!} + \frac{yT}{3!} + \frac{(yT)^2}{4!} + \dots\right)}{\left(1 + \frac{yT}{2!} + \frac{(yT)^2}{3!} + \frac{(yT)^3}{4!} + \dots\right)}.$$

Both numerator and denominator can now be calculated without worrying about loss of precision, and for small yT a high degree of accuracy can be obtained with only a few terms in each approximation.

6. **[Extra credit]** Prove that

(a) For par and premium bonds ($c \geq y > 0$):

i. $D(T)$ is always below D_P.

ii. $D(T)$ is always increasing in T.

iii. $D(T)$ approaches a limit of D_P as $T \to \infty$.

From Equation (2), we can immediately see that

$$D(0) = 0 < D_P, \qquad \text{and}$$
$$\lim_{T \to \infty} D(T) = D_P.$$

In addition, if there is some $T \geq 0$ with $D(T) = D_P$, T must satisfy

$$T(y-c) - 1 = 0, \qquad \text{i.e.,}$$
$$T = \frac{1}{y-c}. \tag{3}$$

For $c \geq y$ this value is negative, so there is no value of $T \geq 0$ with $D(T) = D_P$. Thus $D(T) < D_P$ for all $T \geq 0$.

To prove ii., rewrite Equation (2) in the form

$$D(T) = \frac{\frac{c}{y}\left[1 - e^{-yT}\right] + T(y-c)e^{-yT}}{c + (y-c)e^{-yT}}. \tag{4}$$

Differentiating Equation (4) with respect to T, we obtain

$$\begin{aligned} f(T) &\equiv e^{yT}\left[c + (y-c)e^{-yT}\right]^2 \times \frac{dD(T)}{dT} \\ &= \left[2cy - c^2 - (y-c)cyT\right] + (y-c)^2 e^{-yT}, \end{aligned} \tag{5}$$

where $\frac{dD}{dT}$ has the same sign as f. Setting $T = 0$, we have

$$f(0) = y^2 > 0. \tag{6}$$

Differentiating Equation (5), we get

$$\frac{df}{dT} = y(c-y)\left[c - (c-y)e^{-yT}\right]. \tag{7}$$

As long as $y > 0$, $\frac{df}{dT}$ has the same sign as $(c-y)$. Since $c \geq y$, we have $\frac{df}{dT} \geq 0$, so $f(T)$ is always constant or increasing in T. Since $f(0) > 0$, this implies that $f(T) > 0$ for all $T \geq 0$; hence $D(T)$ is always increasing in maturity.

(b) For discount bonds ($0 < c < y$):

i. There is a unique $T_{max} > 0$ such that $D(T)$ is increasing in T for $T < T_{max}$ and decreasing in T for $T > T_{max}$.

ii. $D(T)$ hits a maximum greater than D_P at $T = T_{max}$.

iii. $D(T)$ approaches a limit of D_P as $T \to \infty$.

From Equation (7), for $c < y$ we have $\frac{df(T)}{dT} < 0$ for all $T \geq 0$. We know $f(0) > 0$, and from Equation (5) it is simple to verify that $\lim_{T\to\infty} f(T) < 0$. Thus, there is a unique $T_{\max} \in (0, +\infty)$ with

$$f(T) \begin{cases} > 0 & \text{for } T \in [0, T_{\max}) \\ = 0 & \text{for } T = T_{\max}, \\ < 0 & \text{for } T \in (T_{\max}, \infty) \end{cases}$$

$D(T)$ is thus increasing for $T < T_{max}$, decreasing for $T > T_{max}$, and hits a maximum at $T = T_{max}$.

Finally, since $\lim_{T\to\infty} D(T) = D_P$ from part a(iii) and $D'(T) < 0$ for $T > T_{\max}$, we must have $D(T) > D_P$ for $T \in [T_{\max}, \infty)$.

7. **Ho and Lee:**

(a) You are given the following zero-coupon bond yields (all annually compounded):

Maturity	Yield
1 year	5.00%
2 years	5.50%
3 years	5.70%
4 years	5.90%
5 years	6.00%
6 years	6.10%

Assuming the volatility of the (annually compounded) one-year rate is 1.5% per year, calibrate a Ho and Lee (1986) tree (as we did in class) to these bond prices, where the interest rate in the tree is the annually compounded one-year rate. Print the tree, and use it to price the following securities and to calculate their spot rate durations.

Match the bond prices sequentially, as in class, to obtain

Time T	0	1	2	3	4	5
Drift m		1.02%	0.14%	0.46%	-0.02%	0.31%
	5.00%	7.52%	9.16%	11.13%	12.61%	14.42%
		4.52%	6.16%	8.13%	9.61%	11.42%
			3.16%	5.13%	6.61%	8.42%
				2.13%	3.61%	5.42%
					0.61%	2.42%
						-0.58%

(b) A *non-prepayable* 6-year mortgage. Assume this is a self-amortizing loan, requiring equal payments at the end of each of the next six years, with a quoted (annual) interest rate of 5.5% and an initial balance of \$100.

The payment for each period can be calculated from the formula

$$C = \frac{100}{\sum_{k=1}^{6} \frac{1}{1.055^k}} = 20.0179.$$

The value of the non-prepayable mortgage can be computed by discounting the periodic payments backwards. We obtain the formula

$$V_{i,j} = \frac{0.5 \cdot (V_{i+1,j} + V_{i+1,j+1}) + C}{1 + r_{i,j}},$$

where $r_{i,j}$ is taken from the tree calibrated above. We get the following table:

6-year non-prepayable mortgage tree					
t=0	t=1	t=2	t=3	t=4	t=5
98.91	80.43	64.25	48.79	33.52	17.50
	87.24	68.69	51.45	34.88	17.97
		73.65	54.36	36.34	18.46
			57.56	37.92	18.99
				39.62	19.55
					20.13

The spot rate duration can be approximated as

$$\begin{aligned} D &= -\frac{1}{P}\frac{P_{1,u}-P_{1,d}}{r_{1,u}-r_{1,d}} \\ &= -\frac{1}{98.91}\frac{80.44-87.24}{7.52\%-4.52\%} \\ &= 2.29 \end{aligned}$$

(c) A *prepayable* 6 year mortgage. This is the same mortgage as above, except that the loan may be prepaid at any time. When a borrower prepays a mortgage, the borrower pays back the remaining principal on the loan, and does not make any of the remaining scheduled payments. Assume that the borrower refinances optimally, incurring no prepayment penalties or other transaction costs upon refinancing.

A prepayable mortgage can be viewed as long a non-prepayable mortgage and short a prepayment option. The non-prepayable mortgage was dealt with in the previous question. The prepayment option can be priced by assuming that the borrower has the option to prepay when the present value of future mortgage payment is higher than the remaining principal amount. The tree for the prepayment option is

Prepayment option					
t=0	t=1	t=2	t=3	t=4	t=5
0.89	0.10	0.00	-	-	-
	1.77	0.22	0.00	-	-
		3.48	0.46	0.01	-
			3.55	0.96	0.01
				2.66	0.57
					1.16

Given the $t = 0$ price of the prepayment option, the initial price of the prepayable mortgage is then \$98.91 − \$0.89 = \$98.02.

Alternatively, we can value the prepayable mortgage in one step. At each period, the borrower can either exercise the prepayment option or not. Denoting $L_i^{scheduled}$ as the outstanding principal in period i, the prepayment option payoff is

$$C_{i,j}^{Exercise} = (V_{i,j}^{nonprepay} - L_i^{scheduled})^+$$

Therefore, the value of the option is determined by $(C_{i,j}^{wait}, C_{i,j}^{Exercise})^+$. The price of the prepayable mortgage is the following:

6-year prepayable mortgage tree					
t=0	t=1	t=2	t=3	t=4	t=5
98.02	80.33	64.25	48.79	33.52	17.50
	87.47	68.47	51.44	34.88	17.97
		70.17	53.90	36.33	18.46
			54.01	36.96	18.97
				36.96	18.97
					18.97

Intuitively, the nodes where prepayment occurs correspond to low interest rate states, where it is optimal to refinance. Applying the same formula as in part a), the spot rate duration is

$$\frac{-1}{98.02} \times \frac{80.33 - 85.47}{7.52\% - 4.52\%} = 1.75.$$

(d) A principal only (PO) security based on the prepayable mortgage above. The holder of this security receives any *principal* payments made by the mortgage holder(s) (either scheduled or unscheduled), but none of the interest.

We need to split each payment into principal vs. interest. The holder of the PO security in each period i is then entitled to the principal payment.

In nodes where prepayment does occur, the holder of the PO security receives $L_i^{outstanding}$. In the nodes where prepayment does not occur, the holder receives

$$P_{i,j} = \frac{0.5 \cdot (P_{i+1,j} + P_{i+1,j+1}) + CF(i+1)}{1 + r_{i,j}}$$

where

$$CF(i+1) = \text{scheduled principal at } i+1$$

The resulting tree is shown below

PO security					
t=0	t=1	t=2	t=3	t=4	t=5
83.19	68.48	56.05	43.71	30.90	16.58
	77.19	60.57	46.35	32.16	17.03
		70.17	49.95	33.98	17.50
			54.01	36.96	18.97
				36.96	18.97
					18.97

With the same formula as in a), the spot rate duration is

$$\frac{-1}{83.19} \times \frac{68.48 - 77.19}{7.52\% - 4.52\%} = 3.49$$

(e) An interest only (IO) security based on the prepayable mortgage above. The holder of this security receives any *interest* payments made by the mortgage holder(s), but none of the principal.

The holder of the IO receives the interest payment each period. In states where prepayment occurs, the holder receives nothing. In the nodes where prepayment does not occur, the holder receives

$$P_{i,j} = \frac{0.5 \cdot (P_{i+1,j} + P_{i+1,j+1}) + CF(i+1)}{1 + r_{i,j}}$$

where

$$CF(i+1) = \text{scheduled interest at } i+1$$

The resulting tree is shown below

PO security					
t=0	t=1	t=2	t=3	t=4	t=5
14.82	11.85	8.20	5.08	2.63	0.91
	8.27	7.89	5.10	2.72	0.94
		-	3.95	2.36	0.96
			-	-	-
				-	-
					-

To get the spot rate duration, we calculate as before

$$\frac{-1}{14.82} \times \frac{11.85 - 8.27}{7.52\% - 4.52\%} = -8.05$$

Note the negative result.

References

Goldberg, David, 1991, What every computer scientist should know about floating-point arithmetic, *Computing Surveys* (March).

Ho, Thomas S. Y., and Sang Bin Lee, 1986, Term structure movements and pricing interest rate contingent claims, *Journal of Finance* 41, 1011–1029.