MFE 230I: Problem Set 1

Due: Monday, June 14, 2021 by 9:30 a.m.

1. You take out a 30-year, fixed-rate mortgage for \$1,000,000. The interest rate on the mortgage loan is 5.25% (APR, compounded monthly), and it requires you to make equal payments at the end of each of the next 360 months (i.e., the first payment is 1 month from today and the last payment is 360 months from today).

   (a) What is the amount of each monthly payment?

   PV of payments, discounted using loan interest rate, must equal \$1,000,000. I.e., if monthly payment amount is c, we have

$$\begin{aligned} 1{,}000{,}000 &= \frac{c}{1+r} + \frac{c}{(1+r)^2} + \ldots + \frac{c}{(1+r)^{360}} \\ &= \frac{c}{r}\left[1 - \frac{1}{(1+r)^{360}}\right], \end{aligned}$$

   where $r = 0.0525/12$. Solving, we get $r = \$5{,}52.04$.

   (b) What is the remaining balance on the loan immediately after the 100th loan payment?

   Remaining balance = PV of remaining 260 payments,

$$\begin{aligned} \frac{c}{1+r} + \frac{c}{(1+r)^2} + \ldots + \frac{c}{(1+r)^{260}} &= \frac{c}{r}\left[1 - \frac{1}{(1+r)^{260}}\right] \\ &= \$856{,}495.04. \end{aligned}$$

   (c) How much interest in total will you pay during year 10 of the mortgage (i.e., what is the total amount of interest included in the 12 payments ending with the payment 120 months from today)?

   Remaining balance at start and end of year 10 (end of months 108 and 120) are

$$\begin{aligned} B_{108} &= \frac{c}{r}\left[1 - \frac{1}{(1+r)^{252}}\right] = \$842{,}076.04, \\ B_{120} &= \frac{c}{r}\left[1 - \frac{1}{(1+r)^{240}}\right] = \$819{,}482.80. \end{aligned}$$

   Total principal paid during the year is the difference between these values, \$22,593.94.
   Total payment during the year is $12 \times 5{,}52.04 = \$66{,}264.44$.
   Of this, total interest is $66{,}264.44 - 22{,}593.94 = \$43{,}670.50$.

2. The annually compounded one-, two- and three-year spot rates ($r_1(0,1)$, $r_1(0,2)$ and $r_1(0,3)$) are 5%, 6% and 7% respectively.

(a) What are the first three annually compounded (one-year) forward rates for years 1, 2 and 3 ($f_1(0,0,1)$, $f_1(0,1,2)$ and $f_1(0,2,3)$)?

We have

$$(1 + f_1(t,T_1,T_2))^{T_2-T_1} = \frac{Z(t,T_1)}{Z(t,T_2)}.$$

Therefore,

$$\begin{aligned}
f_1(0,0,1) &= r_1(0,1) = 0.05. \\
f_1(0,1,2) &= \frac{Z(0,1)}{Z(0,2)} - 1 = \frac{(1 + r_1(0,2))^2}{(1 + r_1(0,1))} - 1 = 0.0701. \\
f_1(0,2,3) &= \frac{Z(0,2)}{Z(0,3)} - 1 = \frac{(1 + r_1(0,3))^3}{(1 + r_1(0,2))^2} - 1 = 0.0903.
\end{aligned}$$

(b) If no explicit forward-rate agreements existed, how would you use a combination of spot lending and borrowing to effectively borrow \$1,000 a year from today, and pay it back 3 years from today (plus interest)?

To borrow \$1,000 for 2 years, starting 2 years from today, lend $\$\frac{1{,}000}{(1+r_1)}$ today for one year and simultaneously borrow the same amount for *three* years:

Year:	0	1	2	3
Lend:	−952.38	1,000		
Borrow:	+952.38			−1,166.71
Total		1,000		−1,166.71

3. The annually compounded one-, two- and three-year forward rates ($f_1(0,0,1)$, $f_1(0,1,2)$ and $f_1(0,2,3)$) are 11%, 13% and 17% respectively. What are the annually compounded one, two and three-year spot rates ($r_1(0,1)$, $r_1(0,2)$ and $r_1(0,3)$)?

We have

- $r_1(0,1) = f_1(0,0,1) = 11.00\%$.
- $(1 + r_1(0,2))^2 = (1+r_1(0,1))(1+f_1(0,1,2)) \Rightarrow r_1(0,2) = \sqrt{1.11 \times 1.13} - 1 = 11.996\%$.
- $(1 + r_1(0,3))^2 = (1 + r_1(0,2))^2(1 + f_1(0,2,3)) \Rightarrow r_1(0,3) = 13.639\%$.

4. A Treasury Bill has 32 days to maturity. If the quoted discount is 5.2%, what is the price of one Bill (assume a \$10,000 face value)?

With a face value of \$10,000, we have

$$d = \frac{360}{n} \times \frac{(10{,}000 - P)}{10{,}000}.$$

With $n = 32$ and $d = 0.052$, we obtain $P = 9{,}953.78$.

5. Prove that for a discount bond, YTM > current yield > coupon rate.

Let the bond's face value be \$1, the coupon rate be c, the YTM be y, and the current yield be y_{current}. Since the bond price $P < 1$, we immediately have that

$$y_{\text{current}} \equiv \frac{c}{P} > c.$$

Assuming annual payments/compounding (the proof is similar for other assumptions), the price of the bond is

$$P = \sum_{i=1}^{T} \frac{c}{(1+y)^t} + \frac{1}{(1+y)^T} = \frac{c}{y}\left[1 - \frac{1}{(1+y)^t}\right] + \frac{1}{(1+y)^T}.$$

Dividing both sides by P gives

$$1 = \frac{1}{P} \cdot \frac{c}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{P}\frac{1}{(1+y)^T},$$

and rearranging terms gives

$$y \cdot \left(\frac{1 - \frac{1}{(1+y)^T} \cdot \frac{1}{P}}{1 - \frac{1}{(1+y)^T}} \right) = \frac{c}{P} = y_{\text{current}}.$$

The quantity in parentheses is positive and less than one if $P < 1$, so

$$y > y_{\text{current}}.$$

6. Here is the start of the Bloomberg "23" yield curve for trade date May 29, 2018 that we studied in class:

Date	Zero Rate	Forward Rate
05/31/2018	—	2.31813000000000
08/31/2018	2.37666301585850	2.33346095337569
11/30/2018	2.37150478655406	2.46702480555569
02/28/2019	2.42393753171055	2.55633514771236

(a) Calculate the discount factor for February 28, 2019 from the *zero rates*.

Start with $(D_1, M_1, Y_1) = (31, 5, 2018)$ and $(D_2, M_2, Y_2) = (28, 2, 2019)$. Using the 30I/360 day-count convention:

- D_1 is the last day of month, so change D_1 to 30.
- D_2 is also the last day of month, but this is the maturity date and M_2 is Feb, so leave $D_2 = 28$.
- Day count $= 360(2019 - 2018) + 30(2 - 5) + (28 - 30) = 268$.

So
$$DF_{2/28/19} = \frac{1}{\left(1 + \frac{.02423938}{2}\right)^{2 \times \frac{268}{360}}} = 0.982224.$$

(b) Calculate the discount factor for February 28, 2019 from the *forward rates*.

- (Actual) days between 5/31/18 and 8/31/18 = 92.
- (Actual) days between 8/31/18 and 11/30/18 = 91.
- (Actual) days between 11/30/18 and 2/28/19 = 90.

So
$$DF_{2/28/19} = \frac{1}{1 + \left(.0231813 \times \frac{92}{360}\right)} \times \frac{1}{1 + \left(.02333461 \times \frac{91}{360}\right)} \times \frac{1}{1 + \left(.02467025 \times \frac{90}{360}\right)} = 0.982224.$$

The next three problems involve fitting yield curves. To obtain the input data for your analysis, start by using Bloomberg to download LIBOR/swap rates from Feb. 28, 2019. **In your analysis, make sure to use *mid-quotes*, i.e., (Bid + Ask) / 2.**

7. Fit a yield curve to the input data you just downloaded, assuming that

   - The *simple interest rate* r_s is a continuous function of maturity, $r_s(t)$.
   - $r_s(t)$ is piecewise linear, with kinks at the maturity dates of the instruments in the input data set.
   - $r_s(t)$ is constant outside the range of maturities in the input data. I.e., if $t_{\min}$ and $t_{\max}$ are, respectively, the shortest and longest maturities in the input sample,
   $$r_s(t) = \begin{cases} r_s(t_{\min}) & \text{if } t \leq t_{\min}, \\ r_s(t_{\max}) & \text{if } t \geq t_{\max}. \end{cases}$$

   [This is Bloomberg's default "method 1" (see Bloomberg, 2012, pp. 7–8).]

   Your goal is to match the zero rates and forward rates calculated by Bloomberg, which you can obtain by selecting `Curve Analysis` → `Forward Analysis` and then exporting to Excel [make sure that `Interpolation` is set to `Piecewise Linear (Simple)` and that `Curve Side` is set to `mid`].

   Using your fitted yield curve, generate a table showing, at 3-month intervals from 3/4/2019 to 12/4/2068 (use the dates in the output spreadsheet you just downloaded from Bloomberg):

   (a) Your fitted discount factor.
   (b) Your fitted zero rate.
   (c) Bloomberg's fitted zero rate
   (d) Your fitted 3-month forward rate

(e) Bloomberg's fitted 3-month forward rate
(f) The difference between your fitted zero rate and Bloomberg's
(g) The difference between your fitted forward rate and Bloomberg's

Report all numbers to 10 decimal places, and report the numbers in columns (b)–(g) as percentages (e.g., 2.6151300000%). **Your numbers should be identical to Bloomberg's to at least this degree of precision. In other words, the numbers in columns (f) and (g) should all be ±0.0000000000%.** If you manage to match only to, say, 7 decimal places, e.g., a difference of 0.000000026%, your calculations are *not* 100% right.

Note: Bloomberg uses the following quoting conventions:

Rate	Compounding	Day count
Zero rates	Semi-annual	30I/360
Forward rates	Quarterly	Actual/360
Simple rates	Simple	Actual/360

Note: this is not easy! Nobody got all the numbers to match exactly last year. In answering this question, make sure to read — and reread — the description in Bloomberg (2012) in detail.

This can be done, but I'm painfully aware of how many days of work it takes. . . . There are a *lot* of conventions and small decisions to get right, and you'll have a much more detailed understanding of both term-structure fitting and interest-rate/day-count calculations at the end of this exercise. A few important things to note:

- Settlement date (two working days after trade date) is 3/4/19 (weekend 3/2–3/3).
- 3-month Eurodollar futures contracts all expire on the 3rd Wednesday of the maturity month, and the loan underlying the contract lasts exactly 91 days (13 weeks). For example, if the "Term" listed in the Bloomberg data for a 3-month futures contract is 20200617, this means that the quoted rate applies over the 91-day period starting on Wednesday, March 18, 2020 and ending on Wednesday, June 17, 2020.
- Fixed payments on a swap are made at 6-monthly intervals, *except when a payment date falls on a weekend or holiday.* In this case, the payment is made on the next business day.[1]
- The size of the fixed payment on a swap is *not* actually fixed! Rather, it depends on the length of the relevant period:

$$\text{Fixed payment} = \$100 \times \text{swap rate} \times \frac{\text{days in payment period (30I/360)}}{360}.$$

  While the year fraction here is usually 1/2, because of weekends and holidays *sometimes it is not.*

[1] In experimenting with this, I discovered that the holiday/business-day code in `pandas` only worked properly up to the year 2030, so I submitted a fix to make it work up to 2200 (see `https://github.com/pandas-dev/pandas/pull/27790`).

8. Fit the 5-parameter model of Nelson and Siegel (1987) to your input data. Generate the same table as in Question 7.

   For this and the next question, the analysis is significantly simplified by noting that, conditional on the value of the τs, the βs can be estimated using OLS. Nonlinear optimization can therefore be performed just over the τs.

9. Repeat Question 8, this time using the model of Svensson (1994).

References

Bloomberg, 2012, Building the Bloomberg interest rate curve: Definitions and methodology, Technical report.

Nelson, C. R., and A. F. Siegel, 1987, Parsimonious modeling of yield curves, *Journal of Business* 60, 473–489.

Svensson, L. E. O., 1994, Estimating and interpreting forward rates: Sweden 1992–4, Working Paper 4871, NBER.