MFE230Q: HW2 Solution

1) Generalized butterfly spread:

1. *Plot the payout of the generalized butterfly spread* $\mathcal{B}(60, 80, 100, 130, 1)$ *at maturity, as a function of the underlying stock-price.*

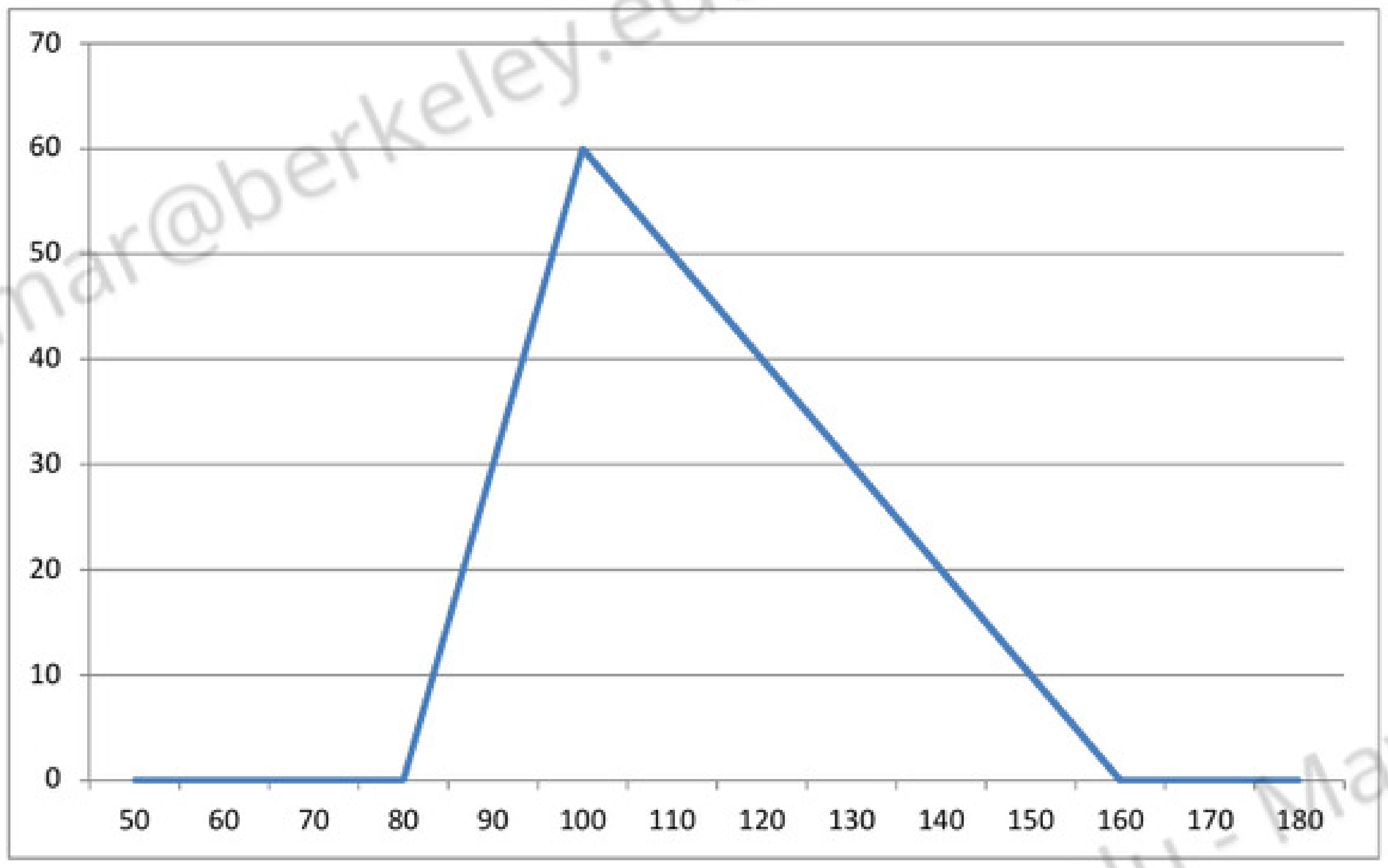

Figure 1: **Payout of generalized butterfly spread.**

2. *Show how the same payout can be generated by a portfolio of put options,*

$$\mathbf{s}_t = \left(P_t^{Euro,K_1,T}, P_t^{Euro,K_2,T}, P_t^{Euro,K_1,T}\right)^T.$$

The payout can be generated by taking an identical position in put options:

$$\mathbf{h} = \left(\frac{L}{K_2 - K_1}, -\frac{L}{K_2 - K_1} - \frac{L}{K_3 - K_2}, \frac{L}{K_3 - K_2}\right)^T.$$

3. *Using put-call parity, verify that the price of the portfolio in (b) is the same as that of the portfolio in (a).*

The value, V_t^C, of the portfolio of call options is

$$\begin{aligned} V_t^C &= \frac{L}{K_2 - K_1} C_t^{Euro,K_1,T} - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right) C_t^{Euro,K_2,T} + \frac{L}{K_3 - K_2} C_t^{Euro,K_3,T} \\ &= \frac{L}{K_2 - K_1}\left(S_t + P_t^{Euro,K_1,T} - \frac{K_1}{R}\right) \end{aligned}$$

$$
\begin{aligned}
& - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right)\left(S_t + P_t^{Euro,K_2,T} - \frac{K_3}{R}\right) \\
& + \frac{L}{K_3 - K_2}\left(S_t + P_t^{Euro,K_3,T} - \frac{K_3}{R}\right) \\
& = \frac{L}{K_2 - K_1} P_t^{Euro,K_1,T} - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right) P_t^{Euro,K_2,T} + \frac{L}{K_3 - K_2} P_t^{Euro,K_3,T} \\
& + \left(\frac{L}{K_2 - K_1} - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right) + \frac{L}{K_3 - K_2}\right) S_t \\
& - \left(\frac{K_1}{K_2 - K_1} - \left(\frac{K_2}{K_2 - K_1} + \frac{K_2}{K_3 - K_2}\right) + \frac{K_3}{K_3 - K_2}\right) \frac{L}{R} \\
& = \frac{L}{K_2 - K_1} P_t^{Euro,K_1,T} - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right) P_t^{Euro,K_2,T} + \frac{L}{K_3 - K_2} P_t^{Euro,K_3,T} \\
& + 0 \times S_t \\
& - (-1 + 1) \times \frac{L}{R} \\
& = \frac{L}{K_2 - K_1} P_t^{Euro,K_1,T} - \left(\frac{L}{K_2 - K_1} + \frac{L}{K_3 - K_2}\right) P_t^{Euro,K_2,T} + \frac{L}{K_3 - K_2} P_t^{Euro,K_3,T} \\
& = V_t^P.
\end{aligned}
$$

2) Consider the trinomial economy below.

$$
\begin{array}{ccc}
 & & \mathit{120,\ \ 105} \\
 & \nearrow & \\
S(0) = 105, \quad B(0) = 100 & \rightarrow & \mathit{105,\ \ 105} \\
 & \searrow & \\
 & & \mathit{100,\ \ 105}
\end{array}
$$

2A) Is the market complete? Why or why not?

No. There are three states of nature, and only two securities. As such, you cannot uniquely identify the AD prices, and hence, cannot price a generic derivative security with CF's = $(CF_{\uparrow}, CF_{\rightarrow}, CF_{\downarrow})$.

2B) Define $(AD_{\uparrow}, AD_{\rightarrow}, AD_{\downarrow})$ as the prices of the Arrow-Debreu securities that pays \$1 iff the up-state, middle state, and down state occur, respectively. Use the stock and bond price dynamics to find two constraints on the values of these AD securities. Use these constraints to write both $AD_{\uparrow}$ and $AD_{\rightarrow}$ solely as a function of $AD_{\downarrow}$.

From the stock and bond dynamics, we have two restrictions on the three AD security prices:

$$
\begin{aligned}
105 &= 120\, AD_{\uparrow} + 105\, AD_{\rightarrow} + 100\, AD_{\downarrow} \\
100 &= 105\, AD_{\uparrow} + 105\, AD_{\rightarrow} + 105\, AD_{\downarrow}. \qquad (1)
\end{aligned}
$$

Subtracting, I find

$$5 \;=\; 15\,AD_{\uparrow} - 5\,AD_{\downarrow}, \tag{2}$$

implying that we can write

$$AD_{\uparrow} \;=\; \frac{1}{3} + \frac{1}{3}\,AD_{\downarrow}. \tag{3}$$

Plugging this back into eq. (1), I find

$$AD_{\rightarrow} \;=\; \frac{13}{21} - \frac{4}{3}AD_{\downarrow}. \tag{4}$$

2C) Identify the range of values of $AD_{\downarrow}$ that are consistent with all three AD prices being positive. Does this economy admit arbitrage opportunities?

The three restrictions are

$$\begin{aligned} 0 \;&<\; AD_{\downarrow} \\ 0 \;&<\; AD_{\uparrow} = \frac{1}{3} + \frac{1}{3}AD_{\downarrow} \\ 0 \;&<\; AD_{\rightarrow} = \frac{13}{21} - \frac{4}{3}AD_{\downarrow}. \end{aligned} \tag{5}$$

Simplifying, we see that all AD prices are positive if

$$0 < AD_{\downarrow} < \frac{13}{28}. \tag{6}$$

Given that there exists at least one solution (in fact, there are an infinite number of solutions) with positive AD prices, it follows that there are no arbitrage strategies in this economy. That is, there is no portfolio $(n_S,\, n_B)$ that has positive CF's in all states of nature and a zero or negative price.

2D) Identify the range of values for the call option with strike $K = 105$

The call option with $K = 105$ is nothing but a piece of paper with CF's $= (15, 0, 0)$. Its price is thus

$$\begin{aligned} C(K = 105) \;&=\; 15\,AD_{\uparrow} \\ &=\; 15\left(\frac{1}{3} + \frac{1}{3}\,AD_{\downarrow}\right) \\ &=\; 5 + 5\,AD_{\downarrow}. \end{aligned} \tag{7}$$

Its range of arbitrage-free prices are thus

$$\begin{aligned} C(K=105)_{low} &= 5+5(0)=5 \\ C(K=105)_{high} &= 5+5\left(\frac{13}{28}\right)=7.32. \end{aligned} \tag{8}$$

2E) Identify the range of values for the call option with strike $K = 100$. Interpret your results.

The call option with $K = 100$ is nothing but a piece of paper with CF's = $(20, 5, 0)$. Its price is thus

$$\begin{aligned} C(K=100) &= 20\,AD_{\uparrow} + 5\,AD_{\rightarrow} \\ &= 20\left(\frac{1}{3}+\frac{1}{3}\,AD_{\downarrow}\right)+5\left(\frac{13}{21}-\frac{4}{3}\,AD_{\downarrow}\right) \\ &= 9.762. \end{aligned} \tag{9}$$

Note that this is a single number, not a range of numbers. Why? Because this CF can be replicated by purchasing one share of stock and shorting $\frac{100}{105}$ shares of bond. This replicating portfolio has a cost of

$$\begin{aligned} V_{repl} &= (1)\,(105)-\left(\frac{100}{105}\right)(100) \\ &= 9.762, \end{aligned} \tag{10}$$

as is necessary to preclude arbitrage.

3) Assume a 2-period, 3-date model as below.

$$\begin{array}{ccccc} & & & \nearrow & u^2S(0),\ B(0)(1+R)^2,\ CF(\uparrow\uparrow) \\ & \nearrow & uS(0),\ B(0)(1+R),\ V(\uparrow) & & \\ & & & \searrow & \\ \text{S(0)},\ \ \text{B(0)},\ \ \text{V(0)} & & & & udS(0),\ B(0)(1+R)^2,\ CF(\uparrow\downarrow) \\ & & & \nearrow & \\ & \searrow & dS(0),\ B(0)(1+R),\ V(\downarrow) & & \\ & & & \searrow & d^2S(0),\ B(0)(1+R)^2,,\ CF(\downarrow\downarrow) \end{array}$$

3A) Assume $S(0) = 100$, $B(0) = 100$, $u = 1.2$, $d = .9$, $R_F = .05$. Using dynamic programming, determine the price **and** the replicating portfolio for the call option with maturity $T = 2$ and strike $K = 95$.

The CF's for the K=95 call option are $C_{\uparrow\uparrow} = \max(0, 144-95) = 49$, $C_{\uparrow\downarrow} = C_{\downarrow\uparrow} = \max(0, 108-95) = 13$, and $C_{\downarrow\downarrow} = \max(0, 81-95) = 0$. To determine the date-0 price, we work backward through the tree. We do not know, at date-0, whether the up-state or down-state will occur, so we must consider both separately:

First consider the case when the up-state occurs at date-1. Our binomial tree looks like

$$
\begin{array}{ccl}
 & & S_{\uparrow\uparrow} = 144,\ B_{\uparrow\uparrow} = 110.25,\ C_{\uparrow\uparrow} = 49 \\
 & \nearrow & \\
S_{\uparrow} = 120,\ B_{\uparrow} = 105,\ C_{\uparrow} = ?? & & \\
 & \searrow & \\
 & & S_{\uparrow\downarrow} = 108,\ B_{\uparrow\downarrow} = 110.25,\ C_{\uparrow\downarrow} = 13
\end{array}
$$

To replicate the CF's of the call option, we choose $n_S(\uparrow)$, $n_B(\uparrow)$ so that if:

$$
\begin{array}{lrcl}
\uparrow\uparrow \text{ occurs}: & 49 & = & 144\, n_S(\uparrow) + 110.25\, n_B(\uparrow) \\
\uparrow\downarrow \text{ occurs}: & 13 & = & 108\, n_S(\uparrow) + 110.25\, n_B(\uparrow).
\end{array} \tag{11}
$$

The solution is

$$
n_S(\uparrow) = 1, \qquad n_B(\uparrow) = -\frac{95}{110.25}. \tag{12}
$$

The arbitrage-free price of the call option in event-$\uparrow$ is thus

$$\begin{aligned} C_{\uparrow} &= \left(1\right)\left(120\right) + \left(-\frac{95}{110.25}\right)\left(105\right) \\ &= 29.52 \end{aligned} \tag{13}$$

Now consider the case when the down-state occurs at date-1. Our binomial tree looks like

$$\begin{array}{ccl} & & S_{\downarrow\uparrow} = 108,\ B_{\downarrow\uparrow} = 110.25,\ C_{\downarrow\uparrow} = 13 \\ & \nearrow & \\ S_{\downarrow} = 90,\ B_{\downarrow} = 105,\ C_{\downarrow} = ?? & & \\ & \searrow & \\ & & S_{\downarrow\downarrow} = 81,\ B_{\downarrow\downarrow} = 110.25,\ C_{\downarrow\downarrow} = 0 \end{array}$$

To replicate the CF's of the call option, we choose $n_S(\downarrow)$, $n_B(\downarrow)$ so that if:

$$\begin{aligned} \downarrow\uparrow \text{ occurs :} \qquad 13 &= 108\, n_S(\downarrow) + 110.25\, n_B(\downarrow) \\ \downarrow\downarrow \text{ occurs :} \qquad 0 &= 81\, n_S(\downarrow) + 110.25\, n_B(\downarrow). \end{aligned} \tag{14}$$

The solution is

$$n_S(\downarrow) = \frac{13}{27}, \qquad n_B(\downarrow) = -\frac{39}{110.25}. \tag{15}$$

The arbitrage-free price of the call option in event-$\downarrow$ is thus

$$\begin{aligned} C_{\downarrow} &= \left(\frac{13}{27}\right)\left(90\right) + \left(-\frac{39}{110.25}\right)\left(105\right) \\ &= 6.19. \end{aligned} \tag{16}$$

Now that we have determined $C_{\uparrow} = 29.52$ and $C_{\downarrow} = 6.19$, we can move back through the tree to date-0. The relevant part of the tree now looks like

$$\begin{array}{ccl} & & S_{\uparrow} = 120,\ B_{\uparrow} = 105,\ C_{\uparrow} = 29.52 \\ & \nearrow & \\ S_0 = 100,\ B_0 = 100,\ C_0 = ?? & & \\ & \searrow & \\ & & S_{\downarrow} = 90,\ B_{\downarrow} = 105,\ C_{\downarrow\downarrow} = 6.19 \end{array}$$

To replicate the CF's of the call option, we choose $n_S(0)$, $n_B(0)$ so that if:

$$\begin{aligned} \uparrow \text{ occurs :} \qquad 29.52 &= 120\, n_S(0) + 105\, n_B(0) \\ \downarrow \text{ occurs :} \qquad 6.19 &= 90\, n_S(0) + 105\, n_B(0). \end{aligned} \tag{17}$$

The solution is

$$n_S(0) = \frac{23.33}{30}, \qquad n_B(0) = -0.6076. \tag{18}$$

The arbitrage-free price of the call option at date-0 is thus

$$\begin{aligned} C_0 &= \left(\frac{23.33}{30}\right)\left(100\right) + (-0.6076)\left(100\right) \\ &= 17.00. \end{aligned} \tag{19}$$

Incidentally, it is worth noting that the so-called **Self Financing** condition automatically holds when we work backwards through the tree. For example, one can show that the portfolio $(n_S(0),\ n_B(0)) = (\frac{23.33}{30},\ -0.6076)$ will provide enough funds at date-1 *regardless* of whether the up-state occurs or the down-state occurs. Here, we consider only the up-state. In this case, the agent's wealth at date $(t+1)^-$ is

$$\begin{aligned} W((t+1)^- | \uparrow) &= 120\, n_S(0) + 105\, n_B(0) \\ &= (120)(\frac{23.33}{30}) + (105)(-0.6076) \\ &= 29.53, \end{aligned} \tag{20}$$

which is just enough to purchase the new portfolio at the current prices:

$$\begin{aligned} W((t+1)^+ | \uparrow) &= 120\, n_S(\uparrow) + 105\, n_B(\uparrow) \\ &= (120)(1) + (105)\left(-\frac{95}{110.25}\right) \\ &= 29.53 \\ &= W((t+1)^- | \uparrow). \end{aligned} \tag{21}$$

Similar arguments hold for all of the other possible events.

3B) Determine the prices of the $K = 95$ call option using the risk-neutral pricing formula

$$C(0) = \left(\frac{1}{1+R_F}\right)^2 E_0^Q\left[C(2)\right] \tag{22}$$

We have shown elsewhere that the one-period risk-neutral probabilities are

$$\begin{aligned} \pi_\uparrow^Q &= \left(\frac{(1+R_F)-d}{u-d}\right) = 0.5 \\ \pi_\downarrow^Q &= \left(\frac{u-(1+R_F)}{u-d}\right) = 0.5. \end{aligned} \tag{23}$$

Therefore, given the assumed independence of the two 'coin-flips' it follows that $\pi_{\uparrow\uparrow}^Q = (0.5)(0.5) = 0.25, \pi_{\uparrow\downarrow}^Q = (0.5)(0.5) = 0.25, \pi_{\downarrow\uparrow}^Q = (0.5)(0.5) = 0.25, \pi_{\downarrow\downarrow}^Q = (0.5)(0.5) = 0.25$. Hence, considering all four paths, we get the same result as above:

$$\begin{aligned} C(0) &= \left(\frac{1}{1+R_F}\right)^2 \mathrm{E}_0^Q\left[C(2)\right] \\ &= \left(\frac{1}{1.05}\right)^2 [(.25)(49) + (.25)(13) + (.25)(13) + (.25)(0)] \\ &= 17.0 \end{aligned} \tag{24}$$

3C) Determine the price of an **American** *put option with strike $K = 110$ and $T = 2$. Recall that an American option allows the holder to exercise the put option early.*

The CF's for the K=110 *European* put option are $P_{\uparrow\uparrow} = \max(0, 110 - 144) = 0$, $P_{\uparrow\downarrow} = P_{\downarrow\uparrow} = \max(0, 110 - 108) = 2$, and $C_{\downarrow\downarrow} = \max(0, 110 - 81) = 29$. To determine the date-0 price, we work backward through the tree. We do not know, at date-0, whether the up-state or down-state will occur, so we must consider both separately:

First consider the case when the up-state occurs at date-1. Our binomial tree looks like

$$
\begin{array}{lcl}
 & & S_{\uparrow\uparrow} = 144,\ B_{\uparrow\uparrow} = 110.25,\ P_{\uparrow\uparrow} = 0 \\
 & \nearrow & \\
S_{\uparrow} = 120,\ B_{\uparrow} = 105,\ P_{\uparrow} = ?? & & \\
 & \searrow & \\
 & & S_{\uparrow\downarrow} = 108,\ B_{\uparrow\downarrow} = 110.25,\ P_{\uparrow\downarrow} = 2
\end{array}
$$

To replicate the CF's of the put option, we choose $n_S(\uparrow)$, $n_B(\uparrow)$ so that if:

$$
\begin{aligned}
\uparrow\uparrow \text{ occurs :} \qquad 0 &= 144\, n_S(\uparrow) + 110.25\, n_B(\uparrow) \\
\uparrow\downarrow \text{ occurs :} \qquad 2 &= 108\, n_S(\uparrow) + 110.25\, n_B(\uparrow).
\end{aligned} \tag{25}
$$

The solution is

$$
n_S(\uparrow) = -\frac{1}{18}, \qquad n_B(\uparrow) = \frac{8}{110.25}. \tag{26}
$$

The arbitrage-free price of the put option's final cash flows in event-$\uparrow$ is thus

$$
\begin{aligned}
P_{\uparrow} &= \left(-\frac{1}{18}\right)\left(120\right) + \left(\frac{8}{110.25}\right)\left(105\right) \\
&= 0.952.
\end{aligned} \tag{27}
$$

Now, what differs between an American and a European put option is that the owner has the option to exercise early. Indeed, if she were to exercise at event-$\uparrow$, the value of the put option would be $\max(0, 110 - 120) = 0$. Since this early-exercise option is less than the value if it is not exercised, the agent will optimally not exercise, and hence its value is in fact 0.952 if the up-state occurs.

Now consider the case when the down-state occurs at date-1. Our binomial tree looks like

$$
\begin{array}{lcl}
 & & S_{\downarrow\uparrow} = 108,\ B_{\downarrow\uparrow} = 110.25,\ P_{\downarrow\uparrow} = 2 \\
 & \nearrow & \\
S_{\downarrow} = 90,\ B_{\downarrow} = 105,\ C_{\downarrow} = ?? & & \\
 & \searrow & \\
 & & S_{\downarrow\downarrow} = 81,\ B_{\downarrow\downarrow} = 110.25,\ P_{\downarrow\downarrow} = 29
\end{array}
$$

To replicate the CF's of the call option, we choose $n_S(\downarrow)$, $n_B(\downarrow)$ so that if:

$$
\downarrow\uparrow \text{ occurs :} \qquad 2 = 108\, n_S(\downarrow) + 110.25\, n_B(\downarrow)
$$

$$\downarrow\downarrow \text{ occurs : } \qquad 29 \;=\; 81\, n_S(\downarrow) + 110.25\, n_B(\downarrow). \tag{28}$$

The solution is

$$n_S(\downarrow) = -1, \qquad n_B(\downarrow) = \frac{110}{110.25}. \tag{29}$$

The arbitrage-free price of the unexercised put in event-$\downarrow$ is thus

$$\begin{aligned} P_\downarrow &= \Big(-1\Big)\Big(90\Big) + \Big(\frac{106}{110.25}\Big)\Big(105\Big) \\ &= 14.76. \end{aligned} \tag{30}$$

Again, the owner of an American put has the option to exercise early. Indeed, if she were to exercise at event-$\downarrow$, the value of the put option would be $\max(0, 110 - 90) = 20$. Since this early-exercise option is higher than the *continuation value*, the agent will optimally exercise. Thus, the value of the American put option is 20 if the down-state occurs.

Now that we have determined $P^A_\uparrow = 0.952$ and $P^A_\downarrow = 20$, we can move back through the tree to date-0. The relevant part of the tree now looks like

$$\begin{array}{lcl} & & S_\uparrow = 120,\ B_\uparrow = 105,\ P_\uparrow = 0.952 \\ & \nearrow & \\ S_0 = 100,\ B_0 = 100,\ C_0 = ?? & & \\ & \searrow & \\ & & S_\downarrow = 90,\ B_\downarrow = 105,\ C_{\downarrow\downarrow} = 20 \end{array}$$

To replicate the CF's of the call option, we choose $n_S(0)$, $n_B(0)$ so that if:

$$\begin{aligned} \uparrow \text{ occurs : } \quad 0.952 &= 120\, n_S(0) + 105\, n_B(0) \\ \downarrow \text{ occurs : } \quad 20.00 &= 90\, n_S(0) + 105\, n_B(0). \end{aligned} \tag{31}$$

The solution is

$$n_S(0) = -\frac{19.048}{30}, \qquad n_B(0) = 0.7347. \tag{32}$$

The arbitrage-free price of the non-exercised put option at date-0 is thus

$$\begin{aligned} P_0 &= \Big(-\frac{19.048}{30}\Big)\Big(100\Big) + (0.7347)\Big(100\Big) \\ &= 9.977. \end{aligned} \tag{33}$$

Again, the agent also has the choice to exercise at date-0, which would be worth $\max(0, 110 - 100) = 10$. Since this is greater than 9.997, it is actually optimal to exercise immediately. Hence, the value of the American Put at date-0 is 10.

3D) Demonstrate that it is never optimal to exercise early a call option on a stock that pays no dividends regardless of the assumed stock dynamics (assuming $R_F > 0$). To show this, assume today (date-t) that $S(t) > K$, (that is, the call option is ***In-the-Money****), implying that by exercising today, the owner of the call would receive $(S(t) - K) > 0$. Investigate two strategies: in the first strategy, the agent exercises the call option today. In the second strategy, the agent keeps the call option "alive", and in addition shorts the stock today and lend K today. Show that:*

1) The CF's today are the same for the two strategies.

2) Regardless of whether the final stock price ends up in-the-money or out-of-the-money (ie., $S(T) > K$ or $S(T) < K$), this second strategy dominates the first strategy.

Since we are assuming the call option is in the money, if the agent were to exercise the option today, she would pay K and receive stock worth $S(t)$, which for convenience we assume she sells, so her CF at date-t would be $(S(t) - K)$.[1] Since she exercises her option today, there are no additional CF's at date-T.

Now, consider strategy 2. By construction of shorting the stock and loaning K, the agent receives CF's of $(S(t) - K)$ at date-t.[2] Now, at date-T, two different possibilities arise that need to be investigated separately: case i) where $(S(T) > K)$ and case ii) where $(S(T) < K)$. For the case $(S(T) > K)$, the option ends up in-the-money, implying it is worth $S(T) - K$. The agent has a liability to give back the stock she shorted, and that will cost her $-S(T)$. Finally her bond matures and she receives $K(1 + R_F)$. Combining, her date-T CF is KR_F in the case if $(S(T) > K)$, which is positive (assuming $R_F > 0$, which is always the case in practice.)

Now consider the case $S(T) < K$. Here, the option ends up out-of-the-money, implying it is worth zero. The agent has a liability to give back the stock she shorted, and that will cost her $-S(T)$. Finally her bond matures and she receives $K(1 + R_F)$. Combining, her date-T CF is $KR_F + (K - S(T))$ in the case $(S(T) < K)$. Clearly, this CF is a sum of two positive numbers, and is thus positive itself.

Putting all of these CF's together, we see that strategy-2 matches the CF's of strategy-1 at date-t, and dominates it at date-T (something positive vs. zero) *regardless* of whether $S(T) > K$ or $S(T) < K$. Thus, it is better to not exercise an American call option on a stock that never pays dividends. As such, the American feature has no value, and thus the American call will go for the same price as the European call.

[1] It may seem that our analysis depends upon whether the agent sells her stock or keeps it at date-t. I claim it does not. The reason is, whatever strategy this agent chooses in scenario-1, she could replicate this strategy in scenario-2, as I discuss in the next footnote.

[2] Had the agent held on to the stock in scenario-1, then her CF's would have been $(-K)$ at date-t and $S(T)$ at some date-T. With this strategy-1, strategy-2 would have just loaned K at date-T to match CF's at date-t, and you can show that CF's at date-T using strategy-2 would still dominate those of strategy-1.

4) An important concept that we will often use is the **Law of Iterated Expectations**. Basically, it says that your expectation today of what the temperature will be in two days should equal your expectation today of your expectation tomorrow of the temperature in two days. Intuitively, if you expect your expectation to change, why not change it now?

As a specific example, assume that two (biased) coins are flipped sequentially. The first coin has probability $= 0.6$ that it will be heads. If the first flip is heads, then the probability of a second heads is 0.7. Instead, if the first flip was tails, then the probability of a second heads is 0.5. The payoffs are $x_{HH} = 100$, $x_{HT} = 80$, $x_{TH} = 60$, $x_{TT} = 40$.

4A) Determine $\mathrm{E}_0[\tilde{x}_2]$, where $\mathrm{E}_0[\cdot]$ implies expectation at date-zero, before either coin is tossed.

We have $\pi_{HH} = (0.6)(0.7) = 0.42$, $\pi_{HT} = (0.6)(0.3) = 0.18$, $\pi_{TH} = (0.4)(0.5) = 0.2$, $\pi_{TT} = (0.4)(0.5) = 0.2$. Hence:

$$\mathrm{E}_0[\tilde{x}_2] = (0.42)(100) + (0.18)(80) + (0.2)(60) + (0.2)(40) = 76.4. \tag{34}$$

4B) Determine $E_H[\tilde{x}_2]$. That is, the expected payout given that the first spin was heads.

$$\mathrm{E}_H[\tilde{x}_2] = (0.7)(100) + (0.3)(80) = 94. \tag{35}$$

4C) Determine $E_T[\tilde{x}_2]$. That is, the expected payout given that the first spin was tails.

$$\mathrm{E}_T[\tilde{x}_2] = (0.5)(60) + (0.5)(40) = 50. \tag{36}$$

4D) Determine $E_0\left[E_1\left[\tilde{x}_2\right]\right]$. Note that, at date-0, $E_1\left[\tilde{x}_2\right] = \Big(E_H[\tilde{x}_2], E_T[\tilde{x}_2]\Big)$ is a random variable encompassing two possible values, depending upon whether the first coin was heads or tails. Compare this result to that found in 3A).

$$\mathrm{E}_0\left[\mathrm{E}_1\left[\tilde{x}_2\right]\right] = (0.6)(94) + (0.4)(50) = 76.4. \tag{37}$$

4E) More generally, consider two random events $\tilde{\omega}_1$, $\tilde{\omega}_2$ whose outcome determines the payoff $x(\omega_1, \omega_2)$. Hence:

$$E_0\left[\tilde{x}_2\right] = \sum_{\omega_1, \omega_2} \pi(\omega_1, \omega_2)\, x_2\left(\omega_1, \omega_2\right).$$

Define the conditional expectation

$$\begin{aligned} E_1\left[\tilde{x}_2\right] &\equiv E[\tilde{x}_2 | \omega_1] \\ &= \sum_{\omega_2} \pi(\omega_2 | \omega_1)\, x_2\left(\omega_1, \omega_2\right). \end{aligned}$$

Then show that the law of iterated expectations holds:

$$E_0 [\tilde{x}_2] \quad = \quad E_0 [E_1 [\tilde{x}_2]] .$$

First note that:

$$\pi(\omega_1, \omega_2) = \pi(\omega_2 | \omega_1)\pi(\omega_1)$$

Then compute $\mathrm{E}_0 [\mathrm{E}_1 [\tilde{x}_2]]$:

$$\begin{aligned}
\mathrm{E}_0 [\mathrm{E}_1 [\tilde{x}_2]] &= \sum_{\omega_1} \pi(\omega_1) \mathrm{E} [\tilde{x}_2 | \omega_1] \\
&= \sum_{\omega_1} \pi(\omega_1) \sum_{\omega_2} \pi(\omega_2 | \omega_1) \, x_2 (\omega_1, \omega_2) \\
&= \sum_{\omega_1, \omega_2} \pi(\omega_2 | \omega_1) \pi(\omega_1) \, x_2 (\omega_1, \omega_2) \\
&= \sum_{\omega_1, \omega_2} \pi(\omega_2, \omega_1) \, x_2 (\omega_1, \omega_2) \\
&= \mathrm{E}_0 [\tilde{x}_2]
\end{aligned}$$