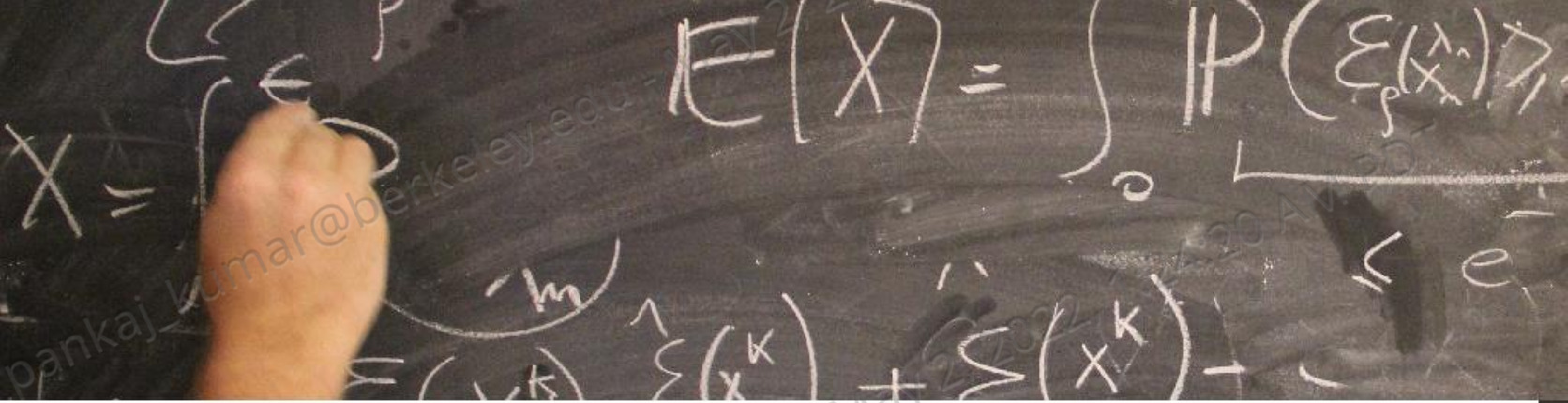

MFE 230Q [Spring 2021] Introduction to Stochastic Calculus GSI Session 1

BerkeleyHaas
MASTER OF FINANCIAL ENGINEERING

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MFE230Q
GSI Duties

- Weekly GSI sessions
  - Focus on problem solving
- Office Hours
  - Q&A format
- Grading
  - Five assignments, midterm and final
- Your **first point of contact** for anything in this class

Email Policies

- Email: simon_xu@haas.berkeley.edu
- Please include [MFE230Q] in the topic
  - Try not to leave questions about assignments or exams until the very last moment
- Longer questions are much easier to answer in office hours than via email

Review

- Basic setup
- Completeness, State-prices and Arbitrage
- Two Types of Arbitrage
- Market Completeness/Incompleteness
- FTAP (First, Second)
- The LOOP
- Risk-Neutral Fundamental Theorems
- Basic Probability Theory
- Sample Problems!

Basic Setup

- N financial assets, M states of the world at $t = 1$
- Price today is a $N \times 1$ vector $\mathbf{s^0}$
- Prices at $t = 1$ summarized by $N \times M$ matrix

Payoff

$$\mathbf{D} = \left[\mathbf{s^1}(\omega_1), \mathbf{s^1}(\omega_2), \ldots, \mathbf{s^1}(\omega_M)\right]$$

- Also define augmented $N \times (M + 1)$ matrix:

$$\bar{\mathbf{D}} = \left[-\mathbf{s^0}, \mathbf{D}\right]$$

- Portfolio holdings given by $N \times 1$ vector $\mathbf{h}$
- $t = 0$ value of portfolio is scalar $\mathrm{V}^0 = \mathbf{h}^{\mathrm{T}}\mathbf{s^0}$

1x1 1xN Nx1

- $t = 1$ value of portfolio is $1 \times M$ vector $\mathbf{V^1} = \mathbf{h}^{\mathrm{T}}\mathbf{D}$

1xN NxM

- Using augmented matrix notation it follows that:

$$\left[-\mathrm{V}^0, \mathbf{V^1}\right] = \mathbf{h}^{\mathrm{T}}\bar{\mathbf{D}}$$

Completeness, State-prices and Arbitrage

- Reachable payoff space $\mathcal{R}$ is row space of $\mathbf{D}$: $N \times M$

$$\mathcal{R} = \{\mathbf{D}^{\mathrm{T}}\mathbf{h} : \mathbf{h} \in \mathbb{R}^{\mathrm{N}}\} \subset \mathbb{R}^{\mathrm{M}}$$

$\mathrm{rank}(D) = M$

- Market is **complete** if $\mathcal{R}$ is whole $\mathbb{R}^{\mathrm{M}}$

$(x_1, x_2, \ldots, x_M)$

- Augmented payoff space $\bar{\mathcal{R}}$ is:

$$\bar{\mathcal{R}} = \{\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{h} : \mathbf{h} \in \mathbb{R}^{\mathrm{N}}\} \subset \mathbb{R}^{\mathrm{M+1}}$$

$\mathrm{rank}(D) \leq \min(N, M)$

- **State-price** is a $N \times 1$ vector ψ such that:

$$\forall [-\mathrm{V}^0, \mathbf{V^1}]^T \in \bar{\mathcal{R}} : \mathrm{V}^0 = \mathbf{V^1}\psi$$

$\Rightarrow N \geq M$

- Corollary: $\mathbf{s^0} = \mathbf{D}\psi$
- Arbitrage portfolio is such $\mathbf{h}$, that $\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{h} > \mathbf{0}$

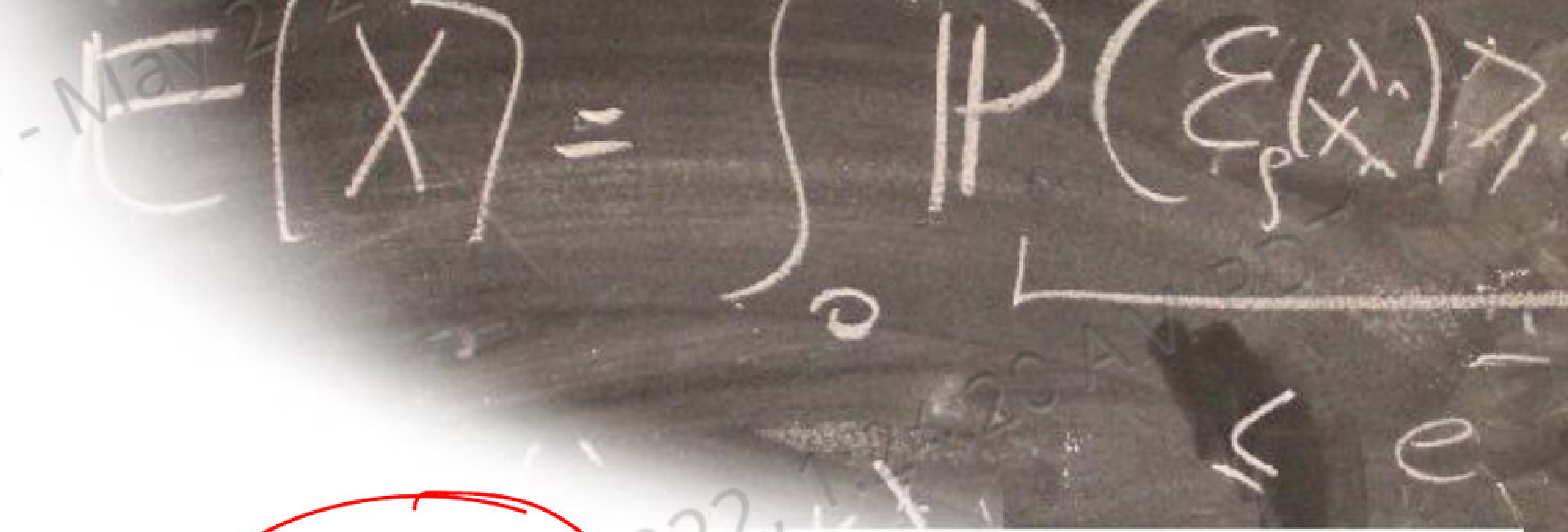
$h^T S_0 = h^T D \psi$

$N \times 1$ $\quad N \times M$ $\quad M \times 1$

D $\quad D^{-1}$ exists

$M \times M$ $\quad \psi = D^{-1} S_0$

Arbitrage



- Arbitrage portfolio is such $\mathbf{h}$, that $\overline{\mathbf{D}}^{\mathrm{T}}\mathbf{h} > \mathbf{0}$
- Definition formalizes economic notion of “free lunch” into a mathematical structure!
- Intuition (Weak interpretation): Arbitrageurs – rational investors – step in and exploit arbitrage opportunities thus removing them from the market. We don’t specifically talk about the “how it happens” and instead focus on the fact that an arbitrage can be exploited. We then propose a position *i.e.* portfolio **h** that provides an arbitrage profit.
- **No-Arbitrage** is a good first-approximation of market behavior in the absence of:
  1. “strong” financial frictions
  2. “many” irrational traders

Types of Arbitrage

- Initial value of portfolio is non-positive. Future value is non-negative with probability one, and positive with a positive probability.
  - $V^0 \le 0$
  - $V^1(\omega) \ge 0$ for all possible states ω
  - $V^1(\omega^*) > 0$ for at least one state ω^*

Type I Arbitrage

- Initial Value of portfolio is negative (i.e., you get paid to take it). Future value is non-negative with probability one.
  - $V^0 < 0$
  - $V^1(\omega) \ge 0$ for all possible states ω

Type II Arbitrage

Types of Arbitrage

Type I Arbitrage

- **Intuition**: You get something **non-negative** at no cost at *t = 0* [TODAY] *and* get something **non-negative with probability 1** and something **positive with positive probability** at t = 1 [TOMORROW].

$t=0$ $t=1$ w_1 w_2 w_3
3 0
≥ 0 ≥ 0 ≥ 0

Type II Arbitrage

- **Intuition:** You get something **positive** at no cost at t = 0 [TODAY], *and* get something **non-negative with probability 1** at t = 1 [TOMORROW]

$t=0 > 0$ W_1 W_2 W_3
≥ 0 ≥ 0 ≥ 0

Type I and Type II Arbitrage

- **Intuition:** You get something **positive** at no cost at t = 0 [TODAY], *and* get something **non-negative with probability 1** and something **positive with positive probability** at t = 1 [TOMORROW].

Market Completeness

- **N** assets, **M** states
- Reachable payoff space $\mathcal{R}$ is row space of $\mathbf{D}$:

$$\mathcal{R} = \{\mathbf{D}^{\mathrm{T}}\mathbf{h} : \mathbf{h} \in \mathbb{R}^{\mathrm{N}}\} \subset \mathbb{R}^{\mathrm{M}}$$

- Market is **complete** if $\mathcal{R}$ is whole $\mathbb{R}^{\mathrm{M}}$

Intuition: market *completeness* means: any possible payoff in the future can be realized by picking some particular portfolio $\mathbf{h}$

Mathematically:

- When $\mathbf{N} = \mathbf{M}$, this is equivalent to matrix **D** being invertible i.e. $\det(D) \neq 0$

Obviously…

- If $\mathbf{N} \geq \mathbf{M}$, market *may be* complete;
- if $\mathbf{N} < \mathbf{M}$, market is *definitely not* complete since we *definitely can't* span $\mathbf{R}^{\mathrm{M}}$ with just **N** rows (*necessary condition* not met)

FTAP (First, Second)

$NA \iff \exists\, \psi \gg 0$ $\psi > 0$

- *First Fundamental Theorem of Asset Pricing*: The market is arbitrage free if and only if there exists a strictly positive state price vector, $\psi \gg 0$.

Note: If there exist *multiple* state price vectors, market **may** still be arbitrage-free.

Example 1:
Assignment 1, *Economy A*

$$\overline{D} = \begin{bmatrix} -1 & 1 & 2 & 3 \end{bmatrix}$$

- *Second Fundamental Theorem of Asset Pricing*: Given a market that admits no arbitrage, then, the state price is unique if and only if the market is complete.

Complete $\iff$ ψ unique

Example 2:

$$\bar{D} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ complete ✓

$\psi = D^{-1} S^0 = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{unique}} \cancel{> 0}$ NA ✗

$(h_1 \ \ h_2)\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 2\end{pmatrix} \Rightarrow$

$h_1 + h_2 \leq 0$

$h_1 + h_2 \geq 0$

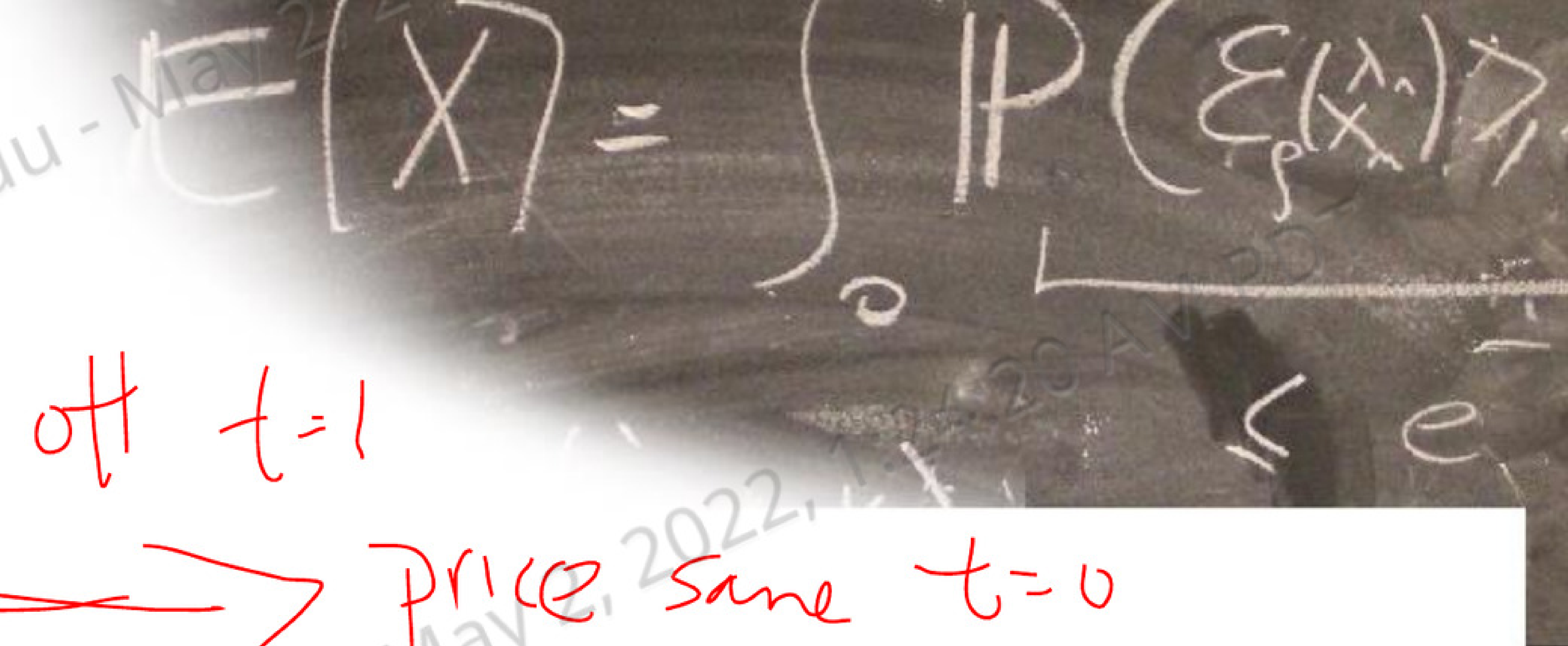
$h_1 + 2h_2 \geq 0$

$\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$V_0 = h^T S_0 = 0$ Type 1

$V_1 = h^T D = (0 \ \ 1) > 0$

LOOP



2 portfolio payoff $t=1$ $\longrightarrow$ price same $t=0$

- The LOOP holds if and only if there exists a state price vector, ψ.

LOOP $\Longleftrightarrow \exists \psi$

- No arbitrage implies LOOP, the opposite not necessarily true.

Type 2 ✗

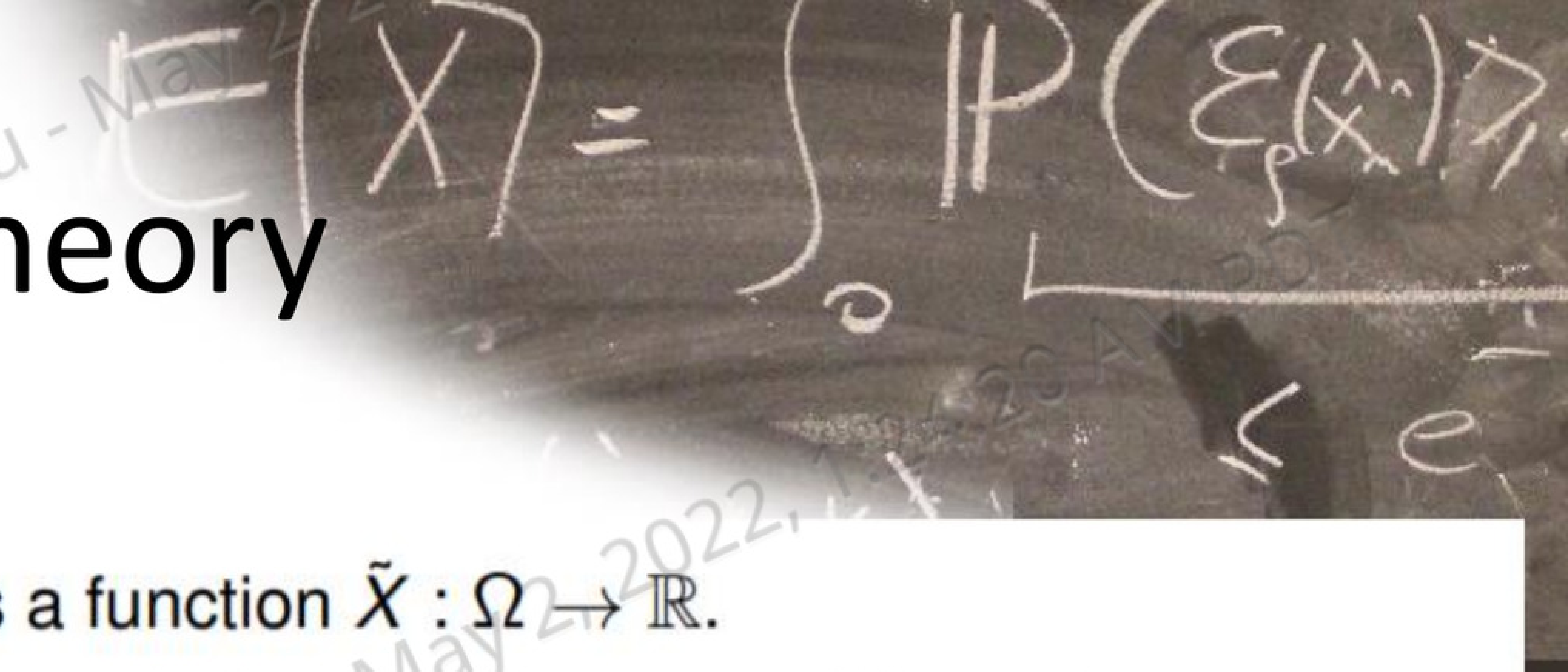
$$\bar{D} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

This price system satisfies LOOP, *but* there is an arbitrage (**Why?**)

Clarification

- **LOOP:** if there exists state-price vector
- NA **Market Completeness**: unique state-price vector
- **No Arbitrage**: a strictly positive state price vector $\psi \gg 0$
- **Relationship:**
  - No-Arbitrage => **LOOP**
  - **No LOOP** =>*Type II* Arbitrage
    - LOOP is *much* **weaker** concept than No-arbitrage.
    - No-arbitrage makes **stronger** assumptions => **stronger** results => results will hold in **LESS** situations (generally, relative to those under *weaker* assumptions)

Basic Prob. Theory



A random variable (r.v.) is a function $\tilde{X} : \Omega \to \mathbb{R}$.

In this part of course, discrete finite sample spaces ($M < \infty$).

The expectation of a r.v. is $E_{\mathbb{P}}[\tilde{X}] = \sum_i \tilde{X}(\omega_i)\mathbb{P}(\omega_i)$.

Could have two probability measures defined over same sample space, $\mathbb{P}$ and $\mathbb{Q}$.

- $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* if $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$.
- $\mathbb{Q}$ *absolutely continuous* w.r.t. $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$, if $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$.

If $\mathbb{Q} \ll \mathbb{P}$, *Radon-Nikodym derivative* (or *likelihood ratio*) of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is

$$L(A) = \frac{d\mathbb{Q}}{d\mathbb{P}} \stackrel{\text{def}}{=} \frac{\mathbb{Q}(A)}{\mathbb{P}(A)}, \qquad \mathbb{P}(A) > 0.$$

Follows immediately that for any r.v., $\tilde{X}$, $E_{\mathbb{P}}[L\tilde{X}] = E_{\mathbb{Q}}[\tilde{X}]$.

$$V^0 = \underline{V^1 \psi} \quad (SPV) \qquad \underline{V^0 = S^0} \qquad \underline{V^1 = D}$$

$$= \frac{1}{R} E_{\mathbb{Q}}(V^1)$$

RN / EMM

$$\mathbb{Q} : q_i = \frac{\psi_i}{\hat{\psi}}, \quad \hat{\psi} = \sum_{i=1}^{M} \psi_i$$

$$R = \frac{1}{\hat{\psi}}$$

$$= \sum_i \hat{\psi} q_i V^1(\omega_i)$$

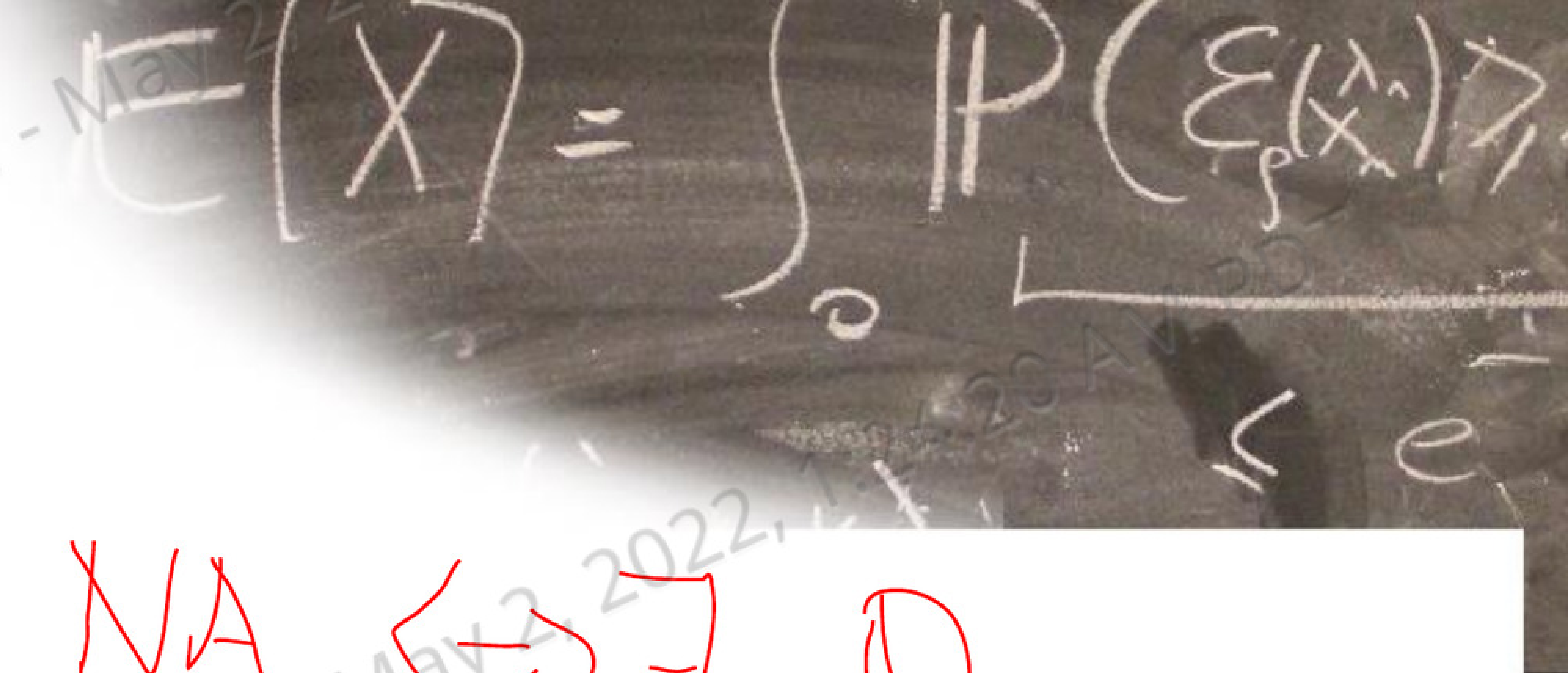
$$= \sum_i \cancel{\hat{\psi}} \cdot \frac{\psi_i}{\cancel{\hat{\psi}}} V^1(\omega_i)$$

$$= SPV$$

$$\mathbb{P} \; P_i$$

$$\psi_i > 0$$

Risk Neutral



Similar to FTAP:

NA $\iff \exists$ Q

- *Risk Neutral First Fundamental Theorem*: The market is arbitrage free if and only if there exists an equivalent martingale measure, $\mathbb{Q}$, such that the price of any assset in $\mathcal{R}$ is given by $V^0 = \frac{1}{R} E_{\mathbb{Q}}\left[\tilde{V}^1\right]$.

Complete $\iff$ unique Q

NA

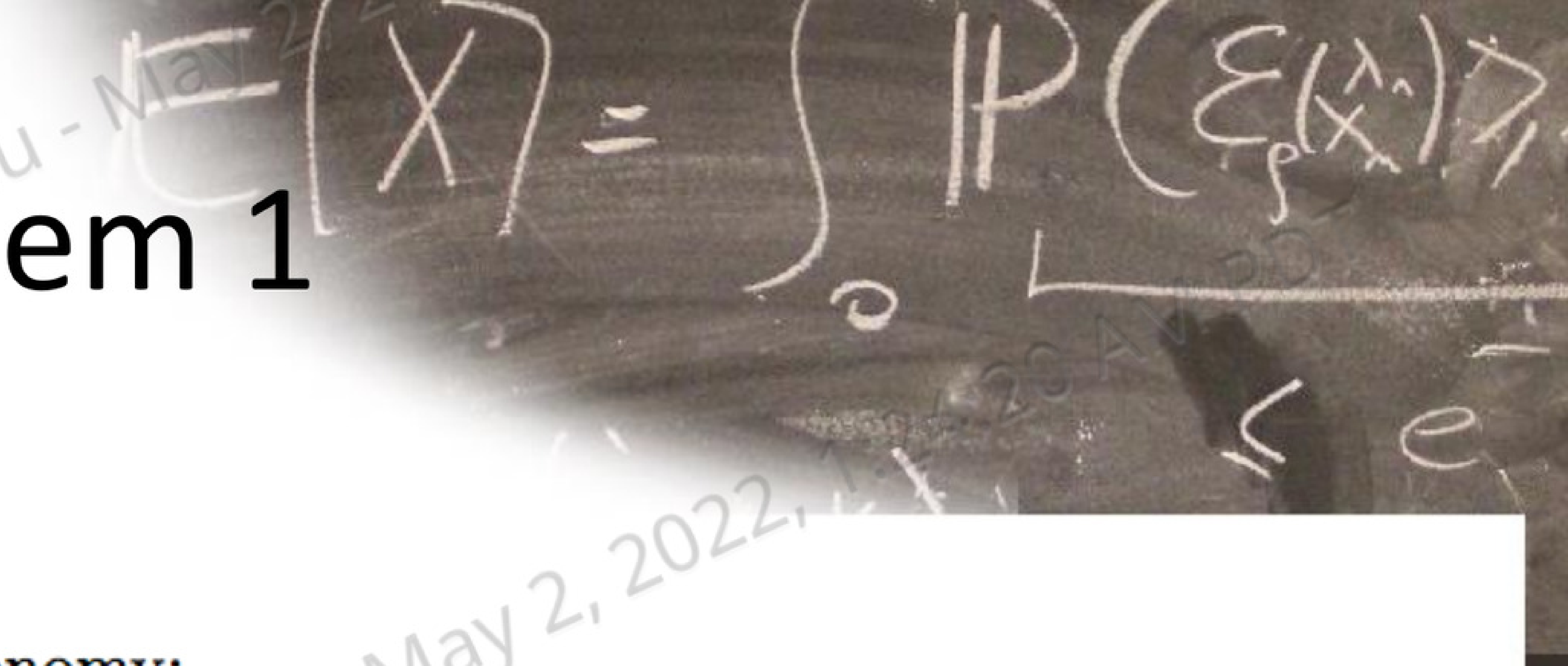
- *Risk Neutral Second Fundamental Theorem*: Given a market that admits no arbitrage, the equivalent martingale measure is unique if and only if the market is complete.

"The **Q**-measure has absolutely nothing to do with true probabilities.
It just offer convenient notation for the linear pricing formula that we know *must exist* from the fundamental theorem." (Walden [2014, pg.36])

Why Risk-Neutral?

- **Intuitively:** Investors are generally *risk-averse*, as investors typically demand more profit for bearing more uncertainty. The calculated, expected values need to be adjusted for an investor's risk preferences.
  - In a complete market with no arbitrage, once a risk-neutral probability measure is found, every asset can be priced by simply taking its expected payoff *under* this measure.
  - We price *as if* investors are risk-neutral.

Sample Problem 1



Consider the following economy:

$$\bar{D} = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 3 & \alpha \end{bmatrix}$$

(a) Compute, if possible, state-prices for $\alpha \in \{0, 5, 6\}$.

(b) For $\alpha \in \{0, 5, 6\}$, determine when there is no-arbitrage.

(c) Compute the risk neutral probabilities for the three cases.

a) $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & \alpha \end{bmatrix}}_{D} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{\alpha - 6} \begin{bmatrix} \alpha - 4 \\ -1 \end{bmatrix}$

$\alpha = 0 \quad \psi = (2/3, 1/6)$

$\alpha = 5 \quad \psi = (-1 \quad 1)$

$\alpha = 6 \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{rank}(D) = 1$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 0\psi_1 + 0\psi_2 = 1$

SPV $\nexists$

b) $\alpha = 0$ NA ✓

$\alpha = 5$ NA ✗

$\alpha = 6$ NA ✗

c) $\alpha = 0 \quad q_1 = 0.8 \quad q_2 = 0.2$

$\alpha = 5$ ✗ RN

$\alpha = 6$ ✗ RN

Sample Problem 2

Consider the following one-period market, with the state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}.$$

(a) Is this market complete?

(b) Is this market arbitrage-free?

(c) What is the risk-free rate ($R = 1 + r$) in this market?

(d) What is the value today of an Arrow-Debreu security paying \$1 in state ω_3?

a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad \text{rank}(D) = 6$

b) $\psi = D^{-1} S^0 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} >> 0$ NA ✓

c) $R = \frac{1}{1.2}$ $\quad r = R - 1$

d) SPV. $\quad S^0 = D\psi = [0\ 0\ 1\ 0\ 0\ 0]\,\psi$

$= \$3$

Extra Problem 1

PART I:

The economy, **Economy** A, is given by

$$s^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 3 & 3 & 3 & 3 & 7 \end{bmatrix}$$

We want to see if **(A)** the market is complete, **(B)** there is an arbitrage opportunity? **(C)** there exists a strictly positive state price $\psi \gg 0$

a) $\underline{N=2}$, $\underline{M=5}$ - incomplete

b) $5^0 = D\psi = D \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_5 \end{bmatrix}$

$1 = 5(\psi_1 + \dots + \psi_5)$ (1)

$1 = 3(\psi_1 + \dots + \psi_4) + 7\psi_5$ (2)

$\underbrace{\frac{1}{5} - \psi_5}_{} = \psi_1 + \dots + \psi_4$

$\psi_2 = \dots = \psi = \frac{1}{100}$

$\psi_1 = \frac{7}{100}$

$\psi_5 = \frac{1}{10}$

NA ✓

c) ✓

Extra Problem 2

PART II:

Consider an economy with $\mathbf{N} = 5$ assets and $\mathbf{M} = 4$ states given by

$$\mathbf{s}^0 = \begin{bmatrix} 1 \\ 2.5 \\ 1.5 \\ 0.75 \\ 0.25 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The augmented payoff matrix is

$$\overline{\mathbf{D}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2.5 & 1 & 2 & 3 & 4 \\ 1.5 & 0 & 1 & 2 & 3 \\ 0.75 & 0 & 0 & 1 & 2 \\ 0.25 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We want to know the following:

(a) Is the market complete?
(b) Are there any redundant assets?
(c) Is the market arbitrage-free?
(d) Can we replicate the Arrow-Debreu security δ^1 from the assets given to us? If yes, find a portfolio with zero units in Asset 1