MFE 230Q – Introduction to Stochastic Calculus
GSI Session 3 Solutions

5 GSI Session 3: Sample Problem 4 - Two Dimensional Itô Process

Consider the following two dimensional Ito process $\bar{X}_t = (X^1, X^2)^T$

$$d\bar{X}(t) = \mu dt + \sigma d\bar{W}(t) \tag{4}$$

where $\bar{W}(t)$ is a 2-dimensional standard independent Wiener process. Here

$$\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \sigma = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \end{bmatrix} \tag{5}$$

1. Define $Y_t = (X^1(t))^2 + (X^2(t))^2 + 2X^1(t)X^2(t)$. Use Itô's lemma to derive $dY(t)$.

2. Note that $Y(t) = (Z(t))^2$ where $Z(t) = X^1(t) + X^2(t)$. Use Itô's lemma for $Y((Z(t))$ to verify that you get the same form as in part *1)*.

3. Rewrite the Itô's Process for X on 'correlated' form (see lecture notes and Bjork chapter 4).and verify again that Itô's lemma on correlated form leads to the same form for dY.

Solutions:

1. Note that [1]

$$\Sigma = \sigma\sigma^T = \begin{bmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} \\ \sigma_{21}\sigma_{11} + \sigma_{22}\sigma_{12} & \sigma_{21}^2 + \sigma_{22}^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{6}$$

Let's apply Ito's lemma to $Y = (X^1(t))^2 + (X^2(t))^2 + 2X^1(t)X^2(t)$

$$\begin{aligned} dY &= Y_{X^1}dX^1 + Y_{X^2}dX^2 + \frac{1}{2}Y_{X^1X^1}(dX^1)^2 + \frac{1}{2}Y_{X^2X^2}(dX^2)^2 + Y_{X^1X^2}(dX^1)(dX^2) \\ &= 2(X^1 + X^2)(dX^1 + dX^2) + (dX^1)^2 + (dX^2)^2 + 2(dX^1)(dX^2) \\ &= 2(X^1 + X^2)(dX^1 + dX^2) + \underbrace{(\sigma_{11}^2 + \sigma_{12}^2)}_{\Sigma_{11}}dt + \underbrace{(\sigma_{21}^2 + \sigma_{22}^2)}_{\Sigma_{22}}dt + 2\underbrace{(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})}_{\Sigma_{21}}dt \\ &= 2(X^1 + X^2)(dX^1 + dX^2) + (\Sigma_{11} + \Sigma_{22} + 2\Sigma_{21})dt \\ &= 2(X^1 + X^2)\left(3dt + \frac{3+\sqrt{3}}{2}dW^1 + \frac{3-\sqrt{3}}{2}dW^2\right) + 6dt \end{aligned}$$

Note that we can solve for X^1 and X^2, since this are simple ODEs, and substitute in the above process of dY.

$$\begin{aligned} X^1 &= t + W^1(t) + W^2(t) \\ X^2 &= 2t + \frac{1+\sqrt{3}}{2}W^1(t) + \frac{1-\sqrt{3}}{2}W^2(t) \end{aligned}$$

[1] I try to use the matrix Σ to simplify, but I believe that it has created some confusion during section. On the top of that, I believe that while writing on the board I forgot one cross term. Have a look on this solutions. I try to reconcile everything.

2. If $Y(t) = (Z(t))^2$ then apply Itô's lemma to $Y(t)$:

$$dY = 2ZdZ + \frac{1}{2}2(dZ)^2 \tag{7}$$

where

$$dZ = dX^1 + dX^2$$

and

$$\begin{aligned}
(dZ)^2 &= (dX^1 + dX^2)^2 \\
&= (dX^1)^2 + (dX^2)^2 + 2dX^1dX^2 \\
&= \underbrace{(\sigma_{11}^2 + \sigma_{12}^2)}_{\Sigma_{11}} dt + \underbrace{(\sigma_{21}^2 + \sigma_{22}^2)}_{\Sigma_{22}} dt + 2\underbrace{(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})}_{\Sigma_{21}} dt \\
&= (\Sigma_{11} + \Sigma_{22} + 2\Sigma_{21})dt
\end{aligned}$$

note that from the second to the third line we use the simplification that we use in part *a)*. Substituting dZ and $(dZ)^2$ into (7) we get

$$\begin{aligned}
dY &= 2(X^1 + X^2)(dX^1 + dX^2) + (\Sigma_{11} + \Sigma_{22} + 2\Sigma_{21})dt \\
&= 2(X^1 + X^2)\left(3dt + \frac{3+\sqrt{3}}{2}dW^1 + \frac{3-\sqrt{3}}{2}dW^2\right) + 6dt
\end{aligned}$$

the same then as above!

3. Finally the correlated form. The goal of this part is to show that we can transform the two-dimensional process into a one dimension, that is, instead of having dX^1 as a function of two independent brownian montions we will have has one brownian montion, dV^1. However dV^1 will now be correlated with dV^2 the dX^2 brownian motion counterpart. Let's redefine dX^1 and dX^2 as:

$$\begin{aligned}
dX^1 &= \mu_1 dt + \bar{\sigma}_1 dV^1 \\
dX^2 &= \mu_2 dt + \bar{\sigma}_2 dV^2
\end{aligned}$$

with

$$
\begin{aligned}
dV^1 &= \frac{\sigma_{11}dW^1 + \sigma_{12}dW^2}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}} \\
dV^2 &= \frac{\sigma_{22}dW^1 + \sigma_{21}dW^2}{\sqrt{\sigma_{22}^2 + \sigma_{21}^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\sigma}_1 &= \sqrt{\sigma_{11}^2 + \sigma_{12}^2} \\
\bar{\sigma}_2 &= \sqrt{\sigma_{22}^2 + \sigma_{21}^2}.
\end{aligned}
$$

Now let's redo part *2)*. Although we need to be careful because dV^1 and dV^2 are now correlated, $(dX^1)^2$, $(dX^2)^2$ and dX^1dX^2 are still the same:

$$
\begin{aligned}
(dX^1)^2 &= (\bar{\sigma}_1)^2dt = (\sigma_{11}^2 + \sigma_{12}^2)dt \\
(dX^2)^2 &= (\bar{\sigma}_2)^2dt = (\sigma_{22}^2 + \sigma_{12}^2)dt \\
dX^1dX^2 &= (\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22})dt
\end{aligned}
$$

If we proceed to substitute out dZ

$$
dY = 2ZdZ + \frac{1}{2}2(dZ)^2.
$$

we get

$$
\begin{aligned}
dY &= 2(X^1 + X^2)(dX^1 + dX^2) + (\Sigma_{11} + \Sigma_{22} + 2\Sigma_{21})dt \\
&= 2(X^1 + X^2)\left(3dt + \frac{3+\sqrt{3}}{2}dW^1 + \frac{3-\sqrt{3}}{2}dW^2\right) + 6dt
\end{aligned}
$$

the same then as above!