MFE230Q: Assignment 2 - Due April 6, 2021

1. Consider the following *Generalized Butterfly Spread*, $\mathcal{B}(L, K_1, K_2, K_3, T)$, where $L > 0$, $K_3 > K_2 > K_1 > 0$, and $T > 0$. The position is generated by:

   - Buying $\frac{L}{K_2-K_1}$ European call options with strike price K_1, $C^{Euro,K_1,T}$
   - Selling $\frac{L}{K_2-K_1} + \frac{L}{K_3-K_2}$ European call options with strike price K_2, $C^{Euro,K_2,T}$
   - Buying $\frac{L}{K_3-K_2}$ European call options with strike price K_3, $C^{Euro,K_3,T}$

   In the notation of the multi-period model (used in class), the position is thus

   $$\mathbf{h} = \left(\frac{L}{K_2 - K_1}, -\frac{L}{K_2 - K_1} - \frac{L}{K_3 - K_2}, \frac{L}{K_3 - K_2}\right)^T$$

   in the assets with value dynamics

   $$\mathbf{s}_t = \left(C_t^{Euro,K_1,T}, C_t^{Euro,K_2,T}, C_t^{Euro,K_3,T}\right)^T.$$

   (a) Plot the payout of the generalized butterfly spread $\mathcal{B}(60, 80, 100, 160, 1)$ at maturity, as a function of the underlying stock-price.

   (b) Show how the same payout can be generated by a portfolio of put options, $\mathbf{s}_t = \left(P_t^{Euro,K_1,T}, P_t^{Euro,K_2,T}, P_t^{Euro,K_3,T}\right)^T$.

   (c) Using put-call parity, verify that the price of the portfolio in (b) is the same as that of the portfolio in (a).

2. Consider the following trinomial economy:

   $$\begin{array}{ccc} & & 120,\quad 105 \\ & \nearrow & \\ \mathrm{S}(0) = 105,\quad \mathrm{B}(0) = 100 & \rightarrow & 105,\quad 105 \\ & \searrow & \\ & & 100,\quad 105 \end{array}$$

   (a) Is the market complete? Why or why not?

   (b) Define $(AD_\uparrow, AD_\rightarrow, AD_\downarrow)$ as the prices of the Arrow-Debreu securities that pays \$1 iff the up-state, middle state, and down state occur, respectively. Use the stock and bond price dynamics to find two constraints on the values of these AD securities. Use these constraints to write both $AD_\uparrow$ and $AD_\rightarrow$ solely as a function of $AD_\downarrow$.

   (c) Identify the range of values of $AD_\downarrow$ that are consistent with all three AD prices being positive. Does this economy admit arbitrage opportunities?

(d) Identify the range of values for the call option with strike $K = 105$

(e) Identify the range of values for the call option with strike $K = 100$. Interpret your results.

3. Assume a 2-period, 3-date model as below.

$$
\begin{array}{ccccc}
 & & & & u^2S(0),\ B(0)(1+R)^2,\ CF(\uparrow\uparrow) \\
 & & & \nearrow & \\
 & & uS(0),\ B(0)(1+R),\ V(\uparrow) & & \\
 & \nearrow & & \searrow & \\
\text{S(0)},\ \text{B(0)},\ \text{V(0)} & & & & udS(0),\ B(0)(1+R)^2,\ CF(\uparrow\downarrow) \\
 & \searrow & & \nearrow & \\
 & & dS(0),\ B(0)(1+R),\ V(\downarrow) & & \\
 & & & \searrow & \\
 & & & & d^2S(0),\ B(0)(1+R)^2,\ CF(\downarrow\downarrow)
\end{array}
$$

(a) Assume $S(0) = 100$, $B(0) = 100$, $u = 1.2$, $d = .9$, $R = .05$. Using dynamic programming, determine the price *and* the replicating portfolio for the call option with maturity $T = 2$ and strike $K = 95$.

(b) Determine the prices of the K = 95 call option using the risk-neutral pricing formula

$$
C(0) = \left(\frac{1}{1 + R_F}\right)^2 \mathrm{E}_0^Q\left[C(2)\right]. \tag{1}
$$

(c) Determine the price of an *American* put option with strike $K = 110$ and $T = 2$. Recall that an American option allows the holder to exercise the put option early.

(d) Demonstrate that it is never optimal to exercise early a call option on a stock that pays no dividends regardless of the assumed stock dynamics (assuming $R_F > 0$). To show this, assume today (date-t) that $S(t) > K$, (that is, the call option is *In-the-Money*), implying that by exercising today, the owner of the call would receive $(S(t) - K) > 0$. Investigate two strategies: in the first strategy, the agent exercises the call option today. In the second strategy, the agent keeps the call option "alive", and in addition shorts the stock today and lend K today. Show that:

- The CF's today are the same for the two strategies.
- Regardless of whether the final stock price ends up in-the-money or out-of-the-money (ie., $S(T) > K$ or $S(T) < K$), this second strategy dominates the first strategy.

4. An important concept that we will often use is the *Law of Iterated Expectations*. Basically, it says that your expectation today of what the temperature will be in two days should equal your expectation today of your expectation tomorrow of the temperature in two days. Intuitively, if you expect your expectation to change, why not change it now?

   As a specific example, assume that two (biased) coins are flipped sequentially. The first coin has probability $= 0.6$ that it will be heads. If the first flip is heads, then the probability of a second heads is 0.7. Instead, if the first flip was tails, then the probability of a second heads is 0.5. The payoffs are $x_{HH} = 100$, $x_{HT} = 80$, $x_{TH} = 60$, $x_{TT} = 40$.

   (a) Determine $\mathrm{E}_0[\tilde{x}_2]$, where $\mathrm{E}_0[\cdot]$ implies expectation at date-zero, before either coin is tossed.

   (b) Determine $\mathrm{E}_H[\tilde{x}_2]$. That is, the expected payout given that the first spin was heads.

   (c) Determine $\mathrm{E}_T[\tilde{x}_2]$. That is, the expected payout given that the first spin was tails.

   (d) Determine $\mathrm{E}_0[\mathrm{E}_1[\tilde{x}_2]]$. Note that, at date-0, $\mathrm{E}_1[\tilde{x}_2] = \Big(\mathrm{E}_H[\tilde{x}_2], \mathrm{E}_T[\tilde{x}_2]\Big)$ is a random variable encompassing two possible values, depending upon whether the first coin was heads or tails. Compare this result to that found in (a).

   (e) More generally, consider two random events $\tilde{\omega}_1, \tilde{\omega}_2$ whose outcome determines the payoff $x(\omega_1, \omega_2)$. Hence:

   $$\mathrm{E}_0[\tilde{x}_2] = \sum_{\omega_1, \omega_2} \pi(\omega_1, \omega_2)\, x_2(\omega_1, \omega_2).$$

   Define the conditional expectation

   $$\begin{aligned} \mathrm{E}_1[\tilde{x}_2] &\equiv \mathrm{E}[\tilde{x}_2 | \omega_1] \\ &= \sum_{\omega_2} \pi(\omega_2 | \omega_1)\, x_2(\omega_1, \omega_2). \end{aligned}$$

   Then show that the law of iterated expectations holds:

   $$\mathrm{E}_0[\tilde{x}_2] = \mathrm{E}_0[\mathrm{E}_1[\tilde{x}_2]].$$