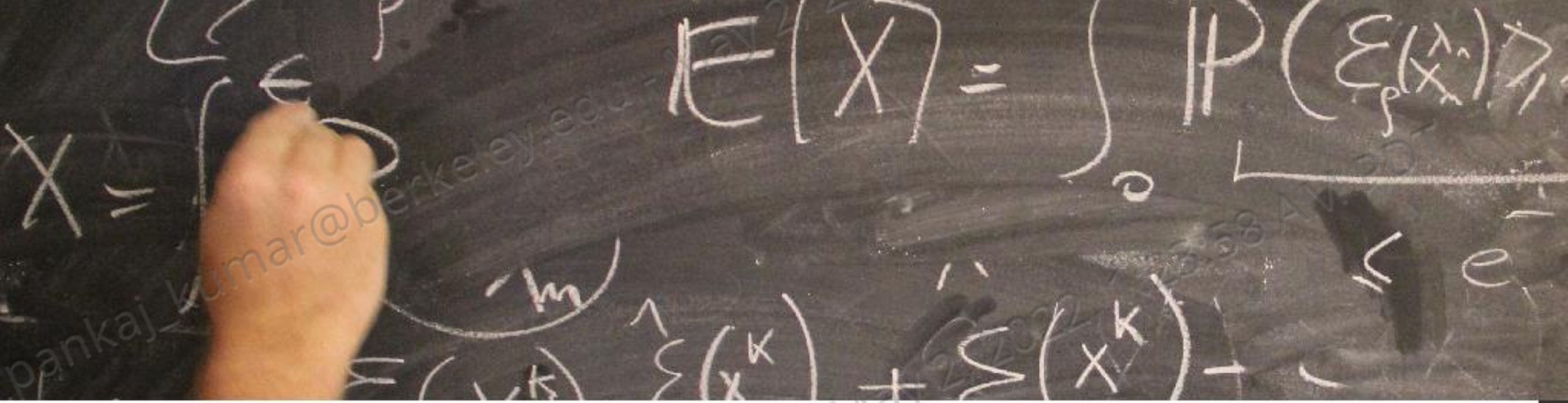

MFE 230Q [Spring 2021]
Introduction to Stochastic Calculus
GSI Session 2

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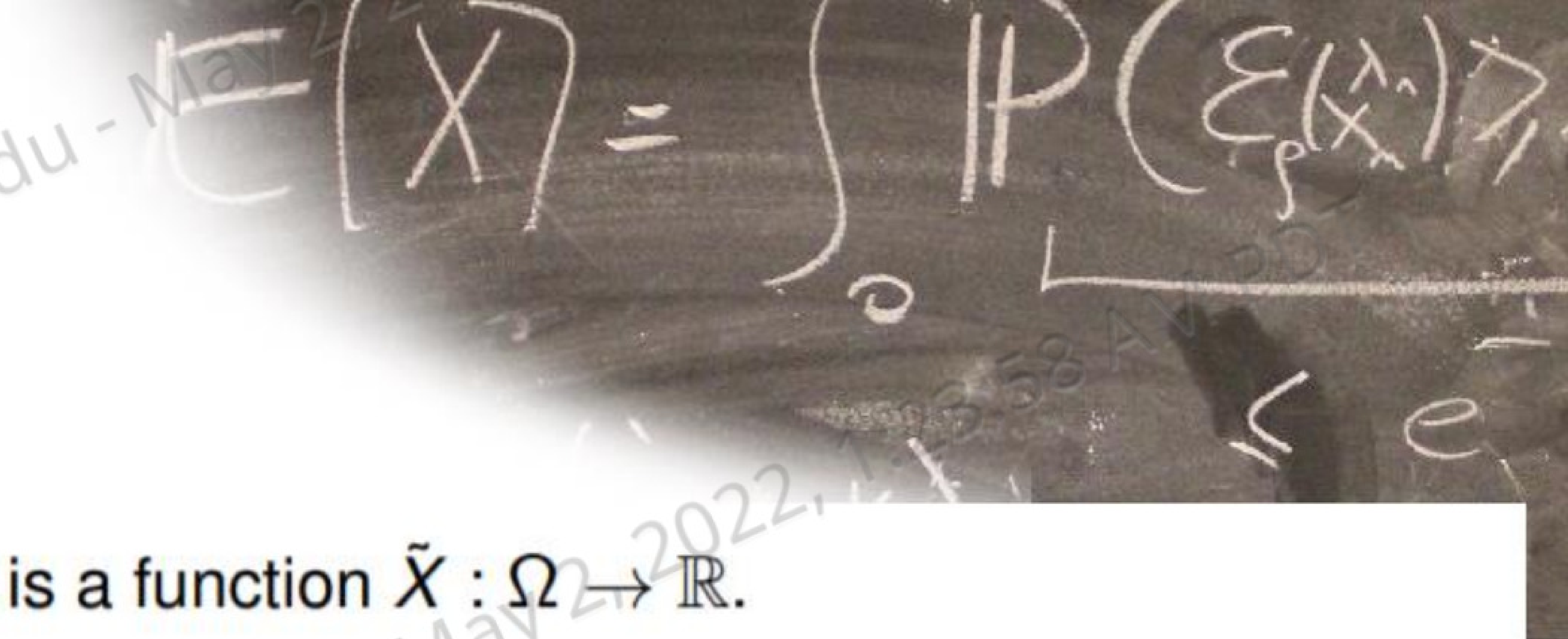
Review

- Risk-Neutral Pricing
- Basic Probability Theory
- The Multi-Period Security market
- Sample Problems!

Risk-Neutral Pricing



A random variable (r.v.) is a function $\tilde{X} : \Omega \to \mathbb{R}$.

In this part of course, discrete finite sample spaces ($M < \infty$).

The expectation of a r.v. is $E_{\mathbb{P}}[\tilde{X}] = \sum_i \tilde{X}(\omega_i)\mathbb{P}(\omega_i)$.

Could have two probability measures defined over same sample space, $\mathbb{P}$ and $\mathbb{Q}$.

- $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* if $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$.
- $\mathbb{Q}$ *absolutely continuous* w.r.t. $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$, if $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$.

If $\mathbb{Q} \ll \mathbb{P}$, *Radon-Nikodym derivative* (or *likelihood ratio*) of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is

$$L(A) = \frac{d\mathbb{Q}}{d\mathbb{P}} \stackrel{\text{def}}{=} \frac{\mathbb{Q}(A)}{\mathbb{P}(A)}, \qquad \mathbb{P}(A) > 0.$$

Follows immediately that for any r.v., $\tilde{X}$, $E_{\mathbb{P}}[L\tilde{X}] = E_{\mathbb{Q}}[\tilde{X}]$.

Pricing Equations

SPV/AD

$$V^0 = \mathbf{V}^1\psi$$

RN

$$= \frac{1}{R}E_{\mathbb{Q}}[\tilde{V}^1], \qquad q_i = \frac{\psi_i}{\hat{\psi}},\ R = \frac{1}{\hat{\psi}},$$

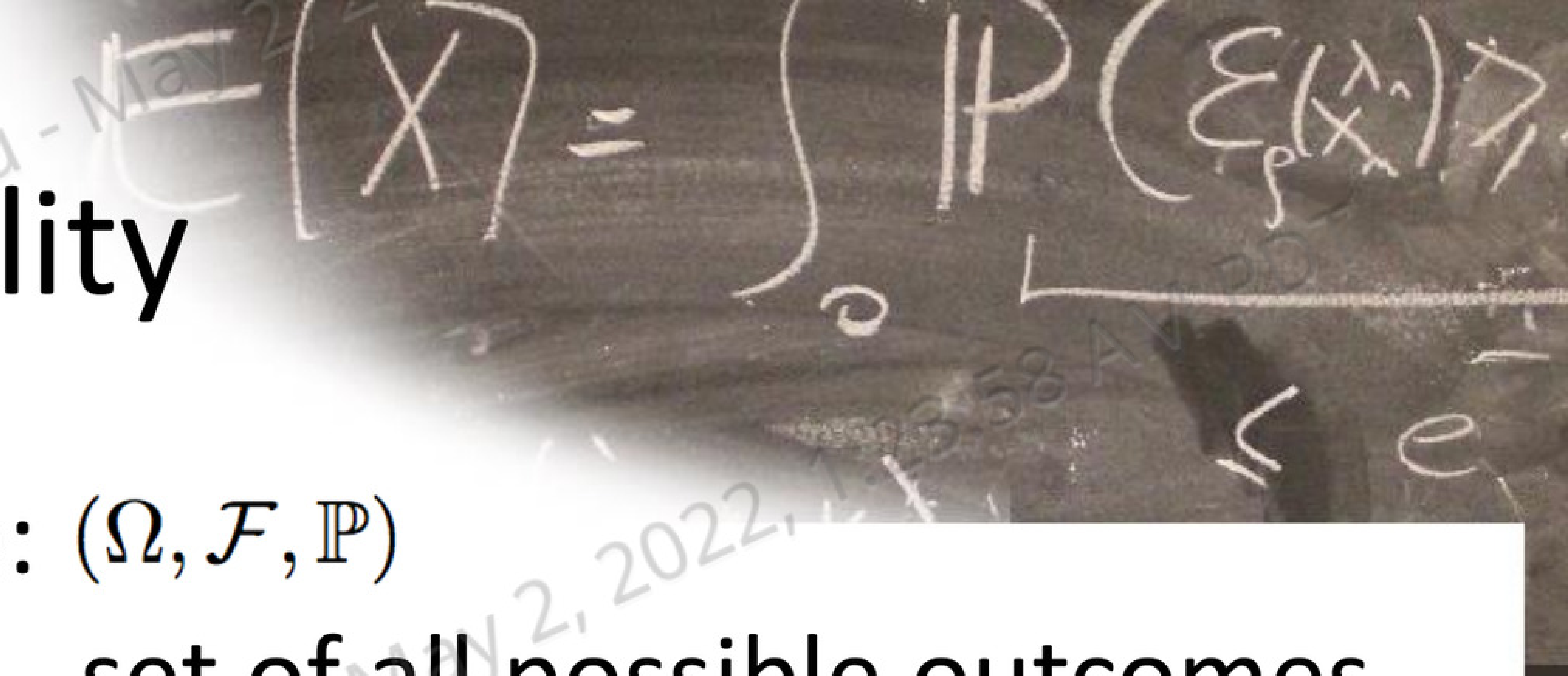
LR

$$= \frac{1}{R}E_{\mathbb{P}}[L\tilde{V}^1], \qquad L_i = \frac{q_i}{p_i},$$

SDF

$$= E_{\mathbb{P}}\left[\frac{M_1}{M_0}\tilde{V}^1\right], \qquad (M_1)_i = \frac{L_i}{R} = \frac{\psi_i}{p_i},\ M_0 = 1.$$

Basic Probability



- Probability Space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Sample space: Ω , set of all possible outcomes
- A set of events: $\mathcal{F}$, where each event is a set containing zero or more outcomes
- The assignment of probability to the events:

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$$

Basic Probability

$\Omega = \{a, b, c, d\}$ $\quad \phi \neq \{\phi\}$

A σ-algebra $\mathcal{F}$ is a set of subsets of Ω such that

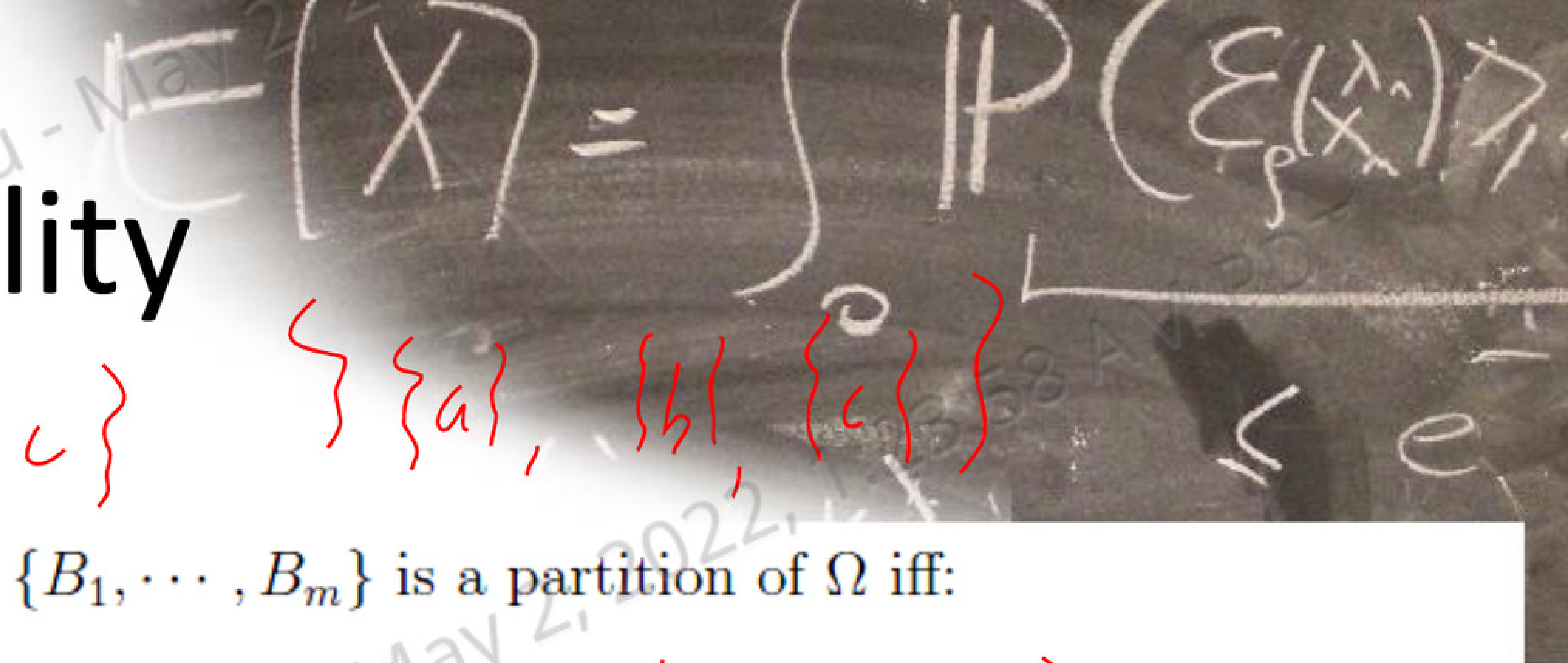
$\mathcal{F} = \{\phi, \{a,b\}, \{c,d\}, \{a,b,c,d\}\}$

1. $\emptyset \in \mathcal{F}$
2. $B \in \mathcal{F} \implies B^c \in \mathcal{F}$
3. $B_1, B_2, \cdots \in \mathcal{F} \implies \cup_{n=1}^{\infty} B_n \in \mathcal{F}$

Properties:

1. $\cap_{n=1}^{\infty} B_n \in \mathcal{F}$
2. $A, B \in \mathcal{F} \implies A \backslash B \in \mathcal{F}$
3. $\{\emptyset, \Omega\} \subset \mathcal{F} \subset 2^{\Omega}$.

Basic Probability



$\Omega = \{a, b, c\}$ $\{\{a\}, \{b\}, \{c\}\}$

Let $\Omega = \{\omega_1, \cdots, \omega_M\}$. Then $\{B_1, \cdots, B_m\}$ is a partition of Ω iff:

1. $\cup_{i=1}^m B_i = \Omega$ ✓ $\{\{a, b\}, \{c\}\}$
2. $B_i \cap B_j = \emptyset$, $i \neq j$. ✓

Note: Each of the B_i's are sets!

Let $\mathcal{B}$ be a non-empty collection of subsets of Ω. Note: Each element of $\mathcal{B}$ is a set. Denote $\sigma(\mathcal{B})$ as the smallest sigma-algebra containing all the elements of $\mathcal{B}$.

Here, smallest means that any sigma-algebra containing the elements of $\mathcal{B}$ would have to contain all the elements of $\sigma(\mathcal{B})$ as well.

generated σ-algebra

Basic Probability

$\Omega = \{a, b, c, d\}$ $\mathcal{B} = (\{a\}, \{b\}, \{c\}, \{d\})$

We will consider sigma-algebras generated by partitions of Ω.

Let $\mathcal{B}$ be a partition of Ω, then $\sigma(\mathcal{B})$ is constructed by taking all possible unions and complements of the elements in $\mathcal{B}$.

Note: $\sigma(\{\{\omega_1\}, \cdots, \{\omega_M\}\}) = 2^{\Omega}$.

$\sigma(\mathcal{B}) = \{\emptyset, \Omega, \{a\}, \{a,b\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}, \ldots\ldots\}$

– *Examples:*

$$\Omega = \{\omega_1, \omega_2, \omega_3\}$$

$$\mathcal{F}_0 = \sigma(\{\Omega\}) = \{\emptyset, \Omega\}$$

$$\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3\}) = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\}$$

$$\mathcal{F}_2 = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}) = 2^{\Omega}$$

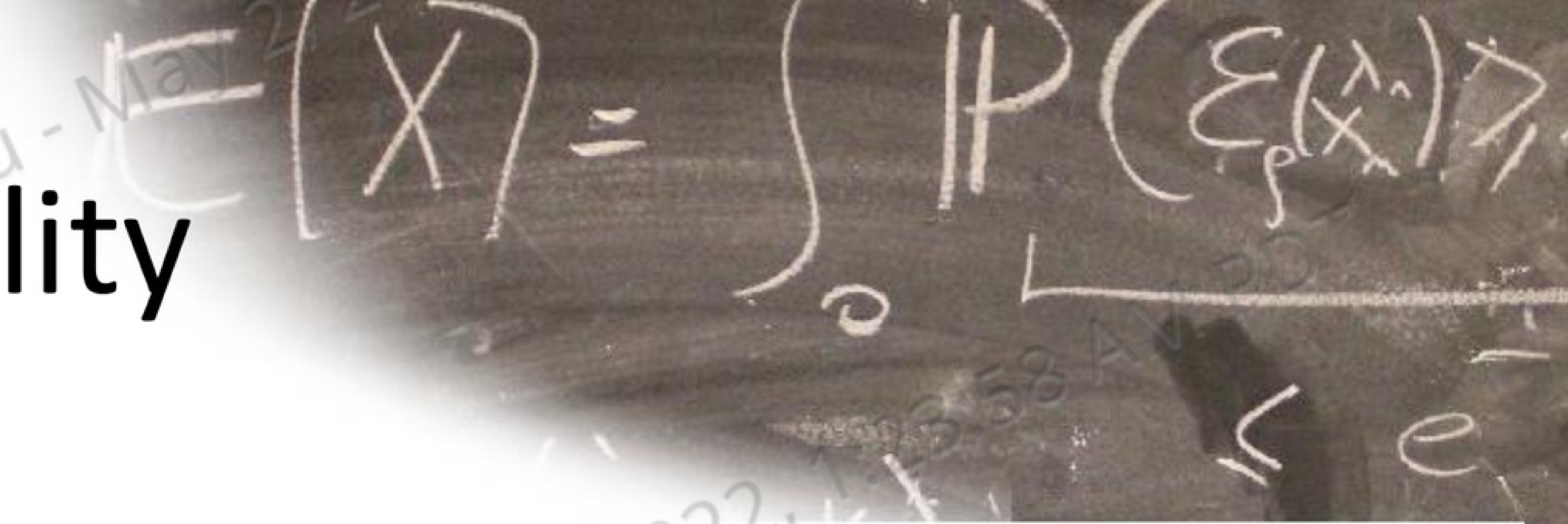
Basic Probability

- Sigma-algebra:

Interpretation: We can view a σ-algebra $\mathcal{F}$ as formalizing the idea of information. More precisely: A σ-algebra $\mathcal{F}$ is a collection of events, and if we assume that we have access to the information contained in $\mathcal{F}$, this means that for every $A \in \mathcal{F}$ we know exactly if A has occured or not.

If we have another σ-algebra $\mathcal{G}$ with $\mathcal{G} \subseteq \mathcal{F}$ then we interpret this as "$\mathcal{G}$ contains less information than $\mathcal{F}$".

Basic Probability



- **Measurability:**

(1) A set $A \subset \Omega$ is said to be $\mathcal{F}-$*measurable*, if $A \in \mathcal{F}$.
Example:
Assume $\Omega = \{\omega_1, \omega_2, \omega_3\}, \mathcal{F} = \sigma(\{\omega_1, \omega_2\}, \{\omega_3\})$
Then: $\{\omega_3\}$ is $\mathcal{F}-$*measurable*, $\{\omega_1\}, \{\omega_2, \omega_3\}$ are not.

(2) For R. V. $\hat{X}$ and finite sample space Ω, $\hat{X}$ is $\mathcal{F}-$*measurable* if $\hat{X}$ is constant for all elements in B_i, for all $i = 1, \cdots, K$, where $\mathcal{F} = \sigma(\{B_1, \cdots, B_K\})$.
Examples:
Assume $\Omega = \{\omega_1, \omega_2, \omega_3\}, \mathcal{F} = \sigma(\{\omega_1, \omega_2\}, \{\omega_3\})$
Then $\widehat{X}(\omega_i) = i$ is not $\mathcal{F}-$*measurable*.
$\widehat{X}(\omega_i) = 1_{i \leq 2}$ is $\mathcal{F}-$*measurable*.

$\{\omega \in \Omega \mid \hat{X}(\omega)$
$\| \quad (-1)$

~~(3)~~ Similarly, for infinite sample space Ω, $\hat{X}$ is $\mathcal{F}-$*measurable* if $\hat{X}^{-1} \in \mathcal{F}$ for every interval, $I \subset \mathcal{R}$

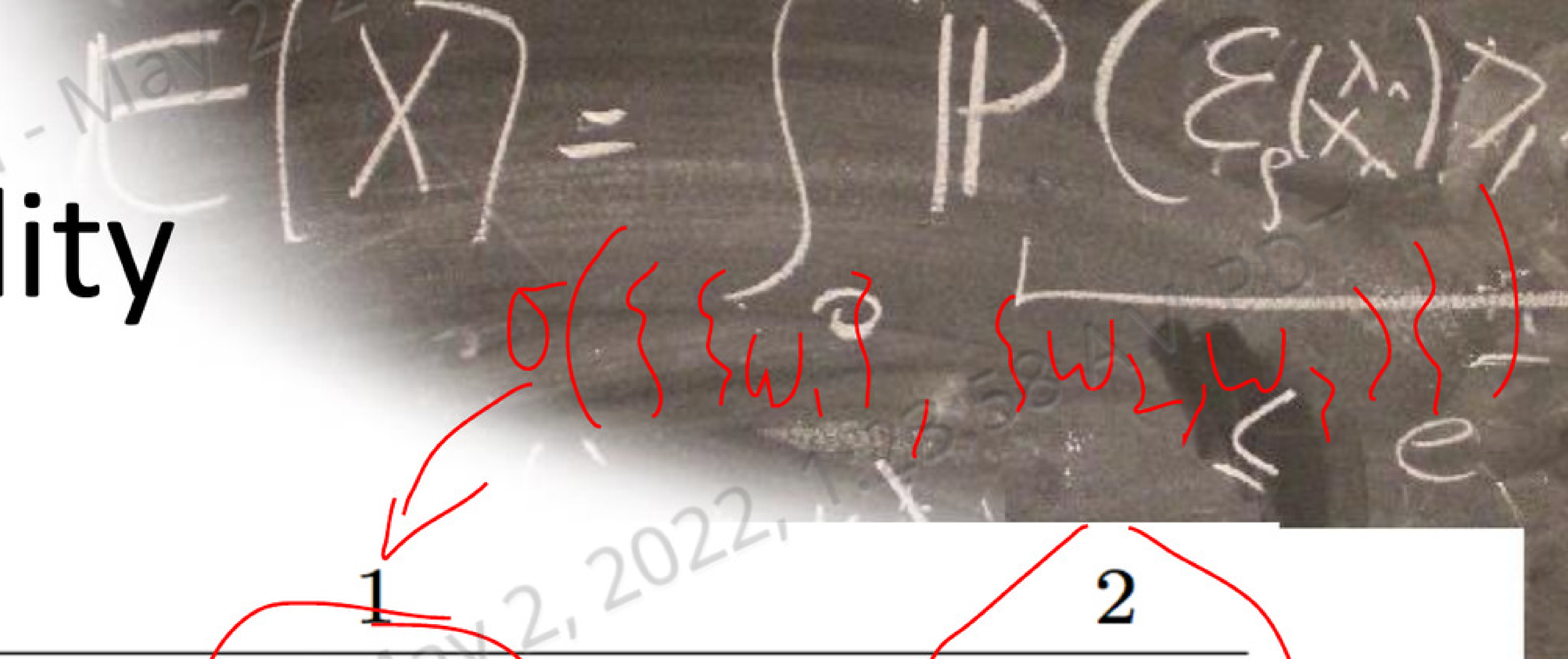
$\hat{X}(\omega_1) = 1$ $\quad B_1 = \{\omega_1, \omega_2\} \quad B_2 = \{\omega_3\}$

$\hat{X}(\omega_2) = 2 \quad X$

$\hat{X}(\omega_1) = 1 \qquad \hat{X}(\omega_3) = 0$

$\hat{X}(\omega_2) = 1$

Basic Probability



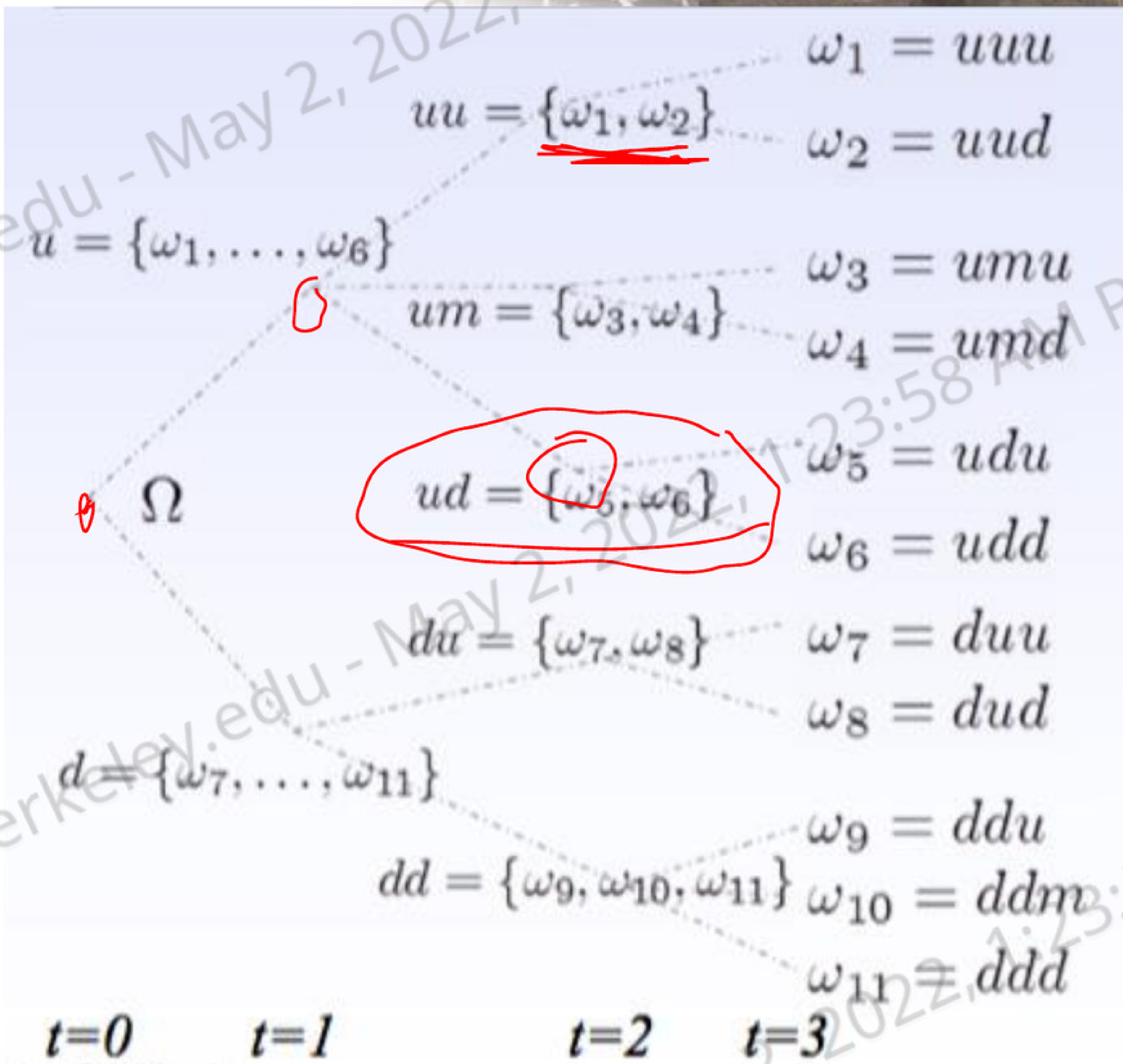
$\sigma(\{\{\omega_1\}, \{\omega_2, \omega_3\}\})$

Ω

T	0		1		2
			$\{\omega_1\}$		$\{\omega_1\}$
		$\nearrow$			
$\mathcal{F}_0$	$\{\Omega\}$		$\mathcal{F}_1$		$\{\omega_2\}$
		$\searrow$		$\nearrow$	
			$\{\omega_2, \omega_3\}$		
				$\searrow$	$\{\omega_3\}$
$\sigma(\{\Omega\}) = \mathcal{F} = \{\emptyset, \Omega\}$			$\{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$		$2^\Omega = \sigma(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\})$

The Multi-Period Market

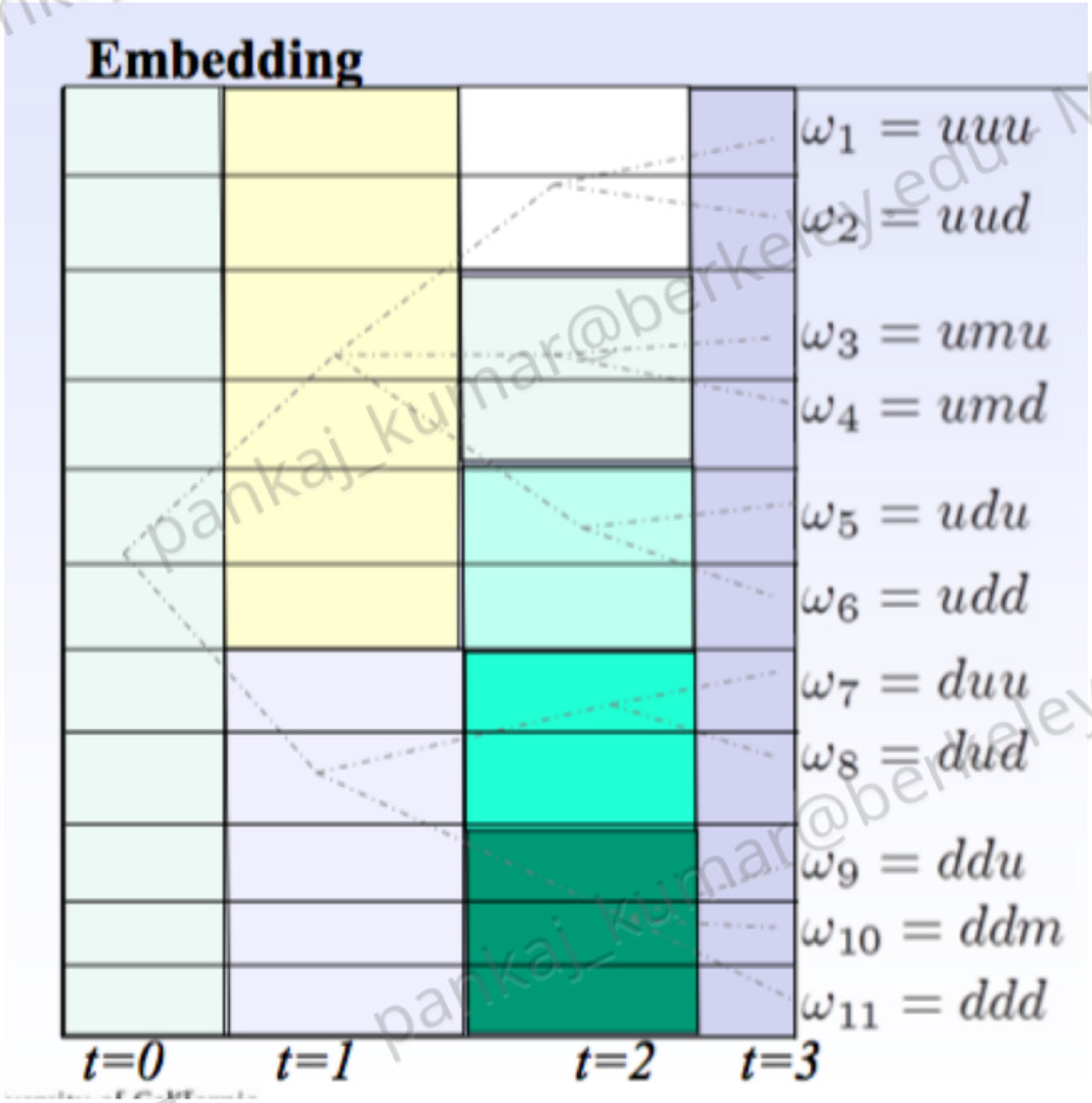
- Filtration
  - Increasing sequence of σ-algebras: $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$
  - Information revelation
  - Think of it as a tree (if it helps)



- Adapted processes
  - X_t is adapted to filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ if for all t, the process X_t is $\mathcal{F}_t$-measurable

The Multi-Period Market

- Adapted Processes:



Which of the following processes are *adapted the filtration* in Example 7 from Lecture 3?

$$f_t : \Omega \to \mathbb{R}$$

1. $f(t,\omega) = i + t$
2. $f(t,\omega) = t$
3. $f(t,\omega) = \left(t^2 - t\right) \mathbf{1}_{\{i \geq 5\}}$
4. $f(t,\omega) = \left(t^2 - t\right) \mathbf{1}_{\{i \geq 6\}}$

$\mathcal{F}_0 = \sigma(\{\Omega\}) = \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \sigma(\{u, d\}) = \{\emptyset, \Omega, u, d\}$

$\mathcal{F}_2 = \sigma(\underline{\{uu, um, ud, du, dd\}}) = \cdots$

$\mathcal{F}_3 = 2^{\Omega}$

---

1. $t = 0$, $f(0, \omega_1) = 1$, $f(0, \omega_2) = 2$

2. $t = 0$, $f(0, \omega_1) = 0$, $f(0, \omega_2) = 0$ $\cdots$

   $t = 1$, $\cdots$

3. $t=0$, $f(0,w) = 0$ - - - - - - ✓

$t=1$, $f(1,w_i) = 0$ - - - - - ✓

$t=2$, $f(2,w_1) = 0$, $f(2,w_2) = 0$ ✓

$f(2,w_3) = 0$ $\quad$ $f(2,w_4) = 0$ ✓

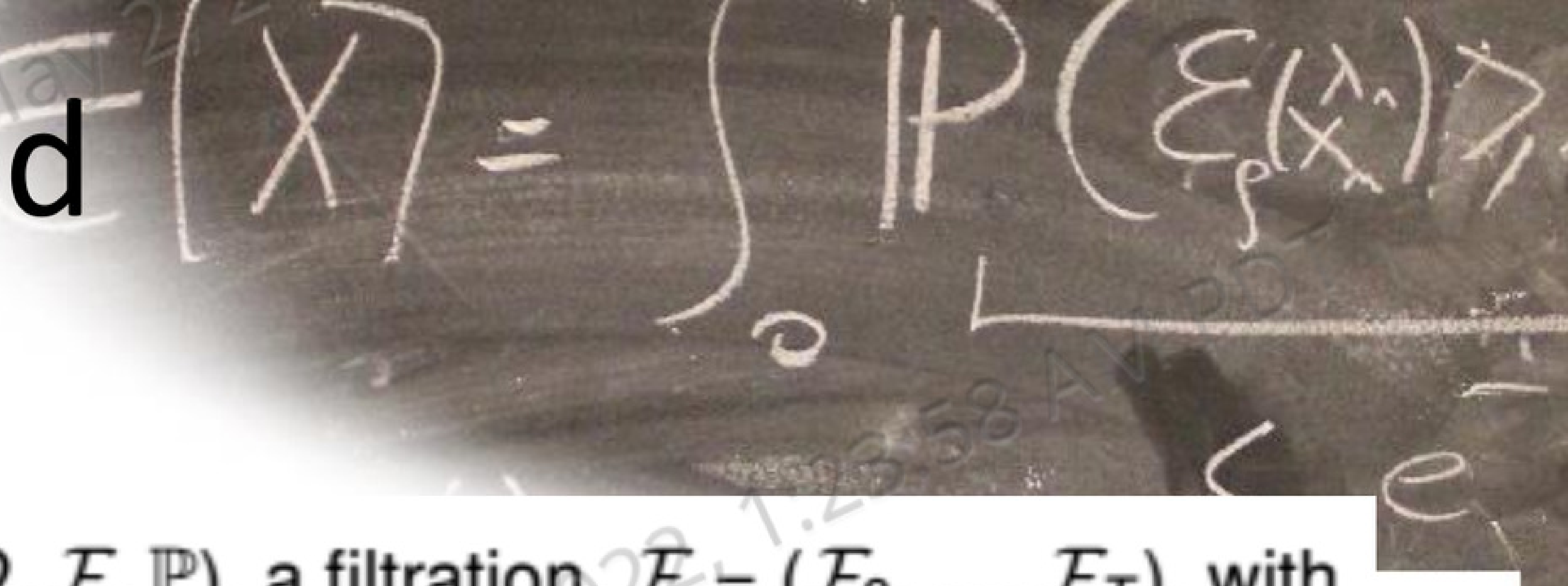
$f(2,w_5) = 2$ $\quad$ $f(2,w_6) = 2$ ✓

$\vdots$

✓

4. $f(2,w_5) = 0 \neq f(2,w_6) = 2$ ✗

The Multi-Period Market



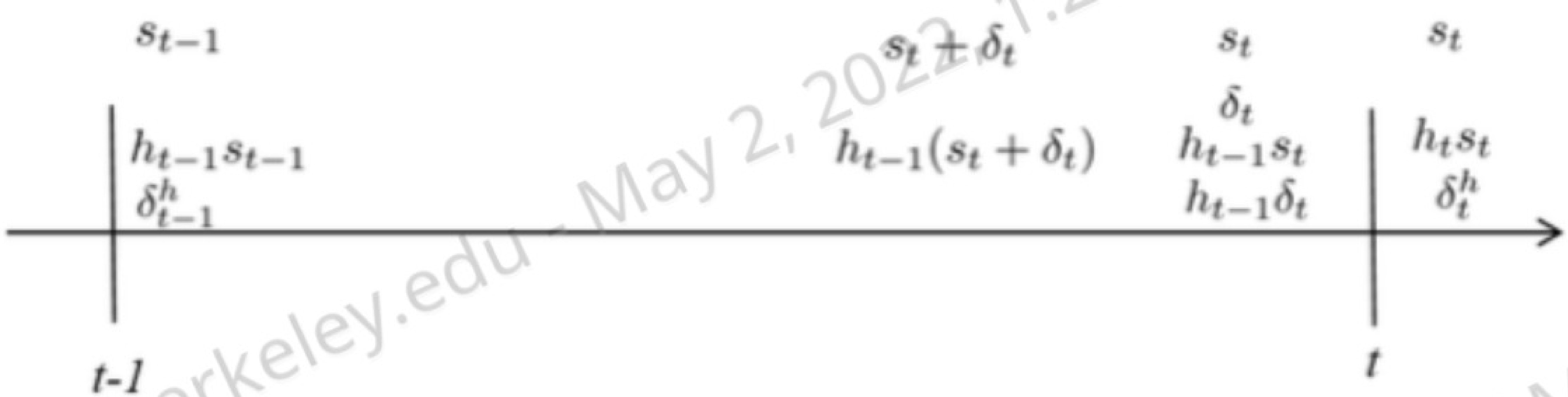
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration, $\underline{\mathcal{F}} = (\mathcal{F}_0, \dots, \mathcal{F}_T)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \sigma(\Omega)$, and a market with N traded assets:

- Security level:
  - *adapted* Dividend process: $\delta_t \in \mathbb{R}^N$;
  - *adapted* (ex Dividend) Price process: $\mathbf{s}_t \in \mathbb{R}^N$;
  - *adapted* Cum dividend process: $\mathbf{s}_t + \delta_t \in \mathbb{R}^N$;

    dividend-price pair $(\delta, \mathbf{s})$
- Portfolio level:
  - *adapted* Trading strategy: $\mathbf{h}_t \in \mathbb{R}^N$, $\mathbf{h}_{-1} = \mathbf{0}$.
  - *adapted* Payoff process (gain process):
    - **Case 1**: dividends: $\delta_t^{\mathbf{h}} = \mathbf{h}'_{t-1}(\mathbf{s}_t + \delta_t) - \mathbf{h}'_t \mathbf{s}_t$.
    - **Case 2:** No dividends: $\delta_t^{\mathbf{h}} = (\mathbf{h}_{t-1} - \mathbf{h}_t)' \mathbf{s}_t = -(\Delta_- \mathbf{h}_t)' \mathbf{s}_t$.
  - *adapted* Value $V_t^{\mathbf{h}} = \mathbf{h}'_t \mathbf{s}_t$.
  - *adapted* Consumption process: $c_t \in \mathbb{R}$,
  - Self-financing trading strategy: $\delta_t^{\mathbf{h}} = c_t$ for $t = 1, \dots, T$.

The Multi-Period Market

- Asset prices
- Asset dividends
- Portfolio value
- Net portfolio pay



- **Several Insights**:
  - **Total payout at time *t*** = *returns from asset holdings – change in portfolio value + dividend generated*.

$$\underbrace{\delta^{\mathbf{h}}_t}_{\text{Total time } t \text{ payout}} = \underbrace{\mathbf{h}'_{t-1}(\Delta_- \mathbf{s}_t)}_{\text{Change in value from asset returns}} - \underbrace{\Delta_- V_t}_{\text{Change in value of portfolio holdings}} + \underbrace{\mathbf{h}_{t-1}\delta_t}_{\text{Dividends}}$$

The Multi-Period Market

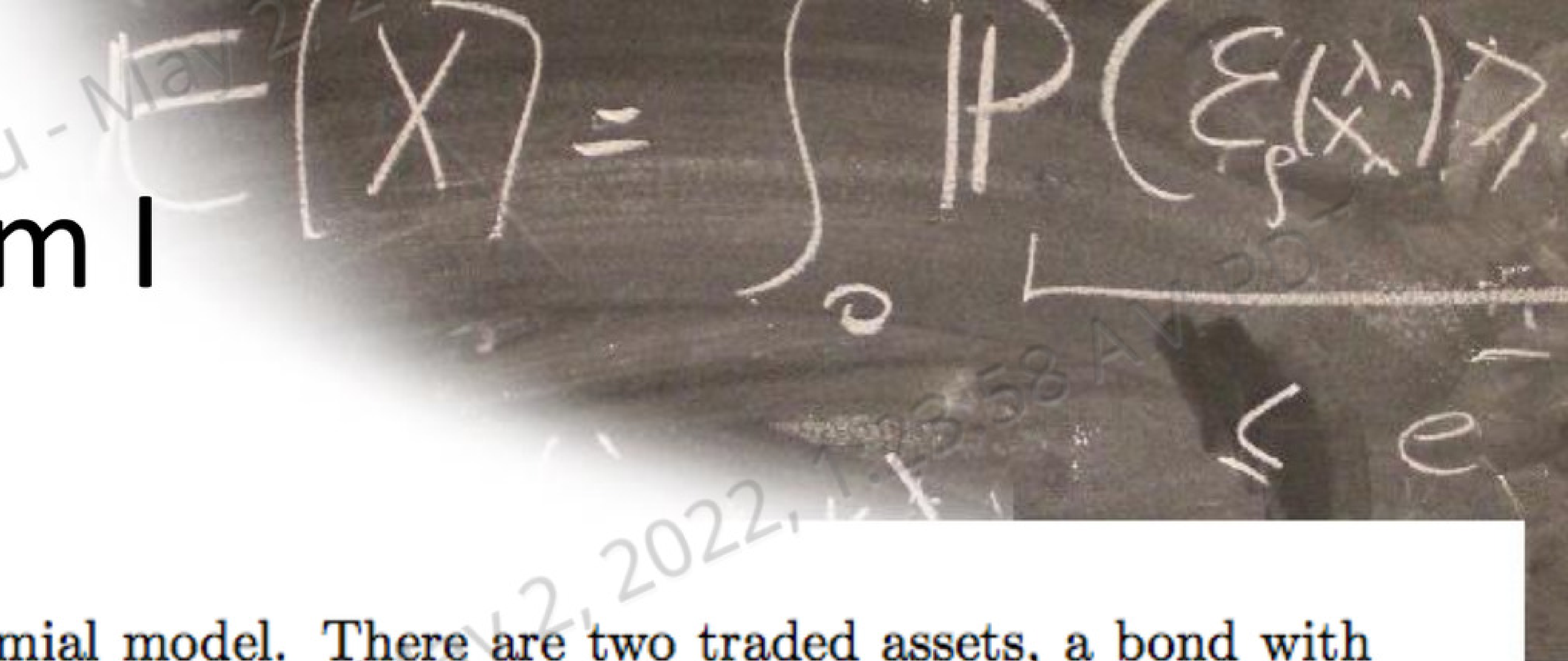
- **Several Insights**:
  - **Change in portfolio value** = *returns from asset holdings* + *change from rebalancing*.

$$\boxed{\Delta_- V_t = \mathbf{h}'_{t-1}\Delta_-\mathbf{s}_t + \Delta_-\mathbf{h}'_t(\mathbf{s}_{t-1} + \Delta_-\mathbf{s}_t).}$$

  - For self-financed portfolio with no dividend and no consumption, *change in portfolio value* = *returns from asset holdings*.

$$\underbrace{\Delta_- V_t}_{\substack{\text{Change in value}\\ \text{of portfolio holdings.}}} = \underbrace{\mathbf{h}'_{t-1}(\Delta_-\mathbf{s}_t).}_{\substack{\text{Change in value}\\ \text{from asset returns}}}$$

Sample Problem I



Consider a simple one-period binomial model. There are two traded assets, a bond with price $B(t)$ and a stock with price $S(t)$. Prices in the possible states of the world are given by.

$T = 0$ $\quad$ $T = 1$

$$B(T) = 105,\ S_{\uparrow}(T) = 110$$

$$B(0) = 100,\ S(0) = 100 \quad \begin{matrix} \nearrow \\ \\ \searrow \end{matrix}$$

$$B(T) = 105,\ S_{\downarrow}(T) = 90$$

A couple of quick questions to warm up:

- If $T = 1$ and interest is compounded once per period, what is the risk free rate in this economy?

$$\underline{R_f = \frac{105}{100} = 1.05}$$

- Is the market complete?

$$\begin{array}{cc} & \xrightarrow{\text{states}} \\ \#\text{asset} \downarrow & D = \begin{bmatrix} 105 & 105 \\ 110 & 90 \end{bmatrix} \end{array}$$

$\text{rank}(D) = 2$ ✓

$S =$ ______

$$\begin{bmatrix} 100 \\ 100 \end{bmatrix} = D \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 5/7 \\ 5/21 \end{pmatrix}$$

Sample Problem I

$$B(T) = 105,\ S_{\uparrow}(T) = 110 \quad 2100$$

$$B(0) = 100,\ S(0) = 100 \quad \begin{matrix} \nearrow \\ \\ \searrow \end{matrix}$$

$$B(T) = 105,\ S_{\downarrow}(T) = 90 \quad 0$$

We wish to find the price of the derivative with the payoff of $\max\{S(T)^2 - K, 0\}$ where $K = 10,000$. This derivative is called a power call. What tools have we learned so far to price this option?

- Replication
- Arrow–Debreu Securities
- Martingale/risk-neutral measure

Rep.

$$n_S \underline{110} + n_B \underline{105} = 2100$$

$$n_S 90 + n_B 105 = 0$$

$$\Rightarrow \quad n_S = 105 \quad n_B = -90$$

LOOP $\Rightarrow 105 \times 100 - 90 \times 100 = 1500$

AD / SPV

$$C = \begin{pmatrix} 2100 & 0 \end{pmatrix} \begin{pmatrix} 5/7 \\ 5/21 \end{pmatrix} = 1500$$

Emr / RN

$$q_i = \frac{\psi_i}{\psi} \Rightarrow q_1 = 3/4 \quad q_2 = 1/4$$

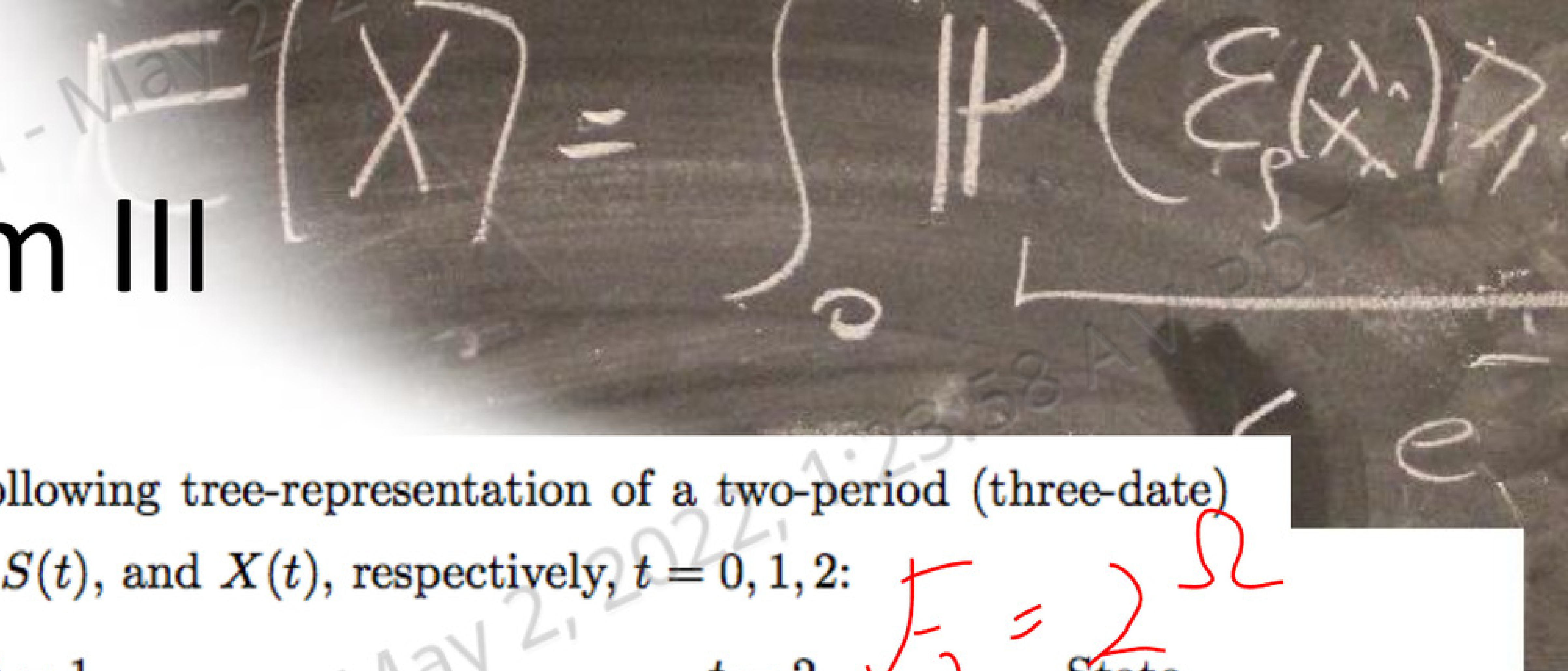
$$C = \frac{1}{1.2} E^Q(V') = \frac{1}{1.05}\left(3/4 \times 2100 + \frac{1}{4} \times 0\right) = 1500$$

Sample Problem II

Current stock price of MFE2014Class Inc. is \$40, over the next two 3-month periods it's expected to go up by 10% or down by 10%. The risk free rate is 12% per year with continuous compounding.

- Compute the price of a 6-month European put option with a strike price of \$42?
- Compute the price of a 6-month American put option with a strike price of \$42?

$$
\begin{tikzcd}[row sep=small]
T=0 & T=3\ months & T=6\ months \\
 & & \$48.40 \\
 & \$44 \arrow[ur] \arrow[dr] & \\
\$40 \arrow[ur] \arrow[dr] & & \$39.60 \\
 & \$36 \arrow[ur] \arrow[dr] & \\
 & & \$32.40
\end{tikzcd}
$$



Sample Problem III

Multi-period model: Consider the following tree-representation of a two-period (three-date) market with two assets, with prices $S(t)$, and $X(t)$, respectively, $t = 0, 1, 2$:

$\mathcal{F}_2 = 2^{\Omega}$

$t = 0$		$t = 1$		$t = 2$	State	
			$\nearrow$	$S(2) = 1.8,\ X(2) = 0.6$	ω_1	0
		$S(1) = 1, X(1) = 1$	$\longrightarrow$	$S(2) = 1.6, X(2) = 1.2$	ω_2	0
	$\nearrow$		$\searrow$			
$S(0) = 1,\ X(0) = 0.8$				$S(2) = 1.2,\ X(2) = 2.4$	ω_3	0
	$\searrow$	$1/3$				
		$S(1) = 1.5,\ X(1) = 1$	$\longrightarrow$	$S(2) = 1,\ X(2) = 1.2$	ω_4	0
			$\searrow$	$S(2) = 3,\ X(2) = 1.2$	ω_5	$1/3$

T_1 $\{\omega_1, \omega_2, \omega_3\}$ 0 T_2 $\{\omega_4, \omega_5\}$

$\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}) = \{\phi, \Omega, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}$

(a) Is the market complete? Motivate!

(b) Is there a risk-free rate process? If so, what is it? If not, why not?

(c) Define the filtration associated with the market.

(d) What is the $t = 0$ price of an Arrow-Debreu security that pays out 1 at time 2 in state ω_5?

$T_2 \quad D = \begin{bmatrix} 1.8 & 1.6 & 1.2 \\ 0.6 & 1.2 & 2.4 \end{bmatrix}$ ✗

$T_3 \quad D = \begin{bmatrix} 1 & 3 \\ 1.2 & 1.2 \end{bmatrix}$ ✓

$T_1 \quad D = \begin{bmatrix} 1 & 1.5 \\ 1 & 1 \end{bmatrix}$ ✓

$T_1 \quad R_f = 1/0.8$ ✓

$T_2 \quad (\alpha \;\; \beta) \begin{pmatrix} 1.8 & 1.6 & 1.2 \\ 0.6 & 1.2 & 2.4 \end{pmatrix} = (r \;\; r \;\; r)$

$T_3 \quad R_f = \cancel{1/1.2}$ ✓

$= 1.2/1$

$\alpha = r/2 \quad \beta = r/6$

$$\frac{r}{2}\times 1 + \frac{r}{6}\times 1 = \frac{2r}{3} \qquad R_f = \frac{r}{\frac{2r}{3}} = \frac{3}{2} = 1.5$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1/3 \end{pmatrix} = 1/3$$

$$\begin{pmatrix} 0 & 1/3 \end{pmatrix} \begin{pmatrix} 2/5 \\ 2/5 \end{pmatrix} = \underline{\underline{2/15}}$$