MFE230Q: Midterm Exam, April 23, 2012

NAME: **ID:**

Please motivate your answers. Please underline your final answers.

1. *One-period model*: Consider the following one-period market, with the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 0.8 \\ 0.5 \\ 0.2 \end{bmatrix}.$$

The probabilities for different events are $\mathbb{P}(\omega_1) = 0.2$, $\mathbb{P}(\omega_2) = 0.3$, $\mathbb{P}(\omega_3) = 0.5$.

(a) (5 points) Is there a risk-free rate ($R = 1 + r$) in this market? If so, what is it?

(b) (5 points) Is the market complete?

(c) (5 points) Brian Debt (who has just started a hedge fund) claims that prices are clearly nonsensical in this market, and that there is an arbitrage opportunity because asset 3 is under-priced. Is he correct?

(d) (5 points) Define a stochastic discount factor, M (i.e., M_1), in this economy.

2. *Itô processes & Itô's lemma*: Throughout this question, assume that W_t is a standard Brownian motion.

(a) (5 points) Consider the Itô process $X_t = e^{W_t^2}$. Calculate the Itô differential of X_t, dX_t.

(b) (5 points) Use Itô's lemma to prove that $\int_0^T W_t^2 dW_t = \frac{1}{3} W_T^3 - \int_0^T W_t dt$.

(c) (5 points) Assume that $dX_t = W_t^2 dt - t^2 dW_t + 2W_t t(dW_t - dt)$. Calculate $\int_0^T dX_t$.

3. *Continuous time trading*: Consider the continuous time economy with three assets, S^1, S^2 and B, and price dynamics

$$\begin{aligned}\frac{dS^1}{S^1} &= \mu_1 dt + \sigma_1 dW_t, \qquad S^1(0) = 1,\\ \frac{dS^2}{S^2} &= \mu_2 dt + \sigma_2 dW_t, \qquad S^2(0) = 1,\\ \frac{dB}{B} &= r dt, \qquad B_0 = 1.\end{aligned}$$

Here, μ_1, μ_2, σ_1, σ_2, and r are positive constants. Note that there is only one Wiener process, W_t, that drives the dynamics of both S^1 and S^2. As in class, we summarize the asset dynamics in the vector $\mathbf{s}_t = (S_t^1, S_t^2, B_t)'$ and a trading strategy in the vector $\mathbf{h}_t = (h_t^1, h_t^2, h_t^3)'$. Neither of the assets pay dividends, Thus, using the notation in class, $\Theta \equiv (0,0,0)'$.

(a) (5 points) Assume that an investor chooses the portfolio investment strategy $\mathbf{h}_t = \left(\frac{a_1}{S_1}, \frac{a_2}{S_2}, 0\right)'$. Derive expressions for the value process, $V_t^{\mathbf{h}}$, and the cumulative dividend process, $F_t^{\mathbf{h}}$ of this strategy.

(b) (5 points) Use your results in (a) to derive a condition on μ_1, μ_2, σ_1, σ_2, and r, that needs to be satisfied for there to be no arbitrage in this economy.

4. *SDEs*:

(a) (5 points) Consider the stochastic differential equation

$$dX_t = \frac{X_t}{2} dt + \sqrt{1 + X_t^2}\, dW_t, \qquad X_0 = 0.$$

where W_t is a Wiener process. What can we say about the general existence and uniqueness of solution(s) to this SDE?

(b) (5 points) Recall the calculus of hyperbolic functions: $\cosh^2(x) - \sinh^2(x) = 1$, $\sinh'(x) = \cosh'(x)$, $\cosh'(x) = \sinh'(x)$, where $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Use these relations to conjecture and verify a solution to the SDE in (a).