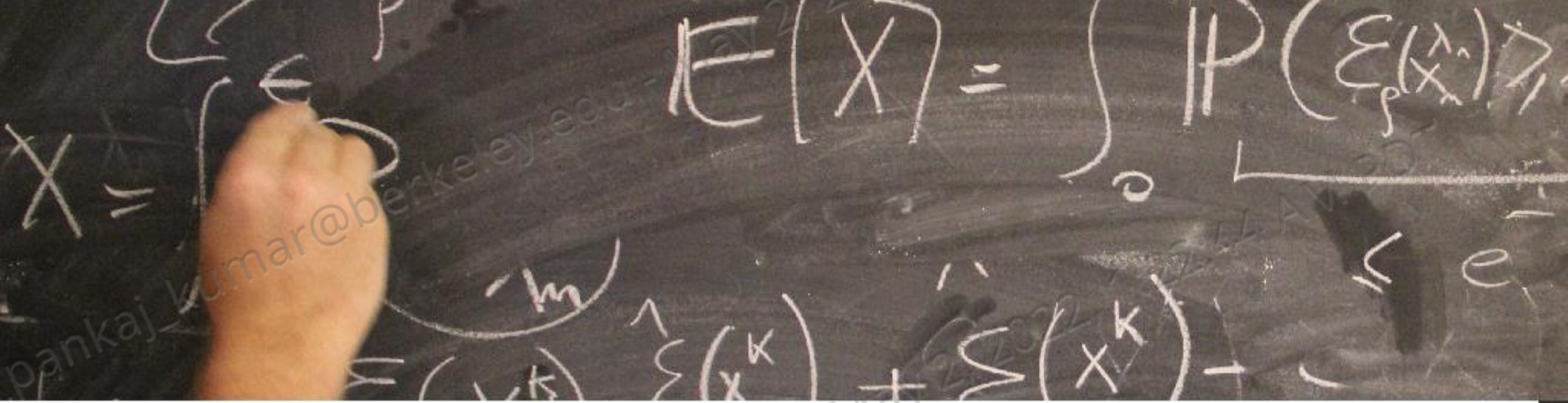


# MFE 230Q [Spring 2021] Introduction to Stochastic Calculus GSI Session 7


GSI: Simon Xu

***Email:***

simon_xu@haas.berkeley.edu

# Sample Problem 1

Suppose $z_1$ and $z_2$ are indices evolving as arithmetic Brownian motions:

$$
\begin{aligned}
dz_1(t) &= \frac{1}{5}dt + dW_1^{\mathbb{P}} \\
dz_2(t) &= \frac{2}{5}dt + 2dW_2^{\mathbb{P}} \\
dW_1^{\mathbb{P}} dW_2^{\mathbb{P}} &= 0 \\
Z_1(0) &= 0 \\
Z_2(0) &= 0
\end{aligned}
$$

where $W_1^{\mathbb{P}}(t)$ and$W_2^{\mathbb{P}}(t)$ are independent Wiener processes under the $\mathbb{P}$-measure. We have the following tradable assets with price processes $B$, $S_1$, and $S_2$ obeying the dynamics

$$
\begin{aligned}
\frac{dS_1(t)}{S_1(t)} &= \frac{1}{5}dt + \frac{1}{3}dz_1(t) + \frac{2}{3}dz_2(t) \\
\frac{dS_2(t)}{S_2(t)} &= \frac{8}{15}dt + \frac{2}{3}dz_1(t) + \frac{1}{2}dz_2(t) \\
S_1(0) &= 1 \\
S_2(0) &= 1
\end{aligned}
$$

What is the price of a **digital index Insurance contract** with payoff function

$$
\Theta(Z_1(T)) = 1000 \cdot \mathbf{1}_{\{z_1(T)<0\}} = \begin{cases} 1000 & \text{if } z_1(T) < 0 \\ 0 & \text{, otherwise} \end{cases}
$$

$\mathbb{P} \Rightarrow \mathbb{Q}$

$$\begin{pmatrix} dW_1^{\mathbb{Q}} \\ dW_2^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} dW_1^{\mathbb{P}} \\ dW_2^{\mathbb{P}} \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} dt$$

$$\begin{pmatrix} dS_1 \\ dS_2 \end{pmatrix} = \underline{\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}} \begin{pmatrix} 8/15 \\ 13/15 \end{pmatrix} dt + \underline{\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}} \begin{pmatrix} 1/3 & 1/4 \\ 2/3 & 1 \end{pmatrix} \underline{\underline{\left( \begin{pmatrix} dW_1^{\mathbb{Q}} \\ dW_2^{\mathbb{Q}} \end{pmatrix} - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} dt \right)}}$$

$$\alpha - \sigma \theta = r$$

$$\begin{pmatrix} 8/15 \\ 13/15 \end{pmatrix} - \begin{pmatrix} 1/3 & 1/4 \\ 2/3 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix}$$

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow dW_1^P = dW_1^Q - 1\,dt$$

$$dW_2^P = dW_2^Q$$

$$dZ_1 = \frac{1}{5}dt + (dW_1^Q - 1)dt$$

$$= -\frac{4}{5}dt + dW_1^Q \qquad Z_1(t) \sim N\left(-\frac{4}{5}t,\ t\right)$$

$$V(0) = e^{-\frac{1}{5}T}\, 1000\, \underbrace{E_0^Q\left(1_{(Z_1(T)<0)}\right)}_{Q(Z_1(T) < 0)}$$

$$V(0) = E_0^Q\left(e^{-rT}\cdot V_T\right)$$

$$\Rightarrow Q\left(\frac{Z_1(T) + \frac{4}{5}T}{\sqrt{T}} < \frac{\frac{4}{5}T}{\sqrt{T}}\right)$$

$$Q\left(Z < \tfrac{4}{5}\sqrt{T}\right) = N\left(\tfrac{4}{5}\sqrt{T}\right)$$

$$V_0 = e^{-\frac{1}{5}T} \, 1000 \cdot N\left(\tfrac{4}{5}\sqrt{T}\right) \underset{T=1}{=} \underline{\underline{645.278}}$$

# Sample Problem 2

The prices of a non-dividend-paying stock and a risk-free bond evolve under the risk-neutral measure $\mathbb{Q}$ as

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\, dW^{\mathbb{Q}}(t)$$

$$\frac{dB(t)}{B(t)} = r\,dt$$

$L > S$

**Part A:** Calculate a closed-form expression for the time-0 price of a **Knock-in European put option** on the stock with exercise price $K > S_0$ and a barrier also at $K$ (i.e. $L = K$).

**Part B:** Assume that $r = 0.1$, $\sigma = 0.16$, $S_0 = 100$, $T = 1$ and $K = 110$. Calculate the time-0 price of a **Knock-in European put option** given these parameters.

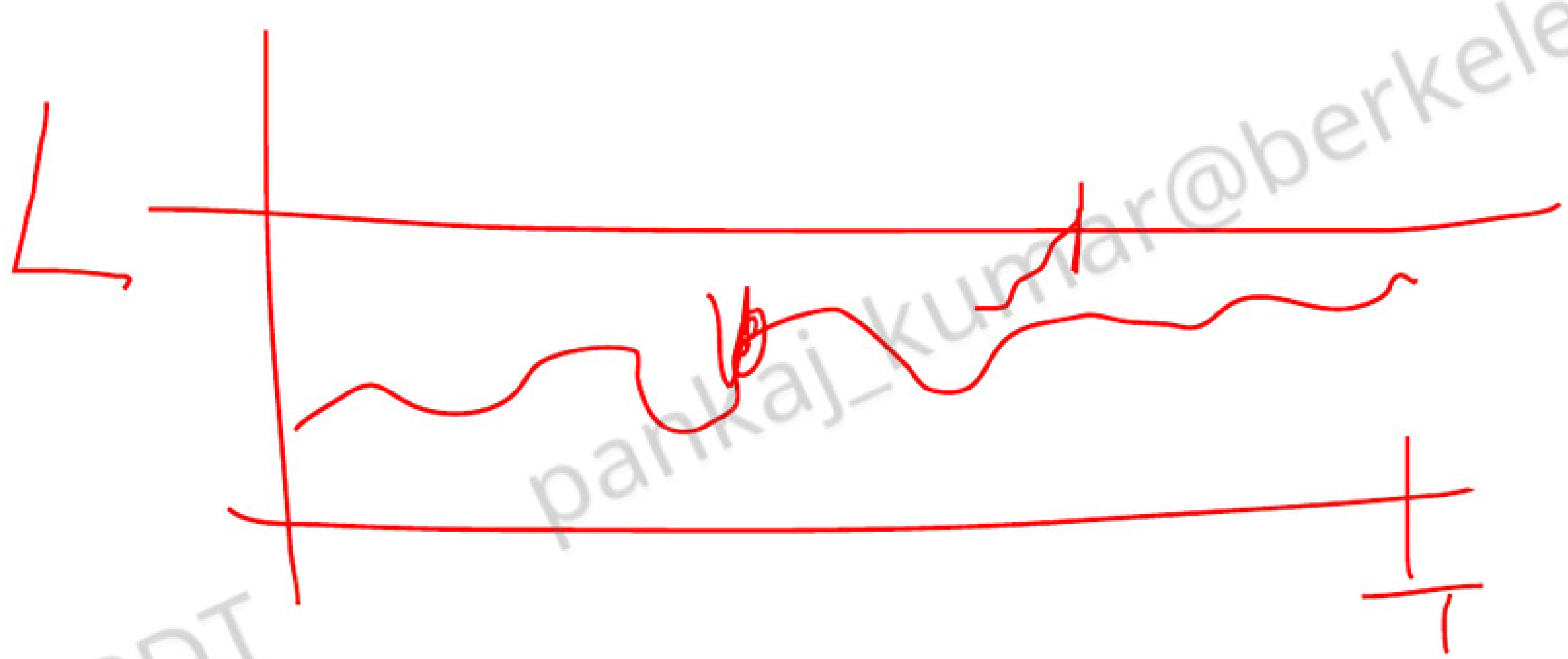
1.80312

Knock-in (up and in option)

fix $L > S_0$ $(Z^{LI})$

- Stock price below $L \Rightarrow 0$ maturity
- hits barrier $L \rightarrow \Phi(S_{T})$ maturity

$$Z^{LI}(T) = \begin{cases} \Phi(S_T) & \text{if } S_t \geq L \text{ for } t \in [0,T] \quad (*) \\ 0 & \text{if } S_t < L \ \ \forall t \in [0,T] \quad (**) \end{cases}$$




$$\Rightarrow F^{LI}(t,s;\Phi) = F(t,s;\Phi_L) + \left(\frac{L}{s}\right)^{2\bar{r}/\sigma^2} F\left(t,\frac{L^2}{s};\Phi^L\right) \qquad (\star)$$

$$s < L, \quad \bar{r} = r - \frac{1}{2}\sigma^2, \quad s = S_t$$

$$\Rightarrow F^{LI}(t,s;\Phi) = F(t,s;\Phi), \quad s \geq L$$

$$\Phi_L = \Phi(x)\,\mathbb{1}(x>L)$$

$$\Phi^L = \Phi(x)\,\mathbb{1}(x<L)$$

(a) $\underline{\Phi} = (K - S_T)^+$

$$Z^{LI} = \begin{cases} (K - S_T)^+ , & (*) \\ 0 , & (**) \end{cases}$$

$\underline{\Phi}_K = (K - S_T)^+ 1_{(S_T > K)} = 0 \Rightarrow F(t, S, \underline{\Phi}_K) = 0$

$\underline{\Phi}^K = \underline{(K - S_T)^+ 1_{(S_T < K)}}$

$= \underline{\Phi}$

$S_0^*$

$$(\forall) = e^{-rT} K N(-\tilde{d}_2) - \left(\frac{K^2}{S_0}\right) N(-\hat{d}_1)$$

$$\hat{d}_1 = \frac{-\ln\left(\frac{K^2}{S_0} \cdot \frac{1}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$-\tilde{d}_2 = -\tilde{d}_1 + \sigma\sqrt{T}$$

# Bond Pricing

A **zero coupon bond** with maturity $T$ (a "$T$-bond") is a contract paying \$1 at the date of maturity $T$.

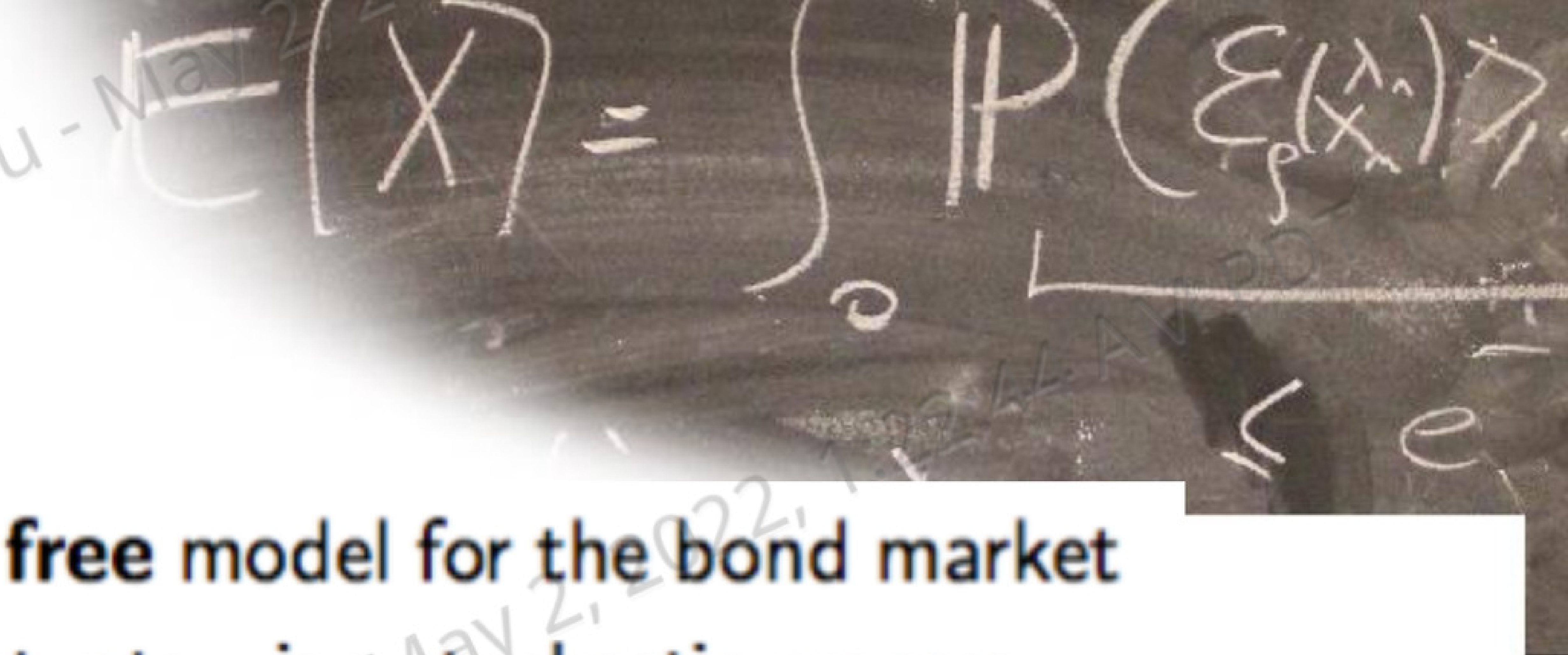
$$p(t,T) = \text{price, at } t, \text{ of a } T\text{-bond}.$$

$$p(T,T) = 1.$$

## Main Problem

- Investigate the **term structure**, i.e. how prices of bonds with different dates of maturity are related to each other.
- Compute arbitrage free prices of interest rate derivatives (bond options, swaps, caps, floors etc.)

# Bond Pricing: Term Structure



A consistent **arbitrage free** model for the bond market

We assume that the short rate $r$ is a stochastic process.

Money in the bank will then grow according to:

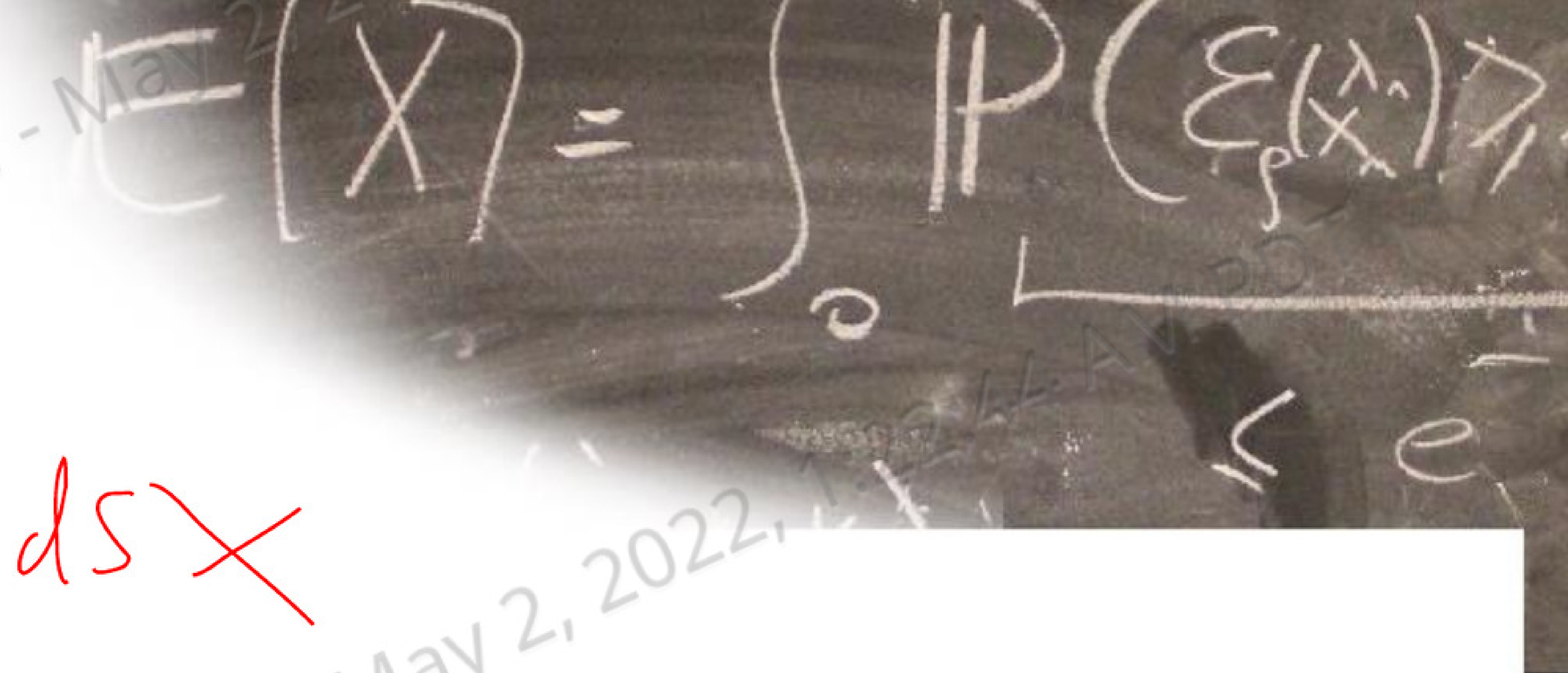
$$\begin{cases} dB_t = r_t B_t dt, \\ B_0 = 1. \end{cases}$$

i.e.

$$B_t = e^{\int_0^t r_s ds}$$

We need a model for the short rate $r$.

# Bond Pricing: Term Structure



dS X

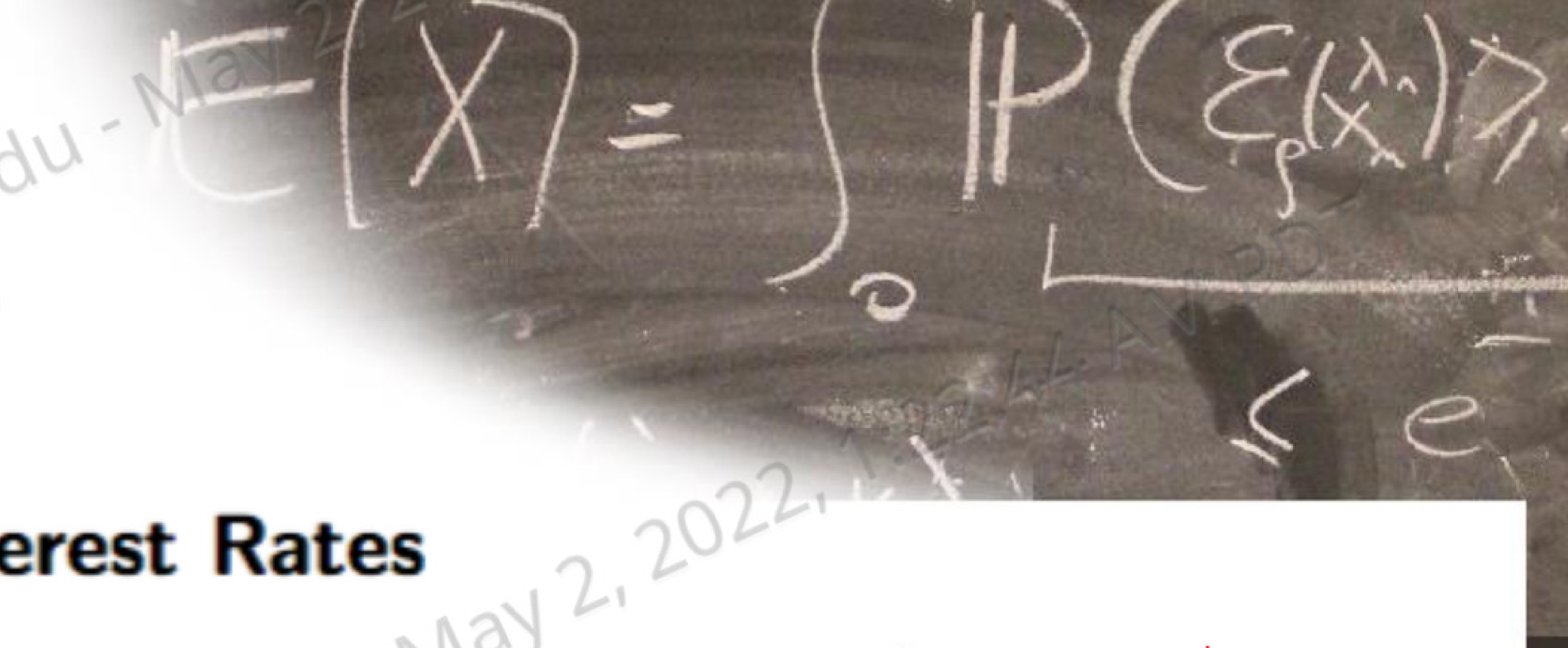
**P-dynamics**

$$
\begin{aligned}
dr_t &= \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \\
dB_t &= r_t B_t dt.
\end{aligned}
$$

**Question:** Are bond prices uniquely determined by the $P$-dynamics of $r$, and the requirement of an arbitrage free bond market?

X No

# Bond Pricing: Term Structure



## Stock Models ~ Interest Rates

**Black-Scholes:**

1 risky asset
1 source of risk

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

**Interest Rates:**

$\mathbb{P}$-dynamics

2 risky assets.
1 source of risk

$$
\begin{aligned}
dr_t &= \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \\
dB_t &= r_t B_t dt.
\end{aligned}
$$

# Bond Pricing: Term Structure

**Stock Models ~ Interest Rates**

**Question:** What is the difference?

**Answer:** The short rate $r$ is **not the price of a traded asset!**

risky.

# Bond Pricing: Term Structure

1. **Rule of Thumb:**

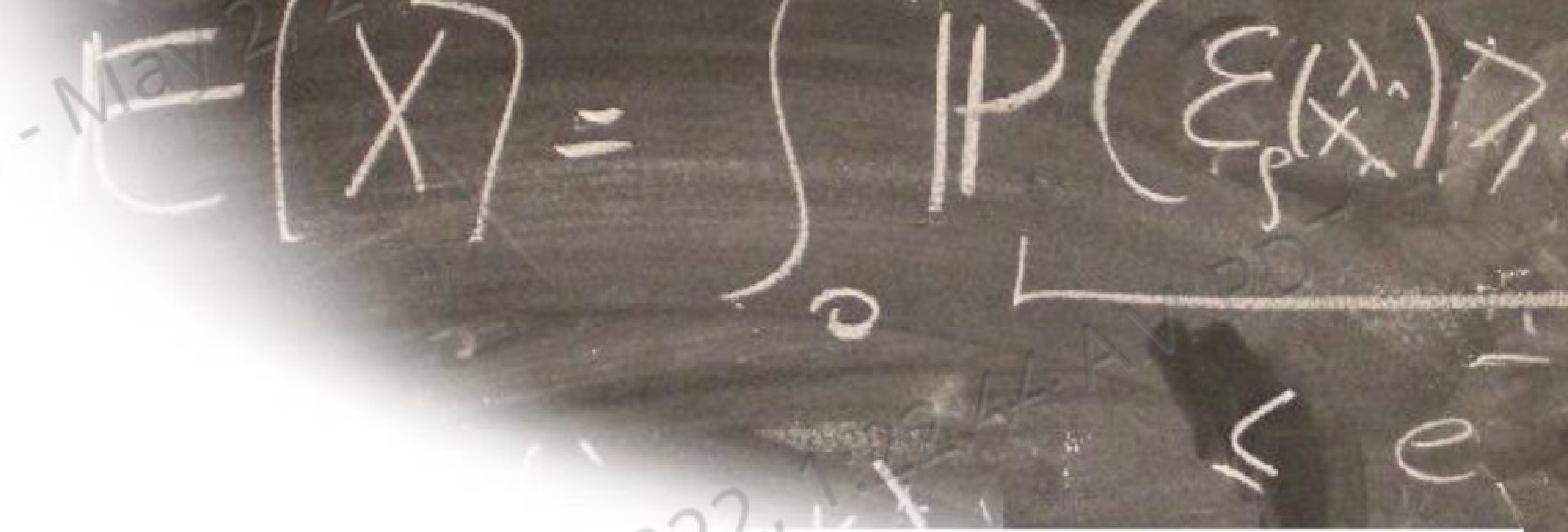
$$
\begin{aligned}
N &= 0, \quad \text{(no risky asset)} \\
R &= 1, \quad \text{(one source of randomness, } W\text{)}
\end{aligned}
$$

   We have $N < R$. The exogenously given market, consisting only of $B$, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price bonds.

# Bond Pricing: Term Structure

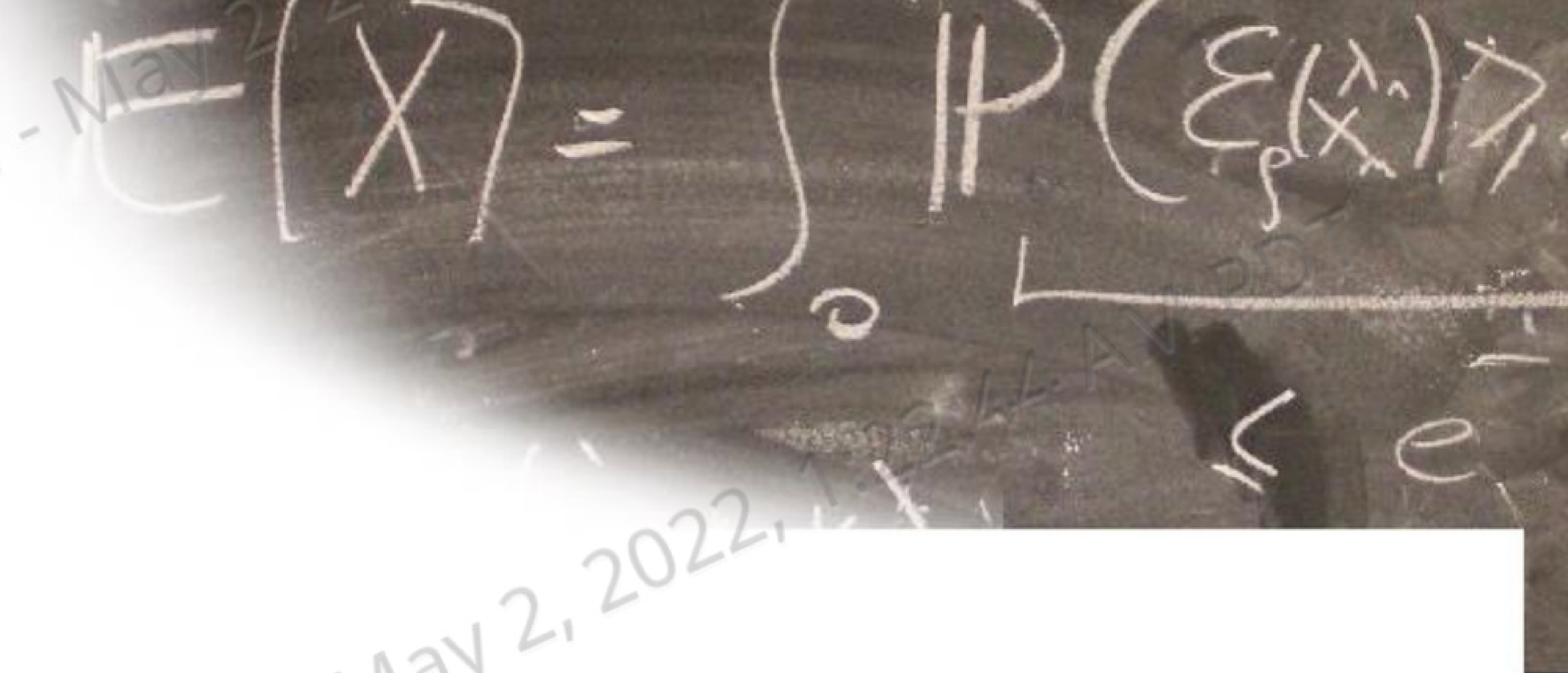


- There is **not** a unique price for a **particular** $T-$bond.

- In order to avoid arbitrage, bonds of **different maturities** have to satisfy internal **consistency** relations.

- If we take **one** "benchmark" $T_0$-bond as given, then all other bonds can be priced **in terms of** the market price of the benchmark bond.

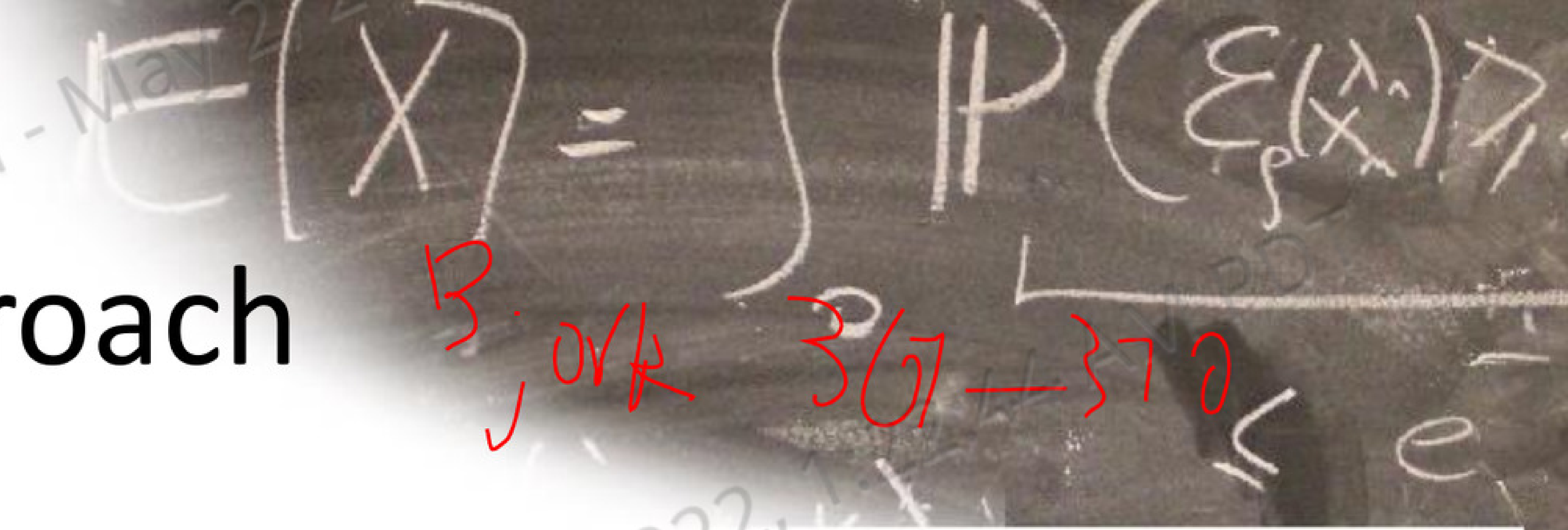
# Bond Pricing: Term Structure



**Assumption:**

$$\begin{aligned} p(t,T) &= F(t, r_t, T) \quad \equiv \\ p(t,T) &= F^T(t, r_t), \\ F^T(T,T) &= 1. \end{aligned}$$

# Bond Pricing: Replication Approach



Björk 367–370

- Form portfolio based on $T$ and $S$ bonds. Use Itô on $F^T(t, r_t)$ to get bond- and portfolio dynamics.

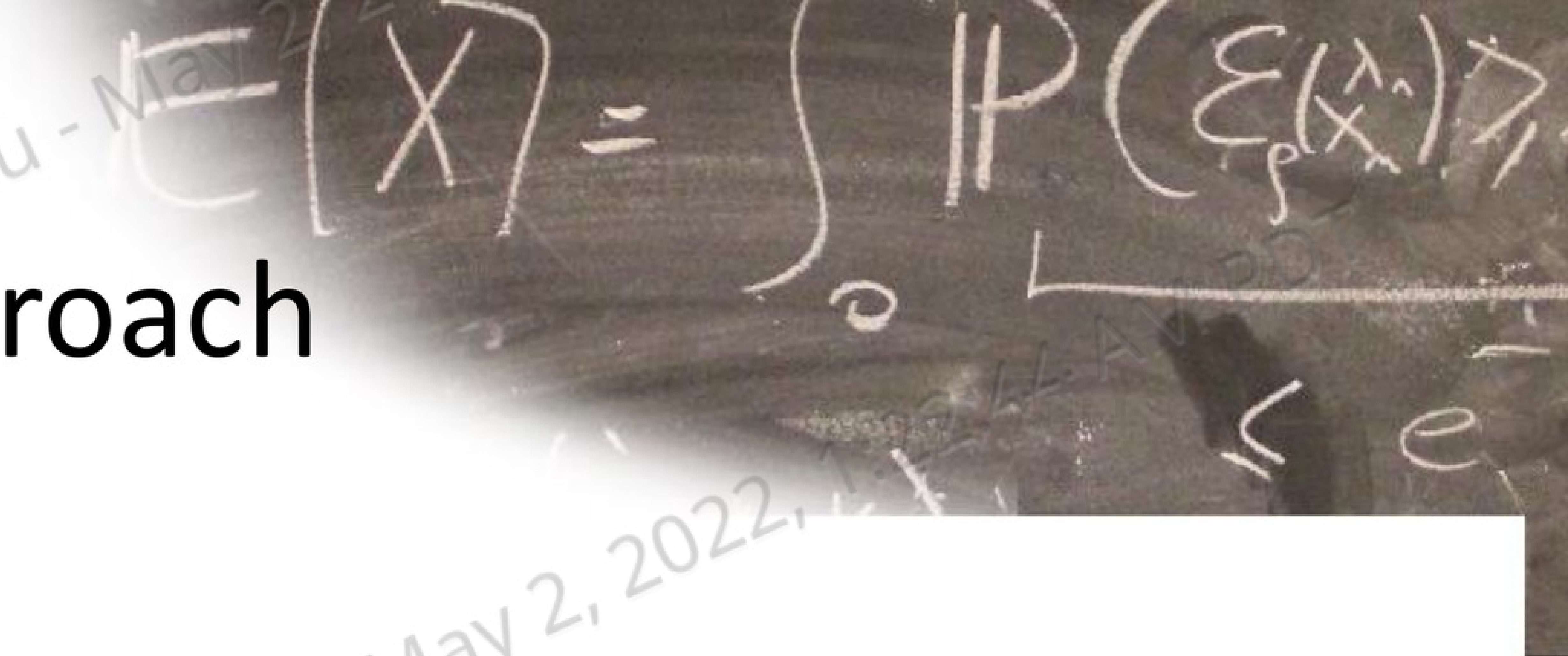
$$dV = V\left\{u^T\frac{dF^T}{F^T} + u^S\frac{dF^S}{F^S}\right\}$$

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot kdt,$$

("synthetic bank" with $k$ as the short rate)

# Bond Pricing: Replicating Approach

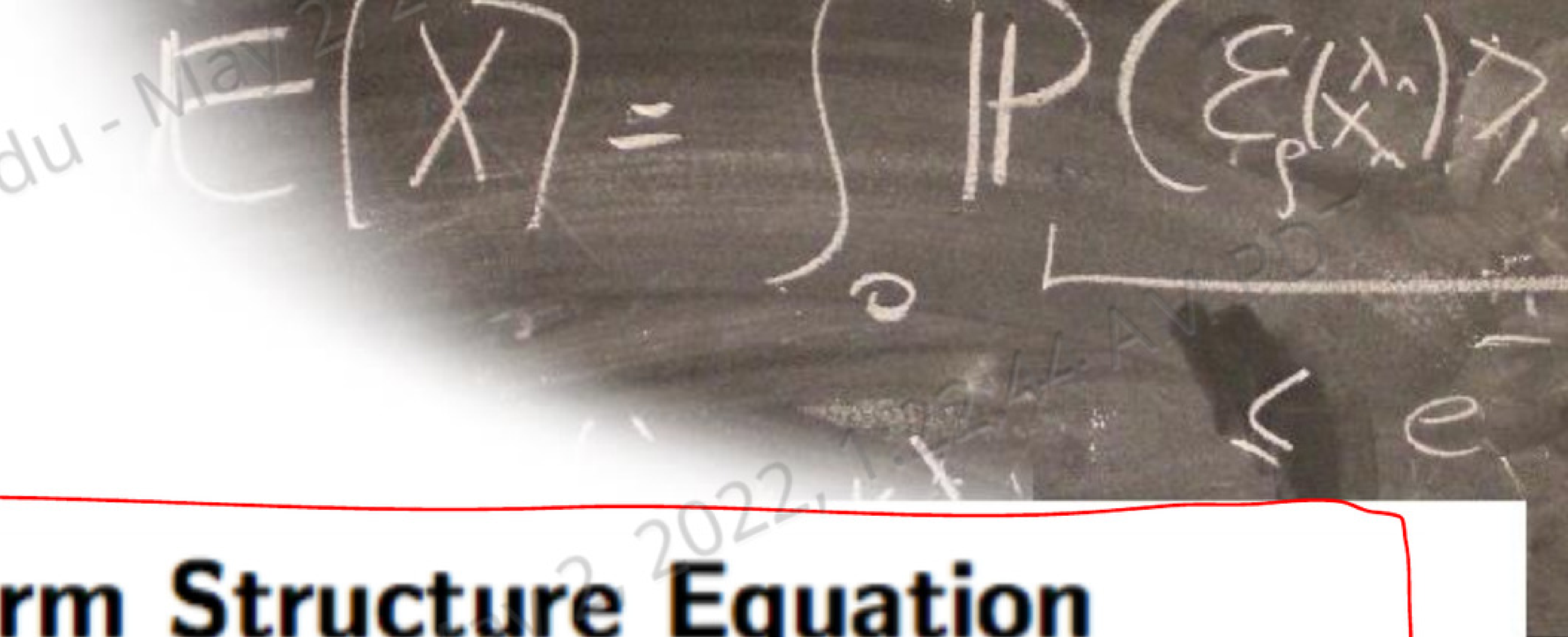


- Absence of arbitrage $\Rightarrow k = r$ .
- Read off the relation $k = r$!

Notation:

$$F_t = \frac{\partial F}{\partial t}, \quad F_r = \frac{\partial F}{\partial r}, \quad F_{rr} = \frac{\partial^2 F}{\partial r^2}$$

# Bond Pricing: PDE Approach



## The Term Structure Equation

≡ 'BS pde'

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T & = & 0, \\ F^T(T,r) & = & 1. \end{cases}$$

$$\chi = \Phi(r(T))$$

$$F_t^T + (\underbrace{\mu}_{\mu(t,r)} - \underbrace{\lambda}_{\lambda(t,r)} \underbrace{\sigma}_{\sigma(t,r)}) F_r^T + \frac{1}{2}\underbrace{\sigma^2}_{\sigma(t,r)} F_{rr}^T - \underbrace{r}_{r(t)} F^T + \underbrace{d}_{d(t,r)} = 0$$

$$F^T(T, r) = \Phi(r)$$

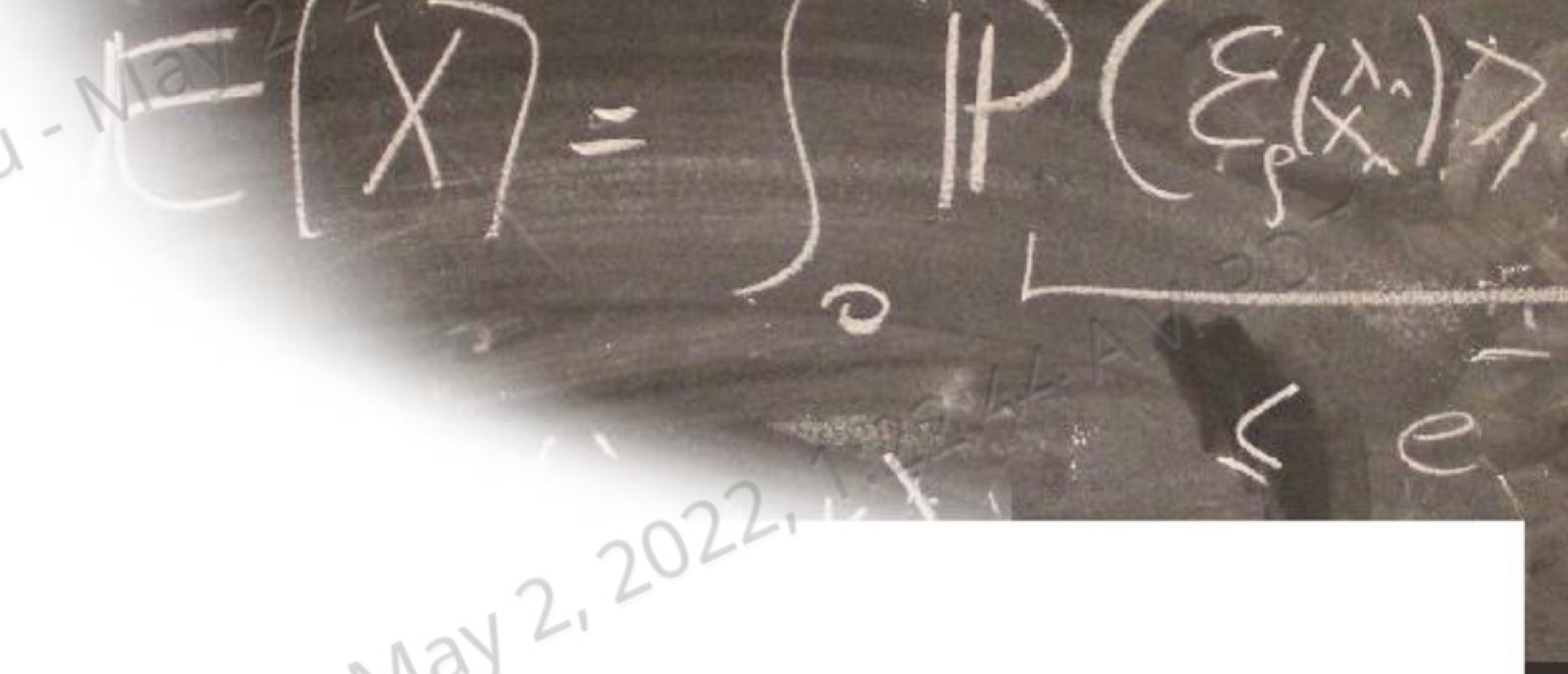
FK #3

$$F^T = E_t^Q\left( e^{-\int_t^T r_s ds} \Phi(r(T)) + \int_t^T d(s, r_s) e^{-\int_t^s r_u du} ds \right)$$

$$dr_s = \underbrace{(\mu - \lambda\sigma)}_{\mu^Q} ds + \sigma dW^Q$$

Q dynamics

# Bond Pricing: PDE Approach



**$P$-dynamics:**

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t.$$

$$\lambda = \frac{\alpha_T - r}{\sigma_T}, \quad \text{for alla } T$$

In order to solve the TSE we need to know $\lambda$.

# Affine Term structure models (ATSM)

- Short rate dynamics of the form
$$dr = (\alpha r + \beta)dt + \sqrt{\gamma r + \sigma}dW_t$$
leads to affine term structure

- Deduce PDE for zero-coupon bonds following standard approach

- Typically, conjecture solution of the form $e^{A(T-t)-B(T-t)r}$, and reduce PDE to a system of ODEs for $A$ and $B$

$$F^T = e^{\underline{A(t,T)} - \underline{B(t,T)}\, r}$$

$A, B$ deterministic functions (of time), $\underline{\underline{ATS}}$

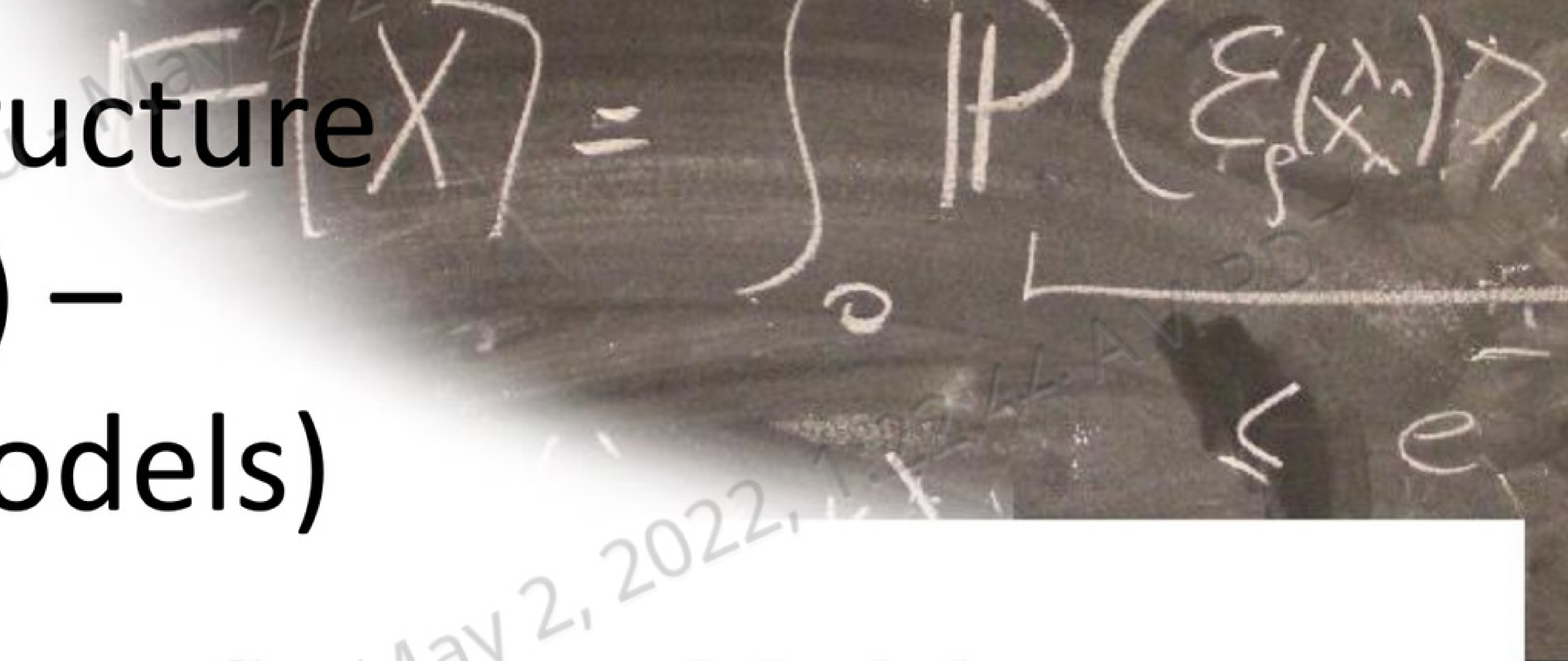
$$\mu = \alpha_t r + \beta_t$$

$$\sigma = \sqrt{\gamma_t r + \delta_t}$$

$$\left\{ \begin{array}{l} B_t + \alpha_t B - \frac{1}{2}\gamma_t B^2 = -1 \\ B(T,T) = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} A_t = \beta_t B - \frac{1}{2}\delta_t B^2 \\ A(T,T) = 0 \end{array} \right.$$

# Affine Term structure models (ATSM) – (One Factor Models)



**Popular 1-factor Term Structure Models:**

1) Vasicek: $dr = \kappa(\theta - r)\,dt + \sigma\,dW^Q$

2) CIR: $dr = \kappa(\theta - r)\,dt + \sigma\sqrt{r}\,dW^Q$

3) Squared Gaussian $dx = \kappa(\theta - x)\,dt + \sigma\,dW^Q$

$$\Rightarrow\; r = x^2 \;(r \text{ not Markov!!})$$

# Affine Term structure models (ATSM) – (One Factor Models)

<u>Comments</u>:

1) Vasicek: spot rate distributed normally
   - permits negative rates.

2) CIR and SG preclude negative rates
   - $\sigma_r \Rightarrow 0$ as $r \Rightarrow 0$.

3) All have mean-reverting behavior
   - $\mathrm{E}\left[r(t=\infty)\right] = \text{Finite}.$

4) Empirically, volatility increases with $r$

# Affine Term structure: Practice

## Practice Final 1 (2011), Question 5

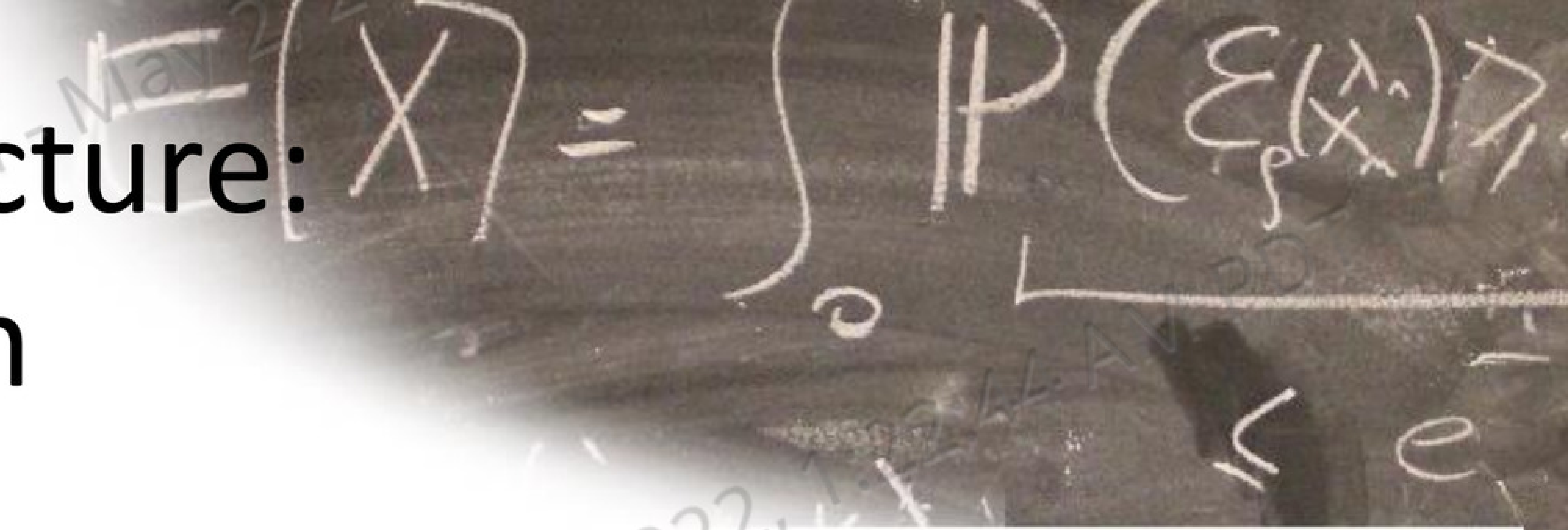
5. *Term structure (20 points)* Assume that the short rate follows the asset pricing dynamics specified by the Vasicek model:

$$dr = (b - ar)dt + \sigma dW^Q, \qquad a > 0.$$

(a) Does the model belong to the class of affine term structure models? Why/Why not?

(b) Derive the formula for the time-$t$ price of a $T$-bond, $p(t, T|r)$.

# Affine Term structure: Practice Solution



(a) If the pricing formula is of the form $p(t, r; T) = e^{A(t,T) - B(t,T)r}$, then the Vasicek model belongs to the class of affine structure models. From class and Björk, we know that short-rates dynamics on the form

$$dr = (\alpha r + \beta)dt + \sqrt{\gamma r + \delta}\, dW_t^Q,$$

leads to affine term structure pricing, so the Vasicek model indeed belongs to this class (with $\alpha = -a$, $\beta = b$, $\gamma = 0$, $\delta = \sigma^2$).

$$dr = (\alpha r + \beta) dt + \sqrt{\gamma r + \delta}\, dW_t^{P}$$

# Affine Term structure: Practice Solution

(b) Let's use the martingale approach, to solve the model:

$$
\begin{aligned}
rpdt &= E^Q[dp] \\
rp &= p_t + p_r(b - ar) + \frac{1}{2}p_{rr}\sigma^2,
\end{aligned}
$$

with boundary condition $p(T, r; T) = 1$, for all $r$. Functions $A$ and $B$ to be determined by

$$
\begin{aligned}
\frac{\partial B}{\partial t} &= aB(t,T) - 1, \\
\frac{\partial A}{\partial t} &= bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T),
\end{aligned}
$$

and the boundary condition $p(T, r; T) = 1$ gives us the conditions $A(T,T) = B(T,T) = 0$.

$$B(t,T) = \int_t^T aB(t,T) - 1 \, ds$$

$$= \text{Riemann integral}$$

# Affine Term structure: Practice Solution

(b)

Solving the ODE will determine the functions $A$ and $B$, which will give us $p(t,T|r)$.

$$B(t,T) \;=\; \frac{1}{a}\left(1-e^{-a(T-t)}\right)$$

$$\begin{aligned}
A(t,T) &= \int_t^T \left(\frac{1}{2}\sigma^2 B^2(s,T) - bB(s,T)\right) ds. \qquad \text{Riemann} \\
&= \int_t^T \left(\frac{\sigma^2}{2a^2}\left(1-e^{-a(T-s)}\right)^2 - \frac{b}{a}\left(1-e^{-a(T-s)}\right)\right) ds. \\
&= \frac{\sigma^2}{2a^2}\left(-\frac{2a(t-T)-4e^{-a(T-t)}+e^{-2a(T-t)}+3}{2a}\right) - \frac{b}{a}\left(\frac{e^{-a(T-t)}-1}{a}+(T-t)\right)
\end{aligned}$$

Part 1: NA, SPU, completeness

Ito's Lemma