MFE230Q: Final Exam, May 23, 2011

NAME: **ID:**

Please motivate your answers. Please underline your final answers.

1. *One-period model* (20 points): Consider the following one-period market, with the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.75 \end{bmatrix}.$$

(a) Is this market arbitrage free?

(b) Is this market complete?

(c) What is the risk-free rate ($R = 1 + r$) in this market?

(d) What is the value of a derivative contract that pays out 1 in state ω_1, 2 in state ω_2 and 3 in state ω_3?

Answers

(a) It is easy to see that the first three assets span the whole market without admitting arbitrage, i.e., that the market is complete with positive state prices given these three assets. This can be done by calculating

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}.$$

However, it also follows immediately from the fact that a portfolio of long one share each of any two assets and short one share of the third defines an A-D like portfolio that pays off 2 units in one state and zero in the others. The cost of such a portfolio is 0.5+0.5-0.5=0.5, so each state price must then be $\psi_i = \frac{1}{2} \times 0.5 = 0.25 > 0$, $i = 1, \ldots, 3$. Now, the price of an asset that pays off 1 in each state is then $\psi_1 + \psi_2 + \psi_3 = 0.75$, which is indeed the price of the fourth asset, so the positive state price vector prices all assets, and it then follows immediately from the fundamental theorem that the market is arbitrage free.

(b) The market is complete, since each A-D security can be replicated. Alternatively, since the state price vector is unique, the second fundamental theorem implies that the market is complete. Note that the argument that the market is complete because there are more assets than states is *not* correct, since it could be that the assets are linearly dependent.

(c) The fourth asset is a risk-free asset, which immediately implies that $R = 1/0.75 = 4/3$.

(d) The value of the derivative is equal to $\psi_1 x_1 + \psi_2 x_2 + \psi_3 x_3 = 0.25(1+2+3) = 1.5$, where x_i is the payoff in state i.

2. *Itô calculus, SDEs and Kolmogorov's equations (25 points):*

(a) Let $y_t = (t + W_t)^3$. Calculate dy_t.

(b)* Define $y_t = \int_0^t (W_s + s)^2 dW_s$, where W_s is a Wiener process under the standard filtration, $\mathcal{F}_t$, $t \geq 0$. Calculate $E[y_t|\mathcal{F}_0]$ and $Var[y_t|\mathcal{F}_0]$.

(c) Solve the partial differential equation:

$$\begin{aligned} F_t + \frac{\sigma^2}{2}F_{xx} + \frac{\delta^2}{2}F_{yy} - rF &= 0, \\ F(T, x, y) &= xy, \\ x &\in \mathbb{R}, \\ y &\in \mathbb{R}, \\ t &\leq T, \end{aligned}$$

where σ, δ and r are strictly positive constants.

(d) Consider the Ornstein-Uhlenbeck process

$$\begin{aligned} dX_t &= \kappa(\theta - X_t)dt + \sigma dW_t, \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned}$$

Derive an expression for the (stationary) long-term probability density function of X (i.e., $p(x, \infty)$, where $P(x, T) = \mathbb{P}(X_T \leq x)$, and $p(x, T) = \frac{dP}{dx}(x, T)$).

Answers:

(a)

$$\begin{aligned} dy_t &= 3(t + W_t)^2 dt + 3(t + W_t)^2 dW_t + \frac{1}{2}6(t + W_t)dt \\ &= \left[3(t + W_t) + 3(t + W_t)^2\right] dt + 3(t + W_t)^2 dW_t \end{aligned}$$

(b) It follows directly from the martingale property of the Ito integral that

$$E[y_t|\mathcal{F}_0] = 0.$$

$$\begin{aligned} Var[y_t|\mathcal{F}_0] &= E\left[\left(\int_0^t (W_s+s)^2 dW_s\right)^2 \middle| \mathcal{F}_0\right] = \\ &= \int_0^t E\left[(W_s+s)^4 \middle| \mathcal{F}_0\right] ds \end{aligned}$$

Now define $Z_t = (W_t + t)$

$$Z_t \sim \mathcal{N}(t,t).$$

If you are familiar with the calculus of moment generating functions (mgf), $M(t) = E[e^{tX}]$, it is straightforward to use the mgf of the normal distribution

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

to solve the problem (note that t does *not* represent time in this expression). Specifically, $E(Z^4) = M^{(4)}(0) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$ so in this case

$$E(Z^4) = t^4 + 6t^3 + 3t^2,$$

and

$$\begin{aligned} Var[y_t|\mathcal{F}_0] &= \int_0^t E[Z_s^4] ds \\ &= \int_0^t (s^4 + 6s^3 + 3s^2) ds \\ &= \frac{t^5}{5} + \frac{3}{2}t^4 + t^3. \end{aligned}$$

Alternatively, you can use Ito's lemma to derive the solution. In class we used Ito's lemma to show the following relation: If $\beta_k(t)$ is defined as

$$\beta_k(t) = E[W_t^k],$$

then since Ito implies that $W_t^k = \int_0^t d(W_s^k) = \int_0^t k W_s^{k-1} dW_s + \frac{1}{2}k(k-1)\int_0^t W_s^{k-2} ds$, it follows (using the martingale property for the first term) that

$$\beta_k(t) = \frac{k(k-1)}{2} \int_0^t \beta_{k-2}(s) ds.$$

Further, since the properties of Wiener processes immediately implies that $\beta_1(t) = 0$ and $\beta_2(t) = t$, the relationship implies that $\beta_3(t) = 0$, and $\beta_4(t) = 3t^2$.

Now, the Ito symmetry implies that

$$\begin{aligned}
Var\left(\int_0^t Z_s^2 dW_s\right) &= E\left[\left(\int_0^t Z_s^2 dW_s\right)^2\right] \\
&= E\left[\int_0^t Z_s^4 ds\right] \\
&= E\left[\int_0^t W_s^4 + 4W_s^3 s + 6W_s^2 s^2 + 4W_s s^3 + s^4 ds\right] \\
&= \int_0^t E[W_s^4]ds + 4\int_0^t E[W_s^3]sds + 6\int_0^t E[W_s^2]s^2 ds + 4\int_0^t E[W_s]s^3 ds + \int_0^t s^4 ds \\
&= \int_0^t \beta_4(s)ds + 0 + 6\int_0^t \beta_2(s)s^2 ds + 0 + \int_0^t s^4 ds \\
&= \int_0^t 3s^2 ds + 6\int_0^t s \times s^2 ds + 0 + \int_0^t s^4 ds \\
&= 3\frac{t^3}{3} + 6\frac{t^4}{4} + \frac{t^5}{5},
\end{aligned}$$

which is the same answer as previously derived.

(c) We use the relationship between PDEs and SDEs that follow from Feynman-Kac's theorem to solve the problem with probabilistic methods. First, note that the underlying processes for X and Y are

$$\begin{aligned}
dX &= \sigma dW_1, \\
dY &= \delta dW_2,
\end{aligned}$$

where W_1 and W_2 are independent Wiener processes. Then note that if we define $Z = XY$ that Z is a martingale, i.e., $E_{z,t}[Z(T)] = z$. Therefore, following Feynman-Kac's theorem,

$$\begin{aligned}
F(t,x,y) &= E_{x,y,t}\left[e^{-r(T-t)}X(T)Y(T)\right] \\
F(t,x,y) &= e^{-r(T-t)}E_{x,y,t}\left[X(T)Y(T)\right] \\
F(t,x,y) &= e^{-r(T-t)}xy,
\end{aligned}$$

which is a solution to the PDE.

(d) First let us solve for X. As we did in class, define $y = Xe^{\kappa t}$. Then

$$\begin{aligned}
dy &= e^{\kappa t}dX + Xe^{\kappa t}\kappa dt \\
&= e^{\kappa t}(\kappa\theta dt + \sigma dW(t))
\end{aligned}$$

and hence

$$\begin{aligned}
y(T) &= y(0) + \int_0^T (e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma dW(t)) \\
&= y(0) + [e^{\kappa T} - 1]\theta + \int_0^T e^{\kappa t}\sigma dW(t)
\end{aligned}$$

Implying that

$$X(T) = X(0)e^{-\kappa t} + [1 - e^{-\kappa T}]\theta + \int_0^T e^{-\kappa(T-t)} \sigma dW(t).$$

The Ornstein-Uhlenbeck process follows a normal distribution (since the dW integral only depends on time), so we only need to define the expected value and the standard deviation to characterize the distribution.

$$\begin{aligned}
E[X(T)] &= X(0)e^{-\kappa t} + [1 - e^{-\kappa T}]\theta \\
Var[X(T)] &= E\left[\left(\int_0^T e^{-\kappa(T-t)} \sigma dW(t)\right)^2\right] \\
&= E\left[\int_0^T e^{-2\kappa(T-t)} \sigma^2 dt\right] \\
&= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})
\end{aligned}$$

so

$$X(\infty) \sim N\left(\theta, \frac{\sigma^2}{2\kappa}\right).$$

The long-term distribution is then

$$p(x, \infty) = \frac{1}{\sqrt{2\pi(\sigma^2/2\kappa)}} e^{-(x-\theta)^2/(2(\sigma^2/2\kappa))} = \sqrt{\frac{\kappa}{\pi\sigma^2}}\, e^{-\kappa(x-\theta)^2/\sigma^2}.$$

Alternatively, one can solve the time-independent Fokker-Planck equation (which is an ODE in this case), using the condition that $\int_{-\infty}^{\infty} p(x, \infty)dx = 1$ to nail down the specific solution.

3. *Girsanov* (15 points): Consider the following economy in which there are two risky assets and one risk-free asset:

$$\begin{aligned}
\frac{dB}{B} &= rdt, \qquad r = 0.1, \\
\frac{dS_1}{S_1} &= 0.2dt + 0.2dW_1 + 0.7dW_2, \\
\frac{dS_2}{S_2} &= 0.3dt + 0.5dW_1 + 0.3dW_2.
\end{aligned}$$

Here, W_1 and W_2 are independent Wiener processes. Calculate the time 0 price, P, of a contingent claim that pays out $-W_2(t)$ at $t = 3$.

Answers:

First define

$$\sigma = \begin{bmatrix} 0.2 & 0.7 \\ 0.5 & 0.3 \end{bmatrix} \text{ and } \mu = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}.$$

Then compute the market price of risk vector

$$\theta = \sigma^{-1}\left(\mu - r\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.3793 \\ 0.0345 \end{bmatrix}.$$

Girsanov's theorem now implies that $W_i^Q(t) = W_i + \theta_i t$, $i = 1, 2$, is a martingale under the Q-measure, and because of normality $W_i(t)^Q \sim N(0, \sigma_i^2 t)$. Thus, $W_2(t) = W_2^Q(t) - \theta_2 t \sim N(-\theta_2 t, \sigma^2 t)$ under the Q-measure, and the pricing formula becomes

$$P = e^{-0.1\times 3} E^Q\left[-W_2(3)\right] = e^{-0.1\times 3} \times 0.0345 \times 3 = 0.0767.$$

4. *Barrier option (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$\begin{aligned} \frac{dB}{B} &= rdt, \\ \frac{dS}{S} &= rdt + \sigma dW^Q. \end{aligned}$$

(a) What is the price of a 3-year up-and-in call option with strike price $K = 100$ and barrier $L = 150$, given the parameters $r = 0.1$, $\sigma = 0.3$, and current stock price $S = 110$?

(b) What is the price of a 3-year up-and-out call option with the same parameters as in (a)?

Answers:

(a)

$$F^I(S; L, \Phi) = F(S; \Phi_L) + \left(\frac{L}{S}\right)^q F(L^2/S; \Phi^L), \qquad q = \frac{2r}{\sigma^2} - 1,$$

$$\begin{aligned} \Phi_L(S,K) &= \Phi(S,L) + (L-K)H(S,L) \\ \Phi^L(S,K) &= \Phi(S,K) - \Phi(S,L) - (L-K)H(S,L) \end{aligned}$$

where H is a digital option, with payoff $I\{x > L\}$, using the linearity of the pricing function

$$
\begin{aligned}
F^I(S;L,\Phi) &= F(S;\Phi(S,L)) + (L-K)\mathbf{H}(S,L) \\
&+ \left(\frac{L}{S}\right)^q \left(F(L^2/S;\Phi(L^2/S,K)) - F(L^2/S;\Phi(L^2/S,L)) - (L-K)\mathbf{H}(L^2/S,L)\right)
\end{aligned}
$$

where we know that the price of the digital option is

$$
\mathbf{H}(S,L) = e^{-r(T-t)} N\left[\frac{\ln(S/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right].
$$

Plugging everything in we get that $F^I(S;L,\Phi) = \$26.25$.

(b) Since the payoff of the vanilla call option, which has price \$41.84, is equal to the sum of the payoffs of the up-and-in and up-and-out call options, the price of the up-and-out call option is equal to \$41.84-\$26.25=\$15.59.

5. *Term structure (20 points)* Assume that the short rate follows the asset pricing dynamics specified by the Vasicek model:

$$
dr = (b - ar)dt + \sigma dW^Q, \qquad a > 0.
$$

(a) Does the model belong to the class of affine term structure models? Why/Why not?

(b) Derive the formula for the time-t price of a T-bond, $p(t,T|r)$.

Answers:

(a) If the pricing formula is of the form $p(t,r;T) = e^{A(t,T)-B(t,T)r}$, then the Vasicek model belongs to the class of affine structure models. From class and Björk, we know that short-rates dynamics on the form

$$
dr = (\alpha r + \beta)dt + \sqrt{\gamma r + \delta}\; dW_t^Q,
$$

leads to affine term structure pricing, so the Vasicek model indeed belongs to this class (with $\alpha = -a$, $\beta = b$, $\gamma = 0$, $\delta = \sigma^2$).

Let's use the martingale approach, to solve the model:

$$
\begin{aligned}
rpdt &= E^Q[dp] \\
rp &= p_t + p_r(b - ar) + \frac{1}{2}p_{rr}\sigma^2,
\end{aligned}
$$

with boundary condition $p(T, r; T) = 1$, for all r. Functions A and B to be determined by

$$\frac{\partial B}{\partial t} = aB(t,T) - 1,$$
$$\frac{\partial A}{\partial t} = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T),$$

and the boundary condition $p(T, r; T) = 1$ gives us the conditions $A(T,T) = B(T,T) = 0$.

Solving the ODE will determine the functions A and B, which will give us $p(t, T|r)$.

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

$$\begin{aligned} A(t,T) &= \int_t^T \left(\frac{1}{2}\sigma^2 B^2(s,T) - bB(s,T)\right) ds. \\ &= \int_t^T \left(\frac{\sigma^2}{2a^2}\left(1 - e^{-a(T-s)}\right)^2 - \frac{b}{a}\left(1 - e^{-a(T-s)}\right)\right) ds. \\ &= \frac{\sigma^2}{2a^2}\left(-\frac{2a(t-T) - 4e^{-a(T-t)} + e^{-2a(T-t)} + 3}{2a}\right) - \frac{b}{a}\left(\frac{e^{-a(T-t)} - 1}{a} + (T-t)\right). \end{aligned}$$

6. *Perpetual dividend paying contract (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$\frac{dB}{B} = rdt,$$
$$\frac{dS}{S} = rdt + \sigma dW^Q.$$

(a) What is the value of a contract that makes constant continuous dividend payment of δ per unit time in perpetuity, i.e., the instantaneous dividend payment is δdt.

(b)* Now, consider a perpetual contract that pays δdt, but only at points in time, t, when $S_t > K$, for some constant $K > 0$. Derive a formula for the value of such a contract as a function of the stock price, S (and other parameters in the economy). Hint: You may use the fact (which is easy to show) that the value is a continuously differentiable function of S everywhere, even though the dividend payments change discontinuously at $S = K$.

Answers:

(a) Note that the payments do not depend on S, so the formula is especially simple:

$$P(0) = \int_0^\infty e^{-rs}\delta ds = \frac{\delta}{r}.$$

(b) Using the PDE approach, we need to solve for two pricing functions, one when the stock price is above K and another when S is below K. When $S > K$ we have (from Feynman-Kac's theorem on time independent form, or equivalently from the generalized B-S equations that include portfolio dividends, also on time independent form):

$$\begin{aligned} rP &= \frac{1}{dt}\mathrm{E}_t^Q\left[dP + \delta\, dt\right] \\ &= rVP_s + \frac{\sigma^2}{2}S^2 P_{ss} + \delta. \end{aligned}$$

The *general solution* to this ODE is: $P(S) = AS + BS^{-q}$ for some constants A and B, where the constant $q = \frac{2r}{\sigma^2}$. One *particular solution* is $P = \frac{\delta}{r}$.

When $S < K$ we have:

$$\begin{aligned} rP &= \frac{1}{dt}\mathrm{E}_t^Q\left[dP\right] \\ &= rVP_s + \frac{\sigma^2}{2}S^2 P_{ss}. \end{aligned}$$

The *general solution* to this ODE is the same as above: $P(S) = CS + DS^{-q}$ for some constants C and D. So we have

$$P(S) = \begin{cases} AS + BS^{-q} + \frac{\delta}{r}, & S \geq K, \\ CS + DS^{-q}, & S < K. \end{cases}$$

To pin down A, B, C and D we use the following boundary conditions

$$\begin{aligned} P(+\infty) &= \frac{\delta}{r}, \\ P(0) &= 0, \\ P_-(K) &= P_+(K), \\ P_-^{'}(K) &= P_+^{'}(K). \end{aligned}$$

The first and the second conditions imply that $A = 0$ and $D = 0$ respectively. The third and the fourth pin down B and C:

$$\begin{aligned} BK^{-q} + \frac{\delta}{r} &= CK \\ -qBK^{-q-1} &= C \end{aligned}$$

The solution is

$$\begin{aligned} B &= -\frac{\delta K^q}{r(1+q)}, \\ C &= \frac{q\delta}{Kr(1+q)}, \end{aligned}$$

leading to

$$P(S) = \begin{cases} \frac{\delta}{r} - \frac{\delta}{r(1+q)} \left(\frac{K}{S}\right)^q, & S \geq K, \\ \frac{q\delta}{r(1+q)} \frac{S}{K}, & S < K. \end{cases}$$

Alternatively, we could have used the risk neutral approach, but the integrals this approach leads to are quite difficult to solve. Thus, as is often the case with perpetual contracts, the PDE approach is more straightforward for this problem.