MFE230Q: Midterm Exam, April 23, 2012

NAME: **ID:**

Please motivate your answers. Please underline your final answers.

1. *One-period model*: Consider the following one-period market, with the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 0.8 \\ 0.5 \\ 0.2 \end{bmatrix}.$$

The probabilities for different events are $\mathbb{P}(\omega_1) = 0.2$, $\mathbb{P}(\omega_2) = 0.3$, $\mathbb{P}(\omega_3) = 0.5$.

(a) (5 points) Is there a risk-free rate ($R = 1 + r$) in this market? If so, what is it?

(b) (5 points) Is the market complete?

(c) (5 points) Brian Debt (who has just started a hedge fund) claims that prices are clearly nonsensical in this market, and that there is an arbitrage opportunity because asset 3 is under-priced. Is he correct?

(d) (5 points) Define a stochastic discount factor, M (i.e., M_1), in this economy.

Answers:

(a) Yes. The first security is the risk free asset, since it pays the same return in all three states of the world. $R = 1/0.8 = 1.25$.

(b) Yes, the three available assets span R^3 (since the 3-by-3 matrix $\mathbf{D}$ is invertible).

(c) He's wrong. It just so happens that state 3 is a state of the world in which the state price is very small. One can show that there's no arbitrage by showing that all state prices are positive:

$$\Psi = \mathbf{D}^{-1}\mathbf{s}^0 = (0.32, 0.46, 0.02)' \gg \mathbf{0}.$$

What Mr. Debt might actually mean is that there is mispricing in the sense of a "statistical arbitrage," but this is not a true arbitrage as defined in asset pricing theory.

(d) $V^0 = E_P[MV^1]$. Using the state prices we can pin down the SDF:

$$\Psi_i = P(\omega_i) \times M_i \times 1$$

so $M = \left(\frac{0.32}{0.2}, \frac{0.46}{0.3}, \frac{0.02}{0.5}\right)' \approx (1.6, 1.53, 0.04)'$.

2. *Itô processes & Itô's lemma*: Throughout this question, assume that W_t is a standard Brownian motion.

(a) (5 points) Consider the Itô process $X_t = e^{W_t^2}$. Calculate the Itô differential of X_t, dX_t.

(b) (5 points) Use Itô's lemma to prove that $\int_0^T W_t^2 dW_t = \frac{1}{3}W_T^3 - \int_0^T W_t dt$.

(c) (5 points) Assume that $dX_t = W_t^2 dt - t^2 dW_t + 2W_t t(dW_t - dt)$. Calculate $\int_0^T dX_t$.

Answers:

(a) $dX_t = 2W_t X_t dW + X_t(1 + 2W_t^2)dt$

(b) Apply Ito's to $Z = \frac{1}{3}W_t^3$, to get: $dZ_t = W_t^2 dW_t + W_t dt$, which is equivalent to $\frac{1}{3}W_T^3 = \int_0^T W_t^2 dW_t + \int_0^T W_t dt$. Rearrange and you get the result.

(c) Apply Ito's to $Z_t = W_t^2 t$ and to $Y_t = W_t t^2$, to get $dZ_t = W_t^2 dt + 2W_t t dW_t + t dt$ and $dY_t = 2tW_t dt + t^2 dW_t$, rearrange to get:

$$\begin{aligned} dX_t &= W_t^2 dt + 2W_t t dW_t - 2tW_t dt - t^2 dW_t \\ &= dZ_t - dY_t - t dt \end{aligned}$$

equivalent to

$$X(T) = W(T)^2 T - W(T)T^2 - \frac{T^2}{2}.$$

3. *Continuous time trading*: Consider the continuous time economy with three assets, S^1, S^2 and B, and price dynamics

$$
\begin{aligned}
\frac{dS^1}{S^1} &= \mu_1 dt + \sigma_1 dW_t, \qquad S^1(0) = 1, \\
\frac{dS^2}{S^2} &= \mu_2 dt + \sigma_2 dW_t, \qquad S^2(0) = 1, \\
\frac{dB}{B} &= r dt, \qquad B_0 = 1.
\end{aligned}
$$

Here, μ_1, μ_2, σ_1, σ_2, and r are positive constants. Note that there is only one Wiener process, W_t, that drives the dynamics of both S^1 and S^2. As in class, we summarize the asset dynamics in the vector $\mathbf{s}_t = (S_t^1, S_t^2, B_t)'$ and a trading strategy in the vector $\mathbf{h}_t = (h_t^1, h_t^2, h_t^3)'$. Neither of the assets pay dividends, Thus, using the notation in class, $\Theta \equiv (0,0,0)'$.

(a) (5 points) Assume that an investor chooses the portfolio investment strategy $\mathbf{h}_t = \left(\frac{a_1}{S_1}, \frac{a_2}{S_2}, 0\right)'$. Derive expressions for the value process, $V_t^{\mathbf{h}}$, and the cumulative dividend process, $F_t^{\mathbf{h}}$ of this strategy.

(b) (5 points) Use your results in (a) to derive a condition on μ_1, μ_2, σ_1, σ_2, and r, that needs to be satisfied for there to be no arbitrage in this economy.

Answers:

(a) $V_t^h = \frac{a_1}{S_1} a_1 + \frac{a_2}{S_2} a_2 = a_1 + a_2$.

For dF_t, we can use the formula $dV_t + dF_t = \mathbf{h}' d\mathbf{s}$ (since the assets pay no dividends), and since we know that V_t is constant, $dV_t = 0$, this reduces to

$$dF_t = \mathbf{h}' d\mathbf{s} = \frac{a_1}{S_1} S_1(\mu_1 dt + \sigma_1 dW_t) + \frac{a_1}{S_2} S_2(\mu_2 dt + \sigma_2 dW_t) = (a_1\mu_1 + a_2\mu_2)dt + (a_1\sigma_1 + a_2\sigma_2)dW_t.$$

Alternatively, we can use the formula $dF_t^h = -dh_t'(s_t + ds_t)$, and calculate

$$
\begin{aligned}
d\mathbf{h}_t^i &= -\frac{a_i}{S_i^2} dS_i + \frac{a_i}{S_i^3} dS_i^2 \\
&= -\frac{a_i}{S_i}(\mu_i dt + \sigma_i dW_t) + \frac{a_i}{S_i}\sigma_i^2 dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d\mathbf{h}_t^i S_t^i &= \left(a_i\sigma_i^2 - a_i\mu_i\right) dt - a_i\sigma_i dW_t, \\
d\mathbf{h}_t^i dS_t^i &= -a_i\sigma_i^2 dt.
\end{aligned}
$$

So, again, $dF_t^h = (a_1\mu_1 + a_2\mu_2)dt + (a_1\sigma_1 + a_2\sigma_2)dW_t$

(b) Since the two risky assets depend on the same risk-factor, we can choose a portfolio (a_1, a_2) that generate risk-free payoffs. Given that the market permits no arbitrage, the value of this portfolio must grow at the risk-free rate. Specifically, given our solution in question (a), if

$$a_1\sigma_1 + a_2\sigma_2 = 0$$

(or, equivalently, $a_1 = -a_2\frac{\sigma_2}{\sigma_1}$) then $dF = (a_1\mu_1 + a_2\mu_2)dt$, and $V_t = a_1 + a_2$. However, it also follows that the strategy of investing $V_t = a_1 + a_2$ (dollars) in the risk-free asset will generate instantaneous payoffs of $dF_t = r(a_1 + a_2)dt$. So noarbitrage implies that

$$a_1\mu_1 + a_2\mu_2 = r(a_1 + a_2),$$

which when plugging in $a_1 = -a_2\frac{\sigma_2}{\sigma_1}$ and rearranging leads to

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}.$$

This is very intuitive: Two assets that depend on the same risk-factor will have the same market price of risk, i.e., the same so-called Sharpe ratio.

4. *SDEs*:

(a) (5 points) Consider the stochastic differential equation

$$dX_t = \frac{X_t}{2}dt + \sqrt{1 + X_t^2}\, dW_t, \qquad X_0 = 0.$$

where W_t is a Wiener process. What can we say about the general existence and uniqueness of solution(s) to this SDE?

(b) (5 points) Recall the calculus of hyperbolic functions: $\cosh^2(x) - \sinh^2(x) = 1$, $\sinh'(x) = \cosh'(x)$, $\cosh'(x) = \sinh'(x)$, where $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Use these relations to conjecture and verify a solution to the SDE in (a).

Answers:

(a) One needs to check that the two boundedness conditions in the existence theorem are satisfied, and the regularity conditions. If these are satisfied, then existence and uniqueness follows immediately.

First, note that $\mu(x) = \frac{x}{2}$, and $\sigma(x) = \sqrt{1 + x^2}$. Clearly, these functions are continuously differentiable for all x, so the regularity condition is satisfied.

Since this is a one-dimensional equation, the norm is just the absolute value: $\|x\| = |x|$. We note that the function $\sigma(x) = \sqrt{1+x^2}$ has derivative $|\sigma'(x)| = \frac{|x|}{\sqrt{1+x^2}} \leq 1$, immediately implying that $|\sigma(x) - \sigma(y)| \leq |x-y|$, as well as $|\sigma(x)| \leq 1 + |x|$.

Second, we therefore have:

$$\begin{aligned} \|\mu(x) - \mu(y)\| + \|\sigma(x) - \sigma(y)\| &= \frac{\|x-y\|}{2} + \|\sigma(x) - \sigma(y)\| \\ &\leq \frac{\|x-y\|}{2} + \|x-y\| \\ &\leq 1.5\|x-y\|. \end{aligned}$$

Third, it follows immediately that

$$\begin{aligned} \|x\| + \|\sqrt{1+x^2}\| &\leq \frac{\|x\|}{2} + 1 + \|x\| \\ &\leq 1.5(1 + \|x\|). \end{aligned}$$

The conditions together guarantee uniqueness and existence.

(b) Conjecture the solution $X_t = \sinh(W_t)$. The differential is $dX_t = \cosh(W_t)dW_t + \frac{1}{2}\sinh(W_t)dt$, so if we substitute $X_t = \sinh(W_t)$ in the right-hand side, we get

$$\begin{aligned} dX &= \frac{1}{2}\cosh(W_t)dt + \sqrt{1 + \sinh(W_t)^2}dW_t \\ &= \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dW_t. \end{aligned}$$

Finally, $\sinh(0) = 0$, so the conjectured solution satisfies the initial condition, and $X_t = \sinh(W_t)$ is therefore indeed the unique solution to the SDE.