MFE230Q: Assignment 3 - Due April 13, 2021

1. (*Exercises 4.1, 4.2 Björk*) Compute the stochastic differential dx when $W(s)$ is a standard Brownian motion (a Wiener process) and

   (a) $x(t) = e^{at}$

   (b) $x(t) = \int_0^t g(s)\, dW(s)$

   (c) $x(t) = e^{\alpha W(t)}$

   (d) $x(t) = e^{\alpha y(t)}$, where $dy = \mu\, dt + \sigma\, dW$

   (e) $x(t) = y^2(t)$, where $dy = \alpha y\, dt + \sigma y\, dW$

   (f) $x(t) = \frac{1}{y(t)}$, where $dy = \alpha y\, dt + \sigma y\, dW$

2. (*Exercise 4.3 Björk*): Let σ_t be a deterministic function of time and define the process X by

$$X(t) = \int_0^t \sigma_s\, dW_s. \tag{1}$$

Show that the so-called *Characteristic Function* of X_t is given by

$$\mathrm{E}_0\left[e^{iuX(t)}\right] = e^{-\frac{u^2}{2}\int_0^t ds\, \sigma_s^2} \tag{2}$$

Here, i is the so-called 'imaginary number' in that $i^2 = -1$. Characteristic functions are important since their *Fourier Transform* gives the probability density $\pi\left(X(t)|\, X(0)\right)$. They are also closely related to the so-called *Moment Generating Function*, which allows us to determine mean, variance, etc.

3. (*Exercise 4.4 Björk*): Suppose the dynamics of X_t follows

$$dX = \alpha X\, dt + \sigma_t\, dW, \tag{3}$$

where α is any real number and σ_t is any stochastic process. Identify $\mathrm{E}_0\left[X(t)\right]$.

4. (*Exercise 4.8 Björk*): Suppose the dynamics of (X_t, Y_t) follow the processes:

$$dX = \alpha X\, dt - Y\, dW \tag{4}$$

$$dY = \alpha Y\, dt + X\, dW. \tag{5}$$

(let me emphasize that there is only one BM here). Assume X_0, Y_0 are given constants. Identify the process dR, where $R(X, Y) = X^2 + Y^2$.

5. (*Exercise 5.1 Björk*): You are given the initial value X_0 and the X dynamics

$$dX = \alpha X \, dt + \sigma \, dW, \tag{6}$$

where α and σ are constants.

(a) Determine the dynamics of dY, where $Y(X, t) = e^{-\alpha t} X$

(b) Using 5a), show that we can formally integrate the dynamics of X via

$$X(t) = e^{\alpha t} X(0) + \sigma \int_0^t e^{\alpha(t-s)} \, dW_s. \tag{7}$$

(c) Intuitively argue why $X(t)|\, \mathcal{F}_0$ is normally distributed.
Then determine $\mathrm{E}_0 \left[X(T) |\, X(0) \right]$ and $\mathrm{Var}_0 \left[X(T) |\, X(0) \right]$.