MFE230Q: Assignment 4 - Due May 4, 2021
Numerical Option Pricing

Consider the Black-Scholes economy

$$\frac{dS}{S} = r\,dt + \sigma\,dW^Q, \quad (1)$$

$$\frac{dB}{B} = r\,dt. \quad (2)$$

Here, the risk free rate r, and volatility σ are all constant. Further, consider the digital option that pays out 1 dollar at time T if $S_T \geq K$, for some constant $K > 0$. Thus, the option is a simple contingent claim with payout function $\Phi(S_T) = 1_{S_T \geq K}$. Our goal is to study how well two of the fundamental types of numerical methods (binomial tree and Monte Carlo) perform for this claim. Assume that $r = 0.1$, $\sigma = 0.16$, $S_0 = 100$, $T = 1$ and $K = 110$.

1. Use the Black-Scholes methodology to derive a closed form expression for the value of the digital option.

2. Use the N-period binomial tree model to determine the price of the digital option. Plot the estimated value as a function of number of steps. What value of N is needed to nail the price down to an error of less than one cent?[1]

3. Use a Monte-Carlo approach to price the option, both with and without antithetic paths. In each case, how many sample paths, M, are needed to nail the price down to an error of less than a cent, with 95% likelihood?

4. Compare the two numerical methods. Is one superior to the other for this problem? Which dimensions are important when comparing the methods?

[1] Specifically, find the smallest N such that for any $N' \geq N$, the difference between the true price and the approximated price with an N'-step binomial tree is less than a cent.