MFE230Q: In-Class Quiz # 2

1. **Ito's lemma**: Define $X(W_t) = W_t^2$, where W_t is a standard Brownian motion.

   (a) Determine dX.

   (b) Determine $\mathrm{E}_0\left[X(W_T)\right]$.

2. **More Ito's lemma**: Define $Y(X_t) = X_t^2$, where X_t follows an arithmetic Brownian motion process

$$dX = \mu\, dt + \sigma\, dW.$$

   Here, both μ and σ are constants.

   (a) Determine dY.

   (b) Is dY distributed normally? If so, how is this possible, given that Y is bounded above by zero?

   (c) Is Y Markov in itself? That is, can the drift and diffusion of dY be expressed as a function of only Y? Explain intuitively why or why not. (This is a bit tricky)

3. **Continuous time model:** Using the notation in the continuous time model of the Lecture Notes, part 1.2b, consider the economy with one stock with GBM dynamics for prices and constant dividend-yield, α, and one risk free asset with constant returns:

$$\begin{aligned} \frac{dS}{S} &= \hat{\mu} dt + \sigma dW, \qquad S_0 = 1, \\ \frac{dB}{B} &= r dt, \qquad B_0 = 1 \\ d\Theta &= (\alpha S dt, 0)', \end{aligned}$$

   $\alpha > 0$, μ and $\sigma > 0$ constant.

   Derive the value and cumulative dividend processes, V_t and $F_t^{\mathbf{h}}$, for each of the following three trading strategies:

   (a) $\mathbf{h}_t = (0, e^{-rt})'$,

   (b) $\mathbf{h}_t = (e^{\alpha t}, 0)'$,

   (c) $\mathbf{h}_t = \left(\frac{1}{\sigma S_t}, -\frac{1}{\sigma B_t}\right)'$, assuming $\alpha = 0$.

4. **Kolmogorov equations:** Solve the PDE (i.e., find $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ that satisfies the PDE and initial condition),

$$f_t + \mu f_x = \frac{\sigma^2}{2} f_{xx},$$

$$
\begin{aligned}
f(0,x) &= \delta_{x_0}(x), \\
t &\geq 0, \\
x &\in \mathbb{R}.
\end{aligned}
$$

Here, μ, $\sigma > 0$ are constants, and $\delta_{x_0}(x)$ is the Dirac "delta"-function defined in the distributional sense as $\delta_{x_0}(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \varphi\left(\frac{x - x_0}{\epsilon}\right)$, for some smooth nonnegative function, φ, with compact support that satisfies $\int \varphi(x) dx = 1$.

Solutions

1. :

   (a) $X = W^2$ implies that $X_W = 2W$, $X_{WW} = 2$. Thus, we have

$$\begin{aligned} dX &= X_W \, dW + \frac{1}{2} X_{WW} \, dW^2 \\ &= 2W \, dW + dW^2 \\ &= dt + 2W \, dW. \end{aligned} \tag{1}$$

   In particular, in contrast to standard calculus,

$$dX = d\left(W^2\right) \neq 2W \, dW. \tag{2}$$

   (b) Formally integrating, we have

$$X(T) = X(0) + \int_0^T dt + \int_0^T 2W \, dW. \tag{3}$$

   Using $X(0) = W^2(0) = 0$ and $\mathrm{E}_0\left[\int_0^T 2W \, dW\right] = 0$ we have

$$\mathrm{E}_0\left[X(T)\right] = \mathrm{E}_0\left[W^2(T)\right] = T. \tag{4}$$

2. :

   (a) $Y = X^2$ implies that $Y_X = 2X$, $Y_{XX} = 2$. Thus, we have

$$\begin{aligned} dY &= Y_X \, dX + \frac{1}{2} Y_{XX} \, dX^2 \\ &= 2X\left(\mu \, dt + \sigma \, dW\right) + \sigma^2 \, dt \\ &= \left(2\mu X + \sigma^2\right) dt + 2\sigma X \, dW. \end{aligned} \tag{5}$$

   (b) Ito's lemma states that *every differential* $dY(X)$ is distributed normally (assuming that dX does not jump). Note: this does not say that all variables over *finite* intervals $Y(T)|\mathcal{F}_0$ are distributed normally. Clearly, $Y_T = X_T^2$ cannot be distributed normally, since Y_T can only take on positive values. But Ito's lemma does state that

$$Y_{t+dt}\big|\,\mathcal{F}_t \overset{dt \to 0}{\sim} N\left(Y_t + \left(2\mu X_t + \sigma^2\right) dt,\, 4\sigma^2 X_t^2 \, dt\right). \tag{6}$$

   The emphasis is that dY is normally distributed only because we are looking forward over an infinitesimal time dt.

(c) One might be tempted to argue that $Y = X^2$ so that $X = \sqrt{Y}$, and then write dY dynamics as

$$dY \overset{?}{=} \left(2\mu\sqrt{Y} + \sigma^2\right) dt + 2\sigma\sqrt{Y}\, dW, \tag{7}$$

but this is incorrect. Why? Because $X = \pm\sqrt{Y}$, and this makes all the difference. For example, let's say that $\mu = 1$, $\sigma = 1$ and $Y_t = 4$. There are two possibilities: $X_t = 2$ and $X_t = -2$. If $X_t = 2$, then $\mathrm{E}\,[dY] = (2)(1)(2) + 1 = 5$, so Y is expected to increase. In contrast, if $X_t = -2$, then $\mathrm{E}\,[dY] = (2)(1)(-2) + 1 = -3$, so Y is expected to decrease. In this case, it would be helpful to know the recent past values of Y, because if they had been increasing, it is more likely that $X_t = 2$, but if they had been decreasing, it is more likely that $X_t = -2$. The fact that knowing past values helps you predict future values better implies that Y by itself is not Markov. Note, however, that X is Markov in itself (since μ and σ are constants) and that $\{X, Y\}$ are jointly-Markov.

3. From the lecture notes, we know that $S_t = e^{\mu t + \sigma W_t}$, $B_t = e^{rt}$, $\mu = \hat{\mu} - \sigma^2/2$.

(a) $V_t = \mathbf{h}_t'\mathbf{s}_t = (0, e^{-rt})(S_t, e^{rt})' \equiv 1$, $dF_t^{\mathbf{h}} = -d\mathbf{h}_t'(\mathbf{s}_t + d\mathbf{s}_t) + \mathbf{h}'d\Theta_t = -(0, -re^{-rt}dt)(S_t + dS_t, e^{rt} + re^{rt}dt)' + (0, e^{rt})(\alpha S_t dt, 0)' = rdt$. So, $F_t^{\mathbf{h}} = \int_0^t r dt = rt$.

The interpretation is that this trading strategy is equivalent to depositing money in the bank and in each "period" collecting the interest payments so that the money on the bank never grows. By this rebalancing, portfolio "dividends" are thus generated, even though the bank deposit/bond does not make dividend (coupon) payments.

(b) $V_t = \mathbf{h}_t'\mathbf{s}_t = (e^{\alpha t}, 0)(e^{\mu t + \sigma W_t}, B_t)' = e^{(\mu+\alpha)t + \sigma W_t}$, $dF_t^{\mathbf{h}} = -d\mathbf{h}_t'(\mathbf{s}_t + d\mathbf{s}_t) + \mathbf{h}'d\Theta_t = -(\alpha e^{\alpha t}dt, 0)(S_t + dS_t, B_t + dB_t)' + (e^{\alpha t}, 0)(\alpha S_t dt, 0)' = 0$. So, $F_t^{\mathbf{h}} \equiv 0$. It also follows immediately that V_t satisfies the SDE

$$\frac{dV_t}{V_t} = (\hat{\mu} + \alpha)dt + \sigma dW_t.$$

The interpretation is that this trading strategy reinvests all dividends, so that it is self-financing, thereby achieving a higher growth rate, $\hat{\mu} + \alpha$. Thus, this strategy does the opposite of the strategy in (a).

(c) $V_t = \mathbf{h}_t'\mathbf{s}_t = \left(\frac{1}{\sigma S_t}, -\frac{1}{\sigma B_t}\right)(S_t, B_t)' \equiv 0$,

$$d\mathbf{h}' = \frac{1}{\sigma}\left(-\frac{dS_t}{S_t^2} + \frac{1}{2}\frac{2}{S_t^3}(dS_t)^2, \frac{dB_t}{B_t^2}\right) = \frac{1}{\sigma}\left(\frac{1}{S_t}\left(-\hat{\mu}dt - \sigma W_t + \sigma^2 dt\right), \frac{r}{B_t}dt\right).$$

So,

$$dF_t^{\mathbf{h}} = -d\mathbf{h}_t'(\mathbf{s}_t) - d\mathbf{h}_t' d\mathbf{s}_t = \left(\frac{\hat{\mu} - r}{\sigma}dt + dW_t - \sigma\, dt\right) + (\sigma\, dt) = \lambda\, dt + dW_t,$$

where $\lambda = \frac{\hat{\mu} - r}{\sigma}$, leading to $F_t^{\mathbf{h}} = \lambda t + W_t$.

The interpretation is that at each point in time, the expected profit of taking on one unit of dW_t risk is $\lambda\, dt$. In other words, the *market price of W-risk* is λ. $F_t^{\mathbf{h}}$ represents the cumulative profits of financing such risky bets with borrowed money.

4. The PDE satisfies the Fokker-Planck equations for the SDE

$$dX = \mu dt + \sigma dW_t, \qquad X_0 = x_0.$$

The solution to this SDE is a Brownian motion with distribution $X_t \sim N(\mu t + x_0, \sigma^2 t)$. The pdf of X_t is thus $p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}}$.

Therefore, via the Fokker-Planck formula, $p(t,x)$ is the solution to the PDE.

As an extra test, you can verify that $p(t,x)$ indeed satisfies Fokker-Planck's PDE by differentiating p and plugging the terms into the PDE.