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Chapter 8

Completeness and Hedging

Exercise 8.1 In the given model we have

- Two random sources: the Wiener process W and the jump process N (since it has fixed jump size λ).
- One traded assets (excluding the risk-free asset), which is the stock

Using the so called "Meta-Theorem" we can conclude that:

(a) The market is free of arbitrage.

(b) But it is not complete.

(c) Consequently, there will not be a unique price for a given European Call Option.

(d) It is, however, possible to replicate the call maturing in January using a portfolio consisting of the underlying stock, a call maturing in December and the risk free asset. This is possible, because if we take one option as given, the we implicitly assume two traded assets as given (stock and the given ooption) and the model becomes complete.

Exercise 8.2

(a) To be done!

(b) We have $F(t, s, z)$ be defined by

$$\begin{aligned} F_t + r \cdot s \cdot F_s + \frac{1}{2}\sigma^2 s^2 F_{ss} + gF_z &= rF \\ F(T, s, z) &= \Phi(s, z) \end{aligned}$$

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and the dynamics under Q for S and Z

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ dZ(u) &= g(u, S(u))du \end{aligned}$$

We want to show that $F(t, S(t), Z(t)) = e^{-r(T-t)} E^Q_{t,s,z} [\Phi(S(T), Z(T))]$.

For that we find , by Itô, the dynamics of $\Pi(t) = F(t, S(t), Z(t))$, the arbitrage free pricing process

$$\begin{aligned} d\Pi(t) &= F_t dt + F_s [(rS(t)dt + \sigma S(t)dW(t)] + F_z \cdot g(t, S(t))dt + \frac{1}{2} F_{ss}\sigma^2 S^2(t)dt \\ &= \underbrace{\left[F_t + r \cdot S(t) \cdot F_s + \frac{1}{2}\sigma^2 S^2(t) F_{ss} + g(t, S(t))F_z\right]}_{r\Pi(t)} + \sigma S(t) F_s dW(t) \end{aligned}$$

Integrating we have

$$\Pi(T) = \Pi(t) + r \int_t^T \Pi(u)du + \sigma \int_t^T S(u)F_s dW(u)$$

Hence

$$E^Q_{t,z,s} [\Pi(T)] = \Pi(t) + r \int_t^T E^Q_{t,z,s} [\Pi(u)] \, du$$

So, using the usual "trick" of setting $m(u) = E^Q_{t,z,s} [\Pi(u)]$ and using techniques of ODE we finally get

$$\Pi(t) = F(t, S(t), Z(t)) = e^{-r(T-t)} E^Q_{t,s,z} [\Phi(S(T), Z(T))] .$$

(Remember that $\Pi(T) = F(T, S(T), Z(T)) = \Phi(S(T), Z(T))$.)

Exercise 8.3 The price arbitrage free price is given by (note that this time our claim is *not* simple, i.e. it is not of the form $\mathcal{X} = \Phi(S(T))$).

$$\begin{aligned} \Pi(t) &= e^{-r(T_2-t)} E^Q_t [\mathcal{X}] \\ &= e^{-r(T_2-t)} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E^Q_t [S(u)] \, du \end{aligned}$$

We know that under Q

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ S(t) &= s \end{aligned}$$

So,

$$\Rightarrow E^Q_t [S(u)] = se^{r(u-t)}$$

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} se^{r(u-t)} du = \frac{1}{T_2 - T_1} \frac{s}{r} \left[e^{r(T_2-t)} - e^{r(T_1-t)}\right]$$

The price to the "mean" contract is thus

$$\Pi(t) = \frac{s}{r(T_2 - T_1)} \left[1 - e^{-r(T_2-T_1)}\right] .$$

Exercise 8.4 To be done!!!

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Chapter 9

Properties of the Pricing Formulas

Exercise 9.1 The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} K, & \text{if } S(T) \leq A \\ K + A - S(T), & \text{if } A < S(T) < K + A \\ 0, & \text{otherwise.} \end{cases},$$

has then following contract function (recall that $\mathcal{X} = \Phi S(T)$)

$$\Phi(x) = \begin{cases} K, & \text{if } x \leq A \\ K + A - x, & \text{if } A < x < K + A \\ 0, & \text{otherwise.} \end{cases},$$

which can be decomposed into the following "basic" contract functions written

$$\Phi(x) = K \cdot \underbrace{1}_{\Phi_B(x)} - \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - A - K]}_{\Phi_{c,A+K}(x)}.$$

Having this T-claim $\mathcal{X}$ is then equivalent to having the following (replicating) portfolio at time T:

* K in monetary units
* short (position in) a call with strike A
* long (position in) a call with strike $A + K$

Given the decomposition of the contract function Φ into basic contract functions, we immediately have that the arbitrage free pricing process Π is

$$\Pi(t) = K \cdot \overbrace{e^{-r(T-t)}}^{B(t)} - c(s, A, T) + c(s, A + K, T)$$

where $c(s, A, T)$ and $c(s, A + K, T)$ stand for the prices of European call options on S and maturity T with strike prices A and $A + K$, respectively. The Black-Scholes formula give us both $c(s, A, T)$ and $c(s, A + K, T)$.

The hedge portfolio thus consists of a reverse position in the above components, i.e., borrow $e^{-r(T-t)}K$, buy a call with strike K and sell a call with strike $A + K$.

Exercise 9.2 We apply the exact same technique as before. First note that since our T-claim $\mathcal{X}$ is given by

$$\mathcal{X} = \begin{cases} K - S(T), & \text{if } 0 \leq S(T) \leq K \\ S(T) - K, & \text{if } K < S(T) \end{cases},$$

it has a contract function Φ that can be written as

$$\Phi(x) = \underbrace{\max[0, x - K]}_{\Phi_{c,K}(x)} + \underbrace{\max[0, K - x]}_{\Phi_{p,K}(x)}.$$

Having the *straddle* is then equivalent to having the following constant(replicating) portfolio at time T:

* (long position in) a call option with strike K
* (long positions in) a put option with the same strike K

The arbitrage free pricing process Π follows immediately from the decomposition of our contract function Φ, thus,

$$\Pi(t) = c(s, K, T) + p(s, K, T)$$

where $c(s, K, T)$ and $p(s, K, T)$ stand, respectively, for the prices of European call and put options on S with maturity T and strike price K . So the Black-Scholes formula give us $c(s, K, T)$ and using the put-call parity for instance we get $p(s, K, T)$.

The hedge portfolio thus consists of a reverse position in the above components, i.e., sell both a call and a put on S, with maturity T and strike K.

Exercise 9.3 We apply, once again, the exact same technique. Since the T-claim $\mathcal{X}$ is given by:

$$\mathcal{X} = \begin{cases} B, & \text{if } S(T) > B \\ S(T), & \text{if } A \leq S(T) \leq B \\ A, & \text{if } S(T) < A. \end{cases},$$

the implied contract function Φ can be written as

$$\Phi(x) = A \cdot \underbrace{1}_{\Phi_B(x)} + \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} - \underbrace{\max[0, x - B]}_{\Phi_{c,B}(x)}.$$

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Having this *bull spread* is then equivalent to having the following constant(replicating) portfolio at time T:

* A in monetary units
* long (position in) a call option with strike A
* short (position in) a call option with strike B

The arbitrage free pricing process Π follows once again immediately from the decomposition of the contract function

$$\Pi(t) = A \cdot \overbrace{e^{-r(T-t)}}^{B(t)} + c(s, A, T) - c(s, B, T)$$

where $c(s, A, T)$ and $c(s, B, T)$ stand for the prices of European call options on S, with maturity T and strike prices A and B, respectively, and can be computed using the Black-Scholes formula.

The hedge portfolio consists of a reverse position in the above components, i.e., borrow $Ae^{-r(T-t)}$, sell a call with strike A and buy another with strike B.

Exercise 9.4 We apply, once again, the exact same technique. The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} 0, & \text{if } S(T) < A \\ S(T) - A, & \text{if } A \leq S(T) \leq B \\ C - S(T), & \text{if } B < S(T) \leq C \\ 0, & \text{if } S(T) > C. \end{cases}$$

where $B = \frac{A+C}{2}$, has a contract function Φ that can be written as

$$\Phi(x) = \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - C]}_{\Phi_{c,C}(x)} - 2\underbrace{\max[0, x - B]}_{\Phi_{c,B}(x)}$$

Having this *butterfly* is then equivalent to having the following constant(replicating) portfolio at time T:

* long (position in) a call option with strike A
* long (position in) a call option with strike C
* short (position in) a call option with strike B

The arbitrage free pricing process Π follows immediately from the decomposition of the contract function Φ and is given by

$$\Pi(t) = c(s, A, T) + c(s, C, T) - 2c(s, B, T)$$

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where $c(s, A, T)$, $c(s, B, T)$ and $c(s, C, T)$ stand for the prices of European call options on S, with maturity T and strike prices A, B and C, respectively, and can be computed using the Black-Scholes formula.

The hedge portfolio consists of a reverse position in the above components, i.e., sell two call options one with strike A and other with strike B and buy other two both with strike B.

Exercise 9.5 We have a portfolio P and two derivatives F and G. In order to delta-hedge our portfolio we need to combine the two derivatives in a way such that

$$u_F.\Delta_F + u_G.\Delta_G = -\Delta_P,$$

since, in addition we what to gamma-hedge we also need

$$u_F.\Gamma_F + u_G.\Gamma_G = -\Gamma_F$$

Where u_F and u_G are the quantities of the derivatives F and G, respectively, that should be bought (or sold if negative in value).

So we just need to solve the system

$$\begin{cases} -u_F + 5u_G = -2 \\ 2u_F - 2u_G = -3 \end{cases}$$

$$\begin{cases} u_F = -\dfrac{19}{8} \\ u_G = -\dfrac{7}{8} \end{cases}$$

the hedging strategy is then to short $\frac{19}{8}$ of derivative F and $\frac{7}{8}$ of derivative G.

Exercise 9.6 To be done!!!

Exercise 9.7 To be done!!!

Exercise 9.8 Consider the value (pricing function) of the portfolio $P(t, s)$. Recall that

$$\begin{aligned} \Theta_P &= \frac{\partial P}{\partial t} \\ \Delta_P &= \frac{\partial P}{\partial s} \\ \Gamma_P &= \frac{\partial^2 P}{\partial s^2} \end{aligned}$$

So the following two equations are equivalent and are just the Black-Scholes equation

$$\begin{aligned} \Theta_P + \Delta_P + rs\Delta_P + \frac{1}{2}\sigma^2 s^2 \Gamma_P &= rP \\ \frac{\partial P}{\partial t} + rs\frac{\partial P}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P}{\partial s^2} &= rP. \end{aligned}$$

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Exercise 9.9 To be done!!!

Exercise 9.10 From the put-call parity we have that

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - S$$

where $p(t,s)$ and $c(t,s)$ stand for the price of a put and a call option on S with maturity T and strike price K.

The *delta* measures the variation in the price of a derivative with respect to changes in the value of the underlying. Differentiating the put-call parity w.r.t. S we have

$$\begin{aligned}\underbrace{\frac{\partial p(t,s)}{\partial S}}_{\Delta_{put}} &= \frac{\partial}{\partial S}\left(Ke^{-r(T-t)}\right) + \underbrace{\frac{\partial c(t,s)}{\partial S}}_{\Delta_{call}} - \frac{\partial S}{\partial S} \\ \Delta_{put} &= \Delta_{call} - 1\end{aligned}$$

Since,$\Delta_{call} = N[d1] \Rightarrow \Delta_{put} = N[d1] - 1$.

To find the result on the *gamma* we differentiate one more time (so two times) w.r.t. S the put-call parity and get

$$\underbrace{\frac{\partial^2 p(t,s)}{\partial S^2}}_{\Gamma_{call}} = \underbrace{\frac{\partial^2 c(t,s)}{\partial S^2}}_{\Gamma_{put}}$$

From the fact that $\Gamma_{call} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$ it follows that $\Gamma_{put} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$.

Exercise 9.11 To be done!!!

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Chapter 13

Multidimensional Models: Classical Approach

Exercise 13.1 The equivalence follows from the next five steps

- (i) $\Rightarrow$ (ii)

  If
$$\Pi(t) = e^{-r(T-t)} E^Q_{t,s}\left[\Pi(T)\right] \quad \Leftrightarrow \quad e^{-rt}\Pi(t) = e^{-rT} E^Q_{t,s}\left[\Pi(T)\right]$$
  And this implies that
$$e^{-rT}\Pi(T) = e^{-rt}\Pi(t) + \int_t^T (\cdots) dW(s)$$
  Thus,
$$d\left(e^{-rt}\Pi(t)\right) = (\cdots)dW(t) \quad \Rightarrow \quad d\Pi(t) = r\Pi(t)dt + (\cdots)dW(t)$$

- (ii)$\Rightarrow$ (i)

  If $d\Pi(t) = r\Pi(t)dt + (\cdots)dW(t)$, then integrating we get
$$\Pi(T) = \Pi(t) + \int_t^T r\Pi(s)ds + \int_t^T (\cdots)dW(s)$$
  Taking expectations, under the martingale measure Q, and conditional on the information at time t we see
$$E^Q_t\left[\Pi(T)\right] = \Pi(t) + \int_t^T rE^Q_t\left[\Pi(s)\right] ds.$$
  Setting as usual $m(T) = E^Q_t\left[\Pi(T)\right]$ and differentiating w.r.t T we get the following ODE
$$\begin{cases} \dfrac{\partial m(T)}{\partial T} = m(T) \\ m(t) = \Pi(t) \end{cases}$$

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whose soltuion is given by

$$\begin{aligned} m(T) &= m(t)e^{r(T-t)} \\ E_t^Q[\Pi(T)] &= \Pi(T)e^{r(T-t)} \quad \Leftrightarrow \quad \Pi(t) = e^{-r(T-t)}E_t^Q[\Pi(T)] \end{aligned}$$

Up to now we showed,

$$\boxed{\text{(i)} \Leftrightarrow \text{(ii)}}$$

- (ii) $\Rightarrow$ (iii)

  Since $dB(t) = rB(t)dt$ under any measure (afterall B is deterministic). If we have $d\Pi(t) = r\Pi(t)dt + (\cdots)dW(t)$ then, by Itô we also have

$$\begin{aligned} d\left(\frac{\Pi(t)}{B(t)}\right) &= \frac{1}{B(t)}\left[r\Pi(t)dt + (\cdots)dW(t)\right] - \frac{\Pi(t)}{B(t)^2}rB(t)dt \\ &= 0dt + (\cdots)dW(t). \end{aligned}$$

  From the 0 drift we immediately see that $\frac{\Pi(t)}{B(t)}$ is indeed a martingale.

- (iii) $\Rightarrow$ (ii)

  This follows also immediately from Itô, but not take as given that $\frac{\Pi(t)}{B(t)}$ is a Q-martingale and hence its dynamics under Q are given by

$$d\left(\frac{\Pi(t)}{B(t)}\right) = (\cdots)dW(t)$$

  and the bank account dynamics $dB(t) = rB(t)dt$. Then we immediately get

$$d\Pi(t) = r\Pi(t)dt + (\cdots)dW(t).$$

  In the last two points we showed

$$\boxed{\text{(ii)} \Leftrightarrow \text{(iii)}}$$

- From (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) we immediatley get

$$\boxed{\text{(i)} \Leftrightarrow \text{(iii)}}$$

Exercise 13.2 To be done!!!

Exercise 13.3

(a) The exchange option is a T-claim defined by, $\Phi(s_1, s_2) = \max[S_1(T) - S_2(T), 0]$. However, using the fact that this contract function is homogeneous of degree 1 we can write it in terms of $z = \frac{s_1}{s_2}$

$$\begin{aligned}\Phi(s_1, s_2) &= \max[s_1 - s_2; 0] = s_2 \max\left[\underbrace{\frac{s_1}{s_2}}_{z} - 1; 0\right] \\ &= s_2 \underbrace{\max[z-1;0]}_{\Psi(z)}.\end{aligned}$$

We also know that any derivative that depends on both S_1 and S_2 must satisfy the following PDE (which has no second-order cross derivative because the two wiener processes are uncorrelated),

$$\begin{cases} F_t + rs_1F_1 + es_2F_2 + \frac{1}{2}\sigma_1^2F_{11} + \frac{1}{2}\sigma_2^2F_{22} - rF = 0 \\ F(T, s_1, s_2) = \max[s_1 - s_2, 0] \end{cases}$$

From above we see that

$$F(t, s_1, s_2) = s_2.G\left(t, \frac{s_1}{s_2}\right)$$

and from

$$\begin{aligned} F_t &= s_2G_t \\ F_1 &= s_2G_z\frac{\partial z}{\partial s_1} = s_2G_z\frac{1}{s_2} = G_z \\ F_2 &= G + s_2G_z\frac{\partial z}{\partial s_2} = G + s_2G_z\left(-\frac{s_1}{s_2^2}\right) = G - zG_z \\ F_{11} &= G_{zz}\frac{\partial z}{\partial s_1} = \frac{1}{s_2}G_{zz} \\ F_{22} &= G_z\frac{\partial z}{\partial s_2} - \left(-\frac{\partial z}{\partial s_2}G_z + zG_{zz}\frac{\partial z}{\partial s_2}\right) = \frac{1}{s_2}z^2G_{zz} \\ F_{12} &= G_{zz}\frac{\partial z}{\partial s_2} = -\frac{1}{s_2}zG_{zz} \end{aligned}$$

we realize that G satisfies the (much easier) PDE,

$$\begin{cases} G_t(t,z) + \frac{1}{2}z^2G_{zz}(t,z)(\sigma_1^2 + \sigma_2^2) \\ G(T, z) = \max[z-1, 0] \end{cases}$$

This last PDE is, in fact, the pricing equation of an European call option with strike price $K = 1$ written on a single stock with volatility $\sqrt{\sigma_1^2 + \sigma_2^2}$, in a world with short rate $r = 0$. So, we can use Black-Scholes formula to concretize the pricing function

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$\Pi(t) = F(t, S_1(t), S_2(t))$:

$$\begin{aligned}
\Pi(t) &= S_2(t)G(t, Z(t)) \\
&= S_2(t)\left\{\overbrace{\frac{S_1(t)}{S_2(t)}}^{Z(t)} N\left[d_1(t,z)\right] - N\left[d_2(t,z)\right]\right\} \\
&= S_1(t)N\left[d_1(t,z)\right] - S_2(t)N\left[d_2(t,z)\right]
\end{aligned}$$

for $z = \frac{s_1}{s_2}$ and

$$\begin{aligned}
d_1(t,z) &= \frac{1}{\sqrt{(\sigma_1^2+\sigma_2^2)(T-t)}}\left\{\ln z + \frac{1}{2}(\sigma_1^2+\sigma_2^2)(T-t)\right\}, \\
d_2(t,z) &= d_1(t,z) - \sqrt{(\sigma_1^2+\sigma_2^2)(T-t)}
\end{aligned}$$

(b) If now we have

$$d\bar{W}_1.d\bar{W}_2 = \rho dt.$$

The only difference is that the PDE that the price of any derivative that depend simultaneously on S_1 and S_2 must satisfy is

$$\begin{cases} F_t + rs_1F_1 + es_2F_2 + \frac{1}{2}\sigma_1^2F_{11} + \frac{1}{2}\sigma_2^2F_{22} + \rho\sigma_1\sigma_2F_{12} - rF = 0 \\ F(T, s_1, s_2) = \max\left[s_1 - s_2, 0\right] \end{cases}$$

So, in this case G, defined by $F(t, s_1, s_2) = s_2G(t, z)$ solves the PDE instead

$$\begin{aligned}
G_t + \frac{1}{2}z^2G_{zz}(\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho) &= 0 \\
G(T,z) &= \max\left[z-1, 0\right],
\end{aligned}$$

that we again recognize as the Black-Scholes equation and, therefore we have that G is the price of a call with underlying Z, exercise price $K = 1$ and with, $r = 0$, and we we take the volatility of the undelying to be $\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$. Thus G can be obtaind directly from the Black-Scholes formula, and after some minor calculation we can also obtain F

$$F(t, s_1, s_2) = s_1N\left[d_1'(t,z)\right] - s_2N\left[d_2'(t,z)\right]$$

where

$$d_1'(t,z) = \frac{\ln\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)(T-t)}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\sqrt{T-t}}$$

and $d_2'(t,z) = d_1'(t,z) - \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\sqrt{T-t}$.

Exercise 13.4 For this "special" maximum option we have $\mathcal{X} = \max\left[aS_1(T); bS_2(T)\right]$

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where S_1 and S_2 are defined by

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)d\bar{W}_1$$

$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)d\bar{W}_2$$

and $\bar{W}_1$ is independent from $\bar{W}_2$.

The contract function Φ is homogeneous of degree 1 and can be rewritten in terms of the normalized variable $z = \frac{s_1}{s_2}$:

$$\begin{aligned}
\Phi(s_1, s_2) &= \max\left[as_1; bs_2\right] = bs_2 \max\left[\frac{as_1}{bs_2}; 1\right] \\
&= = bs_2\left\{\max\left[\frac{as_1}{bs_2} - 1; 0\right] + 1\right\} \\
&= bs_2\left\{\frac{a}{b}\max\left[\overbrace{\frac{s_1}{s_2}}^{z} - \frac{b}{a}; 0\right] + 1\right\} \\
&= bs_2 + as_2 \underbrace{\max\left[z - \frac{b}{a}; 0\right]}_{\Psi(z)}
\end{aligned}$$

Thus, the pricing function F is given by $F(t, s_1, s_2) = bs_2 + as_2 G(t, z)$ where G solves the boundary problem So we have

$$\begin{aligned}
G_t + \frac{1}{2}z^2 G_{zz}(\sigma_1^2 + \sigma_2^2) &= 0 \\
G(T, z) &= \max\left[z - \frac{b}{a}, 0\right].
\end{aligned}$$

Once again, the above expression is the Black-Scholes equation, G is the price of a call option on Z with strike price $K = \frac{b}{a}$, $r = 0$, and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ We can use Black-Scholes formula to concretize the pricing function $\Pi(t) = F(t, S_1(t), S_2(t)$:

$$\begin{aligned}
\Pi(t) = bS_2(t) + aS_2(t)G(t, Z(t)) &= bS_2(t) + aS_2(t)\left\{\frac{S_1(t)}{S_2(t)}N\left[d_1'\right] - \frac{b}{a}N\left[d_2'\right]\right\} \\
&= bS_2(t) + aS_1(t)N\left[d_1'\right] - bS_2(t)N\left[d_2'\right] \\
&= (1 - N\left[d_2'\right])bS_2(t) + aN\left[d_1'\right]S_1(t)
\end{aligned}$$

where

$$d_1' = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T - t}}\left\{\ln\left(\frac{aS_1(t)}{bS_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)(T - t)\right\}$$

and $d_2' = d_1' - \sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T - t}$.

Exercise 13.5 To be done!!!

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Chapter 15

Incomplete Markets

Exercise 15.1 Given a claim $\mathcal{X} = \Pi(X(T)))$ and since the dynamics of X under the Q-measure are

$$dX(t) = [\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))]\, dt + \sigma(t, X(t))dW^Q(t),$$

we can find the Q-dynamics of the pricing function $F(t, X(t))$ using the Ito formula

$$\begin{aligned}
dF(t, X(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\
&= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right] dt + \sigma(t, X(t))dW^Q(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t))dt \\
&= \underbrace{\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t))\right]}_{rF(t,X(t))} dt \\
&\quad + \sigma(t, X(t))dW^Q(t) \\
&= rF(t, X(t)) + \sigma(t, X(t))dW^Q(t)
\end{aligned}$$

where the last step results from the fact that F has to satisfy the pricing PDE:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t)) = rF.$$

Exercise 15.2 To be done!!!

Exercise 15.3 To be done!!!

Exercise 15.4 To be done!!!

Exercise 15.5 To be done!

Exercise 15.6 To be done!

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Chapter 16

Dividends

Exercise 16.1 To be done!!!

Exercise 16.2 We know that when there is a continuous dividend δ being paid we have the following dynamics under Q for the asset S and the dividen structure D

$$\begin{aligned} dS(t) &= [r - \delta(S(t))]\, S(t)dt + \sigma(S(t))\, S(t)dW(t) \\ dD(t) &= S(t)\delta(S(t))\, dt \end{aligned}$$

Just by rewritting and rearranging terms we have

$$\begin{aligned} dS(t) &= rS(t)dt - \underbrace{\delta(S(t))\, S(t)dt}_{dD(t)} + \sigma(S(t))\, S(t)dW(t) \\ e^{-r\cdot t}dS(t) - e^{-r\cdot t}rS(t)dt &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma(S(t))\, S(t)dW(t) \\ d\left[e^{-r\cdot t}S(t)\right] &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma(S(t))\, S(t)dW(t). \end{aligned}$$

Integratting the last expression we get

$$\begin{aligned} e^{-r\cdot t}S(t) &= e^{-r\cdot 0}S(0) - \int_0^t e^{-ru}dD(u) + e^{-r\cdot t}\int_0^t \sigma(S(u))S(u)dW(u) \\ S(0) &= e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u) - e^{-r\cdot t}\int_0^t \sigma(S(u))S(u)dW(u), \end{aligned}$$

and taking $E_0^Q[.]$ expectations we finally get the results

$$S(0) = E_0^Q\left[e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u)\right].$$

Exercise 16.3 To be done!!!

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Exercise 16.4 Consider the *normalized gain process*

$$G(t) = \underbrace{e^{-rt}}_{X}\underbrace{S(t)}_{Y} + \underbrace{\int_0^t e^{-r\tau} dD(\tau)}_{P}.$$

We know that the Q dynamics are (note that D is deterministic)

$$\begin{aligned} dS(t) &= (r-\delta)\, S(t)dt + \sigma S(t) dW^Q(t) \\ dD(t) &= \delta S(t) dt. \end{aligned}$$

Note that the process G can be seen as $G = X \cdot Y + Z$ (and X, Y and P are as described above) . Applying Ito we get

$$\begin{aligned} dG(t) &= Y(t)dX(t) + X(t)dY(t) + dP(t) \\ &= \underbrace{S(t)}_{Y(t)}\left(-re^{-rt}dt\right) + \underbrace{e^{-rt}}_{X(t)}\left[(r-\delta)\, S(t)dt + \sigma S(t) dW^Q(t)\right] + e^{-rt}\underbrace{\delta S(t)dt}_{dD(t)} \\ &= 0 \cdot dt + e^{-rt}\left[\sigma S(t) dW^Q(t)\right]. \end{aligned}$$

Since the drift term is zero the normalized gain process is a martingale under Q.

Exercise 16.5 We have to show that $F_\delta(t,s) = F_0(t, se^{-\delta(T-t)})$, i.e, that the pricing function when we have a continuous dividend process δ is just the "standard" (no-dividend) price function where we replace s by $se^{-\delta(T-t)}$. To see this note that

$$\begin{aligned} F_0(t,s) &= e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S(T))\right] \quad \text{with} \quad dS(t) = rS(t)dt + \sigma S(t) dW^Q(t) \\ F_\delta(t,s) &= e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S(T))\right] \quad \text{with} \quad dS(t) = (r-\delta)\, S(t)dt + \sigma S(t) dW^Q(t). \end{aligned}$$

Define $Z(t) = se^{-\delta(T-t)}$, then by Ito the dynamics of Z under Q are given by

$$dZ(t) = (r-\delta)\, Z(t)dt + \sigma Z(t) dW^Q(t)$$

and the dynamics of the pricing function $F_0(t, \underbrace{se^{-\delta(T-t)}}_{z})$ are given by

$$\begin{aligned} dF_0(t, Z(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial z}dZ(t) + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}\left(dZ(t)\right)^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial z}\left[(r-\delta)\, Z(t)dt + \sigma Z(t) dW^Q(t)\right] + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}\sigma^2 Z^2(t)dt \\ &= \left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial z}(r-\delta)\, Z(t) + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}\sigma^2 Z^2(t)\right\} dt + \frac{\partial F}{\partial z}\sigma Z(t) dW^Q(t). \end{aligned}$$

In the case of $F_\delta(t,s)$ we get

$$\begin{aligned} dF_0(t, S(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial s}dS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}\left(dS(t)\right)^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial s}\left[(r-\delta)\, S(t)dt + \sigma S(t) dW^Q(t)\right] + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}\sigma^2 S^2(t)dt \\ &= \left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial s}(r-\delta)\, S(t) + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}\sigma^2 S^2(t)\right\} dt + \frac{\partial F}{\partial s}\sigma S(t) dW^Q(t). \end{aligned}$$

Note that both SDE are equal, thus we must have

$$\begin{aligned} F_0(t,z) &= F_\delta(t,s) \\ F_0(t, se^{-\delta(T-t)}) &= F_\delta(t,s). \end{aligned}$$

Exercise 16.6 In the Black-Scholes model with a constant continuous dividend yield δ we have, under the objective probability measure P, the following dynamics

$$\begin{aligned} dS(t) &= (\mu+\delta)\, S(t)dt + \sigma S(t)dW(t) \\ dB(t) &= rB(t)dt, \end{aligned}$$

which implies that under the Q-measure we have

$$dS(t) = (r-\delta)\, S(t)dt + \sigma S(t)dW^Q(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t,x) = c(t,x) - \Pi(t, S(T)) + e^{-r(T-t)}K.$$

But we also know that

$$\begin{aligned} \Pi(t, S(T)) &= e^{-r(T-t)} E_t^Q\left[S(T)\right] \\ &= e^{-r(T-t)} S(t) e^{(r-\delta)(T-t)} \\ &= S(t)e^{-\delta(T-t)}. \end{aligned}$$

So,

$$p(t,x) = c(t,x) - S(t)e^{-\delta(T-t)} + e^{-r(T-t)}K.$$

Exercise 16.7 To be done!

Exercise 16.8 In order to investigate completeness of the Black-Scholes model with a constant continuous dividend yield δ, we first note that if the market is complete, then any contigent claim $\mathcal{X} = \Phi(S(T))$ can be replicated by a self-financing portfolio V of the risky asset S and the riskless asset B, where

$$V(t) = h^B(t)B(t) + h^S(S(t)) = u^B V(t) + u^S V(t)$$

so,

$$dV(t) = V(t)\left\{u^B(t)r + (\alpha+\delta)u^S(t)\right\} dt + V(t)u^S\sigma dW(t)$$

and $V(t) = F(t,(t))$.

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Hence, by Ito we have

$$\begin{aligned} dV(t) &= \left\{ \frac{\partial F}{\partial t} + (\alpha+\delta)S(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2} \right\} + \sigma S(t)\frac{\partial F}{\partial s} dW(t) \\ dV(t) &= V(t)\left\{ \frac{\frac{\partial F}{\partial t} + (\alpha+\delta)S(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}}{V(t)} \right\} dt + V(t)\frac{\sigma S(t)\frac{\partial F}{\partial s}}{V(t)} dW(t) \end{aligned}$$

Them we know that

$$u^S = \frac{\sigma S(t)\frac{\partial F}{\partial s}(t, S(t))}{V(t)} = \frac{\sigma S(t)\frac{\partial F}{\partial s}(t, S(t))}{F(t, S(t))}$$

and since we have the "modified" Black-Scholes equation

$$\frac{\partial F}{\partial t} + (r-\delta)s\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} = rF,$$

after some manipulation we get

$$\begin{aligned} u^B(t)rdt + (\alpha+\delta)u^S(t) &= \frac{\frac{\partial F}{\partial t} + (\alpha+\delta)S(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}}{V(t)} \\ u^B(t) &= \frac{F(t, S(t)) - S(t)\frac{\partial F}{\partial s}(t, S(t)}{F(t, S(t))}. \end{aligned}$$

Exercise 16.9 To be done!

Exercise 16.10 To be done!

Exercise 16.11 To be done!

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Chapter 17

Currency Derivatives

Exercise 17.1 We have, under the objective probability measure, the following processes for the spot exchange rate X (units of domestic currency d per foreign currency unit f), and the domestic , B_d, and foreign, B_f, riskless assets:

$$\begin{aligned} dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t) \\ dB_d(t) &= r_d B(t)_d dt \\ dB_f(t) &= r_f B(t)_d dt, \end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. Hence the Q-dynamics of X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t,x) = c(t,x) - \Pi(t, X(T)) + e^{-r_d(T-t)}K.$$

But we also know that

$$\begin{aligned} \Pi(t, X(T)) &= e^{-r_d(T-t)} E_t^Q [X(T)] \\ &= e^{-r_d(T-t)} X(t) e^{(r_d - r_f)(T-t)} \\ &= X(t) e^{-r_f(T-t)}. \end{aligned}$$

So,

$$p(t,x) = c(t,x) - X(t)e^{-r_f(T-t)} + e^{-r_d(T-t)}K.$$

(Remark: Compare this exercise with exercise 11.6 in the previous section, and see that the foreign risk-free rate can be treated as a continuous dividend-yield on the spot exchange rate.)

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Exercise 17.2 The *binary option* on the exchange rate X is a T-claim, Z, of the form

$$\mathcal{Z} = 1_{[a,b]}\left(X(T)\right),$$

i.e. if $a \leq X(T) \leq b$ then the holder of this claim will obtain one unit of domestic currency, otherwise gets nothing. The dynamics under Q of the exchange rate X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic.

Integrating the above expression we get

$$X(T) = X(t) + \int_t^T (r_d - r_f)X(u)du + \int_t^t \sigma X(u)dW(u)$$

and so we have (by solving the SDE)

$$X(T) = X(t)e^{\left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t) + \sigma_x(W(T)-W(t))},$$

and we see from taking the logarithm that

$$\ln(X(T)) \sim N\left[\ln\left(X(t)\right) + \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t), \sigma_x\sqrt{T-t}\right].$$

So, the price, $\Pi(t)$ of the binary option is given by

$$\begin{aligned}
\Pi(t) &= e^{-r_d(T-t)}E_t^Q\left[\mathcal{Z}\right] \\
&= e^{-r_d(T-t)}Q\left(a \leq X(T) \leq b\right) \\
&= e^{-r_d(T-t)}Q\left(\ln(a) \leq X(T) \leq \ln(b)\right) \\
&= e^{-r_d(T-t)}Q\left(d_a \leq z \leq d_b\right)
\end{aligned}$$

where $z \sim N[0,1]$ and $d_a = \frac{\ln\left(\frac{a}{X(t)}\right) - \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and $d_b = \frac{\ln\left(\frac{b}{X(t)}\right) - \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and so we have that the price of the binary exchange option $\mathcal{Z}$ is:

$$\Pi(t) = e^{-r_d(T-t)}\left[N\left(d_b\right) - N\left(d_a\right)\right].$$

Exercise 17.3 Under the objective probability measure we have the following dynamics for the domestic stock S_d, the foreign stock S_f, the exchange rate X and the domestic B_d and foreign B_f riskless assets

$$\begin{aligned}
dS_d(t) &= \alpha_d S_d(t)dt + \sigma_d S_d(t)dW_d(t), \\
dS_f(t) &= \alpha_f S_f(t)dt + \sigma_f S_f(t)dW_f(t), \\
dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t), \\
dB_d(t) &= r_d B_d(t)dt, \\
dB_f(t) &= r_f S_d(t)dt,
\end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. The domestic stock denominated in terms of the foreign currency is given by $\tilde{S_d} = \frac{S_d}{X}$, then by Ito

$$\begin{aligned} d\tilde{S_d}(t) &= \frac{1}{X(t)}dS_d(t) - \frac{S_d(t)}{X^2(t)}dX(t) + \frac{S_d(t)}{X^3(t)}\left(dX(t)\right)^2 - \frac{-1}{X^2(t)} \underbrace{dS_d(t)dX(t)}_{0 \text{ for } W_d \text{ and } W \text{ indep.}} \\ &= \alpha_d \tilde{S}_d(t)dt + \sigma_d \tilde{S}_d(t)dW_d(t) - \alpha_x \tilde{S}_d(t)dt - \sigma_x \tilde{S}_d(t)dW(t) + \sigma_x^2 \tilde{S}_d(t)dt \\ &= \left(\alpha_d - \alpha_x + \sigma_x^2\right)\tilde{S}_d(t)dt + \sigma_d \tilde{S}_d(t)dW_d(t) - \sigma_x \tilde{S}_d(t)dW(t). \end{aligned}$$

The dynamics above are under the objective probability measure. Under Q_f the drift term of all assets denominated in the foreign currency must have r_f (the risk-free rate on the foreign economy) as the drift. Also, we can use the properties of the Wiener processes to normalize the diffusion part.

It follows that

$$\tilde{S}_d(t) = r_f \tilde{S}_d(t)dt + \sqrt{\sigma_d^2 + \sigma_x^2}\tilde{S}_d(t)dW_f(t).$$

Exercise 17.4 The *exchange option* which gives the right to exchange one unit of domestic equity for one unit of foreign equity is a T-claim, $\mathcal{Z}^d$ of the form

$$\begin{aligned} \mathcal{Z}^d &= \max\left[X(T)S_f(T) - S_d(T), 0\right] \\ &= S_d(T) \cdot \max\left[\underbrace{\frac{X(T)S_f(T)}{S_d(T)}}_{Z(T)} - 1, 0\right] \end{aligned}$$

.

So, our T-claim, some proportion $(S_d(T))$ of $\max\left[Z(T) - 1, 0\right]$, which is a call option with strike 1 on the underlying $Z = \frac{XS_f}{S_d}$.

We also know the dynamics, under P, the domestic stock S_d, the foreign stock S_f, the exchange rate X and the domestic B_d and foreign B_f riskless assets

$$\begin{aligned} dS_d(t) &= \alpha_d S_d(t)dt + \sigma_d S_d(t)dW_d(t), \\ dS_f(t) &= \alpha_f S_f(t)dt + \sigma_f S_f(t)dW_f(t), \\ dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t), \\ dB_d(t) &= r_d B_d(t)dt, \\ dB_f(t) &= r_f S_d(t)dt, \end{aligned}$$

So, under the objective probability measure, we have by Ito,the following dynamics for Z (recall that by assumption the processes S_d, S_f and X are uncorrelated)

$$dZ(t) = \frac{S_f(t)}{S_d(t)}dX(t) + \frac{X(t)}{S_d(t)}dS_f(t) - \frac{X(t)S_f(t)}{S_d^2(t)}dS_d(t) + \frac{X(t)S_f(t)}{S_d^3(t)}\left(dS_f(t)\right)^2$$

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$$
\begin{aligned}
&= Z(t)\left[\alpha_d dt + \sigma_x dW(t)\right] + Z(t)\left[\alpha_f dt + \sigma_f dW_f(t)\right] - Z(t)\left[\alpha_d dt + \sigma_d W_d(t)\right] + Z(t)\sigma_d^2 dt \\
&= \left(\alpha_x + \alpha_f - \alpha_d + \sigma_d^2\right) Z(t)dt + \sigma_x Z(t)dW(t) + \sigma_f Z(t)dW_f(t) - \sigma_d Z(t)dW_d(t)
\end{aligned}
$$

By noting that the variable Z is denominated on the domestic currency so its drift term must be r_d under the Q measure and rearranging the diffusion term where $\bar{W}$ is a Q Wiener process, we have

$$
dZ(t) = r_d Z(t)dt + \left(\sigma_x + \sigma_f - \sigma_d\right) d\bar{W}(t).
$$

Using this we know that the arbitrage free price of our claim is

$$
\begin{aligned}
\Pi(t) &= S_d(t) \cdot c(t, z, 1) \\
&= S_d(t)\left\{\frac{X(t)S_f(t)}{S_d(t)} N\left[d_1(t,z)\right] - N\left[d_2(t,z)\right]\right\} \\
&= X(t)S_f(t)N\left[d_1(t,z)\right] - S_d(t)N\left[d_2(t,z)\right]
\end{aligned}
$$

where

$$
d_1(t,z) = \frac{\ln\left(\overbrace{\frac{X(t)S_f(t)}{S_d(t)}}^{z}\right) + \left(r_d + \frac{1}{2}\left(\sigma_x + \sigma_f - \sigma_d\right)^2\right)(T-t)}{\left(\sigma_x + \sigma_f - \sigma_d\right)\sqrt{T-t}}
$$

and $d_2(t,z) = d_1(t,z) - \left(\sigma_x + \sigma_f - \sigma_d\right)\sqrt{T-t}$.

Exercise 17.5 To be done!!!

Exercise 17.6 To be done!!!

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Chapter 20

Bonds and Interest Rates

Exercise 20.1 *Forward Rate Agreement*

(a) Note that the cash flow to the lender's in a FRA ($-K$ at time S and $Ke^{R^*(T-S)}$ at time T) can be replicated by the following portfolio:

* sell K S-bonds

* buy $Ke^{R^*(T-S)}$ T-bonds

So, at time $t < S$, the value $\Pi(t)$, on the lender's cash flow in a FRA has to equal to the value of the replicating portfolio and is given buy

$$\Pi(t) = Ke^{R^*(T-S)}p(t,T) - Kp(t,S).$$

(b) If we have that $\Pi(0) = 0$ we must have

$$\begin{aligned} Ke^{R^*(T-S)}p(0,T) - Kp(0,S) &= 0 \\ e^{R^*(T-S)}p(0,T) - p(0,S) &= 0 \\ R^*(T-S) &= \ln\left(\frac{p(0,S)}{p(0,T)}\right) \\ R^* &= \frac{\ln(p(0,S)) - \ln(p(0,T))}{T-S} \end{aligned}$$

Since by definition the forward rate $R(t;S,T)$ is given by

$$R(t;S,T) = -\frac{\ln(p(t,T)) - \ln(p(t,S))}{T-S},$$

we have that $R^* = R(0;S,T)$.

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Exercise 20.2 Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t),$$

then we have, by Ito, that $\ln(p(t,T))$ has dynamics that are given by

$$\begin{aligned} d\ln(p(t,T)) &= m(t,T)dt + v(t,T)dW(t) - \frac{1}{2}v(t,T)v(t,T)dt \\ &= [m(t,T) - v(t,T)v(t,T)]\,dt + v(t,T)dW(t), \end{aligned}$$

and integrating we have

$$\ln(p(t,T)) = \ln(p(0,T)) + \int_0^t m(s,T) - v(s,T)v(s,T)ds + \int_0^t v(s,T)dW(s).$$

Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$, we also have

$$f(t,T) = \underbrace{-\frac{\partial \ln(p(0,T))}{\partial T}}_{f(0,T)} - \int_0^t \left(\underbrace{\frac{\partial m(s,T)}{\partial T}}_{m_T(s,T)} - v(s,T) \underbrace{\frac{\partial v(s,T)}{\partial T}}_{v_T(s,T)} \right) ds - \int_0^t \frac{\partial v(s,T)}{\partial T} dW(s)$$

and from its differential form we finally get the result

$$df(t,T) = \left[\underbrace{v(t,T)v_T(t,T) - m_T(t,T)}_{\alpha(t,T)} \right] dt - \underbrace{v_T(t,T)}_{\sigma(t,T)} dW(t).$$

Exercise 20.3 The coupon, at each maturity date T_i is given by $C_i = rK(T_i - T_{i-1})$. At the last maturity, T_n, we also have the payment of the face value K. The value at time t of such a coupon is given by

$$p(t) = Kp(t,T_n) + \sum_{i=1}^{n} p(t,T_i)rK(T_i - T_{i-1}).$$

If the price of the coupon bond is equal to the face value at time t, we have $p(t) = K$, hence

$$\begin{aligned} K &= Kp(t,T_n) + \sum_{i=1}^{n} p(t,T_i)rK(T_i - T_{i-1}) \\ 1 - p(t,T_n) &= r\sum_{i=1}^{n} p(t,T_i)\cdot(T_i - T_{i-1}) \\ r &= \frac{1 - p(t,T_n)}{\sum_{i=1}^{n} p(t,T_i)\cdot(T_i - T_{i-1})} \end{aligned}$$

Exercise 20.4 We will show that if at any point *we borrow 1 unit of money and sell the next maturity zero-coupon bond (and if we are at the last maturity we just borrow 1 unit of money)*, then we replicate the floating LIBOR-indexed coupon bond.

- It is easy to see why our strategy replicates the a LIBOR return at any intermediate maturity date T_i, since we have

$$C_i = 1 - p(T_i, T_{i+1}) \quad \text{Which is, by definition, the LIBOR return.}$$

  So, at each maturity time T_i, we need one bond maturing (equivalent to borrow one unit of money) and to sell the next maturity bond.

  The price at any time $t < T_i$ is given by

$$\Pi(C_i)(t) = p(t, T_i) - p(t, T_{i+1}).$$

- To see why it replicates the final payment (K=1 at T_n), we just need to realize that at maturity T_n-zero-coupon bond, there is no "next maturity", so we just collect our 1 unit of money, whose price as of today is

$$\Pi(K)(t) = p(t, T_n).$$

The simple strategy presented, has thus, the same price as to buy one zero-coupon-bond for each maturity $T_0, T_1, ..., T_n$ and to sell one zero-coupon-bond for each maturity $T_1, ..., T_n$.

Hence the, at time t, the "price" of our LIBOR-indexed floating bond is the price of the replicating strategy and hence

$$p(t) = \sum_{i=0}^{n} p(t, T_i) - \sum_{i=1}^{n} p(t, T_i) = p(t, T_0).$$

Exercise 20.5 Let $\{y(0, T; T \geq 0\}$ denote the zero coupon yield curve.

(a) Then we have that

$$p(0, T) = e^{-y(0,T)\cdot T}.$$

Using the definition of the instantaneous forward rate $f(t, T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and the expression above we get the result

$$\begin{aligned} f(0, T) &= -\frac{\partial \ln(p(0, T))}{\partial T} \\ &= -\frac{\partial \ln\left(e^{-y(0,T)\cdot T}\right)}{\partial T} \\ &= \frac{\partial (y(0, T) \cdot T)}{\partial T} \\ &= y(0, T) + T \cdot \frac{\partial y(0, T)}{\partial T}. \end{aligned}$$

(b) If the zero coupon yield curve is an increasing function of T, then we know that $\frac{\partial y(0,T)}{\partial T} \geq 0$, and using the result from (a) we have that $f(0, T) \geq y(0, T)$ for any $T > 0$.

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It remains to prove that $y_M(0,T) \leq y(0,T)$. This will follow from the price of a coupon bond is given by

$$p_T(t) = Kp(t,T_n) + \sum_{i=1}^{n} c_i p(t,T_i).$$

So in particular we have that

$$p_T(0) = K\underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)}.$$

but it can also we given, since y_M is the yield to maturity of a coupon bond, by

$$p_T(0) = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}.$$

So,

$$K\underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)} = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}$$

By comparing the LHS and the RHS and since $c_i \geq 0$, and $y(0,T_n) \geq y(0,T_{n-1}) \geq \cdots \geq y(0,T_1)$ (by assumption), we must have that $y(0,T_n) \geq y_M(0,T_n)$ since it is valid for any T_n we can write that $y(0,T) \geq y_M(0,T)$ for any T.

Exercise 20.6 We know that the *duration* for a coupon bond, with coupon c_i, price p at $t = 0$ and yield to maturity y is defined by

$$D = \frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i}}{p}$$

We also have that $p = \sum_{i=1}^{n} e^{-yT_i}$. Hence

$$\begin{aligned}
\frac{\partial p}{\partial y} &= \sum_{i=1}^{n} -T_i e^{-yT_i} \\
&= -\underbrace{\frac{\sum_{i=1}^{n} -T_i e^{-yT_i}}{p}}_{D} p \\
&= -Dp
\end{aligned}$$

The convexity $C = \frac{\partial^2 p}{\partial y^2}$ is then

$$\begin{aligned}
\frac{\partial^2 p}{\partial y^2} &= -\left(\frac{\partial D}{\partial y} p + D\frac{\partial p}{\partial y}\right) \\
&= -\frac{\sum_{i=1}^{n} T_i^2 e^{-yT_i} p + Dp \sum_{i=1}^{n} T_i c_i e^{-yT_i}}{p^2} p + D^2 p \\
&= -\sum_{i=1}^{n} T_i^2 e^{-yT_i} - Dp \underbrace{\frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i}}{p}}_{D} + D^2 p \\
&= -\sum_{i=1}^{n} T_i^2 e^{-yT_i}
\end{aligned}$$

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Exercise 20.7 A consol bond, pays one unit of cash at any $T_i = i$ for $i = 1, 2, 3, ...$, hence its price at time $t = 0$ is given by

(a)

$$p_c(0) = \sum_{i=1}^{\infty} p(t,i) = \sum_{i=1}^{\infty} e^{-yi} = e^{-y}\frac{1}{1-e^{-y}} = \frac{1}{e^y - 1}$$

(b) The duration of a consol bond D_c is then

$$D_c = \frac{\sum_{i=1}^{\infty} ie^{-yi}}{p_c}$$

(c) For consol bonds we also have (similarly to Exercise 20.6)

$$\frac{\partial p_c}{\partial y} = -D_c p_c,$$

hence

$$\begin{aligned} -\frac{e^y}{(e^y-1)^2} &= -D_c\frac{1}{e^y-1} \\ D_c &= \frac{e^y}{e^y-1} \end{aligned}$$

(d) The convexity $C = \frac{\partial^2 p_c}{\partial y^2}$ can be obtain in the following way

$$\begin{aligned} \frac{\partial^2 p_c}{\partial y^2} &= -\frac{\partial D_c}{\partial y}p_c - D_c \underbrace{\frac{\partial p_c}{\partial y}}_{-D_c p_c} \\ &= \frac{e^y(e^y-1) - e^{2y}}{(e^y-1)^2}p_c + D_c^2 p_c \\ &= -\frac{e^y}{(e^y-1)^2}p_c + \frac{e^{2y}}{(e^y-1)^2}p_c \\ &= \frac{e^y}{e^y-1}p_c \\ &= \frac{e^y}{(e^y-1)^2} \end{aligned}$$

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Chapter 21

Short Rate Models

Exercise 21.1 To be done!

Exercise 21.2 The object of the exercise is to connect the forward rates to the risk neutral valuation of bond prices.

(a) Recall from the risk-neutral valuation of bond prices that

$$p(t,T) = E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]$$

and hence, using the definition of a instantaneous forward rate (and assuming that we can differentiate under the expectation sign) we get

$$\begin{aligned} f(t,T) &= -\frac{\partial \ln\left(p(t,T)\right)}{\partial T} \\ &= -\frac{\partial}{\partial T}\ln\left(E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]\right) \\ &= \frac{E_t^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]}. \end{aligned}$$

(b) To check that indeed we have $f(t,t) = r(t)$ use the expression above

$$f(t,T) = \frac{E_t^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]},$$

and set $T = t$:

$$f(t,t) = \frac{E_t^Q\left[r(t)\exp\left\{-\int_t^t r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^t r(s)ds\right\}\right]}$$

$$= \frac{E_t^Q\left[r(t)\exp\{0\}\right]}{E_t^Q\left[\exp\{0\}\right]} = r(t).$$

Exercise 21.3 Recall that in a *swap of a fixed rate vs. a short rate* we have:

- A invests K at time 0 and let it grow at a *fixed* rate of interest R over the time interval $[0,T]$. A has thus an amount K_A at time T and at that time (T) pays the surplus $K_A - K$ to B.
- B invests the principal at a *stochastic* short rate of interest over the interval $[0,T]$. B has thus an amount K_B at time T and at that time (T) pays the surplus $K_B - K$ to A.
- The *swap rate* for this contract is defined as the value, R, of the fixed rate which gives this contract value zero at $t = 0$.

At maturity party A has $\Phi(T) = K_B - Ke^{R\cdot T}$ where $K_B = Ke^{\int_0^T r(s)ds}$. Thus, the value of this contract at time 0 is given by

$$\begin{aligned}
\Pi(0) &= E_{0,r}^Q\left[e^{-\int_0^T r(s)ds}\left(\underbrace{Ke^{\int_0^T r(s)ds} - Ke^{R\cdot T}}_{\Phi(T)}\right)\right] \\
&= E_{0,r}^Q\left[K - Ke^{R\cdot T - \int_0^T r(s)ds}\right] \\
&= K\cdot E_{0,r}^Q\left[1 - e^{R\cdot T}e^{-\int_0^T r(s)ds}\right] \\
&= K\cdot\left(1 - e^{R\cdot T}\underbrace{E_{0,r}^Q\left[e^{-\int_0^T r(s)ds}\right]}_{p(0,T)}\right)
\end{aligned}$$

Since we must have $\Pi(0) = 0$ we have

$$\begin{aligned}
K\cdot\left(1 - e^{R\cdot T}p(t,T)\right) &= 0 \\
e^{R\cdot T} &= \frac{1}{p(0,T)} \\
R\cdot T &= -\ln\left(p(0,T)\right) \\
R &= -\frac{\ln\left(p(0,T)\right)}{T}.
\end{aligned}$$

Exercise 21.4 To be done!

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Chapter 22

Martingale Models for the Short Rate

Exercise 22.1 In the Vasicek model we have

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

with $a > 0$.

(a) Using the SDE above and multiplying both sides by e^{at} we have that

$$\begin{aligned} e^{at}dr(t) + r(t)ae^{at}dt &= e^{at}bdt + \sigma e^{at}dW(t) \\ d(e^{at}r(t)) &= e^{at}bdt + \sigma e^{at}dW(t) \\ e^{at}r(t) &= r(0) + \int_0^t e^{as}bds + \sigma \int_0^t e^{as}dW(s) \\ r(t) &= r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] + \sigma e^{-at}\int_0^t e^{a}(s)dW(s). \end{aligned}$$

Looking at the solution of the SDE it is immediate that r is Gaussian, so it is enough to determine the mean and variance.

From the solution of the SDE we see that

$$E\left[r(t)\right] = r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right]$$

and

$$\begin{aligned} V\left[r(t)\right] &= \sigma^2 e^{-2at}\int_0^t e^{2as}ds \\ &= \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \end{aligned}$$

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(b) As $t \to \infty$ we have

$$\lim_{t\to\infty} E\left[r(t)\right] = r(0) \lim_{t\to\infty} e^{-at} + \frac{b}{a}\left[1 - \lim_{t\to\infty} e^{-at}\right] = \frac{b}{a},$$

and

$$\lim_{t\to\infty} V\left[r(t)\right] = \frac{\sigma^2}{2a}\left(1 - \lim_{t\to\infty} e^{-2at}\right) = \frac{\sigma^2}{2a}.$$

So, $N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$ is the limiting distribution of r.

(c) The results in (a) and (b) are based on the assumption that the value $r(0)$ is known. If, instead we have that $r(0) \sim N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$, then for any t we have that

$$E\left[r(t)\right] = E\left[r(0)\right] e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] = \frac{b}{a},$$

and

$$\begin{aligned} V\left[r(t)\right] &= e^{-2at} V\left[r(0)\right] + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= \frac{\sigma^2 e^{-2at}}{2a} + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) = \frac{\sigma^2}{2a}. \end{aligned}$$

(d) To be done!

Exercise 22.2 From Exercise 22.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)e^{-a(u-t)} + \frac{b}{a}\left[1 - e^{-a(u-t)}\right] + \sigma \int_t^u e^{-a(u-s)} dW(s).$$

By definition,

$$p(t,T) = E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right].$$

Let us define $Z(t,T) = -\int_t^T r(u)du$, from exercise 22.1 we know that r is normally distributed, so Z is also normally distributed since

$$\begin{aligned} Z(t,T) &= -r(t)e^{at}\int_t^T e^{-au}du - \frac{b}{a}\left[1 - e^{at}\int_t^T e^{-au}du\right] + \sigma\int_t^T\int_t^u e^{-a(u-s)}dW(s)du \\ &= -\frac{r(t)}{a}\left[1 - e^{-a(T-t)}\right] - \frac{b}{a}(T-t) - \frac{b^2}{a}\left[1 - e^{-a(T-t)}\right] + \sigma\int_t^T\int_t^u e^{-a(u-s)}dW(s)du \end{aligned}$$

and has

$$E^Q_{t,r}\left[Z(t,T)\right] = \underbrace{\left(-\frac{1}{a}\left[1 - e^{-a(T-t)}\right]\right)}_{\text{deterministic function of } t \text{ and } T} r(t) \underbrace{-\frac{b}{a}(T-t) - \frac{b^2}{a}\left[1 - e^{-a(T-t)}\right]}_{\text{deterministic function of } t \text{ and } T}$$

and

$$\begin{aligned} V^Q_{t,r}\left[Z(t,T)\right] &= V\left[\sigma\int_t^u\int_t^T e^{-a(u-s)}dudW(s)\right] \\ &= \underbrace{\sigma\int_t^u\left(\int_t^T e^{-a(u-s)}du\right)^2 ds}_{\text{deterministic function of } t \text{ and } T}. \end{aligned}$$

Since $Z(t,T)$ is normally distributed $\forall_{t,T}$ (from the properties of the normal distribution), we have that

$$\begin{aligned}
p(t,T) &= E^Q_{t,r}\left[e^{Z(t,T)}\right] = e^{E^Q_{t,r}[Z(t,T)]+\frac{1}{2}V^Q_{t,r}[Z(t,T)]} \\
&\Rightarrow \\
\ln(p(t,T)) &= E^Q_{t,r}\left[Z(t,T)\right] + \frac{1}{2}V^Q_{t,r}\left[Z(t,T)\right] \\
&= \underbrace{-\frac{1}{a}\left[1-e^{-a(T-t)}\right]}_{B(t,T)} r(t) + \\
&+ \left(\underbrace{-\frac{b}{a}(T-t) - \frac{b^2}{a}\left[1-e^{-a(T-t)}\right] + \frac{1}{2}\sigma\int_t^u \left(\int_t^T e^{-a(u-s)}du\right)^2 ds}_{A(t,T)}\right).
\end{aligned}$$

So, the Vasicek model has an affine term structure.

Alternative Solution

From Exercise 22.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)\underbrace{e^{-a(u-t)}}_{D(u)} + \underbrace{\frac{b}{a}\left[1-e^{-a(u-t)}\right] + \sigma\int_t^u e^{-a(u-s)}dW(s)}_{F(u)}.$$

where $D(u)$ is deterministic and $F(u)$ is stochastic and both are non dependent on r.

By definition,

$$\begin{aligned}
p(t,T) &= E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right] \\
&= E^Q_{t,r}\left[e^{-\int_t^T [r(t)D(u)+F(u)du]}\right] \\
&= \left[\underbrace{e^{\left(-\int_t^T D(u)du\right)r(t)}}_{B}\underbrace{E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right]}_{A}\right].
\end{aligned}$$

In part A, since $-\int_t^T F(u)du$ is normally distributed (note it is a "sum" of Wiener increments), we know (from the properties of the normal distribution) that there exist a $A(t,T)$ such that

$$E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right] = e^{A(t,T)}.$$

From part B we immediatelly see that $B(t,T) = \int_t^T D(u)du$.

So, the Vasicek model has an affine term structure.

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Exercise 22.3 The problem with the *Dothan's model* is that when r follows a GBM, like

$$dr(t) = ar(t)dt + \sigma r(t)dW(t),$$

the solution to the SDE is, for any $t < u$,

$$r(u) = r(t)e^{\left(a-\frac{1}{2}\sigma^2\right)(u-t)+\sigma(W(u)-W(t))},$$

i.e., in the Dothan's model r is lognormally distributed. Since,

$$p(t,T) = E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right],$$

to use the same procedure as in exercise 22.2 we would have to compute the above expected value, i.e., the expected value of an integral (that is a "sum") of lognormally distributed variables, which is a mess!

Exercise 22.4 To be done!

Exercise 22.5 Consider the *Ho-Lee* model

$$dr(t) = \Theta(t) + \sigma dW(t),$$

where σ is a given constant.

Take also as given the term structure at time $t = 0$: $\{p^*(0,T); t \geq 0\}$.

For the Ho-Lee model we know that the term structure of zero-coupon bonds is affine with

$$A(T,t) = \int_t^T \Theta(s)(s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}$$

and

$$B(T,t) = T - t.$$

And hence since $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}r(t)$. From $A(t,T)$ and $B(t,T)$ we see that

$$\begin{aligned}\frac{\partial A(t,T)}{\partial T} &= \Theta(T) - \left(\int_t^T \Theta(s)ds + T\Theta(T)\right) + \frac{\sigma^2}{2}(T-t)^2 \\ &= -\int_t^T \Theta(s)ds + \frac{\sigma^2}{2}(T-t)^2\end{aligned}$$

$$\frac{\partial B(t,T)}{\partial T} = 1.$$

Since fitting the bond prices is just equivalent to fitting the forward rates we fit the last by imposing

$$f^*(0,T) = \int_0^T \Theta(s)ds - \frac{\sigma^2}{2}T^2 + r(0).$$

Finally, differentiating with respect to time we get the result

$$\begin{aligned}\frac{\partial f^*(t,T)}{\partial T} &= \Theta(T) - \sigma^2 T \\ \Theta(T) &= \frac{\partial f^*(t,T)}{\partial T} + \sigma^2 T.\end{aligned}$$

Exercise 22.6 To be done!

Exercise 22.7 Once again we are considering the *Ho-Lee model* (see also exercises 22.5 and 22.6).

(a) From the ATS (affine term structure) of the Ho-Lee model we know that

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left\{(T-t)f^*(0,T) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)\right\}$$

and, thus $\ln p(t,T) = \ln p^*(0,T) - \ln p^*(0,t) + f(0,T)(T-t) - -\frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)$.

So, since $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}$ we have for the forward rates that

$$f(t,T) = \underbrace{\frac{\partial \ln p^*(0,T)}{\partial T}}_{f^*(0,T)} + f^*(0,T) - \sigma^2 t(T-t) - r(t) = f^*(0,T) - f^*(0,t) - \sigma^2 t(T-t) - r(t).$$

(b) Fix an arbitrary time t, then

$$\lim_{T\to\infty} f(t,T) = \underbrace{\lim_{T\to\infty} f^*(0,T)}_{\text{which by assumption exist}} - f^*(0,t) - \sigma^2 t \lim_{T\to\infty} T - \sigma^2 t^2 - r(t) = \infty.$$

So, the Ho-Lee model is not consistent with the existence of an horizontal asymptote to the forward rate curve.

Exercise 22.8 Take the follwoing CIR model

$$dY(t) = \left(2aY(t) + \sigma^2\right) dt + 2\sigma\sqrt{Y(t)}dW(t), \quad Y(0) = y_0,$$

Then by Ito $Z(t) = \sqrt{Y(t)}$ follows

$$\begin{aligned}dZ(t) &= \frac{1}{2}\left(Y(t)^{-\frac{1}{2}}\right) dY(t) + \frac{1}{2}\left(-\frac{1}{4}\left(Y(t)^{-\frac{3}{2}}\right)\right)(dY(t))^2 \\ &= \frac{1}{2}\left(2aY(t)^{\frac{1}{2}} + \sigma^2 Y(t)^{-\frac{1}{2}}\right) dt + \sigma dW(t) - \frac{1}{2}\sigma^2 Y(t)^{-\frac{1}{2}} \\ &= a\underbrace{Y(t)^{\frac{1}{2}}}_{Z(t)} dt + \sigma dW(t),\end{aligned}$$

which is a linear diffusion.

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Chapter 23

Forward Rate Models

Exercise 23.1 We know that the *Hull-White model*

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t)$$

has an affine term structure, i.e., that

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

and that in this model we have in particular, $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$.

Furthermore, $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t)$, so in this case we have

$$f(t,T) = -\frac{\partial A(t,T)}{\partial T} + e^{-a(T-t)} r(t).$$

By Itô we have that the dynamics under Q of the forward rates is given by

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] dt + ae^{-a(T-t)} r(t)dt + e^{-a(T-t)} dr(t) \\ &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] dt + ae^{-a(T-t)} r(t)dt + e^{-a(T-t)} \left[(\Theta(t) - ar(t))dt + \sigma dW(t)\right] \\ &= \alpha(t,T)dt + e^{-a(T-t)} \sigma dW(t), \end{aligned}$$

where $\alpha(t,T) = -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] + e^{-a(T-t)}\Theta(t)$.

Exercise 23.2 Take as given an HJM model (under Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where the volatility $\sigma(t,T)$ is a deterministic function of t and T.

(a) By the HJM drift condition, $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma'(t,u)du \quad \forall t,T$, which for deterministic $\sigma(t,T)$, means that $\alpha(t,T)$ is also deterministic.

To see the distribution of the forward rates note that

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \\ &= \underbrace{f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma'(s,u)duds}_{\mu(t,T)} + \int_0^t \sigma(s,T)dW(s). \end{aligned}$$

Note that $f(0,T)$ is observable and that the double integral is deterministic, so the only stochastic part is $\int_0^t \sigma(s,T)dW(s)$.

Hence $f(t,T) \sim N\left[\mu(t,T), \int_0^t \sigma^2(s,T)ds\right]$.

Since $r(t) = f(t,t)$ we immediately have that $r(t)$ is also normally distributed.

(b) Since we have $p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$, and in (a) we have already shown that $f(t,s)$ is normally distributed, then $p(t,T)$ is lognormally distributed.

Exercise 23.3 Recall that in between the vectors of market-price of risks of the domestic market and the foreign market there is the following relation:

$$\lambda_f(t) = \lambda_d(t) - \sigma_x'(t).$$

where, $\sigma_x(t)$ is the vector of volatilities of in the SDE for the exchange-rate X(denoted in units of domestic currency per unit of foreign currency).

We also know that *any* process, so in particular forward rates, has under Q (the *domestic* martingale measure), the drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_d(t))$, where $\mu(t,T)$ is the drift term under the objective probability measure and that the volatility term remains the same. So, in particular, for the foreign forward rates must have under the domestic martingale measure, Q,

$$df_f(t,T) = (\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t))\,dt + \sigma_f(t,T)dW(t),$$

hence, $\alpha_f(t,T) = \mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)$.

Likewise, any process, under Q^f (the *foreign* martingale measure),has a drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_f(t))$.

So, in particular, for the foreign forward rates we have under Q^f

$$df_f(t,T) = \underbrace{(\mu_f(t,T) - \sigma_f(t,T)\lambda_f(t))}_{\tilde{\alpha}_f(t,T)}\,dt + \sigma_f(t,T)dW^f(t).$$

So, using the relation between λ_d and λ_f we can also establish a relation between α_f and $\tilde{\alpha}_f$,

$$\tilde{\alpha}_f(t,T) = \mu_f(t,T) - \sigma_f(t,T)\lambda_f(t)$$

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$$
\begin{aligned}
&= \mu_f(t,T) - \sigma_f(t,T)\left[\lambda_d(t) - \sigma_x'(t)\right] \\
&= \underbrace{\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)}_{\alpha_f(t,T)} + \sigma_f(t,T)\sigma_x'(t).
\end{aligned}
$$

We also have that, under the foreign martingale measure, Q^f, the coefficients of the foreign martingale measure must satisfy the standard HJM drift condition so:

$$\tilde{\alpha}_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds.$$

Using teh relation found between α_f and $\tilde{\alpha}_f$ and using the drift condition above we get

$$
\begin{aligned}
\tilde{\alpha}_f(t,T) &= \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) + \sigma_f(t,T)\sigma_x'(t) &= \sigma_f(t,T)\int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) &= \sigma_f(t,T)\left[\int_t^T \sigma_f'(t,s)ds - \sigma_x'(t)\right].
\end{aligned}
$$

<u>Alternatively</u> one can use the change-of-numeraire technique to solve the exercise.

Note that, for domestic processes, under the Q_d-measure the numeraire is B_d.

Likewise, for foreign processes, under the Q_f-measure the numeraire is B_f.

However, for domestic processes, under Q_f-measure the numeraire is actually XB_f.

With this in mind it follows that since under Q_d we have

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW^{Q_d}(t)$$

then, under Q_f we must have

$$df_f(t,T) = \left[\alpha_f(t,T) + \sigma_f(t,T)\sigma_N(t)\right]dt + \sigma_f(t,T)dW^{Q_f}(t)$$

where σ_N stands for "the volatility of the numeraire". Since $Var\left[X(t)B_f(t)\right] = VarX(t)$ it follows that $\sigma_N = \sigma_x$.

$$df_f(t,T) = \left[\underbrace{\alpha_f(t,T) + \sigma_f(t,T)\sigma_x(t)}_{\tilde{\alpha_f}(t,T)}\right]dt + \sigma_f(t,T)dW^{Q_f}(t)$$

From the HJM drift condition we then have

$$
\begin{aligned}
\tilde{\alpha_f}(t,T) &= \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds \\
\alpha_f(t,T) + \sigma_f(t,T)\sigma_x(t) &= \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds \\
\alpha_f(t,T) &= \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \sigma_x(t)\right]
\end{aligned}
$$

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Exercise 23.4 From the relation

$$g(t,T) = f_f(t,T) - f_d(t,T)$$

and using the Q_d-dynamics for f_d and f_f in Exercise 23.3 we can, by Itô find the Q_d dynamics of the yield spread g:

$$\begin{aligned} dg(t,T) &= df_f(t,T) - df_d(t,T) \\ &= \underbrace{[\alpha_f(t,T) - \alpha_d(t,T)]}_{\alpha_g(t,T)} dt + \underbrace{[\sigma_f(t,T) - \sigma_d(t,T)]}_{\sigma_g(t,T)} dW^{Q_d}(t). \end{aligned}$$

From the Exercise 23.3 we have, as well, the modified drift condition

$$\alpha_f(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \sigma_x(t)\right]$$

and for α_d the standard drift condition holds, i.e.

$$\alpha_d(t,T) = \sigma_d(t,T)\int_t^T \sigma_d(t,s)ds.$$

We, thus, have

$$\begin{aligned} \alpha_g(t,T) &= \alpha_f(t,T) - \alpha_d(t,T) \\ &= \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \sigma_x(t)\right] - \sigma_d(t,T)\int_t^T \sigma_d(t,s)ds \\ &= \{\text{since } \sigma_f(t,T) = \sigma_g(t,T) + \sigma_d(t,T)\} \\ &= \sigma_g(t,T)\left[\int_t^T \sigma_g(t,s)ds + \int_t^T \sigma_d(t,s) - \sigma_x(t)\right] + \sigma_d(t,T)\left[\int_t^T \sigma_g(t,s)ds - \sigma_x(t)\right] \end{aligned}$$

Exercise 23.5 From

$$C(t) = \int_t^\infty p(t,s)ds,$$

heuristically we have

$$\begin{aligned} dC(t) &= d\left(\int_t^\infty dp(t,s)ds\right) \\ &= -p(t,t)dt + \int_t^\infty dp(t,s) \\ &= -1dt + \int_t^\infty [p(t,s)r(t)dt + p(t,s)v(t,s)dW(t)]\, ds \\ &= \left[-1 + r(t)\underbrace{\int_t^\infty p(t,s)ds}_{C(t)}\right] dt + \left[\underbrace{\int_t^\infty p(t,s)v(t,s)ds}_{\sigma_C(t)}\right] dW(t) \\ &= [C(t)r(t) - 1]\, dt + \sigma_C(t)dW(t) \end{aligned}$$

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Chapter 24

Change of Numeraire

Exercise 24.1 In the *Ho-Lee model* we have under Q

$$dr(t) = \Theta(t)r(t)dt + \sigma dW(t)$$

And we know that on this model we have an affine term structure for bond-prices

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}$$

with $B(t,T) = T - t$.

If the Q-dynamics of $p(t,T)$ are given by

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

then by Ito we have

$$v(t,T) = -\sigma B(t,T) = -\sigma(T-t).$$

The project is to price an European call option with:

- date of maturity T_1
- strike price K
- where the underlying is a zero-coupon bond with date of maturity T_2

and $T_1 < T_2$.

Note that our Z-claim is given by

$$\mathcal{Z} = \max\left[p(T_1,T_2) - K; 0\right],$$

Hence, we have using a change of measure on the standard arbitrage pricing formula that

$$\begin{aligned}
\Pi(t;\mathcal{Z}) &= E_t^Q\left[e^{-\int_t^{T_1} r(s)ds}\max\left[p(T_1,T_2)-K;0\right]\right] \\
&= p(t,T_1)E_t^{T_1}\left[\max\left[p(T_1,T_2)-K;0\right]\right] \\
&= p(t,T_1)E_t^{T_1}\left[\max\left[\underbrace{\frac{p(T_1,T_2)}{p(T_1,T_1)}}_{Z(T_1)}-K;0\right]\right].
\end{aligned}$$

Note that dividing $p(T_1,T_1)$ in the last step of the equation "changes nothing" (since $p(T,T)=1$ for all T) but has the advantage of seeing our claim as a claim on the process Z, which we know is a martingale under Q^{T_1}, and as long as it has deterministic volatility the Black-Scholes formula can help us. To check this note that

$$Z(t)=\frac{p(t,T_2)}{p(t,T_1)}$$

can also be written as (just using the fact that we have an ATS)

$$Z(t)=\exp\left\{A(t,T_2)-A(t,T_1)-\left[B(t,T_2)-B(t,T_1)\right]r(t)\right\},$$

and, therefore, has the following dynamics under Q(applying Ito formula)

$$dZ(t)=\{\cdots\}\,Z(t)dt+Z(t)\sigma_z(t)dW(t)$$

where $\sigma_z(t)=-\sigma\left[B(t,T_2)-B(t,T_1)\right]=-\sigma\left(T_2-T_1\right)$, will be the same as under Q^{T_1}, and is deterministic.

Since under Q^{T_1},

$$Z(T_1)=Z(t)-\sigma\left(T_2-T_1\right)\int_t^{T_1}dW(s)$$

The conditional (on information at time t) distribution of Z is $Z(T_1)\sim N\left[Z(t),\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)\right]$.

So, (from the BS formula)

$$\begin{aligned}
\Pi(t) &= p(t,T_1)\left\{Z(t)N\left[d_1(t,T_1)\right]-KN\left[d_2(t,T_1\right]\right\} \\
&= p(t,T_2)N\left[d_1(t,T_1)\right]-p(t,T_1)KN\left[d_2(t,T_1\right]
\end{aligned}$$

where

$$d_1(t,T_1)=\frac{\ln\frac{Z(t)}{K}+\frac{1}{2}\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}}=\frac{\ln\frac{p(t,T_2)}{Kp(t,T_1)}+\frac{1}{2}\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}}$$

and $d_2(t,T_1)=d_1(t,T_1)-\sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}$.

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Exercise 24.2 Take as given an HJM model of the form

$$df(t,T) = \alpha(t,T) + \sigma(t,T)dW(t) + \sigma_2 e^{-a(T-t)}dW_2(t)$$

where $\sigma(t,T) = [\,\sigma_1(T-t) \quad \sigma_2 e^{-a(T-t)}\,]$, $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ and we have W_1 and W_2 independent Wiener processes and σ_1 and σ_2 constants.

(a) From the relation between the dynamics of bond prices and forward rates we know that

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)S(t,T)dW(t)$$

with $S(t,T) = -\int_t^T \sigma(t,s)ds$.

In this case we have then

$$\begin{aligned} S(t,T) &= -\int_t^T [\,\sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)}\,]\,ds \\ &= \left[\underbrace{-\frac{\sigma_1}{2}(T-t)^2}_{\sigma_1^p(t,T)} \quad \underbrace{-\frac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)}_{\sigma_2^p(t,T)}\right] \end{aligned}$$

So,

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)dt + p(t,T)S(t,T)dW(t) \\ &= r(t)p(t,T)dt + p(t,T)\,[-\tfrac{\sigma_1}{2}(T-t)^2 \quad -\tfrac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \\ &= r(t)p(t,T)dt + p(t,T)\,[\,\sigma_1^p(t,T) \quad \sigma_2^p(t,T)\,]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \end{aligned}$$

(b) We will use exactly the same technique as in exercise 24.1 to price an European call option with maturity T_0 on a T_1-bond.

Note that in this model the Z process $Z(t) = \frac{p(t,T_1)}{p(t,T_0)}$ is a martingale under Q^{T_0} and its dynamics has deterministic volatility given by

$$\sigma_{T_1,T_0}(t) = S(t,T_1) - S(t,T_0) = -\int_{T_0}^{T_1} [\,\sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)}\,]\,ds$$

Hence (by the same arguments as in exercise 24.1)

$$Z(T_0) \sim N\left[Z(t), \underbrace{\int_t^{T_0} ||\sigma_{T_1,T_0}(s)||^2 ds}_{\Sigma^2_{T_0,T_1}}\right]$$

and the pricing formula is given by

$$\begin{aligned} \Pi(t) &= p(t,T_0)\,\{Z(t)N\,[d_1(t,T_0)] - KN\,[d_2(t,T_0]\} \\ &= \{p(t,T_1)N\,[d_1(t,T_0)] - p(t,T_0)KN\,[d_2(t,T_0]\} \end{aligned}$$

where

$$d_1(t,T_0) = \frac{\ln \frac{Z(t)}{K} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}} = \frac{\ln \frac{p(t,T_2)}{Kp(t,T_1)} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}}$$

and $d_2(t,T_1) = d_1(t,T_1) - \sqrt{\Sigma^2_{T_0,T_1}}$.

Exercise 24.3 To be done!

Exercise 24.4 To be done!

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