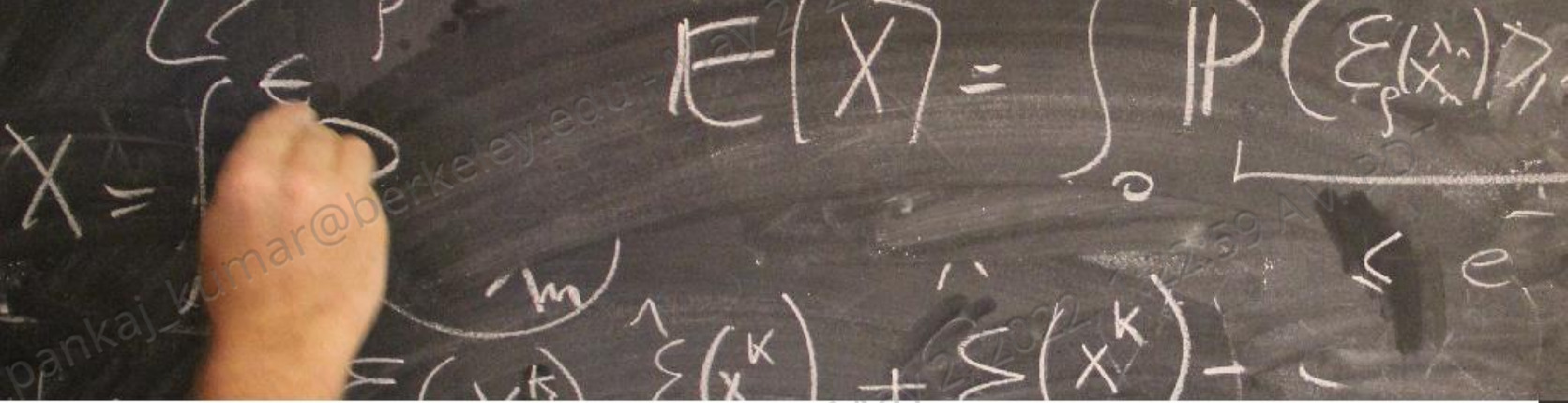

MFE 230Q [Spring 2021]
Introduction to Stochastic Calculus
GSI Session 6

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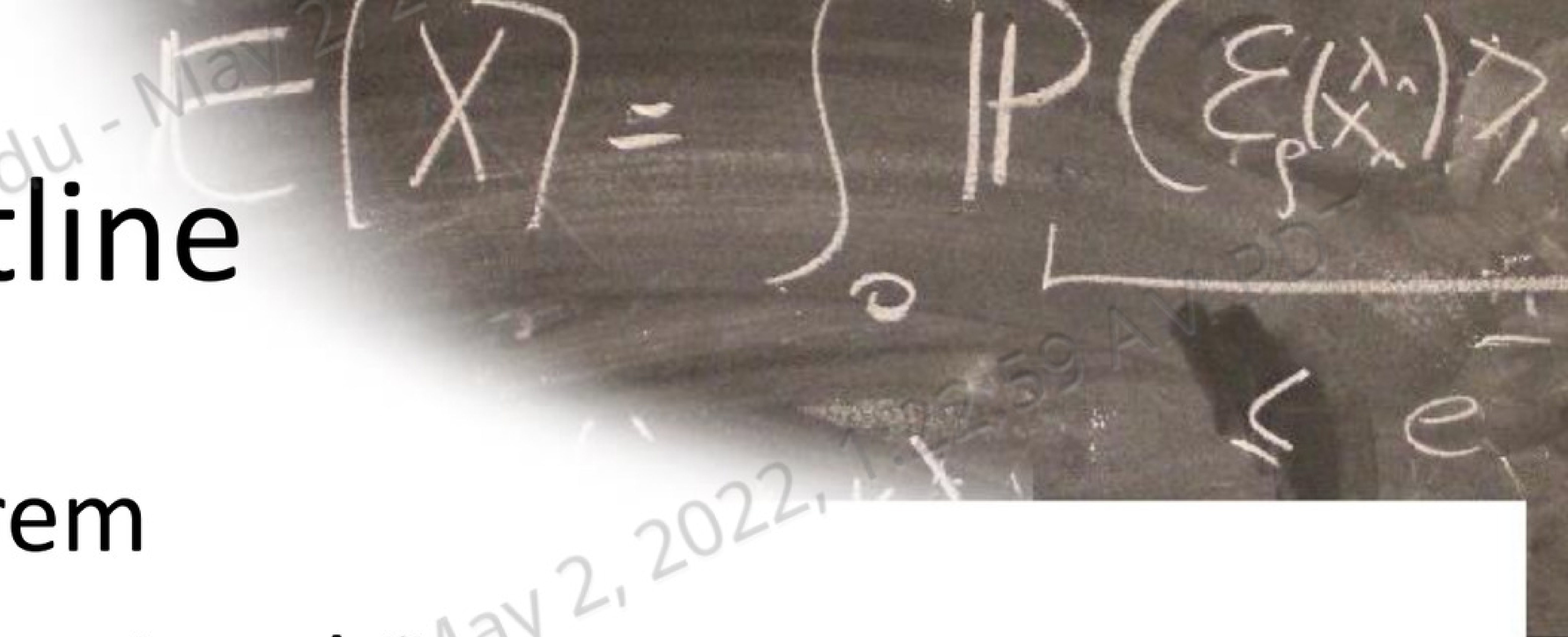
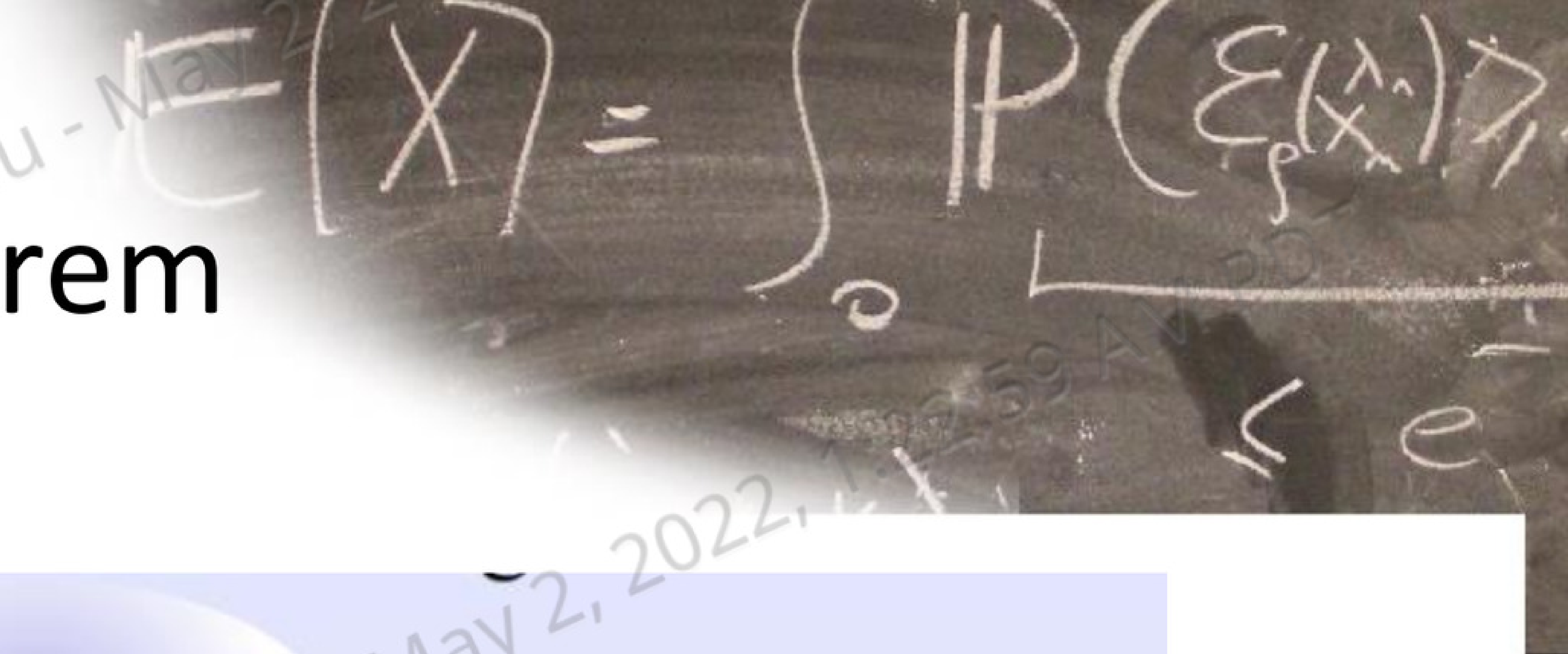
Review - Outline



- Girsanov Theorem
- Sample Problems I and II
- Martingale Property of Radon-Nikodym Process and Application (Extra)

Girsanov Theorem - Motivation



True dynamics:
$\frac{dS}{S} = \mu dt + \sigma dW$ $\mathbb{P}$

Replication

Black Scholes PDE:
$V_t + rSV_S + \frac{\sigma^2}{2} S^2 V_{SS} - rV = 0$

Kolmogorov

Measure transformation:
$\mathbb{P} \mapsto \mathbb{Q}$ via $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$

- Girsanov's theorem provides measure transformation
- Reason for result is still replication, not expectation
- Conceptually, this "route" is more challenging
- New approach easier for many claims, which BS not directly available

Risk neutral dynamics: $\mathbb{Q}$
$V_t = e^{-r(T-t)} E_t^Q [\Phi(S(T))],$
$\frac{dS}{S} = rdt + \sigma dW^Q.$

Multiple asset (N traded risky assets)

No Divs.

$$\frac{dS^i}{S^i} = \mu_t^i\,dt + \sigma_t^i\,dW_t^i, \quad i = 1, \dots, N \quad (\mathbb{P})$$

$$\mathrm{cov}(dW^i, dW^j) = \rho^{ij}\,dt$$

const.

?

$$\frac{dB_t}{B_t} = r_t\,dt \Rightarrow R_{t,T} = e^{\int_t^T r(s)\,ds} = e^{r(T-t)}$$

Defn $\mathbb{Q}$ (EMM). $\mathbb{Q} \sim \mathbb{P}$ iff $S^0, S^1, S^2, \dots, S^N$ have r_t as drift.

traded!

$$\mathbb{Q}: \quad dS^i = S^i r_t\,dt + S^i \sigma_t^i\,dW_t^i$$

S^{N+1}

. . . .

Pricing

$\mathbb{Q}$ EMM: $V_t = E_t^{\mathbb{Q}}\left(R_{t,T}^{-1} V_T\right) = E_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,ds} V_T\right)$

Radon-Nikodym derivative

Q prob measures $\mathbb{Q} \sim \mathbb{P}$ $\exists$ strictly positive rv

ξ st

$$E^{\mathbb{Q}}(Z) = E^{\mathbb{P}}(\xi Z) \quad \forall \text{ rv } Z$$

?

$$\Rightarrow E^{\mathbb{P}}(\xi) = 1$$

$$\mathbb{Q}(A) = E^{\mathbb{Q}}(1_A) = E^{\mathbb{P}}(\xi 1_A)$$

RN derivative X_t

$$E_t^{\mathbb{Q}}(X_T) = E_t^{\mathbb{P}}\left(\frac{\xi_T}{\xi_t} X_T\right) \checkmark$$

where $\xi_t = E_t^{\mathbb{P}}(\xi_T)$ $\checkmark$

$\xi_0 = 1$ $\checkmark$

Pricing Thms

$$V_t = E_t^{\mathbb{Q}}\left(R_{t,T}^{-1} V_T\right) \quad (\text{EMM})$$

$$= E_t^{\mathbb{P}}\left(\frac{\xi_T}{\xi_t} R_{t,T}^{-1} V_T\right) \quad (\text{LR})$$

$$= E_t^{\mathbb{P}}\left(\frac{M_T}{M_t} V_T\right) \quad (\text{SDF})$$

$$M_s = \frac{\xi_s}{R_{0,s}}, \quad 0 \le s \le T, \quad M_0 = 1, \quad R_{0,0} = 1$$

Girsanov Theorem
- Measure Transformation

The Radon-Nikodym derivative for continuous state spaces:

- Given two equivalent probability measures on a sample space, Ω, (with associated σ-algebra $\mathcal{F}$), $P \sim Q$, there is a strictly positive measurable r.v. ξ, such that $E^Q[Z] = E^P[\xi Z]$ for all (measurable) r.v., Z, such that $E^Q[|Z|] < \infty$.
  - Specifically, $E^P[\xi] = 1$.
- Similarly, for a strictly positive $\mathcal{F}$-measurable r.v., ξ, such that $E^P[\xi] = 1$, an equivalent measure, $Q \sim P$ is defined by $\mathbb{Q}(A) = E^Q[1_A] = E^P[\xi 1_A]$ for all sets $A \in \mathcal{F}$, where 1_A is the indicator function on A.

Girsanov Theorem - Bjork's Treatment

Setup

Let W be a P-Wiener process and fix a time horizon T. Suppose that we want to change measure from P to Q on $\mathcal{F}_T$. For this we need a P-martingale L with $L_0 = 1$ to use as a likelihood process, and a natural way of constructing this is to choose a process g and then define L by

$$\begin{cases} dL_t &= g_t dW_t \\ L_0 &= 1 \end{cases}$$

Girsanov Theorem - Bjork's Treatment

This definition does not guarantee that $L \geq 0$, so we make a small adjustment. We choose a process φ and define L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

The process L will again be a martingale and we easily obtain

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

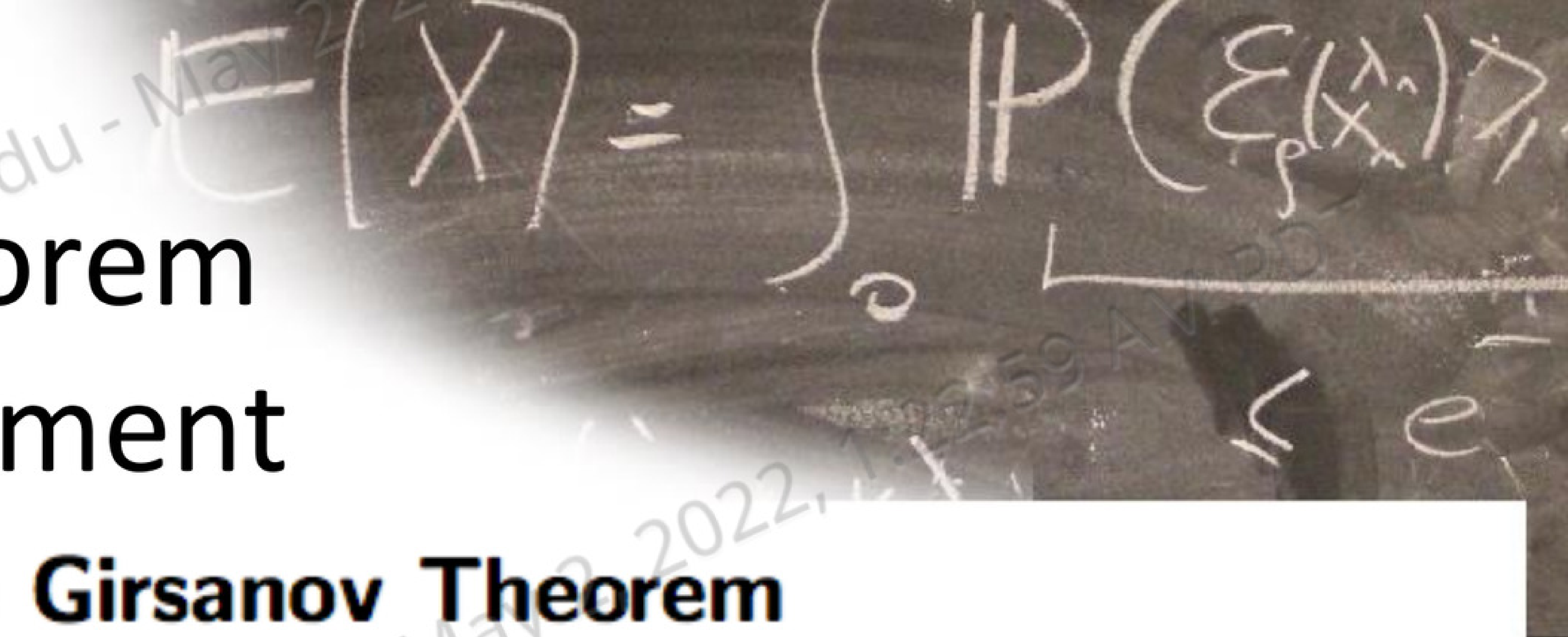
Girsanov Theorem - Bjork's Treatment

Thus we are guaranteed that $L \geq 0$. We now change measure form P to Q by setting

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\, 0 \leq t \leq T$$

The main problem is to find out what the properties of W are, under the new measure Q. This problem is resolved by the **Girsanov Theorem**.

Girsanov Theorem
- Bjork's Treatment



The Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem: Choose an adapted process φ, and define the process L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

Assume that $E^P[L_T] = 1$, and define a new mesure Q on $\mathcal{F}_T$ by

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

Girsanov Theorem - Bjork's Treatment

The Girsanov Theorem

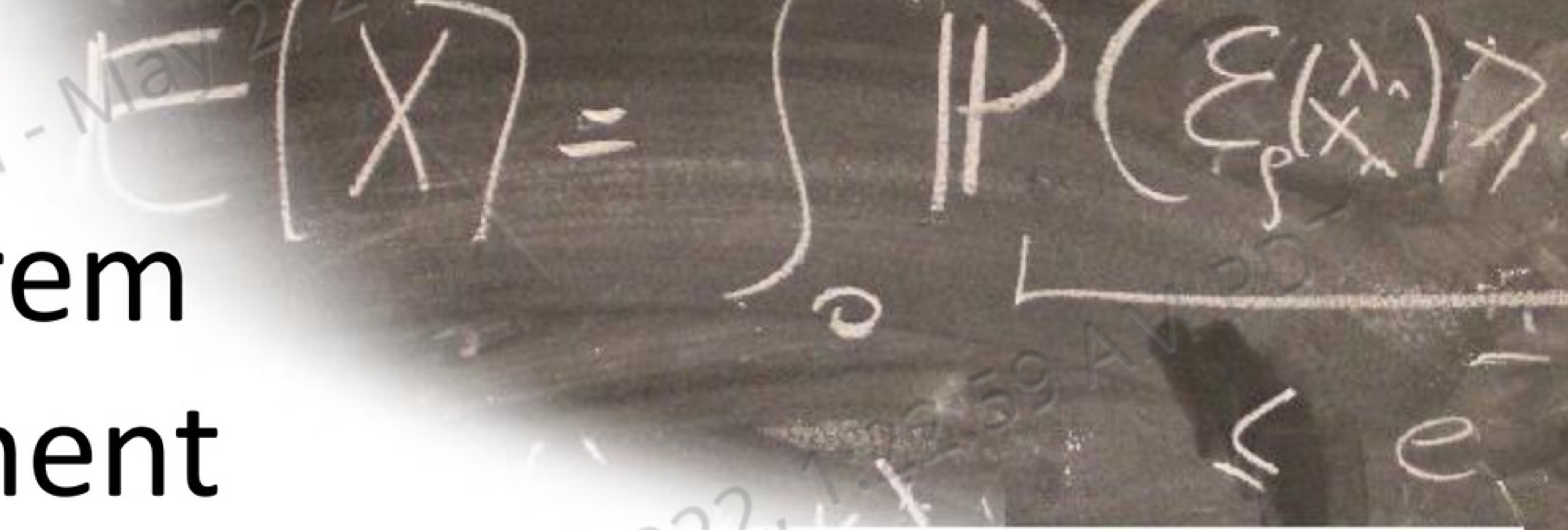
Then $Q << P$ and the process W^Q, defined by

$$W_t^Q = W_t - \int_0^t \varphi_s ds$$

is Q-Wiener. We can also write this as

$$dW_t = \varphi_t dt + dW_t^Q$$

Girsanov Theorem - Bjork’s Treatment



The Converse Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem. Assume that:

- $Q << P$ on $\mathcal{F}_T$, with likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t \; 0, \leq t \leq T$$

Girsanov Theorem - Bjork's Treatment

The Converse Girsanov Theorem

- The filtation is the **internal** one .i.e.

$$\mathcal{F}_t = \sigma\left\{W_s;\ 0 \le s \le t\right\}$$

Then there exists a process φ such that

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

Girsanov's Thm (univariate)

θ_t, $0 \le t \le T$ adapted, Novikov's condition $E\left(e^{\frac{1}{2}\int_0^T \theta_s^2 ds}\right) < \infty$

Define:

① $W_t^{Q} = W_t + \int_0^t \theta_s ds \Rightarrow dW^Q = dW + \theta_t dt$

② $\xi_t = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \Rightarrow \xi_0 = 1$ ✓

③ $E_0^{Q}(Z_T) = E_0^{P}(\xi_T Z_T)$ $\forall$ rv Z_T

Ito's Lemma

② $\Rightarrow d\xi_t = \underbrace{-\xi_t \theta_t dW_t}_{\text{drift}=0} \Rightarrow \xi_t$ P martingale

$\Rightarrow \xi_t = E_t^{P}(\xi_T \mid F_t)$ ✓

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t \quad \mathbb{P}\text{-process.}$$

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}$$

?

$$dX_t = r_t\, dt + \sigma_t\, dW_t^{\mathbb{Q}} \quad \mathbb{Q}\text{-process}$$

$$dW_t^{\mathbb{Q}} = dW_t + \theta_t\, dt \Rightarrow dW_t = dW_t^{\mathbb{Q}} - \theta_t\, dt$$

Multivariate Girsanov

$\underset{k\times 1}{\theta_t}$, $\underset{k\times 1}{W_t}$, $\mathbb{E}\left(e^{\frac{1}{2}\int_0^t \theta_s'\theta_s\,ds}\right) < \infty$

(1) $\underset{k\times 1}{W_t^Q} = \underset{k\times 1}{W_t} + \int_0^t \underset{k\times 1}{\theta_s}\,ds$ element wise $\Rightarrow$ $dW^Q = dW + \theta_t\,dt$

(2) $\underset{1\times 1}{\xi_t} = e^{-\int_0^t \underset{1\times k}{\theta_s'}\,\underset{k\times 1}{dW_s} - \frac{1}{2}\int_0^t \underset{1\times k}{\theta_s'}\,\underset{k\times 1}{\theta_s}\,ds}$

(3) Same

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t \qquad \mathbb{P}\text{ process}$$

$N\times 1 \quad N\times 1 \quad 1\times 1 \quad N\times N \quad N\times 1$

$$\Leftrightarrow \begin{bmatrix} dX_t^1 \\ \vdots \\ dX_t^N \end{bmatrix} = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^N \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & & \sigma_{NN} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^N \end{bmatrix}$$

$$\theta_t = \sigma_t^{-1} (\mu_t - V_t)$$

$N\times 1 \quad N\times N \quad N\times 1 \quad N\times 1$

$$dX_t = V_t \, dt + \sigma_t \, dW_t^{\mathbb{Q}} \qquad \mathbb{Q}\text{-process}$$

Sample Problem 1

1 Up-and-In Down-and-In Perpetual Barrier Option

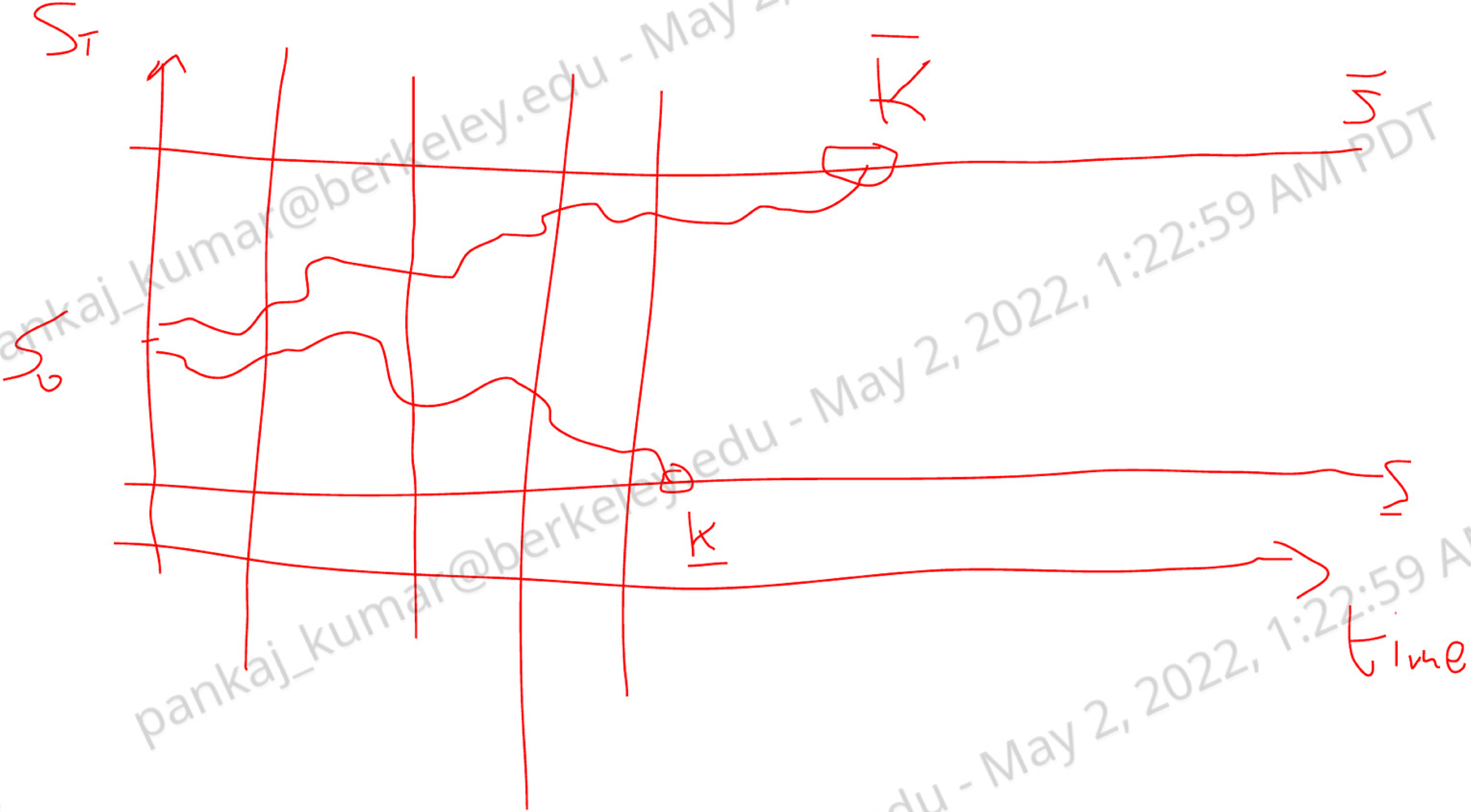
There are a stock and money market account with Q dynamics

$$dS = rS\,dt + \sigma S dW_t^Q$$
$$dB = rB\,dt.$$

Someone offers you an up-and-in, down-and-in perpetual barrier option that pays off $\overline{K}$ if the stock hits $\overline{S}$ or $\underline{K}$ if it hits $\underline{S}$ (the option 'dies' once it pays off at either of the barriers). What is the price of the option?

$\overline{K} - \overline{S}$

$\underline{S} - \underline{K}$



S_T
$\bar{K}$
$\bar{S}$
S_0
$\underline{S}$
$\underline{K}$
time

Sample Problem 1

$C_t = 0$

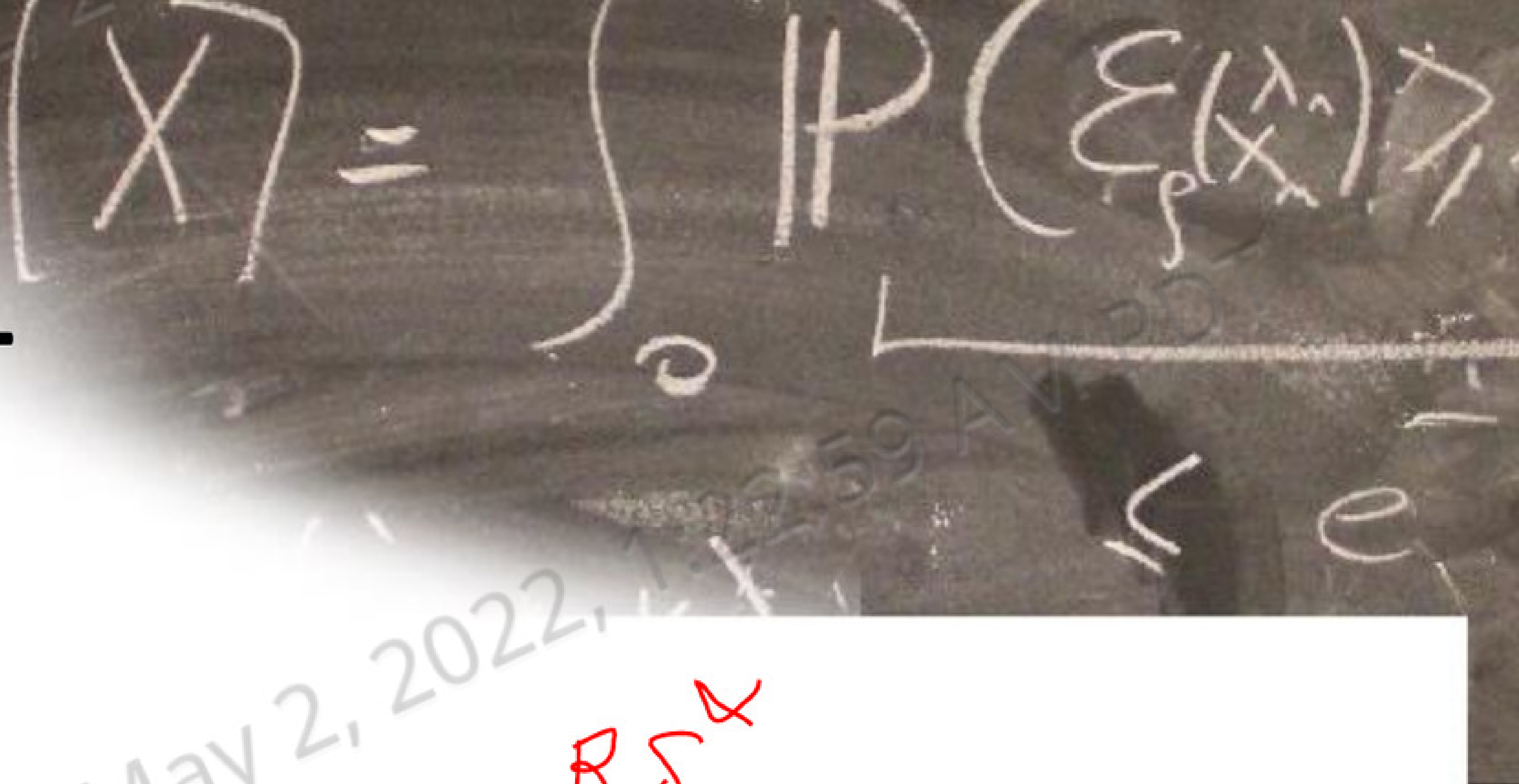
Since the option is perpetual and the dynamics of the state variables do not depend explicitly on time, we know that the price will not depend on t. Therefore, we know that as long as $S \in [\underline{S}, \overline{S}]$ it satisfies the usual Black-Scholes PDE

$$C_S r S + \frac{1}{2} C_{SS} \sigma^2 S^2 - rC = 0.$$

The boundary conditions are

$$C(\underline{S}) = \underline{K} \quad (1)$$
$$C(\overline{S}) = \overline{K}. \quad (2)$$

Sample Problem 1



BS^{α}

Let's conjecture trial solutions of the form $C(S) = S^{\alpha}$, where α is a constant to be determined. Plugging into the PDE, we see that α must satisfy

$$\alpha^2\left(\frac{\sigma^2}{2}\right) + \alpha\left(r - \frac{\sigma^2}{2}\right) - r = 0.$$

Solving this, we get $\alpha = 1$ or $\alpha = -\frac{2r}{\sigma^2}$.

$C(S) = S^{\alpha} \neq 0$

$C_S = \alpha S^{\alpha-1}$

$C_{SS} = \alpha(\alpha-1) S$

$$\alpha S^{\alpha-1} r \cancel{S} + \frac{\sigma^2}{2}\alpha(\alpha-1) S^{\cancel{\alpha-2}} \cancel{S^2} - r S^{\alpha} = 0$$

$$\alpha r + \frac{\sigma^2}{2}\alpha(\alpha-1) - r = 0$$

Quad. α

$$\Longrightarrow \alpha^2 \frac{\sigma^2}{2} + \alpha\left(r - \frac{\sigma^2}{2}\right) - r = 0$$

$$\Longrightarrow \alpha^{+} = 1, \quad \alpha^{-} = -\frac{2r}{\sigma^2}$$

Sample Problem 1

S^{α^+}, S^{α^-}

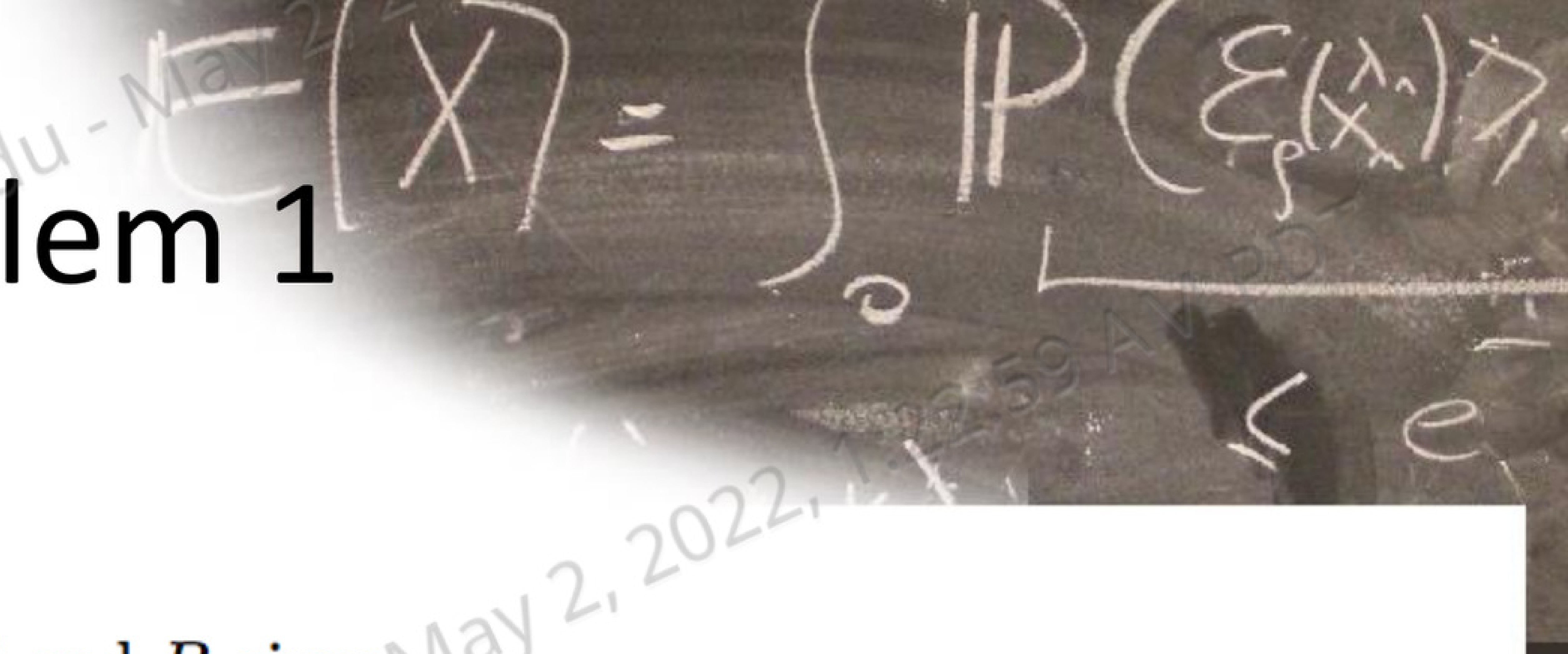
particular
solns

So, the general solution must be of the form $C(S) = AS + BS^{-2r/\sigma^2}$ for some constants A and B. We can determine the constants from the boundary conditions.

$$C(\underline{S}) = \underline{K} \iff A\underline{S} + B\underline{S}^{-2r/\sigma^2} = \underline{K} \quad (1)$$

$$C(\overline{S}) = \overline{K} \iff A\overline{S} + B\overline{S}^{-2r/\sigma^2} = \overline{K} \quad (2)$$

Sample Problem 1



Solving these equations for A and B gives

$$A = \frac{\underline{K}\,\overline{S}^{-2r/\sigma^2} - \overline{K}\,\underline{S}^{-2r/\sigma^2}}{\underline{S}\,\overline{S}^{-2r/\sigma^2} - \overline{S}\,\underline{S}^{-2r/\sigma^2}}$$

$$B = \frac{\overline{K}\,\underline{S} - \underline{K}\,\overline{S}}{\underline{S}\,\overline{S}^{-2r/\sigma^2} - \overline{S}\,\underline{S}^{-2r/\sigma^2}}$$

Now that we know A and B, we know the call price $C(S) = AS + BS^{-2r/\sigma^2}$.

Sample Problem 2

2 Change of Measure

not traded

Consider an economy with two traded assets, V and B, and **another asset** whose price is P and is characterized by its claim to cash flows of x. The dynamics of V , B and x are

$$\frac{dV}{V} = \mu dt + \sigma dW_t$$

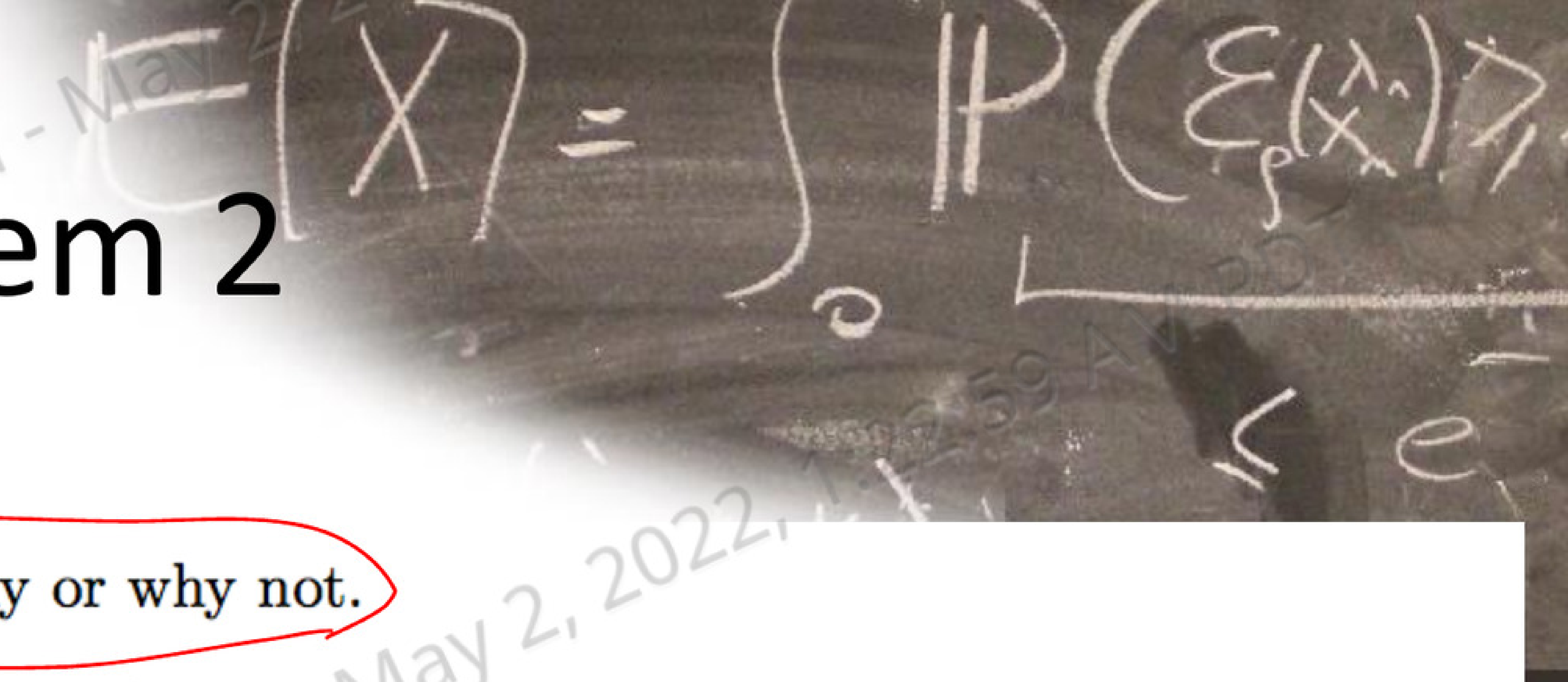
$$\frac{dB}{B} = rdt$$

$$\frac{dx}{x} = \alpha dt + \nu dW_t$$

Same.

Here, μ, σ, r, α and ν are all constants. I emphasize that the Brownian motion dW_t that drives the stock value V is the same Brownian motion that drives the dividend process x.

Sample Problem 2



1. Is this economy complete? Why or why not.
2. Specify the risk-neutral dynamics of x to be

$$\frac{dx}{x} = \alpha^Q dt + \nu^Q dW_t^Q$$

   Determine α^Q and ν^Q.

3. The value of the claim to dividend can be written as

$$P(t) = E_t^Q \left[\int_t^\infty e^{-r(s-t)} x(s) ds \right]$$

   Determine the value of the claim.

4. Find the PDE that $P(t)$ statisfies. Show that the solution to the PDE is the same as what you got in the previous part.

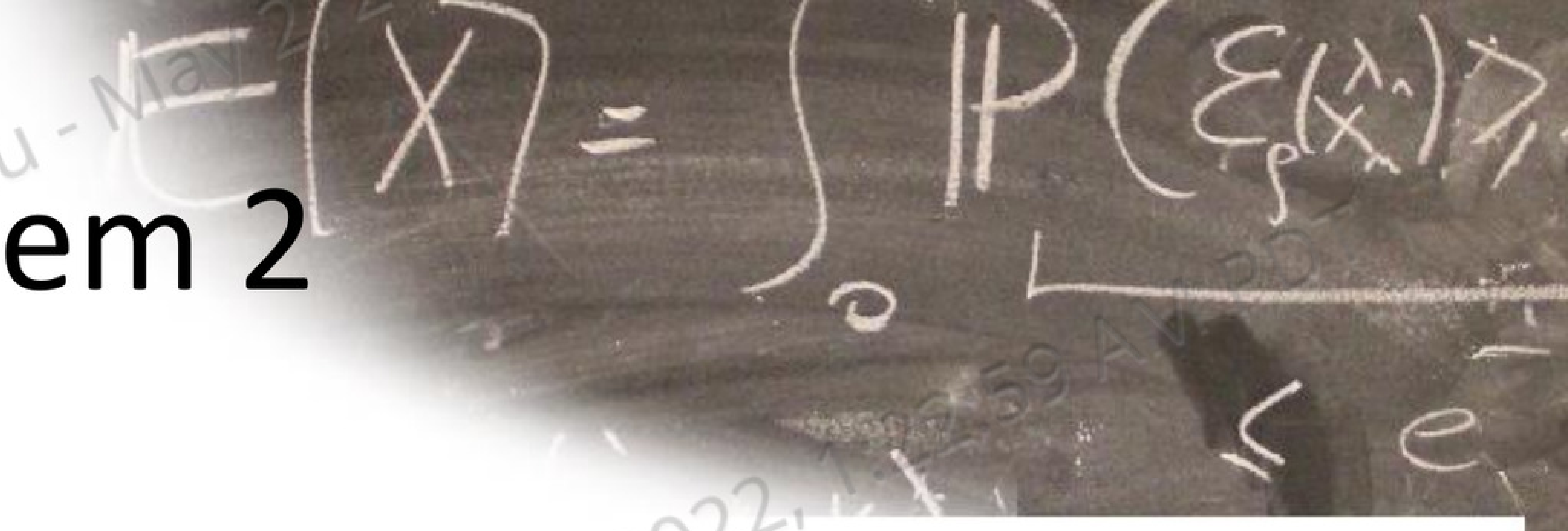
Sample Problem 2

Bjork 'Meta-Thm'

1. Yes, the economy is complete since there is one source of risk and there is one risk traded asset and a bond. The number of sources of risk is equal to the number of brownian motions.

1 BM = 1 traded risky assets.

Sample Problem 2



2. Because V is a traded asset, we know that its expected return under the risk-neutral measure is r. That is:

$$\theta = \frac{\mu - r}{\sigma}$$

$$\begin{aligned} \frac{dV}{V} &= \mu dt + \sigma dW_t \\ &= r dt + \sigma dW_t^Q \end{aligned}$$

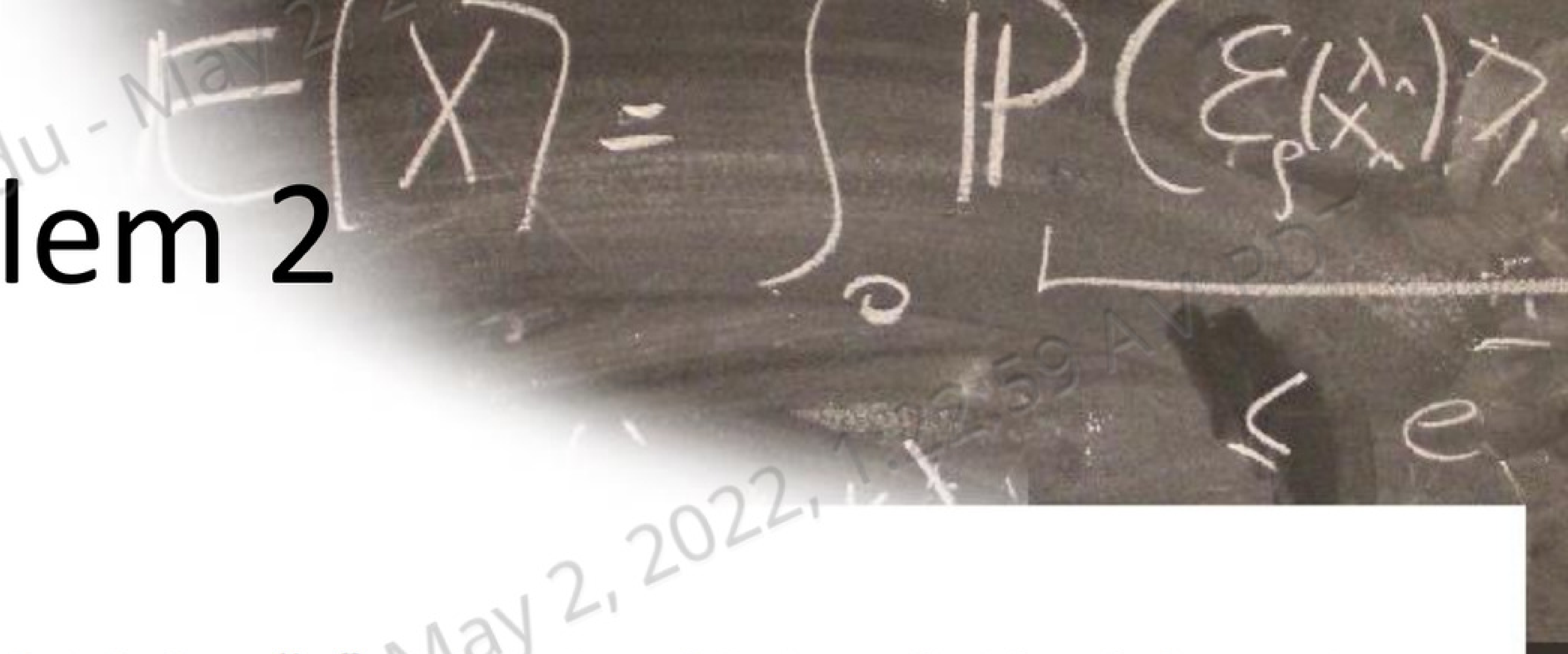
traded!

Thus we find that

Girsanov's.

$$dW_t = dW_t^Q - \theta dt$$

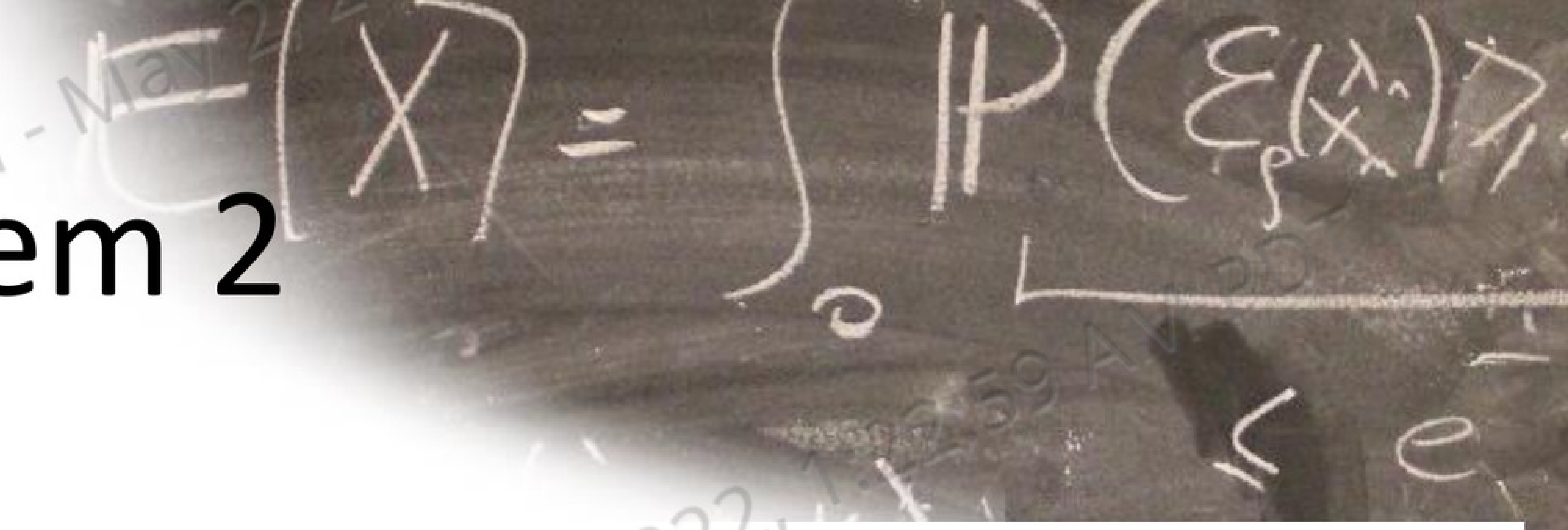
Sample Problem 2



where I define the market price of risk $\theta = \frac{\mu - r}{\sigma}$. Plugging this into dividend dynamics, we find:

$$
\begin{aligned}
\frac{dx}{x} &= \alpha dt + \nu dW_t \\
&= \alpha dt + \nu(dW_t^Q - \theta dt) \\
&= \underbrace{(\alpha - \nu\theta)}_{\alpha^Q} dt + \nu dW_t^Q
\end{aligned}
$$

Sample Problem 2



3. We can re write the value of the claim to dividend as:

$$E_t^Q\left(\int_t^\infty e^{-r(s-t)} x_s\, ds\right) \quad P(t) = \int_t^\infty e^{-r(s-t)} E_t^Q\left[x(s)\right] ds$$

α

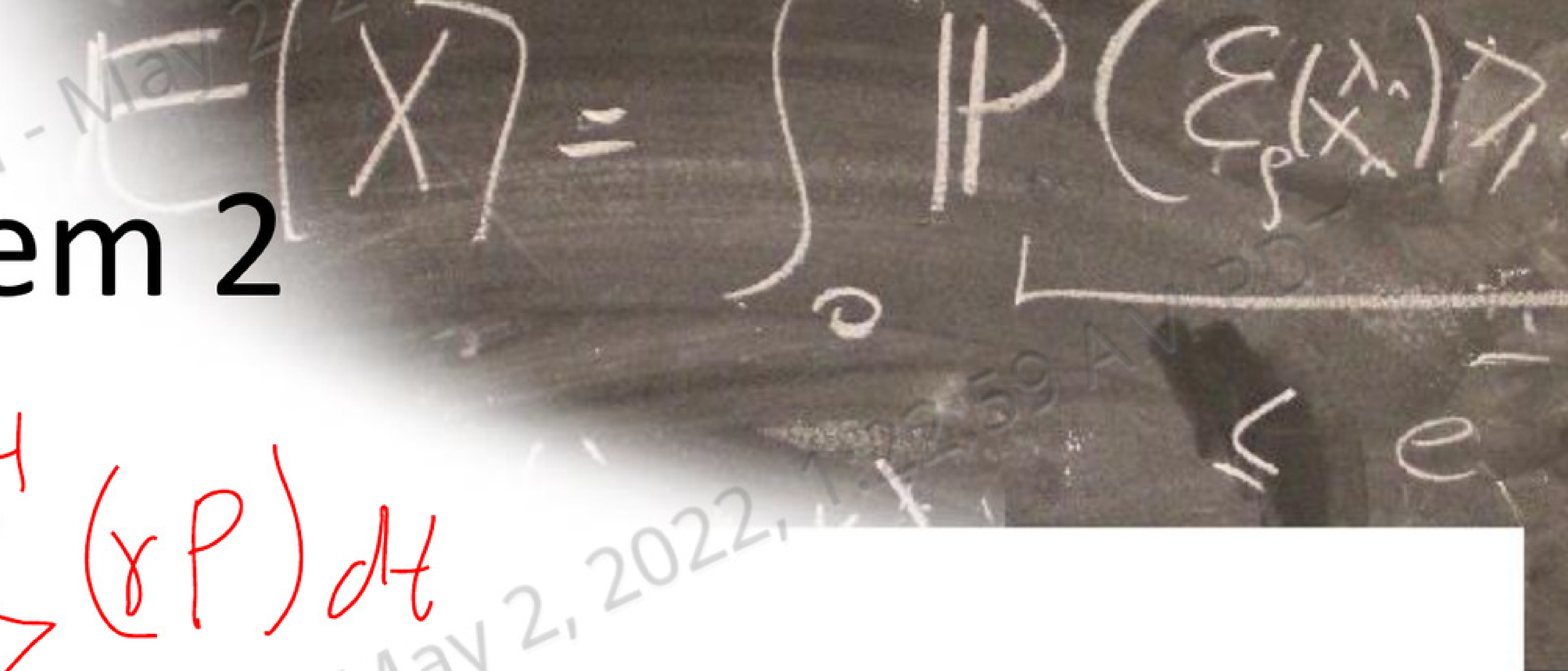
Since $x(s)$ follows a geometric brownian motion we should be able to show that

$$E_t^Q\left[x(s)\right] = x(t)e^{\alpha^Q(s-t)}$$

implying that

$$P(t) \;= \int_t^\infty e^{-(r-\alpha^Q)(s-t)} ds \;= \frac{x(t)}{r-\alpha^Q}$$

Sample Problem 2



P

0 dt (rP) dt

4. We can use the fact that under the risk-neutral measure expected returns (capital gains plus dividends) equal the risk free rate:

$$\boxed{rPdt} = E_t^Q\left[dP + xdt\right] = \left(\alpha^Q x P_x + \frac{\nu^2 x^2}{2} P_{xx} + x\right) dt$$

where I used the fact that the security price has no explicitly function of time (i.e., $P_t = 0$). It is clear that this ODE is consistent with the solution in the previous part - you can double check this.

$$\left(\alpha^Q x P_x + \frac{\nu^2 x^2}{2} P_{xx} + x - rP\right) = 0$$

$\boxed{\text{FK}}$ #3

Useful Martingale Properties (Extra)

Assume W_t is a brownian motion, then:

$$W_t$$

$$W_t^2 - t$$

$$e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$$

are all martingales, which are useful in problem solving, especially in interviews.