MFE 230Q: In-Class Quiz #3

1. A stock price follows geometric Brownian motion (under the risk neutral measure)

$$\frac{dS}{S} = r\,dt + \sigma dW^Q.$$

   (a) Define $s = \log S$. Use Ito's lemma to find its dynamics.

   (b) A forward contract that matures at date-T has a value of date-T equal to $(S(T) - F)$ for some fixed forward price F. The date-t value of this contract can be written as

$$\begin{aligned} V(t, S(t)) &= \mathrm{E}_t^Q\left[e^{-r(T-t)}\left(S(T) - F\right)\right] \\ &= \mathrm{E}_t^Q\left[e^{-r(T-t)}\left(e^{s(T)} - F\right)\right]. \end{aligned}$$

   Use what you know about expectations of normal variables to directly determine its price today. Interpret your result in terms of a static replication.

2. Assume that x follows the Markov process

$$dx = \kappa\left(\theta - x\right)\,dt + \sigma\,dW^Q,$$

where θ and σ are constants. Also, think of $\Phi(X(t))$ as some 'payoff' at date-t that depends only upon the date-T value of $x(T)$.

   (a) Define

$$y(t, x(t)) = \mathrm{E}_t^Q\left[e^{-r(T-t)}\;\Phi(X(T))\right].$$

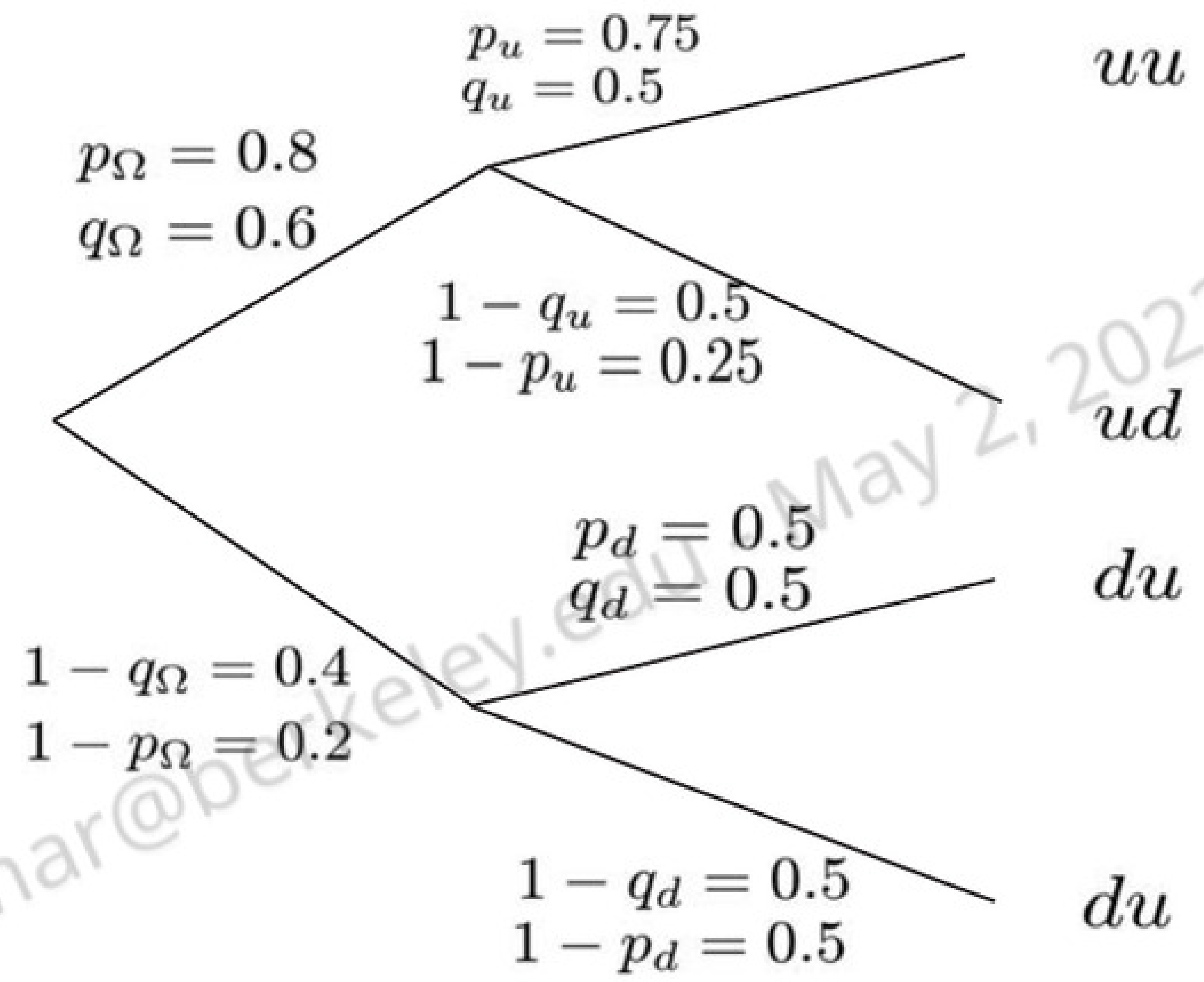
   Is $y(t, x(t))$ a Q-martingale? If so, find the PDE that y satisfies. If not, find a function of y that is a martingale, and then identify the PDE that y satisfies.

   (b) Define

$$y(t, x(t)) = \mathrm{E}_t^Q\left[e^{-\int_t^T r(x(s))\,ds}\;\Phi(X(T))\right].$$

   Is $y(t, x(t))$ a Q-martingale? If so, find the PDE that y satisfies. If not, find a function of y that is a martingale, and then identify the PDE that y satisfies.

3. Consider the following two period, three date ($t = 0, 1, 2$) market with four states ($\Omega = \{uu, ud, du, dd\}$, which was introduced in the lecture notes, part 2.2. As in the lecture notes, we assume that the risk-free rate process is $R_{t,t+1} \equiv 1$. We wish to verify the relationships between the EMM, the Radon-Nikodym derivative, and the SDF in this market:



$$V_t = E_t^Q\left[\frac{1}{R_{t,T}}V_T\right] = E_t^P\left[\frac{\xi_T}{\xi_t}\frac{1}{R_{t,T}}V_T\right] = E_t^P\left[\frac{M_T}{M_t}V_T\right]. \tag{1}$$

(a) From the risk-neutral and true probabilities, calculate the Radon-Nikodym derivative, $\frac{d\mathbb{Q}}{d\mathbb{P}}$ on 2^Ω.

(b) Use the one-period conditions to define random variables η_1 ($\mathcal{F}_1$-measurable) and η_2 ($\mathcal{F}_2$-measurable), such that the pricing formulas $V_0 = E_0^P[\eta_1 V_1]$, and $V_1 = E_1^P[\eta_2 V_2]$ are satisfied.

(c) Use the law of iterated expectations to derive a pricing formula for V_0 in terms of η_1, η_2, and the P-expectations operator, $E_0^P[\cdot]$.

(d) Define $\xi_t = E_t^P\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. Show that $\xi_0 = 1$, $\xi_1 = \eta_1$ and $\xi_2 = \eta_1\eta_2$.

(e) Use the previous results to deduce that (1) holds.

Solutions

1. (a) $s = \log S$. Hence, $s_S = S^{-1}$, $s_{SS} = -S^{-2}$. Thus, from Ito's lemma we have

$$\begin{aligned} ds &= s_t\, dt + s_S\, dS + \frac{1}{2} s_{SS}\, dS^2 \\ &= 0 + \frac{dS}{S} - \frac{1}{2}\left(\frac{dS}{S}\right)^2 \\ &= \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW^Q. \end{aligned} \tag{2}$$

(b) From eq. (2), we have

$$s(T)|\mathcal{F}_t \sim N\left(s(t) + \left(r - \frac{\sigma^2}{2}\right)(T-t),\ \sigma^2(T-t)\right). \tag{3}$$

We thus have

$$\begin{aligned} V(t, S(t)) &= \mathrm{E}_t^Q\left[e^{-r(T-t)}\left(e^{s(T)} - F\right)\right] \\ &= e^{-r(T-t)}\mathrm{E}_t^Q\left[e^{s(T)}\right] - F\, e^{-r(T-t)} \\ &= e^{-r(T-t)} e^{s(t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \frac{\sigma^2}{2}(T-t)} - F\, e^{-r(T-t)} \\ &= S(t) - F\, e^{-r(T-t)}. \end{aligned} \tag{4}$$

Interpreting these results: the CF's at date-T of the forward contract are $(S(T) - F)$. To replicate these CF's using only the stock and bond markets, purchase one share of stock today (at price $S(t)$), and borrow $F\, e^{-r(T-t)}$. At date-T, the stock will pay off $S(T)$, and the loan will cost $-F$.

2. (a) Choose $v < t$. From the definition, we have

$$\begin{aligned} y(t, x(t)) &= \mathrm{E}_t^Q\left[e^{-r(T-t)}\,\Phi(X(T))\right] \\ y(v, x(v)) &= \mathrm{E}_v^Q\left[e^{-r(T-v)}\,\Phi(X(T))\right]. \end{aligned} \tag{5}$$

From the law of iterated expectations (LIE), we find

$$\begin{aligned} \mathrm{E}_v^Q\left[y(t, x(t))\right] &= \mathrm{E}_v^Q\left[\mathrm{E}_t^Q\left[e^{-r(T-t)}\,\Phi(X(T))\right]\right] \\ &\overset{LIE}{=} \mathrm{E}_v^Q\left[e^{-r(T-t)}\,\Phi(X(T))\right] \\ &\neq y(v, x(v)). \end{aligned} \tag{6}$$

Hence, y is not a martingale. However, if we define

$$\begin{aligned} F(t, y(t)) &\equiv e^{-rt}\, y(t) = \mathrm{E}_t^Q\left[e^{-rT}\,\Phi(X(T))\right] \\ F(v, y(v)) &\equiv e^{-rv}\, y(v) = \mathrm{E}_v^Q\left[e^{-rT}\,\Phi(X(T))\right], \end{aligned} \tag{7}$$

we find that F is a martingale in that

$$\begin{aligned} \mathrm{E}_v^Q\left[F(t,y(t))\right] &= \mathrm{E}_v^Q\left[\mathrm{E}_t^Q\left[e^{-rT}\,\Phi(X(T))\right]\right] \\ &\overset{LIE}{=} \mathrm{E}_v^Q\left[e^{-rT}\,\Phi(X(T))\right] \\ &= F(v,y(v)). \end{aligned} \tag{8}$$

Note that $F_t = -re^{-rt}\,y(t)$, $F_y = e^{-rt}$, $F_{yy} = 0$. Thus, from Ito's lemma (or, equivalently, from Feynman-Kac), we get the PDE

$$\begin{aligned} 0 &= \mathrm{E}_t^Q\left[dF(t,y(t))\right] \\ &= \mathrm{E}_t^Q\left[F_t\,dt + F_y\,dy + \frac{1}{2}F_{yy}\,dy^2\right] \\ &= e^{-rt}\left(-ry\,dt + \mathrm{E}_t^Q\left[dy\right]\right). \end{aligned} \tag{9}$$

Now, applying Ito's lemma to dy, we get the PDE

$$0 = -ry + y_t + \kappa\left(\theta - x\right)y_x + \frac{\sigma^2}{2}y_{xx} \tag{10}$$

subject to the 'final condition' $y(t,x(t)) = \Phi(X(T))$

(b) Choose $v < t$. From the definition, we have

$$\begin{aligned} y(t,x(t)) &= \mathrm{E}_t^Q\left[e^{-\int_t^T ds\,r(x(s))}\,\Phi(X(T))\right] \\ y(v,x(v)) &= \mathrm{E}_v^Q\left[e^{-\int_v^T ds\,r(x(s))}\,\Phi(X(T))\right]. \end{aligned} \tag{11}$$

From the law of iterated expectations (LIE), we find

$$\begin{aligned} \mathrm{E}_v^Q\left[y(t,x(t))\right] &= \mathrm{E}_v^Q\left[\mathrm{E}_t^Q\left[e^{-\int_t^T ds\,r(x(s))}\,\Phi(X(T))\right]\right] \\ &\overset{LIE}{=} \mathrm{E}_v^Q\left[e^{-\int_t^T ds\,r(x(s))}\,\Phi(X(T))\right] \\ &\neq y(v,x(v)). \end{aligned} \tag{12}$$

Hence, y is not a martingale. However, if we define

$$\begin{aligned} F(t,y(t)) &\equiv e^{-\int_0^t ds\,r(x(s))}\,y(t) = \mathrm{E}_t^Q\left[e^{-\int_0^T ds\,r(x(s))}\,\Phi(X(T))\right] \\ F(v,y(v)) &\equiv e^{-\int_0^v ds\,r(x(s))}\,y(v) = \mathrm{E}_v^Q\left[e^{-\int_0^T ds\,r(x(s))}\,\Phi(X(T))\right], \end{aligned} \tag{13}$$

we find that F is a martingale in that

$$\begin{aligned} \mathrm{E}_v^Q\left[F(t,y(t))\right] &= \mathrm{E}_v^Q\left[\mathrm{E}_t^Q\left[e^{-\int_0^T ds\,r(x(s))}\,\Phi(X(T))\right]\right] \\ &\overset{LIE}{=} \mathrm{E}_v^Q\left[e^{-\int_0^T ds\,r(x(s))}\,\Phi(X(T))\right] \\ &= F(v,y(v)). \end{aligned} \tag{14}$$

Note that $F_t = -r(x(t))\, e^{-\int_0^t ds\, r(x(s))}\, y(t)$, $F_y = e^{-\int_0^t ds\, r(x(s))}$, $F_{yy} = 0$. Thus, from Ito's lemma (or, equivalently, from Feynman-Kac), we get the PDE

$$\begin{aligned} 0 &= \mathrm{E}_t^Q \left[dF(t, y(t)) \right] \\ &= \mathrm{E}_t^Q \left[F_t\, dt + F_y\, dy + \frac{1}{2} F_{yy}\, dy^2 \right] \\ &= e^{-\int_0^t ds\, r(x(s))} \Big(-r(x(t))\, y\, dt + \mathrm{E}_t^Q [dy] \Big). \end{aligned} \tag{15}$$

Now, applying Ito's lemma to dy, we get the PDE

$$0 = -r(x(t))\, y + y_t + \kappa\, (\theta - x)\, y_x + \frac{\sigma^2}{2} y_{xx} \tag{16}$$

subject to the 'final condition' $y(t, x(t)) = \Phi(X(T))$.

3. (a)

	p	q	$\frac{d\mathbb{Q}}{d\mathbb{P}}$
uu	0.6	0.3	0.5
ud	0.2	0.3	1.5
du	0.1	0.2	2
dd	0.1	0.2	2

(b) From one-period conditions:

$$V_0 = 0.6 \times V_{1u} + 0.4 \times V_{1d} = 0.8 \times 0.75 \times V_{1u} + 0.2 \times 2 \times V_{1d} = E_0^P[\eta_1 V_1],$$

where $\eta_{1u} = 0.75$, $\eta_{1d} = 2$.

$$\begin{aligned} V_{1u} &= 0.5 \times V_{2uu} + 0.5 \times V_{2ud} = 0.75 \times 0.667 \times V_{2uu} + 0.25 \times 2 \times V_{2ud} = E_1^P[\eta_2 V_2 | u], \\ V_{1d} &= 0.5 \times V_{2du} + 0.5 \times V_{2dd} = 0.5 \times 1 \times V_{2du} + 0.5 \times 1 \times V_{2dd} = E_1^P[\eta_2 V_2 | d]. \end{aligned}$$

where $\eta_{2uu} = 0.667$, $\eta_{2ud} = 2$, $\eta_{2du} = 1$, $\eta_{2dd} = 1$.

(c) $V_0 = E_0^P[\eta_1 V_1] = E_0^P[\eta_1 E_1^P[\eta_2 V_2]] = E_0^P[\eta_1 \eta_2 V_2]$.

(d) ξ_2: Note that $\xi_2 = \frac{d\mathbb{Q}}{d\mathbb{P}}$, since $\mathcal{F}_2 = 2^\Omega$. Now, $\xi_2(uu) = \frac{\mathbb{Q}(uu)}{\mathbb{P}(uu)} = \frac{q_\Omega q_u}{p_\Omega p_u} = \frac{q_\Omega}{p_\Omega} \times \frac{q_u}{p_u} = \eta_{1u}\eta_{2uu} = (\eta_1\eta_2)_{uu}$. Similar relations hold for ud, du and dd states. Thus $\xi_2 = \eta_1 \eta_2$.

ξ_1: $E_1^P[\xi_2] = E_1^P[\eta_1\eta_2] = \eta_1 E_1^P[\eta_2] = \eta_1 E_1^Q[1] = \eta_1$.

ξ_0: $\xi_0 = E_0^P[\xi_2] = E_0^Q[1] = 1$.

(e) Since $R \equiv 1$, $M_t \equiv \xi_t$, the only relationship we need to verify is that $E_t^Q[V_2] = E_t^P\left[\frac{\xi_2}{\xi_t} V_2\right]$, $t = 0, 1, 2$.

- The relationship trivially holds at $t = 2$.
- At $t = 1$, the one period relationship gives: $E_1^Q[V_2] = E_1^P[\eta_2 V_2] = E_1^P\left[\frac{\eta_1\eta_2}{\eta_1} V_2\right] = E_1^P\left[\frac{\xi_2}{\xi_1} V_2\right]$.
- At $t = 0$, we have $E_0^Q[V_2] = E_0^P[\eta_1\eta_2 V_2] = E_0^P\left[\frac{\xi_2}{\xi_0} V_2\right]$.