MFE230Q: Final Exam, May 20, 2021

NAME:

This exam is open book and open notes. You are allowed to use the text book, slides, and other material mentioned in the syllabus, but you should provide a reference in your solution if you do so (e.g., "see page 72 of slides named ...1B"). You do not need to rederive formulas we have shown in class, if not explicitly asked to do so. You may use a calculator or Excel, but nothing else (not Matlab, etc.). You should not submit other documents than your clear hand-written solutions, which you should scan or provide high-quality snapshots of, and which you should upload via bcourses.

You are *not* allowed to use the Internet for help or guidance, *in any way.* Moreover, you are *not* allowed to communicate—*in any way*—with anyone (including other students in the class), except for the professor and GSI.

Please keep your camera on during the exam, and the microphone off. If possible, keep the sound on at a low volume, so that the professor can alert you if needed. If you need to ask clarifying questions, please send an email to the professor (`walden@haas.berkeley.edu`) and/or GSI (`simon_xu@haas.berkeley.edu`). Please do *not* use Zoom for questions, since this may distract other students.

Please underline your final answers. All answers must be justified.

I swear on my honor that I have followed the rules states stated above. Especially, I have neither given nor received aid on this exam

Signature **Date**

1. *One-period model (25 points)*: Consider the following one-period market, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1.5 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

   (a) Is this market arbitrage free?

   (b) Is this market complete?

   (c) Find all stochastic discount factors consistent with no arbitrage.

   (d) Find all possible risk-free rates, R^f.

   Solution

   (a) Let $\boldsymbol{\psi} = [\psi_1, \ldots, \psi_6]^\intercal$. The linear system

$$\mathbf{D}\boldsymbol{\psi} = \mathbf{s}^0$$

   has at least one strictly-positive solution $\boldsymbol{\psi} = [1/9, 1/9, 2/15, 2/15, 2/15, 2/15]^\intercal$.

   (b) No, because there are 6 states but only 3 assets.

   (c) The set of all admissible SDFs is given by

$$\mathcal{A} := \quad \Big\{\boldsymbol{\psi} : \quad \psi_1 = \frac{1}{3} - 2\psi_2,\ \psi_3 = \frac{1}{3} - 1.5\psi_4,\ \psi_5 = \frac{1}{3} - 1.5\psi_6, \\ \psi_2 \in \left(0, \frac{1}{6}\right),\ \psi_j \in \left(0, \frac{2}{9}\right),\ j = 4, 6\Big\}.$$

   (d)

$$\begin{aligned} R_f \in \left\{ \frac{1}{\sum_{i=1}^6 \psi_i} \middle| \boldsymbol{\psi} \in \mathcal{A} \right\} &= \left\{ \frac{1}{1 - \psi_2 - \frac{1}{2}\psi_4 - \frac{1}{2}\psi_6} \middle| \psi_2 \in \left(0, \frac{1}{6}\right),\ \psi_j \in \left(0, \frac{2}{9}\right),\ j = 4, 6 \right\} \\ &= \left(1, \frac{18}{11}\right). \end{aligned}$$

   The lower and upper bounds are attained at $\boldsymbol{\psi} = [1/3, 0, 1/3, 0, 1/3, 0]^\intercal$ and $\boldsymbol{\psi} = [0, 1/6, 0, 2/9, 0, 2/9]^\intercal$, respectively.

2. *Binomial tree (25 points)*: Consider a two-period binomial tree, with parameters $u = 1.5$, $d = 0.8$, $R = 1.2$. The initial price of the stock (which pays no dividends) is $S_0 = 100$.

   (a) Calculate the price of a European put option on the stock with strike price $K = 100$ and expiration at $T = 2$.

(b) Calculate the price of an American put option on the stock with strike price $K = 100$ and expiration at $T = 2$.

Solution

(a) The risk-neutral probability of an upper move is given by

$$q = \frac{R-d}{u-d} = \frac{4}{7}.$$

The put option is in-the-money solely on the event $\{dd\}$ with payoff 36 and thus its price is given by

$$P_0^{\mathrm{E}} = 36 \times \left(\frac{3}{7}\right)^2 \frac{1}{1.2^2} = \boxed{\tfrac{225}{49} \approx 4.59.}$$

(b) The price of the American option satisfies the following backward recursion equation:

$$\begin{cases} P_1^{\mathrm{A}}(S_1) & = \frac{1}{R} E^Q \left[\max[K - S_2, 0]\right], \\ P_0^{\mathrm{A}}(S_0) & = \frac{1}{R} E^Q \left[\max\left[K - S_1, P_1^{\mathrm{A}}(S_1)\right]\right]. \end{cases}$$

Accordingly, we get

$$\begin{cases} P_1^{\mathrm{A}}(S_1 = 150) & = 0, \\ P_1^{\mathrm{A}}(S_1 = 80) & = 36 \times \frac{1}{1.2} \times \frac{3}{7} = \frac{90}{7}, \end{cases}$$

and

$$P_0^{\mathrm{A}}(S_0 = 100) = \frac{3}{7}\frac{1}{1.2} \max\left(20, \frac{90}{7}\right) = \boxed{\tfrac{50}{7} \approx 7.14.}$$

3. *Continuous time model* (20 points): Consider an economy with one stock with GBM dynamics for prices, and one risk free asset with constant returns:

$$\begin{aligned} \frac{dS}{S} &= \hat{\mu} dt + \sigma dW, \\ \frac{dB}{B} &= r dt, \end{aligned}$$

$\hat{\mu}$, r, and $\sigma > 0$ constants. The bond makes no dividend payments, but the stock makes instantaneous payments proportional to the stock price. Thus, $d\Theta_t = (\alpha S_t dt, 0)$, is the instantaneous dividend process of the two assets, using the notation from class, where $\alpha > 0$ is a constant.

Consider the trading strategy

$$\mathbf{h}_t = (h_t^S, h_t^B)' = \left(2e^{(r+\sigma^2-2\alpha)(T-t)} S_t, -e^{(r+\sigma^2-2\alpha)(T-t)} \frac{S_t^2}{B_t}\right)', \qquad T > 0.$$

(a) What is the value process, $V_t^{\mathbf{h}}$, for this trading strategy?

(b) What is the cumulative dividend processes, $F_t^{\mathbf{h}}$, of this trading strategy?

(c) What asset pricing conclusions can be drawn from your results in (a) evaluated at $t=0$ and $t=T$, and (b)? Remember to justify your answer.

Solution

(a) The value process is

$$\begin{aligned} V_t &= \mathbf{h}_t'(S_t, B_t)' \\ &= \left(2e^{(r+\sigma^2-2\alpha)(T-t)}S_t, -e^{(r+\sigma^2-2\alpha)(T-t)}\frac{S_t^2}{B_t}\right)(S_t, B_t)' \\ &= \boxed{e^{(r+\sigma^2-2\alpha)(T-t)}S_t^2.} \end{aligned}$$

or if we substitute in for $S_t^2 = S_0^2 e^{(2\hat{\mu}-\sigma^2)t+2\sigma W_t}$:

$$\begin{aligned} V_t &= e^{(r+\sigma^2-2\alpha)(T-t)}S_0^2 e^{(2\hat{\mu}-\sigma^2)t+2\sigma W_t}. \\ &= S_0^2 e^{(r+\sigma^2-2\alpha)T+(2\hat{\mu}+2\alpha-r-2\sigma^2)t+2\sigma W_t}. \end{aligned}$$

(b) From the law of motion for the portfolio process,

$$dV_t + dF_t^{\mathbf{h}} = \mathbf{h}_t' d(S_t, B_t)' + \mathbf{h}_t' d\Theta_t, \tag{1}$$

we can calculate the instantaneous portfolio dividends, $dF^{\mathbf{h}}$. From the formula for the value process in (a), using Ito we get

$$dV_t = e^{(r+\sigma^2-2\alpha)(T-t)}\left(-(r+\sigma^2-2\alpha)S_t^2 dt + 2S_t^2(\hat{\mu}dt + \sigma dW_t) + \sigma^2 S_t^2 dt\right).$$

We also have

$$\mathbf{h}_t' d(S_t, B_t)' = e^{(r+\sigma^2-2\alpha)(T-t)}S_t^2(2(\hat{\mu}dt + \sigma dW) - rdt),$$

and

$$\mathbf{h}_t' d\Theta_t = 2\alpha e^{(r+\sigma^2-2\alpha)(T-t)}S_t^2 dt.$$

Plugging these expressions into eq. (1), we get $\boxed{dF_t^{\mathbf{h}} = 0}$, i.e., the portfolio is self financed.

(c) We have shown that we can replicate the value process $V_t = e^{(r+\sigma^2-2\alpha)(T-t)}S_t^2$ with a self financed portfolio strategy. The value of this strategy at time T is then S_T^2, and the value at time 0 is $\boxed{e^{(r+\sigma^2-2\alpha)T}S_0^2.}$

Thus, if the market admits no arbitrage, the time 0 price of a contingent claim that pays S_T^2 at time T must be $e^{(r+\sigma^2-2\alpha)T}S_0^2$. As a side note, such a claim can be interpreted as a power option with zero strike price.

Note that this price differs from the "naive" price of S_0^2 for such a claim, that one might conjecture. This is the price of buying S_0 shares of the stock at time 0. However, such a portfolio, if passively held, will be worth $S_0 S_T$ at time T, not S_T^2. Moreover, it will generate dividends between time 0 and T, which the power option does not. Thus, the "naive" price is typically incorrect.

4. *Barrier option (25 points)*: Consider the following Black-Scholes economy:

$$\frac{dB}{B} = r\,dt,$$
$$\frac{dS}{S} = \hat{\mu}\,dt + \sigma\,dW,$$

where the constants $r > 0$, $\hat{\mu} > 0$, $\sigma > 0$.

(a) Assume $S_0 \in (1, 2)$. Derive a formula for the price, $P(S_0)$, of a perpetual barrier option that makes a terminal payment of 3 at time t, if the stock price reaches the low barrier $S_t = 1$, and 1 at t if the price reaches the high barrier $S_t = 2$.

Solution Since it is a perpetual option, the option price satisfies the following ODE

$$rS\frac{dP}{dS} + \frac{\sigma^2}{2}S^2\frac{d^2P}{dS^2} - rP = 0$$

with the boundary conditions $P(2) = 1$ and $P(1) = 3$.

Suppose

$$P(S) = aS + bS^\alpha$$

where

$$\alpha = -\frac{2r}{\sigma^2}.$$

Routine calculations verifies that P satisfies the above ODE. The boundary conditions imply the following linear equation system

$$\begin{cases} P(1) &= a + b = 3, \\ P(2) &= 2a + b2^\alpha = 1. \end{cases}$$

Solving the above system gives

$$\begin{cases} a &= 3 - \frac{5}{2-\xi} \\ b &= \frac{5}{2-\xi}, \end{cases}$$

where $\xi = 2^\alpha$. Putting all these together yields

$$\boxed{P(S_0) = \left(3 - \frac{5}{2-\xi}\right) S_0 + \frac{5}{2-\xi} S_0^\alpha}.$$

with $\boxed{\xi = 2^\alpha \text{ and } \alpha = -\frac{2r}{\sigma^2}.}$

5. *Contingent claim (25 points)*: Two telecommunications companies, with share-prices S_1 and S_2, each have infrastructure and handset divisions, and are therefore exposed to shocks in these two sectors. Their stock dynamics (which are linearly independent), together with that of a risk free bond, are:

$$\frac{dS_1}{S_1} = \hat{\mu}_1\, dt - \sigma_{11}\, dW_1 - \sigma_{12}\, dW_2,$$
$$\frac{dS_2}{S_2} = \hat{\mu}_2\, dt - \sigma_{21}\, dW_1 - \sigma_{22}\, dW_2,$$
$$\frac{dB}{B} = r\, dt.$$

Here, dW_1 represents shocks to the infrastructure sector and dW_2 shocks to the handset sector, and both are standardized Wiener processes. For simplicity, we assume that W_1 and W_2 are independent.

A trading desk plans to introduce a contract that allows one to hedge infrastructure risk. Specifically, the contract would pay $e^{W_1(T)}$ at time T.

(a) What is the time 0 market price of this contract?

(b) How could the payoff of the contract be replicated synthetically? What would be the replicating portfolio at time 0, $\mathbf{h} = (a, b, c)'$, where a, b, and c denote the number of shares invested in stock 1, 2, and the bond, respectively?

Solution

(a) Let

$$\Sigma = \begin{bmatrix} -\sigma_{11}, -\sigma_{12} \\ -\sigma_{21}, -\sigma_{22} \end{bmatrix}$$

and

$$\boldsymbol{\mu} = \begin{bmatrix} \hat{\mu}_1, \\ \hat{\mu}_2 \end{bmatrix}.$$

Then the Girsanov kernel is given by

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1, \\ \theta_2 \end{bmatrix} = \Sigma^{-1}\boldsymbol{\mu}.$$

And thus, $W_1^Q(t) := W_1(t) + \theta_1 t$ is a Q-Brownian Motion and accordingly,

$$X_t = \exp\left[W_1^Q(t) - \frac{t^2}{2}\right]$$

is a Q-martingale.

It follows that value of the contingent claim at time $t \in [0, T)$ is given by

$$\begin{aligned}
V_t &= e^{-r(T-t)} E^Q \left[\exp\left[W_1(T)\right] \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E^Q \left[\exp\left[W_1^Q(T) - \theta_1 T\right] \middle| \mathcal{F}_t\right] \\
&= \exp\left[-(r+\theta_1)(T-t) + W_1^Q(t) - \theta_1 t\right] E^Q \left[\exp\left[W_1^Q(T) - W_1^Q(t)\right] \middle| \mathcal{F}_t\right] \\
&= \exp\left[-\left(r + \theta_1 - \frac{1}{2}\right)(T-t) + W_1(t)\right],
\end{aligned}$$

where the last equality follows by the martingale property of X_t. Specially, we have

$$\boxed{V_0 = e^{-(r+\theta_1-0.5)T}.}$$

(b) The first step is to express $W_1(t)$ in terms of $S_1(t)$ and $S_2(t)$. To this end, we note that

$$\log\frac{S_i(t)}{S_i(0)} = \tilde{\mu}_i t - \sigma_{i1}W_1(t) - \sigma_{i2}W_2(t),\ i = 1, 2,$$

where $\tilde{\mu}_i = \hat{\mu}_i - \frac{1}{2}\left(\sigma_{i1}^2 + \sigma_{i2}^2\right)$. Solving the above linear system gives

$$\begin{aligned}
W_1(t) &= \frac{-\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\left[\log\frac{S_1(t)}{S_1(0)} - \tilde{\mu}_1 t\right] + \frac{\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\left[\log\frac{S_2(t)}{S_2(0)} - \tilde{\mu}_2 t\right] \\
&= f(t, S_1(t), S_2(t)).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{\partial f}{\partial S_1} &= \frac{-\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\frac{1}{S_1(t)}, \\
\frac{\partial f}{\partial S_2} &= \frac{\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\frac{1}{S_2(t)}.
\end{aligned}$$

We recall from Part (a) that

$$V_t = e^{-(r+\theta_1-0.5)(T-t)+W_1(t)} = e^{-(r+\theta_1-0.5)(T-t)} e^{f(t,S_1(t),S_2(t))}.$$

Then it follows from Prop. 8.6 of Bjork's book (2nd Edition) that the deltas are given by

$$\begin{aligned} h_{S_1}(t) &= \frac{\partial V_t}{\partial S_1} = \frac{-\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} \frac{V_t}{S_1(t)}, \\ h_{S_2}(t) &= \frac{\partial V_t}{\partial S_2} = \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} \frac{V_t}{S_2(t)}. \end{aligned}$$

and

$$h_B(t) = \frac{V_t - \sum_{i=1}^2 S_i(t) \frac{\partial V_t}{\partial S_i}}{B(t)} = \left(1 - \frac{-\sigma_{22}+\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\right) \frac{V_t}{B(t)}.$$

Plugging $t = 0$ into the above displays yields

$$\begin{aligned} h_{S_1}(0) &= \frac{-\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} \frac{e^{-(r+\theta_1-0.5)T}}{S_1(0)}, \\ h_{S_2}(0) &= \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} \frac{e^{-(r+\theta_1-0.5)T}}{S_2(0)}, \\ h_B(0) &= \left(1 - \frac{-\sigma_{22}+\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\right) e^{-(r+\theta_1-0.5)T}. \end{aligned}$$

6. *Term structure (15 points)*: Assume that the short rate process satisfies the following dynamics:

$$dr_t = -\mu r_t \, dt + \sigma \, dW^Q.$$

   (a) Derive an expression for the yield curve, $y(t, T|r)$, $T > t \geq 0$.

   (b) How can the short rate, r_t, be inferred from the yield curve?

Solution

   (a) **This is a special case of the Vasicek model covered in class, with $b = 0$, which in turn belongs to the class of affine term structure models. The same derivation as in class yields the price for a T-bond**

$$P(t, T|r) = e^{A(T-t)-B(T-t)r},$$

where A and B satisfy the ODEs:

$$\begin{cases} B' &= -\mu B + 1,\ B(0) = 0, \\ A' &= \frac{\sigma^2}{2} B^2,\ A(0) = 0, \end{cases}$$

with solutions

$$\begin{aligned} B(\tau) &= \frac{1}{\mu}\left(1 - e^{-\mu\tau}\right), \\ A(\tau) &= \frac{\sigma^2}{2}\int_0^\tau B^2(s)ds \\ &= \frac{\sigma^2}{2\mu^2}\int_0^\tau \left(1 - e^{-\mu s}\right)^2 ds \\ &= \boxed{-\tfrac{3\sigma^2}{4\mu^3} + \tfrac{\sigma^2}{2\mu^2}\tau + \tfrac{\sigma^2}{\mu^3}e^{-\mu\tau} - \tfrac{\sigma^2}{4\mu^3}e^{-2\mu\tau}.} \end{aligned}$$

Finally, we get

$$y(t, T|r) = -\frac{\ln p(t, T|r)}{T - t} = \boxed{\tfrac{1}{T-t}\left[B(T-t) - A(T-t)r\right].}$$

(b) **It is the short end of the yield curve, $r_t = y(t, t)$. This can e.g., be seen from the informal argument for the time t price of a $t+dt$-bond, $P(t, t+dt) = e^{-r_t dt} = e^{-y(t,t)dt}$.**