MFE230Q: Midterm Exam, April 27, 2016

Solution

1. *One-period model*:

   (a) We are given two assets with linearly independent payoff vectors and equal prices. There is no type one arbitrage: Every portfolio with zero value today must buy some number of one asset and sell the same number of the other asset. (as they have same prices). Hence, WLOG we can focus on a portfolio that holds only one unit of some asset and sells one unit of the other asset. But from the asset matrix we see that depending on the state realization either asset can perform better than the other. Thus type 1 arbitrage is excluded. There is no type 2 arbitrage: In order to have a portfolio that costs a negative price today (we get paid to buy it) we need to buy a multiple of portfolio $h_1' = (\alpha, -1)$, for some $0 \leq \alpha < 1$ or a multiple of portfolio $h_2' = (-1, \beta)$ for some $0 \leq \beta < 1$. But calculating the vector $V = h'D$ for either portfolio h_1 or h_2 gives at least one negative entry in some state.

   Alternatively, find strictly positive state price vector:

   (b) Market is incomplete as we have three states but only two assets.

   (c) By no arbitage and the First Fundamental Theorem of Asset Pricing we know there exists a strictly positive state price vector where each entry is the price of the corresponding Arrow-Debreu security. The Call option has value zero in state one and two and has value 50 in state 3. By no arbitrage pricing, the time zero value of the Call satisfies

   $$C = 50\,\psi_3 > 0,$$

   which is a lower bound. Moreover, we can replicate the risky and risk-less asset using the Arrow-Debreu securities and obtain

   $$\begin{aligned} 100 &= 50\psi_1 + 100\psi_2 + 150\psi_3 \qquad (1) \\ 100 &= 100(\psi_1 + \psi_2 + \psi_3) \qquad (2) \end{aligned}$$

   This yields $1 = \psi_1 + \psi_2 + \psi_3$, solving this for ψ_1 and plugging in equation 1 we obtain

   $$50 = 100\psi_3 + 50\psi_2 \qquad (3)$$

   For $\psi_2 > 0$ we obtain the inequality $50 > 100\psi_3$, i.e.,

   $$\psi_3 < \frac{1}{2}.$$

   Therefore, we obtain the upper bound

   $$C = 50\,\psi_3 < 25. \qquad (4)$$

2. *Binomial tree model:*

(a) The filtration $\underline{\mathcal{F}} := \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ is the increasing sequence of σ-algebras generated by the partitions at each time t. It is given by the following :

$$\begin{aligned}
\mathcal{F}_0 =& \sigma\left(\{\Omega\}\right)\\
=& \{\emptyset, \Omega\}\\
\mathcal{F}_1 =& \sigma\left(\{uuu, uud, udu, udd\}, \{duu, dud, ddu, ddd\}\right)\\
=& \sigma\left(\left\{u^3, u^2d, ud^2\right\}, \left\{du^2, d^2u, d^3\right\}\right)\\
=& \left\{\emptyset, \Omega, \left\{u^3, u^2d, ud^2\right\}, \left\{du^2, d^2u, d^3\right\}\right\}\\
\mathcal{F}_2 =& \sigma\left(\{uuu, uud\}, \{udu, udd\}, \{duu, dud\}, \{ddu, ddd\}\right)\\
=& \sigma\left(\left\{u^3, u^2d\right\}, \left\{u^2d, ud^2\right\}, \left\{du^2, d^2u\right\}, \left\{d^2u, d^3\right\}\right)\\
\mathcal{F}_3 =& \sigma\left(\{uuu\}, \{uud\}, \{udu\}, \{udd\}, \{duu\}, \{dud\}, \{ddu\}, \{ddd\}\right)\\
=& \sigma\left(\left\{u^3\right\}, \left\{u^2d\right\}, \left\{d^2u\right\}, \left\{d^3\right\}\right)\\
=& 2^{\Omega}
\end{aligned}$$

(b) At time t, in state x, after a total of k up moves and $t-k$ down moves, we have

$$L_x = \left(\frac{q}{p}\right)^k \times \left(\frac{1-q}{1-p}\right)^{t-k} = \frac{\binom{t}{k} q^k (1-q)^{t-k}}{\binom{t}{k} p^k (1-p)^{t-k}} \stackrel{\text{def}}{=} \frac{A_i}{B_i}.$$

For events that contain multiple outcomes, $X = \cup_i \{x_i\}$, we have $L_X = \frac{\sum_i A_i}{\sum_i B_i}$.

(c) The price of a bond is $\frac{1}{R^3}$, the price of a "u^2d" AD security is $\frac{3q^2(1-q)}{R^3}$, so the price of the derivative is the difference between the two:

$$V = \frac{1}{R^3} - \frac{3q^2(1-q)}{R^3}.$$

(d) We can solve this using a recursive formulation.

Recursive formulation: The price today is the RN discounted value of tomorrow's payoff:

$$V = \frac{1}{R}\left(q + (1-q)V\right).$$

Solve for V to get

$$V = \frac{q}{R-(1-q)}.$$

(e) Similar as in (d), recursive formulation gives

$$V = \frac{1}{R}\left(quS + (1-q)dV\right).$$

Solve for V gives

$$V = \frac{quS}{R-(1-q)dS} = \frac{(R-d)u}{(u-d)\left(R - \frac{u-R}{u-d}d\right)}S = \frac{(R-d)u}{Ru-du}S = S.$$

This is perfectly intuitive. The asset will pay off at some point in time (since $p > 0$ and $T = \infty$), and at that point it pays off the value of the stock. It is thus equivalent to stock ownership, although the liquidation time is random.

(f) Given the argument in (e), it is clear that the strategy of purchasing the stock at time 0 and then selling it after the first up-move replicates the payoff of the derivative.

3. *Ito calculus:*

(a) We have

$$d[W^4T^4] = 4W^3T^4dW + 4W^4T^3dt + 6W^2T^4dt,$$

leading to

$$W_t^4T^4 = 4\int_0^T W_t^3t^4dW + 4\int_0^T W_t^4t^3dt + 6\int_0^T W_t^2t^4dt.$$

Direct algebraic manipulations then imply that $h(t, W_T) = \frac{W^4T^4}{4}$, and $g(t, W_t) = -\left(W_t^4t^3 + \frac{3}{2}W_t^2t^4\right)$.

(b) Define $m_t = E[W_t^4]$. From (a), that $E[W_t^2] = t$, and the martingale property, it follows that

$$\begin{aligned} m_TT^4 &= 4E\left[\int_0^T W_t^3T^4dW\right] + 4\int_0^T m(t)T^3dt + 6\int_0^T t \times t^4dt \\ &= 4\int_0^T m(t)t^3dt + 6\int_0^T t^5dt, \end{aligned}$$

which by differentiation leads to the ODE

$$m_T'T^4 + 4m_TT^3 = 4m_TT^3 + 6T^5,$$

which after simplification implies that

$$m_T' = 6T.$$

Together with the obvious initial condition $m_0 = 0$, this then gives us $m(T) = \int_0^T 6tdt = 3T^2$. We have arrived at

$$E[W_T^4] = m(T) = 3T^2.$$

4. *Continuous time model*:

(a) The value process is

$$\begin{aligned} V_t &= \mathbf{h}_t'(S_t, B_t)' \\ &= \left(2e^{(r+\sigma^2-2\alpha)(T-t)}S_t, -e^{(r+\sigma^2-2\alpha)(T-t)}\frac{S_t^2}{B_t}\right)(S_t, B_t)' \\ &= e^{(r+\sigma^2-2\alpha)(T-t)}S_t^2. \end{aligned}$$

or if we substitute in for $S_t^2 = S_0^2 e^{(2\hat{\mu}-\sigma^2)t+2\sigma W_t}$:

$$
\begin{aligned}
V_t &= e^{(r+\sigma^2-2\alpha)(T-t)} S_0^2 e^{(2\hat{\mu}-\sigma^2)t+2\sigma W_t}. \\
&= S_0^2 e^{(r+\sigma^2-2\alpha)T+(2\hat{\mu}+2\alpha-r-2\sigma^2)t+2\sigma W_t}.
\end{aligned}
$$

(b) From the law of motion for the portfolio process,

$$
dV_t + dF_t^{\mathbf{h}} = \mathbf{h}_t' d(S_t, B_t)' + \mathbf{h}_t' d\Theta_t, \tag{5}
$$

we can calculate the instantaneous portfolio dividends, $dF^{\mathbf{h}}$. From the formula for the value process in (a), using Ito we get

$$
dV_t = e^{(r+\sigma^2-2\alpha)(T-t)} \left(-(r+\sigma^2-2\alpha)S_t^2 dt + 2S_t^2(\hat{\mu}dt + \sigma dW_t) + \sigma^2 S_t^2 dt\right).
$$

We also have

$$
\mathbf{h}_t' d(S_t, B_t)' = e^{(r+\sigma^2-2\alpha)(T-t)} S_t^2 (2(\hat{\mu}dt + \sigma dW) - r dt),
$$

and

$$
\mathbf{h}_t' d\Theta_t = 2\alpha e^{(r+\sigma^2-2\alpha)(T-t)} S_t^2 dt.
$$

Plugging these expressions into eq. (5), we get $dF_t^{\mathbf{h}} = 0$, i.e., the portfolio is self financed.

(c) We have shown that we can replicate the value process $V_t = e^{(r+\sigma^2-2\alpha)(T-t)} S_t^2$ with a self financed portfolio strategy. The value of this strategy at time T is then S_T^2, and the value at time 0 is $e^{(r+\sigma^2-2\alpha)T} S_0^2$.

Thus, if the market admits no arbitrage, the time 0 price of a contingent claim that pays S_T^2 at time T must be $e^{(r+\sigma^2-2\alpha)T} S_0^2$. As a side note, such a claim can be interpreted as a power option with zero strike price.

Note that this price differs from the "naive" price of S_0^2 for such a claim, that one might conjecture. This is the price of buying S_0 shares of the stock at time 0. However, such a portfolio, if passively held, will be worth $S_0 S_T$ at time T, not S_T^2. Moreover, it will generate dividends between time 0 and T, which the power option does not. Thus, the "naive" price is typically incorrect.

5. *Kolmogorov equations:*

(a) Either use Feynman-Kac's theorem directly, together with knowledge about expected value of OU-process, or conjecture solution on form $F(t,x) = e^{At}(Bx + C) + D$, to get $F_t = Ae^{At}(Bx + C)$, $F_x = Be^{At}$, $F_{xx} = 0$, and plug into PDE and terminal (boundary) condition, to get

$$
\begin{aligned}
Ae^{At}(Bx + C) + a(b - x)Be^{At} &= 0, \\
e^{AT}(Bx + C) + D &= x,
\end{aligned}
$$

which must hold for all $x \in \mathbb{R}$ and $0 \le t \le T$.

The second equation yields that $C = -De^{-AT}$ and $B = e^{-AT}$. Plugging this into the first equation yields:

$$Ae^{At}(e^{-AT}x - De^{-AT}) + a(b-x)e^{-AT}e^{At} = 0.$$

Identifying coefficients yields:

$$\begin{aligned} A &= a, \\ D &= b, \\ C &= -be^{-AT}, \end{aligned}$$

altogether leading to

$$F(t,x) = b + e^{a(t-T)}(x-b).$$

(b) Identifying the coefficients $a = \kappa$, $b = \theta$, and $c = \frac{\sigma^2}{2}$, Feynman-Kac's theorem relates the PDE with the expected value of the OU-process:

$$F(t,x) = E_t\left[X_T \middle| X_t = x\right].$$