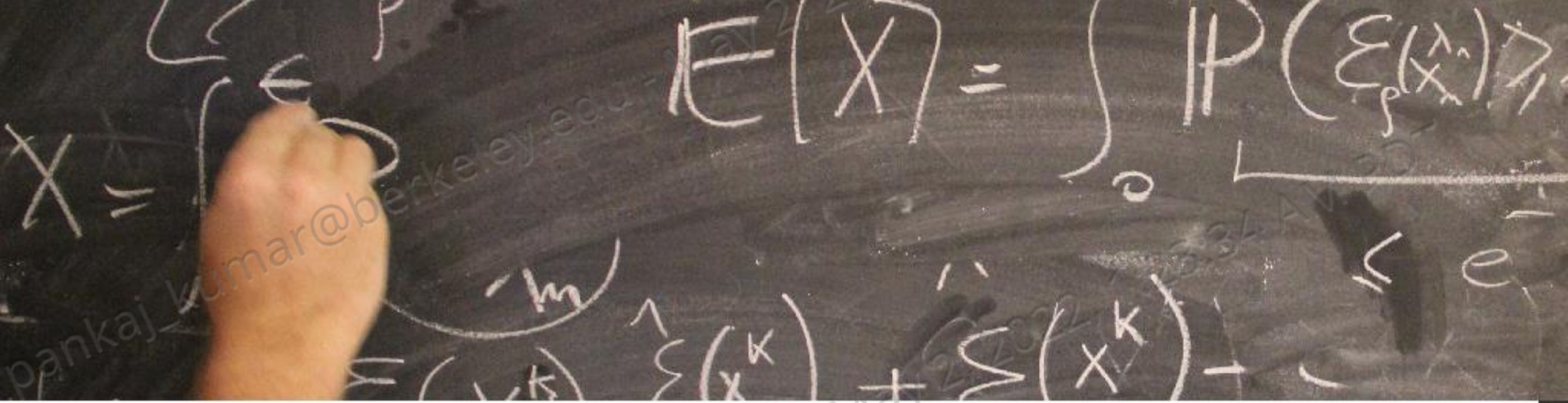

MFE 230Q [Spring 2021]
Introduction to Stochastic Calculus
GSI Session 3

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MASTER OF FINANCIAL ENGINEERING

The Ito Integral

Itô integral: Assume that W_t is a Brownian motion, and a_t is an adapted process with a.s. continuous sample paths, such that $E\left[\int_0^T a_t^2\,dt\right] < \infty$.
Then

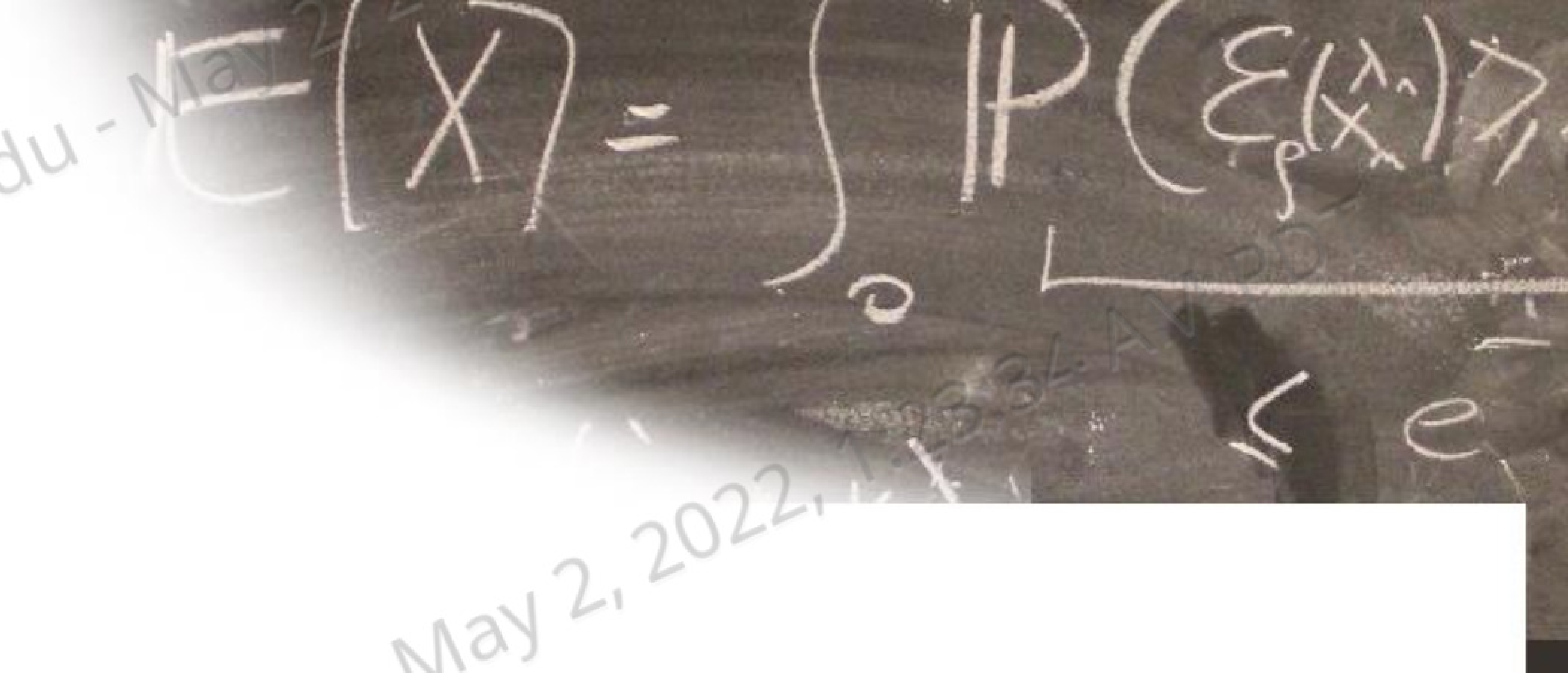
RV

$$\underbrace{\oint_{t=0}^{T} a_t\,dW}_{\text{Itô}} = \lim_{n\to\infty} \sum_{k=1}^{n} a_{(k-1)T/n}\left(W_{kT/n} - W_{(k-1)T/n}\right).$$

↑ left endpoint

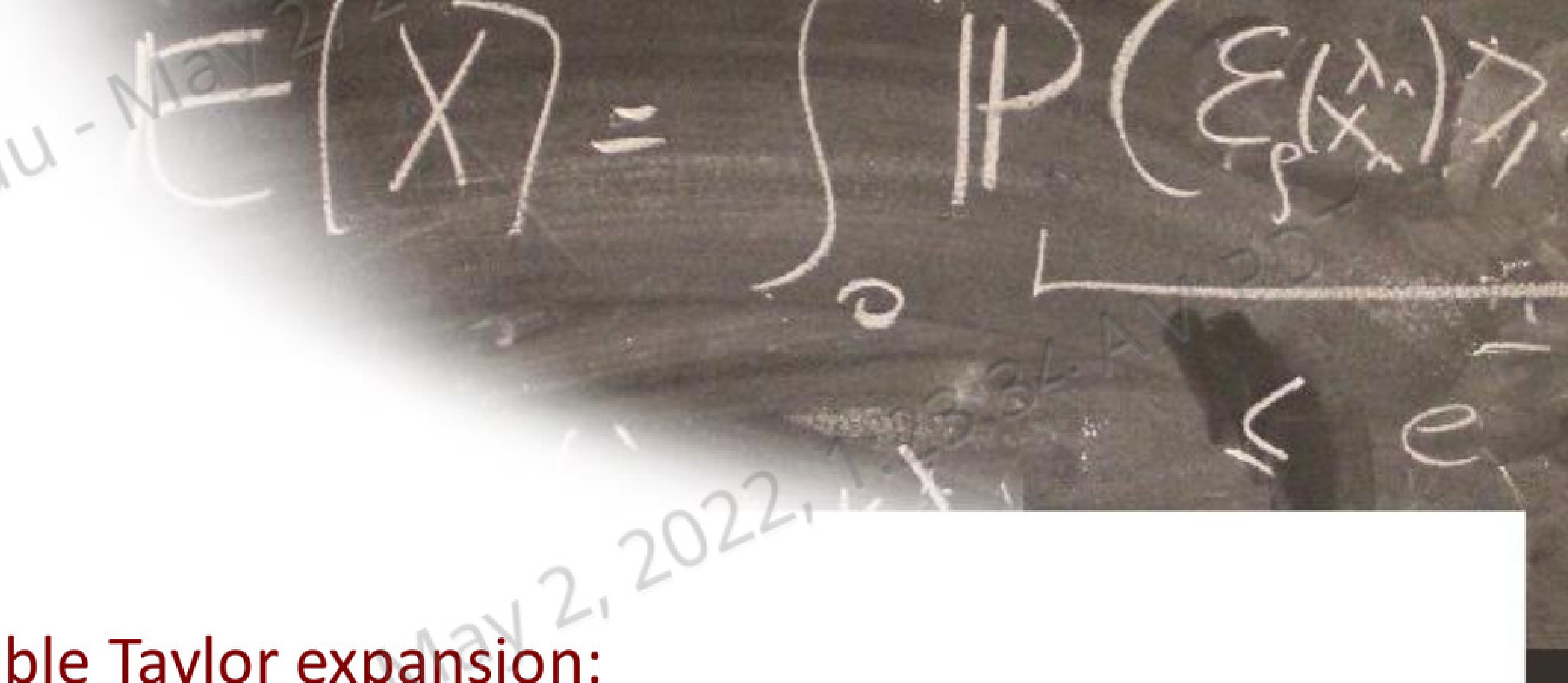
- Follows that $\oint_0^T W_t\,dW_t = \frac{1}{2}W^2(T) - \frac{T}{2}$.
- Definition can be extended to general square integrable functions.
- If $E\left[\int_0^T a_t^2\,dt\right] < \infty$, we write $a_t \in L^2([0,T])$.

Ito’s Integral Properties



$$
\begin{aligned}
E[\oint_0^T a_t dW_t] &= 0, \text{(Martingale property)},\\
E[(\oint_0^T a_t dW_t)^2] &= \int_0^T E[a_t^2]dt, \text{(Ito isometry)}.
\end{aligned}
$$

Ito’s Lemma
Taylor Series



Have a look at typical two-variable Taylor expansion:

$$
\begin{aligned}
dF(x,y) &= \frac{\partial F}{\partial x}(x,y)dx + \frac{\partial F}{\partial y}(x,y)dy \\
&+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(x,y)dx^2 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(x,y)dy^2 + \frac{\partial^2 F}{\partial x \partial y}(x,y)dxdy + \cdots
\end{aligned}
$$

Higher order items such as dx^2 decays much faster so typically we use:

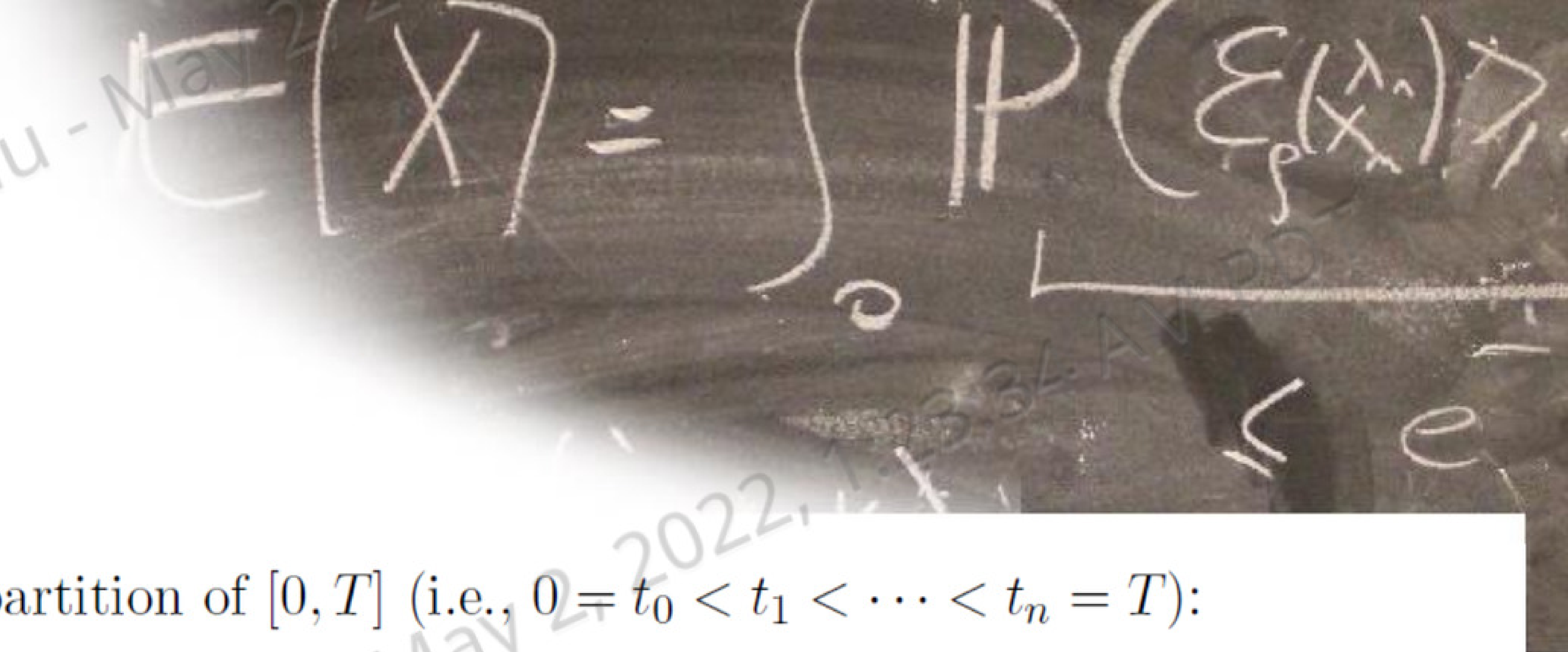
$$
dF(x,y) \approx \frac{\partial F}{\partial x}(x,y)dx + \frac{\partial F}{\partial y}(x,y)dy
$$

Set t as x, Brownian motion W(t) as y, then:

$$
\begin{aligned}
dF(t,W) &= \frac{\partial F}{\partial t}(t,W)dt + \frac{\partial F}{\partial W}(t,W)dW \\
&+ \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(t,W)dt^2 + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}(t,W)dW^2 + \frac{\partial^2 F}{\partial t \partial W}(t,W)dtdW + \cdots
\end{aligned}
$$

However…

Ito's Lemma
Taylor Series



Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0, T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$):

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) \quad = \quad 0$$

$$\text{By: } |(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)| \quad \leq \quad \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)|(t_{j+1} - t),$$

$$\text{W(t) is continuous, so: } \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \quad \to 0 \quad \text{as } ||\Pi|| \to 0$$

So dW*dt = 0!

$$\sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \quad \leq \quad \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \sum_{j=0}^{n-1} (t_{j+1} - t_j) = ||\Pi|| T$$

$$||\Pi|| \to 0 \quad \Rightarrow \quad ||\Pi|| T \to 0$$

So dt*dt = 0!

But due to Quadratic Variation: dW*dW = dt!

Ito’s Lemma

After filtering out decaying items, we have:

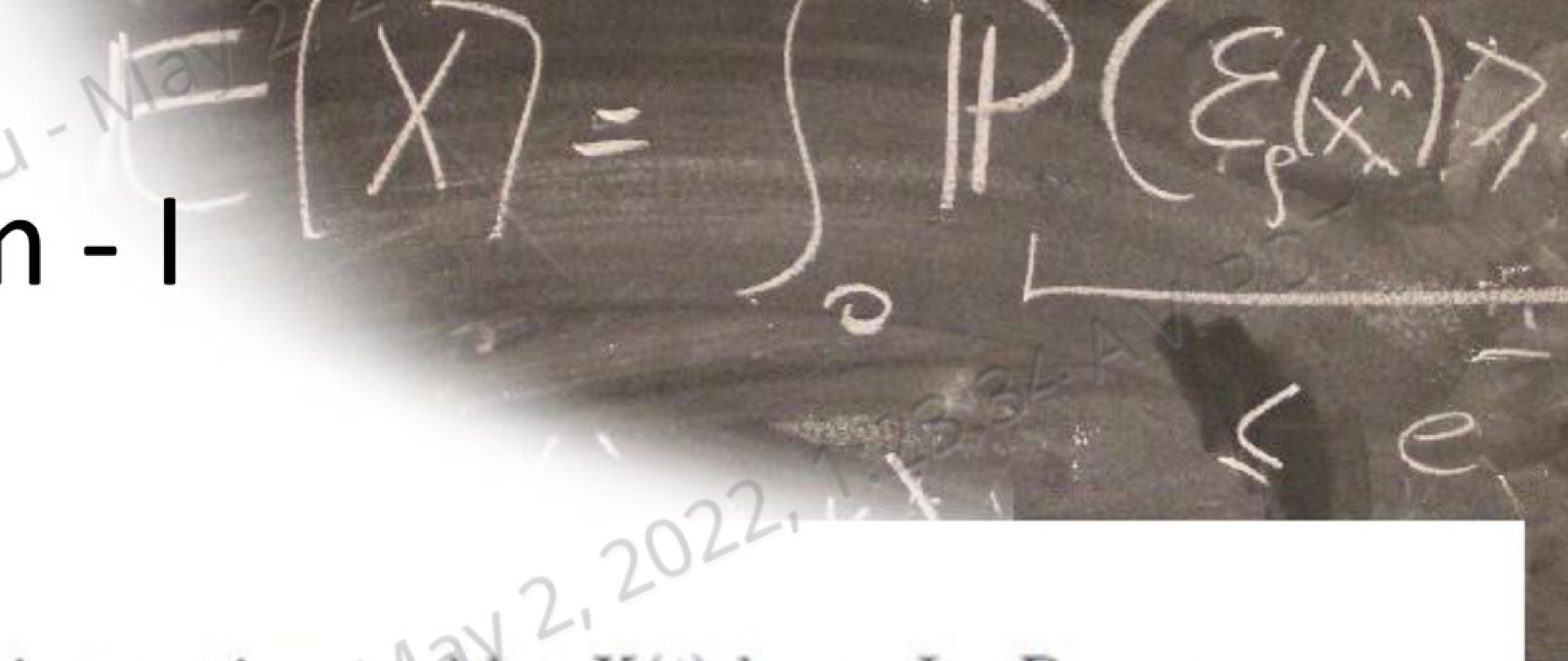
$$dF(t, W) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W} dW + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} dt$$

① X_t : $\boxed{dX_t = \mu_t dt + \sigma_t dW_t}$ SDE

$Y_t = g(t, X_t)$, Y_t : $dY_t = g_t dt + g_x dX_t + \frac{1}{2} g_{xx} dX_t^2$

$$\boxed{(dt)^2 = 0,\ dt\, dW_t = 0,\ (dW_t)^2 = dt}$$

$$\boxed{X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s}$$



Sample Problem - I

Let $W(t)$ be a standard Brownian motion, and let $X(t)$ be an Ito Process

$$dX_t = \overset{\mu X}{\mu(t,\omega)}\, dt + \overset{\sigma X}{\sigma(t,\omega)}\, dW(t). \qquad (1)$$

1.1 Example: $F(t,x) := t \sin x$

Let the dynamics of X be given by (1). Suppose $F(t,x) := t\sin x$. What is dF?

$$F_t = \sin x$$

$$F_x = t\cos x$$

$$F_{xx} = -t\sin x$$

$$dF = \sin X_t\, dt + t\cos X_t\,(\mu dt + \sigma dW_t)$$
$$- \frac{1}{2} t \sin(X_t)\,(\mu dt + \sigma dW_t)^2$$

$$(\mu dt + \sigma dW)(\mu dt + \sigma dW) = \sigma^2 dt$$

$$\Rightarrow dF = \left[\left(1 - \frac{t}{2}\sigma_t^2\right)\sin(X_t) + t\mu\cos(X_t)\right] dt$$
$$+ \sigma_t\, t\cos(X_t)\, dW_t$$

Sample Problem – II

Geometric Brownian Motion

1.2 Example: $F(t, x) := \ln x$

Let the dynamics of X be given by (1), but take $\mu(t, X_t) = \mu X_t$ and $\sigma(t, X) = \sigma X$ where μ and σ are constants. Suppose $F(t, x) := \ln x$. What is dF? How can we express $F(t, X)$ in integral form? Can we use the resulting expression for $F(t, X)$ to express $X(t)$ in an alternate way?

$$F_t = 0$$
$$F_x = \frac{1}{x}$$
$$F_{xx} = -\frac{1}{x^2}$$

$$dF = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t$$

$$\Rightarrow F_t = F_0 + \int_0^t \mu - \frac{1}{2}\sigma^2 \, ds + \sigma \int_0^t 1 \, dW_s$$

$$= F_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \left(W_t - W_0\right)$$

$$\Rightarrow \ln X_t = \ln X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \left(W_t - \underbrace{W_0}_{=0}\right)$$

$$\Rightarrow \boxed{X_t = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right]}$$

GBM: $E(X_t) = X_0 \exp(\mu t)$

$$V(X_t) = X_0^2 \exp(2\mu t)\left[\exp(\sigma^2 t) - 1\right]$$

② Correlated Multi-d Ito Lemma.

$$dX_t^i = \mu_i(t)\,dt + \sigma_i(t)\,dV_t^i \qquad i=1,\dots,k$$

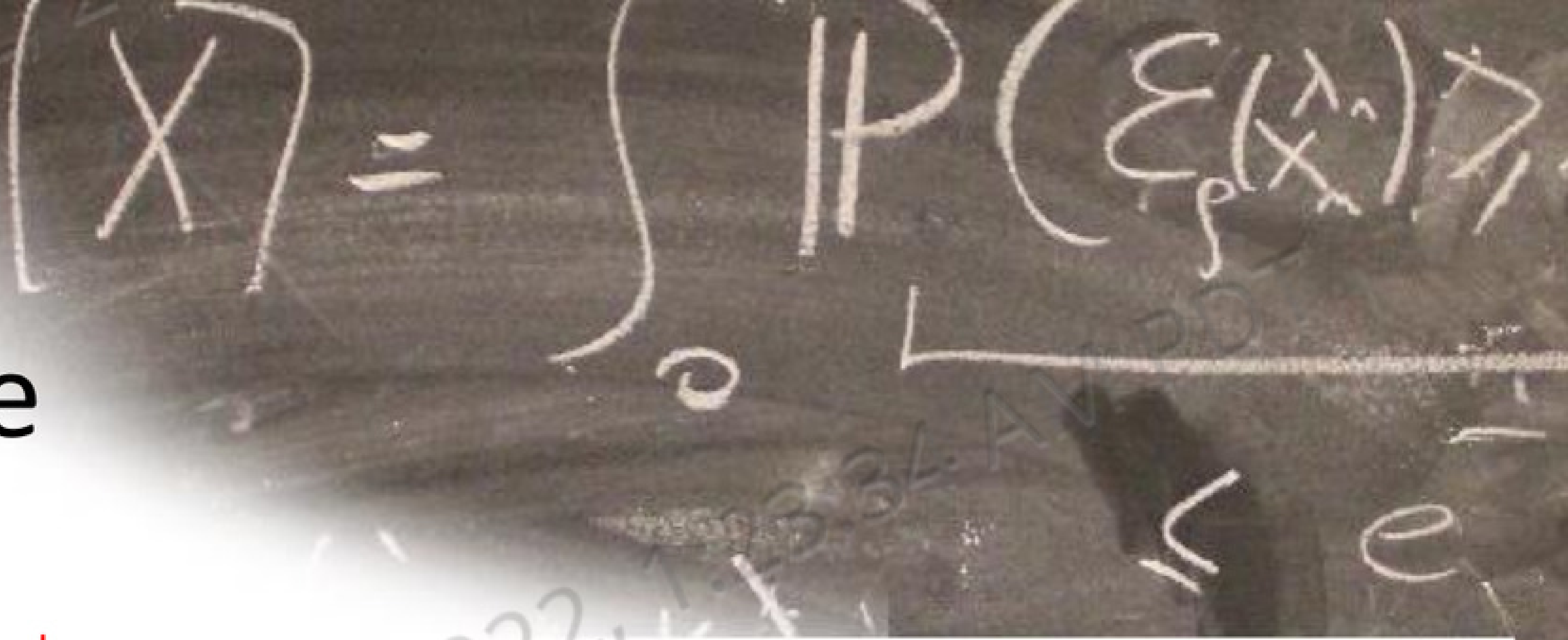
$$\underset{1\times 1}{Y_t} = g(\underset{1\times 1}{t}, \underset{k\times 1}{\underset{\sim}{X_t}}) \qquad \operatorname{cov}(dV_t^i, dV_t^j) = \rho_{ij}\,dt \qquad i=j \quad \rho_{ij}=1$$

$$dY_t = g_t\,dt + \sum_{i=1}^{k} g_{x_i}\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^{k} g_{x_i x_j}\,dX_t^i\,dX_t^j$$

$$(dt)^2 = 0, \quad dt\,dV^i = 0, \quad dV^i dV^j = \rho_{ij}\,dt$$

Sample Problem – III

Ito’s version of Leibniz Rule



$dX_t = \mu_X X\,dt + \sigma_X X\,dW_t$

1.3 Example: Itô Product Rule

Let the dynamics of X be given by (1), and for simplicity take $\mu(t, X) = \mu_X X$ and $\sigma(t, X) = \sigma_X X$ where μ_X and σ_X are constants. Suppose there also exists a process Y (perfectly correlated with X) with dynamics

$$dY(t) = \mu_Y Y\,dt + \sigma_Y Y\,dW(t).$$

Take $F(t, x, y) := xy$. What is dF?

$F_t = 0$

$F_x = y, \; F_y = x$

$F_{xx} = 0, \; F_{yy} = 0$

$F_{xy} = 1$

$$dF = Y_t\,dX_t + X_t\,dY_t + \underbrace{\tfrac{1}{2}dX_t\,dY_t + \tfrac{1}{2}dY_t\,dX_t}_{= dX_t\,dY_t}$$

Note: $dW_t \cdot dW_t = \rho\,dt = 1\,dt$

$$\Rightarrow dF = X_t Y_t(\mu_X + \mu_Y + \sigma_X\sigma_Y)\,dt + X_t Y_t\,(\sigma_X + \sigma_Y)\,dW_t$$

3 Uncorrelated Multi-d Ito Lemma

$$\underset{k\times 1}{d\tilde{X}_t} = \underset{k\times 1}{\mu_t}\, \underset{1\times 1}{dt} + \underset{k\times d}{\sigma_t}\, \underset{d\times 1}{dW_t}$$

$$dX_1 = \mu_1 dt + \sigma_{11} dW_1 + \cdots + \sigma_{1d} dW_d$$

$$\vdots$$

$$dX_k = \mu_1 dt + \sigma_{k1} dW_1 + \cdots + \sigma_{kd} dW_d$$

$$\underset{1\times 1}{Y_t} = g(t, X_t), \quad dY_t = g_t dt + \sum_{i=1}^{k} g_{x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{k} g_{x_i x_j} dX_i dX_j$$

$$dt^2 = 0, \quad dt\, dW^i = 0, \quad (dW^i)^2 = 1\, dt, \quad dW_i dW_j = 0 \quad \forall i \neq j$$

Sample Problem – IV

Multidimensional Ito's lemma

$dX_1 = \mu_1 dt + \sigma_{11} dW_1 + \sigma_{12} dW_2$

Consider the following two dimensional Ito process $\bar{X}_t = (X^1, X^2)^T$ $k \times 1$

$dX_2 = \mu_2 dt + \sigma_{21} dW_1 + \sigma_{22} dW_2$

$$d\bar{X}(t) = \mu dt + \sigma d\bar{W}(t) \tag{4}$$

2×1

where $\bar{W}(t)$ is a k-dimensional standard independent Wiener process. Here

2×1

$$\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \sigma = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \end{bmatrix} \tag{5}$$

$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

1. Define $Y_t = (X^1(t))^2 + (X^2(t))^2 + 2X^1(t)X^2(t)$. Use Itô's lemma to derive $dY(t)$.

2. Note that $Y(t) = (Z(t))^2$ where $Z(t) = X^1(t) + X^2(t)$. Use Itô's lemma for $Y((Z(t))$ to verify that you get the same form as in part *1)*.

3. Rewrite the Itô's Process for X on 'correlated' form (see lecture notes and Bjork chapter 4).and verify again that Itô's lemma on correlated form leads to the same form for dY.

1. $dY = Y_{x^1} dX^1 + Y_{x^2} dX^2 + \frac{1}{2} Y_{x^1x^1} (dX^1)^2 + \frac{1}{2} Y_{x^2x^2} (dX^2)^2$

$+ Y_{x^1x^2} (dX^1)(dX^2)$

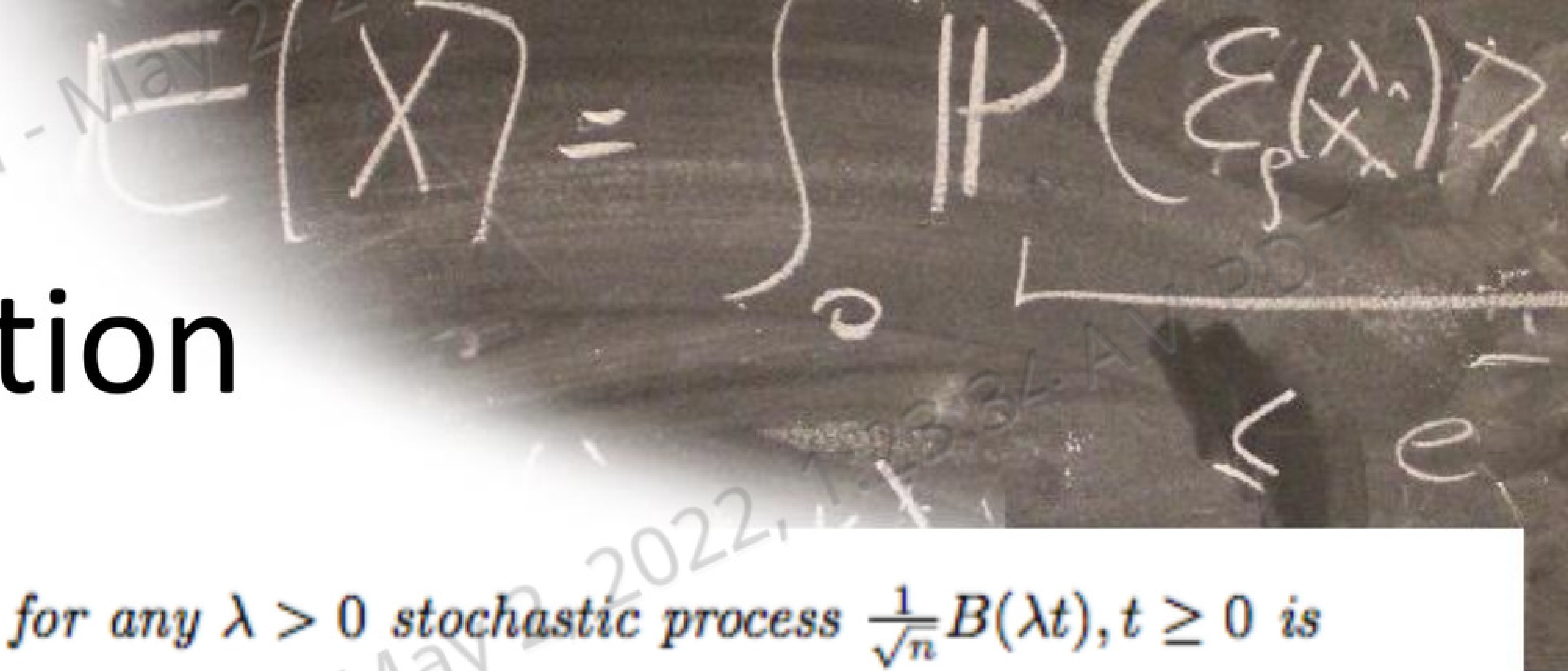
$= 2(X^1 + X^2)(dX^1 + dX^2) + (dX^1)^2 + (dX^2)^2$

$+ 2\, dX^1 dX^2$

$$= 2(X^1 + X^2)\left(3dt + \frac{3+\sqrt{3}}{2} dW^1 + \frac{3-\sqrt{3}}{2} dW^2\right)$$

$+ 6dt$

Appendix: Brownian Motion



Theorem 2. *If $B(0) = 0$ then for any $\lambda > 0$ stochastic process $\frac{1}{\sqrt{n}}B(\lambda t), t \geq 0$ is a Brownian motion.*

Proof. First, let us notice that $X(t) = \frac{1}{\sqrt{\lambda}}B(\lambda t)$ is a Gaussian process, i.e. for any set $t_1 < t_2 < \cdots < t_n$ the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is a multivariate Gaussian distribution. This property is clearly inherited from Brownian motion properties.

Since normal distribution is characterized by its mean and covariance we have to check that the mean and covariance of the process $X(t)$ coincide with those of Brownian motion. It is easy:

$$\mathbb{E}X(t) = \mathbb{E}\frac{1}{\sqrt{\lambda}}B(\lambda t) = \frac{1}{\sqrt{\lambda}}\mathbb{E}B(\lambda t) = 0,$$

and for $s < t$

$$\mathbb{E}X(s)X(t) = \mathbb{E}\frac{1}{\sqrt{\lambda}}B(\lambda s)\frac{1}{\sqrt{\lambda}}B(\lambda t) = \frac{1}{\lambda}\mathbb{E}B(\lambda s)B(\lambda t) = \frac{1}{\lambda}\min(\lambda s, \lambda t) = s.$$

Thus $\frac{1}{\sqrt{\lambda}}B(\lambda t)$ has the same distribution as a Brownian motion.

Appendix: Ito's Lemma
ODE Tricks

Example 2: $$y' - \frac{2y}{x} = 0.$$

First, we have $P(x) = \dfrac{-2}{x}$ Then: $M(x) = e^{\int P(x)\, dx}$

We now have: $M(x) = e^{\int \frac{-2}{x}\, dx} = e^{-2\ln x} = \left(e^{\ln x}\right)^{-2} = x^{-2}$ $\quad M(x) = \dfrac{1}{x^2}.$

Multiplying both sides by $M(x)$ we obtain

$$\frac{y'}{x^2} - \frac{2y}{x^3} = 0$$

$$\frac{y'x^3 - 2x^2y}{x^5} = 0$$

$$\frac{x(y'x^2 - 2xy)}{x^5} = 0$$

$$\frac{y'x^2 - 2xy}{x^4} = 0.$$ => (By separation of variables) $\quad y(x) = Cx^2.$

Appendix: Brownian Motion

Theorem 3. *If $B(t)$ is a Brownian Motion starting at 0 then so is the process defined by $X(0) = 0$ and $X(t) = tB(1/t)$ for $t > 0$.*

Proof. Fix $t_1 < t_2 < \cdots < t_n$. Then clearly $X(t_1) = t_1 B(1/t_1), \ldots, X(t_n) = t_n B(1/t_n)$ has a multivariate Gaussian distribution. We just have to check that it has the same mean and covariance structure. First of all,

$$\mathbb{E}X(t) = t\mathbb{E}B(1/t) = t \cdot 0 = 0.$$

Also, for $0 < s < t$

$$\mathbb{E}X(s)X(t) = st\mathbb{E}B(1/s)B(1/t) = st\min(1/s, 1/t) = st \cdot 1/t = s$$

Thus $X(t)$ is a Brownian motion.

Appendix: Ito's Lemma
ODE Tricks

Separation of Variables:

Suppose a differential equation can be written in the form

$$\frac{d}{dx} f(x) = g(x) h(f(x)), \qquad (1)$$

which we can write more simply by letting $y = f(x)$:

$$\frac{dy}{dx} = g(x) h(y).$$

As long as $h(y) \neq 0$, we can rearrange terms to obtain:

$$\frac{dy}{h(y)} = g(x) dx,$$

Solution:

$$\int \frac{1}{h(y)} dy + C_1 = \int g(x) \, dx + C_2,$$

because a single constant $C = C_2 - C_1$ is equivalent.)

Appendix: Ito's Lemma
ODE Tricks

Example 1: $$\frac{d}{dx}f(x) = f(x)(1 - f(x))$$

Solution:

Step 1: Denote y = f(x): $$\frac{dy}{dx} = y(1-y).$$

Step 2: Separate variables: $$\frac{dy}{y(1-y)} = dx.$$

Step 3: Integrate both side: $$\int \frac{dy}{y(1-y)} = \int dx,$$

Step 4: Basic algebra: $$\int \frac{1}{y}\,dy + \int \frac{1}{1-y}\,dy = \int 1\,dx,$$

$$\ln|y| - \ln|1-y| = x + C$$ *C* is an arbitrary constant

Step 5: After arrangement: $$y = \frac{1}{1+Be^{-x}}.$$ *B* is an arbitrary constant

Appendix: Ito's Lemma
ODE Tricks

Integrating factor:

$$y' + P(x)y = Q(x)$$

Factor: $$M(x) = e^{\int_{s_0}^{x} P(s)ds}$$

Multiply both side: $$y'e^{\int_{s_0}^{x} P(s)ds} + P(x)e^{\int_{s_0}^{x} P(s)ds}y = Q(x)e^{\int_{s_0}^{x} P(s)ds}$$

Also we have: $$y'e^{\int_{s_0}^{x} P(s)ds} + P(x)e^{\int_{s_0}^{x} P(s)ds}y = \frac{d}{dx}(ye^{\int_{s_0}^{x} P(s)ds})$$

Equate RHS of above 2 Eqns.: $$\frac{d}{dx}(ye^{\int_{s_0}^{x} P(s)ds}) = Q(x)e^{\int_{s_0}^{x} P(s)ds}$$

Integrate and we have: $$ye^{\int_{s_0}^{x} P(s)ds} = \int_{t_0}^{x} Q(t)e^{\int_{s_0}^{t} P(s)ds}dt + C$$

For homogeneous ODE: $Q(x) = 0$

We have: $$y = \frac{C}{e^{\int_{s_0}^{x} P(s)ds}}$$