MFE230Q: Assignment 5 - Due May 11, 2021
Black-Scholes Economy

1. Consider the Black-Scholes economy

$$\frac{dS}{S} = (r-\delta)\,dt + \sigma\,dW^Q \quad (1)$$

$$\frac{dB}{B} = r\,dt, \quad (2)$$

Here, the risk free rate r, dividend yield δ and volatility σ are all constant. Determine the arbitrage-free price of a European Put Option with strike K and maturity T.

2. Consider the Black-Scholes economy

$$\frac{dS}{S} = \hat{\mu}\,dt + \sigma\,dW^P \quad (3)$$

$$\frac{dB}{B} = r\,dt, \quad (4)$$

$\hat{\mu}$, r, and σ constant.

   (a) Define $y(t) = \log S(t)$. Determine the date-0 price of an A/D security that pays $\frac{1}{\epsilon}$ if $y(T)$ ends up between a value y and $y+\epsilon$. Take the limit of $\epsilon \Rightarrow 0$.

   (b) Use these A/D prices to determine the price of a call option with strike K. (Of course, you should get the same answer as B/S, since it is the same model).

3. Consider the Black Scholes economy where the stock pays a constant dividend yield δ,

$$\frac{dS + \delta S\,dt}{S} = \hat{\mu}\,dt + \sigma\,dW^P$$

$$\frac{dB}{B} = r\,dt.$$

where μ, σ, and r are all constant. Now, consider a call option that has no maturity date, but has strike K and will be exercised the first time the stock price reaches S^*. Hence, the cash flow when this *first hitting time* occurs is $(S^* - K)$ (Clearly, S^* has been chosen such that it is greater than K).

   (a) Determine the differential equation that the value of this call, $C(S)$ satisfies. Argue why the call price is not an explicit function of time.

   (b) Solve for the call price. Hint: look for solutions of the form $C(S) = S^\alpha$ for some α, and use the linearity of the differential equations involved which implies that if f_1 and f_2 are solutions to a linear differential equation, then $Af_1 + Bf_2$ is also a solution, to write down a general solution. Use your boundary conditions at $S = 0$ and $S = S^*$ to identify A and B. Note: the values for α are complicated by the existence of dividends, but so what, they are just constants.

(c) Now assume that the agent is allowed to choose for herself the S^* at which she will elect to exercise her option. Determine this optimal S^*. What happens as $\delta \Rightarrow 0$? Interpret.

4. *Corporate finance*: Consider a firm whose value V evolves via the risk-neutral dynamics

$$\frac{dV}{V} = r\,dt + \sigma\,dW^Q. \tag{5}$$

There are two claimants to this firm value: debt $D(V)$ and equity $E(V)$. Thus, $V = D(V) + E(V)$. Debt will receive a constant coupon $(C\,dt)$ for all dates-t until the first time the default boundary V_B is reached. When it is reached, equity defaults on the debt issue, and thus the coupon flow stops. However, the debtholders become owners of the firm. Thus, when firm value reaches V_B for the first time, the equity claim is zero and the debt claim is V_B.

(a) Determine the value of the claim $P(V)$ that receives \$1 the first time V reaches V_B. Argue why this claim is not an explicit function of time.

(b) The value of debt stems from two components: the first component is the present value of the coupon claim, a claim which receives coupons continuously until the first time V reaches V_B. The second component is the present value of receiving the firm (i.e., something worth V_B) the first time V reaches V_B. Determine the present value of these two claims (it is easier to do them separately). Interpret your results with respect to $P(V)$ found in 2a.

(c) The value of equity is $E(V) = V - D(V)$. The benefit of remaining solvent is that they have claim to the firm value V. The cost of remaining solvent is that they are paying the coupon C out of their own pocket. Above, V_B was taken as exogenous. Identify the optimal level of the default boundary V_B where it is best for equity to declare bankruptcy and hand the firm over to the debtholders so that they can stop making coupon payments.