

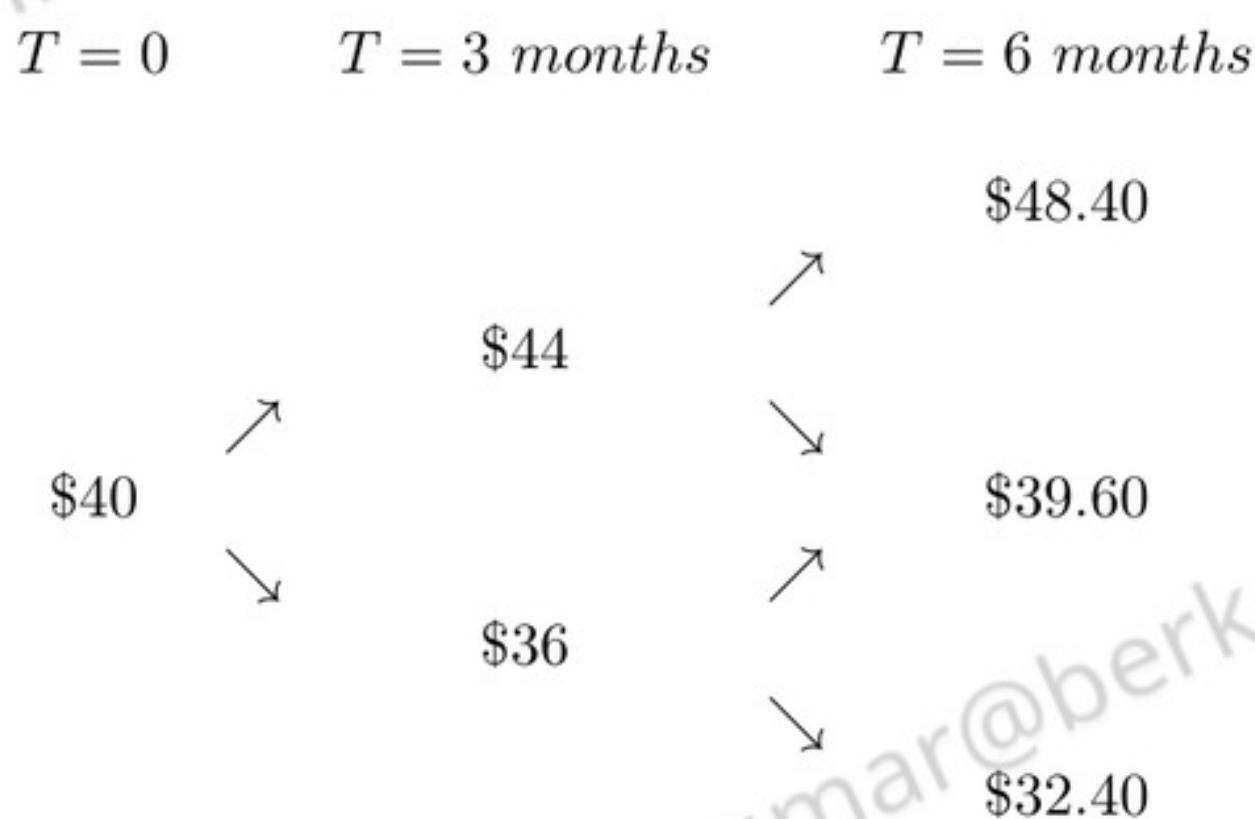
MFE 230Q – Introduction to Stochastic Calculus  
GSI Session 2 Solutions

**2 GSI Session 2: Sample Problem 2 - A two-period, three-date economy American put option**

Current stock price of MFE2016Class Inc. is \$40, over the next two 3-month periods it's expected to go up by 10% or down by 10%. The risk-free rate is 12% per year with continuous compounding.

- (a) Compute the price of a 6-month **European put option** with a strike price of \$42?  
(b) Compute the price of a 6-month **American put option** with a strike price of \$42?

**2.1 Solutions:**



(a) Compute the price of a 6-month European put option with a strike price of \$42

In this part I will essentially use the risk neutral probabilities to price the European put. Firstly, compute the risk neutral for each one period binomial tree probabilities using:

$$T = 0 \quad T = 3 \text{ months}$$

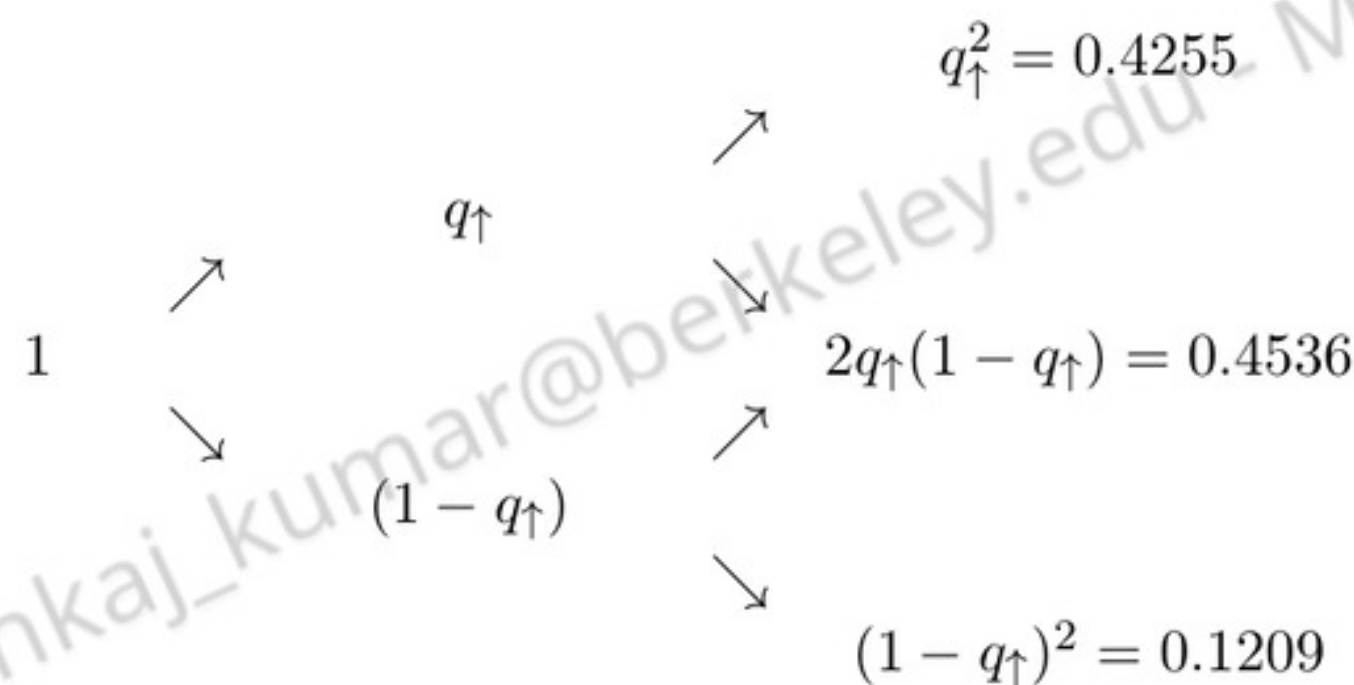


$$q_{\uparrow}u + (1 - q_{\uparrow})d = R_f = e^{0.12 \frac{3}{12}} = 1.0305$$

$$q_{\uparrow} = 0.6523$$

Then we compute the risk neutral probabilities in each state at  $T = 6 \text{ months}$ :

$$T = 0 \quad T = 3 \text{ months} \quad T = 6 \text{ months}$$

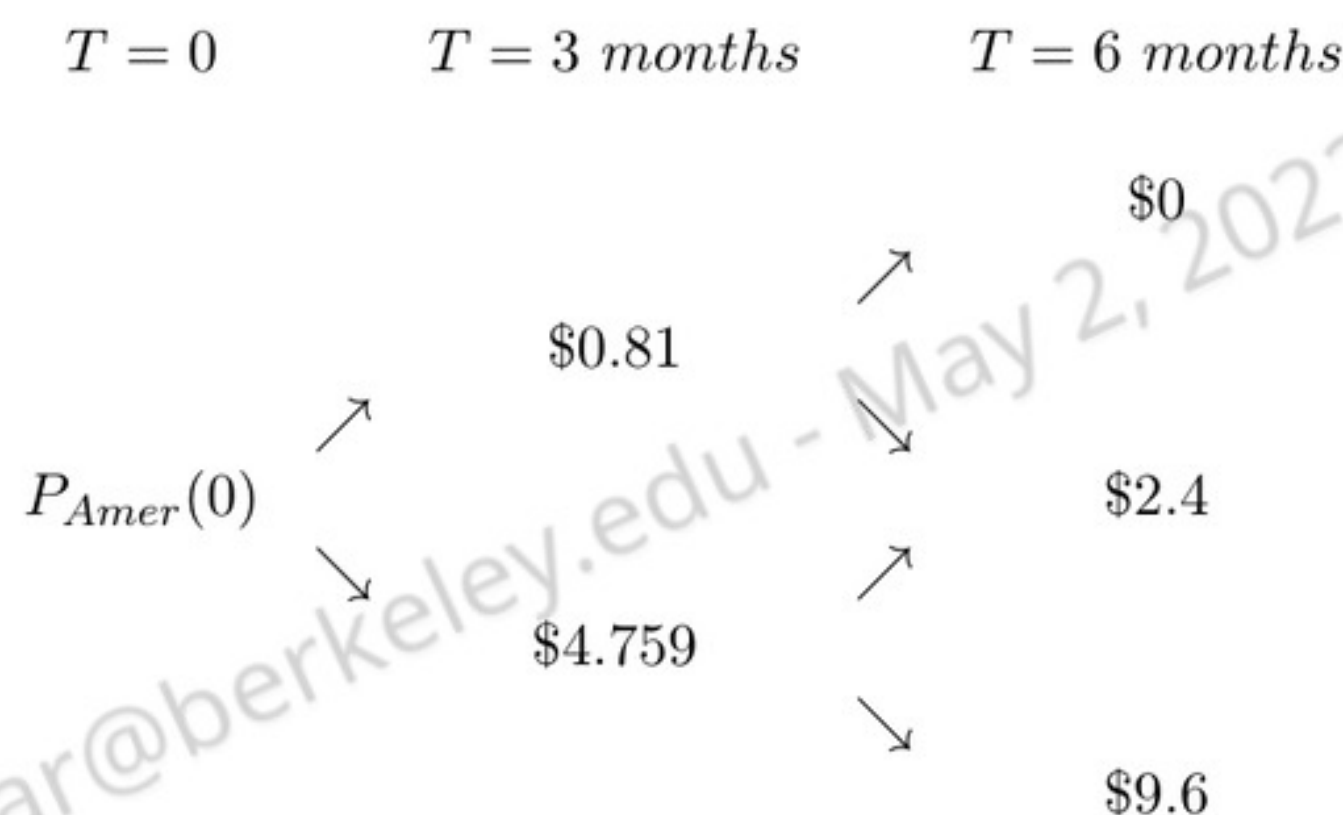


Finally we can price the **European put option** using the risk neutral probabilities:

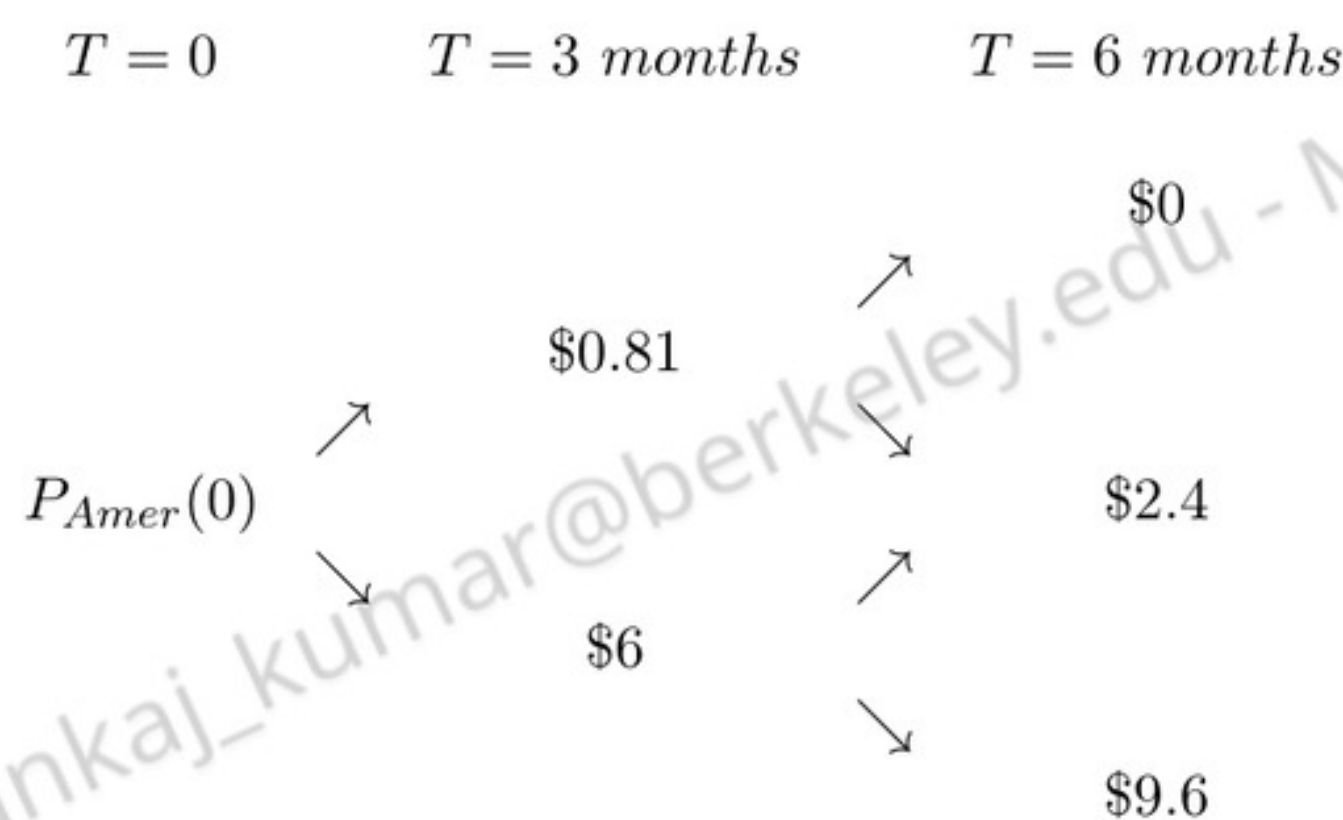
$$\begin{aligned} P_{Euro}(0) &= \frac{1}{R_f^2} \mathbb{E}^{\mathbb{Q}} [P_{Euro}(2)] \\ &= \frac{1}{R_f^2} (0.4255 \times 0 + 0.4536 \times \$2.4 + 0.1209 \times \$9.6) \\ &= \$2.188 \end{aligned}$$

(b) Compute the price of a 6-month European put option with a strike price of \$42

Below the payoffs of the American put at  $T = 6$  months, and the no-arbitrage prices using the risk neutral probabilities.



Note that at  $T = 3$  months, in the down state, it is optimal to exercise the American put since  $\$42 - \$36 = \$6 > \$4.759$ . We should then recompute the value in the down state and consequently recompute the price today.



Price today is then

$$\begin{aligned}
 P_{Amer}(0) &= \frac{1}{R_f} \mathbb{E}^{\mathbb{Q}} [P_{Euro}(1)] \\
 &= (0.6523 \times \$0.81 + (1 - 0.6523) \times \$6) \\
 &= \$2.537
 \end{aligned}$$



### 3 EXTRA: Change of measure (Radon-Nikodym Derivative)

Consider an  $M$ -state, single-period economy. The possible states of the world are  $\Omega = \{\omega_j : j = 1, \dots, M\}$ , and these have associated *physical* probabilities given by some measure  $\mathbb{P}$ . Assume that there is no arbitrage in the model so that there exist risk-neutral probabilities given by  $\mathbb{Q}(\omega_j)$ .

We call the ratio of probabilities  $L(\omega_j) := \frac{\mathbb{Q}(\omega_j)}{\mathbb{P}(\omega_j)}$  the *Radon-Nikodym derivative* of the measure  $\mathbb{Q}$  with respect to the measure  $\mathbb{P}$ <sup>1</sup>.  $L$  describes the ‘rate of change’ of  $\mathbb{Q}$  as  $\mathbb{P}$  changes. Notice that we can treat  $L(\cdot)$  as a random variable  $\tilde{L}$  since its value changes depending on which state  $\omega_j$  is realized.

For our purposes, the key fact about the Radon-Nikodym derivative is that if  $\tilde{X}$  is any random variable defined on  $\Omega$ , then the following identity holds

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}] = \mathbb{E}^{\mathbb{P}}[\tilde{L}\tilde{X}] \quad (10)$$

Why is eq. (10) important? It turns out that it is easier to evaluate the right-hand side of (10) directly, rather than use brute-force on the  $\mathbb{Q}$ -expectation.<sup>2</sup>

Let’s do an example to make things concrete. Consider the one-period binomial model:

$$\begin{array}{ccc} & \nearrow & B(T) = 105, S_{\uparrow}(T) = 110 \\ B(0) = 100, S(0) = 100 & & \\ & \searrow & B(T) = 105, S_{\downarrow}(T) = 90 \end{array}$$

After some calculations, we obtain the risk-neutral probabilities  $\mathbb{Q}(\omega_u) = 3/4, \mathbb{Q}(\omega_d) = 1/4$ . Assume that the physical probabilities are given by  $\mathbb{P}(\omega_u) = \mathbb{P}(\omega_d) = 1/2$ .

Our goal is to price a plain vanilla call option with  $K = 100$  using the risk-neutral valuation formula

$$C_0 = \frac{1}{1+R} \mathbb{E}^{\mathbb{Q}}[\tilde{C}(T)].$$

<sup>1</sup>Since we’re in a finite-state world,  $L$  is simply a ratio of two numbers and doesn’t look like a derivative in the typical calculus sense. However, once we start working in a continuous-state setting, the terminology should become clear. (Actually, even in the finite-state case, it is a derivative in an appropriate generalized sense, but we won’t get into this. . .)

<sup>2</sup>Typically, either expectation in (10) is easy to compute in the a finite-state model. However, introducing  $L$  in this setting helps us develop our intuition for the continuous-time models we will study in the future.

By eq. (10) we can compute the expectation by first computing  $L$  and then taking expectations under the measure  $\mathbb{P}$ . The values for  $L$  in each state are

$$\begin{aligned} L(\omega_u) &= \frac{Q(\omega_u)}{P(\omega_u)} = \frac{3/4}{1/2} = 3/2 \\ L(\omega_d) &= \frac{Q(\omega_d)}{P(\omega_d)} = \frac{1/4}{1/2} = 1/2 \end{aligned}$$

Hence, the call value is

$$\begin{aligned} C(0) &= \frac{1}{1+R} \mathbb{E}^Q[\tilde{C}(T)] \\ &= \frac{1}{1.05} \mathbb{E}^P[\tilde{L}\tilde{C}(T)] \\ &= \frac{1}{1.05} [\mathbb{P}(\omega_u)L(\omega_u)C(T)(\omega_u) + \mathbb{P}(\omega_d)L(\omega_d)C(T)(\omega_d)] \\ &= \frac{1}{1.05} \left[ \frac{1}{2} \times \frac{3}{2} \times 10 + \frac{1}{2} \times \frac{1}{2} \times 0 \right] \\ &= \frac{1}{1.05} \times \frac{15}{2} \\ &\approx 7.14. \end{aligned}$$

You can easily check that this is the same as the value we would obtain if we computed the risk-neutral expectation directly.