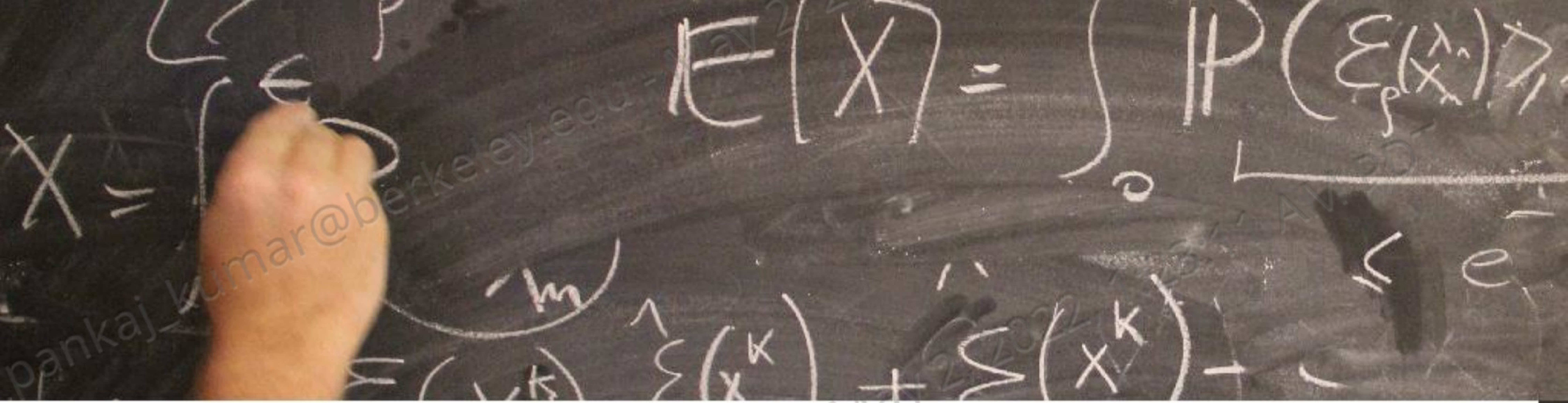


MFE 230Q [Spring 2021] Introduction to Stochastic Calculus GSI Session 5

BerkeleyHaas
MASTER OF FINANCIAL ENGINEERING

GSI: Simon Xu

Email:

simon_xu@haas.berkeley.edu

Outline

- Black-Scholes
- Numerical methods
- Example on Black-Scholes
  - Geometric Average Asian Call

Black-Scholes PDE

$dB = r(t, S_t)\, B(t)\, dt$

- Feynman-Kac: link between PDEs and ODEs
- Can use it to price a contingent claim as a discounted expected value

$dS = S\alpha\, dt + S\sigma\, dW_t \Leftarrow \mathbb{P}$ measure

Black-Scholes PDE: The value of any simple contingent claim with payout function $C_T = \Phi(S_T)$ satisfies the PDE:

$$C_t + rSC_S + \frac{1}{2}\sigma^2 C_{SS} S^2 - rC = 0,$$

$$C(T) = \Phi(S(T)).$$

$dS = S\underline{\underline{r}}\, dt + S\sigma\, dW_t^{\mathbb{Q}} \Leftarrow \mathbb{Q}$ process

$$F(t,s) = e^{-r(T-t)} E^Q\left(\Phi(S_T) \mid S_t = s \right)$$

Black-Scholes
Via Replicating Portfolios

$dF = -dh'(S + dS) = 0$

Black-Scholes replicating portfolio (Björk Theorem 8.5):Consider a simple contingent claim with time T-payoff $\Phi(S(T))$. Define $F(t, S)$ as the solution to the PDE

$$F_t + rSF_S + \frac{1}{2}\sigma^2 F_{SS} S^2 - rF = 0,$$
$$F(T, S) = \Phi(S).$$

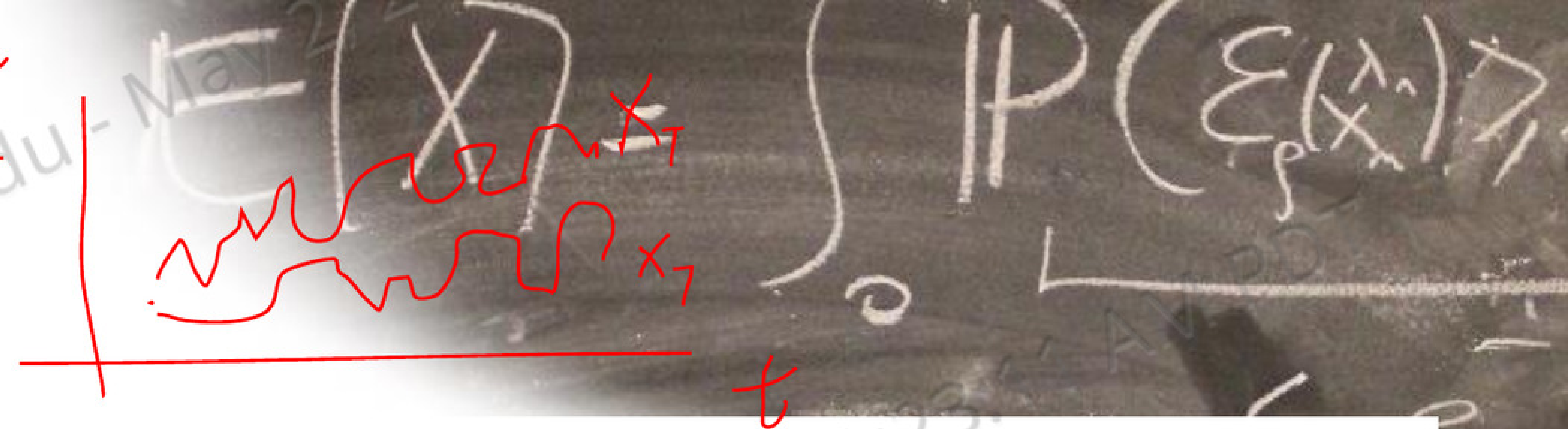
Then the contingent claim can be replicated by a self-financed portfolio with time t value $V^{\mathbf{h}}(t, S_t) = F(t, S_t)$. The self-financed replicating portfolio trading strategy is defined as

$$\mathbf{h}_t = (h^B, h^S)' = ((F - SF_S)/B, F_S)'.$$

One can also show that we can replicate any simple ($\Phi(S_T)$) or general (e.g. path-dependent) contingent claims

⇒ Black-Scholes market is complete

Numerical – Monte Carlo



The Euler discretization scheme for the stochastic differential equation

$$dX_t = b(X_t)dt + a(X_t)dW_t$$

μ σ

t_k t_{k+1}

is given by

$$\hat{X}^N_{t_{k+1}} = \hat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + \int_{t_k}^{t_{k+1}} a(X_s)dW_s$$
$$\simeq \hat{X}^N_{t_k} + b(\hat{X}^N_{t_k})(t_{k+1} - t_k) + a(\hat{X}^N_{t_k})(W_{t_{k+1}} - W_{t_k}).$$

$\sim N(0, t_{k+1} - t_k)$

In particular, when X_t is the geometric Brownian motion given by

$$dX_t = rX_tdt + \sigma X_tdW_t$$

we get

$$\hat{X}^N_{t_{k+1}} = \hat{X}^N_{t_k} + r\hat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma\hat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}),$$

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

(1) $\mathbb{E}\left(\Phi(X_T)\right)$

(2) $t_k = k\,\Delta t = k\,\dfrac{T}{M^{max}}$

# of points in time (interval)

N^{max} : # of sample trajectories

$t=0$, $\hat{X}_{t=0}$

$j = 1, 2, \ldots, N^{max}$

$k = 1, \ldots, M^{max}$

gen $Z^j \sim N(0,1)$

$$\hat{X}^j_{t_{k+1}} = \hat{X}^j_{t_k} + \mu(t_{k+1} - t_k) + \sigma\sqrt{t_{k+1} - t_k}\, Z^j$$

$$t_{k+1} = t_k + \Delta t$$

$$\Phi^j = \Phi(\hat{X}^j)$$

$$\hat{\theta} = \frac{1}{N^{max}} \sum_{j=1}^{N^{max}} \Phi^j \to E(\Phi)$$

Sampling error

$$\hat{\sigma}^2 = \frac{1}{N^{max} - 1} \sum_{j=1}^{N^{max}} \left(\Phi^{j} - \hat{\theta} \right)^2$$

Numerical – Monte Carlo w. Antithetic Paths

Variance -reduction

Construct two samples of S(T):

$$S_j^+(T) = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_j \right)$$

$\sim N(0,1)$

$\sim N(0,1)$

$$S_j^-(T) = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}(-Z_j) \right)$$

$Corr(Z_j, -Z_j) = -1$

With these, get two payoff samples:

$$V_j^+ = \exp(-rT)\max[S_j^+(T) - K, 0]$$
$$V_j^- = \exp(-rT)\max[S_j^-(T) - K, 0]$$

This is your estimator:

$$\overline{V} = \frac{1}{n}\sum_{j=1}^{j=n}\frac{1}{2}(V_j^+ + V_j^-)$$

$Z_j \overset{iid}{\sim} N$

$-Z_j \overset{iid}{\sim} N$

$(Y_1^+, Y_1^-), (Y_2^+, Y_2^-), \ldots, (Y_{N^{max}}^+, Y_{N^{max}}^-)$

$1, 2, \ldots \to N^{max}$ } irvs.

$(Y_i^+, Y_i^-) \sim F(\cdot)$ same dist.

$$\hat{V} = \frac{1}{2N^{max}} \sum_{j=1}^{N^{max}} (Y_i^+ + Y_i^-)$$

$$= \frac{1}{N^{max}} \sum_{j=1}^{N^{max}} \left(\frac{Y_i^+ + Y_i^-}{2} \right)$$

$$\hat{\sigma}^2_{AV} = V\left(\frac{Y_j^+ + Y_j^-}{2}\right) = \frac{1}{4}\left(V(Y_j^+) + V(Y_j^-) + 2\,\mathrm{cov}(Y_j^+, Y_j^-)\right)$$

$$= \frac{1}{4}\left(2V(Y_j^+) + 2\,\mathrm{cov}(\cdots)\right)$$

$$= \frac{1}{2}V(Y_j^+) + \frac{1}{2}\mathrm{cov}(Y_j^+, Y_j^-)$$

$$= \frac{1}{2}V(Y_j^+) + \frac{1}{2}\underbrace{\mathrm{Corr}(Y_j^+, Y_j^-)}_{<0}\,V(Y_j^+)$$

$$< \hat{\sigma}^2_{MC}$$

Sample Problem 1: Black-Scholes

1 Geometric-average Asian call

Consider the standard Black-Scholes setup

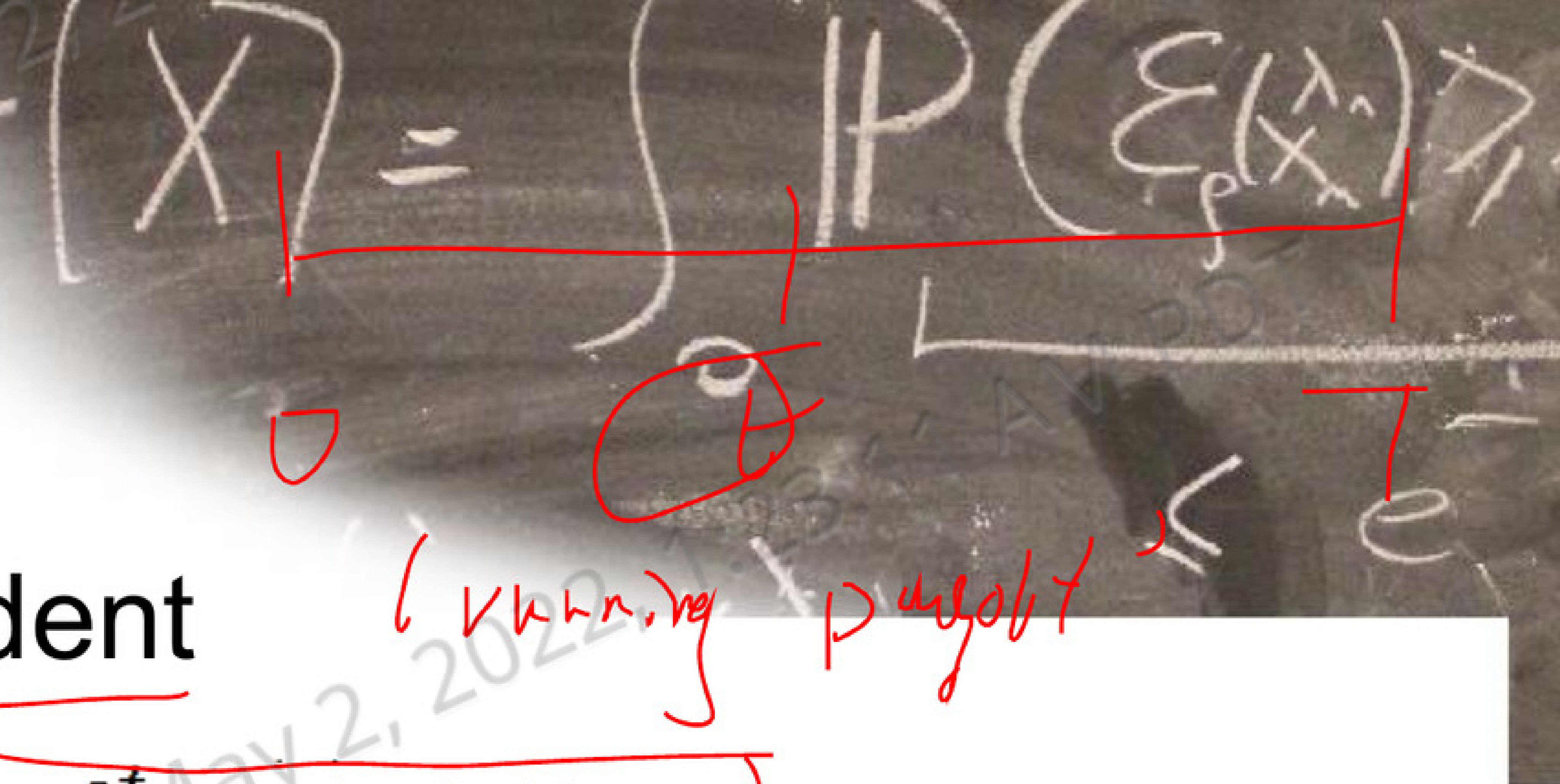
$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW^Q(t)$$

GBM

$$\frac{dB(t)}{B(t)} = r\,dt.$$

Find the arbitrage free price of an option with payoff $\max[\frac{1}{T}\int_0^T \log(S(u))du - K, 0]$.

Sample Problem 1: Black-Scholes



The payoff is path dependent

(running payoff)

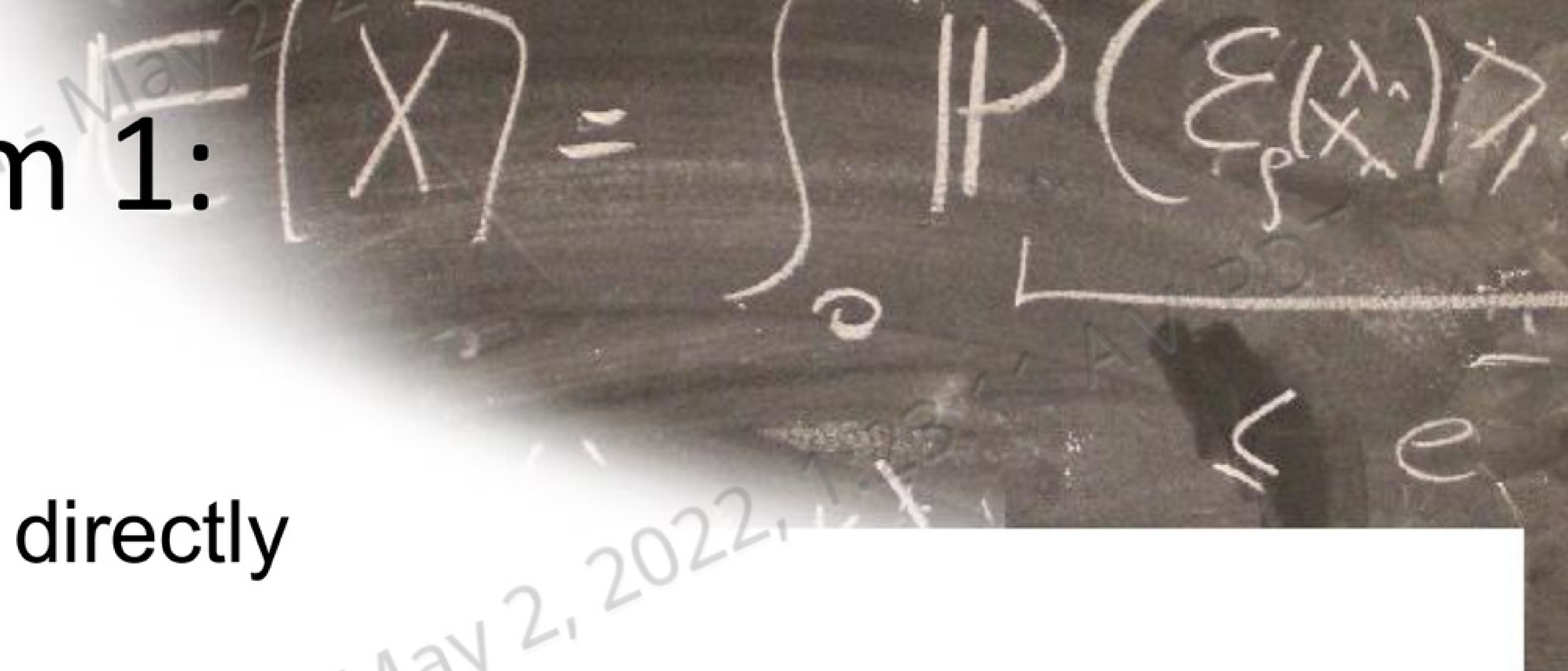
$$\text{Define } x(t) := \int_0^t \log(S(u))du$$

$$C(t, S(t), x(t)) = E_t^Q\left[e^{-r(T-t)} \max\left(\frac{x(T)}{T} - K, 0\right)\right]$$

Two ways to tackle it (without using simulation)

Evaluate the above expectation directly

Find the PDE for $C(t, S(t), x(t))$ → solve it numerically



Sample Problem 1: Black-Scholes

1. Evaluate expectation directly

We know $S(t)$ follows GBM,

$$S(u) = S(t)\exp\left\{(r - \frac{1}{2}\sigma^2)(u-t) + \sigma(W^Q(u) - W^Q(t))\right\}$$

So we can write out x(t) as

$$x(T) = \int_0^T \log(S(u))\,du \quad = \underbrace{\int_0^t \log(S(u))du}_{x(t)} + \int_t^T \log(S(u))du$$

$$= x(t) + \int_t^T \log(S(u))\,du$$

fixed

$$= x(t) + \int_t^T \left(\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)(u-t) + \sigma(W^Q(u) - W^Q(t))\right) du$$

$$= x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2} + \sigma \int_t^T \boxed{(W^Q(u) - W^Q(t))}\, du$$

$$= x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2} + \sigma \int_t^T \boxed{\int_t^u dW^Q(s)}\, du$$

$$= x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2} + \sigma \int_t^T \int_s^T 1\, du\, dW^Q(s)$$

$$= x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2} + \sigma \int_t^T (T-s)dW^Q(s),$$

Fubini's Thm
for Ito integrals

prove ?

$$\int_t^T W^Q(s) - W^Q(t)\, ds = \int_t^T (T-s)\, dW_s$$

$$= \int_t^T T\, dW - \int_t^T s\, dW$$

$$\int_t^T 1\, dW = \lim_{n\to\infty} \sum_{k=0}^{n-1} 1 \cdot \left(W_{t_{k+1}} - W_{t_k} \right)$$

$$= TW_T - TW_t - \int_t^T s\, dW$$

$f(t,w) = tw$ Ito's Lemma

$f_t = w$

$f_w = t$

$f_{ww} = 0$

$df = w\,dt + t\,dw$

$d(tw) = W\,dt + t\,dW$

$$\int_t^T d(sW) = \int_t^T W\, ds + \int_t^T s\, dW$$

$$TW_T - tW_t = \int_t^T W\,ds + \int_t^T s\,dW$$

$$\int_t^T s\,dW = TW_T - tW_t - \int_t^T W\,ds$$

$$TW_T - TW_t - \left(TW_T - tW_t - \int_t^T W\,ds\right)$$

$$= \underline{tW_t - TW_t} + \int_t^T W_s\,ds$$

$$W_t(t-T) = -W_t(T-t) = -W_t\int_t^T 1\,ds$$

$$\int_t^T \underbrace{W_s - W_t}_{\Delta W} \, ds$$

Sample Problem 1: Black-Scholes

fixed

X ~ normal

- We have got $x(T) = x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2} + \sigma\int_t^T (T-s)\,dW^Q(s)$, which we know is normally distributed.
- Let's write $g(t)\widetilde{X} \overset{d}{=} \int_t^T (T-s)\,dW_s^Q$ where $\tilde{X}$ is a standard normal variable

$E\left(\int_t^T T-s\,dW\right)=0$

$$g(t)^2 \overset{d}{=} Var\left[\int_t^T (T-s)\,dW_s\right] = E\left[\left(\int_t^T (T-s)\,dW_s\right)^2\right] = \int_t^T (T-s)^2\,ds = \frac{(T-t)^3}{3}$$

- For convenience, also define $H(t) = x(t) + \tau\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{\tau^2}{2}$ where $\tau = T - t$
- So the equality at the top becomes

$$x(T) \overset{d}{=} \underbrace{x(t) + (T-t)\log(S(t)) + \left(r - \frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{2}}_{H(t)} + \sigma g(t)\widetilde{X}$$

Sample Problem 1: Black-Scholes

- Now we can compute the expectation directly,

$$\left(\frac{X(T)}{T} - K, 0\right)^+ \geq 0$$

$$\begin{aligned}
C(t, S(t), x(t)) &= E_t^Q\left[e^{-r(T-t)} \max\left(\frac{x(T)}{T} - K, 0\right)\right] \\
&= \frac{e^{-r(T-t)}}{T} E_t^Q\left[\left(\boxed{x(T)} - KT\right) \mathbf{I}_{\{x(T) - KT \geq 0\}}\right] \\
&= \frac{e^{-r(T-t)}}{T} E_t^Q\left[\left(\boxed{H(t) + \sigma g(t) \tilde{X}} - KT\right) \mathbf{I}_{\left\{\tilde{X} \geq \frac{KT - H(t)}{\sigma g(t)}\right\}}\right] \\
&= \frac{e^{-r(T-t)}}{T} \left(E_t^Q\left[H(t) \mathbf{I}_{\{\cdot\}}\right] + \sigma g(t) E_t^Q\left[\tilde{X} \mathbf{I}_{\{\cdot\}}\right] - KT E_t^Q\left[\mathbf{I}_{\{\cdot\}}\right]\right) \\
&= \frac{e^{-r(T-t)}}{T} \left[H(t) Q\left(\tilde{X} > \frac{KT - H(t)}{\sigma g(t)}\right) + \sigma g(t) \int_{\frac{KT - H(t)}{\sigma g(t)}}^{\infty} x f(x)\, dx - KT Q\left(\tilde{X} > \frac{KT - H(t)}{\sigma g(t)}\right)\right]
\end{aligned}$$

$E(1_A) = P(A)$

$\Longleftrightarrow x(T) \geq KT$

(2)

$\sigma g(t)$

where $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the density of a standard normal random variable, and $Q(A)$ denotes the probability of event A under the Q measure.

- Define $d(t) = -\frac{KT - H(t)}{\sigma g(t)}$. Note also

- So we can write out the call option price:

cdf

$$C(t, S(t), x(t)) = \frac{e^{-r(T-t)}}{T}\left[H(t) N(d(t)) + \frac{\sigma g(t)}{\sqrt{2\pi}} e^{\frac{-[d(t)^2]}{2}} - KT N(d(t))\right]$$

$$(2) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha(t)}^{\infty} x e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2(t)}{2}}$$

Sample Problem 1: Black-Scholes

Bjork pg 70

- 2. Derive the pricing PDE
  - By **Ito's Lemma**,

$$d\left(e^{-rt}C(t,S(t),x(t))\right) = e^{-rt}(C_S dS(t) + C_x dx + C_t dt + \frac{1}{2}C_{SS}\langle dS, dS\rangle) + C(-re^{-rt}dt)$$

$$= e^{-rt}\underbrace{(rS(t)C_S + C_x \log S(t) + C_t + \frac{1}{2}C_{SS}S(t)^2\sigma^2 - Cr)}_{\text{Drift}}dt + e^{-rt}C_S S\sigma dZ^Q$$

  - Under **risk-neutral measure**, price discounted by risk-free rate must be a **martingale** → drift = 0. Therefore we get the PDE below:

$$rSC_S + C_x \log S + C_t + \frac{1}{2}C_{SS}S^2\sigma^2 - rC = 0,$$

with the terminal condition $C(T,S,x) = \max\left(\frac{x}{T} - K, 0\right)$