MFE230Q: Midterm Exam, April 27, 2016

NAME: **ID:**

Please underline your final answers. All answers must be justified.

1. *One-period model* (20 points): Consider the market represented by the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and following price system:

$$\bar{\mathbf{D}} = \begin{bmatrix} -100 & 50 & 100 & 150 \\ -100 & 100 & 100 & 100 \end{bmatrix}.$$

The first asset is a stock and the second is a risk-free bond.

   (a) Does this market permit arbitrage?

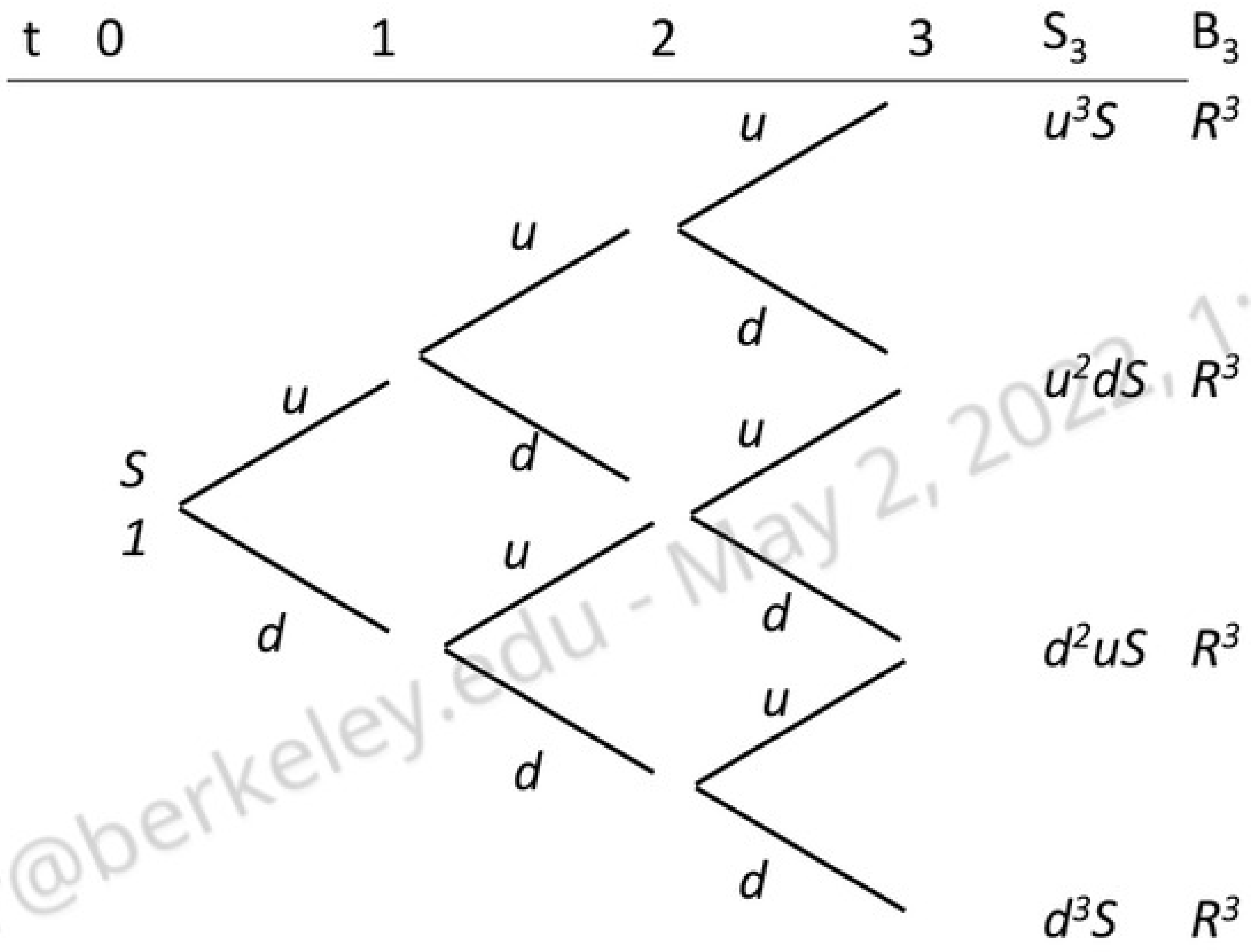
   (b) Is the market complete?

   (c) Consider an at the money call option on the stock (thus with strike price $K = 100$), with time-0 price C in this market. Derive upper and lower bounds on C. Clearly state the assumptions you are making.

2. *Binomial tree model* (30 points): Consider the binomial tree model, with a stock and a risk-free bond, as described in the figure below. The (dollar) prices of the stock and the bond at time 0 are $S_0 = S$ and $B_0 = 1$, respectively, and in each period the stock's price may move from S_t to $S_{t+1} = uS_t$ or $S_{t+1} = dS_t$, whereas the bond's return is always R, $B_{t+1} = RB_t$. The stock pays no dividend and the bond makes no coupon payments, and we also assume that

$$u > R > 1 > d.$$

The probability, p, for an up-move in each period is 50% (which is thus also the probability for a down-move). We initially, in questions (a)-(c), assume that the model lasts for $T = 3$ period, as in the figure.



(a) Define a state space and filtration that are consistent with the information diffusion of the model between $0 \le t \le 3$.

(b) What is the likelihood process $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$ in this market?

(c) Mr. Rando Nidokym "does not like the state u^2d," and he therefore decides to introduce a derivative in the market that pays a dollar in each state of the world at $t = 3$, *except* for in state u^2d, in which it pays 0. *What is the $t = 0$ price of such a derivative?*

In questions (d)-(f), we assume that the model lasts for many periods. Specifically, we assume that T is so large that $T = \infty$ is a reasonable approximation, which we use.

(d) Mr. Nidokym next decides that he wants to "celebrate up-moves" by introducing a derivative that pays a dollar at the first point in time that the market experiences an up-move. Thus, if the stock moves up in the first period (which it does for all $u\cdots$ paths), the derivative will pay 1 at $t = 1$ and then seize to exists, whereas if the stock moves down for four periods followed by an up-move in period five (which it does for all $ddddu\cdots$ paths), the derivative will pay 1 at $t = 5$ and then seize to exist, etc. *What is the $t = 0$ market price of the derivative?*

(e) Mr. Nidokym decides to also introduce another derivative that is similar to the derivative in (d) in that it makes a terminal payment after the first up-move in the market, but instead of making a payment of one dollar, the stock price is paid that point. Thus, with the $u\cdots$ path the derivative would pay uS at $t = 1$, and with the $ddddu\cdots$ price path it would pay d^4uS at $t = 5$. *What is the $t = 0$ market price of this derivative?*

(f) Describe a portfolio strategy that only uses the primitive assets (the stock and the bond) to replicate the payoff of the derivative in (e).

3. *Ito calculus* (20 points): Let W_t, $t \geq 0$, be a Wiener process.

   (a) Derive an expression for the stochastic integral $F(T) \stackrel{\text{def}}{=} \int_0^T W_t^3 t^4 dW_t$ on the form $F(T) = h(t, W_T) + \int_0^T g(t, W_t)dt$, where the right-hand-side of the expression uses the standard Riemann integral.

   (b) Use your result in (a), and the fact that $E_0[W_t^2] = t$, to derive an expression for $E_0[W_T^4]$, without using any further known properties of normal distributions.

4. *Continuous time model* (20 points): Consider an economy with one stock with GBM dynamics for prices, and one risk free asset with constant returns:

$$\begin{aligned} \frac{dS}{S} &= \hat{\mu}dt + \sigma dW, \\ \frac{dB}{B} &= rdt, \end{aligned}$$

   $\hat{\mu}$, r, and $\sigma > 0$ constants. The bond makes no dividend payments, but the stock makes instantaneous payments proportional to the stock price. Thus, $d\Theta_t = (\alpha S_t dt, 0)$, is the instantaneous dividend process of the two assets, using the notation from class, where $\alpha > 0$ is a constant.

   Consider the trading strategy

$$\mathbf{h}_t = (h_t^S, h_t^B)' = \left(2e^{(r+\sigma^2-2\alpha)(T-t)}S_t, -e^{(r+\sigma^2-2\alpha)(T-t)}\frac{S_t^2}{B_t}\right)', \qquad T > 0.$$

   (a) What is the value process, $V_t^{\mathbf{h}}$, for this trading strategy?

   (b) What is the cumulative dividend processes, $F_t^{\mathbf{h}}$, of this trading strategy?

   (c) What asset pricing conclusions can be drawn from your results in (a) and (b)? Remember to justify your answer.

5. *Kolmogorov equations* (15 points): Consider the PDE

$$\begin{aligned} F_t + a(b-x) + cF_{xx} &= 0, \\ F(T,x) &= x, \\ t &\geq 0, \\ x &\in \mathbb{R}. \end{aligned}$$

   (a) What is the solution $F(t,x)$, $0 \leq t \leq T$, $x \in \mathbb{R}$ to this PDE?

   (b) How does the solution in (a), via Feynman-Kac's theorem, relate to properties of the Ornstein-Uhlenbeck process:

$$dX = \kappa(\theta - X)dt + \sigma dW?$$