MFE230Q: In-Class Quiz # 1

1. Consider the following one-period, two-state economy:

$$\mathbf{s}^0 = (2,2)^T, \qquad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 2.5 & 2.5 \end{bmatrix}$$

   What is the price of a put option on asset 1, with a strike price of 2.5?

2. A forward contract on an underlying stock specifies an agreement where the buyer gets to purchase the stock at a pre-specified price, F, in the future. In contrast to a call option, the buyer *must* buy the stock. The forward price, F is set so that the value of the contract today is zero.

   Consider a stock that does not pay dividends, with current price S. The one-period risk-free rate is r, and the gross rate is then $R = 1 + r$. The market admits no arbitrage. Consider a contract that states:

   "*The buyer of this contract shall buy one unit of the stock at price F one period from now.*"

   Derive the forward price, F, that makes the current value of this contract zero.

3. In the HW, you will show that it is never optimal to exercise a call option early on a stock that pays no dividends. However, there are times when it is optimal to exercise a put early, whether or not the firm pays dividends. Intuitively, the difference is that, for a call, you *owe* K if you exercise, and due to the time value of money, it is better to pay later than pay now. In contrast, for a put, you are *owed* K, and due to the time value of money, it is better to receive now than receive later.

   To price American securities, we start off exactly as we do for European securities; namely, we start at the last date and work backwards, where for each possible event we replicate the CF's of the derivative we would like to price. What is new in pricing American options is that, after we determine the so-called **Continuation Value** of the derivative, we then check whether immediate exercise is more valuable than not exercising immediately. Thus, for example, suppose we were examining the value of a put option assuming that the down-state occurs, and we have determined the continuation value $P_{cont,\downarrow}$. The value of the American put will be

$$P_{\downarrow} = \max\left[P_{(cont,\downarrow)}, (K - S_{\downarrow})\right]. \tag{1}$$

   We then proceed working back through the tree as normal.

(a) With that in mind, assume a 2-period, 3-date model as below.

$$
\begin{array}{ccccc}
 & & & & u^2S(0),\ B(0)(1+R)^2,\ CF(\uparrow\uparrow) \\
 & & & \nearrow & \\
 & & uS(0),\ B(0)(1+R),\ V(\uparrow) & & \\
 & \nearrow & & \searrow & \\
\mathrm{S}(0),\ \ \mathrm{B}(0),\ \ \mathrm{V}(0) & & & & udS(0),\ B(0)(1+R)^2,\ CF(\uparrow\downarrow) \\
 & \searrow & & \nearrow & \\
 & & dS(0),\ B(0)(1+R),\ V(\downarrow) & & \\
 & & & \searrow & \\
 & & & & d^2S(0),\ B(0)(1+R)^2,,\ CF(\downarrow\downarrow)
\end{array}
$$

Assume $S(0) = 100$, $B(0) = 100$, $u = 1.1$, $d = .9$, $R_F = .02$. Determine the price of the **European** put option with strike $K = 101$ and $T = 2$ using the risk-neutral pricing formula:

$$C(0) = \left(\frac{1}{1+R_F}\right)^2 \mathrm{E}_0^Q\left[C(2)\right]. \tag{2}$$

(b) Determine the price of an **American** put option with strike $K = 101$ and maturity $T = 2$.

4. Consider a two-period market, $t \in T = \{0, 1, 2\}$, and the following probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}) = \frac{1}{3}$, and

$$
\begin{aligned}
\mathcal{F} &= \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}) \\
&= \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\},
\end{aligned}
$$

with the filtration

$$
\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\}, \\
\mathcal{F}_1 &= \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}, \\
\mathcal{F}_2 &= \mathcal{F}.
\end{aligned}
$$

There are two assets in the economy, and their dynamics are summarized by the price dividend pair $(\delta, \mathbf{s})$. Here $\delta = (\delta_1, \delta_2)^T$ and $\mathbf{s} = (s_1, s_2)^T$, and the functions $\delta_i : T \times \Omega \to \mathbb{R}$, $s_i : T \times \Omega \to \mathbb{R}$, $i = 1, 2$, are defined by

s_1	$t=0$	$t=1$	$t=2$
ω_1	1	1	1
ω_2	1	0.5	1
ω_3	1	0.5	0.5

δ_1	$t=0$	$t=1$	$t=2$
ω_1	0	1	0
ω_2	0	0	0
ω_3	0	0	0

s_2	$t=0$	$t=1$	$t=2$
ω_1	1	1.1	1.1
ω_2	1	1.1	1.21
ω_3	1	1.1	1.21

δ_2	$t=0$	$t=1$	$t=2$
ω_1	0	0	0
ω_2	0	0	0
ω_3	0	0	0

Here, the element on the ith row and jth column represents the value of the function at $t = j$ in state ω_i.

(a) Is the price dividend pair adapted?

(b) How would this economy be represented in the more familiar "tree" view?

(c) Derive the short-term risk-free process.

(d) Is the market complete?

(e) Is the market arbitrage free?

(f) Derive the stochastic discount factor (SDF) process.

(g) Derive the equivalent martingale measure (EMM).

(h) Consider the trading strategy $\mathbf{h} = (h_1, h_2)^T$, defined by

h_1	$t=0$	$t=1$	$t=2$
ω_1	2.2	3.3	0
ω_2	2.2	0	0
ω_3	2.2	0	0

h_2	$t=0$	$t=1$	$t=2$
ω_1	-1	0	0
ω_2	-1	0	0
ω_3	-1	0	0

- What is the dividend process, $\delta^{\mathbf{h}}$?
- Is this process consistent with the equivalent martingale measure?