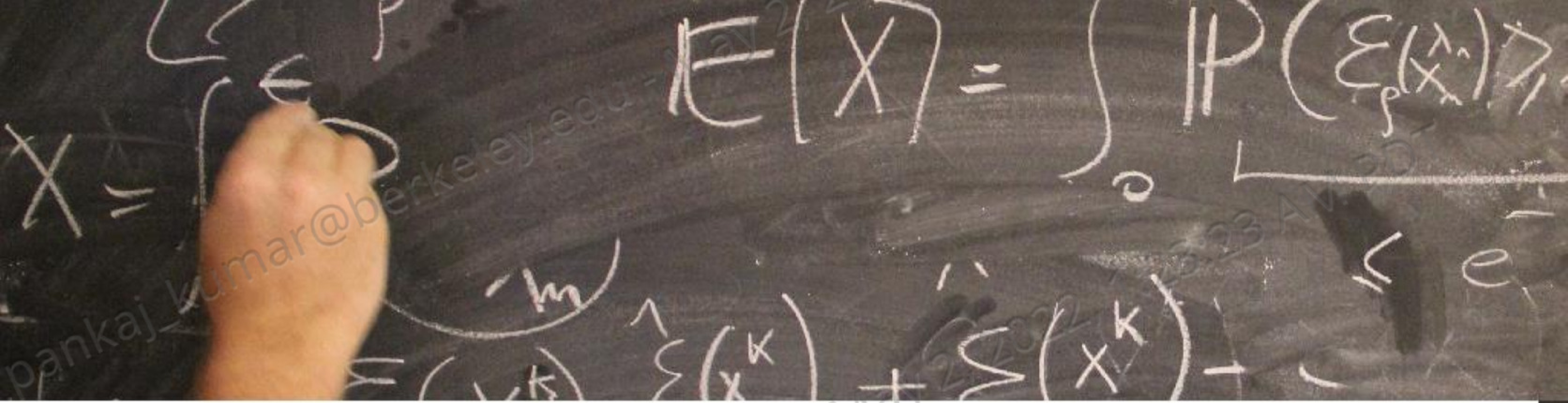

MFE 230Q [Spring 2021] Introduction to Stochastic Calculus GSI Session 4

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Review

- Ito Process and SDEs
- Feynman Kac Theorem
- Backward Kolmogorov Equation
- Forward Kolmogorov Equation

Ito Process

Definition 17 (**Itô) process (stochastic integral)**:

Assume that W_t *is a Brownian motion. An Itō process is a stochastic process (adapted),* $F : [0, \infty) \times \Omega$*, of the form*

$$F(T) = F(0) + \oint_0^t u(t,\omega)dt + \oint_0^T v(t,\omega)dW_t = \oint_0^T (u(t,\omega)dt + v(t,\omega)dW_t), \tag{4.34}$$

where u *and* w *are adapted processes, such that* $\mathbb{P}\left[\int_0^t v(s,\omega)^2 ds < \infty \text{ for all } t \geq 0\right] = 1$, *and* $\mathbb{P}\left[\int_0^t |u(s,\omega)| ds < \infty \text{ for all } t \geq 0\right] = 1$.

Ito Process

Next, we want to develop a calculus for Itō integrals. Specifically, we would like to write differential forms of general adapted processes:

$$F(T) - F(0) = \oint_0^T dF. \tag{4.35}$$

We warn that there will be classical interpretation of $\frac{dF}{dt}$ as a derivative. The correct interpretation of "dF" is through the stochastic process it defines via (4.34): for the pair (u, v), the function $F(T)$ is defined through (4.34), and the differential notation for this relation is

$$\text{“}dF = u\,dt + v\,dW.\text{”}$$

SDEs

- We can also express Ito process as stochastic differential:

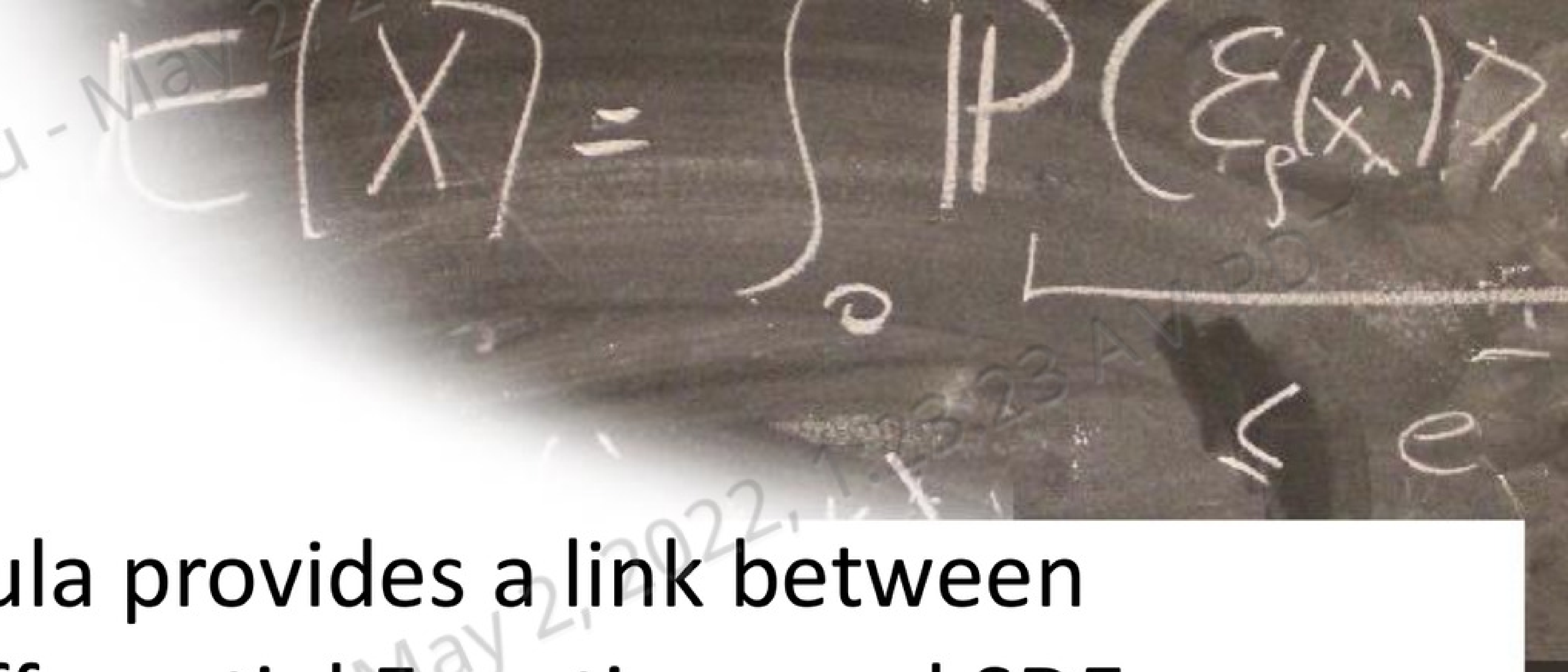
$$dX_t = \mu(t, \omega)dt + \sigma(t, \omega)dW_t$$

- Now assume that instead of some generic outcome ω, we have X_t feeding back into μ and σ functions:

SDE:
$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \Longleftrightarrow \text{PDE}$$

- This is now a Stochastic Differential Equation!
- We might be interested in finding a solution X_t

Feynman-Kac Theorem



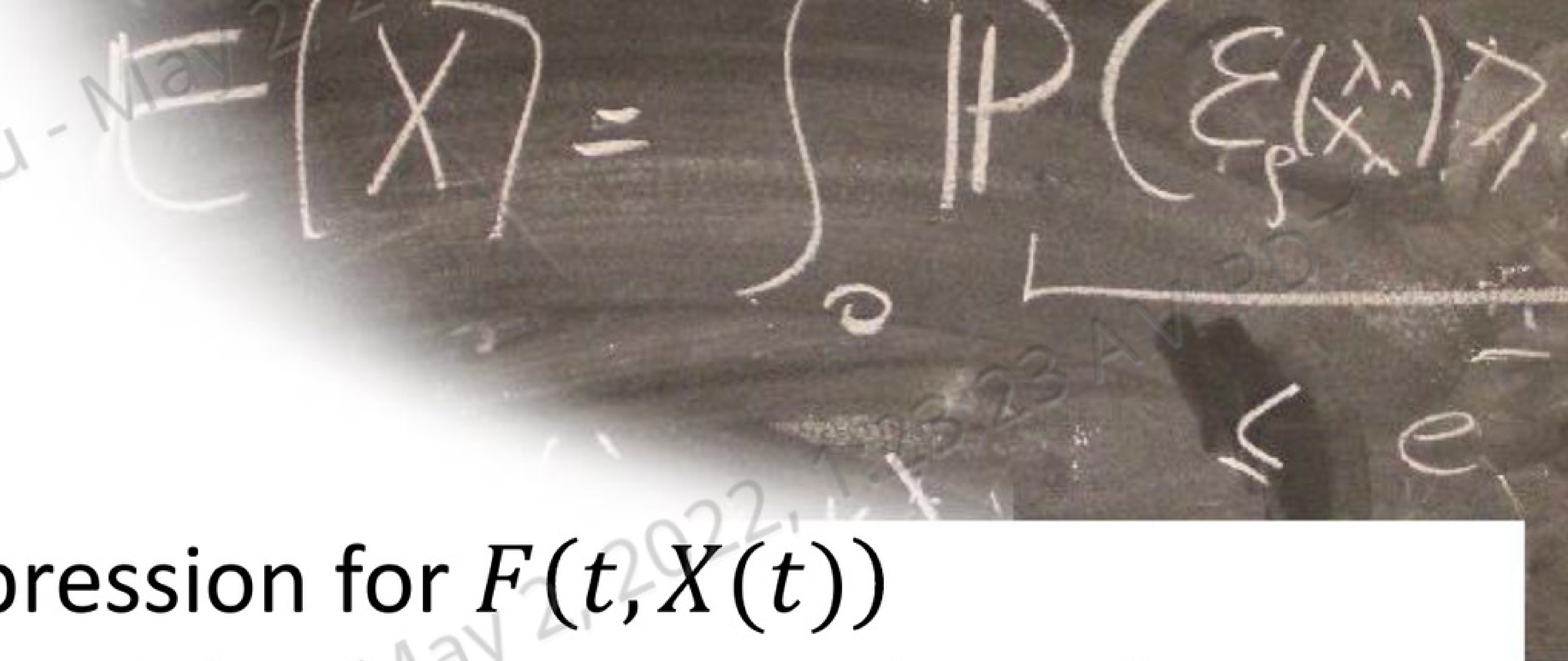
- Feynman-Kac formula provides a link between parabolic Partial Differential Equations and SDEs
- Consider this *parabolic* one-dimensional PDE with boundary condition (aka **boundary value problem**):

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} &= 0 \\ F(T,x) &= \Phi(x) \end{aligned}$$

Fix t and x, and define stochastic process X on $[t,T]$ via SDE:

$$\begin{aligned} dX_s &= \mu(s,X_s)dt + \sigma(s,X_s)dW_s \\ X_t &= x \end{aligned}$$

Feynman-Kac Theorem



- We want to find expression for $F(t, X(t))$
- Apply **Ito's Lemma** on $F(t, x)$ assuming it's a solution:

$$
\begin{aligned}
dF &= F_t dt + F_X dX + \tfrac{1}{2} F_{XX} (dX)^2 \\
&= F_t dt + F_X \mu(t,X) dt + F_X \sigma(t,X) dW + \tfrac{1}{2} F_{XX} \sigma(t,X)^2 dt \\
&= \underbrace{\left[F_t + \mu(t,X) F_X + \tfrac{1}{2}\sigma(t,X)^2 F_{XX}\right]}_{=0} dt + F_X \sigma(t,X) dW \\
&= F_X \sigma(t,X)\ dW
\end{aligned}
$$

Feynman-Kac Theorem

- Now we have

$$dF = F_X \sigma(t, X) dW_t$$

$$F\big(T, X(T)\big) - F\big(t, X(t)\big) = \int_t^T F_X\, \sigma\big(s, X(s)\big) dW_s$$

- Substitute with the boundary condition and take conditional expectation:

$$\Phi\big(X(T)\big) - F\big(t, X(t)\big) = \int_t^T F_X\, \sigma\big(s, X(s)\big) dW_s$$

$$\mathrm{E}_{t,x}\Big[\Phi\big(X(T)\big) - F\big(t, X(t)\big)\Big] = \mathrm{E}_{t,x}\left[\int_t^T F_X\, \sigma\big(s, X(s)\big) dW_s\right]$$

RHS is zero due to martingale property, so we get:

$$\boxed{F\big(t, X(t)\big) = \mathrm{E}_{t,x}\Big[\Phi\big(X(T)\big)\Big]}$$

Feynman Kac Theorem

- So given an Ito process

$$\begin{aligned} dX_s &= \mu(s, X_s)dt + \sigma(s, X_s)dW_s, \\ X_t &= x. \end{aligned}$$

and “payoff” function $F(t,x)$, we have the following:

$$\frac{\partial F}{\partial t} + \mu(t,X)\frac{\partial F}{\partial X} + \frac{1}{2}\sigma(t,X)^2\frac{\partial^2 F}{\partial X^2} = 0$$

$$\Updownarrow$$

Proof via Ito’s Lemma

$$F(t,x) = E_{t,x}[F(T,X(T))]$$

$F(t, X(t))$ is a martingale

Feynman-Kac Theorem

- Generalization:

The general linear parabolic 1-D PDE

$$F_t + \mu(t,x)F_x + \frac{1}{2}\sigma^2(t,x)F_{xx} - r(t,x)F + g(t,x) = 0$$

$$F(T,x) = \Phi(x)$$

Under some regularity conditions,

$$F(t,x) = E_t[\underbrace{e^{-\int_t^T r(s,X_s)ds}}_{\text{Discount factor}}\underbrace{\Phi(X_T)}_{\text{Payoff of the contract}} + \int_t^T \underbrace{g(s,X_s)}_{\text{Dividend/ Coupon}}\underbrace{e^{-\int_t^s r(u,X_u)du}}_{\text{Discount factor}}ds|X_t = x]$$

Where X_t is the solution of the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \qquad X_t = x$$

① FK #1 $(r=0, g=0)$

$F(t,x)$, $x \in \mathbb{R}$ soln PDE

PDE: $$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} = 0$$

BC: $$F(T,x) = \Phi(x)$$

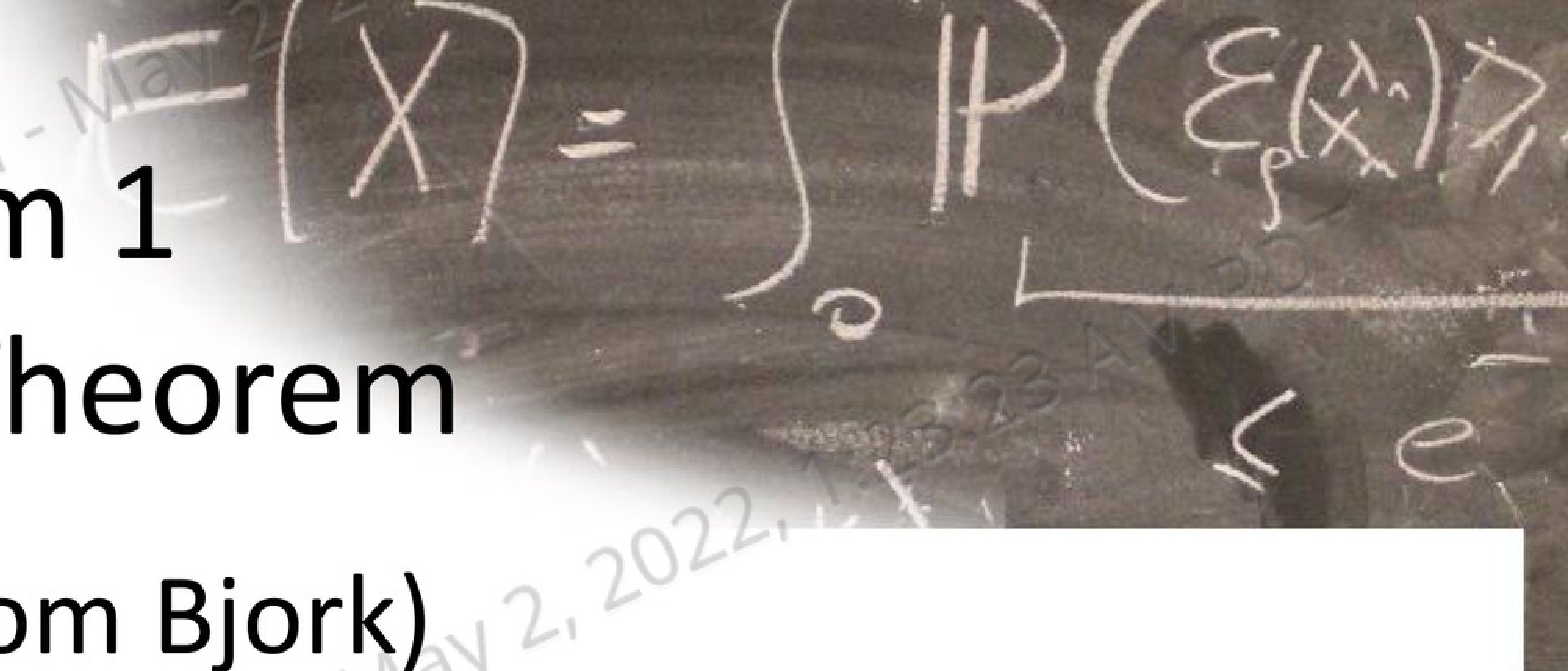
$$\Updownarrow$$

SDE: $$F(t,x) = E\left[\Phi(X_T) \mid X_t = x\right]$$

$$\Rightarrow \boxed{dX_s = \mu(s, X_s)\,ds + \sigma(s, X_s)\,dW_s}$$

$$X_t = x$$

Sample Problem 1
Feynman-Kac Theorem



- Example 1 (5.7 from Bjork)

Solve the PDE

$$F_t + \frac{1}{2}\sigma^2 F_{xx} = 0$$

BC: $$\boxed{F(T, x) = x^2.}$$

where σ is a constant and $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

Ex #1 SDE: $\underline{dX_s = 0\, ds + \sigma dW_s}$

$X_t = x$

$F(t,x) = E[\Phi(X_T) \mid X_t = x]$

$\overset{?}{\Longrightarrow} \quad = E(X_T^2 \mid X_t = x)$

$X_T = (x + \sigma(W_T - W_t))^2$

$\int_t^T dX_s = \int_t^T \sigma\, dW_s$

$\Rightarrow X_T - X_t = \sigma(W_T - W_t)$

$\Longrightarrow X_T - x = \sigma(W_T - W_t)$

$$W_T - W_t \sim N(0, T-t)$$

$$X_T \sim N(x, \sigma^2(T-t))$$

$$V(X_T) = E(X_T^2) - E(X_T)^2$$

$$\sigma^2(T-t) = E(X_T^2) - x^2 \Rightarrow E(X_T^2) = \sigma^2(T-t) + x^2$$

$$F(t, x) = \sigma^2(T-t) + x^2$$

$$BC: F(T, x) = \sigma^2 \cdot 0 + x^2 = x^2$$

FK #2. (r = const. $q = 0$)

$\cancel{r}(t,x)$

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - r\,F(t,x) = 0$$

PDE $\qquad F(T,x) = \Phi(x)$

$$\Updownarrow$$

SDE $\quad X$ follows dynamics $\quad$ FK #1

$$F(t,x) = e^{-r(T-t)}\,E\left[\Phi(X_T) \mid X_t = x\right]$$

FK #3 : (r function, g function)

PDE:

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} - r(t,x)F(t,x) + g(t,x) = 0$$

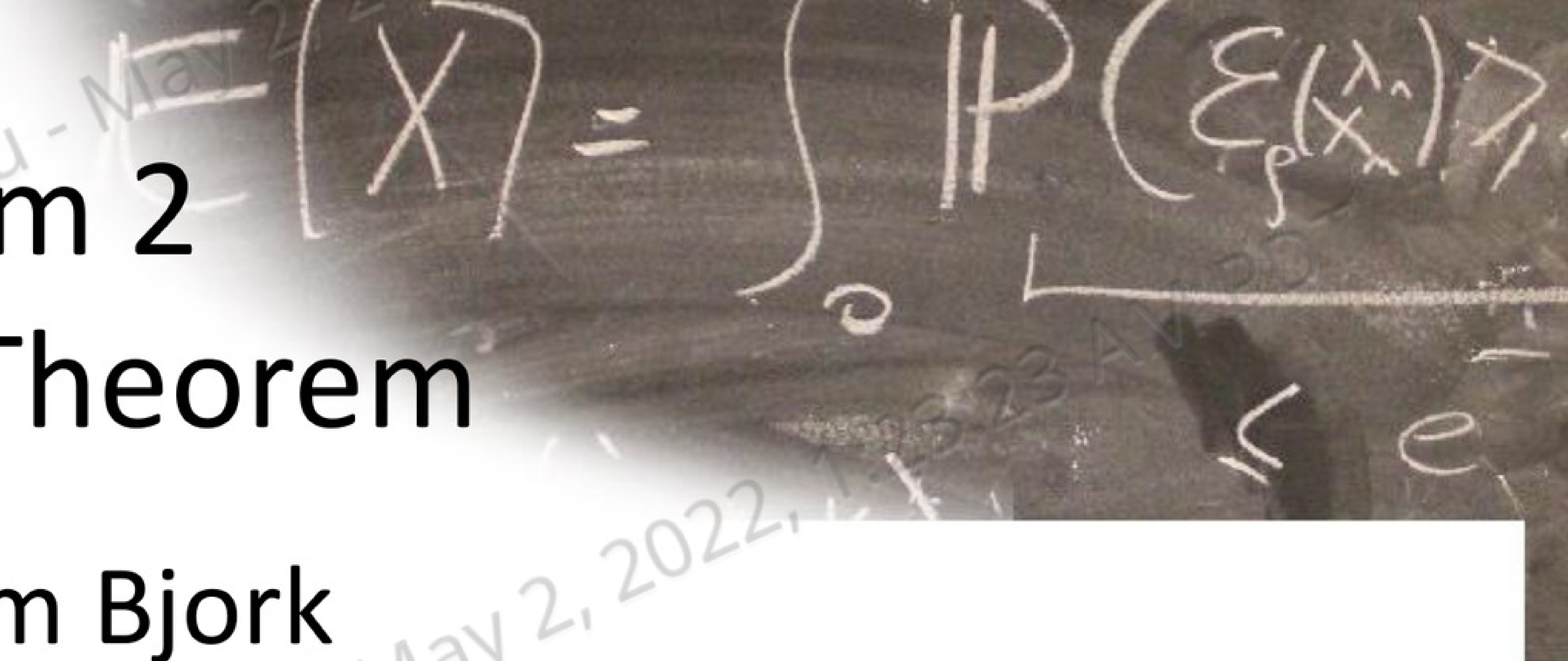
$$F(T,x) = \Phi(x)$$

$$\Updownarrow$$

SDE: X satisfies same dynamics FK #1

$$F(t,x) = E_t\left[e^{-\int_t^T r(s,X_s)ds}\,\Phi(X_T) + \int_t^T g(s,X_s)\,e^{-\int_t^s r(u,X_u)du}\,ds \,\middle|\, X_t = x\right]$$

Sample Problem 2
Feynman-Kac Theorem



- Exercise 5.11 from Bjork

Exercise 5.11 Use the result of the previous exercise in order to solve

$$\frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + x = 0,$$
$$F(T,x) = \ln(x^2).$$

$\mu(t,x) = 0$

$\sigma(t,x) = x$

$g(t,x) = x$

$r(t,x) = 0$

FK#3

$$dX_s = X_s\,dW_s \Big\}\ \text{GBM}\ \mu=0,\ \sigma=1 \qquad E(X_t)=x_0\,e^{\mu t}$$

$$X_t = x \qquad\qquad E(X_t)=x_0$$

$$F(t,x) = E_t\left[\Phi(X_T) + \int_t^T g(s,X_s)\,ds\right]$$

$$= \underbrace{E_t[\ln(X_T^2)]}_{(**)} + \underbrace{E_t\left[\int_t^T X_s\,ds\right]}_{(*)}$$

$$(*) \quad \int_t^T E_t(X_s)\,ds \qquad X_s = X_t\,e^{(\mu-\frac{1}{2}\sigma^2)(s-t)+\sigma^2(W_s-W_t)}$$

$$= \int_t^T x\,ds \qquad \boxed{dX_t = \mu X_t\,dt + \sigma X_t\,dW_t}$$

$$= x(T-t) \checkmark \qquad \mu=0 \qquad \sigma=1$$

$(**) \Rightarrow E_t\left(2\ln(X_T)\right) = 2E_t\left(\ln(X_T)\right)$

$X_T = X_t\, e^{-\frac{1}{2}(T-t) + (W_T - W_t)}$

$= 2E_t\left(\ln(X_t)\right) + 2E_t\left(-\frac{1}{2}(T-t) + \underbrace{(W_T - W_t)}\right)$

$= 2\ln(x) - (T-t) + 0$

$(*) + (**) = 2\ln(x) + (x-1)(T-t)$ ✓

BC: $F(T, x) = 2\ln(x) = \ln(x^2)$ ✓

Ornstein - Uhlenbeck SDE

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t$$

$$X_0 = x$$

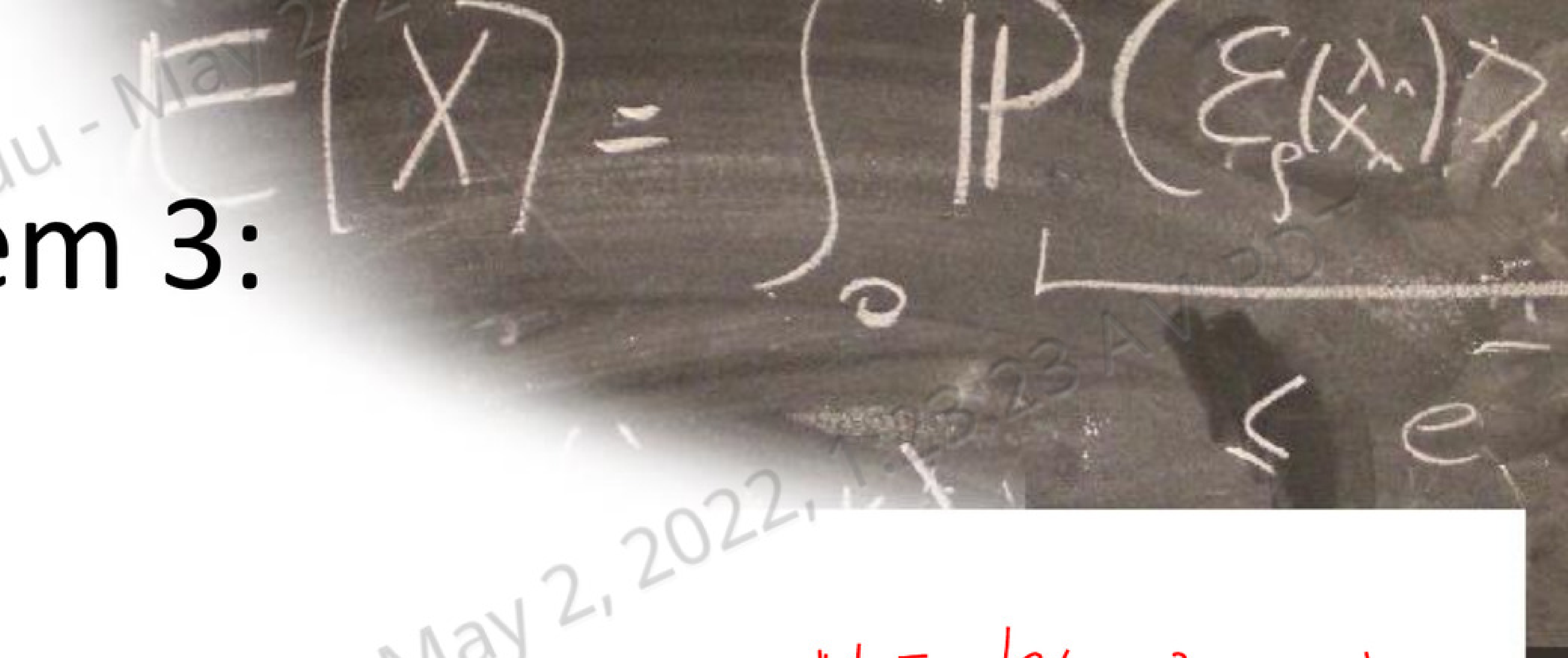
$$X_t = e^{-\kappa t} X_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(s-t)}\,dW_s$$

$$E(X_t) = e^{-\kappa t} X_0 + \theta\left(1 - e^{-\kappa t}\right)$$

$$V(X_t) = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

Sample Problem 3:

FK #3



Feynman Kac

Consider the PDE

$$
\begin{aligned}
0 &= F_t + \kappa(\theta - x)F_x + \frac{\sigma^2}{2}F_{xx} - rF + \beta x, \\
0 &= F(T, x), \\
t &\le T, \\
x &\in \mathbb{R}.
\end{aligned}
$$

$\mu = \kappa(\theta - x)$

$\sigma = \sigma$

$r = r$

$g = \beta x$

Here, θ, κ, σ, β, and r are strictly positive constants. Use what you know about Feynman Kac's theorem and the Ornstein-Uhlenbeck process to find the solution, $F(t, x)$.

$$F(t,x) = \beta E_t\left[\int_t^T X_s e^{-\int_t^s r\,du}\,ds\right]$$

$$dX_s = \kappa(\theta - X_s)\,dt + \sigma\,dW_s$$

$$X_t = x$$

$$\beta \int_t^T E_t\left(X_s e^{-r(s-t)}\right) ds$$

$$X_s = x e^{-\kappa(s-t)} + \theta\left(1 - e^{-\kappa(s-t)}\right) + \boxed{\sigma e^{-\kappa s}\int_t^s e^{\kappa u}\,dW_u}$$

$$E\left(\cdots \int \cdots dW_u\right) = 0$$

$$\beta x e^{(r+\kappa)t} \underline{\int_t^T e^{-(r+\kappa)s} ds} + \beta \theta e^{rt} \underline{\int_t^T e^{-rs} ds}$$

$$- \beta \theta e^{(r+\kappa)t} \underline{\int_t^T e^{-(r+\kappa)s} ds}$$

$$F(t,x) = \beta \left[\frac{x-\theta}{r+\kappa} \left(1 - e^{(r+\kappa)(t-T)}\right) + \frac{\theta}{r} \left(1 - e^{r(t-T)}\right) \right]$$

BC: $F(T,x) = 0$