MFE 230Q – Introduction to Stochastic Calculus
GSI Session 2 Solutions

2 GSI Session 2: Sample Problem 2 - A two-period, three-date economy *American* put option

Current stock price of MFE2016Class Inc. is \$40, over the next two 3-month periods it's expected to go up by 10% or down by 10%. The risk-free rate is 12% per year with continuous compounding.

(a) Compute the price of a 6-month **European put option** with a strike price of \$42?

(b) Compute the price of a 6-month ***American*** **put option** with a strike price of \$42?

2.1 Solutions:

$$
\begin{array}{ccccc}
T=0 & & T=3\ months & & T=6\ months \\
 & & & & \$48.40 \\
 & & & \nearrow & \\
 & & \$44 & & \\
 & \nearrow & & \searrow & \\
\$40 & & & & \$39.60 \\
 & \searrow & & \nearrow & \\
 & & \$36 & & \\
 & & & \searrow & \\
 & & & & \$32.40
\end{array}
$$

(a) **Compute the price of a 6-month European put option with a strike price of \$42**

In this part I will essentially use the risk neutral probabilities to price the European put. Firstly, compute the risk neutral for each one period binomial tree probabilities using:

$$
\begin{array}{ccc}
T=0 & & T=3\ months \\
 & & \$44 \\
 & \nearrow & \\
\$40 & & \\
 & \searrow & \\
 & & \$36
\end{array}
$$

$$
\begin{aligned}
q_{\uparrow}u+(1-q_{\uparrow})d &= R_f=e^{0.12\frac{3}{12}}=1.0305 \\
q_{\uparrow} &= 0.6523
\end{aligned}
$$

Then we compute the risk neutral probabilities in each state at $T=6\ months$:

$$
\begin{array}{ccccc}
T=0 & & T=3\ months & & T=6\ months \\
 & & & & q_{\uparrow}^2=0.4255 \\
 & & & \nearrow & \\
 & & q_{\uparrow} & & \\
 & \nearrow & & \searrow & \\
1 & & & & 2q_{\uparrow}(1-q_{\uparrow})=0.4536 \\
 & \searrow & & \nearrow & \\
 & & (1-q_{\uparrow}) & & \\
 & & & \searrow & \\
 & & & & (1-q_{\uparrow})^2=0.1209
\end{array}
$$

Finally we can price the **European put option** using the risk neutral probabilities:

$$
\begin{aligned}
P_{Euro}(0) &= \frac{1}{R_f^2}\mathbb{E}^{\mathbb{Q}}\left[P_{Euro}(2)\right] \\
&= \frac{1}{R_f^2}\left(0.4255\times 0+0.4536\times \$2.4+0.1209\times \$9.6\right) \\
&= \$2.188
\end{aligned}
$$

(b) **Compute the price of a 6-month European put option with a strike price of \$42**

Below the payoffs of the American put at $T = 6$ months, and the no-arbitrage prices using the risk neutral probabilities.

$$
\begin{array}{ccccc}
T=0 & & T=3\ months & & T=6\ months \\
 & & & & \$0 \\
 & & & \nearrow & \\
 & & \$0.81 & & \\
 & \nearrow & & \searrow & \\
P_{Amer}(0) & & & & \$2.4 \\
 & \searrow & & \nearrow & \\
 & & \$4.759 & & \\
 & & & \searrow & \\
 & & & & \$9.6
\end{array}
$$

Note that at $T = 3$ months, in the down state, it is optimal to exercise the American put since $\$42 - \$36 = \$6 > \$4.759$. We should then recompute the value in the down state and consequently recompute the price today.

$$
\begin{array}{ccccc}
T=0 & & T=3\ months & & T=6\ months \\
 & & & & \$0 \\
 & & & \nearrow & \\
 & & \$0.81 & & \\
 & \nearrow & & \searrow & \\
P_{Amer}(0) & & & & \$2.4 \\
 & \searrow & & \nearrow & \\
 & & \$6 & & \\
 & & & \searrow & \\
 & & & & \$9.6
\end{array}
$$

Price today is then

$$
\begin{aligned}
P_{Amer}(0) &= \frac{1}{R_f}\mathbb{E}^{\mathbb{Q}}\left[P_{Euro}(1)\right] \\
&= (0.6523 \times \$0.81 + (1 - 0.6523) \times \$6) \\
&= \$2.537
\end{aligned}
$$

3 EXTRA: Change of measure (Radon-Nikodym Derivative)

Consider an M-state, single-period economy. The possible states of the world are $\Omega = \{\omega_j : j = 1, \ldots, M\}$, and these have associated *physical* probabilities given by some measure $\mathbb{P}$. Assume that there is no arbitrage in the model so that there exist risk-neutral probabilities given by $\mathbb{Q}(\omega_j)$.

We call the ratio of probabilities $L(\omega_j) := \frac{\mathbb{Q}(\omega_j)}{\mathbb{P}(\omega_j)}$ the *Radon-Nikodym derivative* of the measure $\mathbb{Q}$ with respect to the measure $\mathbb{P}$[1]. L describes the 'rate of change' of $\mathbb{Q}$ as $\mathbb{P}$ changes. Notice that we can treat $L(\cdot)$ as a random variable $\widetilde{L}$ since its value changes depending on which state ω_j is realized.

For our purposes, the key fact about the Radon-Nikodym derivative is that if $\widetilde{X}$ is any random variable defined on Ω, then the following identity holds

$$\mathbb{E}^{\mathbb{Q}}[\widetilde{X}] = \mathbb{E}^{\mathbb{P}}[\widetilde{L}\widetilde{X}] \tag{10}$$

Why is eq. (10) important? It turns out that it in some situations is is easier to evaluate the right-hand side of (10) directly, rather than use brute-force on the $\mathbb{Q}$-expectation.[2]

Let's do an example to make things concrete. Consider the one-period binomial model:

$$\begin{array}{ccc} & & B(T) = 105,\ S_{\uparrow}(T) = 110 \\ & \nearrow & \\ B(0) = 100,\ S(0) = 100 & & \\ & \searrow & \\ & & B(T) = 105,\ S_{\downarrow}(T) = 90 \end{array}$$

After some calculations, we obtain the risk-neutral probabilities $\mathbb{Q}(\omega_u) = 3/4, \mathbb{Q}(\omega_d) = 1/4$. Assume that the physical probabilities are given by $\mathbb{P}(\omega_u) = \mathbb{P}(\omega_d) = 1/2$.

Our goal is to price a plain vanilla call option with $K = 100$ using the risk-neutral valuation formula

$$C_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[\widetilde{C}(T)].$$

[1]Since we're in a finite-state world, L is simply a ratio of two numbers and doesn't look like a derivative in the typical calculus sense. However, once we start working in a continuous-state setting, the terminology should become clear. (Actually, even in the finite-state case, it is a derivative in an appropriate generalized sense, but we won't get into this…)

[2]Typically, either expectation in (10) is easy to compute in the a finite-state model. However, introducing L in this setting helps us develop our intuition for the continuous-time models we will study in the future.

By eq. (10) we can compute the expectation by first computing L and then taking expectations under the measure $\mathbb{P}$. The values for L in each state are

$$L(\omega_u) = \frac{\mathbb{Q}(\omega_u)}{\mathbb{P}(\omega_u)} = \frac{3/4}{1/2} = 3/2$$
$$L(\omega_d) = \frac{\mathbb{Q}(\omega_d)}{\mathbb{P}(\omega_d)} = \frac{1/4}{1/2} = 1/2$$

Hence, the call value is

$$\begin{aligned}
C(0) &= \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[\widetilde{C}(T)] \\
&= \frac{1}{1.05}\mathbb{E}^{\mathbb{P}}[\widetilde{L}\widetilde{C}(T)] \\
&= \frac{1}{1.05}\left[\mathbb{P}(\omega_u)L(\omega_u)C(T)(\omega_u) + \mathbb{P}(\omega_d)L(\omega_d)C(T)(\omega_d)\right] \\
&= \frac{1}{1.05}\left[\frac{1}{2}\times\frac{3}{2}\times 10 + \frac{1}{2}\times\frac{1}{2}\times 0\right] \\
&= \frac{1}{1.05}\times\frac{15}{2} \\
&\approx 7.14.
\end{aligned}$$

You can easily check that this is the same as the value we would obtain if we computed the risk-neutral expectation directly.