MFE230Q: Final Exam, May 23, 2011

NAME: **ID:**

Please motivate your answers. Please underline your final answers. A table with values of the cumulative normal distribution is provided at the end of this exam.

Question	Score
1	
2	
3	
4	
5	
6	
Total:	

1. *One-period model (20 points)*: Consider the following one-period market, with the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, as defined in class:

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{s}^0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.75 \end{bmatrix}.$$

   (a) Is this market arbitrage free?

   (b) Is this market complete?

   (c) What is the risk-free rate ($R = 1 + r$) in this market?

   (d) What is the value of a derivative contract that pays out 1 in state ω_1, 2 in state ω_2 and 3 in state ω_3?

2. *Itô calculus, SDEs and Kolmogorov's equations (25 points):*

   (a) Let $y_t = (t + W_t)^3$. Calculate dy_t.

   (b)* Define $y_t = \int_0^t (W_s + s)^2 dW_s$, where W_s is a Wiener process under the standard filtration, $\mathcal{F}_t$, $t \geq 0$. Calculate $E[y_t|\mathcal{F}_0]$ and $Var[y_t|\mathcal{F}_0]$.

   (c) Solve the partial differential equation:

$$\begin{aligned} F_t + \frac{\sigma^2}{2}F_{xx} + \frac{\delta^2}{2}F_{yy} - rF &= 0, \\ F(T, x, y) &= xy, \\ x &\in \mathbb{R}, \\ y &\in \mathbb{R}, \\ t &\leq T, \end{aligned}$$

   where σ, δ and r are strictly positive constants.

   (d) Consider the Ornstein-Uhlenbeck process

$$\begin{aligned} dX_t &= \kappa(\theta - X_t)dt + \sigma dW_t, \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned}$$

   Derive an expression for the (stationary) long-term probability density function of X (i.e., $p(x, \infty)$, where $P(x, T) = \mathbb{P}(X_T \leq x)$, and $p(x, T) = \frac{dP}{dx}(x, T)$).

3. *Girsanov (15 points)*: Consider the following economy in which there are two risky assets and one risk-free asset:

$$\begin{aligned} \frac{dB}{B} &= rdt, \qquad r = 0.1, \\ \frac{dS_1}{S_1} &= 0.2dt + 0.2dW_1 + 0.7dW_2, \\ \frac{dS_2}{S_2} &= 0.3dt + 0.5dW_1 + 0.3dW_2. \end{aligned}$$

   Here, W_1 and W_2 are independent Wiener processes. Calculate the time 0 price, P, of a contingent claim that pays out $-W_2(t)$ at $t = 3$.

4. *Barrier option (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$\begin{aligned}\frac{dB}{B} &= rdt,\\ \frac{dS}{S} &= rdt + dW^Q.\end{aligned}$$

(a) What is the price of a 3-year up-and-in call option with strike price $K = 100$ and barrier $L = 150$, given the parameters $r = 0.1$, $\sigma = 0.3$, and current stock price $S = 110$?

(b) What is the price of a 3-year up-and-out call option with the same parameters as in (a)?

5. *Term structure (20 points)* Assume that the short rate follows the asset pricing dynamics specified by the Vasicek model:

$$dr = (b - ar)dt + \sigma dW^Q, \qquad a > 0.$$

(a) Does the model belong to the class of affine term structure models? Why/why not?

(b) Derive the formula for the time-t price of a (zero coupon) T-bond, $p(t, T|r)$.

6. *Perpetual dividend paying contract (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$\begin{aligned}\frac{dB}{B} &= rdt,\\ \frac{dS}{S} &= rdt + \sigma dW^Q.\end{aligned}$$

(a) What is the value of a contract that makes constant continuous dividend payment of δ per unit time in perpetuity, i.e., the instantaneous dividend payment is δdt.

(b)* Now, consider a perpetual contract that pays δdt, but only at points in time, t, when $S_t > K$, for some constant $K > 0$. Derive a formula for the value of such a contract as a function of the stock price, S (and other parameters in the economy). Hint: You may use the fact (which is easy to show) that the value is a continuously differentiable function of S for all $S > 0$, even though the dividend payments change discontinuously at $S = K$.

x	N(x)	x	N(x)
0	0.5000	1.45	0.926
0.05	0.5199	1.5	0.933
0.1	0.5398	1.55	0.939
0.15	0.5596	1.6	0.945
0.2	0.5793	1.65	0.951
0.25	0.5987	1.7	0.955
0.3	0.6179	1.75	0.960
0.35	0.6368	1.8	0.964
0.4	0.6554	1.85	0.968
0.45	0.6736	1.9	0.971
0.5	0.6915	1.95	0.974
0.55	0.7088	2	0.977
0.6	0.7257	2.05	0.980
0.65	0.7422	2.1	0.982
0.7	0.7580	2.15	0.984
0.75	0.7734	2.2	0.986
0.8	0.7881	2.25	0.988
0.85	0.8023	2.3	0.989
0.9	0.8159	2.35	0.991
0.95	0.8289	2.4	0.992
1	0.8413	2.45	0.993
1.05	0.8531	2.5	0.994
1.1	0.8643	2.55	0.995
1.15	0.8749	2.6	0.995
1.2	0.8849	2.65	0.996
1.25	0.8944	2.7	0.997
1.3	0.9032	2.75	0.997
1.35	0.9115	2.8	0.997
1.4	0.9192	2.85	0.998

Table 1: Table of the Standard Normal Cumulative Distribution Function $N(z)$. For negative values, the symmetry $N(-x) = 1 - N(x)$ can be used.