MFE230Q: Final Exam, May 21, 2012

NAME: **ID:**

Please motivate your answers. Please underline your final answers.

Question	Score
1	
2	
3	
4	
Total:	

1. *Discrete model (25 points)*: Consider the following three-date market, with three assets, B, S, P, which can be interpreted as a risk-free bond, a stock, and a put option on the stock, respectively. The assets are traded at $t = 0, 1, 2$. There are five states, $\omega_1, \dots, \omega_5$. At time $t = 1$, at which points the assets are traded, it is determined whether the economy is in a boom (u) or in a recession (d). Given that the economy is in a boom, the stock price can move up (in state ω_1) or down (in state ω_2). In either case the option expires out of the money. If the economy is in a recession, on the other hand, there are three possible outcomes for the stock. It can go up, moderately down, or significantly down to the point that the underlying firm defaults and becomes worthless. These are states ω_3, ω_4, and ω_5, respectively. In states ω_4 and ω_5, the option expires in the money. The probabilities for the different states are $\mathbb{P}(\omega_1) = 0.1$, $\mathbb{P}(\omega_2) = 0.2$, $\mathbb{P}(\omega_3) = 0.5$, $\mathbb{P}(\omega_4) = 0.1$, and $\mathbb{P}(\omega_5) = 0.1$. The prices of the three assets in different states and times are summarized below.

B(0)	*S(0)*	*P(0)*		*B(1)*	*S(1)*	*P(1)*		*B(2)*	*S(2)*	*P(2)*	*State*
							↗	144	240	0	ω_1
			u	120	150	0					
			↗				↘	144	120	0	ω_2
100	100	20									
			↘				↗	120	200	0	ω_3
			d	120	100	40	→	120	100	20	ω_4
							↘	120	0	120	ω_5

(a) Define the filtration for this market.

(b) Is the market complete?

(c) An insurance company is considering introducing a credit default swap on the firm's default risk. Specifically, they would sell an insurance contract that pays a hundred dollars at $t = 2$ in the case that the (stock-S) firm defaults (in state ω_5), and makes no payment otherwise. What is the $t = 0$ market price of such a credit default swap?

(d) Assume that the insurance company wishes to hedge the risk of the credit default swap in the market. How could it do this with a dynamic portfolio trading strategy?

Solution:

(a)

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \Omega\}, \\ \mathcal{F}_2 &= 2^{\Omega}. \end{aligned}$$

(b) Remember from class that the (finite horizon) multi-period market is complete if is complete period-by-period and state-by-state. It follows immediately that the market is complete between $t = 0$ and $t = 1$, and that it is complete at $t = 1$ after an up move (in state $\{\omega_1, \omega_2\}$), since two states are spanned by one risky and one risk-free asset. What remains is to show that the market is complete at $t = 1$ after a down move. Since there are three states and three assets, a necessary and sufficient condition is that the payoff matrix

$$\mathbf{D} = \begin{bmatrix} 120 & 120 & 120 \\ 200 & 100 & 0 \\ 0 & 20 & 120 \end{bmatrix}$$

nonsingular. An easy way to verify nonsingularity is to check that the determinant is nonzero, $|\mathbf{D}| \neq 0$. Specifically, for the determinant of this 3×3 matrix, we have:

$$\begin{aligned} |\mathbf{D}| = \left| \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \right| &= d_{31} \left| \begin{bmatrix} d_{12} & d_{13} \\ d_{22} & d_{23} \end{bmatrix} \right| - d_{32} \left| \begin{bmatrix} d_{11} & d_{13} \\ d_{21} & d_{23} \end{bmatrix} \right| + d_{33} \left| \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \right| \\ &= 0 - 20(120 \times 0 - 200 \times 120) + 120(120 \times 100 - 200 \times 120) \\ &= -960,000 \neq 0. \end{aligned}$$

So, the market is indeed complete.

(c) Compute the state prices between period 0 and 1

$$\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix},$$

and the state prices between period 1 and 2 in the down state in period 1

$$\begin{pmatrix} \psi_{\omega_3} \\ \psi_{\omega_4} \\ \psi_{\omega_5} \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix},$$

Then the price of the CDS is equal to $\psi_d \times \psi_{\omega_5} \times 100 = 1/2 \times 1/4 \times 100 = 12.5$.

Alternatively, you could use to results from (d) to immediately determine the price of the portfolio as $h_0^P \times P(0) = 5/8 \times 20 = 12.5$.

(d) After a down move, the portfolio $(h_1^{d,B}, h_1^{d,S}, h_1^{d,P})'$ should be chosen such that the payoffs in states ω_3, ω_4 and ω_5 should be 0, 0, and 100, respectively. This leads to the equations

$$\begin{aligned} 120h_1^{d,B} + 200h_1^{d,S} + 0h_1^{d,P} &= 0 \\ 120h_1^{d,B} + 100h_1^{d,S} + 20h_1^{d,P} &= 0 \\ 120h_1^{d,B} + 0h_1^{d,S} + 120h_1^{d,P} &= 100 \end{aligned}$$

From the third equation, $1.2h_1^{d,P} = 1 - 1.2h_1^{d,B}$. Similarly, from the first equation, $2h_1^{d,S} = -1.2h_1^{d,B}$. Plugging these into the second equation leads to $h_1^{d,B} = -5/12$, which in turn leads to

$$(h_1^{d,B}, h_1^{d,S}, h_1^{d,P})' = (-5/12, 1/4, 5/4)'.$$

The price of this portfolio at $t = 1$ is $-5/12 \times 120 + 1/4 \times 100 + 5/4 \times 40 = 25$.

The payoffs in states ω_1 and ω_2 are 0, so trivially, a replicating portfolio after an up move is

$$(h_1^{u,B}, h_1^{u,S}, h_1^{u,P})' = (0, 0, 0)'$$

(since the price of the option is 0 after an up move, arbitrary other positions in the option are of course also possible).

Now, at 0 a hedging portfolio $(h_0^B, h_0^S, h_0^P)'$ should be chosen such that the payoffs 0 and 25 should be replicated after an up and down move, respectively. Since there are more assets than states, there are again several ways of doing this, but since the option between $t = 0$ and $t = 1$ is basically an Arrow-Debreu security on the down state, it is clear that

$$(h_0^B, h_0^S, h_0^P)' = (0, 0, 5/8)$$

is an especially simple such portfolio.

2. *Black-Scholes (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$
\begin{aligned}
\frac{dB_t}{B_t} &= r dt, \\
\frac{dS_t}{S_t} &= \hat{\mu} dt + dW.
\end{aligned}
$$

All the standard assumptions (no transaction costs, no arbitrage, etc.) are satisfied. Assume that an investor wishes to create a simple contingent claim with payoff $\Phi(S_T)$ at time T, by using dynamic portfolio trading.

(a) Formulate a dynamic, self-financing, trading strategy, $\mathbf{h}_t = (h_t^B, h_t^S)'$, $0 \le t \le T$, that allows the investor to replicate the payoff of the contingent claim, by trading in the bond and the stock.

(b) Prove that the trading strategy in (a) does indeed replicate the payoff of the contingent claim.

(c) What is the time-0 price in this market of a so-called "power" contingent claim, that makes the terminal payoff $\Phi(S_T) = S_T^2$?

Solution:

(a) This is basically Theorem 8.5 in Björk (see also the no-arbitrage section in part 1.2 of the slides). The theorem states that if F solves the PDE

$$
\begin{aligned}
F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - rF &= 0, \\
F(T, S) &= \Phi(S),
\end{aligned}
$$

then the portfolio strategy

$$
\mathbf{h}_t = (h_t^B, h_t^S)' = \left(\frac{F - SF_S}{B}, F_S\right)'
$$

is self-financed, has value $V(t, S_t) = F(t, S_t)$, and replicates the payoff $V_T = \Phi(S_T)$ at T.

(b) That $V(t, S_t) = F(t, S_t)$ follows immediately from $h_t^B B_t + h_t^S S_t = F - SF_S + SF_S = F$, which of course also ensures that the final payout is correct, $V_T = V(T, S_T) = F(T, S_T) = \Phi(S_T)$.

What remains to show is that the strategy is self financed, i.e., that the instantaneous cash flows $dF^{\mathbf{h}}$ generated by the portfolio is 0. We have (see, e.g., the continuous time portfolio model section in Part 1.2 of slides)

$$
dF^{\mathbf{h}} = \mathbf{h}_t'(dB, dS)' - dV = \frac{F - SF_S}{B} dB + F_S dS - dV.
$$

Now, since $V = F$, it follows that $dV = dF = F_t dt + F_S dS + \frac{1}{2}\sigma^2 S^2 F_{SS} dt$, which when plugged into the equation yields

$$\begin{aligned}
dF^{\mathbf{h}} &= \frac{F - SF_S}{B} dB + F_S dS - \left(F_t dt + F_S dS + \frac{1}{2}\sigma^2 S^2 F_{SS} dt \right) \\
&= (F - SF_S) r dt - \left(F_t dt + \frac{1}{2}\sigma^2 S^2 F_{SS} dt \right) \\
&= -(F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - rF) dt \\
&= -0 \times dt.
\end{aligned}$$

So, the portfolio strategy is indeed self financed and the proposition follows.

(c) The easiest way to calculate the price is by using the risk neutral expectation:

$$P_0 = E_0^Q \left[e^{-rT} \Phi(S_T) \right] = e^{-rT} E_0^Q \left[S_T^2 \right].$$

We rewrite $S_T = S_0 e^{y_T}$, where under the risk neutral probability measure $y_T \sim N((r - \sigma^2/2)T, \sigma^2 T)$. Also, we define $x_T = 2y_T \sim N(2(r - \sigma^2/2)T, 4\sigma^2 T)$ and we then have $S_T^2 = S_0^2 e^{x_T}$. Standard formulas of log-normal distributions imply that

$$E_0^Q[e^{x_T}] = e^{2(r - \sigma^2/2)T + \frac{1}{2}4\sigma^2 T} = e^{2rT + \sigma^2 T}.$$

So, in total we get

$$\begin{aligned}
P_0 = e^{-rT} E_0^Q \left[S_T^2 \right] &= e^{-rT} S_0^2 e^{2rT + \sigma^2 T} \\
&= S_0^2 e^{(r + \sigma^2)T}.
\end{aligned}$$

Note that this is a higher price than the "naive" price $P_0 = S_0^2$, which would be the price of buying S_0 shares of the stock at $t = 0$. Such a strategy would lead to the terminal payoff $S_0 S_T$, not S_T^2.

One could also find the solution by solving the Black-Scholes PDE in (a). Specifically, we verify that the function $F(t, S_t) = S_t^2 e^{(r+\sigma^2)(T-t)}$ satisfies the PDE in (a). We have $P_0 = F(0, S_0) = S_0^2 e^{(r+\sigma^2)T}$, and $P_T = F(T, S_T) = S_T^2$. Further, the PDE is satisfied, since

$$\begin{aligned}
F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - rF &= -(r + \sigma^2)F + rS2Se^{(r+\sigma^2)(T-t)} + \frac{1}{2}\sigma^2 S^2 2e^{(r+\sigma^2)(T-t)} - rF \\
&= -(r + \sigma^2)F + 2rF + \sigma^2 F - rF \\
&= 0.
\end{aligned}$$

Thus, F is a solution to the PDE, and P_0 is therefore the price of the power claim.

3. *Dividends (20 points)*: Consider the "Black Scholes" economy where the stock pays a constant dividend yield δ.

$$\begin{aligned}\frac{dB}{B} &= r\,dt, \qquad r > 0,\\ \frac{dS + \delta S\,dt}{S} &= \hat{\mu}\,dt + \sigma\,dW.\end{aligned}$$

where $\hat{\mu}$, σ, and r are all constant. Now, consider a call option that has no maturity date, but has strike K and will be exercised the first time the stock price reaches S^*. Hence, the cash flow when this *first hitting time* occurs is $(S^* - K)$ Here, S^* has been chosen such that it is greater than K.

(a) Determine the differential equation that the value of this call, $C(S)$, satisfies.

(b) Solve for the call price.

(c) Now assume that the buyer of the option is allowed to choose for herself the S^* at which she will elect to exercise it. Determine this optimal S^*.

Solution:

(a) Under the risk-neutral measure, the stock price process follows

$$dS = (r - \delta)\,S\,dt + \sigma\,S\,dz.$$

Since the call pays no dividend, we have

$$\begin{aligned} rC &= \mathrm{E}_t^Q\left[dC\right]\\ &= C_t + (r-\delta)\,SC_S + \frac{\sigma^2}{2}S^2C_{SS}. \end{aligned} \tag{1}$$

However, since there is no explicit time-dependence in the state variable dynamics, and no explicit time-dependence in the payoff, it follows that this call will have the same value each time the same value of S is reached. It thus follows that $C_t = 0$, implying that its dynamics reduce to

$$rC = (r-\delta)\,SC_S + \frac{\sigma^2}{2}S^2C_{SS}. \tag{2}$$

(b) Assuming $C(S) \sim S^\alpha$, we find

$$rS^\alpha = (r-\delta)\,\alpha S^\alpha + \alpha(\alpha-1)\frac{\sigma^2}{2}S^\alpha \tag{3}$$

or equivalently that

$$0 = \left(\frac{\sigma^2}{2}\right)\alpha^2 + \alpha(r - \delta - \frac{\sigma^2}{2}) - r \tag{4}$$

with solutions

$$\alpha_{\pm} = \left(\frac{1}{\sigma^2}\right)\left[-(r-\delta-\frac{\sigma^2}{2}) \pm \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2r\sigma^2}\right]. \tag{5}$$

Note that since $r > 0$, the term inside the square root is larger than the term outside. As such, we have $\alpha_+ > 0$, $\alpha_- < 0$. Furthermore, $\alpha_+ = 1$ when the dividend payout $\delta = 0$. Moreover, it is straightforward to demonstrate that α_+ is increasing in δ in that $\frac{\partial\alpha_+}{\partial\delta} > 0$.

Thus, we know that the call price is of the form

$$C(S) = AS^{\alpha_+} + BS^{\alpha_-} \tag{6}$$

The boundary conditions are

$$\begin{aligned} C(S=0) &= 0 && (7) \\ C(S=S^*) &= S^* - K && (8) \end{aligned}$$

Solving for A and B, we find

$$C(S) = (S^* - K)\left(\frac{S}{S^*}\right)^{\alpha_+}. \tag{9}$$

(c) We solve the first-order condition:

$$\begin{aligned} 0 &= \frac{\partial C}{\partial S^*} \\ &= (1-\alpha_+)(S^*)^{-\alpha_+} + \alpha_+ K (S^*)^{-\alpha_+ - 1} && (10) \end{aligned}$$

implying that

$$S^* = K\left(\frac{\alpha_+}{\alpha_+ - 1}\right) \tag{11}$$

As a side note: recall that $\alpha_+ > 1$ when $\delta > 0$, but approaches one as $\delta \Rightarrow 0$. Thus, as $\delta \Rightarrow 0$, we find $S^* \Rightarrow \infty$, implying that it is always better to wait. This is consistent with the fact that, for finite maturity American call options, it is never optimal to exercise early if the dividend is zero.

4. *Term structure (20 points)*: Assume that the short rate follows the asset pricing dynamics specified by the CIR model:

$$dr_t = a(b-r)dt + \sigma\sqrt{r}dW^Q.$$

This model belongs to the class of affine term structure models, implying that the price of a T-bond is on the form

$$p(t,T|r_t) = e^{A(T-t)-B(T-t)r_t}.$$

(a) State the ODEs that determine the functions $B(\cdot)$ and $A(\cdot)$, respectively.

(b) Verify that the functions

$$\begin{aligned} B(x) &= \frac{2(e^{qx}-1)}{(q+a)(e^{qx}-1)+2q}, \\ A(x) &= \frac{2ab}{\sigma^2}\ln\left(\frac{2qe^{(q+a)x/2}}{(q+a)(e^{qx}-1)+2q}\right), \end{aligned}$$

where $q=\sqrt{a^2+2\sigma^2}$, solve the ODEs stated in (a).

Solution:

(a) Using the same derivation as in class (i.e., plugging in the conjectured form of the solution into the one-factor term structure PDE), it follows that A and B need to satisfy the two ODEs:

$$B' + Ba + \frac{1}{2}B^2\sigma^2 = 1,$$
$$-A' - Bab = 0.$$

The boundary conditions are $B(0)=0$, $A(0)=0$.

(b) We begin by noting that given the conjectured solutions for B and A,

$$\begin{aligned} B'(x) &= \frac{4q^2e^{qx}}{((q+a)(e^{qx}-1)+2q)^2}, \\ A'(x) &= \frac{2ab}{\sigma^2}\left[\frac{q+a}{2} - \frac{(q+a)qe^{qx}}{(q+a)(e^{qx}-1)+2q}\right]. \end{aligned}$$

We first verify the first ODE.

$$\frac{4q^2e^{qx}}{((q+a)(e^{qx}-1)+2q)^2} + a\frac{2(e^{qx}-1)}{(q+a)(e^{qx}-1)+2q} + \frac{1}{2}\sigma^2\left(\frac{2(e^{qx}-1)}{(q+a)(e^{qx}-1)+2q}\right)^2 = 1$$

$$\begin{aligned} 4q^2e^{qx} + a2(e^{qx}-1)((q+a)(e^{qx}-1)+2q) + \frac{1}{2}\sigma^2(2(e^{qx}-1))^2 &= ((q+a)(e^{qx}-1)+2q)^2 \\ &= (q+a)^2(e^{qx}-1)^2 + (2q)^2 \\ &+ 2(q+a)(e^{qx}-1)2q \\ 2a(e^{qx}-1)((q+a)(e^{qx}-1)+2q) + \frac{1}{2}\sigma^2(2(e^{qx}-1))^2 &= (q+a)^2(e^{qx}-1)^2 \\ &+ 2a(e^{qx}-1)2q \\ (2aq+2a^2)(e^{qx}-1)^2 + \frac{1}{2}\sigma^2(2(e^{qx}-1))^2 &= (q^2+2qa+a^2)(e^{qx}-1)^2 \\ (2aq+2a^2)+2\sigma^2 &= (q^2+2qa+a^2) \end{aligned}$$

In the last equation we use the fact that $q^2 = a^2 + 2\sigma^2$ and we are done. Now let's verify the second ODE:

$$\begin{aligned}
\frac{2ab}{\sigma^2}\left[\frac{q+a}{2} - \frac{(q+a)qe^{qx}}{(q+a)(e^{qx}-1)+2q}\right] + ab\frac{2(e^{qx}-1)}{(q+a)(e^{qx}-1)+2q} &= 0\\
\left[\frac{q+a}{2} - \frac{(q+a)qe^{qx}}{(q+a)(e^{qx}-1)+2q}\right] + \frac{\sigma^2(e^{qx}-1)}{(q+a)(e^{qx}-1)+2q} &= 0\\
\frac{q+a}{2}((q+a)(e^{qx}-1)+2q) - (q+a)qe^{qx} + \sigma^2(e^{qx}-1) &= 0\\
(q+a)^2(e^{qx}-1) - 2(q+a)q(e^{qx}-1) + 2\sigma^2(e^{qx}-1) &= 0\\
(q^2+2qa+a^2)(e^{qx}-1) - 2(q+a)q(e^{qx}-1) + 2(q^2-a^2)(e^{qx}-1) &= 0\\
(q^2+2qa+a^2) - 2(q+a)q + (q^2-a^2) &= 0\\
0 &= 0.
\end{aligned}$$

Finally, it is immediately verified that $B(0) = 0/(2q) = 0$, and $A(0) = \frac{2ab}{\sigma^2}\ln(1) = 0$. We are done.