18

BARRIER OPTIONS

The purpose of this chapter is to give a reasonably systematic overview of the pricing theory for those financial derivatives which are, in some sense, connected to the extremal values of the underlying price process. We focus on barrier options, ladders and lookbacks, and we confine ourselves to the case of one underlying asset.

18.1 Mathematical Background

In this chapter we will give some probability distributions connected with barrier problems. All the results are standard, see e.g. Borodin–Salminen (1997).

To start with some notational conventions, let $\{X(t);\ 0 \le t < \infty\}$ be any process with continuous trajectories taking values on the real line.

Definition 18.1 *For any $y \in R$, the* **hitting time** *of y, $\tau(X, y)$, sometimes denoted by $\tau(y)$ or τ_y, is defined by*

$$\tau(y) = \inf\{t \ge 0 \,|X(t) = y\}.$$

The X-process **absorbed at** *y is defined by*

$$X_y(t) = X(t \wedge \tau)$$

where we have used the notation $\alpha \wedge \beta = \min[\alpha, \beta]$.

The **running maximum** *and* **minimum** *processes, $M_X(t)$ and $m_X(t)$, are defined by*

$$M_X(t) = \sup_{0 \le s \le t} X(s),$$
$$m_X(t) = \inf_{0 \le s \le t} X(s),$$

where we sometimes suppress the subscript X.

We will be mainly concerned with barrier problems for Wiener processes, so naturally the normal distribution will play a prominent role.

Definition 18.2 *Let $\varphi(x; \mu, \sigma)$ denote the density of a normal distribution with mean μ and variance σ^2, i.e.*

$$\varphi(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

The standardized density $\varphi(x;0,1)$ is denoted by $\varphi(x)$, and the cumulative distribution function of $\varphi(x)$ is as usual denoted by $N(x)$, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz.$$

Let us now consider a Wiener process with (constant) drift μ and (constant) diffusion σ, starting at a point α, i.e.

$$dX(t) = \mu dt + \sigma dW_t, \tag{18.1}$$
$$X(0) = \alpha. \tag{18.2}$$

We are primarily interested in the one-dimensional marginal distribution for $X_\beta(t)$, i.e. the distribution at time t of the X-process, absorbed at the point β. The distribution of $X_\beta(t)$ is of course a mixed distribution in the sense that it has a point mass at $x = \beta$ (the probability that the process is absorbed prior to time t) and a density. This density has its support on the interval (β, ∞) if $\alpha > \beta$, whereas the support is the interval $(-\infty, \beta)$ if $\alpha < \beta$. We now cite our main result concerning absorption densities.

Proposition 18.3 *The density $f_\beta(x;t,\alpha)$ of the absorbed process $X_\beta(t)$, where X is defined by (18.1)–(18.2), is given by*

$$f_\beta(x;t,\alpha) = \varphi(x;\mu t + \alpha, \sigma\sqrt{t}) - \exp\left\{-\frac{2\mu(\alpha-\beta)}{\sigma^2}\right\} \varphi(x;\mu t - \alpha + 2\beta, \sigma\sqrt{t}).$$

The support of this density is the interval (β,∞) if $\alpha > \beta$, and the interval $(-\infty,\beta)$ if $\alpha < \beta$.

We end this section by giving the distribution for the running maximum (minimum) processes.

Proposition 18.4 *Consider the process X defined by (18.1)–(18.2), and let M (m) denote the running maximum (minimum) processes as in Definition 18.1. Then the distribution functions for $M(t)$ ($m(t)$) are given by the following expressions, which hold for $x \geq \alpha$ and $x \leq \alpha$ respectively.*

$$F_{M(t)}(x) = N\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) - \exp\left\{2\cdot\frac{\mu(x-\alpha)}{\sigma^2}\right\} N\left(-\frac{x-\alpha+\mu t}{\sigma\sqrt{t}}\right),$$
$$F_{m(t)}(x) = N\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) + \exp\left\{2\cdot\frac{\mu(x-\alpha)}{\sigma^2}\right\} N\left(\frac{x-\alpha+\mu t}{\sigma\sqrt{t}}\right).$$

18.2 Out Contracts

In this section we will undertake a systematic study of the relations between a "standard" contingent claim and its different "barrier" versions. This will provide us with some basic insights and will also give us a number of easy formulas to use when pricing various barrier contracts. As usual we consider the standard Black–Scholes model

$$\begin{aligned} dS &= \alpha S dt + \sigma S d\bar{W}, \\ dB &= rBdt, \end{aligned}$$

with fixed parameters α, σ and r.

We fix an exercise time T and we consider as usual a contingent claim $\mathcal{Z}$ of the form

$$\mathcal{Z} = \Phi(S(T)). \tag{18.3}$$

We denote the pricing function of $\mathcal{Z}$ by $F(t, s; T, \Phi)$, often suppressing the parameter T. For mnemo-technical purposes we will also sometimes use the notation $\mathbf{\Phi}(t, s)$, i.e. the pricing function (as opposed to the function defining the claim) is given in bold.

18.2.1 *Down-and-out Contracts*

Fix a real number $L < S(0)$, which will act as the barrier, and consider the following contract, which we denote by $\mathcal{Z}_{LO}$:

- If the stock price stays above the barrier L during the entire contract period, then the amount $\mathcal{Z}$ is paid to the holder of the contract.
- If the stock price, at some time before the delivery time T, hits the barrier L, then the contract ceases to exist, and nothing is paid to the holder of the contract.

The contract $\mathcal{Z}_{LO}$ is called the "down-and-out" version of the contract $\mathcal{Z}$ above, and our main problem is to price $\mathcal{Z}_{LO}$. More formally we can describe $\mathcal{Z}_{LO}$ as follows.

Definition 18.5 *Take as given a T-contract $\mathcal{Z} = \Phi(S(T))$. Then the T-contract $\mathcal{Z}_{LO}$ is defined by*

$$\mathcal{Z}_{LO} = \begin{cases} \Phi(S(T)), & \text{if } S(t) > L \text{ for all } t \in [0, T] \\ 0, & \text{if } S(t) \le L \text{ for some } t \in [0, T]. \end{cases} \tag{18.4}$$

Concerning the notation, L as a **sub**script indicates a "down"-type contract, whereas the letter O indicates that we are considering an "out" claim. You may also consider other types of barrier specifications and thus construct a "down-and-in" version of the basic contract $\mathcal{Z}$. A "down-and-in" contract **starts** to exist the first time the stock price hits a lower barrier. Going on we may then consider up-and-out as well as up-and-in contracts. All these types will be given precise definitions and studied in the following sections.

In order to price $\mathcal{Z}_{LO}$ we will have use for the function Φ_L, which is the original contract function Φ in (18.3) "chopped off" below L.

Definition 18.6 *For a fixed function Φ the function Φ_L is defined by*

$$\Phi_L(x) = \begin{cases} \Phi(x), & \text{for } x > L \\ 0, & \text{for } x \le L. \end{cases} \tag{18.5}$$

In other words, $\Phi_L(x) = \Phi(x) \cdot I\{x > L\}$, where I denotes the indicator function.

For further use we note that the pricing functional $F(t, S; \Phi)$ is linear in the Φ-argument, and that the "chopping" operation above is also linear.

Lemma 18.7 *For all reals α and β, and all functions Φ and Ψ, we have*

$$F(t, s; \alpha\Phi + \beta\Psi) = \alpha F(t, s; \Phi) + \beta F(t, s; \Psi),$$
$$(\alpha\Phi + \beta\Psi)_L = \alpha\Phi_L + \beta\Psi_L.$$

Proof For F the linearity follows immediately from the risk neutral valuation formula together with the linearity of the expectation operator. The linearity of the chopping operation is obvious. □

Our main result is the following theorem, which tells us that the pricing problem for the down-and-out version of the contract Φ is essentially reduced to that of pricing the nonbarrier claim Φ_L. Thus, if we can price a standard (nonbarrier) claim with contract function Φ_L then we can also price the down-and-out version of the contract Φ.

Theorem 18.8 (Pricing down-and-out contracts) *Consider a fixed T-claim $\mathcal{Z} = \Phi(S(T))$. Then the pricing function, denoted by F_{LO}, of the corresponding down-and-out contract $\mathcal{Z}_{LO}$ is given, for $s > L$, by*

$$F_{LO}(t, s; \Phi) = F(t, s; \Phi_L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; \Phi_L\right). \tag{18.6}$$

Here we have used the notation

$$\tilde{r} = r - \frac{1}{2}\sigma^2.$$

Proof Without loss of generality we may set $t = 0$ in (18.6). Assume then that $S(0) = s > L$, and recall that S_L denotes the process S with (possible) bsorption at L. Using risk neutral valuation we have

$$\begin{aligned}
F_{LO}(0,s;\Phi) &= e^{-rT}E^Q_{0,s}\left[\mathcal{Z}_{LO}\right] = e^{-rT}E^Q_{0,s}\left[\Phi(S(T))\cdot I\left\{\inf_{0\le t\le T} S(t) > L\right\}\right] \\
&= e^{-rT}E^Q_{0,s}\left[\Phi_L(S_L(T))\cdot I\left\{\inf_{0\le t\le T} S(t) > L\right\}\right] \\
&= e^{-rT}E^Q_{0,s}\left[\Phi_L(S_L(T))\right].
\end{aligned}$$

It remains to compute the last expectation, and we have

$$E^Q_{0,s}\left[\Phi_L(S_L(T))\right] = \int_L^\infty \Phi_L(x)h(x)dx,$$

where h is the density function for the stochastic variable $S_L(T)$.

From standard theory we have

$$S(T) = \exp\left\{\ln s + \tilde{r}T + \sigma W(T)\right\} = e^{X(T)},$$

where the process X is defined by

$$\begin{aligned}
dX(t) &= \tilde{r}dt + \sigma dW(t), \\
X(0) &= \ln s.
\end{aligned}$$

Thus we have

$$S_L(t) = \exp\left\{X_{\ln L}(t)\right\},$$

so we may write

$$E^Q_{0,s}\left[\Phi_L(S_L(T))\right] = \int_{\ln L}^\infty \Phi_L(e^x)f(x)dx,$$

where f is the density of the stochastic variable $X_{\ln L}(T)$. This density is, however, given by Proposition 18.3 as

$$\begin{aligned}
f(x) &= \varphi\left(x;\tilde{r}T + \ln s, \sigma\sqrt{T}\right) \\
&\quad - \exp\left\{-\frac{2\tilde{r}(\ln s - \ln L)}{\sigma^2}\right\}\varphi\left(x;\tilde{r}T - \ln s + 2\ln L, \sigma\sqrt{T}\right) \\
&= \varphi\left(x;\tilde{r}T + \ln s, \sigma\sqrt{T}\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\varphi\left(x;\tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right).
\end{aligned}$$

Thus we have

$$
\begin{aligned}
E^Q_{0,s}\left[\Phi_L(S_L(T))\right] &= \int_{\ln L}^{\infty} \Phi_L(e^x) f(x)dx \\
&= \int_{\ln L}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln s, \sigma\sqrt{T}\right) dx \\
&\quad - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{\ln L}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right) dx \\
&= \int_{-\infty}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln s, \sigma\sqrt{T}\right) dx \\
&\quad - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{-\infty}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right) dx.
\end{aligned}
$$

Inspecting the last two lines we see that the density in the first integral is the density of $X(T)$ under the usual martingale measure Q, given the starting value $S(0) = s$. The density in the second integral is, in the same way, the density (under Q) of $X(T)$, given the starting point $S(0) = L^2/s$. Thus we have

$$
E^Q_{0,s}\left[\Phi_L(S_L(T))\right] = E^Q_{0,s}\left[\Phi_L(S(T))\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} E^Q_{0,\frac{L^2}{s}}\left[\Phi_L(S(T))\right]
$$

which gives us the result. □

We again emphasize the point of this result.

The problem of computing the price for a down-and-out claim reduces to the standard problem of computing the price of an ordinary (related) claim without a barrier.

For future use we also note the fact that down-and-out pricing is a linear operation.

Corollary 18.9 *For any contract functions Φ and Ψ, and for any real numbers α and β, the following relation holds.*

$$
F_{LO}(t, s; \alpha\Phi + \beta\Psi) = \alpha F_{LO}(t, s; \Phi) + \beta F_{LO}(t, s; \Psi).
$$

Proof The result follows immediately from Theorem 18.8 together with the linearity of the ordinary pricing functional F and the linearity of the chopping operation. □

18.2.2 *Up-and-out Contracts*

We again consider a fixed T-contract of the form $\mathcal{Z} = \Phi(S(T))$, and we now describe the up-and-out version of $\mathcal{Z}$. This is the contract which at the time of delivery, T, will pay $\mathcal{Z}$ if the underlying price process during the entire contract period has stayed below the barrier L. If, at some time during the contract period, the price process exceeds L, then the contract is worthless. In formal terms this reads as follows.

Definition 18.10 *Take as given the T-contract $\mathcal{Z} = \Phi(S(T))$. Then the T-contract $\mathcal{Z}^{LO}$ is defined by*

$$\mathcal{Z}^{LO} = \begin{cases} \Phi(S(T)), & \text{if } S(t) < L \text{ for all } t \in [0,T] \\ 0, & \text{if } S(t) \geq L \text{ for some } t \in [0,T]. \end{cases} \tag{18.7}$$

The pricing functional for $\mathcal{Z}^{LO}$ is denoted by $F^{LO}(t,s;\Phi)$, or according to our earlier notational convention, by $\mathbf{\Phi}^{LO}(t,s)$.

L as a **super**script indicates an "up"-type contract, whereas the superscript O indicates that the contract is an "out" contract. As in the previous sections we will relate the up-and-out contract to an associated standard contract. To this end we need to define, for a fixed contract function Φ, the function Φ^L, which is the function Φ "chopped off" above L.

Definition 18.11 *For a fixed function Φ the function Φ^L is defined by*

$$\Phi^L(x) = \begin{cases} \Phi(x), & \text{for } x < L \\ 0, & \text{for } x \geq L. \end{cases} \tag{18.8}$$

In other words, $\Phi^L(x) = \Phi(x) \cdot I\{x < L\}$.

The main result of this section is the following theorem, which is parallel to Theorem 18.8. The proof is almost identical.

Theorem 18.12 (Pricing up-and-out contracts) *Consider a fixed T-claim $\mathcal{Z} = \Phi(S(T))$. Then the pricing function, F^{LO}, of the corresponding up-and-out contract $\mathcal{Z}^{LO}$ is given, for $S < L$, by*

$$F^{LO}(t,s,\Phi) = F\left(t,s,\Phi^L\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}, \Phi^L\right) \tag{18.9}$$

Parity: $F - F^{LI}$ $F(\ \Phi^L) + F(\cancel{\Phi_L}) - [F(\cancel{\Phi_L}) + (\ )F(\Phi^L)]$ ok

where we have used the notation

$$\tilde{r} = r - \frac{1}{2}\sigma^2.$$

18.2.3 *Examples*

In this section we will use Theorems 18.8 and 18.12, together with the linearity lemma 18.7, to give a systematic account of the pricing of a fairly wide class of barrier derivatives, including barrier call and put options. Let us define the following standard contracts, which will be the basic building blocks in the sequel.

Definition 18.13 *Fix a delivery time T. For fixed parameters K and L define the claims ST, BO, H and C by*

$$ST(x) = x, \ \forall x \tag{18.10}$$

$$BO(x) = 1, \ \forall x \tag{18.11}$$

$$H(x; L) = \begin{cases} 1, & \text{if } x > L \\ 0, & \text{if } x \leq L \end{cases} \tag{18.12}$$

$$C(x; K) = \max\left[x - K, 0\right]. \tag{18.13}$$

The contract ST (ST for "stock") thus gives the owner (the price of) one unit of the underlying stock at delivery time T, whereas BO is an ordinary zero coupon bond paying one at maturity T. The H-contract (H stands for the Heaviside function) gives the owner one if the value of the underlying stock exceeds L at delivery time T, otherwise nothing is paid out. The C-claim is of course the ordinary European call with strike price K. We note in passing that $H(x; L) = H_L(x)$.

We now list the pricing functions for the standard contracts above. The value of ST at time t is of course equal to the value of the underlying stock at the same time, whereas the value of BO at t is $e^{-r(T-t)}$. The value of C is given by the Black–Scholes formula, and the value of H is easily calculated by using risk neutral valuation. Thus we have the following result.

Lemma 18.14 *The contracts (18.10)–(18.13) with delivery time T are priced at time t as follows (with the pricing function in bold).*

$$\begin{aligned}
\mathbf{ST}(t, s) &= s, \\
\mathbf{BO}(t, s) &= e^{-r(T-t)}, \\
\mathbf{H}(t, s; L) &= e^{-r(T-t)} N\left[\frac{\tilde{r}(T-t) + \ln\left(\frac{s}{L}\right)}{\sigma\sqrt{T-t}}\right], \\
\mathbf{C}(t, s; K) &= c(t, s; K),
\end{aligned}$$

where $c(t, s; K)$ is the usual Black–Scholes formula.

We may now put this machinery to some use and start with the simple case of valuing a down-and-out contract on a bond. This contract will thus pay out 1 dollar at time T if the stock price is above the level L during the entire contract

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A direct application of Theorem 18.8 gives us the formula

$$F_{LO}\left(t,s;BO\right)=F\left(t,s;BO_L\right)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};BO_L\right).$$

Obviously we have $BO_L(x)=H(x;L)$ for all x so we have the following result.

Lemma 18.15 *The down-and-out bond with barrier L is priced, for $s>L$, by the formula*

$$\mathbf{BO}_{LO}(t,s)=\mathbf{H}(t,s;L)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\mathbf{H}\left(t,\frac{L^2}{s};L\right), \tag{18.14}$$

where $\mathbf{H}(t,s;L)$ is given by Lemma 18.14.

We continue by pricing a down-and-out contract on the stock itself (no option is involved). Thus we want to compute $F_{LO}(t,s;ST)$ and Theorem 18.8 gives us

$$F_{LO}(t,s;ST)=F(t,s;ST_L)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};ST_L\right). \tag{18.15}$$

A quick look at a figure gives us the relation

$$ST_L(x)=L\cdot H(x;L)+C(x;L).$$

Substituting this into (18.15) and using linearity (Lemma 18.7) we get

$$\begin{aligned}
&\mathbf{ST}_{LO}(t,s)\\
&\quad=F_{LO}(t,s;ST)\\
&\quad=F\left(t,s;LH(\star,L)+C(\star,L)\right)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};LH(\star;L)+C(\star;L)\right)\\
&\quad=L\cdot F\left(t,s;H(\star;L)\right)-L\cdot\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};H(\star;L)\right)\\
&\qquad+F\left(t,s;C(\star;L)\right)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};C(\star;L)\right).
\end{aligned}$$

Summarizing we have the following.

Lemma 18.16 *The down-and-out contract on the underlying stock is given by*

$$\mathbf{ST}_{LO}(t,s) = L \cdot \mathbf{H}\left(t,s;L\right) - L \cdot \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{H}\left(t, \frac{L^2}{s}; L\right)$$
$$+ \mathbf{C}\left(t,s;L\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{C}\left(t, \frac{L^2}{s}; L\right),$$

where **H** *and* **C** *are given by Lemma 18.14.*

We now turn to a more interesting example—a down-and-out European call with strike price K. From the main proposition we immediately have

$$F_{LO}\left(t,s;C(\star;K)\right) = F\left(t,s;C_L(\star;K)\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; C_L(\star,K)\right), \tag{18.16}$$

and we have to treat the two cases $L < K$ and $L > K$ separately. The result is as follows.

Proposition 18.17 (Down-and-out call) *The down-and-out European call option is priced as follows.*
For $L < K$:

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$$\mathbf{C}_{LO}(t,s;K) = \mathbf{C}(t,s;K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{C}\left(t, \frac{L^2}{s}; K\right). \tag{18.17}$$

For $L > K$:

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$$\mathbf{C}_{LO}(t,s;K) = \mathbf{C}(t,s;\cancel{K}) + (L-K)\mathbf{H}(t,s;L)$$
$$- \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left\{ \mathbf{C}\left(t, \frac{L^2}{s}; \cancel{K}\right) + (L-K)\mathbf{H}\left(t, \frac{L^2}{s}; L\right)\right\}. \tag{18.18}$$

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Proof For $L < K$ it is easily seen (draw a figure) that $C_L(x,K) = C(x,K)$, so from (18.16) we get

$$\mathbf{C}_{LO}(t,s;K) = F\left(t,s;C(\star;K)\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; C(\star,K)\right),$$

which proves (18.17).

For $L > K$ the situation is slightly more complicated. Another figure shows that

ok $$C_L(x;K) = C(x;L) + (L-K)H(x;L).$$

Putting this relation into (18.16), and using the linear property of pricing, we get (18.18). □

As we have seen, almost all results are fairly easy consequences of the linearity of the pricing functional. In Section 9.1 we used this linearity to prove the standard put–call parity relation for standard European options, and we can now derive the put–call parity result for down-and-out options.

Drawing a figure we see that $P(x;K) = K - x + C(x;K)$, so, in terms of the standard contracts, we have

$$P(x;K) = K \cdot BO(x) - ST(x) + C(x;K).$$

Using Corollary 18.9 we immediately have the following result. Note that when $L = 0$ we have the usual put–call parity.

Proposition 18.18 (Put–call parity) *The down-and-out put price $\mathbf{P}_{LO}$, and call price $\mathbf{C}_{LO}$, are related by the formula*

$$\mathbf{P}_{LO}(t,s;K) = K \cdot \mathbf{B}_{LO}(t,s) - \mathbf{ST}_{LO}(t,s) + \mathbf{C}_{LO}(t,s;K). \tag{18.19}$$

Here $\mathbf{B}_{LO}$ and $\mathbf{ST}_{LO}$ are given by Lemmas 18.15 and 18.16, whereas $\mathbf{C}_{LO}$ is given by Proposition 18.17.

We end this section by computing the price of a European up-and-out put option with barrier L and strike price K.

Proposition 18.19 (Up-and-out put) *The price of an up-and-out European put option is given by the following formulas.*
If $L > K$ then, for $s < L$:

$$\mathbf{P}^{LO}(t,s;K) = \mathbf{P}(t,s;K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{P}\left(t, \frac{L^2}{s}; K\right). \tag{18.20}$$

If $L > K$, then for $s < L$:

$$\begin{aligned}\mathbf{P}^{LO}(t,s;K) = {}& \mathbf{P}(t,s;L) - (K-L)\mathbf{H}(t,s;L) \\ & - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left\{\mathbf{P}\left(t,\frac{L^2}{s};L\right) - (K-L)\mathbf{H}\left(t,\frac{L^2}{s};L\right)\right\} \\ & + \left\{1 - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\right\}(K-L)e^{-r(T-t)}. \end{aligned} \tag{18.21}$$

Proof If $L > K$ then $P^L(s;K) = P(s;K)$, and then (18.20) follows immediately from Proposition 18.12.

If $L < K$ then it is easily seen that

$$P^L(x) = P(x;L) + (K-L) \cdot BO(x) - (K-L) \cdot H(x;L).$$

Linearity and Proposition 18.12 give us (18.21). □

18.3 In Contracts

In this section we study contracts which will **start** to exist if and only if the price of the underlying stock hits a prespecified barrier level at some time during the contract period. We thus fix a standard T-claim of the form $\mathcal{Z} = \Phi(S(T))$, and we also fix a barrier L. We start by studying the "down-and-in" version of $\mathcal{Z}$, which is defined as follows:

- If the stock price stays above the barrier L during the entire contract period, then nothing is paid to the holder of the contract.
- If the stock price, at some time before the delivery time T, hits the barrier L, then the amount $\mathcal{Z}$ is paid to the holder of the contract.

We will write the down-and-in version of $\mathcal{Z}$ as $\mathcal{Z}_{LI}$, and the formal definition is as follows.

Definition 18.20 *Take as given the T-contract $\mathcal{Z} = \Phi(S(T))$. Then the T-contract $\mathcal{Z}_{LI}$ is defined by*

$$\mathcal{Z}_{LI} = \begin{cases} 0, & \text{if } S(t) > L \text{ for all } t \in [0,T] \\ \Phi(S(T)), & \text{if } S(t) \leq L \text{ for some } t \in [0,T]. \end{cases} \tag{18.22}$$

The pricing function for $\mathcal{Z}_{LI}$ is denoted by $F_{LI}(t,s;\Phi)$, or sometimes by $\mathbf{\Phi}_{LI}(t,s)$.

Concerning the notation, L as a **sub**script indicates a "down" contract, whereas the subscript I denotes an "in" contract. Pricing a down-and-in contract turns out to be fairly easy, since we can in fact price it in terms of the corresponding down-and-out contract.

Lemma 18.21 (In–out parity)

$$F_{LI}(t,s;\Phi) = F(t,s;\Phi) - F_{LO}(t,s;\Phi), \quad \forall s.$$

Proof If, at time t, you have a portfolio consisting of a down-and-out version of $\mathcal{Z}$ as well as a down-and-in version of $\mathcal{Z}$ (with the same barrier L) then obviously you will receive exactly $\mathcal{Z}$ at time T. We thus have

$$F(t,s;\Phi) = F_{LI}(t,s;\Phi) + F_{LO}(t,s;\Phi).$$

□

We can now formulate the basic result.

Proposition 18.22 (Pricing down-and-in contracts) *Consider a fixed T-contract $\mathcal{Z} = \Phi(S(T))$. Then the price of the corresponding down-and-in contract $\mathcal{Z}_{LI}$ is given by*

$$F_{LI}(t,s;\Phi) = F(t,s;\Phi^L) + \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; \Phi_L\right).$$

Proof From the equality $\Phi = \Phi_L + \Phi^L$ we have

$$F(t, s; \Phi) = F(t, s; \Phi_L) + F(t, s; \Phi^L).$$

Now use this formula, the lemma above, and Theorem 18.8. □

The treatment of "up-and-in" contracts is of course parallel to down-and-in contracts, so we only give the basic definitions and results. We denote the up-and-in version of $\mathcal{Z}$ by $\mathcal{Z}^{LI}$, and the definition of $\mathcal{Z}^{LI}$ is as follows:

- If the stock price stays below the barrier L during the entire contract period, then nothing is paid to the holder of the contract.
- If the stock price, at some time before the delivery time T, hits the barrier L, then the amount $\mathcal{Z}$ is paid to the holder of the contract.

Corresponding to Lemma 18.21, we have

$$F^{LI}(t, s; \Phi) = F(t, s; \Phi) - F^{LO}(t, s; \Phi), \quad \forall s, \tag{18.23}$$

and from this relation, together with the pricing formula for up-and-out contracts, we have an easy valuation formula.

Proposition 18.23 (Pricing up-and-in contracts) *Consider a fixed T-contract $\mathcal{Z} = \Phi(S(T))$. Then the price of the corresponding up-and-in contract $\mathcal{Z}^{LI}$ is given by*

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$$F^{LI}(t, s; \Phi) = F(t, s; \Phi_L) + \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; \Phi^L\right).$$

We end this section by giving, as an example, the pricing formula for a down-and-in European call with strike price K.

Proposition 18.24 (Down-and-in European call) *For $s > L$ the down-and-in European call option is priced as follows.*
For $L < K$:

$$\mathbf{C}_{LI}(t, s; K) = \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{C}\left(t, \frac{L^2}{s}; K\right).$$

For $L > K$:

$$\mathbf{C}_{LI}(t, s; K) = \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left\{\mathbf{C}\left(t, \frac{L^2}{s}; \overset{L}{\cancel{K}}\right) + (L - K)\mathbf{H}\left(t, \frac{L^2}{s}; L\right)\right\} - (L - K)\mathbf{H}(t, s; L).$$

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18.4 Ladders

Let us take as given

- A finite increasing sequence of real numbers

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N.$$

  This sequence will be denoted by α.
- Another finite increasing sequence of real numbers

$$0 = \beta_0 < \beta_1 < \cdots < \beta_N.$$

  This sequence will be denoted by β.

Note that the number N is the same in both sequences. The interval $[\alpha_n, \alpha_{n+1})$ will play an important role in the sequel, and we denote it by D_n, with D_N defined as $D_N = [\alpha_N, \infty)$. For a fixed delivery time T we will now consider a new type of contract, called the "(α, β)-ladder", which is defined as follows.

Definition 18.25 *The* (α, β)**-ladder** *with delivery time T is a T-claim $\mathcal{Z}$, described by*

$$\mathcal{Z} = \sum_{n=0}^{N} \beta_n \cdot I\left\{\sup_{t \le T} S(t) \in D_n\right\}. \tag{18.24}$$

In other words, if the realized maximum of the underlying stock during the contract period falls within the interval D_n, then the payout at T is β_n. A typical ladder used in practice is the **forward ladder call** with strike price K. For this contract α is exogenously specified, and β is then defined as

$$\beta_n = \max\left[\alpha_n - K, 0\right]. \tag{18.25}$$

The α-sequence in this case acts as a sequence of barriers, and the ladder call allows you to buy (at time T) the underlying asset at the strike price K, while selling it (at T) at the highest barrier achieved by the stock price during the contract period. The ladder call is intimately connected to the lookback forward call (see the next section), to which it will converge as the α-partition is made finer.

The general (α, β)-ladder is fairly easy to value analytically, although the actual expressions may look formidable. To see this let us define the following series of up-and-in contracts.

Definition 18.26 *For a given pair (α, β), the series of contracts $\mathcal{Z}_0, \ldots, \mathcal{Z}_N$ is defined by*

$$\mathcal{Z}_0 = \beta_0 \cdot I\left\{\sup_{t \le T} S(t) \ge \alpha_0\right\},$$

$$\mathcal{Z}_n = (\beta_n - \beta_{n-1}) \cdot I\left\{\sup_{t \le T} S(t) \ge \alpha_n\right\}, \quad n = 1, \ldots, N.$$

The point of introducing the $\mathcal{Z}_n$-contracts is that we have the following obvious relation

$$\mathcal{Z} = \sum_{n=0}^{N} \mathcal{Z}_n.$$

Thus a ladder is simply a sum of a series of up-and-in contracts. We see that in fact $\mathcal{Z}_n$ is an up-and-in contract on $\beta_n - \beta_{n-1}$ bonds, with barrier α_n. Thus we may use the results of the preceding sections to value $\mathcal{Z}_n$. The result is as follows.

Proposition 18.27 (Ladder pricing formula) *Consider an (α, β)-ladder with delivery time T. Assume that $S(t) = s$ and that $M_S(t) \in D_m$. Then the price, $\Pi(t)$, of the ladder is given by*

$$\Pi(t) = \beta_m + \sum_{n=m+1}^{N} \gamma_n F^{\alpha_n I}(t, s; BO),$$

where $\gamma_n = \beta_n - \beta_{n-1}$, and

$$F^{\alpha_n I}(t, s; BO) = \left(\frac{\alpha_n}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} e^{-r(T-t)} + e^{-r(T-t)} N\left[\frac{\tilde{r}(T-t) + \ln\left(\frac{s}{\alpha_n}\right)}{\sigma\sqrt{T-t}}\right]$$
$$- e^{-r(T-t)} \left(\frac{\alpha_n}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} N\left[\frac{\tilde{r}(T-t) - \ln\left(\frac{s}{\alpha_n}\right)}{\sigma\sqrt{T-t}}\right].$$

Proof Exercise for the reader. □

18.5 Lookbacks

Lookback options are contracts which at the delivery time T allow you to take advantage of the realized maximum or minimum of the underlying price process over the entire contract period. Typical examples are

$$S(T) - \min_{t \le T} S(t) \qquad \text{lookback call}$$

$$\max_{t \le T} S(t) - S(T) \qquad \text{lookback put}$$

$$\max\left[\max_{t \le T} S(t) - K, 0\right] \quad \text{forward lookback call}$$

$$\max\left[K - \min_{t \le T} S(t), 0\right] \quad \text{forward lookback put.}$$

We will confine ourselves to give a sketch of the pricing of a lookback put; for further results see the Notes below.

From general theory, the price of the lookback put at $t = 0$ is given by

$$\begin{aligned}\Pi(0) &= e^{-rT}E^Q\left[\max_{t\leq T} S(t) - S(T)\right] \\ &= e^{-rT}E^Q\left[\max_{t\leq T} S(t)\right] - e^{-rT}E^Q\left[S(T)\right].\end{aligned}$$

With $S(0) = s$, the last term is easily obtained as

$$e^{-rT}E^Q\left[S(T)\right] = s,$$

and it remains to compute the term $E^Q\left[\max_{t\leq T} S(t)\right]$. To this end we recall that $S(t)$ is given by

$$S(t) = \exp\left\{\ln s + \tilde{r}t + \sigma W(t)\right\} = e^{X(t)},$$

where

$$\begin{aligned} dX &= \tilde{r}dt + \sigma dW, \\ X(0) &= \ln s. \end{aligned}$$

Thus we see that

$$M_S(T) = e^{M_X(T)},$$

and the point is of course that the distribution for $M_X(T)$ is known to us from Proposition 18.4. Using this proposition we obtain the distribution function, F, for $M_X(T)$ as

$$F(x) = N\left(\frac{x - \ln s - \tilde{r}T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{2\tilde{r}(x - \ln s)}{\sigma^2}\right\} N\left(-\frac{x - \ln s + \tilde{r}T}{\sigma\sqrt{T}}\right),$$

for all $x \geq \ln s$. From this expression we may compute the density function $f = F'$, and then the expected value is given by

$$E^Q\left[\max_{t\leq T} S(t)\right] = E^Q\left[e^{M_X(T)}\right] = \int_{\ln s}^{\infty} e^x f(x)dx.$$

After a series of elementary, but extremely tedious, partial integrations we end up with the following result.

Proposition 18.28 (Pricing formula for lookback put) *The price, at $t = 0$, of the lookback put is given by*

$$\Pi(0) = -sN[-d] + se^{-rT}N\left[-d + \sigma\sqrt{T}\right] + s\frac{\sigma^2}{2r}N[d] - se^{-rT}\frac{\sigma^2}{2r}N\left[-d + \sigma\sqrt{T}\right],$$

where

$$d = \frac{rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

18.6 Exercises

In all exercises below we assume a standard Black–Scholes model.

Exercise 18.1 An "all-or-nothing" contract, with delivery date T, and strike price K, will pay you the amount K, if the price of the underlying stock exceeds the level L at some time during the interval $[0, t]$. Otherwise it will pay nothing. Compute the price, at $t < T$, of the all-or-nothing contract. In order to avoid trivialities, we assume that $S(s) < L$ for all $s \leq t$.

Exercise 18.2 Consider a binary contract, i.e. a T-claim of the form

$$\mathcal{X} = I_{[a,b]}(S_T),$$

where as usual I is the indicator function. Compute the price of the down-and-out version of the binary contract above, for all possible values of the barrier L.

Exercise 18.3 Consider a general down-and-out contract, with contract function Φ, as descibed in Section 18.2.1. We now modifiy the contract by adding a fixed "rebate" A, and the entire contract is specified as follows:

- If $S(t) > L$ for all $t \leq T$ then $\Phi(S(T))$ is paid to the holder.
- If $S(t) \leq L$ for some $t \leq T$ then the holder receives the fixed amount A.

Derive a pricing formula for this contract.
Hint: Use Proposition 18.4.

Exercise 18.4 Use the exercise above to price a down-and-out European call with rebate A.

Exercise 18.5 Derive a pricing formula for a down-and-out version of the T contract $\mathcal{X} = \Phi(S(T))$, when S has a continuous dividend yield δ. Specialize to the case of a European call.

18.7 Notes

Most of the concrete results above are standard. For barrier options we refer to Rubinstein and Reiner (1991), and the survey in Carr (1995). Two standard papers on lookbacks are Conze and Viswanathan (1991), and Goldman *et al.* (1979). See also Musiela and Rutkowski (1997). The general Theorem 18.8 and its extensions were first published in Björk (1998).