MFE230Q: Final Exam, May 21, 2012

NAME: **ID:**

Please motivate your answers. Please underline your final answers.

Question	Score
1	
2	
3	
4	
Total:	

1. *Discrete model (25 points)*: Consider the following three-date market, with three assets, B, S, P, which can be interpreted as a risk-free bond, a stock, and a put option on the stock, respectively. The assets are traded at $t = 0, 1, 2$. There are five states, $\omega_1, \dots, \omega_5$. At time $t = 1$ (at which point the assets are traded), it is determined whether the economy is in a boom (u) or in a recession (d). Given that the economy is in a boom, the stock price can move up (in state ω_1) or down (in state ω_2). In either case the option expires out of the money. If the economy is in a recession, on the other hand, there are three possible outcomes for the stock. It can go up, moderately down, or significantly down to the point that the underlying firm defaults and becomes worthless. These are states ω_3, ω_4, and ω_5, respectively. In states ω_4 and ω_5, the option expires in the money. The probabilities for the different states are $\mathbb{P}(\omega_1) = 0.1$, $\mathbb{P}(\omega_2) = 0.2$, $\mathbb{P}(\omega_3) = 0.5$, $\mathbb{P}(\omega_4) = 0.1$, and $\mathbb{P}(\omega_5) = 0.1$. The prices of the three assets in different states and times are summarized below.

B(0)	*S(0)*	*P(0)*		*B(1)*	*S(1)*	*P(1)*		*B(2)*	*S(2)*	*P(2)*	*State*
							$\nearrow$	144	240	0	ω_1
			u	120	150	0					
			$\nearrow$				$\searrow$	144	120	0	ω_2
100	100	20									
			$\searrow$				$\nearrow$	120	200	0	ω_3
			d	120	100	40	$\rightarrow$	120	100	20	ω_4
							$\searrow$	120	0	120	ω_5

(a) Define the filtration for this market.

(b) Is the market complete?

(c) An insurance company is considering introducing a credit default swap on the firm's default risk. Specifically, they would sell an insurance contract that pays a hundred dollars at $t = 2$ in the case that the (stock-S) firm defaults (in state ω_5), and makes no payment otherwise. What is the $t = 0$ market price of such a credit default swap?

(d) Assume that the insurance company wishes to hedge the risk of the credit default swap in the market. How could it do this with a dynamic portfolio trading strategy?

2. *Black-Scholes (20 points)*: Consider the standard Black-Scholes economy with a risky and a risk-free asset,

$$\begin{aligned} \frac{dB_t}{B_t} &= r dt, \\ \frac{dS_t}{S_t} &= \hat{\mu} dt + dW. \end{aligned}$$

All the standard assumptions (no transaction costs, no arbitrage, etc.) are satisfied. Assume that an investor wishes to create a simple contingent claim with payoff $\Phi(S_T)$ at time T, by using dynamic portfolio trading.

(a) Formulate a dynamic, self-financing, trading strategy, $\mathbf{h}_t = (h_t^B, h_t^S)'$, $0 \leq t \leq T$, that allows the investor to replicate the payoff of the contingent claim, by trading in the bond and the stock.

(b) Prove that the trading strategy in (a) does indeed replicate the payoff of the contingent claim.

(c) What is the time-0 price in this market of a so-called "power" contingent claim, that makes the terminal payoff $\Phi(S_T) = S_T^2$?

3. *Dividends (20 points)*: Consider the "Black Scholes" economy where the stock pays a constant dividend yield δ.

$$\begin{aligned} \frac{dB}{B} &= r\,dt, \qquad r > 0, \\ \frac{dS + \delta S\,dt}{S} &= \hat{\mu}\,dt + \sigma\,dW. \end{aligned}$$

where $\hat{\mu}$, σ, and r are all constant. Now, consider a call option that has no maturity date, but has strike K and will be exercised the first time the stock price reaches S^*. Hence, the cash flow when this *first hitting time* occurs is $(S^* - K)$ Here, S^* has been chosen such that it is greater than K.

(a) Determine the differential equation that the value of this call, $C(S)$, satisfies.

(b) Solve for the call price.

(c) Now assume that the buyer of the option is allowed to choose for herself the S^* at which she will elect to exercise it. Determine this optimal S^*.

4. *Term structure (20 points)*: Assume that the short rate follows the asset pricing dynamics specified by the CIR model:

$$dr_t = a(b - r)dt + \sigma\sqrt{r}dW^Q.$$

This model belongs to the class of affine term structure models, implying that the price of a T-bond is on the form

$$p(t, T|r_t) = e^{A(T-t) - B(T-t)r_t}.$$

(a) State the ODEs that determine the functions $B(\cdot)$ and $A(\cdot)$, respectively.

(b) Verify that the functions

$$\begin{aligned} B(x) &= \frac{2(e^{qx} - 1)}{(q + a)(e^{qx} - 1) + 2q}, \\ A(x) &= \frac{2ab}{\sigma^2} \ln\left(\frac{2qe^{(q+a)x/2}}{(q + a)(e^{qx} - 1) + 2q}\right), \end{aligned}$$

where $q = \sqrt{a^2 + 2\sigma^2}$, solve the ODEs stated in (a).