MFE230Q: In-Class Quiz # 1

1. Consider the following one-period, two-state economy:

$$\mathbf{s}^0 = (2,2)^T, \qquad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 2.5 & 2.5 \end{bmatrix}$$

   What is the price of a put option on asset 1, with a strike price of 2.5?

2. A forward contract on an underlying stock specifies an agreement where the buyer gets to purchase the stock at a pre-specified price, F, in the future. In contrast to a call option, the buyer *must* buy the stock. The forward price, F is set so that the value of the contract today is zero.

   Consider a stock that does not pay dividends, with current price S. The one-period risk-free rate is r, and the gross rate is then $R = 1 + r$. The market admits no arbitrage. Consider a contract that states:

   "*The buyer of this contract shall buy one unit of the stock at price F one period from now.*"

   Derive the forward price, F, that makes the current value of this contract zero.

3. In the HW, you will show that it is never optimal to exercise a call option early on a stock that pays no dividends. However, there are times when it is optimal to exercise a put early, whether or not the firm pays dividends. Intuitively, the difference is that, for a call, you *owe* K if you exercise, and due to the time value of money, it is better to pay later than pay now. In contrast, for a put, you are *owed* K, and due to the time value of money, it is better to receive now than receive later.

   To price American securities, we start off exactly as we do for European securities; namely, we start at the last date and work backwards, where for each possible event we replicate the CF's of the derivative we would like to price. What is new in pricing American options is that, after we determine the so-called **Continuation Value** of the derivative, we then check whether immediate exercise is more valuable than not exercising immediately. Thus, for example, suppose we were examining the value of a put option assuming that the down-state occurs, and we have determined the continuation value $P_{cont,\downarrow}$. The value of the American put will be

$$P_{\downarrow} = \max\left[P_{(cont,\downarrow)}, (K - S_{\downarrow})\right]. \tag{1}$$

   We then proceed working back through the tree as normal.

(a) With that in mind, assume a 2-period, 3-date model as below.

$$
\begin{array}{ccccc}
 & & & & u^2S(0),\ B(0)(1+R)^2,\ CF(\uparrow\uparrow) \\
 & & & \nearrow & \\
 & & uS(0),\ B(0)(1+R),\ V(\uparrow) & & \\
 & \nearrow & & \searrow & \\
\text{S}(0),\ \ \text{B}(0),\ \ \text{V}(0) & & & & udS(0),\ B(0)(1+R)^2,\ CF(\uparrow\downarrow) \\
 & \searrow & & \nearrow & \\
 & & dS(0),\ B(0)(1+R),\ V(\downarrow) & & \\
 & & & \searrow & \\
 & & & & d^2S(0),\ B(0)(1+R)^2,,\ CF(\downarrow\downarrow)
\end{array}
$$

Assume $S(0) = 100$, $B(0) = 100$, $u = 1.1$, $d = .9$, $R_{_F} = .02$. Determine the price of the **European** put option with strike $K = 101$ and $T = 2$ using the risk-neutral pricing formula:

$$C(0) = \left(\frac{1}{1+R_{_F}}\right)^2 \mathrm{E}_{_0}^{Q}\left[C(2)\right]. \tag{2}$$

(b) Determine the price of an **American** put option with strike $K = 101$ and maturity $T = 2$.

4. Consider a two-period market, $t \in T = \{0, 1, 2\}$, and the following probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}) = \frac{1}{3}$, and

$$
\begin{aligned}
\mathcal{F} &= \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}) \\
&= \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\},
\end{aligned}
$$

with the filtration

$$
\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\}, \\
\mathcal{F}_1 &= \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}, \\
\mathcal{F}_2 &= \mathcal{F}.
\end{aligned}
$$

There are two assets in the economy, and their dynamics are summarized by the price dividend pair $(\delta, \mathbf{s})$. Here $\delta = (\delta_1, \delta_2)^T$ and $\mathbf{s} = (s_1, s_2)^T$, and the functions $\delta_i : T \times \Omega \to \mathbb{R}$, $s_i : T \times \Omega \to \mathbb{R}$, $i = 1, 2$, are defined by

s_1	$t=0$	$t=1$	$t=2$
ω_1	1	1	1
ω_2	1	0.5	1
ω_3	1	0.5	0.5

δ_1	$t=0$	$t=1$	$t=2$
ω_1	0	1	0
ω_2	0	0	0
ω_3	0	0	0

s_2	$t=0$	$t=1$	$t=2$
ω_1	1	1.1	1.1
ω_2	1	1.1	1.21
ω_3	1	1.1	1.21

δ_2	$t=0$	$t=1$	$t=2$
ω_1	0	0	0
ω_2	0	0	0
ω_3	0	0	0

Here, the element on the ith row and jth column represents the value of the function at $t = j$ in state ω_i.

(a) Is the price dividend pair adapted?

(b) How would this economy be represented in the more familiar "tree" view?

(c) Derive the short-term risk-free process.

(d) Is the market complete?

(e) Is the market arbitrage free?

(f) Derive the stochastic discount factor (SDF) process.

(g) Derive the equivalent martingale measure (EMM).

(h) Consider the trading strategy $\mathbf{h} = (h_1, h_2)^T$, defined by

h_1	$t=0$	$t=1$	$t=2$
ω_1	2.2	3.3	0
ω_2	2.2	0	0
ω_3	2.2	0	0

h_2	$t=0$	$t=1$	$t=2$
ω_1	-1	0	0
ω_2	-1	0	0
ω_3	-1	0	0

- What is the dividend process, $\delta^{\mathbf{h}}$?
- Is this process consistent with the equivalent martingale measure?

Solutions

1. Note that the second asset is risk-free, with $R = 1.25$, so this is just our binomial model again. Define $u = \mathbf{D}_{11}/\mathbf{s}_1^0 = 1.5$ and $d = \mathbf{D}_{12}/\mathbf{s}_1^0 = 0.5$. Then the risk neutral probability for an up move is $(R-d)/(u-d) = 0.75$, the payoff of the option is 0 after an up move $(\max(0, 2.5-3))$ and 1.5 after a down move $(\max(0, 2.5-1))$, so the price of the put option, using the risk-neutral valuation formula is $\frac{1}{R}(0.75 \times 0 + 0.25 \times 1.5) = 0.3$.

2. For an arbitrary F, the value of the contract in the next period is $S(1) - F$. This will also be the value in the next period of a portfolio of 1 stock and $-F/R$ dollars invested in bonds (i.e., borrowing F/R). The current value of such a portfolio is $S - F/R$, and by choosing $F = SR$, the current value is zero.

   By the LOOP, this must also be the value of the forward contract. Thus, the forward price is $F = SR$.

3. (a) The risk-neutral probabilities of an up-state and down-state are

$$\pi_\uparrow = \frac{1.02 - 0.9}{1.1 - 0.9} = 0.6, \qquad \pi_\downarrow = \frac{1.10 - 1.02}{1.1 - 0.9} = 0.4.$$

   The payoffs of the European put are

$$P_{\uparrow\uparrow} = \max(0, 101 - 121) = 0, \qquad P_{\uparrow\downarrow} = \max(0, 101 - 99) = 2$$
$$P_{\downarrow\uparrow} = \max(0, 101 - 99) = 2, \qquad P_{\downarrow\downarrow} = \max(0, 101 - 81) = 20.$$

   Hence, the value of the European put is

$$P(0) = \left(\frac{1}{1.02}\right)^2 \Big[(.6)(.6)(0) + (.6)(.4)(2) + (.4)(.6)(2) + (.4)(.4)(20)\Big] = 3.998. \tag{3}$$

   (b) The continuation price of the put option if the up-state occurs is

$$P^A_{cont,\uparrow} = \left(\frac{1}{1.02}\right)[(.6)(0) + (.4)(2)] = \frac{0.8}{1.02}. \tag{4}$$

   In the up-state, the stock is worth $S_\uparrow = 110$, implying that the put holder would lose money $(101 - 110) = -9.0$ if she were to exercise in this event. Clearly, it is optimal not to, so

$$P^A_\uparrow = P^A_{cont,\uparrow} = \frac{0.8}{1.02}. \tag{5}$$

   Similarly the continuation price of the put option if the down-state occurs is

$$P^A_{cont,\downarrow} = \left(\frac{1}{1.02}\right)[(.6)(2) + (.4)(20)] = \frac{9.2}{1.02}. \tag{6}$$

In the down-state, the stock is worth $S_{\downarrow} = 90$, implying that the put holder would make $(101 - 90) = 11.0$ if she were to exercise in this event. Since this is more valuable than the continuation value, it follows that

$$P^A_{\downarrow} = \max\left(\frac{9.2}{1.02}, 11\right) = 11.0. \tag{7}$$

We have thus reduced the two-period model down to a one-period model, with $(P^A_{\uparrow},\, P^A_{\downarrow}) = (\frac{0.8}{1.02}, 11)$. We now work our way back to date-0. The continuation value is

$$P^A_{cont,\,0} \;=\; \left(\frac{1}{1.02}\right)\left[(.6)\left(\frac{0.8}{1.02}\right) + (.4)(11)\right] \;=\; 4.77. \tag{8}$$

At date-0, the stock is worth $S_0 = 100$, implying that the put holder would make $(101 - 100) = 1.0$ if she were to exercise today. Since this is less valuable than the continuation value, it follows that it is not optimal to exercise today, and that the value of the American put is

$$P^A_0 = \max(4.77, 1) = 4.77. \tag{9}$$

Note that this is more valuable than the European put price determined above.

4. (a) Yes, both δ and $\mathbf{s}$ are adapted. Consider s_1, and start with $t = 0$. The finest resolution in $\mathcal{F}_0$ is Ω, and since $s_1(0, \omega) = 1$ for all $\omega \in \Omega$, $s_1(0, \cdot)$ is $\mathcal{F}_0$ measurable. Equivalently, $s_1^{-1}(0, [a, b])$ is either $\emptyset$ (if $1 \notin [a, b]$) or Ω (if $1 \in [a, b]$), so indeed $s_1^{-1}(0, [a, b]) \in \mathcal{F}_0$, using the general definition for a $\mathcal{F}_0$-measurable function (see page 55 of lecture notes). Similarly, for $t = 1$, $s_1(1, \omega_1) = 1$ and $s_1(1, \omega_2) = s_1(1, \omega_3) = 0.5$, so $s_1(1, \cdot)$ is indeed constant on all sets of the finest resolution of $\mathcal{F}_1$ (i.e., on $\{\omega_1\}$ and on $\{\omega_2, \omega_3\}$), and trivially on $\mathcal{F}_3 = 2^{\Omega}$.

   Similar arguments hold for δ_1, s_2 and δ_2.

   (b) From the filtration, it follows that the following tree representation of the economy is valid (where we only write out δ-elements that are nonzero).

$$\begin{array}{ccccc}
 & & s_1(1,\{\omega_1\}),\, s_2(1,\{\omega_1\}),\, \delta_1(1,\{\omega_1\}) & \longrightarrow & s_1(2,\{\omega_1\}),\, s_2(2,\{\omega_1\}) \\
 & \nearrow & & & \\
s_1(0,\Omega),\, s_2(0,\Omega), & & & & s_1(2,\{\omega_2\}),\, s_2(2,\{\omega_2\}) \\
 & \searrow & & \nearrow & \\
 & & s_1(1,\{\omega_2,\omega_3\}),\, s_2(1,\{\omega_2,\omega_3\}) & & \\
 & & & \searrow & \\
 & & & & s_1(2,\{\omega_3\}),\, s_2(2,\{\omega_3\})
\end{array}$$

(c) We note that asset s_2 is a per-period risk-free asset, since an investment in s_2 at t has a deterministic value at $t+1$, for all $t = 0, 1$ and $\omega \in \Omega$. Given the definition of the short-term process, it is then straightforward to derive that $R_{0,1} = 1.1$, $R_{1,2}(\omega_1) = 1$, $R_{1,2}(\omega_2) = R_{1,2}(\omega_3) = 1.1$.

(d) Yes, it follows from the tree representation that the market is period-by-period complete, and thereby the dynamic market is complete.

(e) Yes, the market is free from arbitrage. Again, a period by period argument can be made for why the prices of all Arrow-Debreu securities are positive. Note that in state ω_1, there are two assets at $t = 1$, but since they both generate the same return, there is no abritrage although the number of assets is greater than the number of states at $t = 2$.

(f) The SDF-process is the multi-period version of the one-period SDF, and specifically holds period-by-period. It can therefore be derived by studying the one-period value dynamics of each asset, including dividends, using the one-period noarbitrage argument. Specifically, setting $M_0 = 1$ (the standard normalization), we proceed by finding the values of M in different states that price the Arrow-Debreu securities correctly. For example, between $t = 0$ and $t = 1$, we use the standard binomial model, with $u = 2$, $d = 0.5$, and $R = 1.1$ to derive that the price of an asset that pays one dollar in state ω_1 at $t = 1$ is $\frac{R-d}{R(u-d)} = 0.3636$, and solving

$$0.3636 = \frac{1}{M_0}\left(M_1(\omega_1) \times \frac{1}{3} \times 1\right),$$

leads to $M_1(\omega_1) = 1.0909$.

Similarly, the Arrow-Debreu security that pays off 1 at $t = 1$ in states $\{\omega_1, \omega_2\}$ (remember that at $t = 1$, the filtration is not fine enough to separate between these states, so the "Arrow-Debreu" securities must be defined so that they pay out in both states), has price $\frac{u-R}{R(u-d)} = 0.5455$, and since $\mathbb{P}(\{\omega_2, \omega_3\}) = \frac{2}{3}$, we get

$$0.5455 = \frac{1}{M_0}\left(M_2(\omega_2) \times \frac{2}{3} \times 1\right),$$

leading to $M_1(\omega_2) = M_1(\omega_3) = 0.8182$.

It is immediately clear that the price at $t = 1$ (in ω_1) of an A-D security that pays off 1 in ω_1 at $t = 2$ is 1, leading to $M_2(\omega_1) = M_1(\omega_1) = 1.0909$, and a similar argument as for M_1, using the conditional probabilities $\mathbb{P}(\{\omega_2\}|\{\omega_2, \omega_3\}) = \mathbb{P}(\{\omega_3\}|\{\omega_2, \omega_3\}) = \frac{1}{2}$ leads to $M_2(2, \omega_2) = 0.1488$, $M_2(2, \omega_3) = 1.3388$. Thus, in total

M	$t = 0$	$t = 1$	$t = 2$
ω_1	1	1.0909	1.0909
ω_2	1	0.8182	0.1488
ω_3	1	0.8182	1.3388

(g) We can back out the EMM from the Arrow-Debreu prices of $t = 2$ securities, using the SDF formula. For example, the price at $t = 0$ of an A-D security that pays out at $t = 2$ is $\frac{1}{M_0}\mathbb{P}(\{\omega_1\})M_2(\omega_1) \times 1 = 0.3636$. But the EMM formula then gives $0.3636 = \frac{1}{R_{0,1}R_{1,2}(\omega_2)}\mathbb{Q}(\{\omega_1\}) \Rightarrow \mathbb{Q}(\{\omega_1\}) = 0.4$. A similar argument for ω_2 and ω_3 leads to

	$\mathbb{Q}(\{\omega_i\})$
$i = 1$	0.4
$i = 2$	0.06
$i = 3$	0.54

(h) From the definition of $\delta^{\mathbf{h}}$ (page 68 of lecture notes), it follows immediately that:

h_1	$t = 0$	$t = 1$	$t = 2$
ω_1	-1.2	0	3.3
ω_2	-1.2	0	0
ω_3	-1.2	0	0

From the EMM, it follows that the price at $t = 0$ of an asset that pays 3.3 in ω_1 at $t = 2$ is $\frac{1}{1.1} \times 0.4 \times \times 3.3 = 1.2$ and this is indeed the $t = 0$-investment (minus the dividends) of the trading strategy, so it is indeed consistent.