MFE230: In Class Quiz #4

1. Consider the standard B-S economy,

$$\begin{aligned}\frac{dB}{B} &= rdt, \\ \frac{dS}{S} &= rdt + \sigma\, dW^Q,\end{aligned}$$

r and σ constants. Consider a time-T down-and-out call option with barrier equal to the strike price K. i) What is the price of the option, given parameters: $r = 0.1$, $\sigma = 0.2$, $T = 1$, $K = 100$, $S_0 = 125$? ii) Compare the price with that of a plain vanilla call option (without barrier). iii) What is the price of a down-and-in call with the same barrier and strike price?

2. Consider a BS economy with a stock that pays a constant dividend yield δ:

$$\frac{dS + \delta S\, dt}{S} = \mu\, dt + \sigma\, dW^P \tag{1}$$

$$\frac{dB}{B} = r\, dt. \tag{2}$$

(a) Determine the date-0 price of a forward contract with forward price K via

$$V_{forward} = \mathrm{E}_0^Q\left[e^{-rT}\,(S(T) - K)\right] \tag{3}$$

(b) Using the forward contract, the stock, and the call option, where the call price is

$$C = S(0)\, e^{-\delta T}\, N(x - \delta\sqrt{T}) - K\, e^{-rT}\, N(x - \delta\sqrt{T} - \sigma\sqrt{T}), \tag{4}$$

where as usual $x = \frac{\ln(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$, use a static arbitrage argument to identify the price of the put. Hint: you cannot use the stock in your static hedge as you did when there was no dividend. Why not??

3. Consider the following economy in which there are two indices that evolve according to Brownian motions:

$$\begin{aligned}dz_1 &= \frac{1}{5}dt + dW_1, \\ dz_2 &= \frac{2}{5}dt + 2dW_2,\end{aligned}$$

where $z_1(0) = z_2(0) = 0$. Here, W_1 and W_2 are independent Wiener processes. The indices could, e.g., represent log-price indices, or log-inflation. There is a risk-free asset, which evolves according to

$$\frac{dB}{B} = rdt,$$

$r = \frac{1}{5}$, and two stocks with dynamics

$$\begin{aligned}\frac{dS_1}{S_1} &= \frac{1}{5}dt + \frac{1}{3}dz_1 + \frac{2}{3}dz_2,\\ \frac{dS_2}{S_2} &= \frac{8}{15}dt + \frac{2}{3}dz_1 + \frac{1}{2}dz_2,\end{aligned}$$

$S_1(0) = S_2(0) = 1$. Calculate the price, P, of a digital index insurance contract that pays out a thousand dollars at $T = 1$ if $z_1 < 0$, i.e., the price of a simple contingent claim with payoff function $\Phi(z_1(T)) = 1000 \times (1_{z_1<0})$.

Solutions

1. i) The general formula for the value of a down-and-out contract with payout function Φ and barrier L is

$$F_O(S; L, \Phi) = F(S; \Phi_L) - \left(\frac{L}{S}\right)^q F(L^2/S; \Phi_L), \qquad q = \frac{2r}{\sigma^2} - 1,$$

where $\Phi_L(S) = \Phi(S) \times 1_{S>L}$ and F is the solution to the problem without a barrier, see Björk, page 268. The derivation of the formula follows similar steps as what we have done in class. For this example, since $\Phi(S) = \max(S - K, 0)$, which is zero below the barrier, it follows that $\Phi_K(S) = \Phi(S) = \max(S - K, 0)$. Thus, the solution is

$$F_O(S; K, \Phi) = F(S; \Phi) - \left(\frac{K}{S}\right)^q F(K^2/S; \Phi), \qquad q = \frac{2r}{\sigma^2} - 1,$$

where F is the value we get from the standard BS call option formula. Plugging in the parameters, we get $q = 4$, $x_S = \frac{\ln(1.25)+(0.1+0.04/2)}{0.2} \approx 1.715$, $x_{K^2/S} = \frac{\ln(0.8)+(0.1+0.04/2)}{0.2} \approx -0.516$, so

$$\begin{aligned} F_O(S; K, \Phi) &= 125N(1.715) - 100e^{-0.1}N(1.715 - 0.2) \\ &- 0.8^4(80N(-0.516) - 100e^{-0.1}N(-0.516 - 0.2)) \\ &= 34.99 - 0.8^4 \times 2.79 \\ &= 33.85. \end{aligned}$$

ii) The price of the vanilla call option is the first term in the previous expression, so the discount is 34.99-33.85=1.14 because of the down-and-out-barrier.

iii) A portfolio of the down-and-in call and the down-and-out call together generate the payoff of the vanilla call option, so consequently the price of the down-and-in call is $F_I = F - F_0 = 1.14$. Equivalently, the general formula in Björk, page 276 can be used.

2. (a) You can either solve it directly, using the result we discussed in class for how to adjust B-S for contingent claims when a stock pays constant dividends, or by directly deriving the solution from the risk-neutral expectation formulation. For the first approach, note that the value of the forward contract is the sum of two parts:

$$V_{forward} = e^{-rT}\mathrm{E}_0^Q\left[S(T)\right] - e^{-rT}\mathrm{E}_0^Q\left[K\right]. \tag{5}$$

The first part is a simple contingent claim on $S(T)$, (with the payoff function $\Phi(S(T)) = S(T)$) and since δ is constant, we know from class that the solution is $e^{-rT}\mathrm{E}_0^Q\left[S(T)\right] =$

$e^{-rT}(e^{-\delta T}e^{rT}S(0)) = e^{-\delta T}S(0)$. For the second part, it follows trivially that the value is $-e^{-rT}\mathrm{E}_0^Q[K] = -Ke^{-rT}$, so the total value is

$$V_{forward} \;=\; e^{-\delta T}S(0) - K\,e^{-rT} \tag{6}$$

For the second — brute force — approach, note that the risk neutral dynamics follow

$$\frac{dS + \delta S\,dt}{S} \;=\; r\,dt + \sigma\,dW^Q \tag{7}$$

$$\frac{dB}{B} \;=\; r\,dt. \tag{8}$$

As we have done previously, define $y = \log S$ and use Ito's lemma to get

$$dy = \left(r - \delta - \frac{\sigma^2}{2}\right)dt + \sigma\,dW^Q. \tag{9}$$

Integrating, we get

$$y(T) = y(0) + \left(r - \delta - \frac{\sigma^2}{2}\right)T + \sigma\,W^Q(T). \tag{10}$$

This implies that

$$\begin{aligned}
\mathrm{E}_0^Q[S(T)] &= \mathrm{E}_0^Q\left[e^{y(0)+(r-\delta-\frac{\sigma^2}{2})\,T+\sigma\,W^Q(T)}\right] \\
&= S(0)\,e^{(r-\delta-\frac{\sigma^2}{2})\,T}\,\mathrm{E}_0^Q\left[e^{\sigma\,W^Q(T)}\right] \\
&= S(0)\,e^{(r-\delta)\,T}.
\end{aligned} \tag{11}$$

Again, the date-0 price of a forward contract with forward price K is therefore

$$\begin{aligned}
V_{forward} &= \mathrm{E}_0^Q\left[e^{-rT}\;(S(T) - K)\right] \\
&= e^{-rT}\,\mathrm{E}_0^Q[S(T)] - K\,e^{-rT} \\
&= S(0)\,e^{-\delta T} - K\,e^{-rT}
\end{aligned} \tag{12}$$

(b) If we go long the call and short the put, our final CF's are the same as the forward contract:

$$\begin{aligned}
CF's &= \max(0, S(T) - K) - \max(0, K - S(T)) \\
&= S(T) - K.
\end{aligned} \tag{13}$$

Hence, we can replicate the put by going long the call and short the forward. Thus, the date-0 price of the put is

$$\begin{aligned}
P(0) &= \left[S(0)e^{-\delta T}N(x - \delta\sqrt{T}) - Ke^{-rT}N(x - \delta\sqrt{T} - \sigma\sqrt{T})\right] - \left[S(0)\,e^{-\delta T} - K\,e^{-rT}\right] \\
&= Ke^{-rT}N(-(x - \delta\sqrt{T} - \sigma\sqrt{T})) - S(0)e^{-\delta T}N(-(x - \delta\sqrt{T})),
\end{aligned} \tag{14}$$

where we have used $N(x) = 1 - N(-x)$. We cannot use the stock in the static replication because it pays intermediate dividends, and hence we would not be able to replicate the CF's of the put, which only has CF's at date-T.

3. Since there are two sources of risk and two risky assets (and a risk-free asset), the meta theorem (see Björk, page 122) suggests that the market is complete, and that we therefore can find a unique price of the claim, using standard techniques. We rewrite the stock dynamics in terms of the underlying W risks:

$$
\begin{aligned}
\frac{dS_1}{S_1} &= \frac{1}{5}dt + \frac{1}{3}\left(\frac{1}{5}dt + dW_1\right) + \frac{2}{3}\left(\frac{2}{5}dt + 2dW_2\right), \\
&= \frac{8}{15}dt + \frac{1}{3}dW_1 + \frac{4}{3}dW_2, \\
\frac{dS_2}{S_2} &= \frac{8}{15}dt + \frac{2}{3}\left(\frac{1}{5}dt + dW_1\right) + \frac{1}{2}\left(\frac{2}{5}dt + 2dW_2\right), \\
&= \frac{13}{15}dt + \frac{2}{3}dW_1 + dW_2.
\end{aligned}
$$

We next define the vectors $\mathbf{W} = (W_1, W_2)'$, $\mathbf{S} = (S_1, S_2)'$, $\mu = (8/15, 13/15)'$, and the matrices

$$
\Lambda_S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} 1/3 & 4/3 \\ 2/3 & 1 \end{bmatrix}.
$$

The asset dynamics can then be rewritten on the matrix form

$$
\begin{aligned}
dB &= Br\,dt, \\
d\mathbf{S} &= \Lambda_S \mu\, dt + \Lambda_S \sigma\, d\mathbf{W}.
\end{aligned}
$$

Since σ is invertible, the market is indeed complete, and we can use Girsanov's theorem to create a risk-neutral measure that prices all contingent claims. Specifically, since $z_1 < 0 \Leftrightarrow W_1 < -\frac{1}{5}$, the price of the claim is

$$
P = 1000e^{-1/5} E_0^Q \left[1_{W_1 < -1/5}\right] = 1000e^{-1/5} E_0^P \left[\xi_1 1_{W_1 < -1/5}\right],
$$

where the process ξ_t is chosen such that E^Q is an EMM, i.e., such that the risk-neutral expected returns of all assets are equal to the the risk-free rate, $r = 1/5$. Girsanov's theorem implies that

$$
\xi_t = e^{-\theta' \mathbf{W}_t - t\theta'\theta/2}, \qquad \text{where} \qquad \theta = \sigma^{-1}\left(\mu - r\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -9/5 & 12/5 \\ 6/5 & -3/5 \end{bmatrix}\begin{bmatrix} 5/15 \\ 10/15 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
$$

(see, e.g., part 2.2 of lecture notes).

Putting it together, we arrive at

$$\begin{aligned} P &= 1000\, e^{-1/5} E_0^P \left[e^{-W_1 - 1/2} 1_{W_1 < -1/5} \right] \\ &= 1000\, e^{-1/5} \int_{-\infty}^{-1/5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+2x+1)} d\,x \\ &= 1000\, e^{-1/5} \int_{-\infty}^{-1/5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-(-1))^2} d\,x \\ &= 1000\, e^{-1/5} N\left(-\frac{1}{5} - (-1) \right) \approx 645.3. \end{aligned}$$

Alternatively, since the claim is simple, the price can be calculated by solving the two-dimensional B-S PDE, using the appropriate transformations.