MFE 230Q – Introduction to Stochastic Calculus
GSI Session 1: Extra Problems

PART I:

The economy, **Economy** A, is given by

$$\mathbf{s}^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 3 & 3 & 3 & 3 & 7 \end{bmatrix}$$

We want to see if **(A)** the market is complete, **(B)** there is an arbitrage opportunity? **(C)** there exists a strictly positive state price $\psi \gg 0$

(A)

$\text{rank}(\mathbf{D}) = 2$, so the market is incomplete.

(B)

I will check if the market is arbitrage-free differently from the approach taken in class, which involved trying to find an arbitrage portfolio $\mathbf{h} \in \mathbb{R}^2$ such that

$$V^0 = \mathbf{h}^T \mathbf{s}^0 \leq 0$$
$$\mathbf{V}^1 = \mathbf{h}^T \mathbf{D} > \mathbf{0}$$

$$\left[-V^0, \mathbf{V}^1\right] = \mathbf{h}^T \left[-\mathbf{s}^0 | \mathbf{D}\right] = \mathbf{h}^T \overline{\mathbf{D}}$$
$$> 0$$

I will analyze the state prices i.e. try to see if there exists a strictly positive state price vector $\psi \gg 0$ that solves

$$\mathbf{s}^0 = \mathbf{D}\psi$$

Solution:

$$\mathbf{s}^0 = \mathbf{D}\psi$$

$$\implies \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 3 & 3 & 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix}$$

$$\implies 1 = 5(\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5) \quad (1)$$
$$1 = 3(\psi_1 + \psi_2 + \psi_3 + \psi_4) + 7\psi_5 \quad (2)$$

We manipulate **(1)** and get

$$\frac{1}{5} - \psi_5 = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

We plug in the expression above into **(2)**

$$
\begin{aligned}
1 =& 3\left(\frac{1}{5} - \psi_5\right) + 7\psi_5 \\
\implies \frac{2}{5} =& 4\psi_5 \\
\psi_5 =& \frac{1}{10}
\end{aligned}
$$

We can plug this into **(1)** and get

$$
\begin{aligned}
\frac{1}{5} - \frac{1}{10} =& \psi_1 + \psi_2 + \psi_3 + \psi_4 \\
\implies \psi_1 =& \frac{1}{10} - \psi_2 - \psi_3 - \psi_4
\end{aligned}
$$

We want $\psi \gg 0$, so $\psi_1 > 0$ if we can choose $\psi_2 = \psi_3 = \psi_4 = \frac{1}{100} > 0$, which gives us $\psi_1 = \frac{7}{100}$

As a result, we have found a strictly positive state price vector $\psi \gg 0$:

$$
\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix} = \begin{bmatrix} \frac{7}{100} \\ \frac{1}{100} \\ \frac{1}{100} \\ \frac{1}{100} \\ \frac{10}{100} \end{bmatrix}
$$

By the first FTAP, since there exists a strictly positive state price vector $\psi \gg 0$, the market is arbitrage-free.

(C)

We already found a strictly positive state price vector $\psi \gg 0$!

PART II:

Consider an economy with $\mathbf{N} = 5$ assets and $\mathbf{M} = 4$ states given by

$$
\mathbf{s}^0 = \begin{bmatrix} 1 \\ 2.5 \\ 1.5 \\ 0.75 \\ 0.25 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The augmented payoff matrix is

$$
\overline{\mathbf{D}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2.5 & 1 & 2 & 3 & 4 \\ 1.5 & 0 & 1 & 2 & 3 \\ 0.75 & 0 & 0 & 1 & 2 \\ 0.25 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

We want to know the following:

(a) Is the market complete?

We we have $\mathbf{N} = 5 > \mathbf{M} = 4$ so the *necessary condition* for market completeness is met. Alternatively, $Rank\,(\mathbf{D}) = 4$ is equal to $\mathbf{M} = 4$ and we will be able to span $\mathbb{R}^4$ with the assets and their payoffs given to us. The answer: YES, the market is complete.

(b) Are there any redundant assets?

Since $\mathbf{N} - Rank(\mathbf{D}) = 5 - 4 = 1$, then we have one redundant asset. Asset 2 is a redundant asset.

We can see that the following linear combination generates Asset 2's payoffs (Row 2):

$$1 \cdot \underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}^T}_{\text{Asset 1}} + 1 \cdot \underbrace{\begin{bmatrix}0\\1\\2\\3\end{bmatrix}^T}_{\text{Asset 3}} + 0 \cdot \underbrace{\begin{bmatrix}0\\0\\1\\2\end{bmatrix}^T}_{\text{Asset 4}} + 0 \cdot \underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}^T}_{\text{Asset 5}} = \underbrace{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}^T}_{\text{Asset 2}}$$

(c) Is the market arbitrage-free?

We want to see if there exists an $\psi \gg 0$ so we can apply the first FTAP. We solve the following over-determined system

$$\begin{aligned}
\mathbf{s}^0 &= \mathbf{D}\psi \\
\begin{bmatrix}1\\2.5\\1.5\\0.75\\0.25\end{bmatrix} &= \begin{bmatrix}1&1&1&1\\1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1\end{bmatrix}\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix} \\
\implies 1 &= \psi_1 + \psi_2 + \psi_3 + \psi_4 \\
2.5 &= \psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 \\
1.5 &= 0\psi_1 + 1\psi_2 + 2\psi_3 + 3\psi_4 \\
0.75 &= 0\psi_1 + 0\psi_2 + 1\psi_3 + 2\psi_4 \\
0.25 &= 0\psi_1 + 0\psi_2 + 0\psi_3 + \psi_4
\end{aligned}$$

We solve for ψ recursively given the structure of the equations above:

$$\begin{aligned}
\psi_4 &= 0.25 \\
\psi_3 &= 0.75 - 2\psi_4 = 0.25 \\
\psi_2 &= 1.5 - 2\psi_3 - 3\psi_4 = 1.5 - 0.5 - 0.75 = 0.25 \\
\psi_1 &= 2.5 - 2\psi_2 - 3\psi_3 - 4\psi_4 = 0.25
\end{aligned}$$

Our u*nique* state price vector is in fact strictly positive:

$$\psi = \begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix} = \begin{bmatrix}0.25\\0.25\\0.25\\0.25\end{bmatrix} \gg \mathbf{0}$$

By the first FTAP, the market is arbitrage-free.

What do you notice about ψ and the risk-neutral probabilities $\vec{q}$ that can be obtained from ψ?

NOTE (assuming no redundant assets in the last two cases):

- $\mathbf{N} < \mathbf{M}$: The price system is ***under-determined*** i.e. # unknowns *greater than* # equations.
- $\mathbf{N} = \mathbf{M}$: The price system is ***just-determined*** i.e. # unknowns *equal to* # equations.
- $\mathbf{N} > \mathbf{M}$: The price system is ***over-determined*** # unknowns *less than* # equations.

(d) Can we replicate the Arrow-Debreu security δ^1 from the assets given to us? If yes, find a portfolio with zero units in Asset 1

Since the market is complete, **YES** we can replicate Arrow-Debreu securities: To find this specific portfolio $\mathbf{h}$ that replicates δ^1, we solve the following system:

$$\begin{aligned}
\delta^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= \mathbf{h}^T \mathbf{D} \\
&= \begin{bmatrix} 0 & h_2 & h_3 & h_4 & h_5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\implies 1 &= h_2 \\
0 &= 2h_2 + h_3 \\
0 &= 3h_2 + 2h_3 + h_4 \\
0 &= 4h_2 + 3h_3 + 2h_4 + h_5
\end{aligned}$$

We end up with the portfolio

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

that replicates δ^1!