1. Let $W^P(t)$ be a 1-dimensional Brownian motion. Show that the process $X(t) = e^{\frac{1}{2}t}\cos(W^P(t))$ is a martingale.

   Solution:

   Apply Itô's formula to the function $f(t,w) = e^{\frac{1}{2}t}\cos(w)$. We have $f_t = \frac{1}{2}e^{\frac{1}{2}t}\cos(w)$, $f_w = -e^{\frac{1}{2}t}\sin(w)$, and $f_{ww} = -e^{\frac{1}{2}}\cos(w)$. Plugging into Itô's formula, the dynamics of $X(t)$ are

$$\begin{aligned} dX(t) &= f_t\,dt + f_w\,dW^P(t) + \frac{1}{2}f_{ww}(dW^P(t))^2 \\ &= \frac{1}{2}e^{\frac{1}{2}t}\cos(W^P(t))\,dt - e^{\frac{1}{2}t}\sin(w)\,dW^P(t) - \frac{1}{2}e^{\frac{1}{2}}\cos(w)\,dt \\ &= -e^{\frac{1}{2}t}\sin(w)\,dW^P(t). \end{aligned}$$

   Since there is no dt term, we conclude that $X(t)$ is a martingale.

2. Let $W^P(t)$ be a 2-dimensional Brownian motion, and let the processes $S_i(t), i = 1,2$ have dynamics

$$dS_i(t) = \mu_i S_i(t)\,dt + S_i(t)\sigma_i dW^P(t),$$

   where $\sigma_i = (\sigma_{i,1}, \sigma_{i,2})$ are 2-dimensional row vectors. Compute the dynamics of $S_2(t)/S_1(t)$.

   Solution:

   Apply the two-dimensional version of Itô's formula to the function $f(t,x,y) = \frac{x}{y}$. We have $f_t = 0$, $f_x = \frac{1}{y}$, $f_{xx} = 0$, $f_y = -\frac{x}{y^2}$, $f_{yy} = \frac{2x}{y^3}$, and $f_{xy} = -\frac{1}{y^2}$. Hence,

$$\begin{aligned} d\left(\frac{S_2(t)}{S_1(t)}\right) &= \frac{1}{S_1(t)}\,dS_2(t) - \frac{S_2(t)}{[S_1(t)]^2}\,dS_1(t) + \frac{S_2(t)}{[S_1(t)]^3}\,(dS_1(t))^2 - \frac{1}{[S_1(t)]^2}dS_1(t)\cdot dS_2(t) \\ &= \frac{S_2(t)}{S_1(t)}\left(\mu_2\,dt + \sigma_2 dW^P(t)\right) - \frac{S_2(t)}{S_1(t)}\left(\mu_1\,dt + \sigma_1 dW^P(t)\right) + \frac{S_2(t)}{S_1(t)}\left(\sigma_{1,1}^2 + \sigma_{1,2}^2\right)\,dt \\ &\quad - \frac{S_2(t)}{S_1(t)}\left(\sigma_{1,1}\sigma_{2,1} + \sigma_{1,2}\sigma_{2,2}\right)\,dt \\ &= \frac{S_2(t)}{S_1(t)}\left(\mu_2 - \mu_1 + \sigma_{1,1}^2 + \sigma_{1,2}^2 - \sigma_{1,1}\sigma_{2,1} - \sigma_{1,2}\sigma_{2,2}\right)\,dt + \frac{S_2(t)}{S_1(t)}\left(\sigma_2 - \sigma_1\right)\,dW^P(t) \end{aligned}$$

3. Consider a single stock with dynamics $dS(t) = S(t)r\,dt + S(t)\sigma\,dW^Q(t)$, and assume we also have a risk-free asset with interest rate r. Define $Y(t) := \int_0^t S(u)\,du$. Find the PDE that characterizes the value, V, of an Asian call option on S with strike price K and maturity T. i.e., an option with payoff $\max\{\frac{1}{T}Y(T) - K, 0\}$.

   Solution:

   The stock price cannot be our only state variable since the option is path-dependent, and $S(t)$ does not tell us anything about the history of the stock price (in particular, its sum or average). To address this problem, we augment the state space by also considering the running sum $Y(t)$. The option price will be a function of both S and Y, as well as t.

Apply Itô's formula to $V(t, s, y)$ to get the dynamics of V under the Q measure

$$dV = V_t\,dt + V_s\,dS(t) + \frac{1}{2}V_{ss}(dS(t))^2 + V_y\,dY(t) + \frac{1}{2}V_{yy}\underbrace{(dY(t))^2}_{=0} + V_{sy}\underbrace{dS(t)\cdot dY(t)}_{=0}$$

$$= \left(V_t + rSV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + SV_y\right) dt + \sigma S V_s\,dW^Q(t).$$

We know that the drift must equal rV under the risk-neutral measure. So, equating this with the dt term gives us our PDE

$$rV = V_t + rSV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + SV_y.$$

We also need to specify boundary conditions

$$v(t, 0, y) = e^{-r(T-t)} \max\left\{\frac{y}{T} - K, 0\right\}$$

$$\lim_{y\to 0} v(t, s, y) = 0$$

$$v(T, s, y) = \max\left\{\frac{y}{T} - K, 0\right\}.$$

The first boundary condition might look a little odd at first sight. It requires that if the stock price hits zero ($s = 0$), the value of the option is equal to the discounted value of the payoff under the *current* value of $Y = y$. This is because once the stock hits zero it stays at zero. Hence, for all t after the time τ at which the stock hits zero, we have $Y(t) = Y(\tau)$. In particular, at maturity, we will have $Y(T) = Y(\tau) = y$, which is known at time τ. Therefore, as of time τ the option is risk-free with a known payoff of $\max\left\{\frac{y}{T} - K, 0\right\}$ at time T so we can value it by simply discounting at the risk-free rate.

4. Let the short rate be given by the Cox-Ingersoll-Ross model

$$dr(t) = (a - br(t))\,dt + \sigma\sqrt{r(t)}\,dW^Q(t),$$

where W^Q is a 1-dimensional Brownian motion under the Q measure, and $a, b, \sigma > 0$ are constants. What is the price p of a zero-coupon bond with face value \$1 that matures at time T. A set of ODEs will suffice.

Solution:

We know the price will be a function of t and r. Applying Itô's formula to p gives

$$dp = \left(p_t + (a - br)p_r + \frac{1}{2}\sigma^2 r p_{rr}\right) dt + p_r\sigma\sqrt{r}dW^Q(t).$$

Under the risk-neutral measure, we know that the drift must equal rp so equating them gives the pricing PDE

$$p_t + (a - br)p_r + \frac{1}{2}\sigma^2 r p_{rr} = rp.$$

with terminal (boundary) condition $p(T, r; T) = 1$.

As usual, let's conjecture a solution of the form $p(t,r) = e^{A(t,T)+B(t,T)r}$. Plugging this into the PDE gives

$$\left[-B'(t,T) + bB(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) - 1\right] re^{A(t,T)+B(t,T)r} + \left[A'(t,T) - aB(t,T)\right] e^{A(t,T)+B(t,T)r} = 0,$$

where $'$ denotes differentiation with respect to t. Since this equation must be identically equal to zero for all possible r, we conclude that each of the terms in square brackets must equal zero. This gives us a system of ODEs

$$B'(t,T) = bB(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) - 1$$
$$A'(t,T) = aB(t,T),$$

and the boundary condition from the PDE gives us boundary conditions for A and B: $A(T,T) = B(T,T) = 0$.

It can be shown that these ODEs have a solution, so we have verified that our conjecture for p solves the original PDE. Hence, the bond price is given by $p(t,r;T) = e^{A(t,T)+B(t,T)r}$, where A and B solve the ODEs above.[1]

5. For the economy in the previous question, we now introduce call option on the zero-coupon bond. A European call option with strike price K maturing at time $T_1 < T$ pays off $\max\{p(T_1,r;T) - K, 0\}$ at time T_1. Compute the PDE (and boundary conditions) that the call price must satisfy.

Solution:

Using the same technique as we used for the zero-coupon bond in the previous problem (use Itô's formula on the price and set the drift equal to rc), we conclude that the call price must satisfy the PDE

$$c_t + (a - br)c_r + \frac{1}{2}\sigma^2 r c_{rr} = rc.$$

This is the same PDE as the zero satisfied. The boundary condition is different though. At the maturity date, we require that the call price be equal to the payoff $\max\{p(T_1,r;T) - K, 0\}$, so the boundary condition is

$$c(T_1, r) = \max\{p(T_1,r;T) - K, 0\},$$

which looks like the usual boundary condition for a call on a stock, but with the stock price replaced with the bond price $p(T_1,r;T)$.

There isn't a closed-form solution this time. If you wanted to compute the price of this option, you would solve the PDE numerically.

[1] If you're feeling adventurous you should be able to confirm that the solutions are

$$A(t,T) = -\frac{2a}{\sigma^2}\log\left[\frac{ce^{\frac{1}{2}b(T-t)}}{c\cosh(c(T-t)) + \frac{1}{2}b\sinh(c(T-t))}\right]$$
$$B(t,T) = \frac{\sinh(c(T-t))}{c\cosh(c(T-t)) + \frac{1}{2}b\sinh(c(T-t))},$$

where $c = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$ and sinh and cosh are the hyperbolic sine and hyperbolic cosine functions.

6. Consider an economy with risk-neutral dynamics

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sigma\,dW_1^Q \\ \frac{dV}{V} &= r\,dt + \beta\left(\rho\,dW_1^Q + \sqrt{1-\rho^2}\,dW_2^Q\right) \\ \frac{dB}{B} &= r\,dt, \end{aligned} \tag{1}$$

where $dW_1^Q\,dW_2^Q = 0$.

(a) Determine the dynamics for $\frac{dS}{S}$, $\frac{dV}{V}$, and $\frac{dB}{B}$ under the S-measure – that is, for that probability measure where all assets normalized by S are martingales. Hint: first look at $B^* \equiv \frac{B}{S}$ to identify dW_1^S and then look at $V^* \equiv \frac{V}{S}$ to identify dW_2^S.

(b) Determine the dynamics for $\frac{dS}{S}$, $\frac{dV}{V}$, and $\frac{dB}{B}$ using the PDE approach. That is, use the B/S argument to find the PDE that any derivative $C(t,\,S(t),\,V(t))$ satisfies, and then define $\phi(t,\,S(t),\,V(t))$ implicitly via $C(t,\,S(t),\,V(t)) \equiv S(t)\,\phi(t,\,S(t),\,V(t))$, and then identify the PDE that ϕ satisfies. Finally, use Feynman-Kac to identify the implied dynamics for $\frac{dS}{S}$, $\frac{dV}{V}$, and $\frac{dB}{B}$.

(c) Determine the date-0 price $C(0, S(0),\,V(0))$ of an option that pays at date-T $C(T, S(T),\,V(T)) = S(T)\mathbf{1}_{(S(T)>V(T))}$.

Solution:

a) Applying Ito's lemma to $B^* \equiv \frac{B}{S}$, with $B^*_B = \frac{1}{S}$, $B^*_S = \frac{-B}{S^2}$, $B^*_{BB} = 0$, $B^*_{SS} = \frac{2B}{S^3}$, $B^*_{SB} = \frac{-1}{S^2}$, we find

$$dB^* = \frac{dB}{S} - \frac{B}{S^2}dS + \frac{1}{2}\frac{2B}{S^3}dS^2 + 0 + 0 \tag{2}$$

implying that

$$\begin{aligned} \frac{dB^*}{B^*} &= \frac{dB}{B} - \frac{dS}{S} + \left(\frac{dS}{S}\right)^2 \\ &= \sigma^2\,dt - \sigma\,dW_1^Q \\ &\equiv -\sigma\,dW_1^S, \end{aligned} \tag{3}$$

where the last line should be understood as the definition of dW_1^S. We thus find

$$dW_1^Q = dW_1^S + \sigma\,dt. \tag{4}$$

Thus we can re-express security dynamics as

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sigma\left(dW_1^S + \sigma\,dt\right) \\ &= \left(r+\sigma^2\right)\,dt + \sigma\,dW_1^S \end{aligned}$$

$$\begin{aligned} \frac{dV}{V} &= r\,dt + \beta\rho\left(dW_1^S + \sigma\,dt\right) + \beta\sqrt{1-\rho^2}\,dW_2^Q \\ &= \left(r+\beta\rho\sigma\right)\,dt + \beta\rho\,dW_1^S + \beta\sqrt{1-\rho^2}\,dW_2^Q. \end{aligned}$$

Analogously, applying Ito's lemma to $V^* \equiv \frac{V}{S}$, with $V^*_V = \frac{1}{S}$, $V^*_S = \frac{-V}{S^2}$, $V^*_{VV} = 0$, $V^*_{SS} = \frac{2V}{S^3}$, $V^*_{SV} = \frac{-1}{S^2}$, we find

$$dV^* = \frac{dV}{S} - \frac{V}{S^2} dS + 0 + \frac{1}{2}\frac{2V}{S^3} dS^2 + \frac{-1}{S^2} dS dV \tag{5}$$

implying that

$$\begin{aligned}
\frac{dV^*}{V^*} &= \frac{dV}{V} - \frac{dS}{S} + \left(\frac{dS}{S}\right)^2 - \frac{dS}{S}\frac{dV}{V} \\
&= \left[(r + \beta\rho\sigma)\ dt + \beta\rho\, dW_1^S + \beta\sqrt{1-\rho^2}\, dW_2^Q\right] - \left[\left(r + \sigma^2\right)\ dt + \sigma\, dW_1^S\right] + \sigma^2\, dt - \rho\sigma\beta\, dt \\
&= (\beta\rho - \sigma)\, dW_1^S + \beta\sqrt{1-\rho^2}\, dW_2^Q \\
&\equiv (\beta\rho - \sigma)\, dW_1^S + \beta\sqrt{1-\rho^2}\, dW_2^S
\end{aligned} \tag{6}$$

where the last line should be understood as the definition of dW_2^S. We thus find

$$dW_2^Q = dW_2^S. \tag{7}$$

That is, there is no drift-adjustment for dW_2 under this change of measure.

Putting this altogether, we get

$$\begin{aligned}
\frac{dS}{S} &= \left(r + \sigma^2\right)\ dt + \sigma\, dW_1^S \\
\frac{dV}{V} &= (r + \beta\rho\sigma)\ dt + \beta\rho\, dW_1^S + \beta\sqrt{1-\rho^2}\, dW_2^S \\
\frac{dB}{B} &= r\, dt.
\end{aligned} \tag{8}$$

b) We know that all traded assets have a risk-neutral expected return equal to the risk free rate. Since the option pays no dividend, we have

$$\begin{aligned}
rC &= \frac{1}{dt}\mathrm{E}_t^Q\left[dC\right] \\
&= C_t + rSC_S + rVC_V + \frac{\sigma^2}{2}S^2 C_{SS} + \frac{\beta^2}{2}V^2 C_{VV} + \sigma\beta\rho SVC_{SV}
\end{aligned} \tag{9}$$

Now, from $C(t, S(t), V(t)) \equiv S(t)\ \phi(t, S(t), V(t))$ we find $C_t = S\phi_t$, $C_S = \phi + S\phi_S$, $C_V = S\phi_V$, $C_{SS} = 2\phi_S + S\phi_{SS}$, $C_{VV} = S\phi_{VV}$, $C_{SV} = \phi_V + S\phi_{SV}$. Plugging this into the PDE, we find:

$$rS\phi = S\phi_t + rS\left(\phi + S\phi_S\right) + rVS\phi_V + \frac{\sigma^2}{2}S^2\left(2\phi_S + S\phi_{SS}\right) + \frac{\beta^2}{2}V^2 S\phi_{VV} + \sigma\beta\rho SV\left(\phi_V + S\phi_{SV}\right).$$

Canceling some terms and dividing through by S, we find

$$\begin{aligned}
0 &= \phi_t + rS\phi_S + rV\phi_V + \frac{\sigma^2}{2}S\left(2\phi_S + S\phi_{SS}\right) + \frac{\beta^2}{2}V^2\phi_{VV} + \sigma\beta\rho V\left(\phi_V + S\phi_{SV}\right) \\
&= \phi_t + \left(r + \sigma^2\right)S\phi_S + \left(r + \sigma\beta\rho\right)V\phi_V + \frac{\sigma^2}{2}S^2\phi_{SS} + \frac{\beta^2}{2}V^2\phi_{VV} + \sigma\beta\rho VS\phi_{SV}
\end{aligned} \tag{10}$$

Now, from the Feynman-Kac theorem, we know that the solution to this PDE is also the solution to the expectation

$$\phi(t, S(t), V(t)) = \mathrm{E}_t^* \left[\Phi(S(T), V(T))\right] \tag{11}$$

where, under the *-measure, we have the dynamics

$$\begin{aligned} \frac{dS}{S} &= \left(r + \sigma^2\right) S\,dt + \sigma S\,dW_1^* \\ \frac{dV}{V} &= \left(r + \sigma\beta\rho\right) V\,dt + \beta V \left(\rho\,dW_1^* + \sqrt{1-\rho^2}\,dW_2^*\right). \end{aligned} \tag{12}$$

which can be seen to be identical with dynamics found in 1a. Hence, we can identify this *-measure as that measure where assets discounted by the stock as numeraire are martingales. Indeed, note that $\phi = \frac{C}{S}$, that is, an asset discounted by the stock price, and that eq. (11) shows that it is in fact a martingale under this *-measure.

c) We have shown in class that if all traded assets, normalized by S, are S-martingales, then so is any portfolio of such traded assets. But since derivative securities are simply dynamically-rebalanced portfolios, it follows that derivatives normalized by S are also S-martingales. That is:

$$\begin{aligned} \frac{C(0, S(0), V(0))}{S(0)} &= \mathrm{E}_0^S \left[\frac{C(T, S(T), V(T))}{S(T)}\right] \\ &= \mathrm{E}_0^S \left[\mathbf{1}_{(S(T)>V(T))}\right] \\ &\equiv \pi_0^S \left(S(T) > V(T)\right). \end{aligned} \tag{13}$$

That is, the price of this derivative is equal to $S(0)$ times the S-probability that it ends up in the money. This can be re-written as

$$C(0, S(0), V(0)) = S(0)\,\pi_0^S \left(\log S(T) - \log V(T) > 0\right). \tag{14}$$

As such, it is convenient to define $s(t) = \log S(t)$ and $v(t) = \log V(t)$. As we have done many times previously, we find

$$\begin{aligned} s(T) &= s(0) + \left(r + \frac{\sigma^2}{2}\right) T + \sigma W_1^*(T) \\ v(T) &= v(0) + \left(r + \sigma\beta\rho - \frac{\beta^2}{2}\right) T + \beta\rho W_1^*(T) + \beta\sqrt{1-\rho^2} W_2^*(T). \end{aligned} \tag{15}$$

Thus, we find that the random variable $X|\mathcal{F}_0 \equiv \left(s(T) - v(T)\right)|\mathcal{F}_0$ is normally distributed with mean μ and variance Ω^2 equal to

$$\begin{aligned} \mu &= s(0) + \left(r + \frac{\sigma^2}{2}\right) T - v(0) - \left(r + \sigma\beta\rho - \frac{\beta^2}{2}\right) T \\ &= s(0) - v(0) + \frac{1}{2}\left(\sigma^2 - 2\sigma\beta\rho + \beta^2\right) T \\ \Omega^2 &= \left(\sigma - \beta\rho\right)^2 T + \left(\beta\sqrt{1-\rho^2}\right)^2 T \\ &= \left(\sigma^2 - 2\sigma\beta\rho + \beta^2\right) T. \end{aligned} \tag{16}$$

Thus, the price of the derivative is

$$
\begin{aligned}
C(0, S(0), V(0)) &= S(0) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\Omega^2}} \exp\left\{-\frac{(X-\mu)^2}{2\Omega^2}\right\} \mathbf{1}_{(X>0)} \\
&= S(0) \int_{-\frac{\mu}{\Omega}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \\
&= S(0)\, N\left(\frac{\mu}{\Omega}\right). \qquad (17)
\end{aligned}
$$

7. In discrete time, we argued that an economy was dynamically complete if, at each node, the number N of branches that left the node was equal to the number $M = N$ of traded assets. In continuous time, we have claimed that an economy with N Brownian motions is dynamically complete if there are $M = (N+1)$ traded assets. There seems to be a disparity here, but recall intuitively that $N = 1$ Brownian motion itself is associated with a 2-branch binomial tree.

   We have further claimed that, in continuous time (without jumps), all that matters is mean, variance, and covariance. Now, if we had two independent coins flipped at the same time, and these two coins represent two different stock price movements, it would seem that we would need $2^2 = 4$ branches to capture this model (HH, HT, TH, TT). More generally, for N Brownian motions, where each one can be thought of as a binomial tree with up-value $\left(+\sqrt{dt}\right)$ and down-value $\left(-\sqrt{dt}\right)$, it would seem that we would need 2^N branches. I claim that in fact you only need $(N+1)$ branches to capture mean, variance and covariance. This problem demonstrates this claim for the case $M = 2$.

   Indeed, consider 2 state variables, with dynamics

$$
\begin{aligned}
dx &= \mu\, dt + \sigma\, dW_1 \\
dy &= \alpha\, dt + \nu\, dW_2. \\
dW_1\, dW_2 &= 0. \qquad (18)
\end{aligned}
$$

   Discretizing, we find:

$$
\begin{aligned}
\mathrm{E}_t\left[x(t+dt)\right] &= x(t) + \mu\, dt \\
\mathrm{E}_t\left[y(t+dt)\right] &= y(t) + \alpha\, dt \\
\mathrm{Var}_t\left[x(t+dt)\right] &= \sigma^2\, dt \\
\mathrm{Var}_t\left[y(t+dt)\right] &= \nu^2\, dt \\
\mathrm{Cov}_t\left[x(t+dt),\, y(t+dt)\right] &= 0. \qquad (19)
\end{aligned}
$$

   (a) Show that these five conditions with 2 Brownian motions can be captured with only 3 branches. That is, find the values of $x(t+dt)$ and $y(t+dt)$ on each of the three branches that satisfy these 5 equations. For simplicity, assume each branch has a 1/3-probability of occurring.

   (b) Show how a non-zero correlation $dW_1\, dW_2 = \rho\, dt$ can be accommodated.

Solution:

For simplicity, define

$$
\begin{aligned}
\mu_x &= \mathrm{E}_t\left[x(t+dt)\right] &&= x(t) + \mu\, dt \\
\mu_y &= \mathrm{E}_t\left[y(t+dt)\right] &&= y(t) + \alpha\, dt \\
\sigma_x^2 &= \mathrm{Var}_t\left[x(t+dt)\right] &&= \sigma^2\, dt \\
\sigma_y^2 &= \mathrm{Var}_t\left[y(t+dt)\right] &&= \nu^2\, dt \\
\rho\sigma_x\sigma_y &= \mathrm{Cov}_t\left[x(t+dt),\, y(t+dt)\right] &&= 0.
\end{aligned}
\tag{20}
$$

This last equation implies that in this example $\rho = 0$. We look for a trinomial tree, with each branch having a probability $\pi_{\uparrow} = \pi_{\rightarrow} = \pi_{\downarrow} = \frac{1}{3}$. With free parameters a and b to be determined below, let's look for a solution of the form:

$$
\begin{array}{cl}
 & \mu_x + a\sigma_x, \quad \mu_y + b\sigma_y \\
\nearrow & \\
\rightarrow & \mu_x - a\sigma_x, \quad \mu_y + b\sigma_y \\
\searrow & \\
 & \quad \mu_x, \qquad \mu_y - 2b\sigma_y
\end{array}
$$

With this choice, is is clear that the mean conditions are satisfied:

$$
\begin{aligned}
\mathrm{E}_t\left[x(t+dt)\right] &= \left(\frac{1}{3}\right)(\mu_x + a\sigma_x) + \left(\frac{1}{3}\right)(\mu_x - a\sigma_x) + \left(\frac{1}{3}\right)(\mu_x) \\
&= \mu_x \\
\mathrm{E}_t\left[y(t+dt)\right] &= \left(\frac{1}{3}\right)(\mu_y + b\sigma_y) + \left(\frac{1}{3}\right)(\mu_y + b\sigma_y) + \left(\frac{1}{3}\right)(\mu_y - 2b\sigma_y) \\
&= \mu_y.
\end{aligned}
\tag{21}
$$

The first variance term is

$$
\begin{aligned}
\mathrm{Var}_t\left[x(t+dt)\right] &= \mathrm{E}_t\left[\left(x(t+dt) - \mu_x\right)^2\right] \\
&= \left(\frac{1}{3}\right)(+a\sigma_x)^2 + \left(\frac{1}{3}\right)(-a\sigma_x)^2 + \left(\frac{1}{3}\right)(0)^2 \\
&= \frac{2}{3}a^2\sigma_x^2.
\end{aligned}
\tag{22}
$$

This equals our desired result σ_x^2 if we choose $\frac{2}{3}a^2 = 1$, or $a = \sqrt{\frac{3}{2}}$

The second variance term is

$$
\begin{aligned}
\mathrm{Var}_t\left[y(t+dt)\right] &= \mathrm{E}_t\left[\left(y(t+dt) - \mu_y\right)^2\right] \\
&= \left(\frac{1}{3}\right)(+b\sigma_y)^2 + \left(\frac{1}{3}\right)(+b\sigma_y)^2 + \left(\frac{1}{3}\right)(-2b\sigma_y)^2 \\
&= 2b^2\sigma_y^2.
\end{aligned}
\tag{23}
$$

This equals our desired result σ_y^2 if we choose $2b^2 = 1$, or $b = \sqrt{\frac{1}{2}}$

Finally, our guess automatically generates zero covariance:

$$
\begin{aligned}
\mathrm{Cov}_t\left[x(t+dt),\, y(t+dt)\right] &= \mathrm{E}_t\left[\left(x(t+dt) - \mu_x\right)\left(y(t+dt) - \mu_y\right)\right] \\
&= \left(\frac{1}{3}\right)(+a\sigma_x)\,(+b\sigma_y) + \left(\frac{1}{3}\right)(-a\sigma_x)\,(+b\sigma_y) + \left(\frac{1}{3}\right)(0)\,(-2b\sigma_y) \\
&= 0. \qquad (24)
\end{aligned}
$$

Maybe more useful is that we can approximate 2 independent Brownian motions on a trinomial tree via

$$
\begin{array}{ccc}
 & & \sqrt{\frac{3\,dt}{2}},\ \ \sqrt{\frac{dt}{2}} \\
 & \nearrow & \\
dW_1,\, dW_2 & \rightarrow & -\sqrt{\frac{3\,dt}{2}}\ ,\ \ \sqrt{\frac{dt}{2}} \\
 & \searrow & \\
 & & 0\ ,\ \ -2\sqrt{\frac{dt}{2}}
\end{array}
$$

It is straightforward to show that $\mathrm{E}\left[dW_1\right] = 0$, $\mathrm{E}\left[dW_2\right] = 0$, $\mathrm{Var}\left[dW_1\right] = dt$, $\mathrm{Var}\left[dW_2\right] = dt$, $\mathrm{Cov}\left[dW_1\, dW_2\right] = 0$.

Given this, it is clear how to deal with correlated Brownian motions, since they can always be written as a sum of independent Brownian motions. For example, assume that $dW_1\, dW_3 = \rho\, dt$. Then we can re-write dW_3 as

$$
dW_3 = \rho\, dW_1 + \sqrt{1-\rho^2}\, dW_2, \qquad (25)
$$

where dW_1 and dW_2 are specified as independent. Indeed, we get

$$
\begin{aligned}
\mathrm{E}\left[dW_3\right] &= \rho\,\mathrm{E}\left[dW_1\right] + \sqrt{1-\rho^2}\,\mathrm{E}\left[dW_2\right] \\
&= 0 \\
\\
\mathrm{Var}\left[dW_3\right] &= \left(\rho^2\right)\,\mathrm{Var}\left[dW_1\right] + \left(1-\rho^2\right)\,\mathrm{Var}\left[dW_2\right] + 2\rho\sqrt{1-\rho^2}\,\mathrm{Cov}\left[dW_1,\, dW_2\right] \\
&= dt \\
\\
\mathrm{Cov}\left[dW_1\, dW_3\right] &= \rho\mathrm{Cov}\left[dW_1,\, dW_1\right] + \sqrt{1-\rho^2}\,\mathrm{Cov}\left[dW_1,\, dW_2\right] \\
&= \rho\, dt. \qquad (26)
\end{aligned}
$$

It should also be intuitive that we can model 3 independent Brownian motions via four branches, all with probability = 0.25, with values

$$
\begin{array}{ccc}
\sqrt{a\,dt}, & \sqrt{b\,dt}, & \sqrt{c\,dt} \\
-\sqrt{a\,dt}, & \sqrt{b\,dt}, & \sqrt{c\,dt} \\
0\ , & -2\sqrt{b\,dt}, & \sqrt{c\,dt} \\
0\ , & 0 & -3\sqrt{c\,dt}
\end{array}
$$

A simple calculation shows that $a = 2$, $b = \frac{2}{3}$, $c = \frac{1}{3}$ generates variances equal to dt. From this, it is clear how to generalize to N Brownian motions.