MFE230E Problem Set 1 - Solutions

Question 1, Ruppert Ch.12 Q7

Let Y_t be a stationary AR(2) process,

$$(Y_t - \mu) = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t$$

(a) Without loss of generality, we can assume that $\mu = 0$. Then using the hint:

$$\begin{aligned}\gamma(k) &= Cov(Y_t, Y_{t-k}) = Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t, Y_{t-k}) \\ &= \phi_1 Cov(Y_{t-1}, Y_{t-k}) + \phi_2 Cov(Y_{t-2}, Y_{t-k}) \\ &= \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)\end{aligned}$$

for any $k > 0$. The result follows by dividing through by $\gamma(0)$. (Note that if $k = 0$, there is a non-zero covariance between ϵ_t and Y_{t-k}, which adds another term to the equation.)

(b) Using the result in (a) with $k = 1$, we get $\rho(1) = \phi_1\rho(0) + \phi_2\rho(-1) = \phi_1 + \phi_2\rho(1)$. Using this result with $k = 2$ we get $\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) = \phi_1\rho(1) + \phi_2$.

(c) We have two equations with two unknowns. Plug in $\rho(1) = 0.4$ and $\rho(2) = 0.2$ to find $\phi_1 = 0.3810$ and $\phi_2 = 0.0476$.

It follows that $\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) = 0.3810 \times 0.2 + 0.0476 \times 0.4 = 0.0952$.

Question 2, Ruppert Ch.12 Q14

For difference $d = 0$, the sample path of the original time series y_t exhibits a clear upward trend which suggests that the mean of the process is not constant over time. This violates the first (weak) stationarity criterion. In addition, the ACFs decay to zero only slowly which, indicates either non-stationarity or stationarity with long-memory dependence. Overall, for $d = 0$, we clearly have a non-stationary process.

For first difference, $d = 1$, the sample paths have constant mean and the variance looks constant. In addition, all ACFs are non-significant, which altogether suggests a White-Noise process. Therefore, for $d = 1$, the process looks stationarity.

For second difference, $d = 2$, mean and variance seem constant. ACFs are significant at lags 1,5,6,7 and close to significant at lags 10 and 16 but then decay to zero. The decay of the ACFs is slower here compared to $d = 1$, but the process seems stationary as well. The significant negative ACF at lag 1 is a likely sign of over-differencing (which turns what should be a White-Noise into an MA process).

To further distinguish between $d = 1$ and $d = 2$, a look at the PACFs would be helpful. Additional PACF terms when going from $d = 1$ to $d = 2$ would confirm over-differencing. Independently of having PACFs at hand, $d = 1$ is a better choice since ACFs decay faster and choosing $d = 2$ (without looking at PACFs) would mean taking the likely risk of over-differencing. (Note that in practice we would not stop here but run additional (Augmented) Dickey-Fuller tests.)

Question 3

We can write the process x_t in the following way:

$$x_t = 1.5x_{t-1} - 0.5x_{t-2} + e_t - .5e_{t-1}$$

$$(1 - 1.5L + 0.5L^2)x_t = (1 - 0.5L)e_t$$

$$(1 - 0.5L)(1 - L)x_t = (1 - 0.5L)e_t$$

Pre-multiplying both sided by $(1-0.5L)^{-1}$, which is well-defined, we get:

$$(1-L)x_t = e_t \quad \Leftrightarrow \quad x_t = x_{t-1} + e_t$$

Therefore, x_t is indeed a random walk.

Question 4

(a) If $\phi = \psi$ then:

$(1-\phi L)z_t = (1-\phi L)x_t + (1-\psi L)y_t = e_t + w_t$

Because the sum of two White-Noise processes is itself a White-Noise process, $e_t + w_t$ is a White-Noise and z_t therefore follows an AR(1) process with parameter ϕ.

(b) If $\phi \neq \psi$ then:

$(1-\phi L)(1-\psi L)z_t = (1-\psi L)(1-\phi L)x_t + (1-\phi L)(1-\psi L)y_t = (1-\psi L)e_t + (1-\phi L)w_t$

Let h_t be the process defined by

$h_t \equiv (1-\psi L)e_t + (1-\phi L)w_t$

It is straightforward to verify that h_t is a process with auto-covariance γ_j such that

$\gamma_1 \neq 0$, $\gamma_j = 0\ \forall j > 1$.

If errors are Gaussian, the first two moments (i.e., mean and covariances) of h_t are equal to the first two moments of a MA(1) process. Since the first two moments completely identify the distribution of a Gaussian process, we conclude that h_t follows a MA(1) process. Alternatively, one can show that the sum of an MA(q_1) and an MA(q_2) is MA($\max\{q_1, q_2\}$) even without assuming normal errors (cf. e.g. Hamilton, Section 4.7, p. 106). In conclusion:

$(1-\phi L)(1-\psi L)z_t \sim$ MA(1),

implying that z_t follows an ARMA(2,1) process.

Question 5

We only plot 5 sample paths for an AR(0.9) process, and 5 sample paths for a Random Walk here.

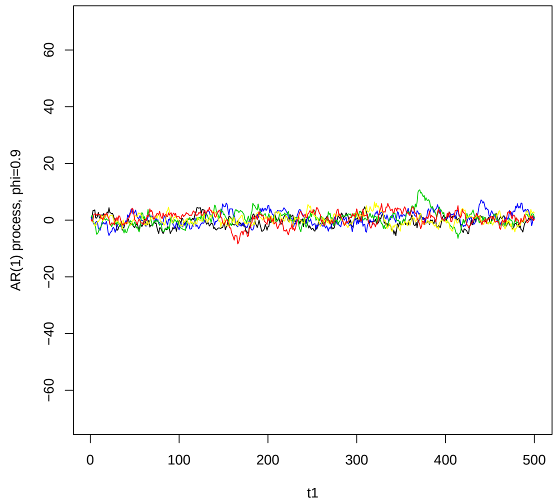

Figure 1: Sample paths for an AR(0.9)

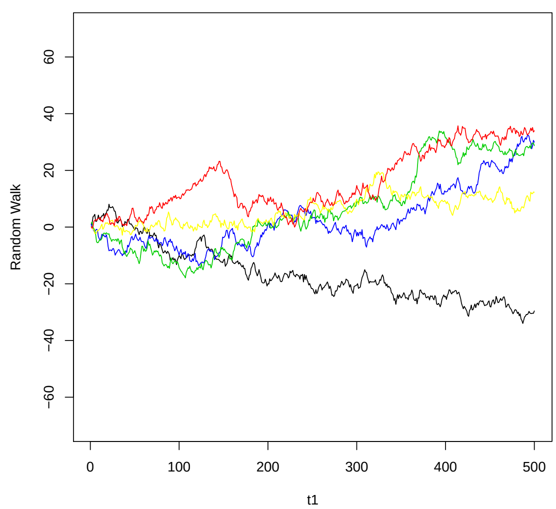

Figure 2: Sample paths for a Random Walk

Note that the scale is the same for both plots. While the AR(0.9) paths clearly exhibit a stationary behavior, i.e. they oscillate around their mean (zero), the plotted Random Walks deviate strongly from their mean and show trends. For those, there is no mean-reversion. This can also be seen in the sample statistics that you were asked to compute.

Question 6

The AR(2) process can be reformulated using lag polynomials:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t$$

$$\Rightarrow (1 - \phi_1 L - \phi_2 L^2) x_t = \varepsilon_t$$

From the Lecture, the AR(2) is stationary if all $|\lambda_i| < 1$, $i = 1, 2$ where the λ_i are such that the process can be formulated as $(1 - \lambda_1 L)(1 - \lambda_2 L^2) x_t = \varepsilon_t$. Let us derive expressions for λ_1 and λ_2 in terms of the parameters ϕ_1 and ϕ_2.

$$(1 - \lambda_1 L)(1 - \lambda_2 L) x_t = \epsilon_t \equiv (1 - \phi_1 L - \phi_2 L^2) x_t = \epsilon_t$$

$$\Leftrightarrow (1 - (\lambda_1 + \lambda_2) L + \lambda_1 \lambda_2 L^2) x_t = \epsilon_t \equiv (1 - \phi_1 L - \phi_2 L^2) x_t = \epsilon_t$$

Equating the coefficients on L and L^2 gives:

$$\lambda_1 + \lambda_2 = \phi_1$$

$$\lambda_1 \lambda_2 = -\phi_2$$

Plugging these equations in one another, if the λ_1 are real, i.e. if $\frac{\phi_1^2}{4} + \phi_2 > 0$, we obtain:

$$\lambda_{1,2} = \frac{\phi_1}{2} \pm \sqrt{\frac{\phi_1^2}{4} + \phi_2}$$

The larger root satisfies:

$$\lambda_2 = \frac{\phi_1}{2} + \sqrt{\frac{\phi_1^2}{4} + \phi_2} < 1 \tag{1}$$

which is equivalent to $\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$ or $\phi_1 + \phi_2 < 1$.

The other root satisfies $|\lambda_1| = \left|\frac{\phi_1}{2} - \sqrt{\frac{\phi_1^2}{4} + \phi_2}\right| < 1$ which is equivalent to

$$\frac{\phi_1}{2} - \sqrt{\frac{\phi_1^2}{4} + \phi_2} < 1 \tag{2}$$

and

$$-\left(\frac{\phi_1}{2} - \sqrt{\frac{\phi_1^2}{4} + \phi_2}\right) < 1 \tag{3}$$

Since 2 is implied by 1, we only need to check under which conditions 3 holds: 3 is identical to $\phi_2 - \phi_1 < 1$.

If $\frac{\phi_1^2}{4} + \phi_2 < 0$, the λ_is are complex (note, this implies $\phi_2 < 0$), and we have:

$$\lambda_2 = \frac{\phi_1}{2} \pm i\sqrt{-\left(\frac{\phi_1^2}{4} + \phi_2\right)}$$

Then $|\lambda_i| < 1$ if $\sqrt{\left(\frac{\phi_1}{2}\right)^2 - \left(\frac{\phi_1^2}{4} + \phi_2\right)} = \sqrt{-\phi_2} < 1$ which is equivalent to $\phi_2 > -1$.

Summarizing the conditions, to ensure the stationarity of the AR(2) process, we need $\phi_2 > -1$, $\phi_1 + \phi_2 < 1$ and $\phi_2 - \phi_1 < 1$.

Ergodicity: We have just demonstrated that the AR(2) process is invertible. Thus, the MA(∞) representation exists and in particular, the coeffcients of the inverse lag polynomial are absolutely summable: we have $(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} = (\theta_0 + \theta_1 L + \theta_2 L^2 + \dots)$ with $\sum_{i=0}^{k} |\theta_i| < \infty$. But we know that every MA(∞) process with absolutely summable coefficients is ergodic. Thus, the AR(2) process is ergodic.

A summary of the conditions is shown in the Figure below. This is taken from Hamilton, in which you have more details about this example on pages 17-18, and 30-32.

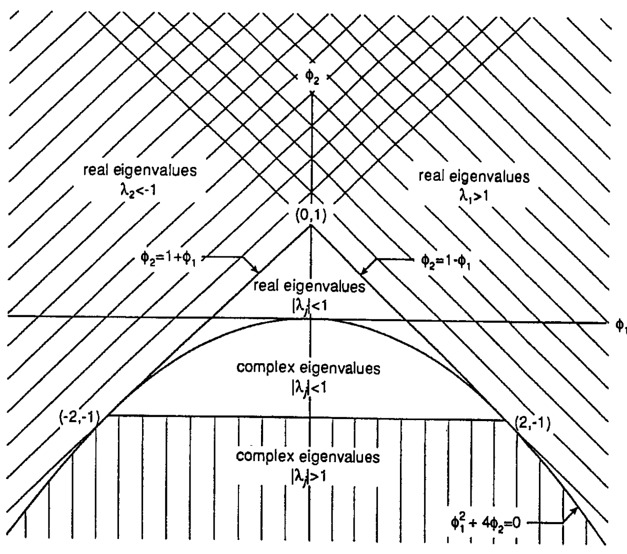

FIGURE 1.5 Summary of dynamics for a second-order difference equation.

Figure 3: Hamilton (1994), p. 17

As a final remark, note that the λs can also be derived in the following equivalent way. From the equation $(1-\lambda_1 L)(1-\lambda_2 L)x_t = e_t$, and as we said in Lecture, the λs are the inverse of the roots of the lag polynomial: $(1-\phi_1 L-\phi_2 L^2)$. Therefore, they can be obtained in two steps.

1. Find the roots of the polynomial by replacing the lag operator by a simple variable z. We want to find the roots of $1-\phi_1 z-\phi_2 z^2$. Using the usual quadratic formulae, this yields:

$$z_1 = \frac{\phi_1 - \sqrt{\phi_1^2+4\phi_2}}{-2\phi_2}$$
$$z_2 = \frac{\phi_1 + \sqrt{\phi_1^2+4\phi_2}}{-2\phi_2}$$

2. Obtain the λs as the inverse of those roots:

$$\lambda_1 = z_1^{-1}$$
$$\lambda_2 = z_2^{-1}$$

You can convince yourself that this is exactly the same results as before, for instance by verifying that $\lambda_1+\lambda_2=\phi_1$ and $\lambda_1\lambda_2=-\phi_2$. Therefore, the rest of the analysis is identical.