1. (15) The put call parity formula relates the prices of a European call to a European put option on the same stock with each option having the same strike price and expiration date. What is this formula, and explain via an arbitrage free argument why it is true?

The put-call parity formula is

$$S + p = c + Xe^{-rt},$$

where S is the price of the stock, p the price of the put option, c the price of the call option, X the strike price, and e^{rt} is the interest rate from now until the option expires at time t. To see that this is an arbitrage free formula, consider two portfolios. Π_1 consists of one share of stock and one put option, and Π_2 consists of one call option and Xe^{-rt} dollars invested so that at expiration this amount grows to $Xe^{-rt}e^{rt} = X$ dollars. At expiration time t there are two possibilities: either the stock is worth more than the strike price or not. In the first case both portfolios are worth S, and in the second case they both are worth X. Since both portfolios are worth the same at time t, they must have equal value at time t. Hence the validity of the above formula.

2. (10) Explain why the stock price model $S_t = S_0 e^{(r-\sigma^2/2)t} e^{\sigma B_t}$ is used when computing the Black-Scholes formula for the price of a European call option.

The idea is to have a price model which satisfies the equation

$$E(S_t) = e^{rt} S_0,$$

where e^{rt} is the interest rate until time t, S_0 the present price of the stock, and $E(S_t)$ is the expected value of the stock price at time t. The reason for this desire is that we can then justify with a replicating portfolio argument that if V_t is the price of an option at time t, then

$$E(V_t) = e^{rt} V_0.$$

Since B_t is Brownian motion we have

$$
\begin{aligned}
E(S_t) &= E\left(S_0 e^{(r-\sigma^2/2)t} e^{\sigma B_t}\right) \\
&= S_0 e^{(r-\sigma^2/2)t} E\left(e^{\sigma B_t}\right) \\
&= S_0 e^{(r-\sigma^2/2)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma\sqrt{t}x} e^{-x^2/2}\, dx \\
&= S_0 e^{(r-\sigma^2/2)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\sigma\sqrt{t})^2/2} e^{\sigma^2 t/2}\, dx \\
&= S_0 e^{rt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\sigma\sqrt{t})^2/2}\, dx = S_0 e^{rt}
\end{aligned}
$$

3. (15) You are given the following bits of data.

day	M	T	W	Th	F
price	58.7	56.85	57	59.6	59
$\ln(S_{i+1}/S_i)$	$-.032$	$.0026$	$.0446$	$-.0101$	

Compute a weekly drift and volatility for this stock. For this example a week consists of 5 days.

We first compute estimations of the expected value and variation of the ln of the stock price ratios.

$$\bar{U} = \frac{1}{4}\sum_{i=1}^{4}\ln(S_{i+1}/S_i) = 0.00127$$

$$S^2 = \frac{1}{3}\sum_{i=1}^{4}\left(\ln(S_{i+1}/S_i) - \bar{U}\right)^2 = 0.00104$$

Since we want a weekly drift and volatility we set $\triangle t = 1/5$. Then

$$\sigma = \frac{S}{\sqrt{\triangle t}} = \frac{\sqrt{0.00104}}{\sqrt{1/5}} = 0.072$$

$$\mu = \frac{\bar{U} + S^2/2}{\triangle t} = \frac{0.00127 + 0.00104/2}{1/5} = 0.00895$$

4. (20) The stock price of Y-Corp is currently $50. What is the price of a European call option which expires in 2 months and which has a strike price of $60? Assume the yearly interest rate is 5.5%, and the yearly volatility of the stock prices is 0.25.

The easiest way to solve this problem is to use the Black-Scholes formula for pricing European calls. First calculate the values d_1 and d_2.

$$d_1 = \frac{\ln(50/60) + (\ln(1.055) + (0.25)^2/2)(1/6)}{\sqrt{1/6}\,(0.25)} \approx -1.6479$$

$$d_2 = d_1 - (0.25)\sqrt{1/6} = -1.711 - (0.25)\sqrt{1/6} \approx -1.7499$$

The price of the call option is

$$
\begin{aligned}
\text{price} &= SN(d_1) - e^{-rt}XN(d_2) \\
&= 50N(-1.6479) - e^{-(\ln 1.055)/6}60N(-1.7499) \\
&= \frac{50}{\sqrt{2\pi}}\int_{-\infty}^{-1.6479}e^{-x^2/2}dx - \frac{59.467}{\sqrt{2\pi}}\int_{-\infty}^{-1.7499}e^{-x^2/2}dx \\
&= \frac{50}{\sqrt{2\pi}}(0.12454) - \frac{59.467}{\sqrt{2\pi}}(0.10042) \\
&= 0.1019
\end{aligned}
$$

5. (25) You have daily closing prices for a stock for the last 6 months. Assume that amounts to 130 bits of data. Explain how to compute the yearly drift and volatility of the stock. Your intention is to build a discrete approximation to a geometric Brownian motion model of the stock prices, and use the discrete approximation to price an option which will expire in one month or 30 days. The yearly interest rate is 6%. Explain how to build the discrete stock price model.

To calculate the yearly drift and volatility of the stock we first have to compute estimations of the expected values and variations of the data $\ln(S_{i+1}/S_i)$, where S_i is one of the daily closing prices of the stock. We will have 129 such terms. The expected values and variations are computed with the following formulas

$$\bar{U} = \frac{1}{129} \sum_{i=1}^{129} \ln(S_{i+1}/S_i)$$

$$S^2 = \frac{1}{128} \sum_{i=1}^{129} \left(\ln(S_{i+1}/S_i) - \bar{U}\right)^2$$

Since we want the annualized drift and volatility and one day is $1/365$ of a year, we set $\triangle t = 1/365$. Thus,

$$\text{drift} = \mu = \frac{\bar{U} + S^2/2}{\triangle t}$$

$$\text{volatility} = \sigma = \frac{S}{\sqrt{\triangle t}}$$

To build our tree of stock prices we need to decide how many levels the tree will have so that we will have a good approximation to the possible stock prices. A large number of steps is advisable, so lets set the number of steps to 90. Thus, each step will represent $1/3$ of a day. That is, $\triangle t = 1/(3*365) = 1/1095$. We next need to define u and d.

$$u = e^{\mu\triangle t + \sigma\sqrt{\triangle t}}$$

$$d = e^{\mu\triangle t - \sigma\sqrt{\triangle t}}$$

All that is left is to decide on a value of S_0, which we can pick to be the latest stock price.

6. (15) You decide to model the price of a stock which is currently selling for \$50 a share with geometric Brownian motion. The yearly interest rate is $e^{0.05}$, and the annualized drift and volatility are $\mu = 0.05$ and $\sigma = 0.2$. What is the probability that the price of the stock will be greater than \$55 in 4 months time?

The model used is

$$S_t = 50e^{(0.05-(0.2)^2/2)t}e^{0.2B_t} .$$

Thus, the probability that the stock price will be greater than \$55 in 4 months is

$$
\begin{aligned}
\text{Prob}\left(S_{1/3} > 55\right) &= \text{Prob}\left(50e^{0.01}e^{0.2\sqrt{1/3}Z} \geq 55\right) \\
&= \text{Prob}\left(Z \geq 5\sqrt{3}\ln(1.1\,e^{-0.01})\right) \\
&= \text{Prob}\left(Z \geq 0.7388\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{0.7388}^{\infty} e^{-x^2/2}\,dx \\
&\approx 0.23
\end{aligned}
$$