b)

The LNM Institute of Information Technology
Jaipur, Rajsthan

Mathematics III (2017), Quiz-II:

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Time: 20 Minutes — Maximum Marks: 10

10

1. Expand $f(z) = \dfrac{1}{z(z^2 - 3z + 2)}$ for the regions $0 < |z| < 1$ and $1 < |z| < 2$. [5]

2. Classify the PDE $z(x+y)z_x + z(x-y)z_y = x^2 + y^2$ and then find the general integral the given PDE. [5]

Q1 $f(z) = \dfrac{1}{z(z-1)(z-2)}$

$f(z)$ has simple Pole at $z=0$, $z=1$ & $z=2$.

Case I given Region

$|z|$ 1

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2(1-z^{-1})} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$f(z) = \frac{(z-1)-(z-2)}{z(z-1)(z-2)} = \frac{1}{z(z-2)} - \frac{1}{z(z-1)}$$

$$\frac{1}{z(z-1)} = \frac{1}{z^2(1-z^{-1})} = \sum_{n=0}^{\infty}\frac{1}{z^{n+2}} \qquad |z|>1$$

$$\frac{-1}{z(1-z)} = -\sum_{n=0}^{\infty} z^{n-1} \qquad |z|<1$$

$$\frac{1}{z(z-2)} = \frac{-1}{2z\left(1-\frac{z}{2}\right)} = -\sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^n}{2z} = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^{n-1} \qquad |z|<2$$

$$\frac{1}{z^2(1-2z^{-1})} = \sum_{n=0}^{\infty}\frac{\left(\frac{2}{z}\right)^n}{z^2} = \frac{2^n}{z^{n+2}} \qquad |z|>2.$$

$$f(z) = -\sum_{n=0}^{\infty} z^{n-1} + \sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}} \qquad 0<|z|<1$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+2}} + \sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}} \qquad 1 = \sum_{n=0}^{\infty}\frac{1}{z^{n+2}} + \frac{z^{n-1}}{2^{n+1}} \qquad 1<|z|<$$

final answer at end of ans 2

Q4(b) | Q5(a) | Q5(b)

Q2 $z(x+y)z_x + z(x-y)z_y = x^2+y^2$

Quasi linear PDE

Applying Lagrange method

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)}$$

$x\,dx - y\,dx = x\,dy + y\,dy \Rightarrow x\,dx - y\,dy = x\,dy + y\,dx$

$d\left(\frac{x^2}{2}\right) - d\left(\frac{y^2}{2}\right) = d(xy)$

Integrating above eq^n

$\frac{x^2}{2} - \frac{y^2}{2} - xy = C_1$

Now $\frac{x\,dx - y\,dy}{zx^2 + xyz - xyz + y^2z} = \frac{z\,dz}{z(x^2+y^2)}$ (multiplying $\frac{dx}{z(x+y)}$ by x both in Numerator & denominator similary dy & z)

$$\frac{x\,dx - y\,dy}{z(x^2+y^2)} = \frac{z\,dz}{z(x^2+y^2)}$$

$\Rightarrow x\,dx - y\,dy = z\,dz$

Integrating both the sides

$\int x\,dx - \int y\,dy = \int z\,dz$

$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = C_2$

$x^2 - y^2 - z^2 = 2C_2 = C_3$

where C_1, C_2 & C_3 are some constants

$f(u, v) = 0$ where $u = u(x, y, z)$, $v = v(x, y, z)$

$u = C_1 = \frac{x^2}{2} - \frac{y^2}{2} - xy$

$v = C_3 = x^2 - y^2 - z^2$

$f\left(\frac{x^2}{2} - \frac{y^2}{2} - xy,\ x^2 - y^2 - z^2\right) = 0$ general solution of PDE

Ques (1) $- \sum_{n=0}^{\infty} z^{n-1}\left(1 + \frac{1}{2^{n+1}}\right)$ $0 < |z| < 1$

$\sum_{n=0}^{\infty} \frac{1}{z^{n+2}} - \frac{z^{n-1}}{2^{n+1}}$ $1 < |z| < 2$

THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MATHEMATICS-III & MTH213
ENDTERM

Time: 3 hours | Date: 27/11/2017 | Maximum Marks: 100

Note: Usual notations are used. Attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **Start a new question on a new page and answer all its parts in the same place.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format.

Question No.			
Page No.			

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1. (a) If $H(z)$ and $K(z)$ are continuous at $z = z_0$, then prove that the functions $G(z) = 5H(z)K(z)$ are also continuous at $z = z_0$ using the definition of continuity ($\delta - \varepsilon$ definition) [6]

   (b) **Prove or Disprove:** The following complex valued functions are analytic at $z = 0$ [8]

   i. $f(z) = e^{-y}\sin x - ie^{-y}\cos x$

   ii. $f(z) = |z|^2 + 3i$

2. (a) Prove the identity $z = \tan\left[\frac{1}{i}\log\left(\frac{iz+1}{iz-1}\right)^{\frac{1}{2}}\right]$. [3]

   (b) Justify whether the function $\sin z$ is a bounded function or not in $\mathbb{C}$. [3]

   (c) Determine the number of zeros, counting multiplicities, of the polynomial $2z^5 - 6z^2 + z + 1$ in the annulus $1 \leq |z| < 2$. [4]

   (d) Find the value of the complex integration $I = \oint_C \frac{Log(z+3) + \cos z}{(z+1)^2} dz$, where C is the closed curve $|z| = 2$ traversed in the counter-clockwise direction. [4]

3. (a) Find the radius of convergence of the power series $\sum a_n(z+1)^n$ where a_n is $\frac{1}{n2^n}$. [6]

   (b) For the function $f(z) = \frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$, determine its isolated singular points and weather these points are poles, removable singular points or essential singular points. Further evaluate $\int_C f(z)dz$, where C is the positively oriented circle centred at 0 with radius $\pi/2$. [8]

4. (a) If isolated singular point z_0 of a function $f(z)$ is a pole of order m, then determine the residue of $f(z)$ at $z = z_0$. [7]

   (b) Use calculus of residue to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$, $(a > b > 0)$. [8]

5. (a) Show that $(x-a)^2+(y-b)^2+z^2=1$ is a complete integral of $z^2(1+p^2+q^2)=1$. Further, find the singular integrals of the given PDE. [6]

(b) Classify the following second order PDE and reduce the equation to canonical form and hence solve it: [10]

$$(n-1)^2 u_{xx} - y^{2n} u_{yy} = n y^{2n-1} u_y,$$

when n is an positive integer.

6. (a) Find the solution of the following problem: [5]

$$\begin{aligned} u_{tt} - u_{xx} - x + t &= 0, & -\infty < x < \infty, t > 0, \\ u(x,0) = x^3, \quad u_t(x,0) &= \cos x & -\infty \le x \le \infty \end{aligned}$$

(b) Classify the following PDE and then solve the problem: [8]

$$\begin{aligned} u_{tt} = 9u_{xx} - u, &\quad 0 < x < \pi,\ t > 0 \\ u(x,0) = x + \sin 2x, u_t(x,0) = 0, &\quad 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, &\quad t \ge 0. \end{aligned}$$

7. (a) Prove that the solution of the following problem, if it exists, is unique: [6]

$$\begin{aligned} u_t - ku_{xx} &= F(x,t), \quad 0 < x < l, t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l \\ u(0,t) &= 0,\ u(l,t) = 0 \quad t \ge 0. \end{aligned}$$

(b) Classify the following PDE and then find the solution: [8]

$$\begin{aligned} u_t &= 4u_{xx}, \quad 0 < x < \pi, t > 0, \\ u(x,0) &= \sin x, \quad 0 \le x \le \pi \\ u(0,t) &= 0,\ u(l,t) = \pi \quad t \ge 0. \end{aligned}$$

Question No	Q1(a)	Q1(b)	Q2(a)	Q2(b)	Q3(a)	Q3(b)	Q4(a)	Q4(b)	Q5(a)	Q5(b)
Page No	1	1	1,2	2	3	3,4	4	5	5,6	6

CODE: ....

S. No. 4512

Ques 6		Ques 7
(a)	(b)	
7	8	8

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2. Candidates shall maintain perfect silence and attend to their Question Paper only. Any conversation or gesticulation or disturbance in the Examination Room / Hall shall be deemed as misbehaviour. If a candidate is found using unfair means or impersonating, it shall be treated as breach of code of conduct and the matter dealt with accordingly.
3. No candidate, without the special permission of the Invigilator concerned, will leave his/her seat or Examination Room until the full duration of the paper is over. Candidate should not leave the room / hall without handing over their Answer Sheets to the Invigilator on duty.
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Name of the student: VISHAKHA DHANWANI.

Roll No. : 16UEC126

Name of Examination : MID TERM

Subject : M3

Day & Date : THURSDAY, 21/09/17

No. of Supplementary Sheets Attached: ....

Student's Signature

Invigilator's Signature

Question No.	Marks Obtained
1	7
2	7
3	05
4	6
5	5
6	7
7	2.5
8	
9	
10	
Total Marks	39.5

Q1 (a). $z = \left(\frac{-1-i}{\sqrt{2}}\right)^{101} = \left[\frac{\sqrt{2}e^{i\left(\frac{\pi}{4}-\pi\right)}}{\sqrt{2}}\right]^{101}$

Principal argument of $(-1-i)$ is $\left(\frac{\pi}{4}-\pi\right) = \tan^{-1}\left(\frac{-1}{-1}\right)-\pi = -\frac{3\pi}{4}$

$z = \left[e^{i\left(-\frac{3\pi}{4}\right)}\right]^{101} = e^{-i\left(\frac{3\pi}{4}\times 101\right)} = e^{-i\left(\frac{303\pi}{4}\right)} = e^{-i\left(75\pi+\frac{3\pi}{4}\right)}$

$z = e^{-i\left(74\pi+\frac{7\pi}{4}\right)} = e^{-i74\pi}\, e^{-i\frac{7\pi}{4}} = (1)\, e^{-i\left(\pi+\frac{3\pi}{4}\right)}$

$z = e^{-i\pi} \cdot e^{-i\frac{3\pi}{4}} = (-1)\, e^{-i\frac{3\pi}{4}} = (-1)\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$

$z = (-1)\left[\cos\left(\pi-\frac{\pi}{4}\right)-i\sin\left(\pi-\frac{\pi}{4}\right)\right]$

$= (-1)\left[\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]$

3

$z = \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}} = a+ib$

$a = \frac{1}{\sqrt{2}}, \quad b = \frac{1}{\sqrt{2}}$

(b) $(1+i)^i = z^c = e^{c\log z}$

where $z = (1+i)$ and $c = i$

$e^{i\log(1+i)} = e^{i\left(\frac{1}{2}\ln 2+i\frac{\pi}{4}\right)}$

A

$\log(1+i) = \ln|1+i| + i\left(\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$

$(1+i)^i = e^{\frac{i}{2}\ln 2} \cdot e^{-\frac{\pi}{4}}$

$(1+i)^i = e^{-\frac{\pi}{4}}\left[\cos\left(\frac{\ln 2}{2}\right)+i\sin\left(\frac{\ln 2}{2}\right)\right]$

Q2 $f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \frac{(\mathrm{Re}\, z)(\mathrm{Im}(z))}{2|z|^2} & \text{if } z \neq 0 \end{cases}$

where $z = x+iy$
$\mathrm{Re}(z) = x$
$\mathrm{Im}(z) = y$
$|z|^2 = x^2+y^2$

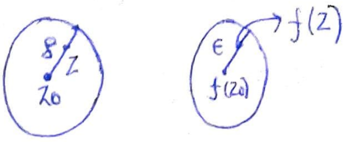
For $z \neq 0$

$f(z) = \frac{xy}{2(x^2+y^2)}$

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{xy}{2(x^2+y^2)}$

Any Complex valued function is said to be continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$ (2)

i.e limit should exist at z_0 and as $z \to z_0$, $f(z)$ should approach $f(z_0)$ and $f(z) = f(z_0)$ for $z \to z_0$. These conditions should be satisfied irrespective of direction of approach to z_0



$$f(z) = \frac{xy}{2(x^2+y^2)}$$

for $z = 0$, $f(z) = 0$

let $z = x$ then $y = 0$

$$\lim_{x \to 0} f(z) = \lim_{x \to 0} \frac{xy}{2(x^2+y^2)} = 0$$

$$\lim_{y \to 0} f(z) = \lim_{y \to 0} \frac{xy}{2(x^2+y^2)} = 0$$

Now let $x = y$

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(z) = \lim_{x \to 0} \frac{x^2}{2(x^2+x^2)} = \lim_{x \to 0} \frac{x^2}{4x^2} = \frac{1}{4}$$

3

this shows that function is not continuous at $z = 0$ as this is Path dependent so limit does not exist.

Q2

(b). $f(z) = i\cos z = u + iv \to (1)$

Entire function: It is a function which is analytic at all the Points i.e $z \ \forall z \in \mathbb{C}$

$$f(z) = i\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) \quad \text{(from eq (1))}$$

$\cancel{u = 0}$, $\cancel{v = \cos z}$

$\cancel{u_x = u_y = 0}$

$v_x =$

$\cos z = \cos(x+iy)$
$= \cos x \cos(iy) - \sin x \sin(i$

$\cos z = \cos x \cosh y - \sin x (i \sinh y)$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$f(z) = i\cos z = \boxed{\sin x \sinh y + i \cos x \cosh y} = u + iv$

continuous and differentiable in entire complex plane.

$\Rightarrow u = \sin x \sinh y \to (2)$; $v = \cos x \cosh y \to (3)$

4

$u_x = \cos x \sinh y \to (4)$ $\qquad v_x = -\sin x \cosh y \to (6)$

$u_y = \sin x \cosh y \to (5)$ $\qquad v_y = \cos x \sinh y \to (7)$

from eq^n (4) & eq^n (7) $\quad u_x = v_y \to (8)$

" eq^n (5) & eq^n (6) $\quad v_x = -u_y \to (9)$

This is true for all values of z in $\mathbb{C}$ as $\cos z$ is Continuous and defined in entire plane.

eq^n 8 & eq^n 9 are Cauchy Reiman equation which are satisfied when $f(z)$ is analytic

Q3(a) $f(z) = |Re(z)\ Im(z)|^{\frac{1}{2}}$ (3)

$f(z) = |xy|^{\frac{1}{2}} = \sqrt{xy}$

$f(z) = u + iv$

$u = \sqrt{xy}$, $v = 0$

~~$u_x = \sqrt{xy}$~~

$u_x(0,0) = \frac{u(x,0) - u(0,0)}{x-0} = \frac{u(x,0) - 0}{x} = 0$ as $\sqrt{xy} = 0$ for $x = 0$

$u_y(0,0) = \frac{u(0,y) - u(0,0)}{y} = 0$

$v_x(0,0) = 0$

$v_y(0,0) = 0$

Partial derivatives of $f(z)$ exists at $z=0$

$f'(z) = \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\Delta x \Delta y)^{1/2}}{\Delta z}$

$f'(0) = \lim_{\Delta z \to 0} \frac{\sqrt{\Delta x \Delta y}}{\Delta x + i\Delta y}$

~~Let $\Delta z = \Delta y$~~ $\boxed{\text{Case I : } \Delta x = \Delta y \text{ path}}$

$f'(0) = \lim_{\Delta x \to 0} \frac{\sqrt{\Delta x . \Delta x}}{\Delta x(1+i)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(1+i)} = \frac{1}{1+i}$ → (1)

Case II $\boxed{\Delta z = \Delta x \text{ path i.e } \Delta y = 0}$

$f'(0) = \lim_{\Delta x \to 0} \frac{\sqrt{\Delta x \Delta y}}{\Delta x + i\Delta y} = 0$ → (2)

eq^n 1 & 2 gives 2 different values for two different paths.

i.e $f(z)$ is continuous but not differentiable at $z = 0$

3

Q3(b) $f(z) = x^2 + iy^2 = u + iv$

$u = x^2$; $v = y^2$

$u_x = 2x$ $v_y = 2y$

$u_{xx} = 2$ $v_{yy} = 2.$

$u_y = 0$ $v_x = 0$

As $u_x = v_y$ only at $z = 0$ $x = y$

so as function satisfies Cauchy Reimann equation

(i.e $u_x = v_y$ & $v_x = -u_y$) only at $z = 0$? $x = y$

differentiable only at $z = 0$

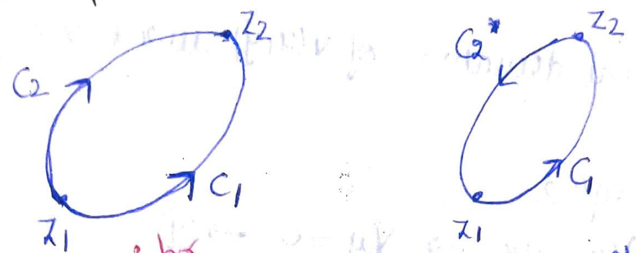
Any function $f(z)$ which is continuous and defined on positively oriented, simple smooth curve in a complex plane having domain D enclosed by curve is said to be analytic in that domain if $f(z)$ is differentiable at each point and in there neighbourhood. (4)

$f(z)$ is said to be analytic at a point z_0 if it is analytic every where in neighbourhood of z_0.

Here function is differentiable only at $z=0$ so $f(z)$ is nowhere analytic.

Q4 (a) $f(z) = z^3+2z+4$

$f(z)$ is defined on a simply connected domain D



why

Given function is analytic and defined on simply connected domain D. So by Cauchy Integral Theorem which states that integration of analytic function in a closed curve defined in simply connected domain is zero.

(3)

i.e $\oint_{z_1}^{z_2} f(z)dz = 0$

Consider two points z_1 and z_2 and two paths C_1 & C_2 such that curve C_1 & C_2 have only z_1 & z_2 as common points and no other common point

Let C_2^* be opposite curve of C_2

~~what given f~~ an

then by Cauchy Integral theorem

$$\int_{z_1, C_1}^{z_2} f(z)dz + \int_{C_2^*} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz = -\int_{C_2^*} f(z)dz$$

$$\Rightarrow \int_{C_1} f(z)dz = -\left(-\int_{C_2} f(z)dz\right)$$

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Hence proved integration of [illegible] of path in D

Q4(b) $f(z) = u(x,y) + i v(x,y)$ (5)

Given $f(z)$ is analytic in Domain D
$u(x,y)$ is constant in D

To prove: $f(z)$ is constant in D

Let $u(x,y) = K$ (where K is some constant)
$\Rightarrow u_x = u_y = 0 \rightarrow (1)$

Now as $f(z)$ is analytic in domain D so it will satisfy Cauchy Reeman (CR) equation i.e $u_x = v_y \rightarrow (2)$
$v_x = -u_y \rightarrow (3)$

where u_x is partial derivative of $u(x,y)$ w.r.t x
u_y " " " " w.r.t y

v_x and v_y are partial derivative of $v(x,y)$ w.r.t x & y respectively

from equation 1 & eqn 2
$u_x = 0$ and $v_y = u_x \Rightarrow v_y = 0 \rightarrow (4)$

from eqn (1) & eqn (3)
$u_y = 0$ and $v_x = -u_y \Rightarrow v_x = 0 \rightarrow (5)$

3

from eqn (4) & eq (5) ??
$v(x,y) =$ Constant let say P
$v(x,y) = P$

$\Rightarrow f(z) = u(x,y) + i v(x,y)$
$= K + iP$
where K and P are constants
$\Rightarrow f(z)$ is constant in D

Q5(a) ML inequality states that
$$\left| \int_C f(z)\, dz \right| \leqslant \int_C |f(z)|\, dz \leqslant M \int_C dz \leqslant ML$$
where $M = |f(z)|$
$L =$ Length of curve

$$\left| \int_{|z|=R} \frac{\log z}{2 z^4}\, dz \right| \leqslant ML$$

Here $f(z) = \dfrac{\log z}{2z^4}$ and $|z| = R \Rightarrow L = 2\pi R$ (6)

$$|f(z)| = \left|\frac{\log z}{2z^4}\right| = \frac{|\log z|}{2|z|^4}$$

$\log z = \log|z| + i\,Arg(z)$

$|\log z| \leq |\log|z|| + |i\,Arg(z)|$

$= \log R + |Arg(z)|$

$\boxed{-\pi < Arg\, z < \pi \Rightarrow |Arg\, z| = \pi}$

M is bound of f(z)

$$\left|\int_{|z|=|R|} \frac{\log z}{2z^4}\right| \leq \underbrace{\left(\frac{\ln R + |Arg(z)|}{2R^4}\right)}_{M} \underbrace{(2\pi R)}_{L}$$

$$\leq \pi\left(\frac{\ln R + \pi}{R^3}\right)$$

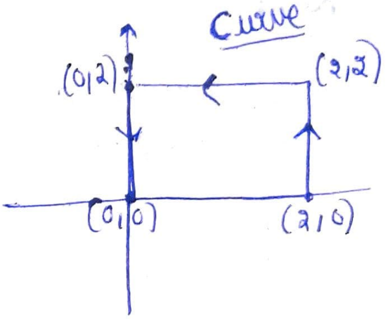
$|f(z)| \leq M \quad |f(z)| \leq \dfrac{|\ln|z|| + |i\,arg\, z|}{2|z|^4} \leq \dfrac{\ln R + \pi}{2R^4} = M$

3

Hence Proved

Q5 (b)

$$I = \oint \frac{\tan\frac{z}{2}}{z^2+4z+3}\, dz$$



$z^2+4z+3 = (z+1)(z+3)$

$$I = \oint \frac{\tan\frac{z}{2}}{(z+1)(z+3)}\, dz$$

where $f(z) = \dfrac{\tan z/2}{(z+1)(z+3)}$ has singularities at $z = -1$ and $z = -3$ $\quad z = (2n+1)\pi$

but $z = -1$ & $z = -3$ does not belong to domain of curve

So by Cauchy's Integral theorem

If $f(z)$ is analytic in a simply connected, positively oriented and closed domain then

$$\oint_C f(z)\, dz = 0$$

traversed in anticlockwise direction

$$\Rightarrow I = \oint \frac{\tan\frac{z}{2}}{z^2+4z+3} = 0$$

2

$$\oint f(z)^n dz = \begin{cases} 2\pi i & \text{for } n=-1 \\ 0 & \text{for } n \neq -1 \text{ and others} \end{cases}$$

Q6 Isolated Singularity

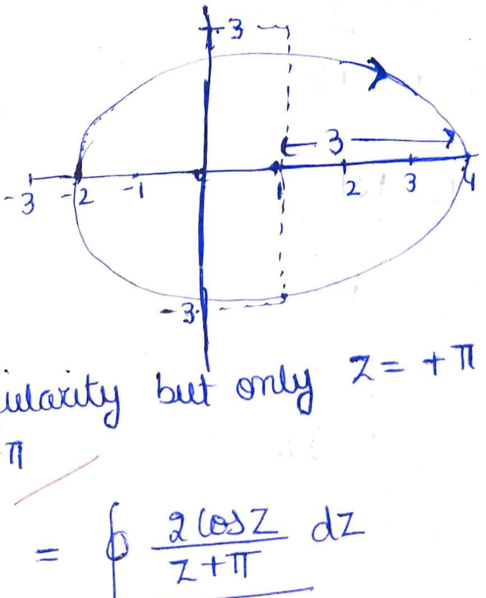
If a function $f(z)$ is singular at any point of domain let say z_0 but is defined and analytic in every neighbourhood of z_0 then z_0 is called a point of Isolated Singularity.

Singular at z_0 means $f(z)$ is not analytic at z_0 i.e not defined at z_0.

$$\boxed{\oint \frac{f(z)}{z-z_0} = 2\pi i f(z_0)}$$

$$I = \oint \frac{2\cos z \, dz}{(z^2-\pi^2)}$$

(a) $C: |z-1| = 3$



$$f(z) = \frac{2\cos z}{z^2-\pi^2}$$

at $z = \pm\pi$ $f(z)$ has singularity but only $z = +\pi$ belongs in domain and not $z = -\pi$

$$= \oint \frac{2\cos z}{(z-\pi)(z+\pi)} dz = \oint \frac{\frac{2\cos z}{z+\pi}}{z-\pi} dz$$

$$= 2\pi i f_1(z)\Big|_{z=\pi} \qquad \text{where } f_1(z) = \frac{2\cos z}{z+\pi}$$

$$= 2\pi i \left(\frac{2\cos(\pi)}{\pi+\pi}\right) = 2\pi i \left(\frac{2\cos\pi}{2\pi}\right) = -2i$$

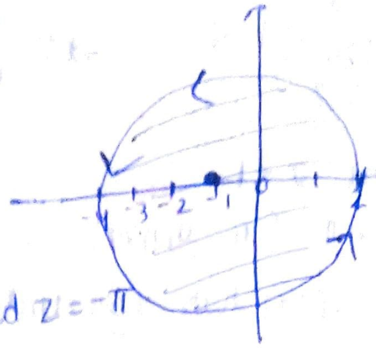
$I = -2i$ for $C: |z-1| = 3$ anticlockwise

clockwise

Q6 (b) $C: |z+1| = 3$

$-4 \leq |z| \leq 2$

(8)



$$f(z) = \frac{2\cos z}{(z-\pi)(z+\pi)}$$

has singularity at $z = \pi$ and $z = -\pi$

$z = -\pi$ belongs to domain

$$f_1(z) = \frac{2\cos z}{z-\pi}$$

$$I = \oint \frac{\left(\frac{2\cos z}{z-\pi}\right)}{z+\pi}\, dz = 2\pi i \left(f_1(z)\right)\Big|_{z=-\pi}$$

$$I = 2\pi i\left[\frac{2\cos(-\pi)}{(-\pi)-\pi}\right] = 2\pi i\left(\frac{-2}{-2\pi}\right) = 2i$$ Counter Clock wise

$I = 2i$ counter Clock wise

Q7 Removable singularity: Let say function $f(z)$ defined in complex plane has singularity at $z = z_0$ then if $\lim_{z \to z_0} f(z)$ exist then ~~z_0 is~~ $f(z)$ is said to have removable singularity at $z = z_0$. eg. $f(z) = \frac{\sin z}{z_0}$ has removable singularity at $z=0$ as $\lim_{z\to 0} \frac{\sin z}{z} = 1$

Pole of a Complex valued function

If in a binomial expansion of $f(z)$ there are finite number of -ve terms then a Pole is said to exist.

$$f(z) = \frac{z\cos\frac{\pi z}{2}}{(z-1)\sin^4 z}$$

$f(z)$ has singularity at $z=1$ and $z = n\pi$ where $n = 0, 1, 2, 3 \ldots$

as $\sin n\pi = 0$

$f(z)$ has a Pole at $z=1$ of ~~order 1~~

To find order of pole

$$\lim_{z\to 1} (z-1)^n f(z) = \text{finite} \neq 0$$

$$\lim_{z\to 1} \frac{(z-1)\cos\frac{\pi z}{2}}{(z-1)\sin^4 z} = \frac{\cos\frac{\pi(1)}{2}}{\sin^4(1)} \cancel{\neq 0} = 0$$