2. Convolution

→ computation of convolution in time-domain will be problem

→ Easy to do in frequency domain

$$f(x) \xrightarrow{F} F[f] = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx$$

Proving following properties:

2.1 Spatial Shift

$$F[f(x-a)](\omega) = e^{-i\omega a} . F[f](\omega)$$

$$F[f(x)](\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}$$

$$F[f(x-a)](\omega) = \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x} dx$$

$$\boxed{\begin{array}{ll} \text{let } x-a = t & \\ dx = dt & x = a+t \end{array}} \quad (1)$$

using (1) in integration

$$= \int_{-\infty}^{\infty} f(t) e^{-i\omega(t+a)} dt = \int_{-\infty}^{\infty} f(t) e^{-i\omega a} . e^{-i\omega t} dt$$

$$= e^{-i\omega a} \underbrace{\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt}_{F[f](\omega)}$$

$$F[f(x-a)](\omega) = e^{-i\omega a} F[f](\omega)$$

Hence Proved!!

2.2 Convolution

$$\mathcal{F}[(k*f)(x)](\omega) = \mathcal{F}[k](\omega)\cdot\mathcal{F}[f](\omega) \rightarrow (1)$$

$$\mathcal{F}[k](\omega) = \int_{-\infty}^{\infty} k(x)e^{-i\omega x}dx$$

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

taking LHS of (1)

$$\mathcal{F}[k(x)*f(x)] = \mathcal{F}\left[\int_{-\infty}^{\infty} k(\tau)f(x-\tau)d\tau\right]$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} k(\tau)f(x-\tau)\,d\tau\right]e^{-i\omega x}dx$$

$$= \int_{-\infty}^{\infty} k(\tau)\left[\int_{-\infty}^{\infty} f(x-\tau)e^{-i\omega x}dx\right]d\tau$$

using $\boxed{\begin{aligned} x-\tau &= a \\ dx &= da \\ x &= \tau+a \end{aligned}}$ in above eqr.

$$= \int_{-\infty}^{\infty} k(\tau)\left[\underbrace{\int_{-\infty}^{\infty} f(a)e^{-i\omega a}da\cdot e^{-i\omega\tau}}\right]d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} k(\tau)\cdot e^{-i\omega\tau}d\tau}_{\mathcal{F}[k](\omega)}\ \underbrace{\int_{-\infty}^{\infty} f(a)e^{-i\omega a}da}_{\mathcal{F}[f](\omega)}$$

$$\mathcal{F}[(k*f)(x)](\omega) = \mathcal{F}[k](\omega)\cdot\mathcal{F}[f](\omega)$$

Hence Proved!!

2.3 Derivative

$$\mathcal{F}\left[\frac{\partial f(x)}{\partial x}\right](\omega) = i\omega \mathcal{F}[f](\omega)$$

$$\mathcal{F} f(x) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{---(1)}$$

$$\mathcal{F}\left[\frac{\partial f(x)}{\partial x}\right](\omega) = \int_{-\infty}^{\infty} \frac{\partial f(x)}{\partial (x)} . e^{-i\omega x} dx \quad \text{---(2)}$$

[using integration of products formula: $\int u\,dv = uv - \int v\,du$

here $u = e^{-i\omega x}$ $\quad v = \frac{\partial f(x)}{\partial x}$

Eqn (2): $\int_{-\infty}^{\infty} \frac{\partial f(x)}{\partial (x)} e^{-i.\omega x} dx = e^{-i\omega x} \int_{-\infty}^{\infty} \frac{\partial f(x)}{\partial (x)} - \int_{-\infty}^{\infty} f'(x)\, dx$

$$= e^{-i\omega x}\left[f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega) e^{-i\omega x} dx$$

$$= \underbrace{\cancel{\left[e^{-i\omega x} f(x)\right]_{-\infty}^{\infty}}}_{0} + i\omega \underbrace{\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx}_{\mathcal{F} f(x)}$$

this gives

$$\mathcal{F}\left[\frac{\partial f(x)}{\partial x}\right](\omega) = i\omega \mathcal{F} f(x)$$

Hence Proved!

3. First and Second order derivative filter in 1D

$$f'[n] = \frac{f[n] - f[n-1]}{\tau} \Longleftrightarrow \frac{1}{\tau}\,\boxed{-1 \mid 1} * \boxed{\dots \mid f[n-2] \mid f[n-1] \mid f[n]}$$

Second order derivative

$$f''[n] = \frac{1}{\tau}\left(f'[n] - f'[n-1]\right)$$

$$= \frac{1}{\tau}\left(\frac{f[n]-f[n-1]}{\tau} - \frac{f[n-1]-f[n-2]}{\tau}\right)$$

$$= \frac{1}{\tau}\left(\frac{f[n] - 2f[n-1] + f[n-2]}{\tau}\right)$$

$$= \frac{1}{\tau^2}\left(f[n] - 2f[n-1] + f[n-2]\right)$$

this corresponds to $\frac{1}{\tau^2}[-1 \;\; 2 \;\; 1]$ kernel

this second derivative kernel is laplacian of Gaussian. which is a high pass filter.

This is used in extracting features (like edge detection) in image processing.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{confirm!}$$