Advanced Mathematics

Lecture in WS 2021/22 at the KFU Graz

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Contents

Chapter 1

Introduction

This lecture covers a number of mathematical aspects, which are important in modern theoretical physics. The largest, and most central, part is group theory, or more precisely the theory of groups and algebras, as it plays a quite important and prominent role in modern physics. The reason is that nature exhibits symmetries, and these are consequences of the symmetry groups in their basic structure. These symmetries can appear in many disguises. The simplest one is that the symmetry can be identified by the naked eye, e. g. in snow flakes or in the form of crystals. Then they manifest themselves in conservation laws, which e. g. forces the orbit of a planet to be planar. And finally, they surface as degeneracies, like the fact that without magnetic field the magnetic levels of the hydrogen atom have all the same energies.

The second part deals with functions of complex variables, and their analytical structure. That is of central importance for many aspects of theoretical physics, but especially features of spectra and scattering processes. The simpelst example are in quantum mechanics the resolvent and the fact that poles of propagators determine the energy spectra. Finally, some basic elements of manifolds and (differential) geometry will be covered, which is necessary to deal with situations in which spaces with curvatures are encountered. Most notable is, of course, general relativity, but also various internal spaces throughtout theoretical physics are of this type. It will be seen that many aspects of group theory reemerge also in these two topics. That is not surprising as in both cases groups enter in the foundation of them. This again justifies the central part played by groups in the following.

The aim of this lecture is not to provide an explanation of any of the physical phenomena. The aim is to provide the mathematical tools to do so. E. g. one of the tools are group and algebra theory. Implementing these tools such as the observed symmetry, conservation, and degeneracy patterns observed in experiment are correctly reproduced by the theoretical description is the role of the corresponding physics lectures. Nonetheless,

many examples will be drawn from physics. Also, the group of rotations and especially of spin will play very central roles, so that many mathematical concepts will have an immediate physical analog.

Chapter 2

Abstract groups

Groups are quite generic structures. A group is essentially any structure which fulfills a numbers of properties, and can be far away from anything one usually imagines under the name of group, e. g. the rotation group encountered in mechanics. It is therefore worthwhile to start out with a suitable abstract definition of what groups are, and what are generic traits of groups.

2.1 Groups

A group G is a structure on a set $\mathcal{G}$ of elements g_i. This set can be finite or infinite. The number of (distinct) elements is called the order of the group. Part of the structure must also be some possibility $\circ$ to combine two elements of the set. This structure is a group if it has has the following properties. Or, any structure fulfilling the following properties is a group, though it may also fulfill many more properties as well. These properties are called group axioms:

- The combination of any two elements is within the set of elements, $g_1 \circ g_2 = g_3$ with $g_i \in \mathcal{G}$
- The combination is associative, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
- There exists a (unique) element e with $e \circ g = g \circ e = g$, i. e. an identity element
- For every g there exists an element g^{-1} with $g \circ g^{-1} = g^{-1} \circ g = e$, i. e. an inverse element

Note that the combination does not need to be commutative, i. e. $g_1 \circ g_2$ may or may not be the same as $g_2 \circ g_1$. If it is the same for all possible combinations of elements,

the combination is commutative, and the group is called Abelian. If not, it is called non-Abelian.

2.2 Creating a standard example

Many of the concepts of group theory are quite abstract. It is therefore useful to create standard examples, which can be used to illustrate in a fixed setting all the concepts in a very explicit way. Of course, other examples will be used to highlight additional aspects.

The standard example to be used during the first part will be the rotations in two and three dimensions, $\mathcal{R}_2$ and $\mathcal{R}_3$, respectively. The elements are then just rotations. These will be characterized by rotation angels, one in two dimensions and three in three dimensions, where for now it does not matter what the precise rotation axes are. The combination of elements is to perform two rotations after each other.

These two sets and combinations are groups. Geometrically, a sequence of rotations remains a rotations, and therefore the set is closed under the combination. Rotations are also associative, as the combination of two rotations is again a uniquely defined rotation. The unit element is just no rotation at all, and the inverse element is the rotation backwards, which compensates a rotation.

From geometry, it follows that two rotations in two dimensions commute, and the group is Abelian. In three dimensions, however, rotations do not commute, and the group is non-Abelian.

Both groups have an infinite number of elements. To obtain a discrete rotation group, the simplest version is to use a set of rotation about a fixed angle, $\mathcal{R}_n^\alpha$, and its multiples, with the angle α being a rational factor of 2π. For the following, it is convenient to chose this angle to be $\pi/2$, providing the groups of discrete rotations in two and three dimensions. Due to the factor requirement, any combination of these group elements is again an element of the group, and the remainder of the group axioms are fulfilled in the same way. In this case, the group elements for the two-dimensional case can be labeled by a single integer n, $g(n)$, which counts the rotation angle α, $n\alpha$, and n runs from 0, the identity, to $2\pi/\alpha - 1$, the maximum rotation. In the three-dimensional case, this requires three indices.

Note that in contrast to what is usually done in linear algebra at no point any kind of matrix was written down, as such a realization - latter to be called a representation - is not necessary to describe the group nature of rotations.

2.3 Subgroups

If in a group G there exists a subset of the elements $\mathcal{G}_s \subset \mathcal{G}$ such that with the same combination this subset also forms a group, this is called a subgroup. A group can have multiple subgroups. Usually each subgroup has the same identity element, but this does not need to be the case; the identity element, and also the inverse element, in a subgroup does not need to be the same as in the full group, though these cases have little relevance for physics. There are always the trivial subgroups of just the identity element and the whole group itself.

A particularly interesting case occurs if a subgroup of a non-Abelian group exists, for which all elements commute with all elements under the combination, i. e.

$$g_s \in \mathcal{G}_s \text{ and } g \in \mathcal{G}_s \implies g_s \circ g = g \circ g_s.$$

Then the subgroup is called the center of the group. The center is then necessarily an Abelian group. There is always the trivial center consisting only of the identity element.

In the standard example, two types of sub-groups appear. One is that the rotation group in two dimensions is a sub-group of the rotations in three dimension. These are just the rotations in a fixed plane, and hence $\mathcal{R}_2 \subset \mathcal{R}_3$, even when no particular plane is specified. The set of discrete rotations is also a sub-group of the corresponding rotations, as every element is also a group element of the full rotations, but the sub-group is closed, $\mathcal{R}_n^\alpha \subset \mathcal{R}_n$.

2.4 Cosets

If there is a subgroup H of a group G, it is possible to define a right/left coset w. r. t. to an element g of a group as the set of all elements hg/gh with $h \in H$. This is in brief also written as Hg/gH. The set of all cosets is denoted as G/H, the coset space. Note that for $g \notin H$, it can be shown that every group element can appear at most in one coset. This implies that for finite groups the dimension of H and the cosets are together the total dimension of the group.

More importantly, if the left and right coset are identical, i. e. for every gh_1 there is some h_2 such that $gh_1 = h_2 g$, or in brief

$$gH = Hg, \tag{2.1}$$

the subgroup H is called a normal or invariant subgroup. Both trivial subgroups are invariant subgroups, but it becomes more interesting if the subgroup is non-trivial. If this

is the case, the set of cosets also forms a group. To see this, define the combination of two cosets as

$$(Hg_1) \circ (Hg_2) = (Hg_1Hg_1^{-1})(g_1g_2) = (Hg_1)(Hg_2).$$

This combination is the combination of two cosets as the coset on the combination of the two group elements in the described way. But if the subgroup is invariant, then

$$Hg_1Hg_1^{-1} = Hg_1g_1^{-1}H = H,$$

where in the last step it was used that H is a subgroup, and the combination of two arbitrary elements of H is again in H. This implies that the combination of two cosets is again a coset. Since $He = H$, there is also a unit element, and thus also the inverse is included, since this is just $Hg_1Hg_1^{-1} = He = H$. Thus, the coset space G/H forms itself a group, the so-called factor group of G by H.

Take for example the group consisting out of the fourth roots of unity, called Z_4. The set containing only the square-root is a subgroup, the group Z_2. Since the group is Abelian, the sub-group is an invariant subgroup, with the coset group Z_2, and thus $Z_4/Z_2 = Z_2$.

In the example of $\mathcal{R}_2^\alpha$, then if 2α is still a factor of 2π, then $\mathcal{R}_2^{2\alpha}$ is a coset. E. g. for $\alpha = \pi/2$, these are the rotations under π. The two possible cosets then differ by the offset of $\pi/2$. These cosets are invariant cosets, because the combination of two rotations of π and $\pi/2$ can always be equally well represented by a combination of two rotations by π and $\pi/2$. This is also true, because the group is Abelian. Both together give therefore the factor group of $\mathcal{R}_2^{\pi/2}$.

2.5 Conjugacy classes and automorphisms

The requirement for an invariant subgroup (2.1) can also be written as

$$gHg^{-1} = H,$$

for any element of the group g, i. e. the subgroup is invariant under a transformation with an element not part of the subgroup. It is useful to generalize this concept to so-called conjugacy classes. A conjugacy class S is a set of group elements, not necessarily a subgroup, which satisfies

$$gSg^{-1} = S$$

for every $g \in G$.Though not immediately obvious, this concept will be useful later on.

These conjugacy classes can also be formulated in a different context. If there exists a map $M : G \to G$ which is 1-to-1[1] and $M(g_1) \circ M(g_2) = M(g_1 \circ g_2)$, i. e. the map preserves the group combination, the map M is called an automorphism of the group.

A special case of an automorphism is

$$M_g(G) = gGg^{-1},$$

with g fixed. This special map is called an inner automorphism. Thus, a conjugacy class S is a set of group elements invariant under all inner automorphisms of the group.

In the case of $\mathcal{R}_2^\alpha$, every combination of elements form conjugacy classes, since the group is Abelian. For $\mathcal{R}_3^\alpha$, this is not the case. The only conjugacy class is the identity transformation, as there is no invariant subset of rotations under arbitrary rotations.

[1]Note that so far 'onto' is not a requirement, which can make a difference for infinite groups.

Chapter 3

Representations

3.1 Definition of representation

So far, groups have been completely abstract, and could be whatever entities they would be, even just the abstract objects, and everything there exists is a (finite of infinite) table declaring the results of any possible combination. Such an abstract group can then be realized by mapping them on a set of linear operators S acting on a vector space V where also a combination of elements is defined which a has a unit element. A particularly important case of linear operators is the special case of matrices. Thus, a representation is a map $D : \mathcal{G} \to S$.

This mapping is called a representation if it maintains the group structure. This is the case if it fulfills the conditions

- $D(e) = 1$, where 1 is the unit element under any defined combination of elements of S, e. g. the unit matrix in a space of matrices with the combination being matrix multiplication
- $D(g_1)D(g_2) = D(g_1 \circ g_2)$, and thus there is a one-to-one relation between the combination in G and in S

Especially, this fulfills the existence of an inverse element for all group elements.

For a representation it is not necessary to map every element g to a different element in S. The trivial representation $D(g) = 1$ certainly fulfills both conditions, and the resulting set also implements trivially a group structure in S. If every element is different in S, then it is called a faithful representation. Note that a faithful representation is not guaranteed for any given pair of group and space. E. g. the set $\{1\}$ is certainly a suitable space when declared together with multiplication to keep the group structure, but except for the trivial group a faithful representation is impossible.

The dimension of the space is then also called the dimension of the representation. The trivial representation has, e. g. dimension 0: A single point does not have a dimension.

A representation is called unitary, if an adjoining operator † is defined on the linear space, and $D^{g\dagger} = D^{g-1}$.

Of course, it is possible to introduce vectors on a linear space S, $|v\rangle$, and corresponding co-vectors $\langle v|$, on which the representation D can act as $|v'\rangle = D|v\rangle$. For any basis $|e_i\rangle$ in the linear space there exist then (basis-dependent) matrix elements

$$D^g_{ij} = \langle e_i|D^g|e_j\rangle$$

which provide an explicit realization of the representation.

For the standard example, a possible representation are the well-known matrix representations, especially for $\mathcal{R}^\alpha_2$

$$D(g(n)) = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}. \tag{3.1}$$

A possibility for $\mathcal{R}^\alpha_3$ would be parametrizations using the Euler angles, or just by the rotation along the three axes,

$$\begin{aligned} D(g(n_1,n_2,n_3)) &= \begin{pmatrix} \cos n_1\alpha & \sin n_1\alpha & 0 \\ -\sin n_1\alpha & \cos n_1\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos n_2\alpha & 0 & \sin n_2\alpha \\ 0 & 1 & 0 \\ -\sin n_2\alpha & 0 & \cos n_2\alpha \end{pmatrix} \times \\ &\quad \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos n_3\alpha & \sin n_3\alpha \\ 0 & -\sin n_3\alpha & \cos n_3\alpha \end{pmatrix}. \end{aligned} \tag{3.2}$$

The trivial representation is just $D(g) = 1$, the one-dimensional unit matrix. This representation fulfills $D(e)$ just being the unit matrix. In case of $\mathcal{R}^\alpha_2$, this implements the group combination just as $D(g(n))D(g(m)) = D(g(n+m))$, as two rotations in two dimensions is just a rotation by the sum of the angles. In three dimensions, this is less obvious, and the group composition in terms of the indices is less trivial due to the non-Abelian nature, but can be implemented as well. The dimensions of these representation are two and three, respectively. Since these matrices are orthogonal, they are automatically also unitary, and therefore this representation is hence unitary as well. The matrix representation is then obtained by using the usual Cartesian coordinates.

3.2 Relations between representations

Because the space on which the group is mapped is linear, it is always possible to modify the representation by a linear transformation without altering the composition rule, i. e. a

basis change. Thus, two representations are called equivalent if it is possible to change from one to the other by applying to all group elements D^g, where g denotes the corresponding group element, the same (invertible) transformation S,

$$D^{g'} = S^{-1} D^g S,$$

since

$$D^{g'_1} D^{g'_2} = S^{-1} D^{g_1} S S^{-1} D^{g_2} S = S^{-1} D^{g_1 \circ g_2} S = D^{(g_1 \circ g_2)'},$$

as required. Of course, the unit element may differ in the new basis, but it is also related to the original one by the same similarity transformation.

In case of the standard example, this translates to the fact that similarity transformations can be applied, without changing the character of being a rotation.

3.3 Reducible and irreducible representations

A representation does not necessarily affects every element of the vector space S. If there is a sub-space, constructed using some projector P, which remains invariant under the group action

$$P D^g P = D^g P^2 = D^g P \tag{3.3}$$

for every group element g, the representation is said to be reducible. However, it should be noted that this can also be regarded as the fact that the group acts trivially on this subspace, i. e. $P D^g P|_P = 1$. In a sense, the reducible representation is actually a direct sum $D^g \oplus 1^g$, i. e. the group acts on the whole space, but with two different representations.

A generalization of this concept is given when the group acts with n, possibly different, representations in n sub-spaces of the whole space, such that the sum of the dimensions of the representations is the total dimensionality of the linear space. Especially, if the basis is such that this tensor representation becomes block-diagonal for every group element g, this is called a completely reducible representation. In this case, the representation of any group element g forms a direct sum $D_1^g \oplus ... \oplus D_n^g$ in the various sub-spaces.

A counter-example can be given that this is not always possible, but requires an infinite group. For finite groups, any reducible representation is also completely reducible[1]. To get a counter-example for infinite groups, consider the (Abelian) group of integers under addition, which form an example of a group. A (non-unitary) 2-dimensional representation

[1]This is stated here without proof. Proving it essentially involves first showing that all representations of finite groups are unitary.

is given by

$$D^x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

since the unit element is just the unit matrix and

$$D^x D^y = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = D^{x+y}.$$

However, there is an invariant sub-space, obtained with the projector

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

Thus, this representation is reducible. It is, however, not completely reducible, as the orthogonal sub-space, constructed by $1 - P$ is not an invariant sub-space. Hence, a block-diagonal decomposition is not possible in this case.

If there is no invariant subspace, and the representation is not block-diagonal for every group element, the representation is called irreducible. The requirement that a completely reducible representation is block-diagonal implies that the blocks must be irreducible representations within the corresponding subspaces. It is said that the reducible representation has been decomposed into its irreducible representations. This also implies that there can be irreducible representations of different dimensionality.

Especially, an irreducible and faithful representation of a non-Abelian group cannot be one-dimensional, as the non-Abelian nature of the group composition cannot be reproduced with numbers. This is also true for finite groups.

A trivial example of a completely reducible representation of the standard examples is given by

$$D^R(g) = \begin{pmatrix} D^I(g) & 0 \\ 0 & 1 \end{pmatrix}.$$

In this way, the two-dimensional or three-dimensional irreducible representation in the upper left corner is extended by a trivial representation. From this completely reducible representation reducible ones can be obtained by an arbitrary similarity transformation. It should be noted that rewriting the two-dimensional irreducible representation to the one-dimensional phase rotation is actually not a change from a reducible to an irreducible representation, as this changes the vector space from a real to a complex one. However, the two-dimensional representation in a complex vector space would be reducible to the one-dimensional one.

3.4 Regular representation

An important possibility of a (matrix) representation is the so-called regular or adjoint representation. It is the most straightforward possibility to generate a representation. It is defined by the action of each generator on the basis vectors of a vector space. Thus, necessarily, the dimensionality of the regular representation needs to coincide with the size of the group. The elements of the representation are then given by

$$D^g_{ij} = \langle e_i | D^g | e_j \rangle, \tag{3.6}$$

where g labels the group elements. By insertion of a unity it can be shown directly that this maps the composition law of the group on the matrix multiplication in this space

$$(D^g D^h)_{ij} = \langle e_i | D^g D^h | e_j \rangle = \langle e_i | D^g | e_k \rangle \langle e_k | D^h | e_j \rangle.$$

Similarly, the other requirements for a representation can be shown.

It should be noted that the adjoint representation is not necessarily irreducible, but can be.

For the standard-example of the rotations, the given two-dimensional and three-dimensional representations are the adjoint representations.

3.5 Some finite groups

In the following, some groups of a finite order will be discussed. These have simpler properties than those of infinite order, and are therefore useful to demonstrate many of the abstract properties. They are also of high physical relevance. E. g. there are groups for crystals which can be used to classify the crystals according to the discrete rotation groups they belong to. In particle physics, e. g. the discrete parity group plays an important role.

3.5.1 Cyclic group

3.5.1.1 Definition

A particular example of a finite groups are the so-called center groups Z_N of order N. These groups have the property that for any element g there is some finite m such that $g^m = e$. E. g. with three elements $\{g, a, b\}$ this structure can be $a^3 = b^3 = e$, and $a^2 = b$, $b^2 = a$, and $ab = ba = e$, and thus an Abelian group. These properties define the group, they cannot be derived.

The same group can be represented by a complex phase with $D(e) = 1$, $D(a) = \exp(i2\pi/3)$, and $D(b) = \exp(i4\pi/3)$, which implements the group structure. The dimension of the (complex) space is thus 1. This representation is also faithful, and irreducible.

3.5.1.2 Parity

A physical example of a cyclic group is the parity transformation in quantum mechanics, which is just the group Z_2. For any Hilbert space, there are only two representations. One is the trivial one $D^g = 1$. The other is $D^1 = 1$ and $D^2 = -1$. Thus, in the completely reducible basis, any one-dimensional subspace can be associated with either of the two representations. Since any element always commutes with the Hamiltonian, as they are always proportional to the unit element, there is a basis where every state has a definite parity. Those in sub-spaces with the trivial representation are said to have even parity, while those in which the other representation acts have odd parity. Thus, the parity of the states are in one-to-one correspondence to the (irreducible) representation of the parity group in their respective sub-spaces.

Note that this does not specify in any way which sub-spaces are associated with which representation. This is a dynamical question, for which a Hamilton operator must be specified.

3.5.2 Permutation group

A permutation group S_n is the (finite) group of all possible sets of arrangements of n elements.

The permutation group S_2, e. g. contains the elements e, $g_1 = 1, 2$, and $g_2 = 2, 1$, i. e. three elements. g_1 and g_2 are the possible interchanges. A composition is defined by the repeated application. There is only one composition not involving the unit element, $g_1 g_2 = g_2 g_1 = e$. This result appears evident, as exchanging 1 and 2 and then 2 and 1 leaves things unchanged.

Things become a bit more interesting for the permutation group S_3. Here, there are six (3!) elements besides the unit element. There are the three transpositions $g_1 = (1, 2)$, $g_2 = (2, 3)$, and $g_3 = (3, 1)$ and the two possible reorderings $g_4 = (1, 2, 3)$ and $g_5 = (3, 2, 1)$. These are compositions of the transpositions, e. g., $g_1 g_2 = g_4$, $g_4 g_3 = g_5$, and so on. However, the group is non-Abelian, as, e. g., $g_2 g_1 = g_5 \neq g_1 g_2$. It is one of the smallest non-Abelian groups.

While the generalization to larger n is straight-forward, the group S_3 can serve as an example for the necessity of matrix representations. A non-trivial one-dimensional

representation over the space of complex numbers is not possible, as then the non-Abelian nature cannot be maintained - $g_1g_2 \neq g_2g_1$ and $g_1g_3 \neq g_3g_1$ and $g_2g_3 \neq g_3g_2$ cannot be realized with numbers except if $g_i = 1$. However, there exists a two-dimensional, then necessarily irreducible, faithful representation of the group,

$$D(g_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D(g_2) = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \qquad D(g_3) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$
$$D(g_4) = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ 1 & -\sqrt{3} \end{pmatrix} \qquad D(g_5) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

and the trivial $e = 1$. The mapping can be proven, e. g., by direct computation. Also, not all of these six matrices can be simultaneously diagonalized, and this is a irreducible representation.

3.6 Characters

A useful tool to characterize representations of groups are characters. They are defined as

$$\chi_D(g) = \mathrm{tr}D(g) = D(g)_{ii}.$$

Hence, they are complex numbers, depending on the representation and the group element. For finite-dimensional representations[2], these are numbers, which are invariant under similarity transformations. Thus, all equivalent representations have the same characters. This allows to identify an equivalence class of representations.

They are also orthonormal in the group,

$$\sum_g \chi_{D_a}(g)^{-1}\chi_{D_b}(g) = N\delta_{ab},$$

where N is the size of the group. This can be seen as follows. Consider the operator

$$A = \sum_g D_a(g^{-1})|a,j\rangle\langle b,i|D_b(g).$$

This operator commutes with any $D_a(h)$, since

$$D_a(h)A = \sum_g D_a(hg^{-1})|a,j\rangle\langle b,i|D_b(g) = \sum_g D_a(hg^{-1})|a,j\rangle\langle b,i|D_b(gh^{-1})D_b(h) = AD_b(h).$$

[2]To be more precise: representations in trace-classes.

According to Schur's lemma[3], it is therefore proportional to $\delta_{ab}\lambda^a_{ij}1$, where the 1 acts in the space of the irreducible representation, as the operator does so, and λ^a_{ij} is the constant of proportionality. The still present δ_{ab} restricts the 1 to be active only in the subspace of the operators. The indices merely label the operator. Taking further the matrix elements yields

$$\sum_g \frac{n_a}{N}(D_a(g)^{-1})_{kj}D_b(g)_{im} = \delta_{ab}\delta_{ij}\delta_{km}. \tag{3.7}$$

This is a very important statement about representations. It states that the representation matrices are orthogonal in the space with coordinates the group elements.

An example is given by the Z_N groups. Its irreducible representations are given by

$$D^N_n(j) = e^{2\pi i\frac{nj}{N}}. \tag{3.8}$$

It can be proven that for Abelian groups there are only one-dimensional irreducible representations. Then the only possibilities are these, which differ by the number of times the unit circle is swept over when traversing the full group. However, this cannot be done arbitrarily often. At one point, all sweeps reduce again back to the original one, if the number n becomes again an integer multiple N. For $N = 2$, e. g., this is possible twice, since n can either be zero (trivial representation) or 1. If it would be two, this returns the trivial representation. For $N = 3$, the possibilities are $n = 0$, $n = 1$ (advance by $2\pi/3$), and $n = 2$ (advance by $4\pi/3$) sweeps. $n = 3$ gives again back the trivial one, etc.. Thus, there are $N-1$ independent irreducible representations. In this case, the characters necessarily coincide with the corresponding representations, i. e. (3.8).

The relation (3.7) then takes the form

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i\frac{nj}{N}}e^{2\pi i\frac{n'j}{N}} = \delta_{n'n}$$

which is nothing but the statement that a Fourier decomposition of the Nth-root of unity is orthogonal when summed over with any possible number of sweeps of the unit circle. Note that the indices at the Ds are otherwise trivial, since it is a one-dimensional representation.

The pre-factor in the sum (3.7) combines the size of the irreducible representations n_a with a normalization factor N, which is the size for a finite group. Thus

$$\sum n_a^2 = N, \tag{3.9}$$

which can be seen after taking a trace over the indices, which will first combine both Ds to a unit matrix, which is then traced to yield a factor n_a. This can be seen as follows.

[3]Which essentially states that only matrices proportional to the unit matrix commute with any other matrix.

Taking the trace of A, not with respect to the indices defining it, but to the outer Hilbert space yields the following. Just taking the trace requires tr$A$ to be proportional to the size of non-vanishing elements of the block-diagonal, and thus yields $\delta_{ab}\lambda^a_{ij}n_a$, where n_a is the size of the representation in question. On the other hand, taking the trace over this full space is equivalent to taking

$$\begin{aligned}\mathrm{tr}A &= \sum_i \langle i| \sum_g D_a(g^{-1})|a,j\rangle\langle b,i|D_b(g)|i\rangle \\ &= \sum_{ig} \langle b,i|D(g)|i\rangle\langle i|D(g)^{-1}|a,j\rangle = \sum_g \langle b,i|a,j\rangle = N\delta_{ab}\delta_{ij},\end{aligned}$$

where N is some measure of the size of the group, and for finite groups the actual size. This identifies $\lambda^a_{ij} = n_a/N$, and therefore yields (3.7). The previous example implemented this already, since $1+1=2$ and $1+1+1=3$ etc..

This result has an important implication for characters of different representations,

$$\frac{1}{N}\sum_g \chi_{D_a}(g)^*\chi_{D_b}(g) = \frac{1}{N}\sum_g D^*_{aii}D_{bjj} = \sum_g \frac{1}{n_a}\delta_{ab}\delta_{ij}\delta_{ji} = \delta_{ab}$$

and thus the characters of two different representations are necessarily different, since otherwise this is a sum of squares, and can therefore not be zero except when all characters would vanish.

Because of the trace nature, the characters are invariant under conjugation,

$$\mathrm{tr}D(h^{-1}gh) = \mathrm{tr}D(h^{-1})D(g)D(h) = \mathrm{tr}D(G), \tag{3.10}$$

and thus constant on conjugacy classes.

For finite groups, there is an important consequence. Take a function $f(g)$ of the group elements, which is constant on a conjugacy class. Any function of the group elements must be of the form

$$f(g) = c^a_{jk}D_a(g)_{jk} = \mathrm{tr}c^aD_a = \mathrm{tr}cD.$$

This is nothing but the statement that a map in the group must be representable. The limitation to finite groups comes from the desire to express this as a sum over irreducible representations, which is only possible if the reducible representations are always completely reducible, which is only true for finite groups.

If the function is constant on conjugacy classes, $f(h^{-1}gh) = f(g)$ then

$$f(h^{-1}gh) = \mathrm{tr}cD(h^{-1}gh) = \mathrm{tr}cD(h^{-1})D(g)D(h) = \mathrm{tr}cD(g) = f(g)$$

For this to be true, it must hold for any h. Thus, it must hold also for a sum over all h, appropriately normalized

$$f(g) = \frac{1}{N}\sum_h f(h^{-1}gh) = \frac{1}{N}c^a_{ij}D_a(h^{-1})_{jk}D_a(h)_{li}D_a(g)_{kl} = \sum_a \frac{1}{n_a}\mathrm{tr}c^a\chi_a(g),$$

where the orthogonality relation (3.7) has been used in the last step. This implies that such constant functions are only characterized by the traces of the matrices c, rather than by the full matrices.

The important insight to be gained from this follows from the fact that the sum is over the number of irreducible representations. All functions which are constant over conjugacy classes can be expressed like this, and since the characters are orthogonal, in general all irreducible representations are necessary. At the same time it would be possible to define functions, which are non-zero only on one conjugacy class, and zero elsewhere. This gives a complete basis of these functions. Since both ways of representing this class of functions must be equal, this implies the dimensionality of both bases must be equal, and hence the number of conjugacy classes must be the same as that of irreducible representations. This insight will play a crucial role later.

This has a further important consequence. Define the matrix

$$V_{ai} = \sqrt{\frac{k_i}{N}}\chi_{D_a}(g_i),$$

where k_i is inserted for later convenience. Then (3.10) implies $V^\dagger V = 1$. But since the matrix is square, this also implies $VV^\dagger = 1$, and thus the characters are also orthogonal over group elements

$$\chi_{D_a}(g_i)^*\chi_{D_a}(g_j) = \frac{N}{k_i}\delta_{ij} \tag{3.11}$$

Which is again quite useful.

As an example for the usefulness of these manipulations, consider the regular representation. Due to the definition (3.6), the character $\chi_R(1) = N$, since the regular representation for a finite group is N-dimensional. At the same time, all of the other representation matrices are necessarily traceless, since otherwise a group element would map some group elements into themselves, which is only possible for the unit element. Hence, the other characters are zero. Now, (3.10) implies that for a general reducible representation r

$$\frac{1}{N}\sum_g \chi_{D_a}(g)^*\chi_{D_r}(g) = \frac{1}{N}\sum_g \chi_{D_a}(g)^*\sum_{a\in R}\chi_{D_a}(g) = \sum_{a\in R} = m_a^r,$$

since the trace of a block-diagonal matrix is just the sum of the traces of the blocks. Hence, m_a^r is the number of times an irreducible representation a appears in the reducible representation r. For the regular representation, it is also true that

$$\frac{1}{N}\sum_g \chi_{D_a}(g)^*\chi_R(g) = \chi_{D_a}(e)^* = n_a,$$

since $\chi_{D_A}(e) = \mathrm{tr} D_A(e) = n_a$, as it is just (unitarily equivalent to) the representation of the unit matrix in the representation a. Thus, $m_a^r = n_a$. Hence, every irreducible representation a appears in the regular representation exactly as often as its dimension.

For the standard example, the characters in the adjoint representations are given in two dimensions by $2\cos\alpha$, with α the rotation angle. Hence, they are 2, 0, -2 and 0 for the subgroup with four rotations. Due to the cyclicity of the trace, these are invariant under any further rotations, which act as similarity transformations.

3.7 Abelian groups

Abelian groups have a number of simplifying properties. First note that $h^{-1}gh = h^{-1}hg = g$, due to the Abelian nature. Hence, each element of an Abelian group is a conjugacy class itself. This implies that for finite groups the number of irreducible representation must be the same as the group order, due to (3.9). Hence, all irreducible representations of Abelian groups are one-dimensional. This statement is actually only true if the considered vector space is complex. For a real vector space, this is no longer necessarily true, as the standard example of rotations in two dimensions shows.

Especially, this implies that all irreducible representations of finite, Abelian groups are automatically diagonal. This has important consequences for a quantum system. If H has some finite, Abelian symmetry, e. g. parity and thus a Z_2 symmetry group, then all the appearing irreducible representations can be simultaneously diagonalized with H. Thus, since they are hermitian, the associated quantum numbers become observable. Note that this is a stronger statement than to have a symmetry. A symmetry can always have a unitary representation, but not every unitary representation is diagonal, and therefore not all eigenvalues are observables. E. g., in the case of spin only the total spin and one component can be measured, but not all components individually, since not all components are simultaneously diagonal. Thus, the diagonalizability of representations is important for the question of possible observables in quantum physics.

3.8 Tensor products

So far, most effort was invested into steering towards smaller, i. e., irreducible representations. Building tensor products is quite the opposite. Assume some representation D_1 on a vector space such that there are n base vectors $|i\rangle$, and a second representation D_2 on a second vector space having m base vectors $|j\rangle$.

It is possible to construct a combined vector space $|i, j\rangle$ just by combining both bases.

This space has $n \times m$ base vectors. Since both representations act on individual sub-spaces, a tensor product $D = D_1 \otimes D_2$ of both representations can be formally defined as being just a representation acting on the two representations individually. Especially, its matrix elements are

$$D_{klij} = \langle k, l|D|i, j\rangle = \langle k|D_1|i\rangle\langle l|D_2|j\rangle, \tag{3.12}$$

which are by construction $(m + n) \times (n + m)$ elements. Of course, even if the original representations D_1 and D_2, which are $n \times n$ and $m \times m$ matrices in their originally subspaces, are irreducible, the tensor product representation D needs no longer be so. In fact, one of the most important tasks in many cases is to deconstruct a given representation in the relevant irreducible representations, as will be seen later in chapter 5.

An important consequence of (3.12) is that characters of a tensor product are products of the characters of the individual characters,

$$\chi_D = \mathrm{tr}D = D_{kiki} = D_{1kk}D_{2ii} = \chi_{D_1}\chi_{D_2}$$

and thus the characters can be used to identify which representation is obtained from the tensor representation, as the characters uniquely characterize a representation.

For rotations, the tensoration is known from, e. g., the coupling of two particles with angular momentum. Different bases are then obtained from the Clebsch-Gordon procedure.

3.9 Symmetry group of the $(2n + 1)$gon

An example for the concepts so far is the symmetry group of the $(2n + 1)$gons, i. e. the regular, closed polygons with $2n+1$ nodes. These objects are transformed into themselves, and therefore symmetric, under rotation by $\pm 2\pi j/(2n + 1)$ for $j = 1...n$. The symmetry group furthermore contains, trivially, the identity, but in addition all $2n + 1$ reflections by planes through one vertex and one edge. Thus, the size of the group is finite and $1 + (2n) + (2n + 1) = 2(2n + 1)$.

This group has furthermore $n + 2$ conjugacy classes. One is just given by the identity. Another one is the set of reflections, as any reflection can be transferred into any reflection by rotations. Thus, a reflection conjugate by a rotation will again yield a (in general different) reflection, and thus the set of all reflections is closed under rotations, and of course, the identity, and hence a conjugacy class. The final set are rotation at fixed j. A rotation by j will become under reflection a rotation by $-j$, and therefore both elements are necessarily in the same conjugacy class. The definition of conjugation requires $g^{-1}sg$ for any element s. Since the rotations in the plane are Abelian, this implies that such a

prescription is: rotate by g, then by s, and then back by g, and thus only by s. Hence, any rotation, together with its mirrored companion, is a conjugacy class of its own, adding another n. Of course, since the vertices are identical, any rotation conjugacy class is equivalent.

This symmetry group has the identical representation. There is a second one-dimensional representation, which maps all rotations again to the identity and all reflections to the negative identity, and which is thus inequivalent. Thus, the character only for reflections are not 1 but -1. Both representations are not faithful. Since the number of conjugacy classes has to be the number of irreducible representations, there are n more necessary. These are given by two-dimensional matrices. There are three base matrices, when the 'odd' point is on the x-axis

$$\begin{aligned} D(e) &= 1 \\ D(P) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ D(R) &= \begin{pmatrix} \cos\frac{2\pi m}{2n+1} & \sin\frac{2\pi m}{2n+1} \\ -\sin\frac{2\pi m}{2n+1} & \cos\frac{2\pi m}{2n+1} \end{pmatrix} \end{aligned}$$

These are the unit matrix, a reflection through the x-axis, changing the sign of all y components, and the smallest possible rotation if the the maximal rotation should cover 2π m times. The latter creates n inequivalent representations, as they perform n different numbers of full rotations. Other elements are obtained by rotated reflection matrices and by replacing the angle by $2j\pi m/2n+1$. The characters are thus 2 for the unit matrix, 0 for all reflections, and twice the cosine of the angle for all rotations.

3.10 Young tableaux

The techniques of Young tableaux is a powerful possibility to talk about permutation symmetries. This is especially useful as with bosons and fermions permutation symmetries play an important part in quantum physics.

Consider first the trivial cyclic group, i. e. S_1. It has only one representation, the trivial one. The first rule to construct a Young tableaux is then to denote a trivial representation as a box.

The next one is S_2, with the elements 1, the identity, and 2, exchange objects 1 and 2. It thus acts on a two-element set. The first group element does nothing, while the second group element interchanges both elements. It is furthermore its own inverse. There are two different irreducible representation: The trivial one and a second one. To construct

the Young tableaux for the trivial representation, the rule is that a row of boxes with as many boxes as group elements will be a trivial representation. Thus, in this case, two boxes. The second representation is special in this case: It is also a cycle, i. e. it permutes a subset of elements (in this case all elements) in a particular way, which cannot be undone by an even number of permutations. In a sense, also the identity is a trivial cycle, since no element is interchanged, and therefore every element is cyclically interchanged with itself. To write down a cycle, set the elements cyclic together in parentheses, yielding the two distinct cycles (1)(2) and (12). Represent now the non-trivial cycle with two boxes over each other as the second rule of Young tableaux.

For S_n, it is important to note the following: Any cycle is also a conjugacy class. This follows, as the requirement of a conjugacy class to be a set S invariant under all group elements g as $g^{-1}Sg$ is automatically satisfied: The permutation and inverse permutations g (which are just an even number of sequences of interchanges) do not change that the cycles remain. Since for a finite group the number of conjugacy classes is the number of irreducible representations, the possible number of cycles is directly equivalent to the number of irreducible representations.

For S_2 there are two cycles and two irreducible representations, each represented by a Young tableaux.

For S_3, there are three cycles. One is the trivial one. Another is given by the possibilities to arrange two elements cyclic, and the third is where all three elements are cycles. These are (1)(2)(3), i. e. one possibility. In the same way as before, this will be three boxes in a row. For all three elements, there are two distinct group elements (123) and (321). However, the important property of a cyclic structure is in both cases identical - there is a distinct order of elements, cyclic and anti-cyclic. Thus, they both belong to the same conjugacy class. It is therefore represented as three boxes over each other. Finally, there are the cycles (12)(3), (13)(2), and (23)(1), three elements, all combination of a two-cycle and a one-cycle. They should therefore be given by a single box and two boxes over each other. By convention, the single box is now attached on the top-right of the two boxes, giving the last Young tableaux, in total 3. Thus, again each Young tableaux presents a distinct irreducible representation of the permutation group.

In S_4, there are 4 1-cycles, (1)(2)(3)(4), 6 2-cycles+2 1-cycles (12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2), 3 times 2 2-cycles (12)(34), (13)(24), (14)(23), 8 3-cycles plus 1 1-cycle(123)(4), (134)(2), (124)(3), (234)(1) plus anticyclic, and 6 4-cycles (1234), (4321), (1243), (2134), (4123), (2413). Following the rules, these are four boxes in a row, 2 boxes over each other and two top-right attached boxes, a square of four boxes, thee boxes over each other and a top-right attached box, and four boxes over each other.

Conversely, from just arranging the boxes, all possible permutations would have been obtained. This can be generalized in an obvious way: For any S_n, all distinct Young tableaux give the irreducible representations, and thus also all cycle structures.

The most powerful option is, however, to create the irreducible representation using a Young tableaux.

Start with S_3. The are three types of Young tableaux. Take first the case of the horizontal line. Then assign to each of the boxes a number, say (123). Now symmetrize this state, i. e. make it symmetric with respect to any exchange,

$$(123) + (132) + (213) + (312) + (231) + (321).$$

Any permutation, say 1 and 2, just returns the state. Thus, this is just the trivial state, as it is reproduced for any application of any group element. Assign next to the vertical line the antisymmetrized state

$$(123) - (132) - (213) + (321) + (231) - (321).$$

This state will become its negative for any permutation. These are the two elements of the last conjugacy class. Finally, for the edge-box, symmetrize again in horizontally and antisymmetrized vertically. There are three different possibilities to do so

$$\begin{aligned}(123) + (213) - (321) - (231)\\ (231) + (321) - (132) - (312)\\ (312) + (132) - (213) - (123)\end{aligned} \quad .$$

Since the sum of all three elements vanishes, only two are linearly independent, and the dimension is thus two, as expected. In this way, another set of explicit irreducible representations is created, essentially a different basis, just from the structure of the Young tableaux. Note that there are three more states, which have the same symmetry properties, but differ only by a minus sign, and are obtained from exchanging the numbers in columns. This is a general feature. The structure is here decisive, telling precisely how many symmetrizations or antisymmetrizations are necessary for each member of a given irreducible representations. It is thus a manual of how to create the irreducible representations of a S_n group.

This also tells that the dimension of the irreducible representations can be obtained from the number of different possible Young tableaux, which can be constructed. It is given by $n!/H$, where $H = \Pi h_i$ is obtained by counting the size of the hooks. For every box, draw a line, which makes a right-turn in the box, and count how many boxes this line goes through. For S_3, there are three cases, and $3! = 6$. For the horizontal line, it

is $3 \times 2 \times 1 = 6$, and thus the representation is one-dimensional. Similarly, the same is true for the vertical line. The final one is $1 \times 3 \times 1 = 3$, and thus the representation is two-dimensional, as required. Also this can be proven in general.

So far, the Young tableaux appear to be quite special to permutation groups. However, symmetrization and anti-symmetrization is a defining property of bosons and fermions, and therefore the concept of Young-tableaux will play a crucial role for the creation of Fock spaces in quantum theories.

Chapter 4

Lie groups

Though discrete groups play an important role in many aspects of physics, an even more important role play continuous groups, as most discrete groups are in the end just restricted continuous groups. Therefore, the next step is to discuss continuous groups, and return to discrete groups when they appear as restriction of continuous groups. Continuous groups, as their name indicates, are groups with a denumerable infinite number of group elements. A simple example are the real numbers with group combination the addition. Any sum of two real numbers is again a real number, the addition is associative, the zero is the unit element, and the negative of the real number is the inverse. Likewise, the real numbers without zero and infinity (or both included when defining $1/0 = \infty$) under multiplication satisfy the group axioms with the unit element now 1. The same is true for complex numbers. Thus, continuous groups are central. Note, however, that the integer numbers are a discrete group under addition, as there is just a denumerable infinite number of them. All of these are Abelian groups.

When it comes to continuous groups, the paradigmatic example, and probably the most important one for physics besides the real and complex numbers, is the class of Lie groups. These are a special class of continuous groups. They will be the central topic of this chapter.

Compared to the numbers, Lie groups have a number of differing properties. In fact, they are more rigid, as they fulfill additional constraints, and more flexible, as they also permit non-commutative structures.

The structure of continuous groups which will be central is that group elements g are no longer counted by an index, but by one or more parameters α, where α is a single, real-valued number or a vector of real-valued numbers, $g(\alpha)$. This relation needs to be one-to-one. Furthermore, any composition of group elements will yield a new group element

chosen as a function f of the two parameters

$$g(\alpha) \circ g(\beta) = g(f(\alpha, \beta))$$

where the choice of the parameter space, which can be some patch or union of patches of the real-valued $\mathbb{R}^n$, together with the composition function determines the group.

Particular cases will be continuous groups where it is possible to define a notion of closeness of two group elements $g(\alpha)$ and $g(\beta)$, which is based on the closeness of α and β. In this case, the group is smooth. However, this will not preclude the possibility that there are disjunct patches of group elements.

It is in principle possible to generalize the concept by not using real numbers to parameterize the group, to more abstract objects, in particular another group. We will return to this, if need be.

The standard example of rotations are continuous groups, and especially Lie groups. In two dimensions, the function f in the group composition is just the addition of the two angles. In three dimensions, the three angels form a three-dimensional vector. In general, the combination of two group elements will not be just by the addition of the angles, as this is not reflecting the non-Abelian nature of rotations in three dimensions. The composition function is therefore more complicated, and can be taken, e. g., from the theory of the Euler representation.

4.1 Generators

For the discrete numbers, the assignment of group element to index was arbitrary. Which element is number 1 and which number 2 did not matter. Essentially, this assignment is only an irrelevant choice of coordinate system. Likewise, the parameters α are vectors in an n-dimensional vector space. Thus, there is a freedom of choice in their basis. However, a useful convention is to define

$$g(0) = e,$$

i. e. the group element for all parameters vanishing is the unit element. Given its important role in the group axioms of section 2.1 this appears appropriate, and will indeed simplify many calculations considerably.

Any representation of a group will now also depend on the parameters. In fact, since there must exist a one-to-one map from the parameters to group elements, this can also be regarded as the representations being a function directly of the parameters, $D(g(\alpha)) = D(\alpha)$. However, since the representation needs not to be one-to-one, neither needs the relation of the representation elements to the parameters to be one-to-one. It

is furthermore useful to transport the convention on the unit element from the group to representations of the group,

$$D(g(0)) = D(0) = 1,$$

where 1 is again the unit element.

A representation is now smooth in a patch of the parameter space, if inside this patch the representation is determined by an analytic function of its parameters, i. e. it can be written as a Taylor series. Especially, if the parameter vector δ is only infinitesimally different from zero, it must be possible to just use the leading term[1]

$$D(\delta) = 1 + i\delta_a X^a + \mathcal{O}(\delta^2), \tag{4.1}$$

where the elements[2] X^a belong to the vector space of the representation, i. e. they are usually matrices as well, and depend on the representation. If the parameters are N-dimensional vectors, there are N elements X_a. This number is unrelated from the actual dimension d of the vector space in which the representation is defined, which can be larger, smaller, or equal to N. These are called the generators of the group. The i appears by convention, and will yield suitable hermiticity and unitary properties of various appearing matrices throughout. Moreover,

$$X_a = -i\frac{\partial}{\partial\delta_a}D(\delta)|_{\delta=0} = -i\partial_a D(0),$$

when considering the representation matrices as functions of the parameters. Finally, if the representation is such as that two different sets of parameters yield two different representation elements, then the generators are necessarily linearly independent. Otherwise, there would exist a basis such that $D(\delta)$ would be independent of some δ_a (so-called not parsimonious), and then the element $D(\delta_a)$ would be linearly dependent on other group elements, and therefore the corresponding generator would be.

For unitary representations, because of

$$1 = D(\delta)D^{-1}(\delta) = D(\delta)D^\dagger(\delta) = (1+i\delta_a X^a)(1-i\delta_a X^{a\dagger})+\mathcal{O}(\delta^2) = 1+i\delta_a(X^a-X^{a\dagger})+\mathcal{O}(\delta^2)$$

it follows that the generators are Hermitian.

[1] In general it may be that a Taylor series has as first term a non-trivial term, and it may also be that the second term is not the linear one. It can be shown that there exists always a parameter redefinition which yields a linear behavior to leading order. This is possible, as there are one-parameter subgroups based on a single generator, under which the composition rule becomes the ordinary addition. These coordinates are called normal.

[2] In most cases, the position of the indices do not matter.

4.2 Algebras

Actually, the generators are more fundamental quantities than the group elements. Especially, it is possible to formally exponentiate the resulting series (4.1) describing a non-infinitesimal group element to obtain

$$D(\delta) = \lim_{N\to\infty}\left(1 + \frac{i\delta_a}{N}X^a\right)^N = \exp(i\delta_a X^a). \tag{4.2}$$

Formally, thus, a given group element is reached by performing an infinite number of infinitesimal steps. This is not a unique definition, as the definition (4.1) is compatible with any function which has a linear term in the Taylor series. However, it is the convenient one. Other definitions do not change anything on a qualitative level. Note, however, that it may still be necessary to chose for different group elements different points for the expansion.

Hence, it would be possible to describe all of group theory on an abstract level only using the generators, and never mentioning the group elements or even their representation. The so-obtained structure is called the (Lie) algebra, the set of all elements

$$X = \delta_a X^a$$

As a consequence, this algebra must be closed under addition, there must be a zero element as an inverse, and a multiplication by (real) numbers[3]. As these are also the criteria for a vector space, the generators do form a vector space. That is different from the group, where especially multiplication by scalars in some representations is usually not mapping a group element into a group element. Thus, the relation (4.2) actually describes the map of a vector space to a group. Especially, as will be seen, it is possible to characterize a Lie algebra by just specifying an abstract relation

$$[X^a, X^b] = if^{abc}X^c, \tag{4.3}$$

with $f_{abc} = -f_{bac}$. Note that since an algebra is a vector space of which the generators form a basis, they are not unique. Just with any other basis, they can be redefined into any other set of linearly independent basis vectors.

As can be shown that, for given so-called structure constants f^{abc}, (4.3) is a unique characterization[4], this relation is also called itself algebra. Actually, not the values of the

[3]It is also possible to enlarge this to complex numbers, but this does not yield anything relevant to physics.

[4]Note that sometimes the pre-factor in the algebra is different from i. Sometimes the i is taken as part of the structure constants, or additional factors like 2 or 1/2 appear, and thus the precise form of the algebra is subject to conventions.

structure constants themselves are unique, as when rescaling both the generators and the structure constants the algebra remains invariant, but the relations between the structure constants.

A fact sometimes relevant in physics, to be discussed later in more detail, is that (4.2) is not a unique map from the algebra to the group. Due to the periodicity, it is possible to obtain different groups, depending on how the δ_a are defined. These groups can differ at most by elements which commute with all other elements, and thus only with respect to the center of the group. This will be taken up later. Until then, it will also be specified explicitly which group is relevant. Thus, the generators and the algebra are, in a sense, more fundamental.

Given this setup, there is a one-parameter family of group elements characterized by a fixed generator X

$$D(\lambda) = e^{i\lambda X},$$

in which the group composition is given by

$$D(\lambda_1)D(\lambda_2) = e^{i(\lambda_1+\lambda_2)X} = D(\lambda_1+\lambda_2) = D(\lambda_2)D(\lambda_1)$$

and thus by addition of the parameters λ_i. This sub-group, as it also includes always with $\lambda = 0$ the unit element and is closed under the group composition, is hence an Abelian subgroup of the group.

When considering now the composition of two different group elements

$$e^{i\alpha_a X^a} e^{i\beta_a X^a} = D(\alpha)D(\beta) = D(\gamma) = e^{i\gamma^a X_a} \tag{4.4}$$

the particular relevance of the algebra becomes manifest. Because the generators may not commute, γ will in general not just be the sum of α and β. For this the Baker-Campbell-Hausdorff formula

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) - \frac{1}{24}[Y,[X,[X,Y]]] + ...\right)$$

becomes relevant. It shows that for the group composition rule (4.4), the commutator of two generators must again be a generator, since otherwise products of generators would remain in contradiction to (4.4). But the algebra ensures that this is the case. Put it differently, the necessity to implement the group composition forces the algebra to obey (4.3).

Of course, since this is an infinite series, it is not always trivial to calculate γ as a function of α and β. Furthermore, the anti-commutativity also implies that derivatives do

act non-trivial on the α^a,

$$\partial_{\alpha^b} e^{i\alpha_a X^a} = i \int_0^1 ds e^{is\alpha_a X^a} X_b e^{(1-s)\alpha_c X_c}$$

which can be obtained similarly from the series expansion. Of course, in the infinitesimal case, this reduces to iX_b.

It should be noted that (4.3) does not take notice of the actual representation of the group, and especially not, whether it is reducible or irreducible, as long as there is at least one distinct element for every group element. But the values of the f^{abc} are nonetheless fixed by the group composition, as can be seen by evaluating (4.4) to second order,

$$e^{i\alpha_a X^a} e^{i\beta_a X^a} = \exp\left(i(\alpha^a + \beta_a)X_a + \frac{\alpha_a \beta_b}{2}[X^a, X^b]\right) = \exp\left(i\left(\alpha^a + \beta^a + if^{abc}\frac{\alpha^b \beta^c}{2}\right)X^a\right),$$

since the parameters determine uniquely the group element, which in turn is uniquely determined by the group composition, and which therefore determines the structure constants. Since in this calculation at no point the properties of the representation entered, the structure constants have, up to normalization, a unique value for all representations, and hence are a structural property of the algebra, and thus group. This is not true for the generators, which change depending on the representation.

There is one more statement, which can be made about the structure constants. If there exists at last one unitary representation, then because of the hermiticity of the generators,

$$-if_{bac}X_c = -[X_b, X_a] = -[X_b^\dagger, X_a^\dagger] = -[X^a, X^b]^\dagger = -i^* f^{abc*} X_c^\dagger = if^{abc*} X_c$$

and thus the Lie algebra relation

$$[X^a, X^b] = if^{abc} X^c, \tag{4.5}$$

which can only be true for $f^{abc} = f^{abc*}$. Thus, when using this form of the algebra with the factor of i explicit, then the structure constants are necessarily purely real for any group which has a unitary representation. Since essentially all continuous groups in physics are of this type, this will be assumed henceforth, except when otherwise noted.

Another identity, which follows directly from the algebra (4.3), is the Jacobi identity

$$\begin{aligned} & [X^a, [X^b, X^c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] \\ = \; & if^{bcd}[X^a, X^d] + if^{cad}[X_b, X_d] + if^{abd}[X_c, X_d] \\ = \; & i(f^{bcd}f^{ade} + f^{cad}f^{bde} + f^{abd}f^{cde})X^e = 0. \end{aligned} \tag{4.6}$$

The last step is a consequence of the the antisymmetry of the structure constants in their first two indices, as can be shown by explicit computation, at least for the cases relevant in this lecture[5]. Using that the generators are independent, the last equation implies actually that already the sum of products of structure constants in the parentheses has to vanish. It is a very useful relation both in the form of the generators and the structure constants.

For the standard example of rotations, the situation is comparatively simple for rotations in two dimensions. Since there rotations commute, all structure constants vanish. For the rotations in three dimensions, this is no longer the case, as they are non-commutative.

A possible realization of the abstract generators of two-dimensional rotations is the single generator

$$R_1^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

which after exponentiation yields precisely the representation of group elements (3.1). An alternative is the one-dimensional generator 1, if the complex one-dimensional representation of two-dimensional rotations is used.

A possible representation of the three-dimensional rotations to yield after exponentiation the group representation (3.2) is obtained by

$$R_1^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

$$R_2^3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{4.8}$$

$$R_3^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \tag{4.9}$$

which therefore is a direct extension of the two-dimensional case. From these, it is straightforward to calculate the structure constants, which turn out to be just the anti-symmetric ϵ tensor, up to some constant. As will be seen later, it is no coincidence that these matrices and structure constants look reminiscent of the Pauli matrices, and that there is a deep relation, due to group theory, between the two groups describing spin 1/2 and spin 1 particles.

Note that there is an alternative way to derive the algebra: If a local representation of the group should exist which maintains the group composition, this implies certain

[5]For infinite-dimensional, or worse, cases, it is not entirely trivial.

analyticity constraints for the continuous representations. It can be shown that they can only be met with an algebra of this structure, where the structure constants satisfy the Jacobi identity.

4.3 Adjoint representation

As the next step it is useful to discuss a very special representation for any Lie algebra: The Jacobi identity can be used to construct the generalization of the adjoint representation of section 3.4 to Lie algebras. Define matrices

$$(t^a)_{bc} = if^{abc}.$$

Then the Jacobi identity can be rewritten, by judiciously replacing some of the structure constants, but not all, as

$$(t^b)_{cd}(t^a)_{de} - (t^a)_{cd}(t^b)_{de} - if^{abd}(t^d)_{ce} = 0,$$

but this is just the Lie algebra (4.3), and thus the t^a are a representation of the generators. For an n-dimensional[6] Lie algebra, these are therefore $n \times n$ matrices, and this is a representation on an n-dimensional, real vector space. Real, as the matrices are all purely imaginary, since the structure constants are real. By exponentiation, which after multiplication with i are purely real matrices, this yields also an $n \times n$-dimensional representation of the group, thus also called the adjoint. Note that the so-obtained adjoint representation is not necessarily irreducible or faithful.

Since the generators form a vector space, it is possible to perform a linear transformation on them without changing the algebra relation,

$$X^{a'} = L_{ab}X^b.$$

However, such a change induces a change of the structure constants to maintain the form of the algebra

$$L_{ad}L_{be}[X^d, X^b] = [X^{a'}, X^{b'}] = if'_{abc}X^{c'} = if'_{abc}L_{cf}X^f$$

implying

$$f'^{abc} = L_{ad}L_{be}(L^{-1})_{cf}f_{def}.$$

Since any linear transformation must be invertible, the new structure constants exist. Note any rescalings, even by complex numbers, are acceptable linear transformations, implying

[6] As a vector space, the dimension of a Lie algebra is given by the number of linearly independent generators.

that different prefactors can, and do in different conventions, appear in the structure constants.

It is an interesting fact that one of the indices does transform differently. Rewriting (4.3) as

$$[X^a, X^b] = f_c^{ab} X^c,$$

would make this more explicit. This is not coincidental that this is reminiscent of covariant and contravariant vectors, as it is possible in general to extend the concept of Lie groups to vector spaces with non-trivial metrics, where there are then indeed covariant and contravariant generators, and the structure constants are tensors of third rank, similar to the Christoffel symbols of general relativity. However, in almost all of physics such more general algebras, and resulting groups, do not play a role, and not at all in this lecture. Therefore, in the following mostly a flat metric in the vector space of generators will be assumed, except when noted otherwise. Then, the position of indices does not matter, and will only be used to enhance readability.

As noted, the generators form in general a vector space. A useful scalar product for the adjoint, and later also for other, representation is defined by

$$\mathrm{tr} t^a t^b = d_{ab}, \tag{4.10}$$

which is symmetric, and yields, due to the imaginarity of the t^a, a real number. In fact, d_{ab} is, by construction, a real and symmetric matrix.

For an arbitrary linear transformation, this changes to

$$L_{ac} L_{bd} \mathrm{tr} t^c t^d = L_{ac} L_{bd} d^{cd},$$

Since arbitrary linear transformations are possible, and d is a real and symmetric matrix, it is possible to diagonalize d, $d = \mathrm{diag} k_1, ..., k_n$, where the k_i are the eigenvalues of d. By choosing for the L a scale factor, it is possible to further transform the k_i always to either 1, 0 or -1. However, since the transformation occurs with the square of L, it is not possible to change the signs, or transform any away from zero.

The values of these eigenvalues actually classify the Lie algebras. The ones most relevant in most of physics are the so-called compact Lie algebras, with $k_i = 1$, i. e. all eigenvalues are positive. These will be the primary concern for now. Another one of particular interest to physics are those with one k_i negative. These non-compact groups contain, e. g., the Lorentz and Poincare group. Groups with any eigenvalue zero are special. Non-compact groups have their name as there exists no non-trivial finite-dimensional representation, while such do always exist for compact groups. This will be seen later.

Returning to compact groups, it is possible to rescale all eigenvalues to some convenient number λ, and thus

$$\mathrm{tr} t^a t^b = \lambda \delta_{ab}.$$

This basis is particular useful, as the structure constants now take conversely the values

$$f_{abc} = \frac{1}{i\lambda}\mathrm{tr}([t^a, t^b]t^c) = \frac{f_{abd}}{\lambda}\mathrm{tr}(t^d t^c),$$

as obtained from rearranging the algebra. Because the trace is cyclic, the structure constants in this basis are now antisymmetric in all three indices, rather than only in the first two. Since this requires the group to be compact, this underlines once more the specialness of the last index of the structure constant in general.

Note that because the generators are then not only imaginary but also antisymmetric, this implies they are hermitian. Hence, the adjoint group representation obtained by exponentiation becomes unitary. Compact groups therefore have always at least one finite-dimensional, unitary representation. At least, if the number of generators is finite.

The standard example of the two-dimensional rotations has the trivial representation, i. e. $t^a = 0$ and thus all group elements being 1, as the adjoint representation, as it is an Abelian group. For the rotations in three dimensions, the adjoint representation is three-dimensional, and coincides with (4.7-4.9).

4.4 Simple algebras and groups

Similar to the case of subgroups in the discrete case in section 2.3, it is possible to construct also for continuous groups subgroups. However, this can now also be done using the algebra.

An invariant subalgebra, containing the generators $\{Y^a\}$, is defined by the fact that for any X^a in the whole algebra

$$[X^a, Y^b] \in \{Y^a\}, \tag{4.11}$$

and thus the invariant subalgebra is closed.

The invariant subalgebra also creates an invariant subgroup, i. e. for any X in the algebra there is a $Z \in \{Y^a\}$ such that

$$e^{iX}e^{iY} = e^{iZ}e^{iX}.$$

or, equivalently,

$$e^{iX}e^{iY}(e^{iX})^{-1} = e^{iZ}$$

This can be seen by expanding both sides to leading non-trivial order in X and Y, yielding

$$(1+iX)(1+iY)(1-iX) = 1+iY-XY+YX+\mathcal{O}(X^2,Y^2) = 1+iY-[X,Y]+\mathcal{O}(X^2,Y^2),$$

If the subalgebra is invariant, then the commutator is again an element of the algebra, and thus the sum of iY and the commutator is again an element of the subalgebra. Therefore, this is again a group element obtained from the invariant subalgebra, and therefore this is indeed an invariant subgroup.

By definition, both the trivial algebra containing only 0 and the full algebra are invariant subalgebras. These are called trivial subalgebras. If there are no non-trivial invariant subalgebras, the algebra, and group, are called simple. As it will turn out, most things about compact Liegroups and Liealgebras can be obtained from the study of simple Liegroups and Liealgebras.

Of course, even if there is no invariant subalgebra, there can still be non-trivial subalgebras, i. e. a set of generators $\{W^a\}$ satisfying

$$[W^a, W^b] \in \{W^a\}$$

but not (4.11). The existence of one or more non-invariant subalgebras is actually the normal case.

Selecting a convenient basis such that there are generators Y^a belonging to an invariant subalgebra and X^a which do not belong, then

$$[X^a, Y^b] = if_{abc}Y^c,$$

by construction. This implies that the f_{abc} have to vanish whenever the indices a and c are both from the algebra, but b is from an invariant subalgebra. Furthermore,

$$[X^a, X^b] \notin \{Y^c\}$$

by definition: (4.11) requires that any X^b which yields with X^a an element of the invariant subalgebra is itself part of the the invariant subalgebra. Thus, any structure constants where the indices mix the invariant subalgebra and the remainder of the algebra have to vanish. It should be noted conversely, that thus the remainder of the algebra necessarily also constitutes at least one other invariant subalgebra.

This result has implications for the reducibility of the adjoint representation. If the algebra would be non-simple, then this implies that generators have non-vanishing submatrices if the index a of the generator t^a belongs to a different subalgebra than the indices ij of the sub-matrices t^a_{ij} of t^a. After exponentiation, this implies a unit matrix in this submatrix. Thus, there would be invariant subspaces in this representation, and thus

the representation would be reducible. However, if there is no invariant sub-space, such sub-matrices do not exist, and therefore the adjoint representation of a simple Liealgebra is irreducible. This can also be seen by the fact the invariant sub-algebras form distinct, invariant sub-spaces, and thus the algebra would be reducible.

An important exception are cases where the is a an invariant sub-group, which consists only out of a single generator. This subalgebra is therefore necessarily Abelian. For an Abelian invariant subgroup the corresponding structure constants vanish. This implies that the generator of the Abelian subalgebra anticommutes with all other generators, and thus

$$\mathrm{tr} t^a t^b = -\mathrm{tr} t^b t^a = \mathrm{tr} t^a t^b$$

which can only be true if the corresponding k^a vanishes. Thus, the existence of an invariant Abelian subalgebra implies that the algebra is not compact, but has flat direction. Thus, compact algebras cannot have invariant Abelian subalgebras. Algebras without such invariant Abelian subgroups are called semisimple. Note that a semisimple group is not necessarily simple, as it can have non-Abelian subgroups. Also, a compact group can have non-Abelian invariant subgroups, and may therefore be not simple. It is semisimple groups which are highly constrained by group theory. These are the elementary objects, and out of them others can be constructed. Thus, unless otherwise noted, in the following only compact, semisimple groups and algebras will be considered. Note that as a consequence the representations of the algebra will be usually (anti-)Hermitian, and of the group unitary.

The standard examples are quite different with respect to this. The two-dimensional rotation is itself Abelian, and appears not compact. However, it has, strictly speaking, no non-trivial invariant subgroup, and therefore is both simple and semisimple. The three-dimensional rotations have no Abelian or non-Abelian invariant subgroups, though they have non-invariant subgroups. Thus, they are compact, simple, and semisimple.

4.5 The simplest case: su(2)

To start out, it is useful to consider the simplest possible compact semisimple and non-Abelian Liealgebra. This will not only illustrate many powerful concepts to be generalized later, but this algebra will also reappear many times over as a building block.

Since this requires to have a non-vanishing three-index antisymmetric tensor, this implies that at least three elements are necessary. Since in this case all other elements are fixed by the single element f_{123}, it is always possible to normalize this element to 1, and

thus $f_{abc} = \epsilon_{abc}$, the 3-dimensional Levi-Civita tensor. The resulting algebra

$$[J^a, J^b] = i\epsilon^{abc} J^c$$

is not guaranteed to exist. However, it is well known in physics, giving the explicit proof that it exists: This is just the algebra realized by the Pauli matrices, and the algebra is the su(2) algebra[7], which after exponentiation yields the SU(2) group, well-known from the physics of spin 1/2 particles.

The first issue is to construct now representations of the group and the algebra. Of course, these are well known in physics, but for the sake of systematics it is useful to follow here the particular route suitable for generalization in group theory.

To this end, the aim is therefore an explicit matrix realization. Since the representation should be either reducible or irreducible, the aim is to find a block-diagonal representation with the least possible elements of blocks. Since only unitary representations will be considered, and it will be currently assumed that the representations are finite-dimensional, it is always possible to diagonalize at least one generator. In fact, the number of diagonal generators is given by the number of generators which commute with each other. For su(2), this is at most one, given the values of the structures constants, which can be seen by explicit calculation. Let this be J^3.

Since the representation of this generator is diagonal, it has eigenvectors with eigenvalues j_3. These may or may not be degenerate, so they may be characterized as well by a second quantity α_{j_3}, counting the degeneracy. Because of the unitarity, the space spanned by the degenerate eigenvalues can be chosen such as that the vectors for the same j_3 and different α_{j_3} are orthogonal. Necessarily, as the representation is unitary, and therefore all eigenvalues finite, there is a largest value of $j_3 = j$.

To proceed, raising and lowering operators $J_\pm$

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$$

are introduced, just as in spin physics. This may now seem very specific, but it will be seen that this concept can be readily generalized later.

Explicit calculation yields

$$[J_3, J_\pm] = \pm J_\pm \tag{4.12}$$

$$[J_+, J_-] = J_3 \tag{4.13}$$

[7]Note that lower case names will always signify the name of an algebra, while uppercase will signify the associated group.

as the form of the algebra in this new basis. In fact, this is a special case of the general base transformation discussed in section 4.3. This implies

$$J_3 J_\pm |j_3, \alpha_{j_3}\rangle = (J_\pm J_3 \pm J_\pm)|j_3, \alpha_{j_3}\rangle = (j_3 \pm 1) J_\pm |j_3, \alpha_{j_3}\rangle, \tag{4.14}$$

and thus $J_\pm$ changes an eigenvector of J_3 into one of an eigenvalue differing by ± 1. So far this is the same as in spin physics.

However, now these operators will be used to construct the irreducible representations and to provide a constructive approach to completely reduce reducible ones, thus putting their action into the perspective of group theory.

Start with the irreducible, finite-dimensional representations. Since all steps made so far are true for both reducible and irreducible representations, nothing has to be adapted.

Since the representation is finite-dimensional, there is a lowest value, which can be achieved by applying J_-. After that, the only possibility is to generate the zero vector, as otherwise (4.14) would be violated, but since it follows directly from the algebra, it must be inviolateable. Can therefore all eigenvectors be reached by applying J_- often enough to $|j_3, \alpha_{j_3}\rangle$? Assume that there would be an eigenstate of J_3, for which this is not the case

$$j_3 |j_3, \alpha_{j_3}\rangle = J_3 |j_3, \alpha_{j_3}\rangle \neq J_3 \sum_{n_i} J_-^{n_i} |j, \alpha_j\rangle,$$

but which is maximal, in the sense that acting with J^+ on it would yield zero. But such a vector would then also be linearly independent of all possible eigenvectors created from the other highest state. Since both sub-spaces remain disconnected, these two towers of state would form invariant subspaces, a contradiction to the assumption that this is an irreducible representation. Thus, this case can be disregarded.

Since the states $|j_3, \alpha_{j_3}\rangle$ therefore form a complete basis, it is sufficient to determine all matrix elements of the generators to obtain the explicit version of the representation. Already the requirement of the irreducibility implies that there cannot be any degeneracy, as no operator changes α_{j_3}, as can be seen from (4.14). Thus, the eigenstates can not be degenerate, as otherwise there would be copies of the subspaces, and therefore the representation would not be irreducible. Therefore, there is no degeneracy, and α_{j_3} can be dropped as well.

It thus remains to determine the explicit representations. To start, note that due to (4.14)

$$J_- |j_3\rangle = N_{j_3} |j - 1\rangle$$

where N_{j_3} is some j_3-dependent number. Now set $j_3 = j$. Then

$$|N_j|^2 \langle j-1|j-1\rangle = \langle j|J_+ J_-|j\rangle = \langle |[J_+, J_-|j\rangle = \langle j|J_3|j\rangle = j\langle j|j\rangle$$

where in the second step it was used that J_+ applied to the highest state yields zero. Since the eigenvector to J_3 can be assumed to be normalized, this provides the value of N_j up to a phase. But since a rescaling by a phase is always possible without changing the normalization, it is admissible to set it by convention to one and thus

$$N_j = \sqrt{j}.$$

Thus, the action of J_- on the highest state is fully specified.

Conversely, this implies

$$J_+|j-1\rangle = \frac{1}{N_j} J_+ J_- |j\rangle = \frac{1}{N_j}[J_+, J_-]|j\rangle = \frac{1}{N_j} J_3 |j\rangle = N_j |j\rangle,$$

using the same trick. Since the only involved relation used in these calculation was the algebra, this would have remained identical, even if α_{j_3} would have been kept, showing that no operator can move a state outside of the subspace created by the ladder starting from the highest state. This confirms the irreducibility argument above.

The same procedure can be used to create a recursion relation for the N_{j_3},

$$\begin{aligned} N_{j-k}^2 &= \langle j-k|J_+ J - |j-k\rangle = \langle j-k|[J_+, J_-] + J_- J_+|j-k\rangle \\ &= \langle j-k|J_3 + J_- N_{j-k+1}|j-k+1\rangle = j-k+N_{j-k+1}^2, \end{aligned}$$

where the freedom in the phase was already used to have real N_{j_3}. This recursion relation can be solved, e. g. using induction, to yield

$$N_{j_3} = \frac{1}{\sqrt{2}}\sqrt{(j+j_3)(j-j_3+1)}$$

and thus the N_{j_3} depend both on j_3 as well as the highest possible value j.

N_{j_3} vanishes for $j_3 = j-1$. Since J_- only lowers j by one, this j_3 is an integer. Since the only action of J_- is to replace a state by another state, the required, and assumed, vanishing of a state for a finite-dimensional representation can only occur if $j = l/2$, with l some positive integer or zero. In this case a solution is possible which is consistent with the assumptions. Of course, for $l = 0$, there is only a single state, and the representation is the one-dimensional trivial one. Note that this does not constitute a proof that there may be more exotic other finite-dimensional representations than the one constructed here, though it may look obvious at the physical level. However, it can be proven, which will be skipped here.

The non-zero different values of l always give an irreducible representation. Since there are $(2j+1)$ states possibly in this way, these are the irreducible representations in $(2j+1)$ dimensions. Hence, the lowest-dimensional, non-trivial representation is two-dimensional.

The explicit matrix elements for these representations, called the spin representation, can now be constructed as

$$\begin{aligned}
(J_a)^j_{kl} &= \langle j, j+1-k|J_a|j, j+1-l\rangle \\
\langle j, j_3|J_3|j, j_3'\rangle &= j_3\delta_{j_3 j_3'} \\
\langle j, j_3|J_+|j, j_3'\rangle &= \sqrt{\frac{(j+j_3+1)(j-j_3)}{2}}\delta_{j_3', j_3+1} \\
\langle j, j_3|J_-|j, j_3'\rangle &= \sqrt{\frac{(j+j_3)(j-j_3+1)}{2}}\delta_{j_3', j_3-1}
\end{aligned} \tag{4.15}$$

where the j are now made explicit to identify the dimensionality of the representations. Of course, in physics it just labels the spin of the particle described by this representation, hence also the name of spin representation.

Notoriously well known in physics is the lowest-dimensional representation with $j = 1/2$,

$$\begin{aligned}
J_1^{\frac{1}{2}} &= \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1 \\
J_2^{\frac{1}{2}} &= \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_2 \\
J_3^{\frac{1}{2}} &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\sigma_3,
\end{aligned}$$

where the special matrices σ_i are known as the Pauli matrices. They fulfill the useful relation

$$\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c.$$

Exponentiating yields the group elements

$$e^{\frac{i\alpha_a\sigma_a}{2}} = \cos|\vec{\alpha}| + i\frac{\alpha_a\sigma_a}{|\vec{\alpha}|}\sin|\vec{\alpha}| \tag{4.16}$$

where for this particular case the exponentiation is explicitly possible.

Since the resulting two-dimensional matrices are unitary, as it is a unitary representation, and of determinant one, this is also called[8] the two-dimensional special unitary group, or SU(2), obtained from the algebra su(2).

[8]Note that the Pauli matrices are also encountered in the context of the so-called quaternions, which are a generalization of the complex numbers.

Since this is the lowest-dimensional non-trivial representation, since the dimensionality is $2j+1$, this is also called the defining or fundamental representation of SU(2).

Irreducible representations of higher dimensionality can be constructed in the same way. E. g. for $j=1$

$$J_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad J_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \qquad J_3^1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

Note that while in the two-dimensional case the resulting group is the basis for all two-dimensional special unitary matrices, this is no true in three dimensions. There, eight matrices would be necessary. This is only a subset of these. They will be constructed later from a different group.

It is also no coincidence that in three dimensions there are three matrices, reminiscent of the three Euler angles. There is a deep relation between the group SU(2) and the rotation group SO(3) in three dimensions. Essentially, SO(3) is part of SU(2) in a particular way, and actually their algebras su(2) and so(3) coincide. This will be returned to later, and has to do with discrete subgroups of continuous groups.

While the representation is irreducible with respect to the group itself, it is possible that the interesting Hilbert space in physics contains further degrees of freedom, and therefore every state can be further characterized by other observables simultaneously diagonalizable. In this case, any state is characterized by a state vector $|j, j_3, \vec{\beta}\rangle$, where $\vec{\beta}$ contains all other quantum numbers. These states can always be chosen normalized in some norm. This has the interesting consequence

$$\langle j', j_3', \vec{\beta}'|J_a|j, j_3, \vec{\beta}\rangle = (J_a^{j'})_{j_3' j_3''}\delta_{jj'}\langle j', j_3'', \vec{\beta}|j, j_3\vec{\beta}\rangle$$

which can be proven by inserting a complete set of states on either side of J_a and then using the orthogonality relation. The orthogonality relations then permit to explicitly evaluate the matrix elements. Since the matrix elements of J are fixed by the group structure, the group structure alone permits to evaluate all expectation values for states for which the expansion are known. This lies at the heart of the Wigner-Eckhardt theorem to be returned to later.

Since the construction principle for this irreducible representation can be generalized to any compact Lie algebra, it is worthwhile to summarize it:

1. Diagonalize one of the generators, here J_3

2. Find the state with the largest eigenvalue

3. Use the lowering operators to construct the other states from this state

4. If there are remaining orthogonal subspaces, repeat from 2.

The fact that several seemingly unique properties of su(2) have been used, like lowering operators, foreshadows the fact that similar structures will be found for other groups.

In this context, the values possible for j_3 are also called weights, and therefore the weights of su(2) in its fundamental representation are $\pm 1/2$. Since the construction starts with the maximal value for j_3, the highest weight, this is also called the highest weight construction. It can be proven that this procedure gives all representations, and that they are all finite-dimensional for compact Lie groups.

4.6 Weights and the Cartan subalgebra

Discussing more general groups will be essentially now an extension of the previous discussion.

The starting point is to first identify a special subset of the generators of a group: Those which all commute with each others, i. e. those which in a suitable representation will be diagonal. This set of commuting generators is called the Cartan subalgebra, or a subalgebra of the Cartan subalgebra if a non-maximal set is chosen. However, the maximal set will be shown to be unique, up to trivial redefinitions, and is therefore usually meant when speaking of a Cartan subalgebra. In the case of su(2), this was J_3.

For su(3), these are J_3 and J_8 in the usual Gell-Mann representation of the generators

$$\begin{aligned}
2J^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 2J^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 2J^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
2J^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & 2J^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & 2J^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
2J^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & 2J^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{aligned} \tag{4.17}$$

which will be used in the following as a second example. The number of such generators is called the rank of the algebra.

Chose a representation with hermitian generators. Then the elements $H_i = H_i^\dagger$ of the Cartan subalgebra satisfy

$$[H_i, H_j] = 0.$$

This is again a linear space, as any linear combinations of the subalgebra is again in the subalgebra.

Because it is such a linear space, it is possible to chose a basis such that

$$\mathrm{tr} H_i H_j = k_D \delta_{ij},$$

and therefore the generators are again orthogonal in the sense of this scalar product.

Since by construction the Cartan subalgebra is the maximal set of commuting generators, it is always possible to find for a given representation a basis in which they are diagonal. As a consequence, any eigenvectors of some H_i is also an eigenvector to any other H_j as well, tough for a possibly different eigenvalue. Thus, given some state, it is characterized by the eigenvalues for all the Cartan elements collected by a vector μ^j, where j runs over the dimensionality n over the subspace in which the representation is

$$H_i|\mu^j, \beta\rangle = \mu_i^j|\mu, \beta\rangle \tag{4.18}$$

and β collects all other quantities needed to specify the vector. The eigenvalues can also be collected in n vectors of a dimensionality equal to the rank. These are called the weights or weight vectors μ_i. Since the generators can always be chosen Hermitian, they are real.

For su(2) with rank 1 in the two-dimensional representation, there are two one-dimensional weight vectors, (1) and (-1).

For su(3) with rank 2 in the three-dimensional Gell-Mann representation, there are three two-dimensional weight vectors, $(1/2, \sqrt{3})$, $(-1/2, \sqrt{3}/6)$, and $(0, -1/\sqrt{3})$ corresponding to the three eigenvectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

It is often useful to construct a vector-of-vector notation, especially

$$\alpha\mu = \alpha_j \mu_j$$

to define a sum over a product of the weight vectors with another set of vectors of dimensionality rank.

4.7 Roots

For the adjoint representation the values of the weights are special. It will be seen that they can be used to uniquely specify the basis of the representation, which will be useful later on.

To calculate them, note that the adjoint representation has the same dimensionality d as the algebra itself. It is therefore possible to select for every generator its own base vector.

Thus, in this basis a linear combination of two states correspond to a linear combination of the corresponding generators.

The action of a generator in this basis is therefore given by

$$X^a|b\rangle = |c\rangle\langle c|X_a|b\rangle = -if_{acb}|c\rangle = |[a,b]\rangle \tag{4.19}$$

In the second step it was used that the adjoint representation has the structure constants as matrix elements. In the third step it was then used that this particular linear combination of base vectors can be characterized by the base vector of the commutator, which is convenient, but just notation.

To give a particular example, choose once more su(2). In this basis, the three generators have a unique state, which can be choose to be the Cartesian base vectors. The resulting matrix representation of the three generators becomes

$$X^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad X^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad X^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which are surely hermitian and traceless, but none of them is diagonal in this basis. Note that the action of the generators in this basis on their corresponding state is to annihilate this state vector, $X^a|a\rangle = 0$.

The important consequence of the appearance of the structure constants is now that the weight vectors vanish if both a and b signify an element of the Cartan subalgebra. Hence, rank m of the weight vectors are zero vectors. However, there are still d-m other weight vectors α, which are non-zero. Since the generators can be chosen Hermitian, their eigenvectors form a complete basis. Thus, there exists a suitable orthogonal basis where all other base vectors are also eigenvectors

$$H_i|\alpha\rangle = \alpha_i|\alpha\rangle$$

of the elements in the Cartan subalgebra.

For the su(2) case, there is only one Cartan element, which has eigenvalues 1, 0, and -1. This takes the form

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The base vectors $|\alpha\rangle$ in this new basis are again the conventional Cartesian unit vectors. However, they are no longer directly associated with the original generators.

As a consequence, because of (4.19), the generators corresponding to these eigenvectors in this basis must satisfy

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{4.20}$$

where the α are the other weight vectors, and thus the other generators can be written such that they are characterized by the d-m m-dimensional weight vectors, which specify their commutation relations with the Cartan subalgebra. Note that this is a change of basis. This permits still to chose a normalization of the generators. It is convenient to select

$$\begin{aligned} \mathrm{tr} H_i^\dagger H_j &= \mathrm{tr} H_i H_j = & \lambda \delta_{ij} \\ \mathrm{tr} E_\alpha^\dagger E_\beta &= & \lambda \Pi \delta_{\alpha_i \beta_i}, \end{aligned}$$

where λ is some overall normalization constant.

As an example consider su(2). The corresponding new basis yields for the E_α

$$E_{(-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad E_{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.21}$$

Note that all matrices remain Hermitian and traceless, though they are no longer imaginary. Furthermore, they act now differently on the base vectors,

$$E_\alpha |\alpha\rangle = 0 \tag{4.22}$$

$$E_\alpha |0\rangle = |\alpha\rangle \tag{4.23}$$

$$E_\alpha |-\alpha\rangle = |0\rangle \tag{4.24}$$

and thus the E_α move the base vectors around. Note that $|0\rangle$ is not the null-vector, but indicates here the vector in the direction of the Cartan algebra.

It is not an accident that $E_\alpha = E_{-\alpha}^\dagger$. Taking the hermitian conjugate of (4.20) implies

$$-[H_i, E_\alpha]^\dagger = [H_i, E_\alpha^\dagger] = -\alpha_i E^\dagger.$$

Here, it has been used that the weight vectors are real as eigenvalues of the Hermitian matrices of the Cartan subalgebra. This implies that

$$E_\alpha^\dagger = E_{-\alpha}$$

Thus, as long as $\alpha \neq 0$, this implies that there is in this basis always a negative weight vector corresponding to a Hermitian generator. The form (4.21) shows this explicitly, but the proof did not make reference anywhere to su(2), and is therefore generally valid. This

also implies that in this basis not all generators are Hermitian. This is not a problem, as the explicit example of su(2) with the basis J_3 and $J_\pm$ shows, where the $J_\pm$ also are not Hermitian, but satisfy $J_\pm^\dagger = J_\mp$.

The weights in this basis are called roots, and their vectors hence root vectors.

There are two interesting features of this particular way of choosing the representation.

One is that the operators $E_{\pm\alpha}$ raise and lower the weights μ of any given state, where γ are any other involved quantum numbers,

$$H_i E_{\pm\alpha}|\mu,\gamma\rangle = ([H_i, E_{\pm\alpha}] + E_{\pm\alpha}H_i)|\mu,\gamma\rangle = (\pm\alpha_i E_{\pm\alpha} + E_{\pm\alpha}H_i)|\mu,\gamma\rangle = (\mu_i \pm \alpha_i)E_{\pm\alpha}|\mu,\gamma\rangle. \tag{4.25}$$

This has already been seen in (4.22-4.24) for the particular case of the state with weights the roots. Especially, this implies that acting with $E_{\pm\alpha}$ on the respective other state $|\mp\alpha\rangle$, these states are annihilated, as $(\pm\alpha \mp \alpha) = 0$.

Due to (4.19), this has implications for the commutator of E_α and $E_{-\alpha}$. This can also be seen in a more explicit way. First, note that by explicit calculation using (4.20)

$$[H_i, [E_\alpha, E_{-\alpha}]] = 0$$

Hence, the commutator must be proportional some linear $\beta_i H_i$ combination of the Cartan generators. As usual, this can be obtained by tracing,

$$\lambda\beta_i = \mathrm{tr} H_i \beta_j H_j = \mathrm{tr} H_i [E_\alpha, E_{-\alpha}] = \mathrm{tr}[H_i, E_\alpha]E_{-\alpha} = \alpha_i \mathrm{tr} E_\alpha^\dagger E_\alpha = \lambda\alpha_i$$

and thus

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i. \tag{4.26}$$

Again, this directly visible from the explicit su(2) case.

Though not explicitly written here, these results are encoding a lot more. There are n generators of which m form the Cartan subalgebra. In the $d(= n$ in the adjoint representation)-dimensional representation, there are thus d weight vectors of dimensionality m. In the adjoint representation, these are either zero or come pairwise with opposite signs. Thus in statements like (4.26) actually α is also a counting index counting the weight vectors, and in a more explicit notation, α would receive an index j running from 1 to $d - n/2$, where the division of 2 comes from the $\pm$-degeneracy, with components α_i^j.

This is once more explicit in the su(2) case. In the 2-dimensional representation, there is one Cartan element, and thus there are two weight vectors which are one-dimensional. In the 3-dimensional adjoint representation, there are three one-dimensional weight vectors, one being zero, and the other two being the negative of each other. For su(3), with a two-dimensional Cartan algebra, the lowest-dimensional representation is three-dimensional.

Thus, there are three two-dimensional weight vectors. The adjoint representation is eight-dimensional, and there are thus eight two-dimensional weight vectors. Of these two are zero, as they correspond to the two Cartan-elements. The other decompose into three pairs, each pair related by a sign.

The number of non-trivial weights is actually the same. In the two-dimensional su(2) case it is two and so it is in the adjoint representation. In the su(3) case, it is in the three-dimensional representation $3 \times 2 = 6$. In the adjoint representation, there are only three independent, non-zero weight vectors, and there are therefore again six independent weights. That does not seem to be a coincidence, and the weights indeed will play an important role in the following.

4.8 The su(2) substructure of arbitrary groups

Now it turns out that any non-zero root vector actually selects an su(2) subalgebra of every algebra. To see this, define generators[9]

$$J_\pm = \frac{1}{|\alpha|} E_{\pm\alpha} \tag{4.27}$$

$$J_3 = \frac{\alpha_i H_i}{|\alpha|^2}. \tag{4.28}$$

Thus while the so-defined $J_\pm$ generators are simple generators, J_3 is an object involving potentially the whole Cartan subalgebra, but still remains an element of the Cartan sub-algebra. Note that in principle there are several such sets of generators, which should be distinguished by having an index α identifying them. If it is not necessary to distinguish two different sub-algebras, this will be suppressed.

In the su(2) case itself, there is only a single root vector, $\alpha = (1)$. Thus the sum for J_3 collapses, and $J_3 = H$, as $\alpha^2 = (1)^2 = 1$. Trivially then $J_\pm = E_{\pm\alpha}$.

In general,

$$[J_3, J_\pm] = \frac{\pm\alpha_i\alpha_i E_{\pm\alpha}}{|\alpha|^3} = \pm\frac{E_{\pm\alpha}}{|\alpha|} = \pm J_\pm \tag{4.29}$$

$$[J_+, J_-] = \frac{[E_\alpha, E_{-\alpha}]}{|\alpha|^2} = \frac{\alpha_i H_i}{|\alpha|^2} = J_3, \tag{4.30}$$

which, by comparison to (4.12-4.13), is just the algebra of an su(2) algebra. Because every generator has been associated with a root vector, this implies that every group has su(2) subalgebras of this form. Moreover, this implies that the number of non-Cartan generators

[9]Note that the operators J_3 represent charge operators in physics, and they satisfy $J_3^3 = J_3$, due to the eigenvalue structure.

of every group is even, as all non-zero root vectors appear pairwise. Especially, this also implies that every irreducible representation of any algebra can therefore be decomposed into irreducible representations of su(2). This will also be a helpful insight in the following.

One question still arising is about the uniques of root vectors, i. e. can there be degenerate root vectors, which correspond to different E_α. Assume that there is a second generator E'_α to the same α. If it is not proportional to the original E_α, its corresponding base vector can always be decomposed into a part proportional and orthogonal to the one of E_α. But, by construction, the base vectors are eigenvectors of the Cartan, which form a full base, labeled by α. Thus, the orthogonal part is either an element of the Cartan, or belong to a different α, and therefore the operator cannot be different.

Furthermore, two different root vectors cannot be proportional to each other, except for sign reversal. Any such proportionality would rescale J_3 by its inverse. But since the representation is three-dimensional, since it is the adjoint, the eigenvalue of J_3 must necessarily be ± 1 or zero. Thus, a constant of proportionality would yield the wrong behavior. Alternatively, because of (4.25), any α different would require further states than the three included in the adjoint representation.

Note that this construction also implies an important statement about generators. The point of the whole construction is that every generator can be rewritten as a sum of generators of su(2). These are traceless, since the raising/lowering operators are traceless to connect different states. The Cartans are also traceless, as any su(2) representation is symmetric, and therefore all Cartan elements have the same positive and negative eigenvalues, and are thus traceless as well. Hence, generators are always traceless. Note that the deconstruction has only be performed for compact Lie groups, so this statement also only applies to compact Lie groups.

4.9 Geometry of weights

The fact that it is possible to decompose the set of generators of every algebra according to (4.27-4.28) into sets of su(2) generators has very important consequences. Especially, since the algebra is a linear space, these combinations can also be used to define a new base of generators, which still satisfy the su(2) algebra (4.12-4.13), even when applied in a different representation.

Select some representation D and some state of the corresponding vector space having weight μ and some other quantum numbers γ. Then select one of the su(2) subalgebras,

characterized by the root vector α, and apply its J_3 to the vector. This yields

$$J_3|\mu,\gamma\rangle_D = \frac{\alpha_i H_i}{|\alpha|^2}|\mu,\gamma\rangle_D = \frac{\alpha_i \mu_i}{|\alpha|^2}|\mu,\gamma\rangle_D. \tag{4.31}$$

Thus the eigenvalue of an arbitrary state of any representation of any J_3 is given by the projection of the weight vector on the root vector. Because J_3 is part of an su(2) algebra, it is necessary that its eigenvalues must be integer or half-integer, since this follows entirely from the algebra as shown in section 4.5.

Since the space is some representation space of the group, the vector is some linear combination of the basis vectors of the contained representation. Consider the case where the highest su(2) representation is j. Now, there is then some $p \geq 0$ such that

$$\begin{aligned}(J_+)^p|\mu,\gamma\rangle_D &\neq 0\\ (J_+)^{p+1}|\mu,\gamma\rangle_D &= 0.\end{aligned}$$

Applying J_3 yields

$$J_3(J_+)^p|\mu,\gamma\rangle_D = \frac{\alpha(\mu+p\alpha)}{\alpha^2}|\mu,\gamma\rangle_D$$

However, this must be an eigenstate of J_3, since all other contributions have been filtered out using the raising generator, and therefore the pre-factor

$$\frac{\alpha(\mu+p\alpha)}{\alpha^2} = j \tag{4.32}$$

must be proportional to j, and thus integer or half-integer, as otherwise another raising operator would not annihilate it.

In the same way the application of lowering operators yields

$$\begin{aligned}(J_-)^q|\mu,\gamma\rangle_D &\neq 0\\ (J_-)^{q+1}|\mu,\gamma\rangle_D &= 0,\end{aligned}$$

for some integer q, but which must then be an eigenvector of J_3 to eigenvector $-j$. Setting both results equal yields

$$2\frac{\alpha\mu}{\alpha^2} + p - q = 0 \rightarrow \frac{\alpha\mu}{\alpha^2} = -\frac{1}{2}(p-q), \tag{4.33}$$

which is a central statement in the following, as it will be the key for classifying Lie algebras. Note that j does no longer appear, and therefore the particular representation does not matter.

Now consider the case that μ is actually also a root vector, say β. This yields

$$\frac{\alpha\beta}{\alpha^2} = -\frac{1}{2}(p-q).$$

But the vector $|\mu, \gamma\rangle_D$ was general. Selecting instead the subalgebra characterized by the root vector β and using $\mu = \alpha$ instead yields

$$\frac{\beta\alpha}{\beta^2} = -\frac{1}{2}(p' - q'),$$

where p' and q' are again some other integers.

The true power of this result is found by multiplying both equations yielding

$$\frac{(p-q)(p'-q')}{4} = \frac{(\alpha\beta)^2}{\alpha^2\beta^2} = \cos^2\theta_{\alpha\beta}$$

This implies that the angle between the root vectors cannot be arbitrary, since the expression on the left-hand-side involves only integers. In fact, $\theta_{\alpha\beta}$ can only have the non-trivial values $\pi/2$, $3\pi/2$, $\pi/4$ or $5\pi/6$ - zero or π only occur if $\alpha = \pm\beta$, and thus correspond to the same su(2) subalgebra. Hence, the root vectors can only appear in very specific geometrical combinations.

4.10 The space of roots

For the su(2) case itself, all of this is actually rather trivial, as with one root vector there can only be 0 or π. For the su(3) case it is more interesting.

For su(3), note first that again the Cartan subalgebra is given by $J_3 = H_1$ and $J_8 = H_2$. In the lowest-dimensional representation these have the eigenvalues $(1/2, -1/2, 0)$ and $(\sqrt{3}/6, \sqrt{3}/6, -\sqrt{3}/3)$, and thus there are three weight vectors, combining two components each. If plotted in a plane of eigenvalues of H_1 and H_2, the weight plane, they form an equilateral triangle, at the corresponding coordinates.

From the remaining 6 generators it is then possible to construct three su(2) subalgebra pairs of raising and lowering operators. They can be enumerated by the root vectors. These root vectors must be the differences of weights, since the raising and lowering operators must move from one root vector to another by virtue of (4.25), and thus shift the eigenvalue by one. Thus

$$J^{\pm}_{\pm 1,0} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{4.34}$$

$$J^{\pm}_{\pm 1/2,\pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(J_4 \pm iJ_5) \tag{4.35}$$

$$J^{\pm}_{\mp 1/2,\pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(J_6 \pm iJ_7). \tag{4.36}$$

In the same plane of eigenvalues, the root vectors therefore form a regular hexagon, with $\pi/6$ angels, as well as two zero vectors from the two Cartans. This did not increase the

number of independent vectors, as two vectors are always trivially connected by a sign, and two vectors are by construction zero. Hence, the figure is again created by just three independent base vectors. However, this ambiguity shows that somehow weights and roots require ordering, as will be discussed next.

Note that the three J_3 generators belonging to the three algebras are given by

$$\begin{aligned} J^3_{\pm 1,0} &= J_3 \\ J^3_{\pm 1/2,\pm\sqrt{3}/2} &= \frac{1}{2}J_3 + \frac{\sqrt{3}}{2}J_8 \\ J^3_{\mp 1/2,\pm\sqrt{3}/2} &= -\frac{1}{2}J_3 + \frac{\sqrt{3}}{2}J_8. \end{aligned}$$

Thus, only for the first root vector they are given by one of the original Cartan generator like in the su(2) case, and are otherwise linear combinations. But these three are not linearly independent, and are just different linear combinations of the Cartan elements.

4.11 Simple roots

The thing which is needed to be implemented is some mechanism that a statement that a given weight is positive makes sense, so that raising and lowering operators due to (4.25) actually do raise or lower. Especially, this is required to define what is meant by highest weight.

The reason that this problem arises is, of course, because there is no ordering relation for vectors. However, in any fixed basis, it is possible to order by components. Introducing a fixed basis may seem suspect at first, as basis invariance is one of the greatest achievements of mathematics. But this will be rectified later, and the results will actually be basis invariant. For the practical calculations, however, it is as often useful to work in a fixed basis.

Once agreed, and given the weights in some basis, the convention will be the following: If the first non-zero component, counting from the top, is positive, then a weight or root is called positive. If it is negative, it is called negative. This also pertains to combinations of roots. E. g. $\mu - \nu$ for two weight vectors μ and ν is positive if the first non-zero component of the result is positive.

In case of the two one-dimensional weight vectors in the lowest-dimensional representation of su(2), (1) and (-1), the first is thus positive and the second negative. For su(3), the weight $(1/2, \sqrt{3}/6)$ is positive and the weights $(-1/2, \sqrt{3}/6)$ and $(0, -\sqrt{3}/3)$ are negative. Of the root vectors, three are positive and three are negative, as can be read from (4.34-4.36). The two zero roots are, of course, neither positive nor negative. Note that

for arbitrary representation the number of positive and negative weights does not need to match, as is visible for su(3). Because of the relation that for every positive root α there is the reversed root $-\alpha$, there is the same number of positive as negative roots. Finally, a change of basis may change the relative number of positive and negative weights, but not of roots.

As is visible in the su(3) case, not all roots are independent. After all, they are all generated from just three vectors. Since the dimensionality of the Cartan algebra, and thus the plane where the roots are located, of these at most two can be independent. Thus, there must be a minimal set of roots of which all others can be constructed. Since the roots are used to construct the group, it would be good to identify them. So the question is, which of them are the independent ones. Well, any set which is linearly independent. But this leaves the problem of identifying a useful one. It turns out that a particular useful statement is that all roots can be created from a set of basic roots, the so-called simple roots. These are defined as all positive roots which cannot be written as sums of other roots. That this is sufficient for any Lie algebra requires a proof, whose construction will also provide further interesting insights. This is actually a stronger requirement than just linear independence, as will be seen.

It is best to do so step by step.

First note that the difference $\alpha - \beta$ of two simple roots α and β cannot be a root. If $\beta > \alpha$, then $\beta - \alpha$ is positive. But then is β a sum of two positive roots, $\alpha - (\beta - \alpha)$, and can therefore be not simple. In the reverse case so is α, and also $\beta \neq \alpha$. If the first non-zero components match, the ordering can always be done according to the second components, and so on, since not all components can be identical. Visually, this can be directly seen in the case of su(2) and su(3).

Because $\alpha - \beta$ is not a root, the corresponding base vectors have to vanish under the action of the corresponding raising and lowering operators,

$$J_\alpha^- |\beta\rangle = 0 = J_\beta^- |\alpha\rangle$$

as they act in the subspace of different su(2) subgroups. In terms of the adjoint representation of su(3), which is eight-dimensional, there are two dimensions being the Cartan, and three pairs which combine with some combination of the Cartans to form three su(2) subspaces of three dimensions each. The lowering and raising operators only act on states in this subspaces.

Using (4.33), this implies for two integer p and p'

$$\begin{aligned} \cos\theta_{\alpha\beta} &= -\frac{\sqrt{pp'}}{2} \\ \frac{\beta^2}{\alpha^2} &= \frac{p}{p'}, \end{aligned}$$

and thus yield an even stronger constraint for simple roots. This implies that

$$\frac{\pi}{2} \leq \theta_{\alpha\beta} < \pi, \tag{4.37}$$

where the upper bound stems from the fact that both roots must be positive.

This helps in establishing that simple roots are linearly independent. Consider an arbitrary linear combination γ of simple roots

$$\gamma = \mu - \nu = \sum x_\alpha \alpha - \sum y_\beta \beta,$$

where x_α and y_β are all positive. Since both sums run over all roots, this is always possible to split the sum into those with positive and negative coefficients and isolate the sign as a prefactor as done here. If the simple roots would be linearly dependent, then there would be a non-trivial possibility to choose all prefactors such that γ vanishes. However, the norm satisfies

$$(\mu - \nu)^2 = \mu^2 + \nu^2 - 2(\mu\nu) > 0$$

due to (4.37) - the cosine of the angle in this range is always negative or zero, and thus the last term is never negative. Since not both μ and ν can have zero norm at the same time, the norm of γ is always non-zero, and hence the simple roots are linearly independent.

On the other hand, by definition, all other positive roots can be obtained by sums of the simple roots with positive, integer coefficient, since negative coefficients are not permitted as this would yield a difference and hence not a root.

This leads to the conclusion that the number of simple roots is actually the dimensionality of the root space, 1 for su(2) and 2 for su(3), and thus there are as many simple roots as the dimensionality of the Cartan. If this would not be the case, there would be some root δ which cannot be determined by the linearly independent roots. But the corresponding Cartan element δH would then commute with all $E_{\pm\alpha}$ due to (4.20). Thus, it would belong to a different disconnected subalgebra then those present, but this is only possible if the group is not simple, in contradiction to the starting point.

Since it is now clear that the simple roots span the space of roots, there should be a constructive way to obtain the reaming roots. The negative roots are trivially obtained by a multiplication from the positive ones, and the zero roots are also known. It thus

requires only to obtain the remaining non-simple positive roots. These are sums of the simple roots, but which?

Due to (4.25), it is sufficient to act with the E_α on the simple roots, until evaluation of (4.33) yields a contradiction. The only question to answer is, if this procedure could miss some positive root γ. This does not happen, since this would imply that acting on any such state with $E_{-\alpha}$ must annihilate it. However, it is still a linear combination of the simple roots, thus

$$\gamma\gamma = \left(\sum k_\alpha\right)\gamma$$

But because of (4.31), this is just the eigenvalue of the state γ, and this needs to be negative. At the same time, this is the norm of the state, which must be positive, leading to a contradiction. Hence, the described procedure generates all positive, and thus all, roots.

Still, this has not yet identified the simple roots. For su(2), this is just (1). For su(3), these are the two roots $(\pm 1/2, 1/(2\sqrt{3}))$, since $(0, -1/\sqrt{3})$ can be written as their sum, and is therefore by definition not simple. Since there must also be two simple roots, and these are the only remaining positive roots, these two are the simple roots.

Note that (4.33) also implies that the lengths of different roots cannot be arbitrary. In fact, simple roots can have at most two different lengths. As a consequence, the ones with the smaller length are called short root, and the other long roots. If all simple roots have the same length the algebra is called simply laced.

4.12 (Re)constructing the algebra

Since the simple roots contain all information about the other roots, it must be possible to reconstruct the entire algebra from it. This is indeed possible. To see how this works, follow the example of the su(3) case, where there are two simple roots.

First, the number of simple roots gives the dimensionality of the Cartan algebra, n. For su(3), this is $2 = n$.

Then, every simple root encodes one su(2) subgroup, satisfying the corresponding algebras (4.29-4.30). So there are at least n, implying at least $3n$ generators. For su(3) $3n = 6$. In addition, every positive root obtained from the simple roots satisfying (4.33) adds another two generators, in total $2y$. For su(3), there is only one more such root, yielding in total $6 + 2 = 8$. The remaining algebra relations can be obtained from (4.25), as well as from the relations leading up to (4.33).

To see how to proceed, note

$$HE_\alpha|\beta\rangle = \frac{\alpha(\alpha+\beta)}{\alpha^2}|[\alpha,\beta]\rangle = \sqrt{2}\frac{\alpha(\alpha+\beta)}{\alpha^2}|\alpha+\beta\rangle \Rightarrow [E_\alpha, E_\beta] = \frac{\sqrt{2}\alpha(\alpha+\beta)}{\alpha^2}E_{\alpha+\beta},$$

up to a phase, chosen by convention. The factor $\sqrt{2}$ appears as $\mathrm{tr}[E_\alpha, E_\beta]^2 = 2\mathrm{tr}E_\gamma^2$, and is thus required to keep a consistent normalization. Thus, for α and β simple roots, this creates the remaining relations. Especially for su(3), this implies

$$[E_{\alpha^1}, E_{\alpha^2}] = \sqrt{2}\left(1-\frac{1}{2}\right)E_{\alpha^1+\alpha^2} = \frac{1}{\sqrt{2}}E_{\alpha^1+\alpha^2}.$$

Since these are the only non-trivial commutators, every other commutator can then be obtained by expanding and using the base commutators (4.29-4.30).

4.13 A non-trivial example: G_2

To exemplify the concepts, another non-trivial example is useful. This will be the Lie algebra g_2, which is important for many reasons. For now, it just pops up out of nowhere, but it will become evident how it comes about later.

This algebra is also a rank 2 algebra, and has thus two simple roots $\alpha^1 = (0,1)$ and $\alpha^2 = (\sqrt{3}/2, -3/2)$. The angle between both simple roots is $5\pi/6$, since

$$\alpha^1\alpha^2 = -\frac{3}{2}.$$

Also, the second simple root has now a length of 3, rather than 1 as before.

Because

$$\begin{aligned}\frac{2\alpha^1\alpha^2}{(\alpha^1)^2} &= -3\\ \frac{2\alpha^2\alpha^1}{(\alpha^2)^2} &= -1,\end{aligned}$$

the lowest states are at magnetic quantum numbers $-3/2$ and $-1/2$: In the adjoint representation, the system contains a spin-3/2 and a spin-1/2 representation of two su(2). This implies that it is possible to act with E_{α_1} three times as a raising operator before hitting eventually zero weight, and thus a Cartan (and afterwards negative roots), while this is possible only once with E_{α_2}. Thus, p_1 can be at most 3 and p_2 can be at most 1. All other roots are of the general form

$$q_1\alpha_1 + q_2\alpha_2$$

for some positive integers q_1 and $q_2 \neq q_1$ (because no multiple of a root is a root), but

$$\frac{2\alpha^i(q_1\alpha_1 + q_2\alpha_2)}{(\alpha^i)^2}$$

must be integer. Inserting, this yields that there are only four combinations

$$\alpha^1 + \alpha^2 \qquad 2\alpha^1 + \alpha^2 \qquad 3\alpha^1 + \alpha^2 \qquad 3\alpha^1 + 2\alpha^2$$

which satisfy the equation, and are thus roots. Note that either $q_i = 0$ will not move into different su(2) subalgebra, and therefore is also not an option. This is a corollary to the fact that (non-trivial) multiple of roots are not roots. Thus, there are in total 2 (Cartan)+4 (simple roots)+8 (other roots)=14 generators in G_2, with a thus not so empty root diagram.

4.14 The Cartan matrix

As the example of g_2 showed, the actual relevant information are encoded in the number of times simple roots p_i and other roots q_i appear. Thus for every weight vector the Cartan for a given simple root applied to its state yields twice its J_3 eigenvalue due to (4.33),

$$2J_3|\mu\rangle = \frac{2H\alpha^i}{(\alpha^i)^2}|\mu\rangle = \frac{2\mu\alpha^i}{(\alpha^i)^2}|\mu\rangle = (q^i - p^i)|\mu\rangle$$

If $\mu = k_i\alpha^i$ is a positive root it can be written as a linear combination of simple roots, and thus

$$q^i - p^i = \frac{2\mu\alpha^i}{(\alpha^i)^2} = k_j\frac{2\alpha^j\alpha^i}{(\alpha^i)^2} = k_jA_{ji} \tag{4.38}$$

and thus the information on the eigenvalue is contained in the entries of a matrix, called the Cartan matrix, and the vector describes the composition of the root in terms of simple roots. Due to the factor two, all entries of A are necessarily integers. By construction the diagonal entries are all 2, and thus the non-trivial information resides in the off-diagonal entries. Since (4.33), the only possible values of $p^i - q^i$ for positive roots are 0, -1, -2, and -3, and thus only these values can appear in the off-diagonal elements. This information, as it is a product between different roots, implies how simple roots fit into the subalgebras described by the other simple roots. Of course, since these are just projections of the simple roots, which are linearly independent, A is invertible.

For the cases so far, the Cartan matrix of su(2) is one-dimensional, and therefore can have only the element 2. The others are

$$A_{\mathrm{su}(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A_{\mathrm{g}_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

confirming once more that the two su(2) subgroups of su(3) are equivalent, but for g_2 they are not: It is the number of times the lowering/raising operators can be applied to the other subalgebras without hitting zero. Thus, in the su(3) case, both simple-root su(2)s are in the spin 1/2 of the other one, while in the g_2 case, they are in different representations.

The Cartan matrix can then be used to simplify calculating the other roots, rather than using every time a geometric check by (4.33). Applying a simple root raising operator J_{α^i} to a root will raise it by one element in this simple root direction, i. e. $k_i \to k_i + 1$. This implies the eigenvalue, according to (4.38) is changed to

$$q^j - p^j + A_{ij}, \tag{4.39}$$

as the corresponding term is increased.

Now, for the simple roots, there is always only one k_i which is different from zero, and has value 1. This is called the $k = 1$ layer. Applying the corresponding lowering operator will reduce this by one, and thus all k_i are zero, which is thus a zero root, and hence belong to the Cartan. It is therefore more interesting what happens when a raising operators is applied. Then, there is only one possibility, i. e. one other k_i is increased by one, since otherwise a non-trivial multiple of a root is obtained, which is not a root. Thus, now two k_i values are non-zero.

Now, consider the following. Every simple root corresponds to a line of the Cartan matrix. The diagonal element is just normalization, and therefore does not matter. The other one represents the lowest value the spin of the su(2) algebra can have in the corresponding representation of the other su(2) subalgebras. Since the simplest root are at the same time constructed to give, according to the derivation of (4.33), the lowest possible values for the corresponding representations, this implies that no other root can exceed these values. Thus, using (4.39), this permits to construct all other roots.

For su(2), this is trivial, since there are no other roots.

For su(3), start with the two simple roots. They have $(2, -1)$ and $(-1, 2)$ as the corresponding matrix rows. Now, no multiple of roots are roots, so the only possibility is to add the roots, yielding $\alpha_1 + \alpha_2$. This implies, according to (4.39), the lines, yielding $(1, 1)$. Since the two -1 indicate that the possible range is $\{-1, 0, 1\}$, this is acceptable, and therefore another root. Forming $\alpha_1 + 2\alpha_2$ yields $(0, 3)$ and $2\alpha_2 + \alpha_1$ yields $(3, 0)$. In both cases, one of the components exceeds the allowed range, and thus neither is a root. Since there is no other possibility to proceed, as shown before since every such step corresponds to applying another raising operator, which yields a complete result, this is finished, and all positive roots for su(3) are constructed, and the negative ones are obtained by inflection.

For g_2, this is more complicated. The matrix rows are $(2,-1)$ and $(-3,2)$, implying the allowed ranges to be different, $[-3,3]$ for the first element and $[-1,1]$ for the second. The first is to add both simple roots, yielding $(-1,1)$, which is valid, and thus another root has been gained. Adding α_2 yields $(-4,3)$, and therefore this is not a root, as it is outside the allowed interval. $(1,0)$ from $2\alpha^1+\alpha^2$ is. Going on yields only one possibility, as $2\alpha_1+2\alpha_2$ is a non-trivial multiple of a root, and thus not a root. Hence, only $3\alpha_1+\alpha_2$ with $(3,-1)$ is obtained, and is valid. Then, $4\alpha^1+\alpha^2$ is $(5,-2)$ is again invalid. $3\alpha^1+2\alpha^2$ has $(1,0)$ and is valid. Since the only remaining options $4\alpha^1+2\alpha^2$ and $3\alpha^1+3\alpha^2$ are again non-trivial multiples, this terminates, and therefore, this created all the positive roots previously found.

Of course, this can be extended to negative roots by subtracting rather than adding, or just by inflection.

4.15 The fundamental representations

So far, most of this applied to the adjoint representation. But the (simple) roots can also be used to make statements about any other (irreducible) representation.

Choose a representation D. Then there are states of highest-weight μ such that

$$E_{\alpha^i}|\mu\rangle = 0$$

for any simple root α^i, and thus for any positive roots, which are only linear combinations of simple roots. In fact, the whole representation will be later constructible from applying combinations of $E_{-\alpha^i}$ to these states.

In (4.33), which was derived for states in arbitrary representation, this implies to set $p=0$, and thus

$$\frac{2\alpha^i\mu}{(\alpha^i)^2} = l^i, \tag{4.40}$$

where the l^i are, because of (4.33), (non-negative) integers. Since the α^i form a complete basis, the μ are uniquely determined by these scalar products. Conversely, every set of l^i defines the representation for which the μ are highest weight states. Since there are rank simple roots, every representation of a simple Lie algebra can be labeled entirely by a set of m non-negative integers. These integers are called Dynkin coefficients. For the adjoint representation, these are given by the projection of the simple roots onto each other. Therefore, the Cartan matrix contains the Dynkin coefficients.

There are weight vectors satisfying

$$\frac{2\alpha^i\mu^j}{(\alpha^i)^2} = \delta_{ij},$$

from which all other weight vectors can be reconstructed by

$$\mu = l^j \mu^j$$

They therefore represent, in an abstract way, the base of the weight space. Note that though the μ^i are highest weights of an irreducible representations, the so constructed weights may not necessarily be irreducible, as they have contributions from different irreducible representations. But they will always contain some irreducible representations, which can be identified using the methods to be discussed in chapter 5.

Since there are rank different such sets of base weight vectors, they correspond to rank different irreducible representations. These are called the fundamental representations of the group. These representations need not be of the same dimensionality, nor need they to be different from other representations. As will be seen, the two fundamental representations of su(3) have both the same dimensionality, and are the lowest ones. For g_2, however, they have different dimensionalities, and one is actually identical with the adjoint representation, while the other has the lowest-possible dimension.

For su(3), the fundamental weights are created from the two vectors (a^i, b^i). Inserting them into (4.40) yields

$$\alpha^j \mu^i = (a^i \mp \sqrt{3} b^i)/2$$

for both simple roots. Appropriately normalized to obtain the fundamental weights yields

$$\begin{aligned} \mu^1 &= (1/2, \sqrt{3}/6) \\ \mu^2 &= (1/2, -\sqrt{3}/6). \end{aligned}$$

To obtain the other weights in the corresponding representation it is sufficient to use the simple roots. Since, by construction, this is a highest weight state, in this case it is necessary to subtract, rather than add them.

These fundamental weights are by construction highest weights, and therefore they correspond in terms of the Cartan matrix to the vectors $(1, 0)$ and $(0, 1)$. To obtain the other weights, it is sufficient to apply thus the lowering operators to them. The only linearly-independent ones are the simple roots, by construction, and thus only they can be used to obtain new weights. Any such created weight is then unique, and therefore the number of these states gives the dimensionality of the representations.

Thus, for μ^1, subtracting α^1 yields $(-1, 1)$, which is an admissible state in su(3), since the components are within the $[-1, 1]$ range. Subtracting α^2 yields $(2, -2)$, which is not. Subtracting $\mu - \alpha^1 - \alpha^2$ yields $(0, -1)$, which is again admissible. Any further subtraction does not work, and therefore the system of weights in this representation is complete. It is hence three-dimensional. Likewise, for the other highest weight, the sequence is μ^2, $\mu^2 - \alpha^2$

and $\mu^2 - \alpha^2 - \alpha^1$, again of dimension three, which due to the symmetries of the simple roots was to be expected. Note that if the application with the other single root would have given an admissible state, the representations would have been four-dimensional. At this point, this would be the maximum possible, since any further application of simple roots would necessarily at most create a known state, if admissible at all, since everything else is obtained by some linear combination of the simple root lowering operators.

Note that once these weights are all known, the remainder to construct the representation explicitly can be done following the outlined procedure for the adjoint representation. This is in general tedious, but unique.

Interestingly, the pattern of weight vectors created from μ^1 go into the weight vectors created by μ^2 under sign reversal. It is said they are conjugated to each other. The reason is that if the generators T^a fulfill the algebra, so do $-T_a^*$,

$$if_{abc}(-T_c)^* = (if_{abc}T_c)^* = ([T_a, T_b])^* = -[-T_a^*, -T_b^*],$$

because the structure constants are real. Given that T^a are in the representation D, then the representation created by the $-T_a^*$ is called the complex conjugate representations, and often denoted by $\overline{D}$. These have the same dimensionality. Since the Cartan generators are Hermitian, this only reverses the signs of its real eigenvalues, and therefore, due to (4.18), the weight vectors reverse the sign.

If there is no similarity transformation which relates the T^a and $-T_a^*$, the representation is called complex, and otherwise real. The fundamental representation of su(3) is complex. For real representations, there is a further distinction. The condition for real representation is that there is some S such that

$$T_a = -ST_a^*S^{-1}.$$

But S can have either the property $S^{-1} = -S^T$ or $S^{-1} = S^T$, since in either case the generators are transformed, up to a minus sign, in the same way, and an additional minus sign, which also reverses the sign of the structure constants, does not spoil the algebra. If $S^{-1} = -S^T$, the representation is called pseudo-real instead of real. The actual difference is that in case of a real representation, the generators are unitarily equivalent to antisymmetric, purely imaginary matrices. Such generators transform under conjugation into itself, since T^a and $-T^{a*}$ coincide. In the pseudo-real case, this is no longer the case, but there is still a similarity transformation relating T^a and $-T^{a*}$.

The fundamental representation of su(2) is pseudo-real, since the Pauli matrices are not equivalent to purely imaginary, antisymmetric matrices, but $\sigma_a = -\sigma_2\sigma_a\sigma_2$ holds, and thus there exists at least a similarity transformation. For g_2, the fundamental representation is real, which is stated here without proof.

Because in the adjoint representation for every root also its negative is present, the adjoint representation is always real. This can be generalized. A representation which has the same set of Dynkin coefficients contains all elements twice, and is therefore always real.

Sometimes the notation is used that a representation is denoted by its dimensionality and therefore the complex conjugate by its dimensions overbarred, e. g. 3 and $\overline{3}$ for su(3). For higher representations, where different representations can have the same dimensionality, this is not always possible. E. g., for SU(4), the fundamental representations are 4, $\overline{4}$, but the third one is 6, which is not a complex representation. For SU(5), these are again four falling into two complex conjugated ones, 5, $\overline{5}$, 10, and $\overline{10}$.

Since the conjugation corresponds to a mapping of group elements into itself, it is an automorphism as discussed in section 2.5. If the representation is not-real, this cannot be undone by a similarity transformation, and therefore is an outer automorphism, while in the real case it is an inner automorphism. Note that there can also be other inner automorphisms besides conjugation.

4.16 Weyl group

A seemingly surprising feature of the weight space has been that they show a high amount of symmetries under reflections. This is actually not a coincidence, but originates from the fact that su(2) algebras and groups have a build-in reflection symmetry. Since the eigenvalue spectrum is symmetric under reflection at 0 - $(-1, 0, 1)$ goes over into $(1, 0, -1)$ - every su(2) subalgebra yields a reflection symmetry. This, of course, is inherited by the weight space. The set of all reflections under which the weight space is invariant is called the Weyl group.

For su(2), this is just a single reflection. But for su(3) in the adjoint representation, there are several reflections, as there are for g_2. Note that if there are different reflection planes, reflections can be combined to generate a discrete rotation group with reflections. E. g., the adjoint weights of su(3) are invariant under rotations by $2\pi/3$.

These reflections also show whether a complex conjugate representation is present. This is the case if two representations are identical under reflection.

4.17 Constructing other representations

To construct other representations, the starting point are always the Dynkin coefficients $(n_1, n_2, ...)$. The values n_i are then chosen to give the highest weights in the correspond-

ing su(2) subalgebras. The other weights are then constructed by applying the lowering operators, i. e. subtracting the simple roots.

E. g., for su(3) the case $(1,1)$ is actually the adjoint representation, as has already been seen before. For g_2, it would be $(3,1)$.

To give examples of other representations, consider again the example of su(3). Start with some arbitrary numbers, e. g. $(2,0)$. Note that now one of the coefficients is actually larger than the largest one in the fundamental or adjoint representation. Inserting this into (4.40) yields

$$2\mu^1 = \left(1, \frac{1}{\sqrt{3}}\right)$$

The possible sequence of weights is then $2\mu_1 - \alpha^1 = (0,1)$, $2\mu_1 - 2\alpha^1 = (-2,2)$, $2\mu_1 - \alpha_1 - \alpha_2 = (1,-1)$, as well as two more, which are the corresponding ones obtained by adding simple roots. Thus, there are 6 weights, and the representation is 6-dimensional. The shape in the weight-plane is a pyramid, and there is hence one non-trivial reflection, implying the presence of the conjugate representation $(0,2)$.

Since it is also very important in physics, another representation should be mentioned, the ten-dimensional $(3,0)$, as well as its complex-conjugate $(0,3)$. This representation also shows another interesting feature. There are two weights, $(2,-1)$ and $(-1,2)$, which are reached as $3\mu_1 - \alpha_1 - \alpha_2$ and $3\mu_1 - 2\alpha_1$, which yields under further application a weight with $(0,0)$, $(3\mu_1 - 2\alpha_1) - \alpha_2$ and $(3\mu_1 - \alpha_1 - \alpha_2) - \alpha_1$. Since, despite appearance, this is not just adding vectors, but just a shorthand for applying non-commuting operators, the question arises, whether the so-reached weight appears actually twice or just once. To answer the question, it is is necessary to explicitly calculate the difference, which is essentially given by the necessary terms introduced to bring one sequence into the other form, while acting on the highest-weight state. In this particular case, both sequences turn out to be identical, since any additional terms turn out to be either identical zero, or vanish when applied to the highest-weight state. There is no general principle to see this, and thus if weights can be reached by different ways, it is necessary to check in every case, whether the so-created weights are linearly dependent or not.

4.18 Dynkin diagrams

As is seen, simple roots are enough to reconstruct any compact Lie algebra. To simplify notation, it is therefore useful to develop a graphical language to encode simple roots. These are the so-called Dynkin diagrams.

Denote a simple root by a dot. Then the angle between two roots is denoted by the

connection between them. No connection implies $\pi/2 = 90°$, $2\pi/3 = 120°$a single line, $3\pi/4 = 135°$a double line, and for $5\pi/6 = 150°$a triple line.

The Dynkin diagram for su(2) is then just a single dot, as there is only one root. For su(3), there are two dots, with an angle of $2\pi/3$, and thus a single line. G_2 would then be two dots with three lines.

4.19 Classification of Lie groups

Probably the most important result in Lie groups is that there is only a denumerable infinite number of them, which can be completely classified by their Dynkin diagrams. The reason is that (4.33) is a very restrictive statement.

To show this, it is best to separate the proof into many smaller steps.

The first step is to introduce the concept of decomposability. A root system (or algebra) is called decomposable, if it separates into two (or more) orthogonal subsets, i. e. into elements which mutually commute with every element of a different subset. This is just the case of semisimple Lie algebras. As a consequence, this implies that the Dynkin diagram of a simple Lie algebra must be simply connected, and any root cannot be orthogonal to every other root. An algebra, which cannot be decomposed, is called indecomposable.

As a consequence, every simple Lie algebra satisfies three constraints

i) The simple roots are linearly independent, as discussed in section 4.11

ii) Because of (4.33), if α and β are two distinct simple roots, $2\alpha\beta/\alpha^2$ is a non-positive integer

iii) The simple root system is indecomposable, as otherwise the Lie algebra would not be simple

From these three constraints the complete classification follows. Any system of vectors satisfying these three constraints is also called a Π-system.

The next step is to notice that if there are only three vectors, there are only two admissible Π systems, corresponding to Dynkin diagrams being chains with at most one double connection. The reason is that the angles enclosed by three vectors need to be less than 2π, because otherwise they are coplanar. Since (4.33) only admits a certain number of discrete angles, there are only two possibilities with three vectors satisfying this condition:

$$\frac{\pi}{2} + 2\frac{2\pi}{3} = \frac{11\pi}{6} < 2\pi$$
$$\frac{\pi}{2} + \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{23\pi}{12} < 2\pi.$$

The other three structure of Dynkin diagrams, which can be created, all have an angular sum of 2π or more, and thus are linear dependent systems, and thus are not Π-systems. That has very far-reaching consequences, as any indecomposable subdiagram of a Dynkin diagram is again an indecomposable Dynkin diagram, and therefore a subalgebra. Therefore, only Dynkin diagrams will correspond to simple Lie algebras which have no other structure then the above will be found to be possible.

This leaves to classify the other Dynkin diagrams with less than three nodes. One node is su(2). Two with a single connected line is su(3), with two connected lines is actually that of so(3), and that with a triply connected line g_2, and therefore all are already known. They will be discussed later on in more detail. Because of the above statements, g_2 is actually the only simple Liealgebra with triply connected nodes, making it rather special, just like su(2).

The next step returns to the construction of smaller Dynkin diagrams from larger ones, which are required to be Π-systems. Take a Dynkin diagram with a singly connected line. Replacing the two nodes attached to the line and the line with a single node, inheriting the previous connections of the two nodes, creates also a Π-system.

Having only a single line between the two simple roots, called α and β, given by the nodes has the following implications. A single line implies an angle of $2\pi/3$. Since furthermore all simple roots are linear independent, then any other simple root γ has to have a vanishing scalar product with either α or β, but not with both. But then either $\gamma(\alpha + \beta) = \gamma\alpha$ or $\gamma(\alpha + \beta) = \gamma\beta$. Thus, replacing α and β by $\alpha + \beta$, and thus removing a node, creates a new set of root vectors with one root vector less, which again satisfies all constraints, and keeps all the angles as before. It is thus again a Π-system. This implies that by removing single lines, it is possible to shrink the system further and further. However, any subset of three vectors will become linearly dependent if it includes more than one double line or a cycle, according to the previous step. Hence, any Dynkin diagram representing a Π-system can have at most one double line (if it consists out of more than two nodes), and can have no cycles.

The next step is gaining insight into what kind of possible ends can be attached to a Π-system. The previous argument ensures the possibility that a single node can be attached, by reversing the process. Furthermore, since it is not possible to append a triple line, the only two other options are a branching into two lines and the addition of a double line.

Consider now the option to append a branch at the last node. Call the two added nodes α and β. Since they are not connected by a line, they are orthogonal and $\alpha\beta = 0$.

They are also connected by a single line to their anchor point, called γ. Thus

$$\frac{2\alpha\gamma}{\alpha^2} = \frac{2\alpha\gamma}{\gamma^2} = \frac{2\beta\gamma}{\beta^2} = \frac{2\beta\gamma}{\gamma^2} = -1.$$

These relations can be used to show that

$$\begin{aligned} \frac{2\gamma(\alpha+\beta)}{\gamma^2} &= -2 \\ \frac{2\gamma(\alpha+\beta)}{(\alpha+\beta)^2} &= -1. \end{aligned}$$

Thus, replacing the branch by a double line with node $\alpha+\beta$ provides again correct values satisfying (4.33). Thus, if a Dynkin diagram ending in a branch is a Π-system then so is the Dynkin diagram with the branch joined to a double line a Π-system.

This also implies that there is no possibility to have branches with more than three twigs. If there would be, they could be contracted to two double lines, which does not form a Π-system. For the same reason can a branch not occur at a node connected by a double line. Furthermore, even if the two branches are starting from different nodes, the Dynkin diagram can be shrunk until an offending diagram is reached. Thus, only Dynkin diagrams with at most one branching can represent Π-systems.

The next step requires to eliminate some special cases, which are not covered by the previous steps. In fact, only four cases have to be treated separately. These four cases are

- A diagram with a central node with three branches of length two nodes each
- A seven-node chain with a branching at the center node with a twig of one node
- An eight-node chain with a single branching at the third-last (or first) node with a twig of length one node
- A five node chain with a double line between nodes 3 and 4 (or 2 and 3)

All of these special cases cannot be shrunk to a contradiction using the previous rules.

However, it turns out that the angles prescribed between the simple roots in these configurations can only occur if the simple roots would be linear dependent, which is a contradiction to the requirement of being simple roots. Thus, these Dynkin diagrams cannot be Π-systems. This can be shown by explicit calculation, which is skipped here.

These results are sufficient to construct now all possible Π systems, and therefore all simple Lie algebras. The best route of action is to construct all possible Dynkin diagrams, starting from a single node, and then adding lines in allowed ways. Here, these diagrams will be constructed, and then afterwards discussed in separate sections in more details.

The simplest possibility is to just create a chain of single nodes. This is the simply laced algebra of special unitary matrices. Since there is an infinite number of such diagrams, there is an infinite family of these.

The next possibility is to have a chain of singly-connected lines, but with a single double connected line at the end. There are now two possibilities, depending on whether the single doubly-connected node at the end is shorter or the others are. If the single one is shorter, this will be the algebra of the special orthogonal rotations in odd dimensions. If the opposite is the case, it is the so-called symplectic group to be discussed in section 4.23.

Since any double line can be turned into a branch, there is always a corresponding diagram with a branch, which turns out to be the group of special orthogonal rotations in even dimensions. However, this group is then also simply laced.

All of these are again infinite families, since the singly-connected chain can be extended arbitrarily. These are the diagrams possible of this type.

Finally, there are five diagrams, which are possible but do not belong to any of the four infinite families. One is the already known group g_2 with two nodes of differing length and a triple connection. Then there is a four-node chain with a double connection in the middle, with the two nodes on either side of the double connection have a different length then on the other side. This algebra is called f_4. Then there are three more simply laced algebras, called e_6, e_7, and e_8, consisting out of a chain with a single branch to a twig of a single node at the third-last node.

All of these algebras will now be discussed in turn.

4.20 Special unitary groups

After knowing there is only a limited set of Lie groups, it is useful to analyze their particularities in turn, since all of these group appear in or the other way in physics.

Probably the most important one in particle physics is the special unitary algebra $\mathrm{su}(N)$ of rank $N-1$, which have as fundamental representation the $N \times N$-dimensional special unitary matrices. Their Dynkin diagram is just a singly-connected straight line of dots.

Using the same normalization of

$$\mathrm{tr} T^a T^b = \frac{1}{2}\delta_{ab}$$

as before for su(2) and su(3), they can be formulated as a straight-forward generalization. Especially, the $N-1$ Cartan generators as generalization from the Gell-Mann form (4.17)

are given by $N \times N$ matrices

$$H^m_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum^m \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right).$$

The remaining generators are as before also Hermitian traceless matrices, and altogether there are $N^2 - 1$ generators. As a consequence, the defining representation is N dimensional.

To construct the remainder requires the N $N-1$-dimensional weight vectors, i. e. vectors of the N eigenvalues of the $N-1$ Cartan generators, given by

$$\mu^j_m = H^m_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^m \delta_{jk} - m\delta_{j,m+1} \right),$$

with no summation implied. Each has norm $(N-1)/(2N)$ and relative scalar product $\nu^i\nu^j = \nu^i_k\nu^k_k = -1/2N$. Thus, they exhibit the same regularity as expected from the Dynkin diagram, forming a regular shape, the so-called $N-1$-simplex, in the $N-1$-dimensional weight space. Since in this counting the nth weight has $n-1$ leading zero entries, it is often useful to chose an inverted ordering where the last positive component is used.

The regularity implies that the $N-1$ roots are obtained as differences between weights,

$$\alpha^i = \mu^i - \mu^{i+1},$$

which then are all of unit lengths and satisfy

$$\alpha^i\alpha^j = \delta_{ij} - \frac{1}{2}\delta_{i,j\pm1},$$

as implied by the Dynkin diagram. The fundamental weights are then given by

$$\phi^j = \sum_{k=1}^j \mu^k.$$

Especially in this form is μ^1 the highest weight of the fundamental representation.

This group is sometimes also called A_n in mathematics.

Except for SU(2), whose fundamental representation is pseudo-real, all other groups have also complex fundamental representations.

4.21 Special orthogonal groups of even dimension

This set of algebras and groups are connected to the rotations in $2n$-dimensional real spaces, the in physics well-known so($2n$) matrices. They have a Dynkin diagram very

similar to the ones of su(N), just that they fork at the end into two elements, rather than just being a chain. In mathematics, they are sometimes called D_n.

The generators are traceless, purely imaginary antisymmetric $2n \times 2n$ matrices, like the example of so(2) in section 4.2 showed. The Cartan generators are given by

$$(H_m)_{jk} = -i(\delta_{j,2m-1}\delta_{k,2m} - \delta_{k,2m-1}\delta_{j,2m}),$$

which is a 2×2 matrix in the form of the second Pauli matrix embedded. Thus, eigenvectors and eigenvalues are just the embedding of those of this Pauli-matrix, i. e. the eigenvalue pairs ±1, together with the corresponding eigenvectors

$$\pm e^k_j = \delta_{j,2k-1} \pm i\delta_{j,2k}.$$

From this it is possible to read off the corresponding weight vectors $e^k_m = \delta_{km}$, as usual.

This creates the root vectors $\pm e^j \pm e^k$ for $j \neq k$, the positive roots $e^j \pm e^k$ for $j < k$ and the n simple roots $e^j - e^{j+1}$ for $j = 1...n-1$ and the cyclic one $e^{n-1} + e^n$.

Interestingly, the Dynkin diagram of so($2n$) cannot be reduced arbitrarily. The algebra so(8) is the last one which has a genuine distinct Dynkin diagram. Removing a further node yields three simply-connected nodes, which is the same as su(4). Thus, the algebras of so(6) and su(4) coincide. Furthermore, an attempt to remove the node in the middle breaks up the diagram into two single nodes. hence, so(4) is not simple, but has the same algebra as su(2)×su(2), which is of fundamental importance in relativity.

4.22 Special orthogonal groups of odd dimension

4.22.1 Generalities

It is at first a little bit surprising that the special orthogonal groups of odd dimension should be different than those in even dimension, but as will be seen later even and odd dimensions often induce quite different structures. As a consequence, this group is called in mathematics sometimes B_n.

In the case of Lie groups, this becomes obvious in the form of the Dynkin diagrams, as this group corresponds to a Dynkin diagram with a double connection between the last two elements of the chain, rather than two connections as is the case for even dimension.

It also becomes more clear when considering the structure of the Cartan generators. Before, they decomposed into 2×2 blocks, which is not possible in odd dimensions. In fact, the last one has eigenvalue zero, and is a zero on the diagonal. This was already visible in the case of SO(3) in section 4.2, even though in this case all generators had zeros

on the diagonal, since this was not a Cartan basis. As a consequence, for the first $2n$ dimensions, the same structure arises as previously. The difference is for the additional odd one. This yields a surplus weight vector $\pm e^x$, another positive root e^x and another simple root e^n. Note that the root $e^{n-1} + e^n$ is therefore not simple, and hence the size of the Cartan is the same as in the corresponding so($2n$) case. This was also seen already for so(2) and so(3) in section 4.2, as also there in both cases only two generators appeared.

Note that the Dynkin diagram of so(3) is the same as that of su(2), and thus is the same algebra.

4.22.2 Spinor representation

The true difference for odd-dimensional rotation groups is seen in the spinor representation, which will also illustrate more on the relation between su(2) and so(3).

The, in a sense, natural representation of these algebras are in a $2n + 1$-dimensional real vector space. However, there also exists a representation in a 2^n-dimensional vector space, and this is actually one of the fundamental representations. The reason comes from the one special simple root. There are in total n different fundamental representations, with fundamental weights

$$\begin{aligned} \mu^j &= \sum_{k=1}^{j} e^k \\ \mu^n &= \frac{1}{2}\sum_{k=1}^{n} e^k \end{aligned}$$

where the first $n - 1$ cases are generated from $j = 1...n - 1$. The last one is different due to the normalization. Using Weyl symmetry, the weights of the last fundamental representation can be constructed

$$\frac{1}{2}\left(\pm e^1 \pm e^2 \pm ... \pm e^n\right).$$

These weights are, by construction, transformed into each other by mirroring, and therefore the spinor representation coincides with its complex conjugate.

Of the weights there are therefore 2^n, and thus the space is 2^n dimensional (e. g. 2-dimensional for so(3) and 4-dimensional for so(5)). Such a space can be characterize as a tensor product of n two-dimensional vector spaces. Then any hermitian generator can be created from a tensor product of Pauli matrices. The Cartan generators are then

$$H_j = \frac{1}{2}\sigma_3^j,$$

where j indicates the two-dimensional space in which the Pauli matrix acts. In a similar way all other generators can be constructed from the Pauli matrices.

These representations now have a few specialties. Since they are self-conjugate, they need to be real. However, there are still two different kinds of real representations, real and pseudo-real. Due to the structure in terms of Pauli-matrices, it depends on the actual number of involved spaces whether the representation are real or pseudo-real. Explicit calculation shows that the spinor representation of $\mathrm{so}(8n+1)$ and $\mathrm{so}(8n+7)$ are real, and of $\mathrm{so}(8n+3)$ and $\mathrm{so}(8n+5)$ are pseudo real. Especially, so(3) has a pseudo-real spinor representation, just like su(2).

In a very similar calculation, it can be shown that the spinor representations of $\mathrm{so}(8n)$ are real and for $\mathrm{so}(8n+4)$ pseudo-real, while the other ones are complex.

As a consequence of the existence of spinor representations, representations of the $\mathrm{SO}(n)$ groups, depending on n, can be real, pseudo-real, or complex.

4.22.3 Connection to Clifford algebras

The fact that there is something called a spinor representation already indicates that there is a relation to spins, and thus also to the Clifford algebra.

A Clifford algebra is an algebra satisfying

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

which should not be confused with the case of the usual Clifford algebra where the Minkowski metric appears on the right-hand side. This version will be discussed later, as it is not connected to a compact Lie algebra.

Given an implementation of the Clifford algebra with n elements, define the operators

$$M_{ij} = \frac{1}{4}\left[\gamma_i, \gamma_j\right]$$

and with $\gamma = (i)^n \gamma_1 ... \gamma_{2l}$

$$M_i = \frac{1}{4}\left[\gamma_i, \gamma\right].$$

It can then be shown that these operators can be used to create spinor representations of the Lie algebras of $\mathrm{so}(n)$, especially for $\mathrm{so}(2l+1)$ with M_i and M_{ij} and for $\mathrm{so}(2l)$ the two inequivalent spinor representations

$$\frac{1}{2}(1 \pm \gamma) M_{ij}$$

Thus, there is an intrinsic relation to the (Euclidean) Clifford algebra.

An explicit example is given by the spinor representation of so(3), which, since $l = 2$, is two-dimensional. Without proof, this turns out to be the just the two-dimensional fundamental representation of su(2), given by (4.16). However, the decisive difference is that the parameters of so(3) are then mapped to twice[10] the one of su(2), and thus to every element of so(3) there corresponds two representation matrices, differing by a sign, showing that this (and actually all) spinor representation of so(3) is double-valued.

4.22.4 Crystallographic subgroups

One of the probably most important applications of special orthogonal groups of odd dimensions are the crystallographic groups, especially of so(3). These are discrete subgroups, which are obtained from so(3) by requiring that only certain rotations are still possible.

Some cases are provided by the previous standard example of discrete rotations. E. g. choosing the only permitted rotation angles to be $\pi/2$ and its direct multiple leads to the crystallographic group of the cube, choosing $2\pi/3$ to the one of the hexagon. There are only very few such subgroups in 3 dimensions, which are closed. This yield the platonic bodies, the tetraeder, the cube, as well as the bodies with eight (octagon), twelve (ikosaeder), and 20 (dodecahedra) surfaces.

This is not not the only possibility to create subgroups. Another possibility is to permit different rotation angles in different directions. This yields eventually the so-called point groups of possible crystal structures.

The same is, of course, possible in lower or higher dimensions, and not restricted to odd dimensions. E. g., in numerical calculations in particle physics the discrete subgroups, and their faithful and reduced representations, of so(4) play an important role, especially for fixed angles of $\pi/2$, the so-called hypercubic groups.

4.23 The symplectic group

The next group appears at first sight somewhat strange, but is actually not unknown. In fact, it is closely related to the phase space of Hamiltonian mechanics. There, it was seen that of the $2n$ dimensions of the phase space n had in the Hamiltonian equations a different sign than the other n. This implied certain transformation properties for rotations in the phase space, which are encoded in the so-called symplectic groups sp($2n$) of even dimensions. The Dynkin diagram is the same as for so($2n + 1$), but length of the simple

[10] Or, alternatively, the factor of 1/2 does not appear in the exponent for so(3), which is correct, since the two-dimensional representation of so(3) is actually given by the σ_i, and not by the J_i.

roots are different. This group is in mathematics sometimes called C_n. In fact, it is possible to consider this group as the norm-preserving rotations in a space of quaternions[11], rather than the spaces of real or complex numbers where the SO(N) and SU(N) groups act.

As a consequence of this structure, every sp($2n$) has as a subalgebra su(n). These are embedded in the $2n$-dimensional representation space as

$$\begin{pmatrix} T^a & 0 \\ 0 & -T_a^* \end{pmatrix},$$

where the T^a are the $n-1$ Cartan generators of su(N). Then, only one more Cartan generator exists, which can be written as

$$H_n = \frac{1}{\sqrt{n}}\sigma_3 \otimes 1$$

and therefore generates a block-diagonal string of the third Pauli matrix σ_3. This Cartan generator also commutes with all the su(N) Cartan generators.

Note that sp(2) has the same Dynkin diagram as su(2), a single node, and therefore both algebras agree, and thus also to the one of so(3). That all three algebras agree, but not the groups has to do with the possibility to have with the same algebra groups with different discrete subgroups, which will be discussed later.

An alternative characterization of the symplectic group is that it covers all $2n$-dimensional matrices M such that

$$M^T J M = J$$

where J is a block-diagonal matrix with every block being the 2×2 matrix

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Another way of regarding the symplectic group is to characterize them as unitary matrices with quaternionic instead of complex entries.

The fundamental representations of symplectic groups are pseudo-real. All other representations are either real or pseudo real.

4.24 Exceptional groups

There are also the five exceptional groups, of which g_2 has already been encountered in section 4.13. Like the previous cases, the exceptional groups can be related to certain

[11] These are 'numbers' of the type $a1 + \vec{b}\vec{\sigma}$, where 1 is the 2×2 unit matrices and the σ_i are proportional to the Pauli matrices. This fact is actually the reason why su(2) is so special, as the quaternions are connected to the group SU(2).

norm-preserving rotations on certain spaces. These spaces are formed by so-called octonions, a generalization of the quaternions, and actually the last extension possible. The reason is that the requirement for fields in which also a division operation and a length can be defined, so-called division algebras, is very restrictive, and there is only a certain limited number of cases possible, which are the real numbers, the complex numbers, the quaternions, and the octonions. Note that the multiplication for octonions, in contrast to the previous cases, is actually not associative.

This gives another perspective of why there is only a limited set of compact Lie algebras: There is only a limited set of division algebras and therefore a limited set of norm-preserving rotations in spaces of division algebras, and compact Lie groups are just the groups implementing these rotations.

To understand better the properties of exceptional algebras, it is useful to discuss the octonions in some detail. Like complex numbers and quaternions, octonions are generated by additional elements,

$$a + b_\alpha i_\alpha$$

where a and the b_i with $i = 1...7$ are real numbers, and the seven i_α generalize the imaginary units, or the Pauli matrices in case of the quaternions. Similarly to the quaternions, they fulfill the multiplication table

$$i_\alpha i_\beta = -\delta_{\alpha\beta} + g_{\alpha\beta\gamma} i_\gamma,$$

where the tensor g is totally antisymmetric with

$$g_{123} = g_{247} = g_{451} = g_{562} = g_{634} = g_{375} = g_{716} = 1$$

in a suitable basis, and all other elements related by index permutations or otherwise are zero. Note that this rule is not associative, since it implies, e. g.,

$$(i_1 i_2) i_7 = i_5 \neq -i_5 = i_1 (i_2 i_7)$$

and thus the octonions are not a group.

The fact that it is a division algebra is signaled by the fact that for the norm

$$|a + b_\alpha i_\alpha| = \sqrt{a^2 + b_\alpha^2}$$

$|AB| = |A||B|$ follows for any two octonions A and B, just like for real and complex numbers and quaternions.

However, they are related to the exceptional Lie groups in a non-trivial way. It starts with the fact that $g_{\alpha\beta\gamma}$ is actually an invariant tensor of g_2, and, as noted above, the other

exceptional groups can be linked to rotations in spaces of octonions. However, this will not be detailed further.

An interesting feature of e_8 is that it is the only simple, compact Lie algebra for which the adjoint representation is also the lowest-dimensional representation, and thus again one of the fundamental representations. Note that in g_2 also one of the fundamental representation coincides with the adjoint, but this is not the lowest-dimensional one.

Except for E_6, which has one complex fundamental representation, the fundamental representations of all exceptional groups are real. This has the consequence that E_6 plays a special role in many physics contexts, where complex representations are often required.

4.25 Subalgebras and Dynkin diagrams

The Dynkin diagrams also offer a straightforward procedure to identify subalgebras. In this context, it is useful to define regular subalgebras as subalgebras where the Cartan generators are a subset of (linear combinations of) the original Cartan algebra. A regular subalgebra is called maximal, if it contains the full Cartan algebra of the original algebra.

Subalgebras can be created from Dynkin diagrams in various ways. The simplest is using any of the the reduction rules found in 4.19. Since this guarantees another simple Lie group, this yields a regular subalgebra. However, since the number of nodes is smaller, this cannot be a maximal one. Furthermore, removing a node will create also a Lie algebra, but since this splits the Dynkin diagram into two, this is no longer a single simple algebra, but a combination of two. Furthermore, since a node is missing, there is besides the Cartan algebra of the two new subalgebras a single Cartan generator, and therefore another u(1) algebra. E. g. the Dynkin diagram of $su(N+M)$ can be cut into two by removing the nth node. This yields a subalgebra of type $su(N)\times su(M)\times u(1)$. The u(1) comes from the removed node, and the total rank is $N-1+M-1+1=N+M-1$, as it should be.

The construction of maximal regular subalgebras is a little bit more involved. This is obtained in the following way. Since the Dynkin diagrams can be grown from a single seed, the lowest root, it is possible to construct a unique extended Dynkin diagram by adding to each simple root this lowest simple root. This is no longer a Π-system, since this system has now a linear dependence, as there is one more simple root. However, the removal of any simple root creates a new Π system. In terms of the Dynkin diagrams, it can be shown that this corresponds to creating a cycle for A_n, adding a branch to the singly-connected end node of B_n, adding another doubly-connected node at the other end of C_n, adding a branch at the other end of D_n, symmetrizing e_6 and elongating all other exceptional groups. Now maximal subalgebras can be generated from removing nodes

from these extended diagrams. This, e. g., shows that so($2N$+1) has a so($2N$) subgroup, as expected from physics.

4.26 General Lie groups

The consequence of the previous sections is that since there is only a countable infinite number of compact, simple Lie groups, and thus there is only a countable infinite number of compact Lie groups.

These are direct product groups involving any of the simple Lie groups, additional Abelian U(1) factors (which generate generators with zero length according to the scalar product of section 4.3), and possibly discrete factor groups in form of center groups. The algebras are combinations of simple groups and the Abelian algebra.

In physics, all gauge theories of Yang-Mills type are of this kind. The only other gauge group encountered in physics are from non-compact groups in form of general relativity. This requires a theory of non-compact groups to be discussed in chapter 7.

4.27 Consequences for discrete groups

As indicated in section 4.22.4, at least some Lie algebras/groups do have also discrete subalgebras/groups, which can necessarily be enumerated. This can be actually generalized. It can be shown that for compact, simple, discrete groups there are only two possibilties. Either they are a discrete subalgebra of a continous Lie algebra, or they belong to a very small, and finite, number of special cases. Those which are discrete subalgebras of the continous ones, similar consequences arise as in section 4.22.4. This yields little new in terms of group theory, and will thus not be pursued here, although these are in practice often very important.

The remaining are 26 so-called sporadic finite groups. They are not subgroups of continous Lie groups. Out of these, twenty are subgroups of a single group, the so-called Monster group. The other five, know as pariahs, are different. The monster goup is the largest one, with about 10^{54} elements. This group has actually a connection to function theory, and a very special function, the so-called j function. This is known as the monstrous moonshine. This function is also connected to so-called conformal field theory, which therefore establishes a connection to physics. There are indications that also the other five pariahs may be connected similar to function theory.

The sporadic groups are separated in three generations. The first generation, the Mathieu groups, are connected to the permutation groups of n points, and consists out

of 5 groups. The second generation, called the Conway groups, contains 7 groups and is connected to the automorphisms of a special lattice, the so-called Leech lattice, in 24 dimensions. The third generation contains the other 7 groups and the monster itself, and are characterized by their relation to the monster.

Chapter 5

Tensor products

It is a rare situation in physics to have only a single symmetry characterizing a system. If multiple symmetries start to play a role, this leads to the concepts of tensorizing groups. This is not limited to compact Lie groups, but it is possible for any groups, no matter whether continuous or discrete.

5.1 Tensorizing states

One particular important situation in physics is that some state is characterized by different representations of two different groups or even the same group. An example is given by atoms with electrons with spin. Then the electrons carry two representations, one of SU(2) and one of SO(3), together with corresponding quantum numbers

In a more general language, a state which belongs to two different representations, possibly of two different groups, can be considered as a product state

$$|j,k\rangle = |j\rangle|k\rangle,$$

where j and k may also be sets of quantum numbers of some group D_1 and D_2. This is called a tensor product of the states. If the representation of D_1 is n-dimensional and of D_2 is m-dimensional, the total dimensionality is nm. This can be thought of as that k runs over all possible values for any fixed value of j and vice versa.

The matrix representation of the groups act then in a tensor representation,

$$D^{1\otimes 2}|j,k\rangle = (D^{1\otimes 2})_{il|jk}|j,k\rangle = (D^1)_{ij}|j\rangle(D^2)_{lk}|k\rangle. \tag{5.1}$$

Thus, this is the statement that each state transforms in its own representation.

Expanding for continuous groups close to the identity this yields

$$(1_{1\otimes 2} + i\alpha^a X_a^{1\otimes 2}) = (1_1 + i\alpha^a X_a^1)(1_2 + i\alpha^a X_a^2) = 1_{1\otimes 2} + i\alpha^a(X_a^1 1_2 + 1_1 X_a^2),$$

where the index at the 1-operator indicates in which space the unit operators act. Leaving the unit operators implicit, this amounts to $X_a^{1\otimes 2} = X_a^1 + X_a^2$, and thus the generators add in a tensor product. Especially, if the states are eigenstates to the generators, this implies that the eigenvalues add. It should be noted that this notation is very compact, as a has to run over the total number of generators $N_g^1 + N_g^2$ of both groups. Formally, the generators of the tensorized group are enumerated such that, e. g. for $N_g^1 = N_g^2 = 2$ this should be read as

$$X = (X_1^1 1_2, X_2^1 1_2, 1_1 X_1^2, 1_1 X_2^2),$$

and thus the explicit scalar product yields

$$\alpha^a (X_a^1 1_2 + 1_1 X_a^2) = \alpha_1 X_1^1 1_2 + \alpha_2 X_2^1 1_2 + \alpha_3 1_1 X_1^2 + \alpha_4 1_1 X_2^2,$$

such that the parameter vector is also $N_g^1 + N_g^2$ dimensional. Alternatively, this can be regarded as

$$e^{i\alpha_a X_a^1} \otimes e^{i\beta_b X_b^2} \approx 1_{1\otimes 2} + i\alpha_a X_a^1 1_2 + i 1_1 \beta_b X_b^2,$$

for α and β both infinitesimal, to make the relation better manifest. This generalizes to tensor products of more than two representations in a straightforward way.

5.2 Clebsch-Gordon construction

This can now be used to obtain the Clebsch-Gordon construction which in the end will turn out to be also a highest-weight construction for higher-dimensional representations. For this note first that

$$J_3^{1\otimes 2} |j j_3, j' j_3'\rangle = (j_3 + j_3') |j j_3, j' j_3'\rangle. \tag{5.2}$$

Thus tensorization can give at most a representation with $j^{1\otimes 2} = j^1 + j^2$. Representations with even higher j can then be obtained by successive tensorization.

As an example, consider to get $j = 3/2$. This can be obtained by tensorizing a $j = 1$ and a $j = 1/2$ representation. The $j = 1$ representation can itself be obtained by tensorizing two $j = 1/2$ representation, but since the $j = 1$ representations are known from orbital angular momentum in quantum mechanics this step will be skipped.

In the spirit of the highest-weight construction, the state with $j_3 = 3/2$ is, by virtue of (5.2) given by

$$|3/2, 3/2\rangle = |1, 1\rangle |1/2, 1/2\rangle = |1, 1/2; 1, 1/2\rangle,$$

where the second notation will be occasionally use for brevity.

To obtain the other states, it will be just proceeded as in the highest-weight construction by applying the tensorized lowering operators, creating

$$\begin{aligned}
|3/2,1/2\rangle &= \frac{1}{N_{1/2}} J_-^{1\otimes 2}|3/2,3/2\rangle = \frac{1}{\sqrt{3}}|1,1/2;1,-1/2\rangle + \sqrt{\frac{2}{3}}|1,1/2;0,1/2\rangle \\
|3/2,-1/2\rangle &= \frac{1}{N_{-1/2}} J_-^{1\otimes 2}|3/2,1/2\rangle = \sqrt{\frac{2}{3}}|1,1/2;0,-1/2\rangle + \sqrt{\frac{1}{3}}|1,1/2;-1,1/2\rangle \\
|3/2,-3/2\rangle &= \frac{1}{N_{-3/2}} J_-^{1\otimes 2}|3/2,1/2\rangle = |1,1/2;-1,-1/2\rangle,
\end{aligned}$$

where the N_i are suitable normalizations. This represents a four-dimensional sub-space of the six-dimensional space obtained by tensorizing the two representations.

The basis for the other two directions is given by orthogonalization to these, e. g. using the Gram-Schmidt procedure. A possible set of base vectors are

$$\begin{aligned}
|1/2,1/2\rangle &= \sqrt{\frac{2}{3}}|1,1/2;1,-1/2\rangle - \sqrt{\frac{1}{3}}|1,1/2;0,1/2\rangle \\
|1/2,-1/2\rangle &= \sqrt{\frac{1}{3}}|1,1/2;0,-1/2\rangle - \sqrt{\frac{2}{3}}|1,1/2;-1,1/2\rangle,
\end{aligned}$$

where the naming is motivated by the fact that they can be obtained from each other by applying $J_\pm^{1\otimes 2}$, and form a representation to $j = 1/2$. Thus, the tensor space is reducible, and furnishes a $3/2$ and a $1/2$ irreducible representation. That the total representation is reducible was expected, as it is just the tensor product of two irreducible representations. However, this permitted nonetheless to construct a new irreducible representation of higher dimension.

This situation is often denoted in short as

$$s \otimes l = j_1 \oplus j_2$$

or in the particular case above

$$1 \otimes 1/2 = 3/2 \oplus 1/2$$

to denote which kind of representations can be derived from tensorizing two representations.

Note that by taking the norm on both sides of the expressions, and using the fact that all states are normalized, expressions like

$$\langle j j_3 | l, s; l_3, m_3\rangle = c_{jlsj_3l_3m_3} \tag{5.3}$$

are obtained. These are called Clebsch-Gordan coefficients. E. g. $c_{3/211/21/21-1/2} = \sqrt{1/3}$. Thus, these coefficients are encoding purely the group-theoretical structure of tensor products, and can therefore be algorithmically calculated. This will be used below in section 5.4.

It is an interesting situation to consider the case of tensorizing twice the same representation, e. g. twice $s = 1/2$. Then

$$|1,1\rangle = |1/2,1/2;1/2,1/2\rangle.$$

This state is necessarily symmetric when the two representations making it up are exchanged. This is a generic feature. Since at the highest weight this is a combination of twice the same state, this is necessarily a symmetric combination. The raising and lowering operator commute with the exchange operator P_{12} for the two representations

$$P_{12}J_{\pm}^{a\otimes a} = P_{12}(J_{\pm}^{a}\otimes 1 + 1\otimes J_{\pm}^{a}) = (1\otimes J_{\pm}^{a} + J_{\pm}^{a}\otimes 1) = (J_{\pm}^{a}\otimes 1 + 1\otimes J_{\pm}^{a})P_{12} = J_{\pm}^{a\otimes a}P_{12},$$

and therefore the symmetry of the state created by the descent operation preserves the symmetry. Hence, all the states remain symmetric.

This is no longer true once the orthogonal other representation is constructed. E. g. for $j = 1$ the orthogonal state is

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|1/2,1/2;1/2,-1/2\rangle - |1/2,1/2;-1/2,1/2\rangle)$$

which changes sign under the exchange of the first and the second representation. Which symmetry it has is a less generic statement, as there may be several lower representations involved. However, they all keep the symmetry of their respective highest-weight state.

Note that the fact that there is a substructure was important for this argument. If a $j = 1$ state is not made up as a tensor product, but has intrinsically this value of j, its symmetry properties may be different. In particle physics, this translates to the fact that composite states made from other particles inherit properties, but elementary particles of the same spin may not need to have the same intrinsic features.

5.3 Tensor operators

Of particular importance for physics are operators which satisfy the algebra

$$[J_a, O_l^s] = O_m^s (J_a^s)_{ml}, \tag{5.4}$$

and which is therefore in a fixed representation s. Operators satisfying this are called tensor operators. Besides their importance for physics, they will be very useful in constructing

further properties of groups. It is therefore worthwhile to investigate them a little closer. Note that in the commutator not yet the representation of J is specified, however, the action of J on O must be known, which is less.

An example is the vector operator in three-dimensional space-time. The commutator is best calculated by using the form $J_a = \epsilon_{abc} r_b p_c$. This yields

$$[J_a, r_b^s] = \epsilon_{acd}[r_b p_d, r_b] = -i\epsilon_{acb} r_c = r_c (J_a^1)_{cb}$$

where in the last step the adjoint representation was identified. However, since for SU(2) the adjoint has been seen to be the irreducible three-dimensional representation, this is necessarily the $j = 1$ representation. Thus, the vector r is also in the $j = 1$ representation. The only alternatives, since a representation to act on a three-dimensional quantity, would have been either a reducible combination of the trivial and the $j = 1/2$ or the totally trivial representation.

It should be noted that the resulting matrices are not the ones obtained using the highest-weight representation. For technical reasons, however, it is usually better to have the J in the explicit highest-weight representation. To achieve this, it is best to find some linear combination O_0^s of the O^s such that

$$[J_3, O_0^s] \sim O_0,$$

since then

$$[J_3, O_0^s] = O_l^s (J_3^s)_{lj},$$

and this collapses to a δ-function multiplied with a constant, since J_3 is diagonal. For the position operator, this is J_3 with $[J_3, r_3] = 0$, and thus $r_3 \sim r_0$. The remaining operators can then be constructed using multiple applications of the commutation relations, e. g.

$$[J^{\pm}, r_0] = \mp \frac{1}{\sqrt{2}} (r_1 \pm i r_2).$$

and thus the explicit form.

A word of caution here when indeed it turns out that the representation for J is found to be reducible. It is then possible to decompose the operators O_l also into sets of operators such that each correspond to an irreducible representation of J. However, there is no general construction principle how to do so, since for operators there is no notion of orthogonality. Thus, the best approach is to first find the linear combinations of operators which commute with J_3, to identify the present j. This is the complicated step, as there is no construction principle available. Then the other operators can be obtained using the raising and lowering operators once more.

Finally, if the operator O is a tensorized operator itself, i. e. a product of two tensor operators in two representations, then it will create an algebra with the tensorized J. So, for O^s_m and O^r_i

$$[J_a, O^s_m O^r_i] = [J_a, O^s_m] O^i_n + O^s_m [J_a, O^i_n] = ((J^s_a)_{mn} \otimes 1_{ij} + 1_{mn} \otimes (J^r_a)_{ij}) O^s_n O^r_i,$$

with the special case

$$[J_3, O^s_m O^r_i] = (m+i) O^s_m O^r_i.$$

5.4 Wigner-Eckart theorem

This can now be used to construct the generalized Wigner-Eckart theorem, a central theorem for the separation of group structure and dynamics. It assumes that the tensor operator in the relevant basis are known. Then it states

$$\langle j', j'_3, \alpha | O^s_l | j, j_3, \beta \rangle = \delta_{j'_3, l+j_3} \langle j, l+j_3 | s, j, l, j_3 \rangle \langle j', \alpha | O^s | j, \beta \rangle. \tag{5.5}$$

where α and β are all other quantum numbers. Visible is the appearance of the Clebsch-Gordan coefficients (5.3). This states that the dependence of the matrix element on j_3, j'_3, and l is trivial and determined solely by the group structure. All the dynamics is completely encoded in the so-called reduced matrix elements $\langle j', \alpha | O^s | j, \beta \rangle$, which does not depend on j_3, j'_3, and l. Of course, this may not be too useful in practice for actually calculating the reduced matrix element, but separates conceptually kinematic effects from dynamic effects. Especially, it implies that if O is a tensor operator, its expectation values are $2s+1$-times degenerate, as there are $2s+1$ different matrix elements on the left-hand side, but only one on the right-hand side, up to cases where Clebsch-Gordan coefficients vanish.

To prove this theorem, notice that in the relation for tensor-operators (5.4) on the right-hand side only the operator itself appears, but the actually value of the J are matrix elements, and thus just numbers. Especially

$$\begin{aligned} J_3 O^s_l |j, j_3, \alpha\rangle &= [J_3, O^s_l] |j, j_3, \alpha\rangle + O^s_l J_3 |j, j_3, \alpha\rangle \\ &= O^s_{j_3 j'_3} (J^s_3)_{j_3 j'_3} |j, j_3, \alpha\rangle + O^s_l (J^j_3)_{j_3 j'_3} |j, j_3, \alpha\rangle = (l + j_3) O^s_l |j, j_3, \alpha\rangle \end{aligned}$$

Acting on this expression now from the left with the corresponding state, J_3 can be applied to it. However, since the representation of J_3 is not explicitly given, this still requires to transform it into an abstract basis. This is done by inserting a unity in the representation s. Then these are the expectation values of J_3 between the two, but they can be translated into each other using the Clebsch-Gordan coefficients, which then finally creates the Wigner-Eckhart theorem (5.5).

5.5 Invariant tensors

It is possible to combine the structure of Lie groups based on weights with the tensor notation.

Since a representation has as many dimensions as its has weight vectors, it is possible to find a representation in which the base vectors correspond to a weight vector. These are the eigenvectors to the Cartan generators, which agree since Cartan generators are simultaneously diagonalizable. The weight vectors are, after all, the eigenvalues of the Cartans for these eigenvectors. Thus, there is a one-to-one correspondence

$$\mu = e_\mu,$$

where the $e_\mu = e$ are the usual Cartesian unit vectors, where the index enumerating the associated weight vector will be suppressed.

An arbitrary generator in this basis will have a matrix representation

$$(T^a)^i_j = \frac{1}{2}\lambda^a_{ij}$$

where the λ are not the actual matrices in the given basis. The position of the indices will become clear soon, but is not coincidentally reminding of the situation in relativity. It should be noted that the range of indices is that of the weights of the corresponding representation. E. g., if this is su(3) and the representation is the fundamental one, the indices run from 1 to 3, while in the adjoint one it would run from 1 to 8. For su(2), it would be 1 to 2 and 1 to 3, respectively. That is important, as this, e. g. limits the number of elements in anti-symmetric quantities in the indices. For su(2) in the fundamental representation, there can be no totally antisymmetric object with three or more indices, as with two different values for the indices there is no possibility to have all three indices different. Furthermore, indices belong to a fixed irreducible representation, and it is not possible to, say, contract indices of different irreducible representations, as they act in different subspaces.

Acting now with T^a on a base vector yields

$$T_a e = (T^a)^j_i e_j,$$

implying a contraction over two indices.

Take now the complex conjugated representation. It lives in the same space, and there will thus be a suitable basis of them. Denote these base vectors by $\bar{e}$ with components e^j to distinguish them from the e and their components e_i. In general, there is a base in which these are again the usual Cartesian unit vectors, but not in the one where these

correspond to the original representation, except when both representations are identical. Then

$$T_a\bar{e} = (-T^a)^\dagger\bar{e} = -(T^a)^i_j e^j$$

and they therefore transform with a minus sign and a contraction over the other index.

Such states can be tensorized, yielding tensors $t^{ijk...}_{abc...} = A^i B^j ... T_a S_b ....$ Acting on them with a generator implies, just as in (5.1), an action on every index, by definition

$$T_a t^{ijk...}_{abc...} = (T_a)^m_{n_l} t^{ijk...}_{ab...n_{l-1}mn_{l+1}...} - (T_a)^{n_l}_m t^{ijk...n_{l-1}mn_{l+1}...}_{abc...}$$

and thus the generators act on all indices simultaneously. An object acting on only one index requires in this convention an explicit definition. The number of indices is the rank of the tensor, as usual. The tensor of highest weight for any rank is the one constructed from tensorizing the single states of highest weight. Such a state is necessarily symmetric in exchange of upper and lower indices within each other, as the combination is symmetric. It also vanishes for any contraction of a upper and lower index, because the necessary Kronecker-δ δ^j_i has the property $\delta^i_j = -\delta^j_i$, since its is also constructed from the (traceless) generators. The features are conserved under application of the raising and lowering operators, and are therefore shared by all states, which are created from this highest-weight state.

Note that it is possible to define tensors, on which the generators act to the left, in principle the usual bra-states. Since they are transformed with the inverse, this implies that they will have the position of indices reversed.

This construction can now be used to characterize the concept of invariant tensors t, i. e. tensors which are invariant under the action of the group/algebra,

$$T_a t = 0$$

Because of the tracelessness of the generators, all algebras have the unit tensor δ^j_i in common,

$$(T_a\delta)^i_j = (T_a)^i_k\delta^k_j - (T_a)^k_j\delta^i_k = (T_a)^i_j - (T_a)^i_j = 0$$

Another always present invariant tensor, again due to the tracelessness, is the Levi-Civita tensor

$$(T_a\epsilon)^{ijk} = (T_a)^i_l\epsilon^{ljk} + (T_a)^j_l\epsilon^{ilk} + (T_a)^l_k\epsilon^{ijl} = (T^a)^l_l\epsilon^{ijk}.$$

This follows, as always all three indices have to be different, and thus whenever there are two different indices, this yields the trace component, yielding thus zero in total. The same is true for ϵ with all indices down, for the same reason. Note that the contraction

with an invariant tensor can be used to raise and lower indices, or to map a tensor of a certain rank to another rank.

There may also be more invariant tensors of different rank. If they cannot be decomposed in these invariant tensors, they are usually a characteristic property of the group. The first example is the Casimir operator defined as

$$T^2 = T^a T^a$$

a concept known from spins: This is the total spin. This operator commutes with all generators,

$$\left[T^2, T^a\right] = T^b \left[T^b, T^a\right] + \left[T^b, T^a\right] T^b = i f^{abc} \left\{T^b, T^c\right\} = 0 \tag{5.6}$$

and is therefore an invariant tensor. because it commutes with all the generators, it has to be proportional to the unit, and the coefficient is called the Casimir. Another way of determining the Casimir for an arbitrary representation is by

$$\begin{aligned} C_j &= g_{ab}\sigma^a\sigma^b \\ g_{bc} &= \mathrm{tr} f_b f_c \end{aligned}$$

where the σ_i are the generators in the representation j, and f are the generators in the fundamental representation, and the value therefore depends on the convention for the normalization of the scalar product of two generators in (4.10). An often useful normalization is $C_F = 1$. This normalization fixes the other Casimirs, but also often different values are assigned to the fundamental Casimirs for different fundamental representations.

An interesting quantity, which can be derived from the Casimirs, is called the index of a representation j, and defined as

$$I_2(j) = \frac{d(j)C_j}{d(F)C_F}$$

where $d(j)$ gives the dimension of the representation j. Except for the spinor representation of the special orthogonal groups of dimension 6 or less, the index is always an integer. This particular combinations appears in many practical calculations.

One further example is the totally antisymmetric tensor d^{abc} defined as

$$d^{abc} = \mathrm{tr} T^a \{T^b, T^c\}$$

which is an invariant tensor of su(3). However, this expression actually vanishes for both su(2) and g_2, and therefore this tensor does not exist in these two cases. That it is invariant for su(3) follows in the same way by explicit calculation.

This can be generalized to arbitrary rank by

$$d^{i_1 \ldots i_n} = \frac{1}{n!} \sum_{\text{permutations } P} \text{tr} T^{P(i_1)} \ldots T^{P(i_n)}$$

where the sum is over all permutations.

This can also be regarded as the case of a generalization of the Casimir concept, by denoting

$$\begin{aligned} C_j^k &= g^{a_1 \ldots a_k} \sigma_{a_1} \ldots \sigma_{a_k} \\ g^{a_1 \ldots a_k} &= \frac{1}{k!} \sum_{\text{permutations } P} \text{tr} T^{P(i_{a_1})} \ldots T^{P(i_{a_k})} \end{aligned}$$

and thus the total symmetric tensor is just a 3rd order Casimir. An interesting side-remark is that only SU(n) groups have a Casimir of order 3, provided $n > 2$.

The invariant tensors play an important role in physics. If a quantity of a certain tensor rank should be constructed which is invariant under the action of the algebra or group, it must be necessarily a linear combination of the invariant tensors of the same rank. Thus, in su(3) any invariant tensor of rank 2 is proportional to the unit matrix, while for rank 3 it is a linear combination of ϵ^{abc} and d^{abc}. This is actually true also beyond compact Lie groups, and is the origins of the tensor decomposition. Together with the statement that the number of invariant tensors for a finite rank is finite, this is known as the primitiveness hypothesis. There is, to the knowledge of the lecturer, no full general proof of it, but it appears to apply to any case relevant in physics.

Moreover, this primitiveness hypothesis implies that when a tensor of higher rank should be constructed, it consists out of tensors constructed from lower rank, e. g. at rank 4 from $\delta^i_j \delta^k_l$ and $\delta^i_l \delta^j_l$ as well as of potentially new invariant tensors. E. g. for su(3) there is also a new invariant totally symmetric tensor at rank 4. It is essentially the statement that any representation of a given dimensionality can be decomposed into reducible and irreducible parts, and so can any vector in this dimensionality.

It is worthwhile to have a few more examples. Take two tensors u and v of rank 1 in the 3-representation of su(3), i. e. their indices run from 1 to 3. Then it is possible to rewrite their tensor product $u \otimes v$ as

$$u^i v^j = \frac{1}{2}(u^i v^j + u^j v^i) + \frac{1}{2} \epsilon^{ijk} \epsilon_{klm} u^l v^m$$

This rewriting is at first not obvious. The logic behind is that the left-hand-side is a dimension 9 object. The first term on the right-hand side is a totally symmetric tensor, and has thus 6 different independent elements, and hence being a 6-dimensional representation.

The second part is an anti-symmetric tensor, multiplying the proper antisymmetric tensor $\epsilon_{klm}v^l v^m$. It has a single lower component, and thus belongs to a conjugate representation. As there are only three elements, this must be a $\overline{3}$ representation. Thus,

$$(1,0)\otimes(0,1) = 3\otimes 3 = 6\oplus\overline{3} = (2,0)\oplus(0,1),$$

and a tensor product of two 3-representation can be decomposed into the two irreducible representations 6 and $\overline{3}$. This decomposition is actually unique. The important step was to note that the only possibilities of decomposing something into other objects was by invoking a decomposition using the invariant tensors. This is not obvious for the first term, but this could be written using the δ-tensor as well. Any attempt to use the symmetric tensor would only result again in the same construction, as this can only create a totally antisymmetric tensor.

Two more examples may be useful,

$$\begin{aligned}(1,0)\otimes(0,1) = 3\otimes\overline{3} &= 8\oplus 1 = (1,1)\oplus(0,0)\\ u^i v_j &= \left(u^i v_j - \frac{1}{3}\delta^i_j u^k v_k\right) + \frac{1}{3}\delta^i_j u^k v_k.\end{aligned}$$

Here, the first term has 8 independent possibilities, again constructed using two different contractions of δ-tensors. The second term has only a single independent object, $u^k v_k$, as the remainder is just an invariant tensor. Thus, this is a decomposition into the adjoint and into the trivial one.

Finally,

$$\begin{aligned}(1,0)\otimes(1,1) = 3\otimes 8 &= 15\oplus\overline{6}\oplus 3 = (2,1)\oplus(0,2)\oplus(1,0)\\ u^i v^j_k &= \frac{1}{2}\left(u^i v^j_k + u^j v^i_k - \frac{1}{4}u^l v^j_l - \frac{1}{4}\delta^j_k u^l v^i_l\right)\\ &\quad + \frac{1}{4}\epsilon^{ijl}\left(\epsilon_{lmn}u^m v^n_k + \epsilon_{kmn}u^m v^n_l\right) + \frac{1}{8}\left(3\delta^i_k u^l v^j_l - \delta^j_k u^l v^i_l\right)\end{aligned}$$

where again the counting of the symmetric and antisymmetric tensors shows the correct dimensionalities. Also note that in all cases for (n,m) $n-m \mod 3 = 0$, which is called the triality. This feature is a property of su(3). In general, it is called n-ality, if for a group $l-k \mod n = 0$ is valid. This is actually not a property of the algebra, and this will be again discussed further later.

The language of tensors now provides also a new possibility to generalize the Wigner-Eckhart theorem. Consider

$$\langle v|W|u\rangle,$$

where the v and u are now arbitrary tensors. This corresponds to matrix elements

$$\langle v^{ij\dots}_{mn\dots}|W^{kl\dots}_{ab\dots}|u^{rs\dots}_{xy\dots}\rangle = \Gamma^{mn\dots kl\dots rs\dots}_{ij\dots ab\dots xy\dots}.$$

But any such object must be decomposable into invariant tensors. Thus, depending on the number of uncontracted indices, this is a sum

$$\langle v|W|u\rangle = \sum_i \lambda_i t_i,$$

where the t_i are the invariant tensors of the corresponding rank. Hence, such matrix elements can be decomposed into a finite sum with a finite number of numbers λ_i which are not determined by the group structure, and invariant tensors of the group. The undetermined numbers λ_i are those which in physics are determined from the dynamics of the system, e. g. the energy levels of hydrogen. This is just a generalization of the Wigner-Eckhart theorem, expressed using the primitiveness assumption above. The evaluation of these matrix elements for $W = 1$ is yielding the Clebsch-Gordan coefficients.

5.6 Young tableaux and su(N)

The concept of Young tableaux from section 3.10 can now be taken up again. The original Young tableaux had the interpretation of every element signifying permutations. For su(N), it will now be taken up again to signify that a Young tableaux represent a tensor which is anti-symmetric in indices which are in columns, and symmetric in indices which are in rows. I. e. exchanging indices in different rows yields a minus sign and exchanging indices in different columns yields a plus sign.

This immediately implies that there are limits to the size of a Young tableaux. Constructing tensors which are tensor products of tensors in the fundamental representations, indices can run at most up to N, and thus the maximal length of a column can be N, while there is no limit for the size of a row. This implies that a tensor with a single index is necessarily a box of the given irreducible representation. The conjugate representation is antisymmetric in the same basis, and is therefore signified by two boxes in a column. Again, a convention is needed for the ordering of the indices, which will be such that the Young tableaux as in section 3.10 do have always a upper-left triangle shape.

Counting the number of independent elements of the tensor formed by the (anti-)symmetry of the indices described by the Young tableaux again yields the dimensionality of the representation, and therefore the representation itself.

The most useful application of Young tableaux is arguably a graphical way how to determine the irreducible representations in a tensor product. The procedure is here

stated without proof. Select one of the starting representations. It is usually best to take the one with the 'larger' Young tableaux. Then label the boxes of the second tableaux with the same letter in every column, but different ones in every row. Then start by redistributing the boxes of the first row in all possible way on the first Young tableau to create new Young tableaux, as long as no more than one box is appended at every column. Then repeat this with the next row, but in addition keep only those Young tableau where there are at least as many previous boxes above the new boxes than there are to the left of it. Especially, this implies if there has only be a single previous box, the next box cannot be append to the right of it.

For su(3), taking the fundamental ones, the previous examples yield that twice the fundamental (a single box) yields a line of two boxes (two symmetric indices, and thus a 6) and two in a column (the conjugate). Similarly the anti-fundamental times the fundamental yields a right angle, being the adjoint with two antisymmetric and one symmetric, and column of three, which is a singlet - there is only a single element in a tensor which is totally antisymmetric and has at most three different values for the indices.

Other applications e. g. give simple counting rules of how often a given original representation resurfaces in the decomposition, and others.

5.7 Deconstructing SU(N)

A particular useful case of deconstructing representations is the question about the SU(M)$\otimes$SU(N)$\otimes$U(1) subgroup of SU($N+M$). Such a subgroup always exists, as it can be generated by having the SU(N) and SU(M) acting on the first N and last M indices of SU($N+M$), respectively, while any U(1) factor will commute with both sub-groups, and is therefore also present. It can be taken to act as N on the first indices, and as $-M$ on the last ones, which certainly commutes with both subgroups. Of course, all this applies as well if either M or N is 1, since this just refers to the case of a single SU subgroup.

Young tableaux can be used to determine the representations of the subgroup. If the representation of SU($N+M$) has $n+m$ boxes in a Young tableaux, these boxes can be split and rearranged for the two SU subgroups.

The simplest case is the defining representation, which has just a single box. Thus, this single box can end up only in either of the subgroups, which implies that the other has to be in the trivial/singlet representation. E. g., for the famous example of SU(5) this implies

$$(5) = (3,1)_2 \oplus (1,2)_{-3}$$

In all other cases, the boxes need to be redistributed. The adjoint representation, which

has 5 boxes, can therefore be rearranged as

$$(24) = (3,2)_5 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (8,1)_0 \oplus (\bar{3},2)_{-5}.$$

Here, it is visible that the U(1) charge of these representations is actually $nM - mN$. To see this, let the indices of both subgroups run over all possible $N + M$ values for the n and m indices. This yields the dimensionality, and hence the representation.

Using similar techniques, it is also possible to deconstruct SU(NM) into representations of SU(N)$\otimes$SU(M).

5.8 Semidirect product

While so far all tensor products have the property that they combine two groups G and H to a new group $G \otimes H$ such that the new group has elements (g, h) with the composition law $(g_1, h_1) \circ (g_2, h_2) = (g_1 g_2, h_1 h_2)$, this is by far not the only possibility, nor the only one relevant for physics.

An alternative are semidirect products, which rather satisfy

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 g_2, h_1 f(g_1) h_2)$$

where $f(g)h$ is some automorphism of H, i. e. a structure-preserving mapping of H into H, but could be both an inner or outer automorphism.

An physically relevant example is the Galileo group including rotations and translations. It is a semidirect product of rotations and translations, since two rotations are just as usually composed, but the direction of the second translation has to be rotated by the first rotation, $(rr', t + rt')$, and thus rotations play the role of G and translations the role of H. The automorphism is then the rotation of the first translation.

Chapter 6

Groups as generators

This brief chapter will analyze some implications of groups and group structures when analyzing the cases of applying group elements to something, usually the vectors of the vector space in which a representation acts. This will be used here synonymously, but it should be noted that this can also be more abstract constructions, since the only requirement is to have some way to define the action of group elements.

6.1 Orbits

The basic concept is that of an orbit. Take a single vector v of the vector space in question. Then an orbit, sometimes also called group-orbit, $\mathcal{O} = \{D(g)v\}$ is defined as the set of all vectors obtained under application of all group elements from this single vector. An example are the hyperspheres which are obtained under the application of the rotation groups on a single vector pointing to the surface of the hypersphere, e. g. in $\mathbb{R}^2$ the orbit of $(1,0)^T$ would be $(\cos\alpha, \sin\alpha)^T$ under the two-dimensional rotation group.

This structure has a number of consequences. First, the group is itself an orbit, as group elements can be applied to group elements by the group composition, always starting from the unit element, and thus the whole group is a single group orbit. Second, any set on which a group acts can be decomposed into orbits, though the number of orbits does not need to be countable. The orbits are also called classes, and membership of an element of the set in an orbit is thus a class relation. Third, group invariants are trivial, i. e. one element, orbits. Fourth, cosets of a subgroup H of a group G, i. e. elements of type gh or gh are orbits as well. Note that in general, except for invariant subgroups, left and right cosets are not identical, and thus the orbits of left and right cosets are not identical.

From the point of physics, it is useful that orbits can also be classified by the invariant tensors. In the simplest case of SO(n) and the vector representation, the invariant tensor

is proportional to the unit matrix. Therefore, the length of the vector, which is invariant under rotation, characterizes the orbit. Since other representations can have more invariants, orbits may have more than one invariant to characterize them. Then, two orbits do only agree if they agree in all the quantities characterized by invariant tensors. E. g. for the (2,2) representation of SU(3) there are two invariant tensors. Therefore, there are two quantities for any orbit which are invariant under the action of the group. One is again the length. The other is a more complicated quantity, which follows from the fact that every matrix of this representation satisfies $d^c_{ab}T_{ab} = 0$, and is hence a traceless, symmetric tensor.

6.2 (Non-)linear representations

Though most of the previous has dealt with linear representations, i. e. representations where group elements g have been mapped to matrix representations $D(g)$ such that they act as

$$D(g)_{ab}x_b$$

on the elements x of the representation (Hilbert) space, this is not necessary.

The only requirement for a representation is actually that it maintains the group composition law. Thus, it is possible to formulate an non-linear representation

$$f_a(x, g),$$

provided

$$f(f(x, g), h) = f(x, g \circ h)$$

where the group composition is maintained. Then, this map is a non-linear representation.

Though non-linear representation of groups (and algebras) are rare, they are sometimes encountered.

6.3 The little group

Given some vector v, it is possible that there is a subgroup H of the original group G for which every subgroup element h satisfies

$$D(h)v = v$$

and thus this particular vector is invariant under a subgroup, and this identifies a suborbit. The subgroup H is then referred to as the little group, sometimes also stability or isotropy

group, of the vector v. Note that a little group of a continuous group can be both discrete and continuous. Orbits with the same subgroup are collected, and called strata.

A straightforward example is the rotation group SO(3) in its three-dimensional representation. The vector $v = e_z$ has then as little group SO(2), all the rotations in the x-y plane. Note that the little groups are representation-dependent. E. g. in the trivial representation, all orbits have the full group as little group. For SU(2) in the fundamental representation, there is only one non-trivial stratum, with the little group being just the trivial group containing only the unit element. The reason is that for every vector in the fundamental representation, there is an SU(2) transformation which transforms it into a unit vector in one direction, and thus all of them are not invariant. In the adjoint representation, there is again only a single non-trivial stratum, but with little group U(1), as there is a phase free.

In physics, this becomes especially important if there is a physical reason, like an external magnetic field in a spin system, which is fixed. Then a spin system, which had previously some higher symmetry group, will have a lower symmetry group, which is the little group defined by the direction of the magnetic field. The little group is called in this context also the residual symmetry group.

In general, there is an infinite number of orbits, but only a finite number of subgroups. As an example, for the group SO(n) and vector representations there is a trivial stratum given by the orbit $v = 0$, which has as little group the original group. All vectors of non-zero length have only SO($n-1$) as little group, the rotations around them. Thus all vectors of non-zero, but fixed, length belong to the same stratum. For symmetric rank two tensor representations, the little groups of SO(n) are different, and the strata can be classified by the number of degenerate eigenvalues, as the spectrum is invariant under rotations, but the ordering is not. Similar, but more complicated, considerations apply to both more complex representations and other groups like SU(n).

Note that the orbits in a stratum for the little group H are in one-to-one correspondence to the coset G/H, as the remaining group elements will transform them into each other.

A general classification of strata and/or little groups is very complicated in general, especially for reducible representations.

The little groups can also be characterized by the invariants of an orbit. Given any orbit ϕ, then an invariant is defined by

$$I(U(g)\phi) = I(\phi),$$

where $U(g)$ is any group element. Thus, the action of an invariant on an orbit is invariant under any group transformation. E. g., once more the length of a vector for SO(n) is such an invariant. As noted, there can be multiple invariants, which can be labeled by an index.

It is possible to ask what are the extrema of the invariants with respect to the orbits,

$$\frac{\partial I(\phi)}{\partial \phi} = 0,$$

that is which orbits maximize the invariant. The solution to this question is known as Morse theory. E. g., for the length of a vector, $I(\phi) = \phi^\dagger \phi$, the only extremum is the trivial vector $\phi = 0$.

Since in physics potentials in the Lagrangian (or Hamiltonian) formulation are invariants of the symmetry groups, often but not always the second-order invariant, Morse theory is actually equivalent to looking for extrema, and thus (metastable) equilibria of potentials. Since potentials often also break some larger group to a smaller group, the question thus turns into the question of finding the little groups, and strata, given some higher-order invariant.

Chapter 7

Continuous groups beyond compact Lie groups

Here, the topic of other groups, especially non-compact Lie groups as well as graded Lie groups will be discussed.

7.1 Topological groups

To discuss more generally continuous groups, it is useful to introduce topological groups. A topological group is a group on which a topology is defined, which gives the notation of closeness a realization. Especially, a topology should imply that if g and h are 'close' to g' and h', then so are gh and $g'h'$, as well as g^{-1} and g'^{-1}. Particular the last statement is non-trivial, as it requires to maintain the concept of closeness even for the inversion.

Besides introducing some kind of distance function between two group elements $d(g, h)$, not much is required for a topology. Alternatively, what is required is that for any group element it is possible to define a map which is locally isomorphic to Euclidean space. The situation based on representations as Taylor series in chapter 4 were just a particular example.

An important subclassification is that of compactness. Topologically compactness occurs if anything can be covered with a finite number of coverings. Local compactness occurs if this is true for a neighborhood. Lie groups are locally compact, as they can be expanded around the unity, but they are not necessarily globally compact, as the example of non-compact Lie groups shows.

Once the group has been equipped with a topology, it is possible to define a path in the group. Since a distance between group elements can be measured, the question can be stated if a group element can be reached from some other group elements by moving along

a continuous path, i. e., whether there is a path such that all group elements along the path are always close. If this is true, the group is connected. If not, then it decomposes into disconnected pieces. All the simple Lie groups are connected. However, e. g. the group $\mathrm{O}(n)$ is not connected, as it is not possible to find a smooth path in the group from group elements with determinant 1 and -1. Thus, these groups decompose into two (or more) disconnected pieces.

Groups, which are not connected, have the property that the piece containing the identity is an invariant subgroup. They are thus not simple. This follows because any two group elements which are continuously connected to the unit element, are also connected via following the path with the unit element as an intermediate stop. On the other hand, for any element g connected to the unit element, all kgk^{-1} are connected to the unit element, since the element g can be reduced to the identity in a continuous path, but then $k^{-1}k = 1$. As a consequence, it may be possible that a not connected group is a (semi-direct) product group. E. g. $\mathrm{O}(n)$=Z_2×$\mathrm{SO}(n)$ for n odd.

The remaining piece disconnected from the invariant subgroup containing the unit element are cosets of the original group, since any two elements g and h from the same disconnected piece fulfill necessarily the requirement that $g^{-1}h$ is in the piece connected to the unit element. As a consequence, the components form a discrete quotient group, and the semi-direct product mentioned above is in many, though not all, cases, the semi-direct product of this quotient group and the invariant subgroup, as was the case for the $\mathrm{O}(2n+1)$ groups.

Groups which have just one component are called simply connected.

Note that for Lie groups, due to the defining requirement of being able to map them to some vector space, the topology for a given representation is automatically induced by the topology of the vector space where this representation is defined.

7.2 Group measures

One useful consequence of topological groups is that it is possible to construct a measure $\mu(g)$ of a group, which is essentially given by the mapping to Euclidean space[1]. In general, this measure will then be locally some curvilinear coordinates. The important point is that such a measure requires a generalization of the concept of translationally invariance of it, in the sense of

$$\int d\mu(g) f(gh) = \int d\mu(g) f(g),$$

[1]Essentially, it is some function which assigns to any set of group elements a number which describes the size of this subset.

i. e. it is invariant under any group transformation. However, since the group needs not be commutative, so neither must be $d\mu f(gh)$, and thus in general $d\mu(g)f(gh) \neq d\mu(g)f(hg)$.

An example where it is invariant is the one for the general linear group on the real numbers of matrices M, which is given by

$$d\mu(g) = \frac{1}{\det M}\Pi dm_{ij}$$

while for the group determined by the matrices

$$L = \begin{pmatrix} e^a & x \\ 0 & e^b \end{pmatrix}$$

it differs, being the left measure

$$d\mu(g)_L = e^{-a}dadbdx$$

and for the right measure

$$d\mu(g)_R = e^{-b}dadbdx.$$

The existence of such an invariant measure is not guaranteed, but can be shown to always exist for locally compact groups, which includes Lie groups. In this case it is called a Haar measure. Furthermore, it is true for any compact groups, and especially for all compact groups

$$\int d\mu(g) < \infty,$$

i. e. the group volume is finite.

A quite useful further example is once more the group SU(2). For the fundamental representation, its elements are given by

$$g(\theta, \eta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\eta & ie^{-i\phi}\sin\frac{\theta}{2}\sin\eta \\ ie^{i\phi}\sin\frac{\theta}{2}\sin\eta & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\cos\eta \end{pmatrix}.$$

The resulting Haar measure is

$$\frac{1}{4\pi^2}\sin^2\frac{\theta}{2}\sin\eta d\theta d\eta d\phi$$

which was normalized such that the integral of the unit is one. Without this normalization, the result would be $16\pi^2$.

7.3 Pseudo groups

While in sections 4.20-4.22 the conventional special unitary and special orthogonal groups have been covered, it is useful to also introduce pseudo-versions of them. These groups are denoted by $\mathrm{SU}(p,q)$ and $\mathrm{SO}(p,q)$ and can be characterized to be the matrices M such that

$$\begin{aligned} M^a G M &= G \\ G &= \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \end{aligned}$$

where a is T for the special orthogonal case and $\dagger$ for the special unitary case. This can be generalized to the corresponding non-unimodular versions.

The relevance of these groups for physics is evident by noting that for the (special) orthogonal case and $p = 3$ and $q = 1$ (or $p = 1$ and $q = 3$) this defines the group of (proper) Lorentz transformations, as then G is the Minkowski metric.

In general, for any non-compact Lie group it is possible to choose the Cartan metric

$$g_{ab} = f^d_{ac} f^c_{bd}$$

such that it becomes $g = \mathrm{diag}(1_p, -1_q)$. As a consequence, all generators can be written as the sum of two generators, $T = K + M$ with $K^T = -K$ and $M^T = M$, such that the algebras created by the two sets of generators K and M obey

$$\begin{aligned} [K,K] &\sim K \\ [K,M] &\sim M \\ [M,M] &\sim -M \end{aligned}$$

and thus form two (noninvariant) subalgebras. This can be used to completely classify all non-compact simple Lie algebras, as the replacement $M \to iM$ turns both algebras into the same form as Lie algebras, and making the Cartan metric positive, and thus that of a compact Lie algebra. However, the process is still more complicated than for compact Lie algebras, and is therefore skipped here. However, this will be very useful in characterizing the Poincare algebra.

7.4 Covering groups

The probably most important consequence of considering topological groups is the insight that the seemingly surprising identification of different Lie algebras in chapter 4 is due to

the fact that the corresponding Lie groups are identical in the patch containing the unit element, and are thus locally isomorphic in this patch. However, the difference is that not all of the respective groups are simply connected. But it can be shown that for every set of locally isomorphic Lie groups there is one unique Lie group which is simply connected. This group is called the covering group.

E. g. for SO(3) and SU(2), the covering group is SU(2), since SO(3) is not simply connected. The explicit mapping for group elements is given by

$$r_{ij} = \mathrm{tr} u^\dagger \sigma_i u \sigma_j \tag{7.1}$$

where u is an arbitrary element of SU(2) and r the corresponding element of SO(3), and the σ_i are the corresponding generators of SU(2). This also shows how the simple connectedness of SU(2) is lost, as the double appearance of the u elements eliminates signs. Especially, elements proportional to the negative unit element of SU(2) are mapped into an element proportional to the positive unit of SO(3). This is generic. Since for any covering the number of continuous parameters needs to be the same, as it would otherwise not be a homomorphism, the different patches can only differ by a discrete group. Furthermore, the difference is exactly the center of the group SU(2), which is the discrete group Z_2, and thus in a sense SO(3)$\sim$SU(2)/Z_2. If the difference would not be the center of the covering group, then the different elements would not commute in the vicinity of the identity, and therefore would correspond to some different group element expanded close to the identity, but this would be continuous in contradiction to the discreteness. Thus, in general for any covering group H, it is homomorphic to its covering group G, up to its center Z, and thus $H \sim G/Z$. In reverse, $H/G \sim Z$. However, the covering group does not need to be a simple Lie group. E. g. for SO(4) the covering group is SU(2)$\times$SU(2), nor does it need to be a simple product group.

Studying (7.1), it is seen that the left-hand-side is a three-dimensional representation, and thus the adjoint representation. This immediately shows that all center elements of a group are mapped to the identity in the adjoint representation, which is therefore not a faithful representation. This is generally true. The reason is that the adjoint representation forms automatically a basis in its representation space, and then the only element commuting with every other element can be the unit matrix, and therefore the whole center is mapped to the unit matrix.

Hence, coinciding Lie algebras can yield different Lie groups, which then differ by connectedness and the center. Conversely, the adjoint representation of the algebra is faithful, if the center is discrete.

It is for this reason interesting to enumerate for the simple Lie groups centers and covering groups. In fact, only the SO(n) groups are not simply connected, but have a

covering group which is different only by a Z_2 factor. Furthermore, the centers of SU(n) are Z_n, of E_7, SO($2n$), and Sp($2n$) Z_2, of E_6 Z_3, and in all other cases trivial.

The concept of covering groups now also gives insight into the somewhat mysterious spinor representations of the SO(n) groups in section 4.22.3. Since the SO(N) are not simply connected, representations exist, which are not faithful, but double-valued, i. e. they map one group element into two different representation elements. The SO(n) are the only such group, and the result are the double-valued spinor representation. The corresponding representation of the covering group is, however, unique. This also explains the example of so(3) in section 4.22.3: In su(2), the elements multiplied by -1 are ordinary group elements. These are factored out when going to the non-connected so(3). Since the spinor representation is equivalent to su(2), these elements nonetheless exist, but as a double-valued 'ghost' image of the su(2) elements.

As can be deduced from this discussion, there are groups for which there exists continuous, unitary irreducible representations, which are either not faithful (adjoint representations for groups with a non-trivial center) or double-valued (spinor representations of the disconnected groups SO(N)). However, these groups all have the same algebra, i. e. they are isomorphic at the algebra level. As a consequence, the group which for a given representation is both faithful and single-valued is called the corresponding true group. E. g. for the isomorphic algebras so(3)$\sim$su(2), for the adjoint representation the group SO(3) is the true group, but for the spinor representation it is the SU(2) group.

7.5 True groups

Th full classification of representations according to the true groups is rather complicated, especially in the non-simple case. Therefore, here only the cases will be considered which are of direct relevance in physics. Here, it becomes especially important that not actually simple groups are the most common case in (particle) physics, but rather just compact Lie groups, with possible Abelian factor groups. The best known example is the standard model of particle physics.

The actual calculation is somewhat involved, and it is best to proceed in several steps.

Note that due to the discussion in chapter 5, it is possible to construct arbitrary representations by tensor products of the fundamental representations.

Start out with the groups without center, i. e. SO($2n+1$), G_2, F_4, and E_8. It is here important that the groups are considered, and not the algebra. Since these groups do not have a center, their adjoint representation is faithful. Furthermore, all tensor representations of them are single-valued.

For the groups with center Z_2, i. e. $SO(2n)$, $Sp(2n)$, and E_7, there are two possibilities for their tensor representations. Either they are build from an even or odd number (rank) of fundamental representations. For odd tensors, permutations of the indices permits to explicitly include the center, while this is not possible for an even rank. Thus, in the latter cases, the true group is G/Z_2 rather than the groups themselves.

The same is also true for the spinor representations of $SO(2n+1)$: If they are build from an even number of fundamental spinor representations, the true group is $SO(2n+1)\times Z_2$, but otherwise the original group.

This shows already how the structure emerges: Since center elements are roots of unity, it is necessary to be able to associate to every element such a factor under permutations of the fundamental representations building a given representation. E. g. for E_6 with center Z_3, the true group is again E_6 if the rank modulo 3 is non-zero, but E_6/Z_3 otherwise.

This effect can be seen as follows: A group element acting on a tensor will act on every fundamental representation simultaneously. In case of Z_2, there are two elements, and thus the values can be at most 1 or -1, and thus

$$\begin{aligned} t_1 &\rightarrow (-1)t_1 \\ t_1\otimes t_2 &\rightarrow (-1)(-1)t_1\otimes t_2 = t_1\otimes t_2 \\ t_1\otimes t_2\otimes t_3 &\rightarrow (-1)(-1)(-1)t_1\otimes t_2\otimes t_3 = (-1)t_1\otimes t_2\otimes t_3 \end{aligned}$$

and therefore a non-trivial action is only possible for an odd number of representations.

If the center is Z_3, the effect is, e. g. for the element $\exp(i2\pi/3)$

$$\begin{aligned} t_1 &\rightarrow (\exp(i2\pi/3))t_1 \\ t_1\otimes t_2 &\rightarrow (\exp(i2\pi/3))(\exp(i2\pi/3))t_1\otimes t_2 = (\exp(i4\pi/3))t_1\otimes t_2 \\ t_1\otimes t_2\otimes t_3 &\rightarrow (\exp(i2\pi/3))(\exp(i2\pi/3))(\exp(i2\pi/3))t_1\otimes t_2\otimes t_3 = (\exp(i2\pi))t_1\otimes t_2\otimes t_3, \end{aligned}$$

and likewise for the other element. Therefore the effect is trivial for the case of modulo 3 representations.

This still leaves the interesting cases of $SO(2n)$ with center Z_2 and $SU(n)$ with center Z_n.

To discuss the $SU(n)$ case, introduce the rank index of a representation

$$t = \sum^{l} s\mu_s \mod n+1$$

where μ_s are the corresponding Dynkin weights of the representation, and the sum is over the involved representations. The weight μ_s counts the number a given representation appears, and s is the number the fundamental representation appears. E. g., for $SU(2)$

$$t = 2j \mod 2+1,$$

where j is the spin of the representation and for SU(3) it is the triality of the representation: The rank index characterizes thus the rank of the representation modulo the size of the center.

This result implies that in an representation identified by rank index t for a group with center Z_n, the application of a center element ω (with $\omega^n = 1$) will yield the total effect ω^t. The complexity arises from the fact that there may still be a subgroup of the center, which is non-trivially realized on this representation. To proceed, note that there is some number f such that

$$\begin{aligned} n &= f n_0 \\ t &= f t_0 \end{aligned}$$

such that n_0 and t_0 are relatively prime. It follows

$$(\omega^t)^{n_0} = \omega^{f t_0 n_0} = (\omega^n)^t = 1 = (\omega^n)^{t_0},$$

and n_0 is the smallest integer for which this is true. Thus, the representation with rank index t carries a representation of the center group $\mathrm{Z}_{n_0} = \mathrm{Z}_n/\mathrm{Z}_f$. The true group is therefore G/Z_f.

Take as an example SU(3). The possible rank indices are 1, 2, and 3. The possible values for f are therefore 1, 1, and 3. Thus, the true group is SU(3) for $t = 1$ and $t = 2$, but $\mathrm{SU}(3)/\mathrm{Z}_3$ for $t = 3$. The latter includes, according to section 5.7, the adjoint representation, which is a totally antisymmetric tensor product of three fundamental representations. The fundamental representation and the 6 representation, being a symmetric tensor product of two fundamental representations, are of the former type. Note that the other in physics particular important representations of SU(3), 10, $\overline{10}$, and 27 also all have rank index 3.

This sequences can be more involved. Consider SU(4), with rank indices 1-4. The values of f are 1, 2, 1, and 4. Thus, the true group can be either SU(4), $\mathrm{SU}(4)/\mathrm{Z}_2$, or $\mathrm{SU}(4)/\mathrm{Z}_4$. For SU(6), the sequence of f is 1, 2, 3, 2, 1 and 6. In general, if n is prime, then f can be either only 1 or n. For the adjoint representation f is always n, and thus the true group for the adjoint representation is always $\mathrm{SU}(n)/\mathrm{Z}_n$.

For the SO($2n$) groups, the situation is a little bit more subtle, but it is along similar lines possible to define a rank index, and then determine the corresponding true groups. It will not be detailed here, but yields that the true groups for the spinor representations are always the covering groups, while in all other cases it is the original group, but sometimes with part, but not all, of the center divided out. Thus, all tensor representations of SO($2n$) contain at least the center Z_2.

7.6 True groups and Abelian subgroups

While so far only the true groups of simple groups have been considered, physics often requires to consider the situation with non-simple groups. Generically, this becomes quite involved. Therefore, the general strategy will be discussed here for the case most relevant to particle physics of groups SU(p)×U(1) and SU(p)×SU(q)×U(1), which cover e. g. the gauge group of the standard model. In the context of physics the quest for the true groups becomes very important, as quantum mechanics requires wave functions to be single-valued. Thus, a meaningful quantum theory can only be built with objects in single-valued representations, and therefore in the true groups.

To start, it is helpful to recall that the continuous unitary irreducible representations of U(1) are $\exp(im\phi)$, where the integer m is both the rank and the Dynkin index of the representation. Note, however, that only $m = 1$ is faithful, as in all other cases multiple group elements are mapped to the same value.

The simplest case is SU(p)×U(1) with p prime. Then there exist two possible groups, SU(p)×U(1) and SU(p)×U(1)/Z_p=U(p). The last equality is obtained by identifying the elements of Z_p in SU(p) with the corresponding ones in U(1). E. g., for SU(2)×U(1) this is achieved by identifying the element diag$(-1,-1)$ from SU(2) with the element (-1) from U(1). This is generalized by identifying the diagonal matrix element with entries ω (with $\omega^p = 1$) with the number ω'.

Another view is illustrated by the fact that the p-dimensional representation of U(p) is the direct product of SU(p) and U(1), and then the identification is manifest: This is just the set of p-dimensional special unitary matrices multiplied by an arbitrary phase, and then there is no distinction between the factor ω from either parts, as both yield a unit matrix with the ω on the diagonal. However, SU(p)×U(1) yields more directly a direct sum of dimension $p+1$, where then the difference of both center elements is manifest: The one is a p-dimensional unit matrix times ω appended with a one-dimensional unit matrix, while the other is a positive p-dimensional unit matrix appended with a one-dimensional diagonal matrix times ω.

Such an identification of elements is only possible if ω^t and ω'^m become one at the same time, and thus $t = m \mod p$. If this is the case, the two representations of the center can be identified, and thus the true group is U(p), and otherwise SU(p)×1. The p and $p+1$ dimensional representations are particular examples of this with $m = t = 1$, and representations $(1,1)$ and $(1,0)+(0,1)$ for (t,m).

In the next case is SU(p)×SU(q)×U(1) with $p \neq q$ and both relative prime. The center elements satisfy $\omega^p = 1$ and $\sigma^q = 1$. There are, of course, again elements ω' and σ' in the U(1) which also exponentiate in the same way to one. With the same argumentation, the

possible true groups are then

- $\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)$
- $\mathrm{U}(p)\times\mathrm{SU}(q)=\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)/\mathrm{Z}_p$, identifying $\omega=\omega'$
- $\mathrm{SU}(p)\times\mathrm{U}(q)=\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)/\mathrm{Z}_q$, identifying $\sigma=\sigma'$
- $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))=\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)/\mathrm{Z}_{p+q}$, identifying both $\omega=\omega'$ and $\sigma=\sigma'$

The last case is actually the one relevant for the standard model of particle physics, as all particles appear to be in the fundamental representation of the full standard model gauge group.

7.7 True groups and little groups

It is possible to show that for any continuous unitary and irreducible representations of a group there are at most a finite number of little groups. Furthermore, there is a smallest, non-trivial little group, in the sense that this little group is a (conjugated) subgroup of all little groups. However, there may be more than one maximal little group, i. e. little groups which are not related as (conjugated) subgroups to other little groups, but still are little groups of the representation in question.

A number of general statements can be made about the strata of these little groups. The first is that the stratum of the minimal little group is open and dense in the representation. It is thus, in a sense, maximal, and therefore also called the generic stratum. In the context of Morse theory, only the second-order invariant has a extremum in the generic stratum, while all other invariants have extrema in other strata, which is stated without proof.

The opposite are, modulo a norm, discrete strata, i. e. strata with a finite number of elements. The orbits in such strata are called critical strata. These strata are often, but not always, associated with maximal subgroups. In fact, it can be shown that the little group is maximal if there is only a single, modulo norm, orbit in its stratum, which is therefore a singlet under the action of this little group. An example were the fixed vectors for $\mathrm{SO}(n)$. It is important that, for a fixed representation, not necessarily a subgroup of the group is also a little group. In fact, it is often not. However, it is usually possible to construct some representation of which a given subgroup of a group is a little group, but this may not necessarily be an irreducible representation, nor easy. In the context of Morse theory, it is possible to show that all invariants, except for the second-order one, have extrema on critical orbits, which is thus an alternative characterization.

Another feature is that the little group may have the same or lesser rank than the original group, or, if it is discrete, even have rank zero. Since the rank is associated with the number of charges, this plays an important role in physics. In this context it is sometimes useful to define the little space as the space which is left invariant by a little group. E. g., for SO(3) this is the two-dimensional subspace left invariant by the SO(2) little group in the vector representation of the group.

7.8 An example: The Lorentz group

Taking up the issue of pseudogroups from section 7.3 again, a natural question is to ask what the representations are. Especially, given that the Lorentz group is a pseudo-group[2].

Considering the cases of SO(1,1) and SO(1,3) first[3], finite-dimensional representations are already known. In two dimensions the representation is given by

$$\Lambda = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

while in four dimensions the group elements can be constructed from six basic elements

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & 0 & \sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\beta & 0 & \cos\beta \end{pmatrix}; \quad J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{pmatrix}$$

$$K_1 = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad K_2 = \begin{pmatrix} \cosh\xi & 0 & \sinh\xi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\xi & 0 & \cosh\xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad K_3 = \begin{pmatrix} \cosh\zeta & 0 & 0 & \sin\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cosh\zeta & 0 & 0 & \sinh\zeta \end{pmatrix}$$

In the first three immediately the group elements of SO(3) are recognized, while the other three constitute the elements of a second SO(3), but with imaginary angels. In this sense, the group splits into the two subgroups as discussed in section 7.3.

However, already the two-dimensional case shows that this representation is not unitary, i. e. $\Lambda^\dagger \neq \Lambda^{-1}$. The reason can be traced back to the fact that the single generator of SO(1,1) is given by

$$\lambda = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

[2]The Poincare group is a tensor product of the Abelian translation group and the Lorentz group, and the latter plays no role in the following.

[3]These are the proper Lorentz group, while O(1,1) and O(1,3) would also include parity and time reversal.

which is not Hermitian. Since furthermore $\mathrm{tr}\lambda^2 < 0$ this implies what was already anticipated: The group is not compact. As shown in section 4.3, this implies that it does not have any finite-dimensional unitary representations.

However, there exists infinite-dimensional unitary representations. In this case, a function space is required. To introduce it, it is useful to have a different view on the Lorentz group.

The Lorentz group consists out of rotations J and boosts K. In general, commutators of J and K do not vanish. However, defining a skew version of these operators

$$\begin{aligned} A &= \frac{1}{2}(J + iK) \\ B &= \frac{1}{2}(J - iK) \end{aligned}$$

this is the case. The Lorentz algebra becomes then a direct product of two SU(2) algebras

$$\begin{aligned} [A_i, A_j] &= \epsilon_{ijk} A_k \\ [B_i, B_j] &= \epsilon_{ijk} B_k \\ [A_i, B_j] &= 0. \end{aligned} \tag{7.2}$$

Hence, any representation of the Lorentz group can be assigned two independent quantum numbers, which are either integer or half-integer. These are connected with well-known physical objects, depending on how the transform under each of the two SU(2) subalgebras. E. g. scalars are then just twice the trivial case, $(0, 0)$. Left-handed and right-handed fermions, however, belong to the $(1/2, 0)$ and $(0, 1/2)$ representations, vectors like the momentum belong to the $(1/2, 1/2)$ representation, and antisymmetric tensors like the generators of angular momentum to the $(1, 0) + (0, 1)$ representation.

Each of the subgroups are themselves compact. But their tensor product giving the Lorentz group is not, as their product is only a semi-direct product. The direct product delivers the conventional $SO(4) \sim SU(2) \otimes SU(2)$, the Euclidean rotation group.

Now, finally the unitary representation is given by a functional space, i. e. a space of functions $\phi^i_{\dot{j}m}(p)$ such that $p^2 = m^2$ and p is an n-dimensional vector with $p^2 = p_0^2 - \vec{p}^2$ living in the n-dimensional non-unitary representation of the Lorentz group. The indices i and $\dot{j}$ are multiindices, which describe the representation the functions form of the two SU(2) subgroups. The action of the Lorentz group is then

$$\Lambda\phi^i_{\dot{j}m}(p) = \lambda^i_k \lambda^{\dot{l}}_{\dot{j}} \phi^k_{\dot{l}m}(\lambda p), \tag{7.3}$$

where the λ are the corresponding non-unitary representations. In this case, $\Lambda^\dagger = \Lambda^{-1}$.

The parameter m classifies the little group of the orbits, where $m > 0$, $m = 0$, and $m < 0$ are the strata for the little groups SO(3), SO(2), and SO(1,2), where only the former two appear in physics. Correspondingly, m is the eigenvalue of the (lowest-order) Casimir of this representation with respect to the orbits, while the spin is the Casimir for the non-unitary finite-dimensional representations.

It is possible to upgrade the Poincare group further to the so-called conformal group. This is done by adding two more space-time transformations, corresponding to 5 more generators in four dimensions, to the group,

$$\begin{aligned} x^\mu &\rightarrow \lambda x^\mu \\ x^\mu &\rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}, \end{aligned}$$

with arbitrary vector a_μ. The first is a scale transformation, which is also called dilatation. The second is the special conformal transformation. The scale symmetry is in so far remarkable in physics, as only scaleless theories can be conformal. Hence, a conformal theory never has any kind of intrinsic mass scale. As a consequence, all infinite-dimensional, unitary representations of the conformal group, which still need to satisfy all the requirements of the representations of the Poincare group as well, can only be those states in (7.3) which satisfy $m^2 = 0$. Note that the finite-dimensional spin representations of the rotation group are not affected.

7.9 Clifford algebra

A closely related subject is the Clifford algebra. The Clifford algebra is defined as

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}1,$$

where $g_{\mu\nu}$ is a non-degenerate, symmetric matrix. The (identical) index range of μ and ν defines the algebra, and therefore this is actually a family of algebras.

The probably best known case is the one where there are four-different values for μ and ν. Then, the lowest-dimensional representation is four-dimensional, and given by the Dirac matrices. If the range is three, the lowest-dimensional representation is two, and the γ_μ coincide with the Pauli matrices. Interestingly, there is no three (and actually no odd) dimensional representation of the Clifford algebra.

Of course, this algebra is well-known in physics, and connected to fermions. Since the property of being a fermion is also connected to the 1/2 representations of the Poincare algebra and the spinor representations of the SO(n) groups, as noted in section 4.22.3, also manifest the Clifford algebra, this suggest a relation between both. This is indeed true,

and the Clifford representations are homomorphic to the representations of the orthogonal and pseudo-orthogonal groups. This becomes immediately obvious by decomposing the representations into their SU(2) content using the Pauli matrices. In fact, the Clifford group is a covering group of these groups, and also called the spin group Spin(n).

Note that the concept of Clifford algebras can be further generalized, but this leads substantially beyond the scope of this lecture, and it is actually rarely encountered.

7.10 Grassmann algebra

A, at first sight, deceptively similar algebra to the Clifford algebra is the Grassmann algebra

$$\{\xi_a, \xi_b\} = 0.$$

However, its properties are very different. The most remarkable fact is that $\xi_a \xi_a = 0$, i. e., the elements are nilpotent. This implies that there is no one-dimensional representation. Higher-dimensional representations are given by nilpotent matrices. However, since nilpotent matrices cannot be inverted, this algebra cannot generate a group under matrix multiplication, in contrast to the Clifford algebra, or any Lie algebra. However, it still forms a semigroup, as all other properties of a group, except the existence of an inverse, and thus neutral element, are satisfied.

However, it is possible to form a group under addition. This can be extended by including also ordinary numbers to form a vector space. This vector space will be sueful to introduce graded algebras below.

Due to the nilpotency, the set $\mathcal{S}$ of independent Grassmann numbers with $a = 1, ..., N$ base numbers are

$$\mathcal{S} = \{1, \alpha^a, \alpha^{a_1}\alpha^{a_2}, ..., \alpha^{a_1} \times ... \times \alpha^{a_N}\},$$

where all a_i are different. This set contains therefore only 2^N elements. There are no more elements, as the square of every Grassmann number vanishes, and by anti-commuting thus any product containing twice the same Grassmann number vanishes. Of course, each element of $\mathcal{S}$ can be multiplied by ordinary complex numbers c, and can be added. This is very much like the case of ordinary complex numbers. Such combinations z are called supernumbers, and take the general form

$$z = c_0 + c_a \alpha^a + \frac{1}{2!} c_{ab} \alpha^a \alpha^b + ... + \frac{1}{N!} c_{a_1 ... a_N} \alpha^{a_1} \times ... \times \alpha^{a_N} = c_0 + c_S. \tag{7.4}$$

Here, the factorials have been included for later simplicity, and the coefficient matrices can be taken to be antisymmetric in all indices, as the product of α^as are antisymmetric.

For $N = 2$ the most general super-number is therefore

$$z = c_0 + c_1\alpha^1 + c_2\alpha^2 + c_{12}\alpha^1\alpha^2,$$

where the antisymmetry has already been used. Sometimes the term c_0 is also called body and the remaining part soul. It is also common to split the super-number in its odd and even (fermionic and bosonic) part. Since any product of an even number of Grassmann numbers commutes with other Grassmann numbers, this association is adequate. For $N = 2$, e. g., the odd or fermionic contribution is

$$c_1\alpha^1 + c_2\alpha^2,$$

while the even or bosonic contribution is

$$c_0 + c_{12}\alpha^1\alpha^2.$$

Since the prefactors can be complex, it is possible to complex conjugate a supernumber. The conjugate of a product of Grassmann-numbers is defined as

$$(\alpha^a ... \alpha^b)^* = \alpha^b ... \alpha^a \tag{7.5}$$

Note that this implies that a product of an even number of Grassmann numbers is imaginary while an odd number is real,

$$\begin{aligned} \alpha^* &= \alpha \\ (\alpha\beta)^* &= \beta\alpha = -\alpha\beta, \end{aligned}$$

due to the anti-commutation when bringing the product back to its original order.

An important property of a super number z is its Grassmann parity $\pi(z)$. It differentiates between numbers which commute or anti-commute, and thus takes the values 0 or 1. Hence, for two super numbers

$$z_1 z_2 = (-1)^{\pi(z_1)\pi(z_2)} z_2 z_1,$$

the Grassmann parity can be used to determine the sign of permutations. Note that only supernumbers with only even or odd numbers of Grassmann numbers have a definite Grassmann parity. Hence, super numbers with definite Grassmann parity 0 or 1 are therefore called even or odd.

Finally, a norm can be defined as

$$|z|^2 = |c_0|^2 + \sum_{k=1}^{\infty} \sum_{\text{Permutations}} \frac{1}{k!} |c_{a_1 ... a_k}|^2,$$

such that it is possible to give meaning to the statement that a super number is small.

To construct groups or representations of Grassmann algebras it is often useful to refer to Grassmannian, in the physics context often also called fermionic, dimensions. Ordinary real coordinates are then referred to as bosonic. This is also called a superspace formalism, especially in the context of supersymmetry to be discussed below.

In superspace, each coordinate is a supernumber instead of an ordinary number. Alternatively, this can be regarded as a product space of an ordinary vector space times a vector space of supernumbers without body. This is very much like the case of a complex vector space, which can be considered as a real vector space and one consisting only of the imaginary parts of the original complex vector space.

A more practical splitting is the one in bosonic and fermionic coordinates. In bosonic coordinates only even products (including none at all) of Grassmann numbers appear, while in the fermionic case there is always an odd number of Grassmann numbers. Denoting bosonic coordinates by β and fermionic ones by ϕ, vectors take the form

$$\left(\begin{array}{c}\beta^i\\\hline\phi^j\end{array}\right).$$

In terms of the coefficients of a super-number in a space with two fermionic and two bosonic coordinates, based on the super-number (7.4), this takes the form

$$\left(\begin{array}{c}c_0\\c_{12}\\\hline c_1\\c_2\end{array}\right).$$

In the latter writing, it is important to notice that the unit vectors are not just bosonic ones, but rather the fermionic α_1 and α_2. This is like introducing the vector space of complex numbers with unit vectors 1 and i - the latter unit vector squares to -1 instead of one, as is the standard version. Similarly, the fermionic coordinates here still anti-commute.

Of course, this permits immediately to construct tensors of higher rank, in particular matrices. To have the same rules for matrix multiplication, it follows that a $(L+K)$-matrix must have the composition

$$\begin{pmatrix} A = \text{bosonic } L\times L & B = \text{fermionic } K\times L\\ C = \text{fermionic } L\times K & D = \text{bosonic } K\times K\end{pmatrix}$$

where bosonic and fermionic refers to the fact whether the entries are bosonic or fermionic (or even and odd, respectively). However, these matrices do have a number of properties

which make them different from ordinary ones. Also, operations like trace have to be modified.

First of all, the transposition operation is different. The sub-product of two fermionic sub-matrices B and C behaves as

$$(BC)_{ik} = B_{ij}C_{jk} = -C_{jk}B_{ij} = -(C^T B^T)_{ki} = (BC)^T_{ki}.$$

Therefore, there appears an additional minus-sign when transposing products of B- and C-type matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T = \begin{pmatrix} A^T & -C^T \\ -B^T & D^T \end{pmatrix}.$$

As a consequence, applying a supertransposition twice does not return the original matrix. Only after applying four times a supertransposition the original matrix is recovered. However, a supertransposed matrix has still the same supertrace and superdeterminant.

Though it is possible to define the inverse of a Grassmann number, there is no direct possibility to determine it explicitly, but products of an even number of Grassmann numbers are ordinary numbers and can therefore be inverted. Therefore, the inverse matrix is rather complicated,

$$M^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

That this is the correct prescription can be checked by explicit calculation,

$$MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} A(A - BD^{-1}C)^{-1} - BD^{-1}C(A - BD^{-1}C)^{-1} & -B(D - CA^{-1}B)^{-1} + B(D - CA^{-1}B)^{-1} \\ C(A - BD^{-1}C)^{-1} - C(A - BD^{-1}C)^{-1} & D(D - CA^{-1}B)^{-1} - CA^{-1}B(D - CA^{-1}B)^{-1} \end{pmatrix}$$

and accordingly for $M^{-1}M$.

This result implies that a super number defined as in (7.4), which is formally a 1×1 matrix, has only an inverse if its pure complex part c_0 is non-zero. If it has, it can be explicitly determined as the infinite series

$$\frac{1}{z} = \frac{1}{c_0} \sum \left(-\frac{c_S}{c_0} \right)^n,$$

though this is rarely needed in practice.

Also the standard operations of trace and determinant get modified. The trace changes to the super-trace

$$\mathrm{str}M = \mathrm{tr}A - \mathrm{tr}D. \tag{7.6}$$

The minus sign is necessary to preserve the cyclicity of the trace

$$\begin{aligned} \mathrm{str} M_1 M_2 &= \mathrm{tr}(A_1 A_2 + B_1 C_2) - \mathrm{tr}(C_1 B_2 + D_1 D_2) \\ &= (A_1)_{ij}(A_2)_{ji} + (B_1)_{ij}(C_2)_{ji} - (C_1)_{ij}(B_2)_{ji} + (D_1)_{ij}(D_2)_{ji}. \end{aligned}$$

The products of A and D matrices are ordinary matrices, and therefore are cyclic. However, the products of the fermionic matrices acquire an additional minus sign when permuting the factors, and renaming the indices,

$$\begin{aligned} & (A_2)_{ij}(A_1)_{ji} + (B_2)_{ij}(C_1)_{ji} - (C_2)_{ij}(B_1)_{ji} + (D_2)_{ij}(D_1)_{ji} \\ = \;& \mathrm{tr}(A_2 A_1 + B_2 C_1) - \mathrm{tr}(C_2 B_1 + D_2 D_1) = \mathrm{str} M_2 M_1. \end{aligned}$$

For the definition of the determinant the most important feature is to preserve the fact that the determinant of a product of matrices is a product of the respective determinants. This can be ensured when generalizing the identity

$$\det A = \exp\left(\mathrm{tr} \ln A\right)$$

of conventional matrices for the definition of the super-determinant

$$\mathrm{sdet} M = \exp\left(\mathrm{str} \ln M\right)$$

This can be proven by the fact that the determinant should be the product of all eigenvalues. Since the trace is the sum of all eigenvalues λ_i, and these are also for a super-matrix bosonic, it follows

$$\exp\left(\mathrm{str} \ln M\right) = \exp\left(\sum_{i \varepsilon A} \ln \lambda_i - \sum_{i \varepsilon D} \ln \lambda_i\right) = \exp \ln \left(\frac{\Pi_{i \varepsilon A} \lambda_i}{\Pi_{i \varepsilon D} \lambda_i}\right) = \frac{\Pi_{i \varepsilon A} \lambda_i}{\Pi_{i \varepsilon D} \lambda_i}. \tag{7.7}$$

Note the important fact that the eigenvalues of the fermionic dimension part D appears in the denominator rather than the numerator. This will play a crucial role in dealing with fermions in quantum theories.

The product rule for diagonalizable matrices follows then immediately. To prove that the super-determinant of the product is the product of the individual determinants also in general requires the Baker-Campbell-Hausdorff formula

$$\exp F \exp G = \exp\left(F + G + \frac{1}{2}[F, G] + \frac{1}{12}\left([[F, G], G] + [F, [F, G]]\right) + ...\right).$$

Set $F = \ln M_1$ and $G = \ln M_2$. Then it follows that

$$\mathrm{str} \ln(M_1 M_2) = \mathrm{str} \ln\left(\exp F \exp G\right) = \mathrm{str}\left(F + G\right) = \mathrm{str}(\ln M_1 + \ln M_2).$$

Here, it was invested that the trace of any commutator of two matrices vanishes due to the cyclicity of the trace. It then follows immediately that

$$\mathrm{sdet}(M_1 M_2) = \exp\left(\mathrm{str}\ln(M_1 M_2)\right) = \exp\mathrm{str}\left(\ln M_1 + \ln M_2\right) = \mathrm{sdet}M_1 \mathrm{sdet}M_2 \qquad (7.8)$$

where the last step was possible as the super-traces are ordinary complex numbers.

To evaluate the super-determinant explicitly, it is useful to rewrite a super-matrix as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

It then follows immediately by the product rule for determinants that

$$\mathrm{sdet}M = \mathrm{sdet}A \mathrm{sdet}\left(D - CA^{-1}B\right).$$

A similar formula an be obtained by isolating the superdeterminant of D rather than A. Since in both cases both factors are purely bosonic, they can be evaluated, and yield a bosonic super-determinant, as anticipated from a product of bosonic eigenvalues.

The construction of representations of continuous groups of Grassmann numbers profits from the fact that the exponential of a Grassmann number ξ with parameter a becomes

$$e^{a\xi} = 1 + a\xi,$$

since the higher powers vanish, since $\xi^2 = 0$. Thus, for Grassmannian generators, the infinitesimal version automatically is already the full representation.

7.11 Graded algebras

A graded algebra is an algebra which satisfies

$$\left[t^a, t^b\right\} = t^a t^b - (-1)^{\eta_a \eta_b} t^a t^b = i f^{abc} t_c,$$

where the η_i are known as gradings of the elements t_a, and are 0 for bosonic and 1 for fermionic generators. Because of the symmetry properties of the left-hand-side, this implies that f^{abc} is zero except when $\eta_c = \eta_a + \eta_b$. Still, if the t_a are Hermitian it follows that $f^*_{abc} = -f_{bac}$. Also a super-Jacobi identity follows

$$(-1)^{\eta_c \eta_a} \left[\left[t_a, t_b\right\}, t_c\right\} + (-1)^{\eta_a \eta_b} \left[\left[t_b, t_c\right\}, t_a\right\} + (-1)^{\eta_b \eta_c} \left[\left[t_c, t_a\right\}, t_b\right\} = 0.$$

It has the usual, but graded, implication for the relation of the structure constants. Note that the grading of a composite operator is in general given by $\left(\sum \eta_i\right) \mod 2$ with the

gradings of the constituent operators η_i. Furthermore, any transformation based on this algebra involves necessarily a mixture of ordinary complex numbers and Grassmann numbers, and therefore the parameters receive also a grading.

Thus, a graded algebra is just a mixture of conventional algebras and Grassmann algebras. Thus, using superspaces, it is possible to construct a group, and then representations in the usual way.

Chapter 8

Complex numbers

Chapter 9

Complex functions

Chapter 10

Cauchy's integration formula

Chapter 11

Series of holomorphic functions

Chapter 12

Singularities

Chapter 13

Manifolds

Chapter 14

Basic Differential Geometry

Chapter 15

Homotopy groups