Math 617 - Assignment 2 - Spring 2019.

For a 5% bonus, submit by Sunday June 2 at 23:59. Otherwise due Sunday June 9 at 23:59. Submit directly to your D2L dropbox.

Rules and Regulations:

1. You are to provide full solutions to the following problems.
2. Solutions must be typeset in a pdf file.
3. You may submit the assignment on your own or in pairs. A 5% bonus will be assigned to individual submissions.
4. If you work closely with another individual, please submit in a group with that individual (in other words, do not attempt to gain the benefit of single submission while working in a group).
5. Any sources must be cited including textbooks, people and online resources.
6. The student must demonstrate that they understand a third-party solution completely, or else the grader will assign them a zero on the question.
7. Direct copying of all or part of another solution (with or without citation) is academic misconduct.

Last name: __________________________ **First name:** __________________________

Student number: __________________________

Each part of each problem will be graded out of 3. A '3' represents a complete or near-complete solution. A '2' represents an otherwise correct solution with moderate logical gaps. A '1' represents demonstration of some insight into the problem.

1. Consider the space V of continuous functions on $[0,1]$ with the 2-norm $\|f\|_2^2 = \int_0^1 |f|^2$. We saw in class that V is an incomplete normed linear space.

   (a) For a continuous function φ on $[0,1]$, define a linear map $M_\varphi : V \to V$ by $M_\varphi f = \varphi f$. Show that M_φ is bounded and calculate its norm.

   (b) Is $\mathcal{A} = \{M_\varphi | \varphi \in C([0,1])\}$ a Banach algebra? Note that $B(V)$ is necessarily incomplete, so it is not enough to prove that $\mathcal{A}$ is closed.

2. Let X be a compact Hausdorff space and Y a subset of X. Let J_Y be the ideal of functions in $C(X)$ vanishing on Y.

   (a) Show that, in general, $\frac{C(X)}{J_Y}$ may not be isomorphic to $C(Y)$.

   (b) Prove or disprove the following statement: If Y is closed in X then $\frac{C(X)}{J_Y}$ is isomorphic to $C(Y)$. If you answer true, is this isomorphism also isometric?

3. Show that any two (Hilbert space) bases of a Hilbert space $\mathcal{H}$ have the same cardinality.

4. Let $\mathcal{H}$ be a Hilbert space with basis $\mathcal{E}$. Show that the set of functionals

   $$\text{span}\{\varphi_e | e \in \mathcal{E}\}$$

   where $\varphi_e(h) = \langle h, e \rangle$ is dense in $\mathcal{H}^*$ (which we identity with $\mathcal{H}$ by the Riesz representation theorem).

5. Show that the closed unit ball in a Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite dimensional. HINT: The closed unit ball must contain any basis.

6. Let $\mathcal{H}$ be a separable Hilbert space with basis $\{e_n\}_{n\in\mathbb{N}}$ and define P_n as the orthogonal projection onto $\text{span}\{e_1, \ldots, e_n\}$.

   (a) A sequence of operators $T_n \in B(H)$ is said to converge strongly to T if $\|Th - T_n h\|$ converges to 0 for all $h \in \mathcal{H}$ (note that strong convergence is actually weaker than operator norm convergence–think of this as the difference between pointwise and uniform convergence). Show that, for any $T \in B(\mathcal{H})$, the sequence $P_n T P_n$ converges strongly to T.

   (b) An operator $F \in B(\mathcal{H})$ is said to be of finite rank if $\text{rank}F := \dim(\text{im}F)$ is finite. If $\text{rank}F = 1$, show that $F(h) = \langle h, g \rangle k$ for some $g, k \in H$ (HINT: Riesz representation theorem).

   (c) Show that the linear span of rank 1 operators is strongly dense in $B(\mathcal{H})$.