Rendering Algorithms

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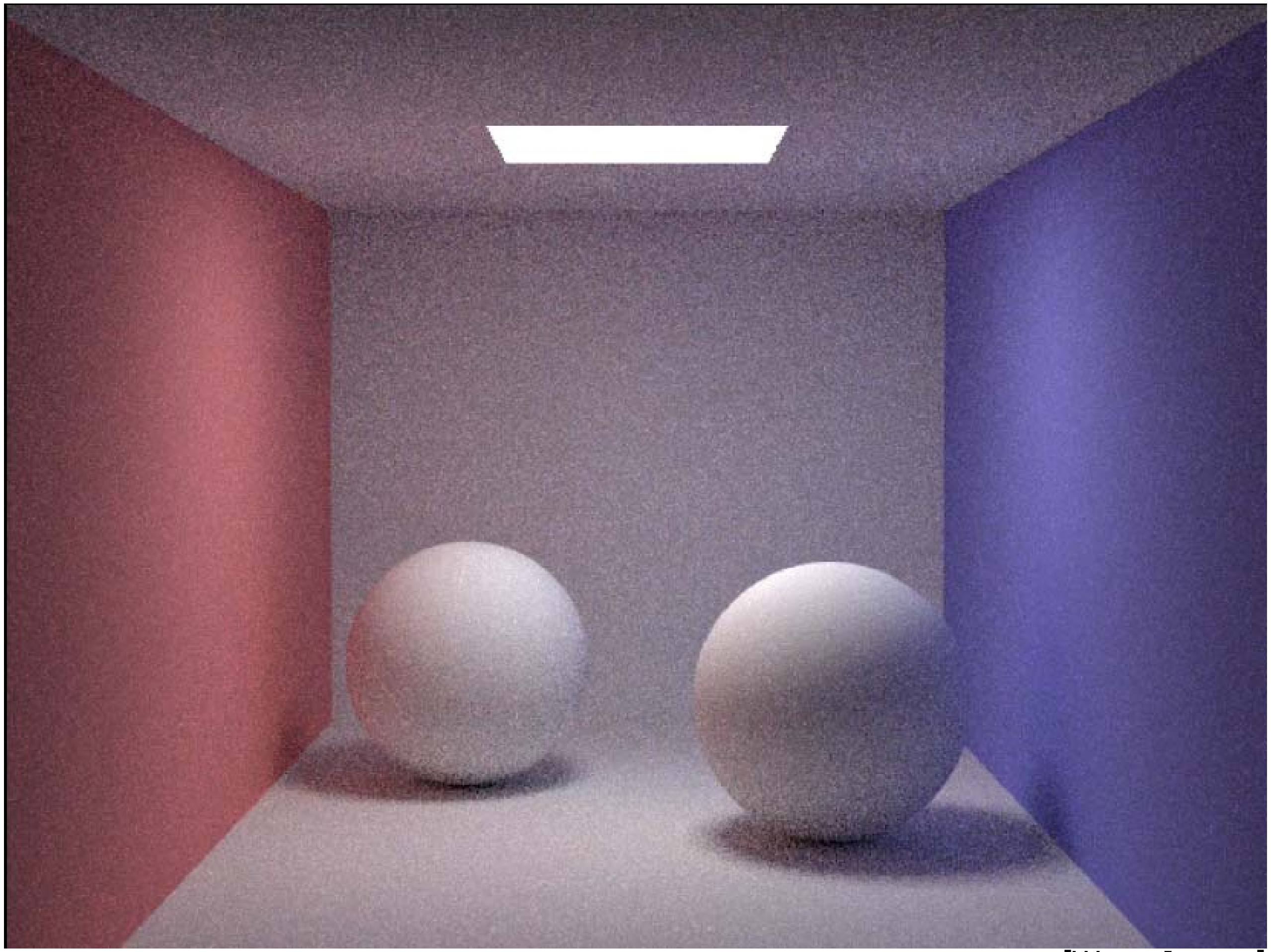
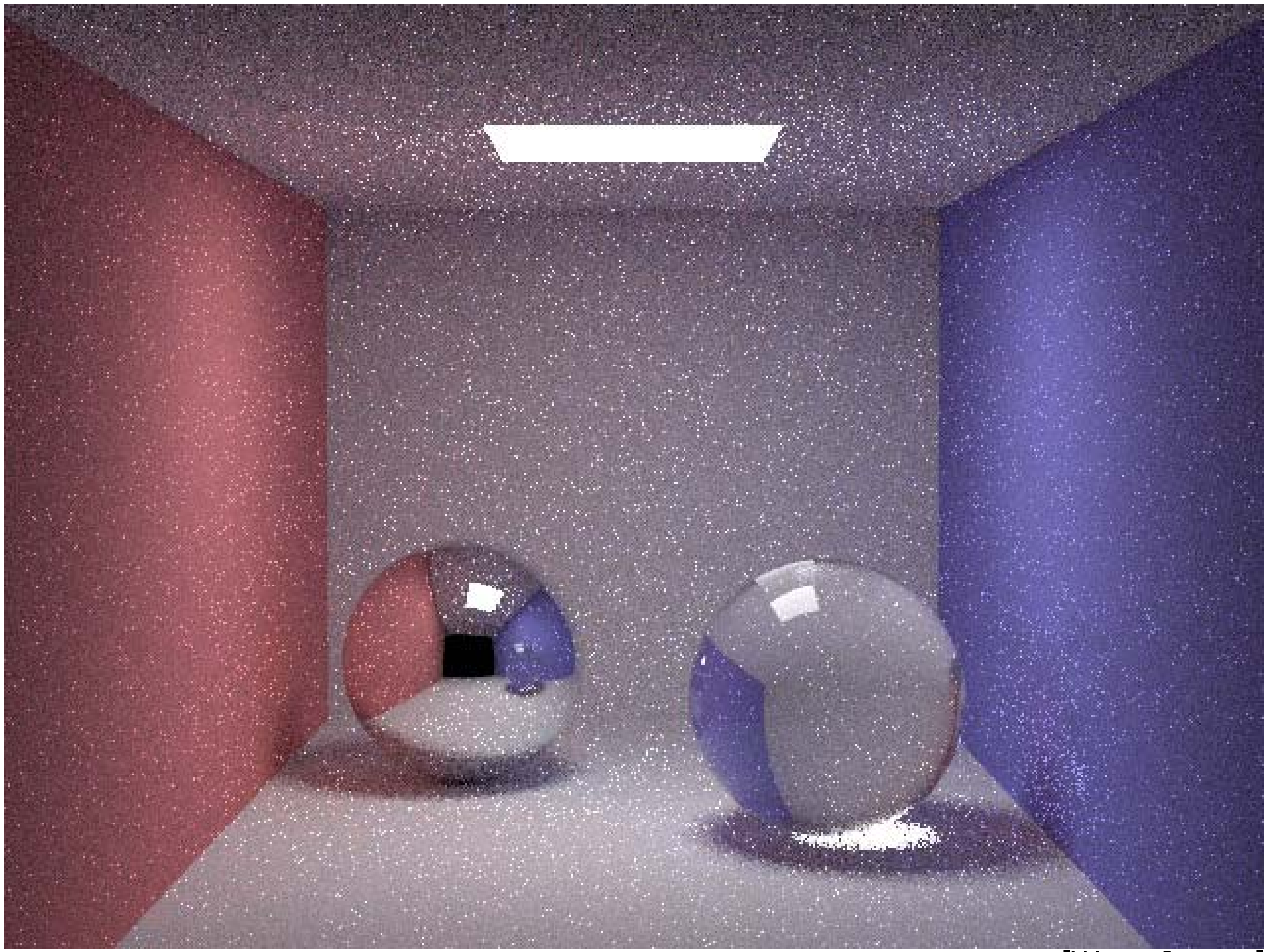
Path tracing


[Wann Jensen]

10 paths/pixel

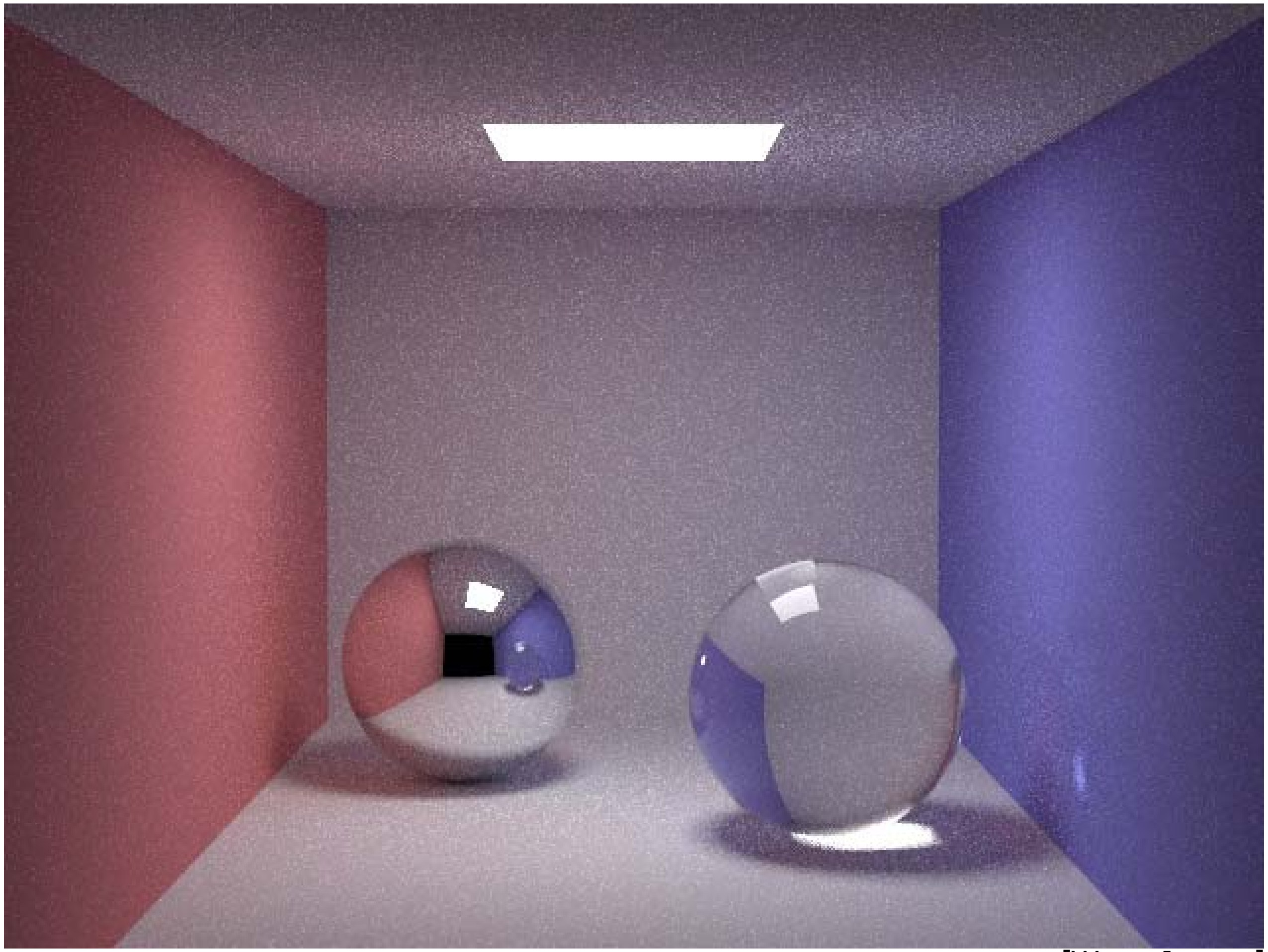
Problems with path tracing



[Wann Jensen]

10 paths/pixel

Problems with path tracing

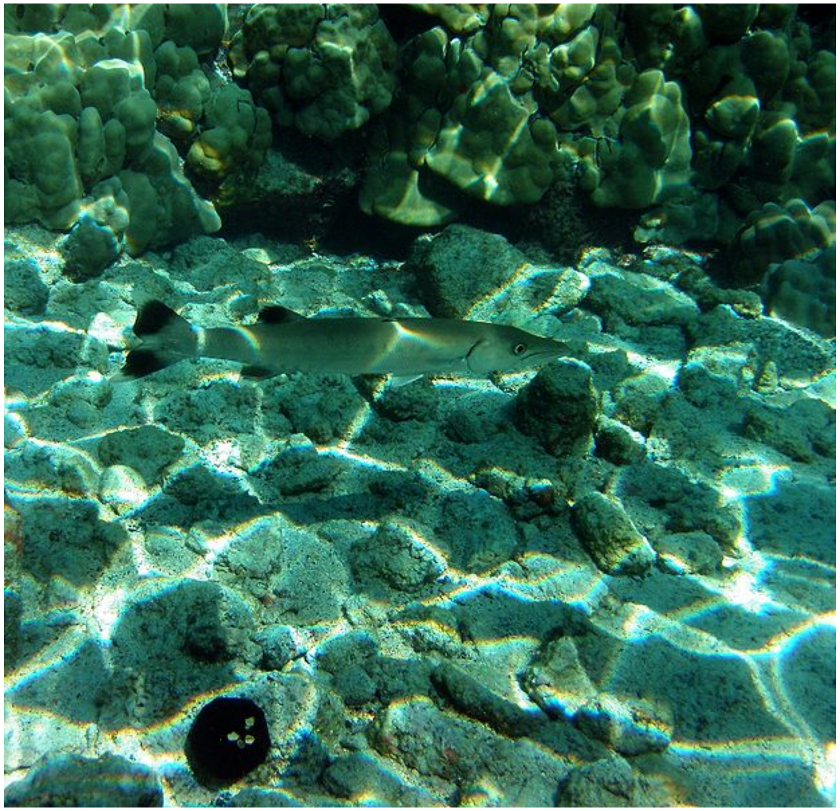


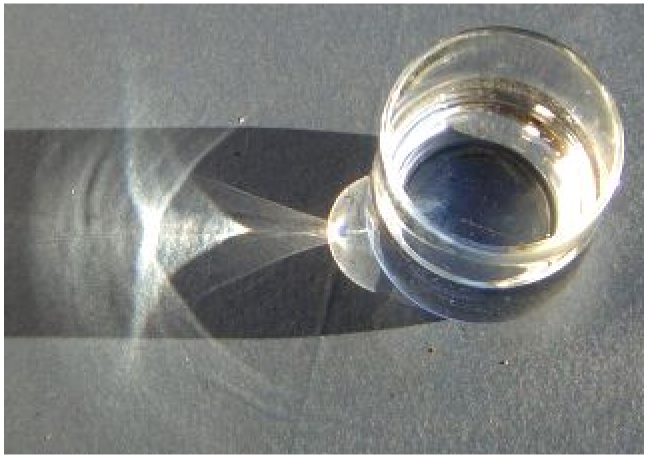
[Wann Jensen]

100 paths/pixel

Caustics

- Focusing of light by reflective and refractive surfaces





http://en.wikipedia.org/wiki/Caustic_(optics)

Light transport notation

- Path tracing not suitable to render caustics
- Light L, diffuse reflection D, specular reflection S, eye E
- Caustics are paths $L\{S\}^{+}DE$
  - $\{S\}^{+}$ means one ore more specular bounces
  - Focusing of light through reflections and refractions on diffuse surfaces

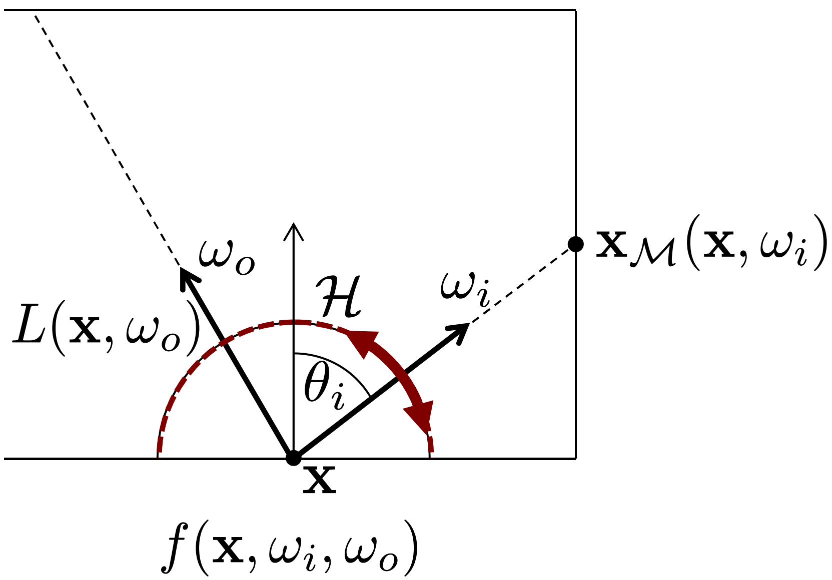
Today

- More sophisticated methods to sample light paths
  - Bidirectional path tracing
  - Photon mapping

Overview

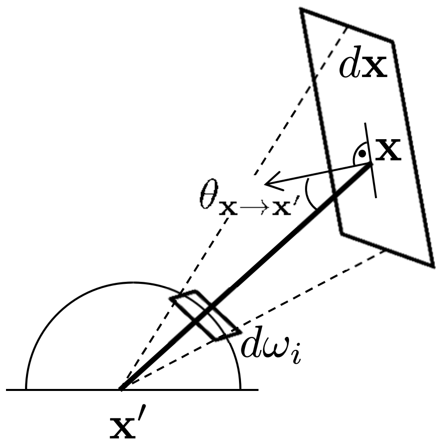
- Background
  - Three-point form of rendering equation
  - Measurement equation
- Reformulation of path tracing
- Bidirectional path tracing
  - MIS weights

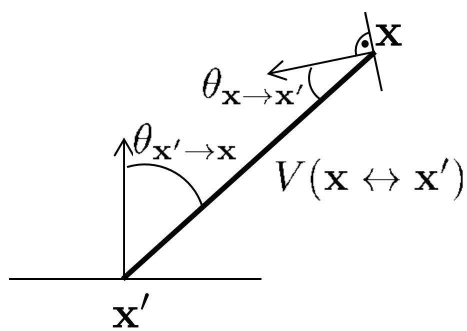
Hemispherical form



$$L(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\mathcal{H}} L(\mathbf{x}_{\mathcal{M}}(\mathbf{x}, \omega_i), -\omega_i) f(\mathbf{x}, \omega_i, \omega_o) \cos(\theta_i) d\omega_i,$$

Integration over surface area





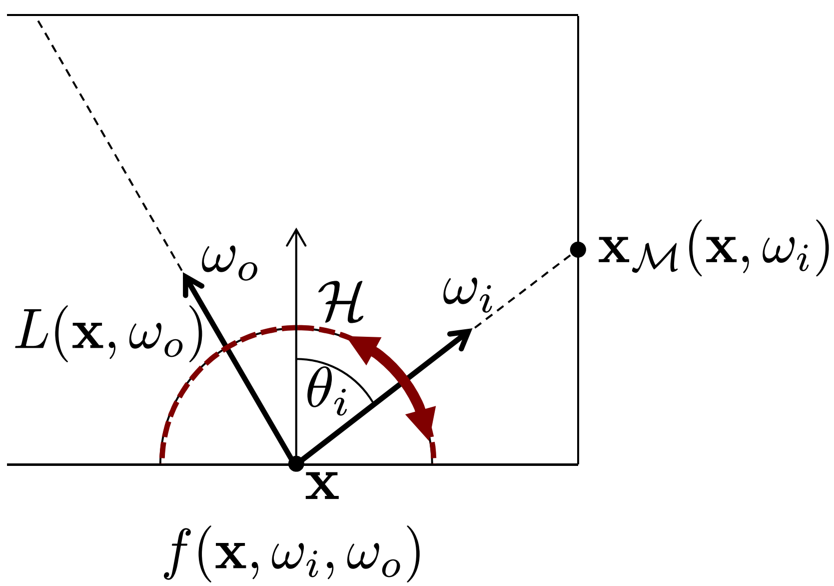
Change of integration variables:

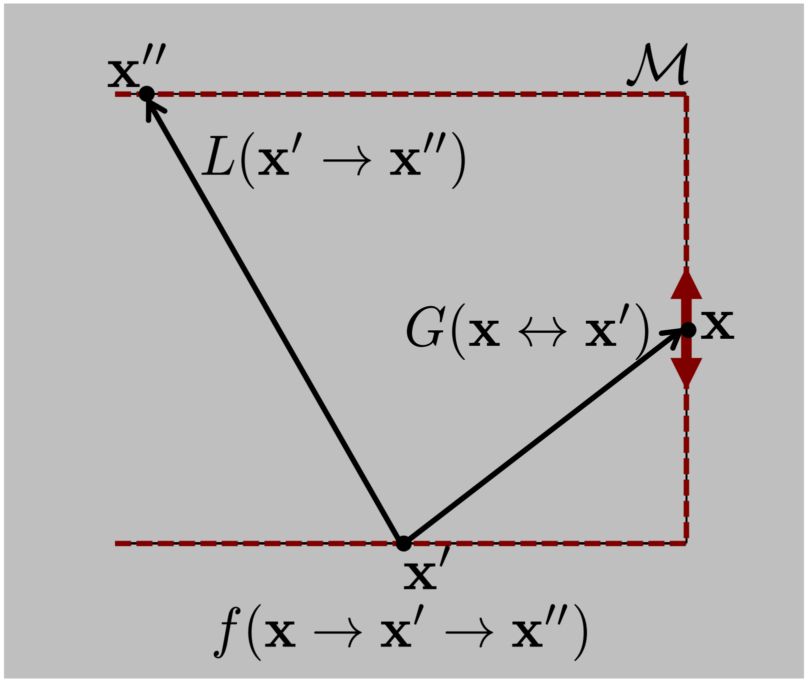
$$d\omega_i = \frac{\cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2} d\mathbf{x}$$

Geometry term:

$$G(\mathbf{x} \leftrightarrow \mathbf{x}') = V(\mathbf{x} \leftrightarrow \mathbf{x}') \cdot \frac{\cos(\theta_{\mathbf{x}'\to\mathbf{x}})\cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2}$$

Three point form





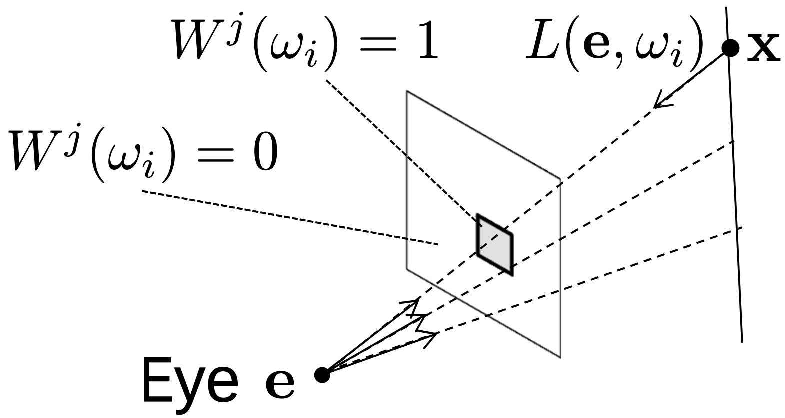
$$L(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\mathcal{H}} L(\mathbf{x}_{\mathcal{M}}(\mathbf{x}, \omega_i), -\omega_i) f(\mathbf{x}, \omega_i, \omega_o) \cos(\theta_i) d\omega_i,$$

$$L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'') + \int_{\mathcal{M}} L(\mathbf{x} \to \mathbf{x}') f(\mathbf{x} \to \mathbf{x}' \to \mathbf{x}'') G(\mathbf{x} \leftrightarrow \mathbf{x}') d\mathbf{x},$$

Three-point form, integration over surface area

Measurement equation

- Expresses pixel value I_j of pixel i



Importance function
W^j for pixel j,
here a box function

$$I_j = \int_{\mathcal{H}} W^j(\omega_i) L(\mathbf{e}, \omega_i) \cos(\theta_i) d\omega_i$$

Hemispherical integral

$$I_j = \int_{\mathcal{M}} W^j(\mathbf{x} \to \mathbf{e}) L(\mathbf{x} \to \mathbf{e}) G(\mathbf{x} \leftrightarrow \mathbf{e}) d\mathbf{x}$$

Surface area integral

Recursive expansion (3-point form)

Initial guess $L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'')$, plug recursively into

$$L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'') + \int_{\mathcal{M}} L(\mathbf{x} \to \mathbf{x}') f(\mathbf{x} \to \mathbf{x}' \to \mathbf{x}'') G(\mathbf{x} \leftrightarrow \mathbf{x}') d\mathbf{x}$$

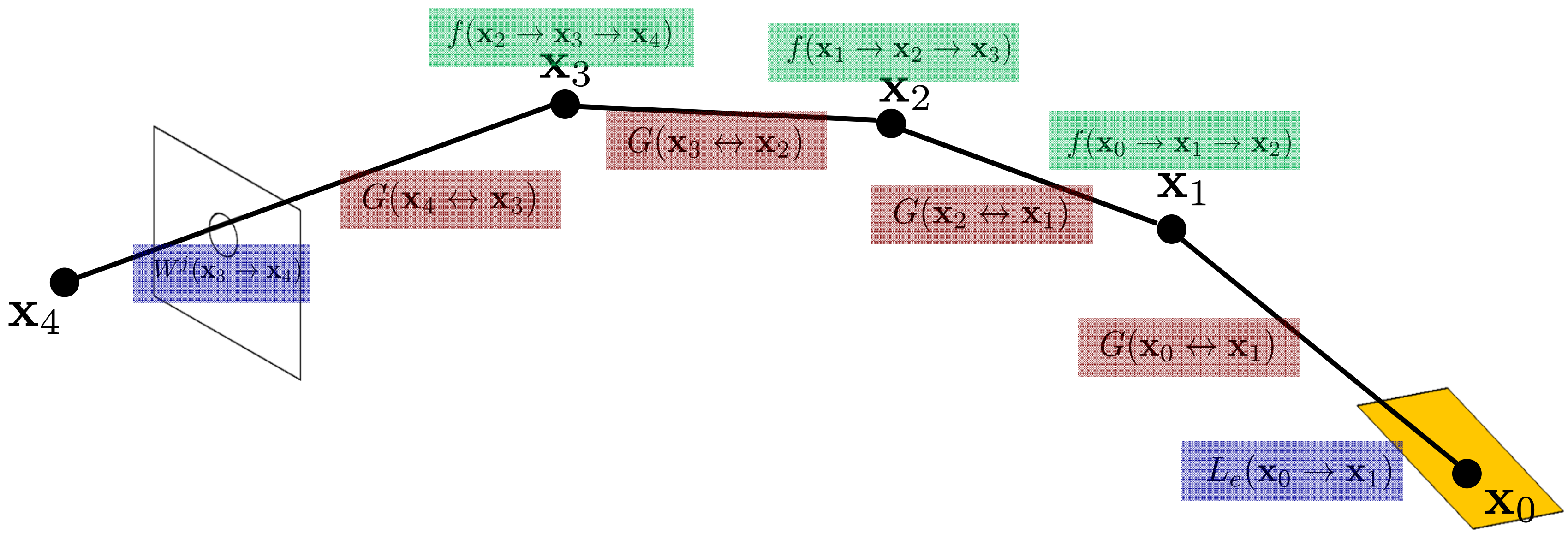
Plug result into $I_j = \int_{\mathcal{M}} W^j(\mathbf{x} \to \mathbf{e}) L(\mathbf{x} \to \mathbf{e}) G(\mathbf{x} \leftrightarrow \mathbf{e}) d\mathbf{x}$

$$I_j = \underbrace{\sum_{k=1}^{\infty}}_{\text{Sum over path lengths } k} \underbrace{\int_{\mathcal{M}^k} L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) \left(\prod_{i=1}^{k-1} f(\mathbf{x}_{i-1} \to \mathbf{x}_i \to \mathbf{x}_{i+1}) G(\mathbf{x}_i \leftrightarrow \mathbf{x}_{i+1}) \right) \cdot W^j(\mathbf{x}_{k-1} \to \mathbf{x}_k) d\mathbf{x}_0 \dots d\mathbf{x}_{k-1}}_{\text{Path contribution function } f_j(\bar{x})}$$

Sum over path lengths k

Path contribution function $f_j(\bar{x})$

Path contribution function



$$f_j(\bar{x}) = L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) f(\mathbf{x}_0 \to \mathbf{x}_1 \to \mathbf{x}_2) G(\mathbf{x}_1 \leftrightarrow \mathbf{x}_2) f(\mathbf{x}_1 \to \mathbf{x}_2 \to \mathbf{x}_3)$$
$$\cdot G(\mathbf{3}_1 \leftrightarrow \mathbf{x}_3) f(\mathbf{x}_2 \to \mathbf{x}_3 \to \mathbf{x}_4) G(\mathbf{x}_3 \leftrightarrow \mathbf{x}_4) W^j(\mathbf{x}_3 \to \mathbf{x}_4)$$

Symmetry!

Monte Carlo integration

- Estimate

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f_j(\bar{X}_i)}{p(\bar{X}_i)}$$

  - Random paths $\bar{X}_i$
  - Path probabilities (conceptually): product of vertex probabilities $p(x_i)$ (area densities!) and probability of $p(k)$ for length k

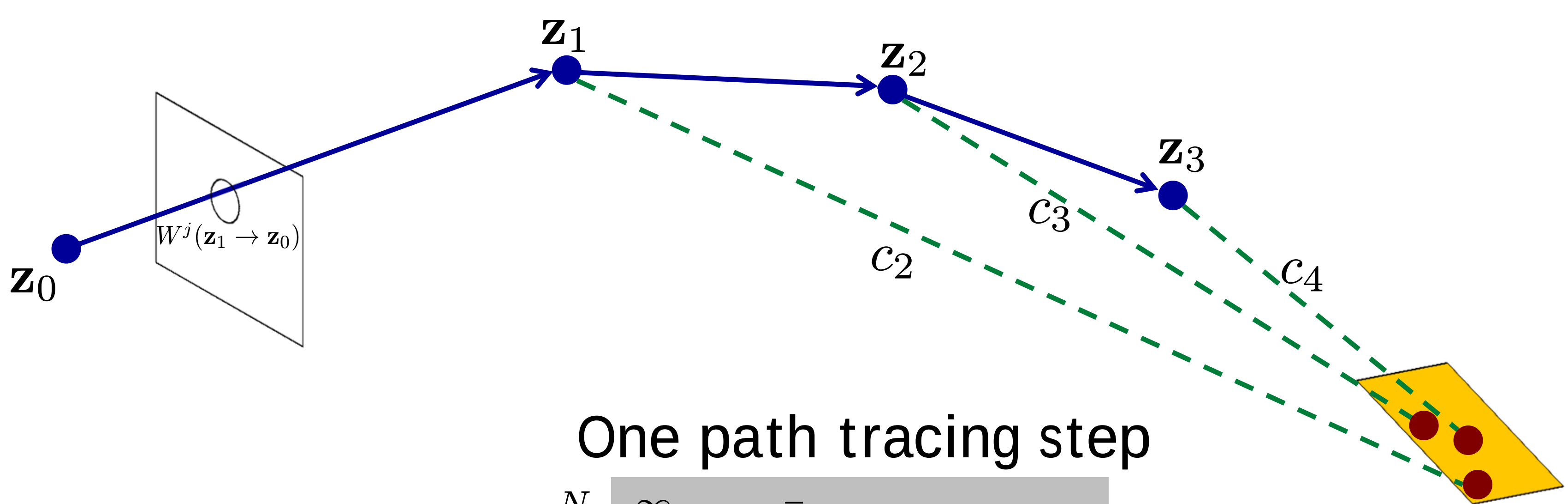
$$p(\bar{X}_i) = \left(\prod_{i=0}^{k} p(\mathbf{x}_i) \right) p(k)$$

- Path contribution

$$\frac{f_j(\bar{X}_i)}{p(\bar{X}_i)} = \frac{L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) \left(\prod_{i=1}^{k-1} f(\mathbf{x}_{i-1} \to \mathbf{x}_i \to \mathbf{x}_{i+1}) G(\mathbf{x}_i \leftrightarrow \mathbf{x}_{i+1}) \right) \cdot W^j(\mathbf{x}_{k-1} \to \mathbf{x}_k)}{\left(\prod_{i=0}^{k} p(\mathbf{x}_i) \right) p(k)}$$

Path tracing

- Sample paths incrementally from eye
- At each step connect to light to obtain path $\bar{X}_{k,i}$ of length k
- Terminate path using Russian roulette, implemented with binary random variable R_k
- Conceptually, each path tracing step evaluates sum over all lengths k=0…∞

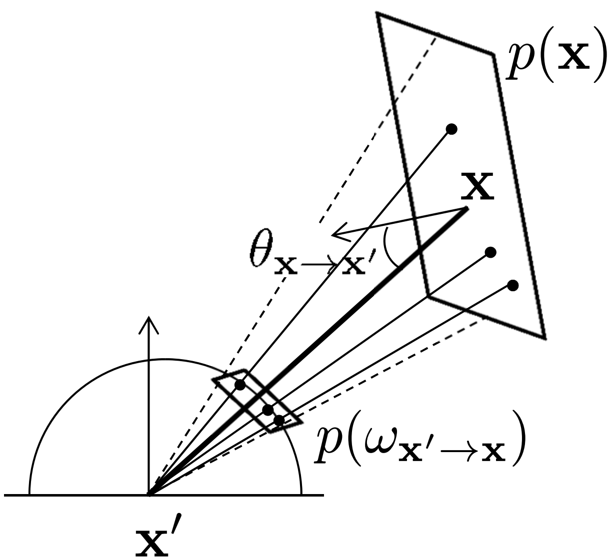


One path tracing step

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \frac{f_j(\bar{X}_{k,i})}{p(\bar{X}_{k,i})} \frac{R_k}{p(R_k = 1)}$$

Relation of densities

- Conversion to area density $p(x)$, if we sampled x by sampling direction $\omega_{x' \to x}$ from x' with density $p(\omega_{x' \to x})$

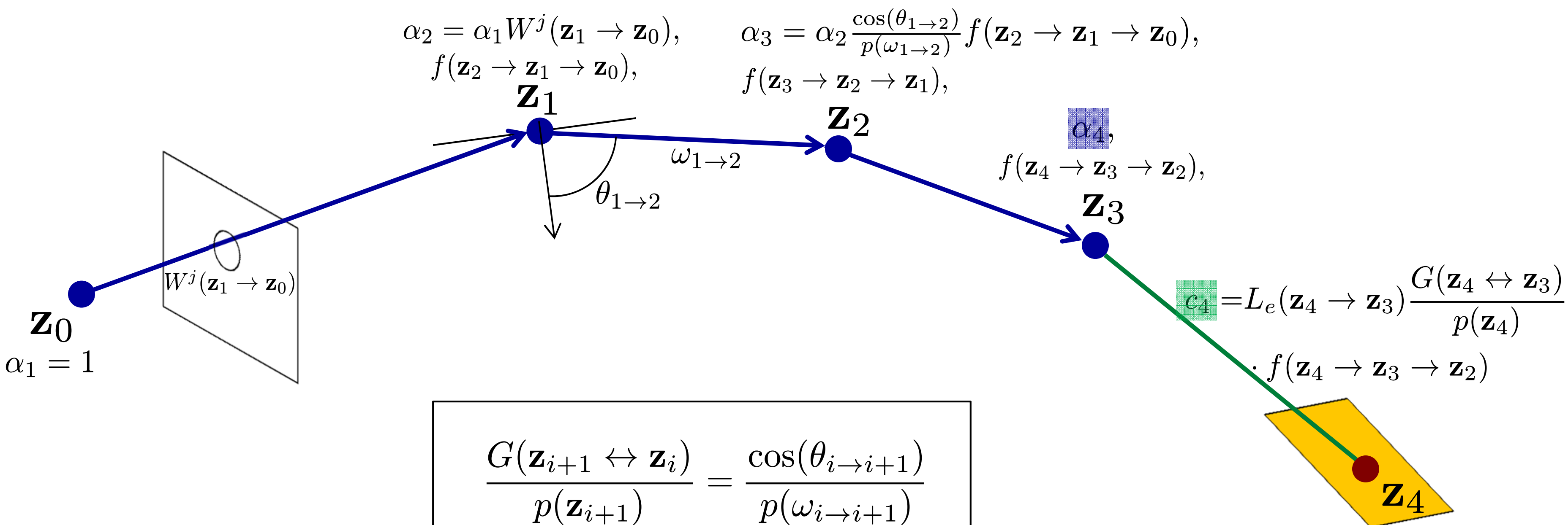


Relation of densities:

$$p(\mathbf{x}) = p(\omega_{\mathbf{x}'\to\mathbf{x}}) \frac{\cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2}$$

$$= p(\omega_{\mathbf{x}'\to\mathbf{x}}) \frac{G(\mathbf{x}' \leftrightarrow \mathbf{x})}{\cos(\theta_{\mathbf{x}'\to\mathbf{x}})}$$

Path tracing



$$\frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} = \frac{\cos(\theta_{i\to i+1})}{p(\omega_{i\to i+1})}$$

$$\frac{f_j(\bar{X}_{k,i})}{p(\bar{X}_{k,i})} = \underbrace{L_e(\mathbf{z}_k \to \mathbf{z}_{k-1}) \frac{G(\mathbf{z}_k \leftrightarrow \mathbf{z}_{k-1})}{p(\mathbf{z}_k)} f(\mathbf{z}_k \to \mathbf{z}_{k-1} \to \mathbf{z}_{k-2})}_{c_k}$$

$$\underbrace{\prod_{i=1}^{k-2} \frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} f(\mathbf{z}_{i+1} \to \mathbf{z}_i \to \mathbf{z}_{i-1}) \cdot \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} \frac{W^j(\mathbf{z}_1 \to \mathbf{z}_0)}{p(\mathbf{z}_0)}}_{\alpha_k}$$

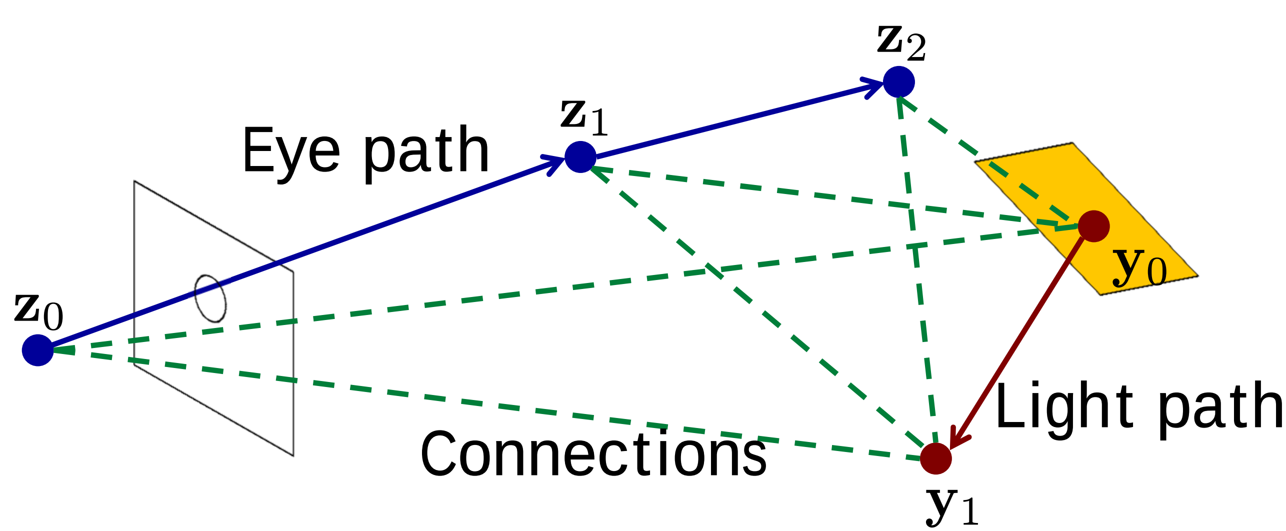
$$\alpha_1 = 1$$

$$\alpha_2 = \alpha_1 \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} W^j(\mathbf{z}_1 \to \mathbf{z}_0) = W^j(\mathbf{z}_1 \to \mathbf{z}_0)$$

$$\alpha_i = \alpha_{i-1} \frac{\cos(\theta_{i-2\to i-1})}{p(\omega_{i-2\to i-1})} f(\mathbf{z}_{i-1} \to \mathbf{z}_{i-2} \to \mathbf{z}_{i-3})$$

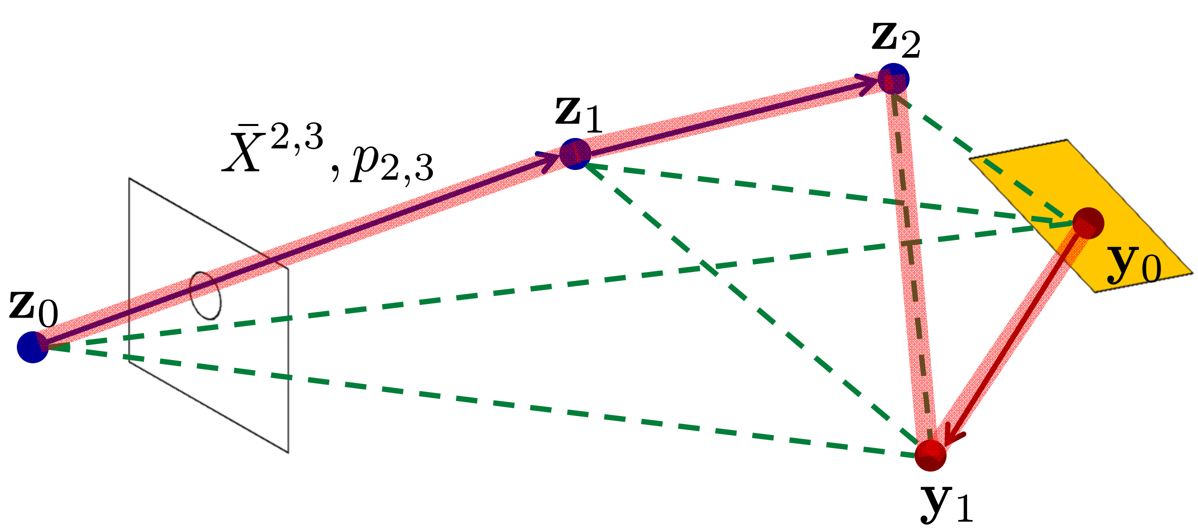
Bidirectional path tracing

- Trace paths from eye and light (eye and light subpaths)
  - Terminate each using Russian roulette
- Evaluate all connections and sum up



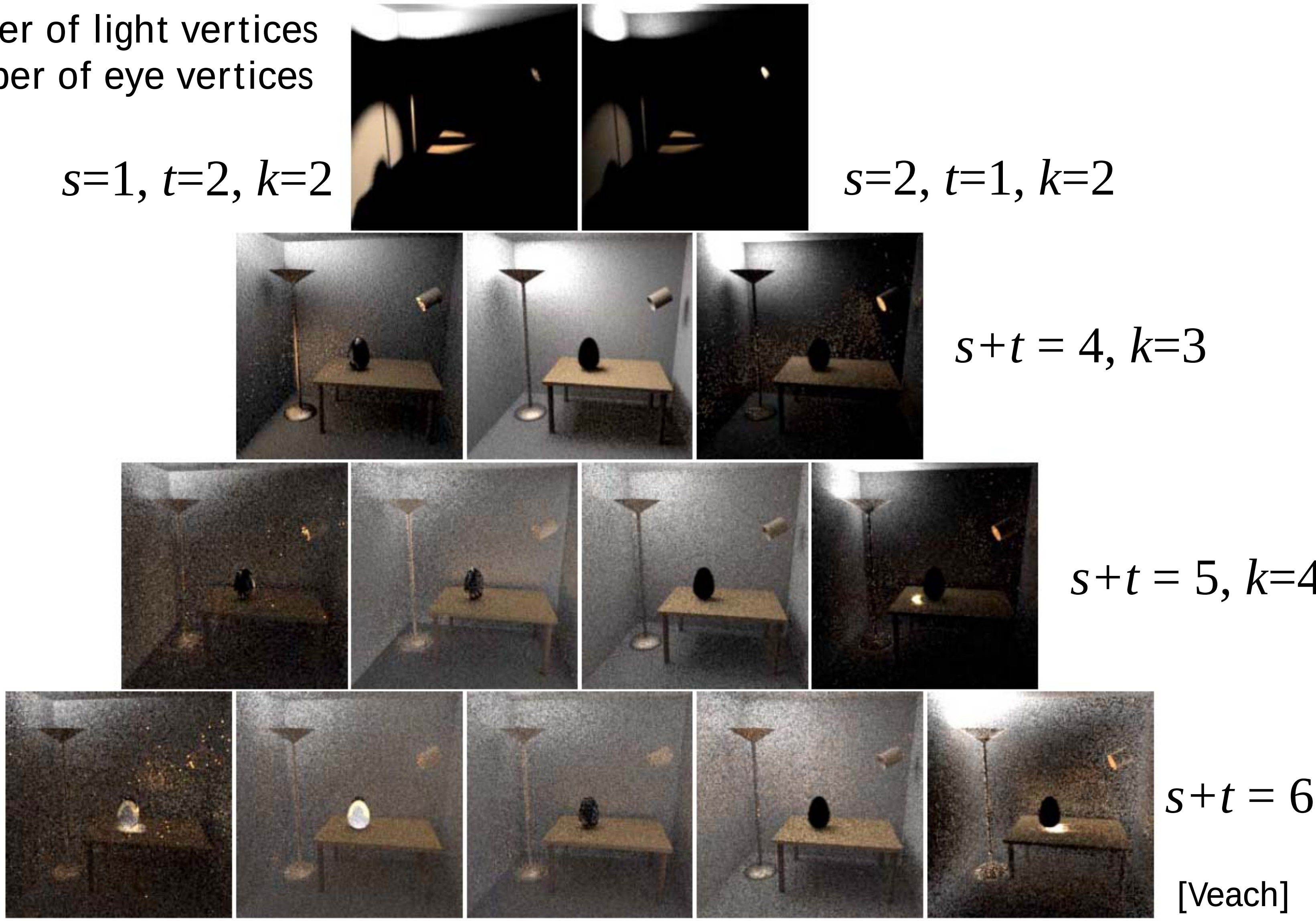
Bidirectional path tracing

- Path of length $k+1$ sampled with s vertices from light, t from eye, denoted $\bar{X}^{s,t}$
- Each length k can be sampled in k+2 ways, or with k+2 "sampling techniques"
  – $s = 0,1,\ldots k+1$ and $t = k+1\text{-}s$
  – Probability density for technique s,t denoted $p_{s,t}$



Example

s:number of light vertices
t: number of eye vertices



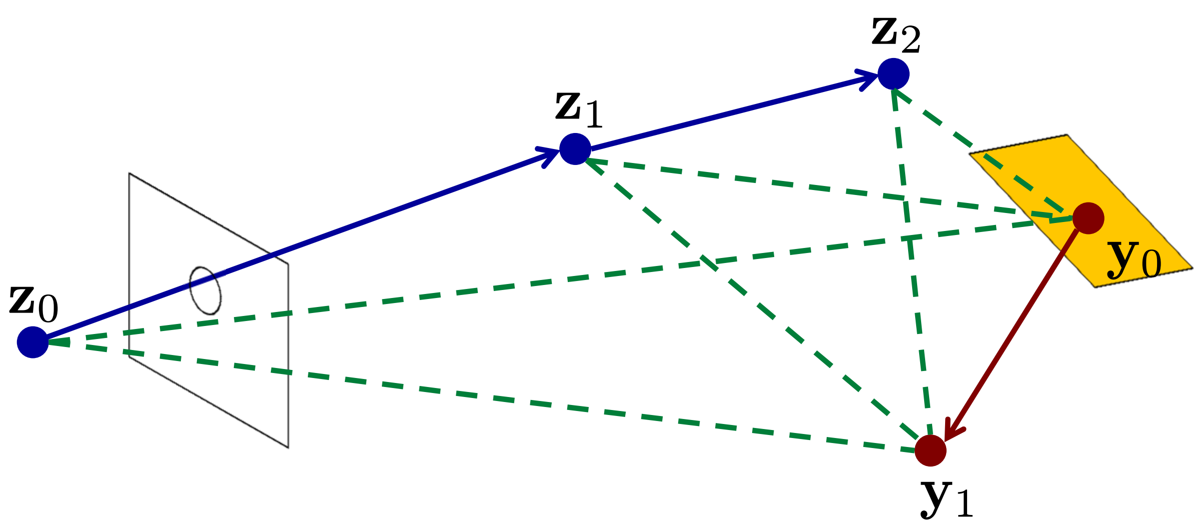
[Veach]

Bidirectional path tracing

- Conceptually, each bidirectional path tracing operation samples all techniques for all path lengths
- In practice, many are not evaluated because of Russian roulette, implemented with binary variable $R_{s,t}$
- Techniques combined with multiple importance sampling, weights w_{st}

One bidirectional path tracing step

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \sum_{s+t-1=k} \frac{R_{s,t}}{p(R_{s,t}=1)} w_{s,t}(\bar{X}_i^{s,t}) \frac{f_j(\bar{X}_i^{s,t})}{p_{s,t}(\bar{X}_i^{s,t})}$$



Path contribution

$$\frac{f_j(\bar{X}^{s,t})}{p(\bar{X}^{s,t})} = \underbrace{\frac{L_e(\mathbf{y}_0 \to \mathbf{y}_1)}{p(\mathbf{y}_0)} \frac{G(\mathbf{y}_0 \leftrightarrow \mathbf{y}_1)}{p(\mathbf{y}_1)} \prod_{i=1}^{s-2} f(\mathbf{y}_{i-1} \to \mathbf{y}_i \to \mathbf{y}_{i+1}) \frac{G(\mathbf{y}_i \leftrightarrow \mathbf{y}_{i+1})}{p(\mathbf{y}_{i+1})}}_{\alpha_s^L}$$

Light subpath

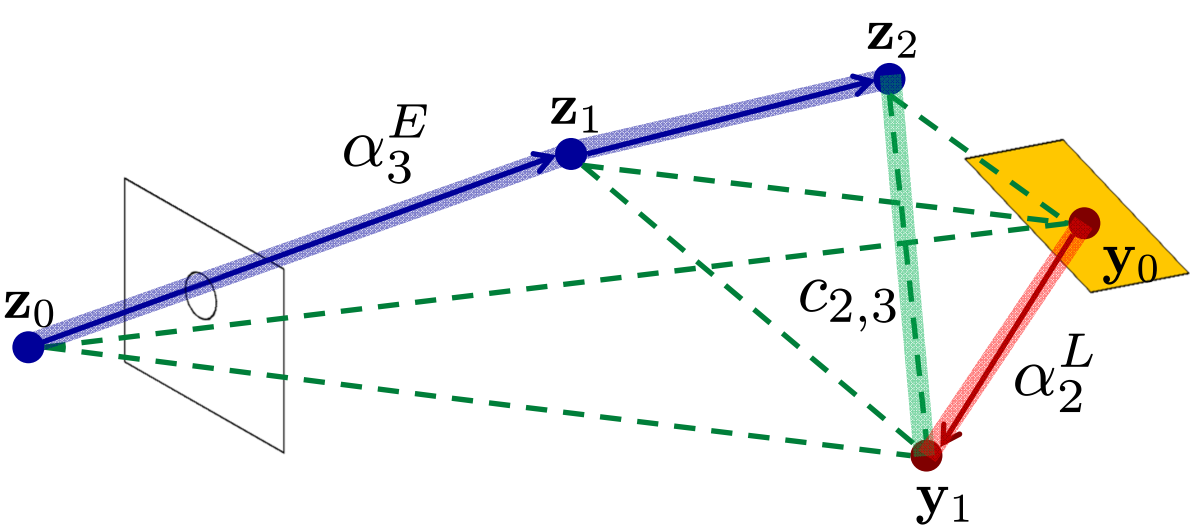
$$\underbrace{f(\mathbf{y}_{s-2} \to \mathbf{y}_{s-1} \to \mathbf{z}_{t-1}) G(\mathbf{y}_{s-1} \leftrightarrow \mathbf{z}_{t-1}) f(\mathbf{y}_{s-1} \to \mathbf{z}_{t-1} \to \mathbf{z}_{t-2})}_{c_{s,t}}$$

Connection

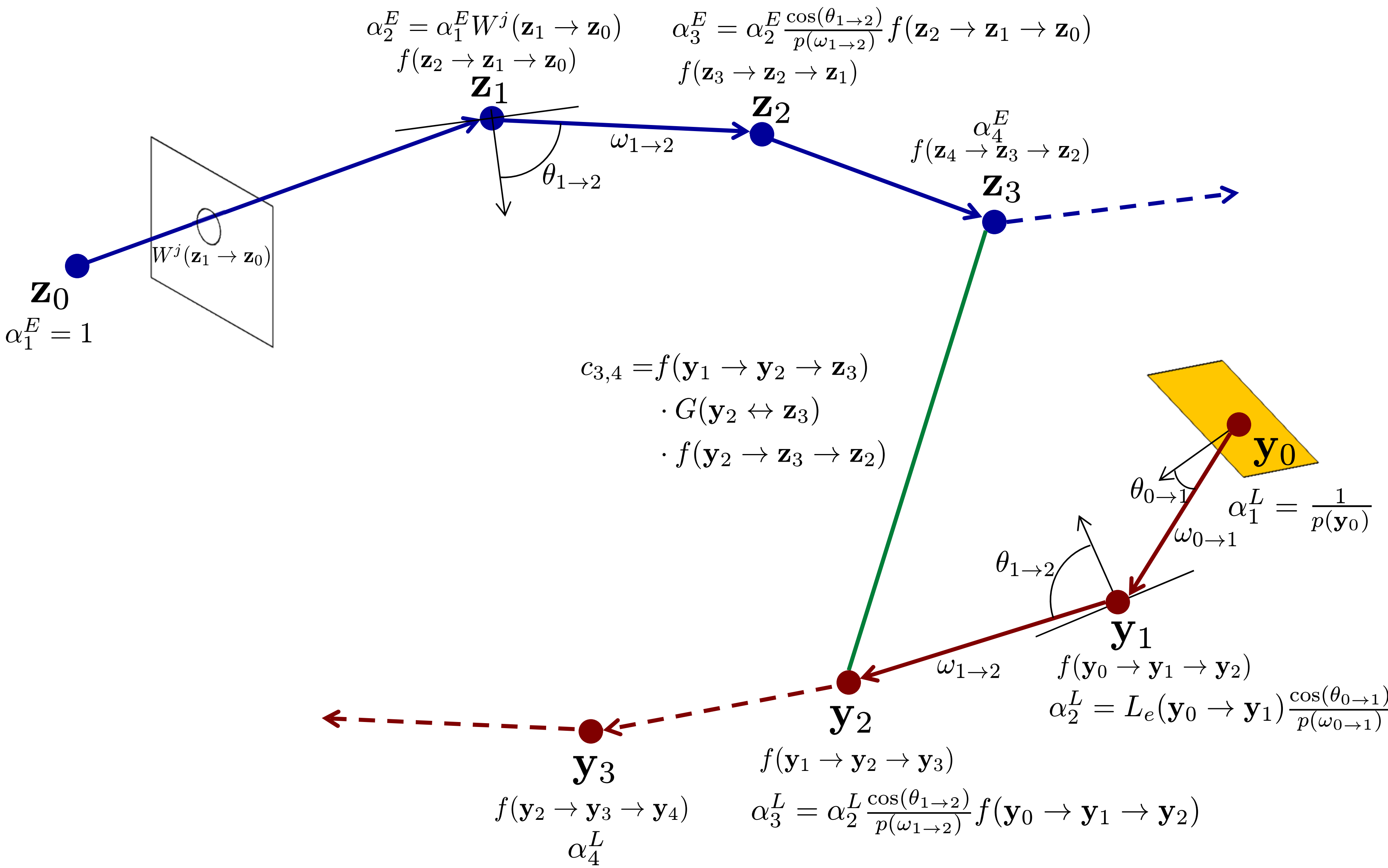
$$\underbrace{\prod_{i=1}^{t-2} \frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} f(\mathbf{z}_{i+1} \to \mathbf{z}_i \to \mathbf{z}_{i-1}) \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} \cdot W^j(\mathbf{z}_1 \to \mathbf{z}_0)}_{\alpha_t^E}$$

Eye subpath

$$= \alpha_s^L c_{s,t} \alpha_t^E$$



Path contribution



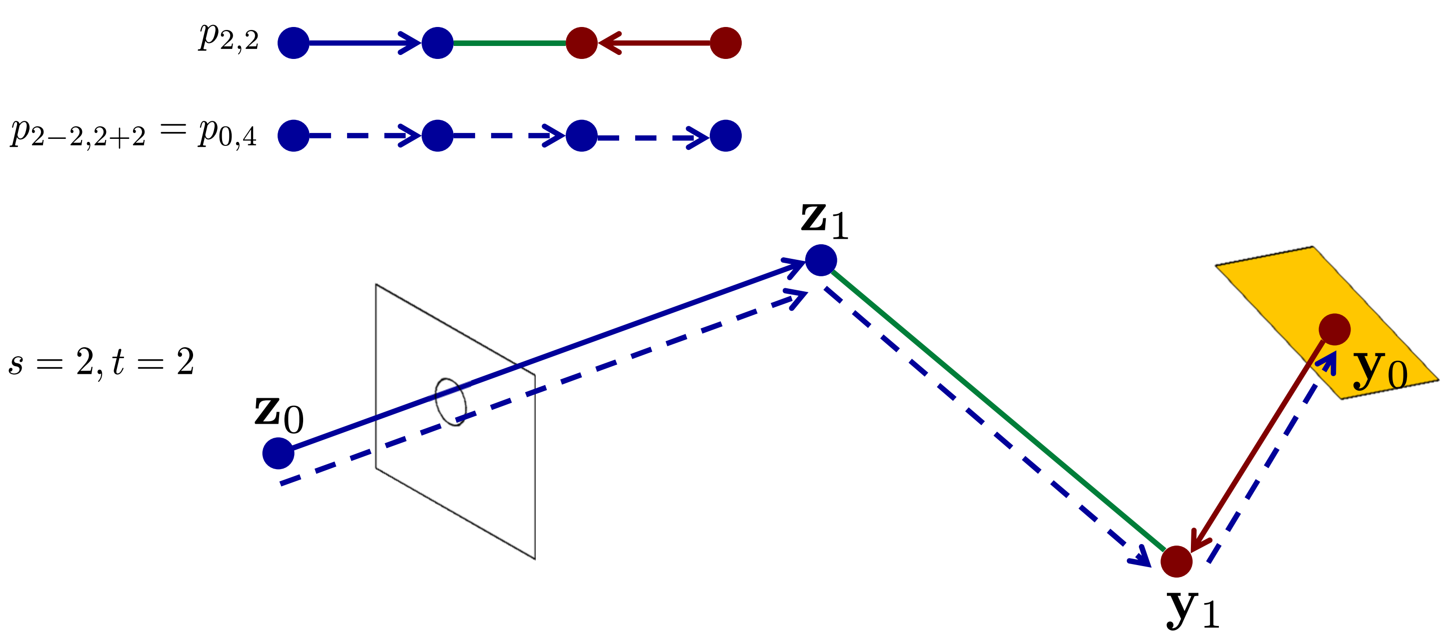
MIS weights

- Balance heuristics

$$w_{s,t} = \frac{p_{s,t}}{\sum_{0<i\leq s} p_{s-i,t+i} + p_{s,t} + \sum_{0<i\leq t} p_{s+i,t-i}}$$
$$= \frac{1}{\sum_{0<i\leq s} \frac{p_{s-i,t+i}}{p_{s,t}} + 1 + \sum_{0<i\leq t} \frac{p_{s+i,t-i}}{p_{s,t}}}$$

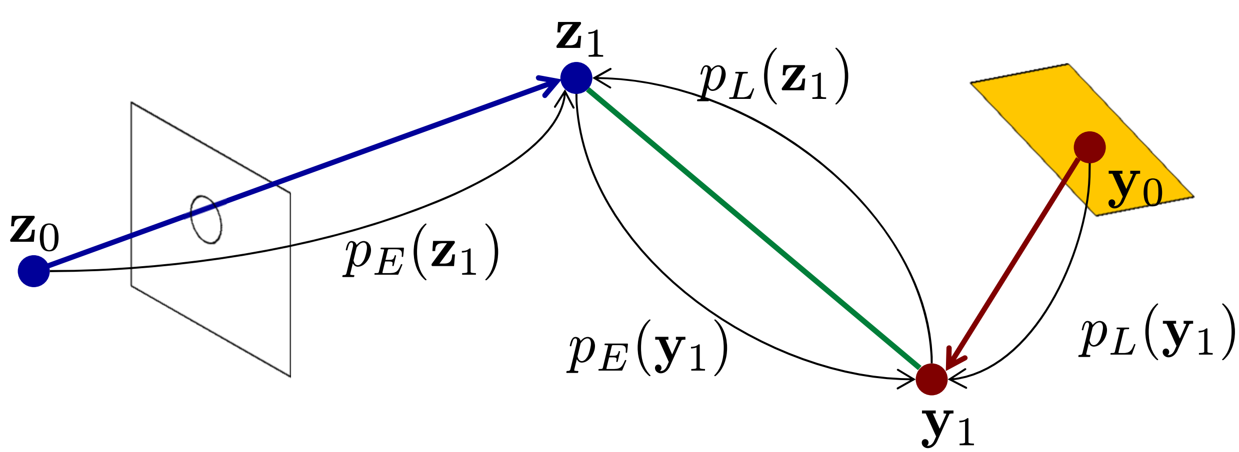
Probabilities to sample path with all other techniques

- Example with s=2, t=2



Notation

- Probability $p_L(\mathbf{y}_i)$ to sample vertex $\mathbf{y}_i$ on light path from light
- Probability $p_E(\mathbf{y}_i)$ to sample from eye
- Similar: $p_E(\mathbf{z}_i)$ and $p_L(\mathbf{z}_i)$



MIS weights

- Note

$$\frac{p_{s-i,t+i}}{p_{s,t}} = \prod_{j=1}^{i} \frac{p_E(\mathbf{y}_{s-j})}{p_L(\mathbf{y}_{s-j})}, \quad 0 < i \leq s$$

„Change i vertices from light to eye“

$$\frac{p_{s+i,t-i}}{p_{s,t}} = \prod_{j=1}^{i} \frac{p_L(\mathbf{z}_{t-j})}{p_E(\mathbf{z}_{t-j})}, \quad 0 < i \leq t$$

„Change i vertices from eye to light“

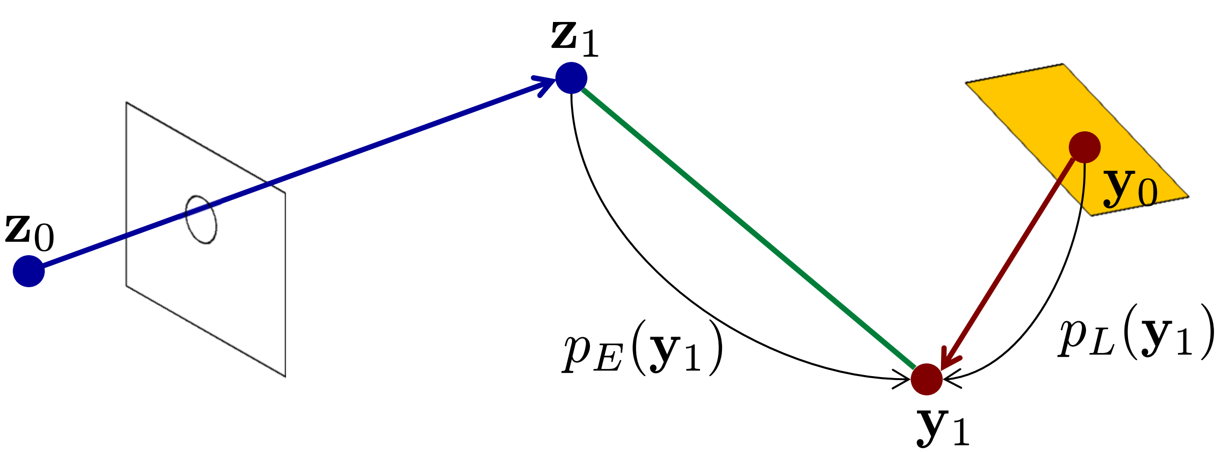
Change 1 vertex from light to eye

$$\frac{p_{s-1,t+1}}{p_{s,t}} = \frac{p_E(\mathbf{y}_{s-1})}{p_L(\mathbf{y}_{s-1})}$$

Change 2 vertices from light to eye

$$\frac{p_{s-2,t+2}}{p_{s,t}} = \frac{p_E(\mathbf{y}_{s-1})}{p_L(\mathbf{y}_{s-1})} \frac{p_E(\mathbf{y}_{s-2})}{p_L(\mathbf{y}_{s-2})}$$

$s = 2, t = 2$



MIS weights

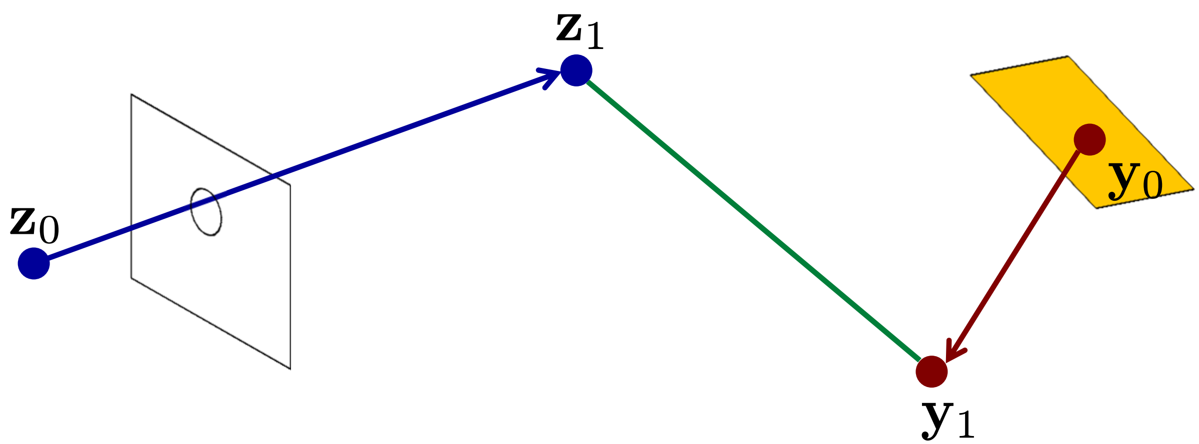
- Ned to compute

$$w_{s,t} = \frac{1}{\sum_{0<i\leq s} \prod_{j=1}^{i} \frac{p_E(\mathbf{y}_{s-j})}{p_L(\mathbf{y}_{s-j})} + 1 + \sum_{0<i\leq t} \prod_{j=1}^{i} \frac{p_L(\mathbf{z}_{t-j})}{p_E(\mathbf{z}_{t-j})}}$$

„Change 1 to s vertices from light to eye“

„Change 1 to t vertices from eye to light“

$s = 2, t = 2$



Example

- Same render time

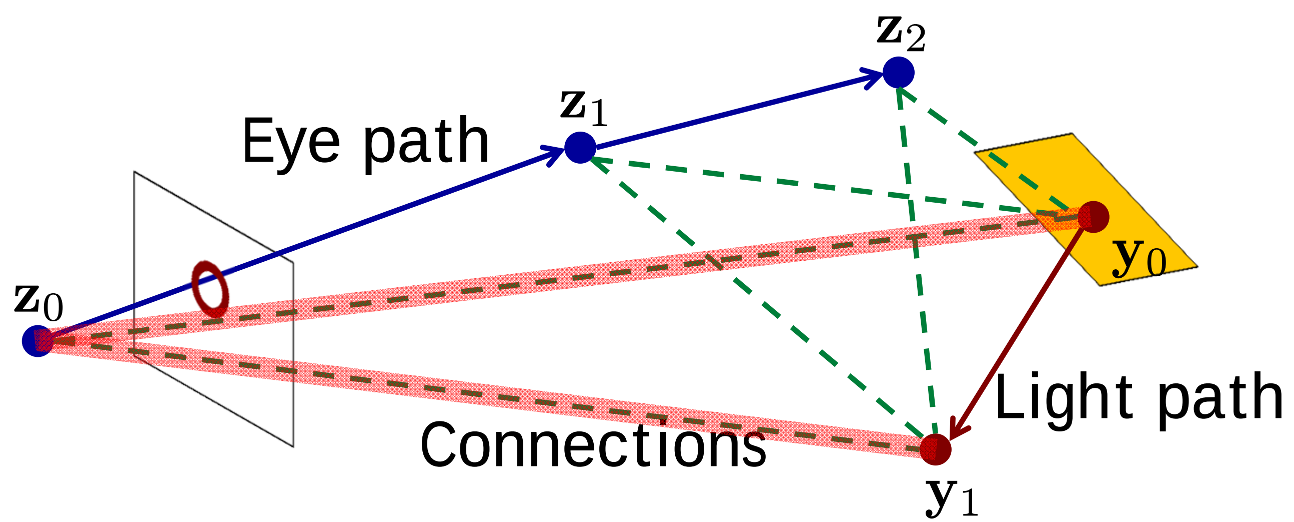
[Veach]

Bidirectional with multiple importance sampling

Standard path tracing

Implementation details

- Short paths ($s,t<=2$) need special attention
  - See definition of α^E, α^L values and connection terms $c_{s,t}$ in document
- For $t=1$, paths do not necessarily go through sampled pixel!
  - Accumulate in separate image buffer, see document



Next time

- Photon mapping