# LAB REPORT: LAB 5

TNM079, MODELING AND ANIMATION

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## Abstract

This report covers level sets and how these can be evolved in time using vector fields or scalar fields, and how different types of equations requires different discretization strategies. It is shown how morphological operations and smoothing can be achieved by fairly simple means and how narrow band optimization can be used to improve performance.

## 1 Background

A three-dimensional level set is the set of all points of a function that produces a certain value (here zero, which is typically the case for computer graphics):

$$S = \{ \mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}) = 0 \}. \tag{1}$$

The sets of points inside and outside the surface can then be defined as

$$S_{\text{inside}} = \{ \mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}) < 0 \} \tag{2}$$
$$S_{\text{outside}} = \{ \mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}) > 0 \} \tag{3}$$

and the level set $S$ is the interface between these two sets. The normal of a level set is given by

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{\|\nabla\phi\|}. \tag{4}$$

To allow the level set change over time, a time parameter, $t$, is introduced in the level set function, $\phi(\mathbf{x}, t)$, making the level set a function of time:

$$S(t) = \{ \mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}, t) = 0 \} \tag{5}$$

To move the level set, let a point $\boldsymbol{\alpha}(t)$ be restricted to the level set, i.e. $\boldsymbol{\alpha}(t) = h$ for all $t$. By differentiating with respect to the time, $t$, the following is obtained:

$$\frac{\partial\phi}{\partial t} = -\nabla\phi \cdot \frac{d\boldsymbol{\alpha}}{dt} \tag{6}$$
$$= -F \|\nabla\phi\| \tag{7}$$

where

$$F = \hat{\mathbf{n}} \cdot \frac{d\boldsymbol{\alpha}}{dt} = \frac{\nabla\phi}{\|\nabla\phi\|} \cdot \frac{d\boldsymbol{\alpha}}{dt} \tag{8}$$

is the level set speed function and describes the speed of $\boldsymbol{\alpha}$ in the normal direction.

To solve the equations numerically, they have to be discretized. The temporal discretization in this lab is simply a forward Euler scheme, but for improved accuracy, the total variation diminishing Runge–Kutta scheme can be used.

The spatial discretization limits the domain of the function $\phi$ to discrete points on a uniform grid with the spacing $\Delta x$. The notation $\phi_{i,j,k}$ is used to mean the value of $\phi$ at the grid position $(i, j, k)$. The rest of the discretization depends depends on the type of the PDE to solve.

To solve for hyperbolic advection, (6) and (7) can be can be written as

$$\frac{\partial\phi}{\partial t} = -\mathbf{V} \cdot \nabla\phi \tag{9}$$
$$= -F \|\nabla\phi\|. \tag{10}$$

These equations describe the advection of a surface along a vector field $\mathbf{V} = [V_x, V_y, V_z]$

and along the normal, respectively. Advecting a surface along its normal can for example be used to dilate or erode an object. This is achieved by setting $F$ equal to a constant that is smaller (erosion) or larger (dilation) than the isovalue zero.

To calculate the gradient, a discretization of the partial spatial derivatives are needed. Since the flow direction is known for the vector field formulation, the upwind scheme can be used, here given for $x$:

$$\frac{\partial\phi}{\partial x} \approx \begin{cases} \phi_x^+ = \left(\phi_{i+1,j,k} - \phi_{i,j,k}\right)/\Delta x, & V_x < 0 \\ \phi_x^- = \left(\phi_{i,j,k} - \phi_{i-1,j,k}\right)/\Delta x, & V_x > 0 \end{cases} \tag{11}$$

For the scalar field formulation (10), the direction of the flow is not explicitly known, but the norm of the gradient needs to be calculated. The norm calculation contains squared partial derivatives, which can be calculated using Godunov's scheme:

$$\left(\frac{\partial\phi}{\partial x}\right)^2 \approx \begin{cases} \max\big(\max(\phi_x^-,0)^2, \\ \qquad \min(\phi_x^+,0)^2\big) & \text{if } F > 0 \\ \max\big(\min(\phi_x^-,0)^2, \\ \qquad \max(\phi_x^+,0)^2\big) & \text{if } F < 0 \end{cases} \tag{12}$$

The $y$ and $z$ derivatives are calculated in the same way.

To ensure stability, the Courant–Friedrichs–Lewy condition relates the time step, $\Delta t$, and the grid spacing, $\Delta x$. For the vector field formulation (9), the condition is given by

$$\Delta t < \Delta x / \max\big(|V_x|, |V_y|, |V_z|\big) \tag{13}$$

and for the formulation given in (10), it is

$$\Delta t < \Delta x / |F| \tag{14}$$

This condition naturally has to be satisfied for all points of the vector/scalar field.

Another operation that can be performed is smoothing; this operation results in a parabolic diffusion equation. The speed function in (7) is set to $F = -a\kappa$, where $\kappa$ is the curvature, and $a$ is a scaling factor that determines how much the object is smoothed.

Since this operation has no flow direction, the previous discretization schemes can not be used—instead a scheme based on central differencing,

$$\frac{\partial\phi}{\partial x} \approx \phi_x^{\pm} = \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x} \tag{15}$$

is used. Using the notation $\phi_x = (\partial\phi)/(\partial x)$, $\phi_{xx} = (\partial^2\phi)/(\partial x^2)$, $\phi_{xy} = (\partial^2\phi)/(\partial xy)$, and so on, the mean curvature can be written as

$$\begin{aligned} \kappa &= \frac{1}{2}\nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{\phi_x^2\left(\phi_{yy} + \phi_{zz}\right) - 2\phi_y\phi_z\phi_{yz}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{3/2}} \\ &+ \frac{\phi_y^2\left(\phi_{xx} + \phi_{zz}\right) - 2\phi_x\phi_z\phi_{xz}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{3/2}} \\ &+ \frac{\phi_z^2\left(\phi_{xx} + \phi_{yy}\right) - 2\phi_x\phi_y\phi_{xy}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{3/2}} \end{aligned} \tag{16}$$

An approximation of the second derivative in the $x$ direction, derived by consecutive central differencing using $\Delta x/2$ as the step size, is given by

$$\frac{\partial^2\phi}{\partial x^2} \approx \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{\Delta x^2}. \tag{17}$$

The mixed derivative $\partial^2\phi/(\partial x\partial y)$, using central differencing, is calculated as

$$\frac{\partial^2\phi}{\partial x\partial y} \approx \frac{\begin{pmatrix} \phi_{i+1,j+1,k} - \phi_{i+1,j-1,k} \\ \quad + \phi_{i-1,j-1,k} - \phi_{i-1,j+1,k} \end{pmatrix}}{4\Delta x\Delta y}. \tag{18}$$

Derivatives in the other directions are calculated analogously. The time step restrictions for this scheme can be shown to be

$$\Delta t < \frac{\Delta x^2}{6a}. \tag{19}$$

It is beneficial to define the level set function such that it is a signed distance function. This means that it satisfies the Eikonal equation

$$\|\nabla\phi\| = 1. \tag{20}$$

When modifying the level set, the size of the gradient can drift, requiring a so called reinitialization to restore the signed distance property. This can be done by applying

$$\frac{\partial \phi}{\partial t} = S(\phi)(1 - \|\nabla \phi\|) \tag{21}$$

where $S(\phi)$ is a smooth approximation of the sign function and is given by

$$S(\phi) = \frac{\phi}{\sqrt{\phi^2 + \|\nabla \phi\|^2 \Delta x^2}}. \tag{22}$$

Equation (21) is a hyperbolic equation and needs to be discretized as such.

Since solving $\phi$ for the entirety of its domain is expensive and typically unnecessary since only the surface and its immediate surroundings are of interest. These observations can be used to form a narrow band scheme. The basic idea is to define two narrow band tubes around the surface:

$$T_\beta = \left\{ \mathbf{x} \in \mathbb{R}^3 : |\phi(\mathbf{x})| < \beta \right\} \tag{23}$$

$$T_\gamma = \left\{ \mathbf{x} \in \mathbb{R}^3 : |\phi(\mathbf{x})| < \gamma \right\} \tag{24}$$

where $0 < \beta < \gamma$, meaning that the $T_\beta$ is tighter that $T_\gamma$. Then the level set equations can be solved only inside these tubes. Since using a hard cutoff could introduce oscillations, a cutoff function is used to smoothen the change between the tubes:

$$c(\phi) = \begin{cases} 1 & \text{if } |\phi| \leq \beta \\ \frac{(2|\phi|+\gamma-3\beta)(|\phi|-\gamma)^2}{(\gamma-\beta)^3} & \text{if } \beta < |\phi| \leq \gamma \\ 0 & \text{if } |\phi| > \gamma \end{cases} \tag{25}$$

The shape of this cutoff function can be seen in Figure 1.

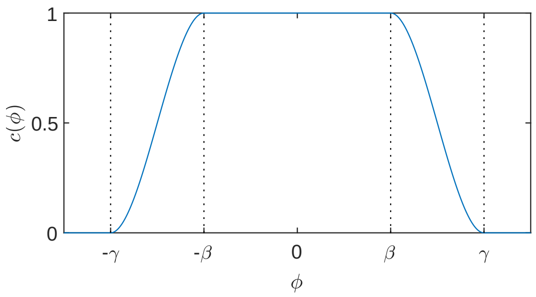



*Figure 1:* The narrow band cutoff function.

# 2 Results

The results of the lab are presented here.

## 2.1 The Signed Distance Property

Figure 2 demonstrates the reinitialization operator. It can be seen how a function that deviates significantly from the signed distance property can be "fixed" by running the reinitialization operator, as can be seen by comparing the visual grid with the wrap-around color scale. Many iterations were needed in this case—but typically $\|\nabla \phi\|$ should not deviate too far from one before running the reinitialization, as the signed distance property should be maintained continuously.

## 2.2 Surface Advection

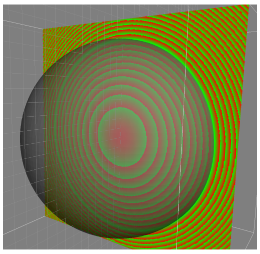
Advection of a surface along a vector field is demonstrated in Figure 3. In this lab a vortex vector field was used, but the vector field could also be generated by for example some simulation. This makes it a powerful method to advect the surface.

## 2.3 Erosion and Dilation

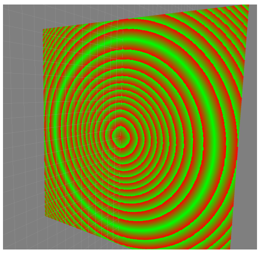
Advecting a level set using a constant scalar field to achieve morphological operations is shown in Figure 4. The erosion and dilation operators can also be used in series to achieve morphological opening or closing which, for example, can be used to remove unwanted spikes.

## 2.4 Mean Curvature Flow

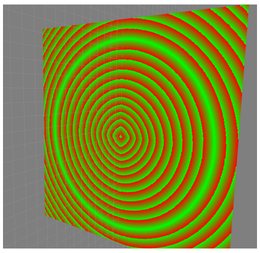
Figure 5 demonstrates how the curvature of a level set can be used to smoothen a surface. By using the curvature to scale the speed along the normal, sharp edges can be smoothened considerably, without affecting flat surfaces. As seen in the figure, this is quite effective.



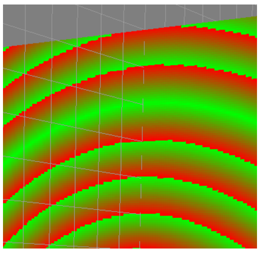
*(a)* The starting point.



*(b)* After running the reinitialization for 40 iterations.



*(c)* After running the reinitialization for 70 iterations.



*(d)* Zoomed in version of Figure 2c.

*Figure 2:* An example of running the reinitialization operation on a level set function that is not a signed distance function. The values of $\phi$ in a plane are visualized using a repeating color scale with period of 0.1. The visual grid has a grid spacing of 0.1.

As described in the background, this technique results in a parabolic diffusion equation and is solved using central differencing as opposed to the upwind scheme.

### 2.5 Narrow Band Optimization

As seen in Table 1, the time needed to perform the erosion without the narrow band optimization, grows very quickly. With the narrow band optimization, the times are more manageable, while still growing significantly with the grid size. Of course the number of points on the grid has cubic complexity with respect to the spatial resolution $1/\Delta x$, so it makes sense that the time increases quickly. When using the narrow band optimization, not all of the points of the domain are actually active, greatly reducing the computations needed.

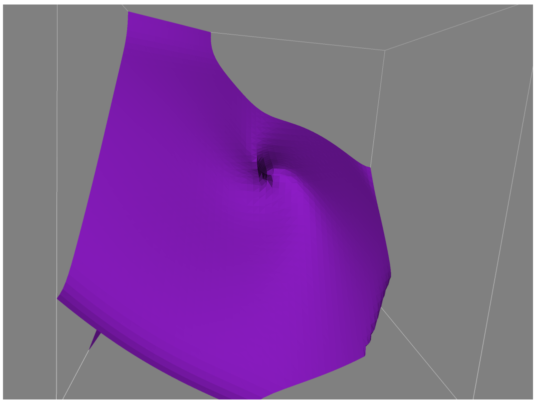


*Figure 3:* A simple plane level set advected with a vortex vector field.

*Table 1:* Comparision between times with and without narrow band optimization for one erosion iteration on a sphere using different grid sizes.

| $\Delta x$ | Time without narrow band | Time with narrow band |
|---|---|---|
| 0.05 | 0.006322 | 0.003441 |
| 0.03 | 0.025742 | 0.008934 |
| 0.01 | 0.507706 | 0.118989 |
| 0.007 | 1.37826 | 0.256237 |
| 0.005 | 5.58634 | 0.925619 |
| 0.003 | 35.6769 | 5.17844 |

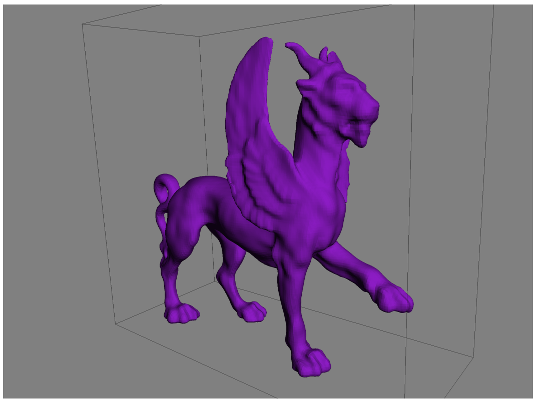
## 3 Conclusion

The level set representation of surfaces provides powerful ways to mutate a surface using scalar fields as well as vector fields. These fields can be used to achieve many things, such as morphological operations, and smoothing, and much more.

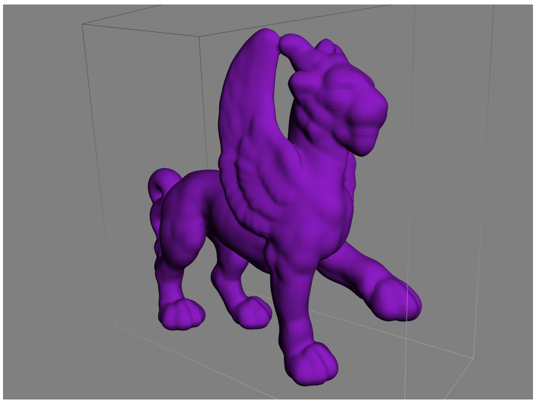
The performance of level set operations can be greatly improved by narrow band optimization which only resolves the level set function close to the surface.

## Lab Partner and Grade

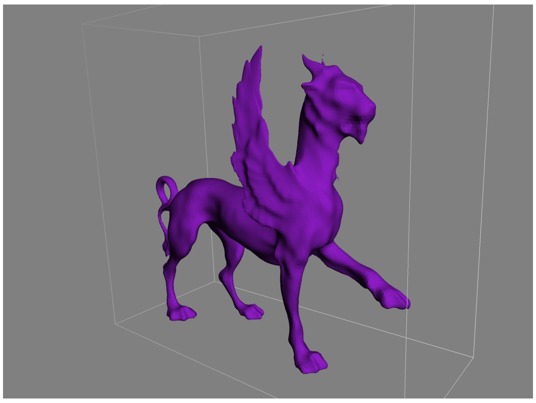
The lab was done together with Viktor Sjögren. All lab tasks marked 3, 4, or 5b were finished and the report aims for grade 5.



*(a)* Original model.

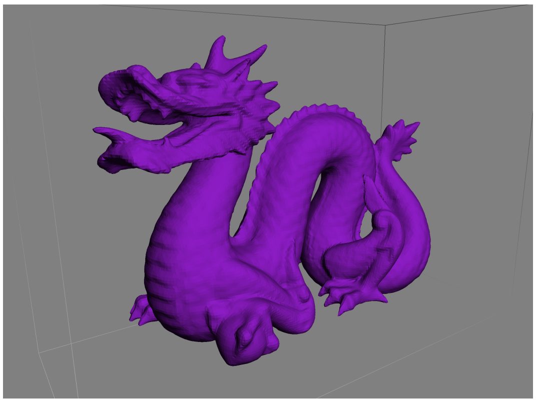


*(b)* After dilation.

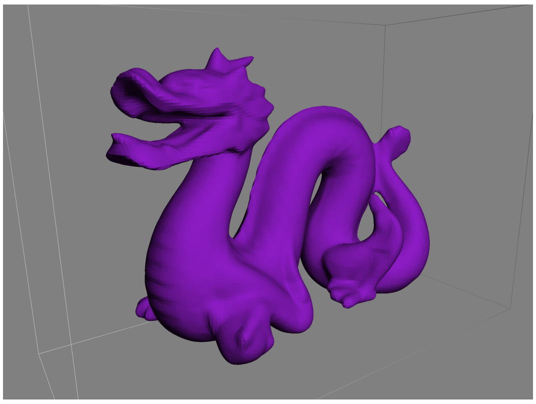


*(c)* After erosion.

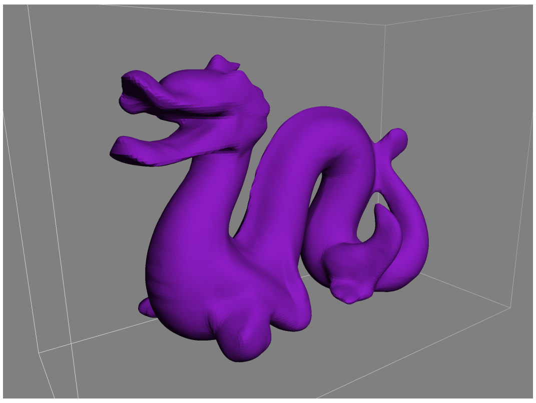
*Figure 4:* A demonstration of the dilation and erosion operators.



*(a)* Original model.



*(b)* 5 iterations of smoothing



*(c)* 10 iterations of smoothing

*Figure 5:* A demonstration of the smoothing operator.