ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS
DEPARTMENT OF STATISTICS

Windings of planar Brownian Motion and Applications in Finance

by

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A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfilment of the requirements
for the degree of Master of Science in Statistics

Supervising Professor: Stavros Vakeroudis

August 2023

Declaration of Authorship

I, Panagiotis Symianakis, declare that this thesis titled, ”Windings of planar Brownian Motion and Applications in Finance” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Signed:

Date: 29/08/2023

The road up and the road down is one and the same.

(ὁδὸς ἄνω κάτω μία καὶ ὡυτή)

Heraclitus
Fragment 60’

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Abstract

School of Information Sciences and Technology
Department of Statistics

M.Sc. in Statistics

by Panagiotis G. Symianakis

This thesis delves into the study of "Windings of Planar Brownian Motion and its Applications in Finance," exploring the fascinating world of stochastic analysis in economics. Beginning with an insightful historical review of Brownian motion, we establish essential notations and penetrate the main properties of one-dimensional Brownian motion, including the Strong Markov Property, the Reflection Principle and the Martingale property. Furthermore, we investigate Itô's formula and Bougerol's identity building a strong theoretical foundation for our subsequent analyses. The study then looks into the basic properties of Planar Brownian Motion, introducing the skew product representation, exit times, and Laplace transformations of hitting times, while examining the significance of Williams' "pinching method" and Spitzer's theorem. In the realm of financial mathematics, we explore the implications of Windings of Planar Brownian Motion, with a focus on Asian options and exponential functionals of Brownian motion, opening up new possibilities for financial modeling and risk assessment. Finally, we validate our theoretical findings through simulations,where we calculate the price of Asian call options for commodities and stocks, reinforcing the practical applicability of our research in real-world scenarios. This comprehensive investigation contributes significantly to the understanding of stochastic analysis and its tangible benefits in financial decision-making.

ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

Περίληψη

Σχολή Επιστημών Και Τεχνολογίας της Πληροφορίας
Τμήμα Στατιστικής

Μεταπτυχιακό στη Στατιστική

Παναγιώτης Γ. Συμιανάκης

Abstract in Greek

Αυτή η διατριβή εμβαθύνει στη μελέτη των ¨Περιελίξεων της δισδιάστατης Κίνησης Brown και των εφαρμογών τους στα Οικονομικά¨, διερευνώντας τον συναρπαστικό κόσμο της στοχαστικής ανάλυσης στις οικονομικές επιστήμες. Ξεκινώντας με μία ιστορική ανασκόπηση εισάγουμε βασικούς συμβολισμούς και διεισδύουμε στις κύριες ιδιότητες της μονοδιάστατης Κίνησης Brown, , συμπεριλαμβανομένης της Ισχυρής Μαρκοβιανής Ιδιότητας της Αρχής της Ανάκλασης και της ιδιότητας Martingale. Επιπλέον, διερευνούμε τη φόρμουλα του Itô και την ταυτότητα του Bougerol δημιουργώντας μια ισχυρή θεωρητική βάση για τις επόμενες αναλύσεις μας. Στη συνέχεια, εξετάζονται βασικές ιδιότητες της Κίνησης Brown σε δύο διαστάσεις, εισάγοντας την αναπαράσταση ¨στρεβλού/λοξού γινομένου (skew product)", τους χρόνους εξόδου και τους μετασχηματισμούς Laplace των χρόνων αυτών, ενώ αναφέρονται και μελετώνται η μέθοδοςυ "pinching" του Williams και το θεώρημα του Spitzer. Στη σφαίρα των οικονομικών μαθηματικών, διερευνούμε τις επιπτώσεις των περιελίξεων της επίπεδης Κίνησης Brown, με έμφαση στα ασιατικά δικαιώματα προαίρεσης και τις εκθετικά συναρτησιακά της Κίνησης Brown, ανοίγοντας νέες δυνατότητες για οικονομική μοντελοποίηση και εκτίμηση κινδύνου. Τέλος, επιβεβαιώνουμε τα θεωρητικά μας ευρήματα μέσω προσομοιώσεων, υπολογίζοντας την τιμή ασιατικών δικαιωμάτων αγοράς για εμπορεύματα και μετοχές, ενισχύοντας την πρακτική εφαρμογή της έρευνάς μας σε πραγματικά σενάρια. Αυτή η ολοκληρωμένη έρευνα συμβάλλει σημαντικά στην κατανόηση της στοχαστικής ανάλυσης και των απτών οφελών της στη λήψη οικονομικών αποφάσεων.

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Acknowledgements

First of all, I would like to express my heartfelt gratitude to my supervisor, Stavros Vakeroudis, for his unwavering guidance, patience, and invaluable expertise. Their insightful feedback and constructive criticism have played a pivotal role in shaping the direction of my research and improving the quality of this work.

Moreover I would like to express my gratitude to the members of the evaluation committee, professor Athanasios Yannacopoulos and professor Michael Zazanis for considering my thesis. Their time, effort, and expertise in evaluating this project are truly commendable. Thank you for your dedication in identifying and recognizing potential corrections.

The faculty members and academic staff at Athens University of Economics and Business who have provided me with an enriching educational environment throughout my academic journey. Their dedication to teaching and research has inspired me to strive for knowledge and excellance.

My father, Giorgos, my mother, Sofia, my sister, Matina , friends, and loved ones deserve my utmost gratitude for their unwavering support, encouragement, and understanding throughout this challenging journey. Their love, patience, and belief in my abilities have been a constant source of motivation, enabling me to overcome obstacles and persevere until the completion of this thesis.

In conclusion, the completion of this thesis would not have been possible without the support and contributions of the aforementioned individuals. I am deeply grateful for their guidance, assistance, and encouragement. However, any errors or shortcomings in this work are entirely my own.

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Abbreviations

BM	**B**rownian **M**otion
MG	**M**artin**G**ale
SMP	**S**trong **M**arkov **P**roperty
WPM	**W**illiams' "**P**inching" **M**ethod
sup	**sup**remum
inf	**inf**imum
NG	**N**atural **G**as
USD	**U**nited **S**tates **D**ollar

To my beloved family:
Giorgos, Sofia & Matina.

A planar Brownian path

Chapter 1

Introduction

1.1 Historical Review

Robert Brown, a British botanist, became interested in the physical Brownian motion around 1828. By the use of microscope, he observed minute particles, known as amyloplasts and spherosomes ejected from the pollen grains in to water, executing a continuous anxious movement while examining grains of pollen of the plant Clarkia pulchella. However, the mathematical study of this observation began in 1900 when the French mathematician Louis Bachelier guessed a few significant properties, including a weak form of the Markov property, and the Gaussian distribution of Brownian motion at a fixed time. In 1905, Albert Einstein came up with more rigorous derivation of the Gaussian character of the one-dimensional marginals. He was the man who fully realized the link of BM with the kinetic theory and atomism. In his biography [8], Einstein clearly exposed his motivation for studying the Brownian Motion:
"My major aim in this was to find facts which would guarantee the existence of atoms"
The first complete construction of Brownian motion as a continuous stochastic process is due to Norbert Wiener in 1923. Later, in collaboration with Paley and Zygmund, Wiener demonstrated the non-differentiability of the Brownian ways, which had been hypothesised by the French physicist Perrin.
The greatest contribution to the study of Brown motion has been given by Paul Lévy. Lévy brought to light numerous notable sample path properties, and a great number of significant distributions connected with Brownian motion. Moreover, he introduced the local times of linear Brownian motion. Since Lévy's work, the linear BM has been examined broadly, especially with the use of Itô's stochastic calculus, which produce a really plain construction of local times. Multidimensional Brownian was not ignored but a few questions raised by Lévy were not cleared out until very recently. During the last decades, many properties of planar BM, such as geometric properties of sample path, asymptotic distributions or multiple points

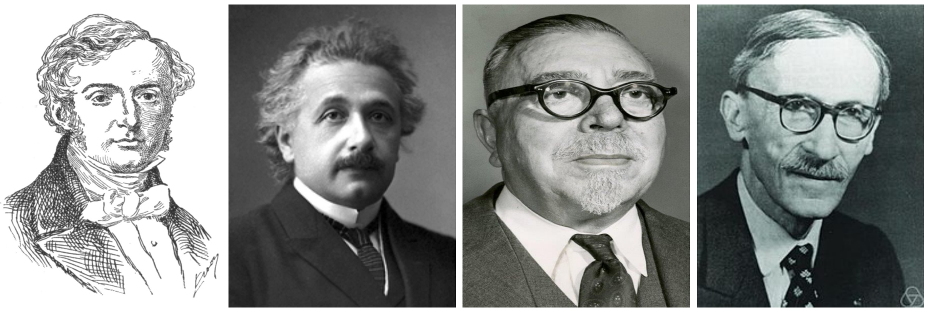

FIGURE 1.1: R. Brown, A. Einstein, N. Wiener, P. Lévy

and intersection problems, have attracted the interest of several scientists.
In this project, we are going to study the windings of 2-dimensional case, and especially of planar Brownian motion. In general terms, planar BM has a great number of properties which disappear when we move to higher dimensions. The reason why it happens is the connection between BM and holomorphic functions and the fact that planar Brownian path, on the time interval $[0, m)$, is dense in the plane although it does not hit a given point.

1.2 Main Notation

The theory of probability is based upon the notion of a probability space or probability triple. Firstly, we have a non-empty set Ω, and in many actual cases it is possible to regard each element $\omega \in \Omega$ as a parameter indexing realizations of the random phenomenon in question. Next we take a family $\mathbf{B}$ of subsets of Ω satisfying the following three conditions[14]:

1. $\Omega \in \mathbf{B}$
2. If $B_n \in \mathbf{B}, n = 1, 2, \ldots,$ then $\bigcup_n B_n \in \mathbf{B}$
3. If $B \in \mathbf{B}$, then $B^c \in \mathbf{B}$, where $B^c = \Omega \setminus B$

In other words $\mathbf{B}$ forms a σ-field or σ-algebra of subsets of Ω. Finally we have a countably additive set function P defined on $\mathbf{B}$ satisfying the following conditions

1. $0 \leq P(B) \leq 1$ for every $B \in \mathbf{B}$
2. If $B_n \in \mathbf{B}, n = 1, 2, \ldots$, are such that $B_i \cap B_j =$ when $i \neq j$, then

$$P\left(\bigcup_n B_n\right) = \sum_n P(B_n)$$

3. $P(\Omega) = 1$

A triple $(\Omega, \mathbf{B}, P)$ is called a *probability space* if each component satisfies the conditions stated above. Elements $B \in \mathbf{B}$ are called *events*, and $P(B)$ is called the *probability* of the event B.
Now let's delve into the mathematical expression of Brownian motion starting from the definition.

Definition 1.1. A stochastic process $\{B(t, \omega), t \geq 0\}$ defined on a probability space $(\Omega, \mathbf{B}, P)$ is called a (linear) *Brownian motion* or a *Wiener process* if it satisfies:

a. $B(0, \omega) = x$ for almost all ω

b. The process has independent increment, i.e. for all time $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments $B(t_n, \omega) - B(t_{n-1}, \omega), \ldots, B(t_2, \omega) - B(t_1, \omega)$ are independent random variables.

c. For any $t \geq 0$ and $h > 0$ the increments $B(t + h, \omega) - B(t, \omega)$ are normally distributed with expectation zero and variance h.
Or simpler,
$$B(t + h, \omega) - B(t, \omega) \sim N(0, h)$$

d. Almost surely, the function $(t, \omega) \rightarrow B(t, \omega)$ is continuous.

In addition, we say that $\{B(t, \omega) : t \geq 0\}$ is a standard Brownian Motion if $x = 0$.
The parameter t denotes time and usually takes values in the interval $[0, T]$ or $[0, \infty]$. Here it is important to look at some technical points. We have defined Brownian motion as a *stochastic process* $\{B(t) : t \geq 0\}$ which is just a family of (uncountably many) random variables $\omega \rightarrow B(t, \omega)$ defined on a single probability space $(\Omega, \mathbf{B}, P)$. Simultaneously, a stochastic process can also be explained as a random function with the sample functions defined by $t \rightarrow B(t, \omega)$. The sample path properties of a stochastic process are the properties of these random functions, and it is these properties we will be most interested in this thesis. For brevity, a BM is also denoted as $\{B(t) : t \geq 0\}$ instead of $\{B(t, \omega) : t \geq 0\}$.

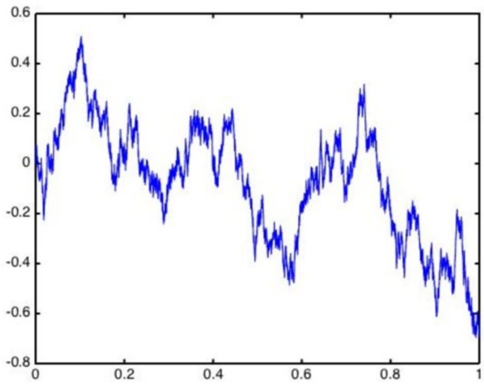

FIGURE 1.2: The graph of a Brownian Motion

If asked to clarify the distribution of $\{B(t), t \geq 0\}$, it is enough to know only the covariance function. It happens because the process is Gaussian and has identically zero mean. For the covariances we have the following:

$$\mathrm{Cov}(B(t), B(s)) = \min\{t, s\}$$

It is known that for a BM we have $B(t) \sim N(0, t)$ as $B(t) - B(0) \sim N(0, t)$.
Indeed, for $0 \leq s < t$, then:

$$\mathrm{Cov}(B(t), B(s)) = \mathbb{E}(B(t)B(s)) - \mathbb{E}(B(t))\mathbb{E}(B(s))$$

$$= \mathbb{E}[B(s)(B(t) - B(s) + B(s))] - 0$$

$$= \mathbb{E}[B(s)(B(t) - B(s)) + B^2(s)]$$

$$= \mathbb{E}[B(s)(B(t) - B(s))] + \mathbb{E}(B^2(s))$$

$$= \mathbb{E}(B(s))\mathbb{E}(B(t-s)) + \mathrm{Var}(B(s)) + \mathbb{E}(B(s))^2$$

$$= 0 + s + 0 = s$$

As $B(t)$,$B(t-s)$ are independent and $\mathbb{E}(X^2) = \mathrm{Var}(X) + \mathbb{E}(X)^2$.
Similarly, for $0 \leq t < s$, we have

$$\mathrm{Cov}(\mathrm{B(t)}, \mathrm{B(s)}) = t$$

Therefore, the result implies.

1.3 Main Properties of One-dimensional Brownian Motion

Theorem 1.2 (Wiener,1923)**.** *Standard Brownian motion exists.*

A proof of Wiener's theorem is in the Appendix A.1.
Noteworthy is the fact that several natural sets resulting from paths of BM are in some sense random fractals. An instinctive approach to fractals is that they are sets which have a nontrivial geometric structure at all scales.
A key role in this behaviour is played by the very simple *scaling invariance* property of Brownian motion, which we now formulate. It recognizes a transformation on the space of functions, which changes the individual Brownian random functions but leaves their distribution unchanged.

Lemma 1.3 (Scaling invariance). *Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion and let $c > 0$. Then the process $\{X(t) : t \geq 0\}$ defined by $X(t) = \frac{1}{c}B(c^2t)$ is also a standard Brownian motion.*

Proof. Clearly, $X(0) = B(0) = 0$.
If $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, then $0 \leq c^2t_1 \leq c^2t_2 \leq ... \leq c^2t_n$. By definition of Brownian motion,

$$B(c^2t_n) - B(c^2t_{n-1}), B(c^2t_{n-1}) - B(c^2t_{n-2}), ..., B(c^2t_2) - B(c^2t_1)$$

are independent random variables. It then follows that

$$\frac{1}{c}B(c^2t_n) - \frac{1}{c}B(c^2t_{n-1}), \frac{1}{c}B(c^2t_{n-1}) - \frac{1}{c}B(c^2t_{n-2}), ..., \frac{1}{c}B(c^2t_2) - \frac{1}{c}B(c^2t_1)$$

are independent random variables, i.e. the stochastic process $\{X(t) : 0 \leq t\}$ has independent increments.
The fact that $X(t+h) - X(t)$ is normally distributed follows immediately from the fact that $B(c^2t + c^2h) - B(c^2t)$ is normally distributed. Furthermore,

$$\mathbb{E}[X(t+h) - X(t)] = \mathbb{E}[\frac{1}{c}B(c^2t + c^2h) - \frac{1}{c}B(c^2t)] = \frac{1}{c}\mathbb{E}[B(c^2t + c^2h) - B(c^2t)] = 0$$

where the last equality follows from the definition of Brownian Motion. To show that the variance equals h, observe that

$$\mathrm{Var}(X(t+h) - X(t)) = \mathrm{Var}(\frac{1}{c}B(c^2t + c^2h) - \frac{1}{c}B(c^2t)) =$$

$$\frac{1}{c^2}\mathrm{Var}(B(c^2t + c^2h) - B(c^2t))$$

$$= \frac{1}{c^2}c^2h = h$$

Because the function $t \to B(t)$ is almost surely continuous, the function $t \to X(t) = \frac{1}{c}B(c^2t)$ is the composition of (almost surely) continuous functions and is therefore almost surely continuous. $\square$

The above theorem (with $c = -1$) implies that the standard BM is symmetric about 0. Simpler, if $\{B(t) : t \geq 0\}$ is a standard BM and $t \geq 0$, then $B(t)$ has the same distribution as $-B(t)$. Moreover, BM is translation invariant, i.e. if $\{B(t) : t \geq 0\}$ is a BM and $x \in \mathbb{R}$, then $\{B(t)+x : t \geq 0\}$ is also a Brownian Motion. The following theorem shows that Brownian motion is also time-shift invariant.

Theorem 1.4 (Time inversion)**.** *Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Then the process $\{X(t) : t \geq 0\}$ defined by:*

$$X(t) = \left\{ \begin{array}{ll} 0 & ,t = 0 \\ tB(1/t) & ,t > 0 \end{array} \right\}$$

is also a standard Brownian motion.

Proof. Recall that the finite dimensional marginals $(B(t_1), ..., B(t_n))$ of Brownian motion are Gaussian random vectors and are therefore characterised by $\mathbb{E}[B(t_i)] = 0$ and $\mathrm{Cov}(B(t_i), B(t_j)) = t_i$ for $0 \leq t_i \leq t_j$.
Obviously, $\{X(t) : t \geq 0\}$ is also a Gaussian process and the Gaussian random vectors $(X(t_1), ..., X(t_n))$ have expectation zero. The covariances, for $t, h > 0$ are given by

$$\mathrm{Cov}(X(t+h), X(t)) = (t+h)t\ \mathrm{Cov}(B(1/(t+h)), B(1/t)) = t(t+h)\frac{1}{t+h} = t.$$

Hence the law of all the finite dimensional marginals

$$(X(t_1), X(t_2), ..., X(t_n)), \text{for } 0 \leq t_1 \leq ... \leq t_n,$$

are the same as for Brownian motion. The paths of $t \rightarrow X(t)$ are clearly continuous for all $t \geq 0$ and in $t = 0$ the following two facts are used:
First, the distribution of X on the rationals $\mathbb{Q}$ is the same as for Brownian motion, hence

$$\lim_{\substack{t \to 0 \\ t \in \mathbb{Q}}} X(t) = 0 \ \text{ almost surely.}$$

Second, X is alost surely continuous on $(0, \infty)$, so that

$$\lim_{\substack{t \to 0 \\ t \in \mathbb{Q}}} X(t) = \lim_{t \to 0} X(t) \ \text{ almost surely.}$$

Hence $\{X(t) : t \geq 0\}$ has almost surely continuous paths, and is a Brownian motion. □

Theorem 1.5 (Markov Property)**.** *Suppose that $\{B(t) : t \geq 0\}$ is a s a Brownian motion started at $x \in \mathbb{R}^d$. Let $s > 0$, then the process $\{B(t+s) - B(s) : t \geq 0\}$ is a standard Brownian motion independent of the process $\{B(t) : 0 \leq t \leq s\}$*

Proof. Properties (a) and (b) follow from the cancellation of the $B(s)$ terms and the fact that $\{B(t) : t \geq 0\}$ is a Brownian motion. Because the map $t \rightarrow B(t+s) - B(s)$ is the composition of (almost surely) continuous functions, the map $t \rightarrow B(t+s) - B(s)$ is continuous. Finally, $\{B(t+s) - B(s) : t \geq 0\}$ is standard BM as $B(0+s) - B(s) = 0$.

Recall that two stochastic processes $\{X(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$ are said to be independent if for any sets of times $t_1, t_2, ..., t_n \geq 0$ and $s_1, s_2, ..., s_m \geq 0$ the vectors $(X(t_1), X(t_2), ..., X(t_n))$ and $(Y(s_1), Y(s_2), ..., Y(s_m))$ are independent. Let $s \geq s_1, s_2, ..., s_m \geq 0$. Because Brownian motion has independent increments, it follows that $(B(t_1 + s) - B(s), B(t_2 + s) - B(s), ..., B(t_n+s)-B(s)$ and $(B(s_1), B(s_2), ..., B(s_m))$ are independent random vectors. □

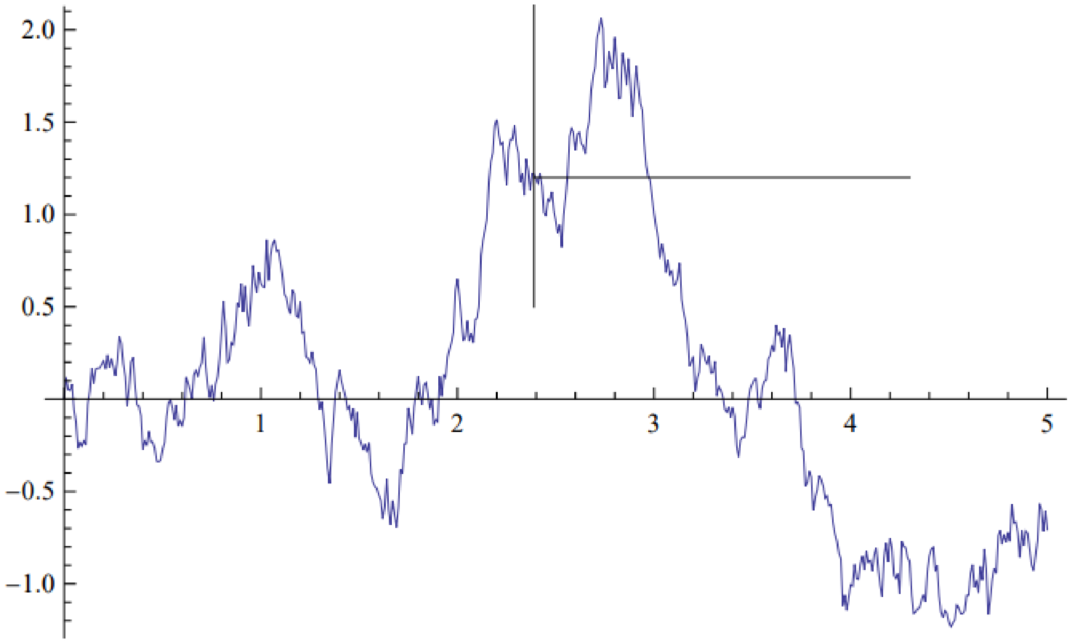

FIGURE 1.3: Markov property of Brownian motion.
Markov property (1.5) says that by placing new axes at $(s, B(s))$, the rightward progression of the new axis system is standard Brownian motion and independent of the past.

1.4 Strong Markov Property and the Reflection Principle

Definition 1.6 ([7]). A collection of subsets of S is called a σ-algebra (or Borel field), denoted by $\mathcal{B}$, if it satisfies the following three properties:

1. $\emptyset \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$)
2. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).
3. If $A_1, A_2, ... \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions)

The empty set $\emptyset$ is a subset of any set. Thus, $\emptyset \subset S$. Property (1.) states that this subset is always in a sigma algebra. Since $S = \emptyset^c$, properties (1.) and (2.) imply that S is always in $\mathcal{B}$ also.

Definition 1.7. A **filtration** o a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\mathcal{F}(t)|t \geq 0)$ of σ-algebras such that $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}, \quad \forall\, s < t$. A probability space together with a filtration is called a filtered probability space. A stochastic process $\{X(t) : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called adapted if, for any $t \geq 0,\ X(t)$ is measurable with respect to $\mathcal{F}(t)$.

Suppose we have a Brownian motion $\{B(t) : t \geq 0\}$ defined on some probability space. We define the filtration $(\mathcal{F}^0(t) : t \geq 0)$ by letting $\mathcal{F}^0(t)$ be the $\sigma-$algebra generated by the random variables $\{B(s) : 0 \leq s \leq t\}$. Clearly the Brownian motion $\{B(t) : t \geq 0\}$ is adopted to this filtration $\mathcal{F}^0(t)$. Instinctively, the $\sigma-$algebra $\mathcal{F}^0(t)$) contains all the information available from observing the Brownian motion up to time t.
Moreover, the Markov property pf Brownian motion implies that the process $\{B(t+s) - B(s) : t \geq 0$ is independent of $\mathcal{F}(s)$. In the next theorem, we improve this statement of independence by showing that the process is independent of the slightly larger $\sigma-$algebra $\mathcal{F}^+(t) := \cup_{s>t}\mathcal{F}^0(s)$. Clearly, the family $(\mathcal{F}^+(t) : t \geq 0)$ is also a filtration and $\mathcal{F}^0(t) \subset \mathcal{F}^+(t)$ for any time $t \geq 0$. By instinct, the $\sigma-$algebra $\mathcal{F}^+(t)$ contains all the information available from observing the Brownian motion up to time t plus the information available from an additional infinitesimal glance beyond time t.

Theorem 1.8. *For every $s \geq 0$, the process $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. By continuity, $B(t+s) - B(s) = \lim_{n\to\infty} B(s_n + t) - B(s_n)$ for a strictly decreasing sequence $s_n \searrow s$. For any $t_1, ..., t_m \geq 0$, the vector $(B(t_1+s) - B(s), B(t_2+s) - B(s), ..., B(t_m+s) - B(s)) = \lim_{j\to\infty}(B(t_1+s_j) - B(s_j), ..., B(t_m - s_j) - B(s_j))$. By the Markov property 1.5 of Brownian motion, $(B(t_1+s_j) - B(s_j), ..., B(t_m+s_j) - B(s_j))$ is independent of $\mathcal{F}^+(s)\ \ \forall\, j$. Hence,$(B(t_1+s) - B(s), ..., B(t_m - s) - B(s))$ is independent of $\mathcal{F}^+(s)$, which then implies that the process $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$. □

With a filtration, we can also define stopping times, a certain type of random time that will be essential to describe the strong Markov property of Brownian motion.

Definition 1.9. A random variable T with values in $[0, \infty]$ defined on a probability space with filtration $(\mathcal{F}(t) : t \geq 0)$ is called **stopping time** if $\{T < t\} \in \mathcal{F}(t)$ for every $t \geq 0$.
It is called a **strict stopping time** if $\{T \leq t\} \in \mathcal{F}(t)$ for every $t \geq 0$.

Intuitively, a random time T is a stopping time if, for any time t, we can decide whether the event that $T < t$ occurs by knowing the process up to time t. While it is easy to show that a strict stopping time is always a stopping time, it is not always the case that a stopping time is a strict stopping time. However, strict stopping times will be equivalent to stopping times if we are defining stopping times with respect to the filtration $\mathcal{F}^+(t) : t \geq 0$.

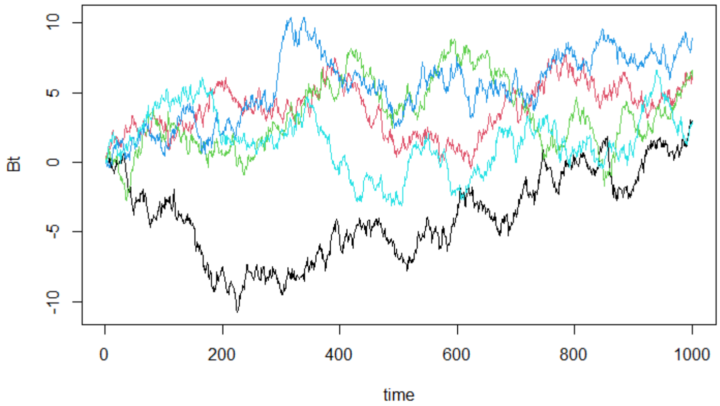

FIGURE 1.4: Graphs of five sampled Brownian motions

Definition 1.10. Let T be a stopping time. Define the $\sigma-$algebra $\mathcal{F}^+(T) := \{A \in \mathcal{A} : A \cap \{T < t\} \in \mathcal{F}^+(t), \ \ \forall\, t \geq 0\}$.

Intuitively, $\mathcal{F}^+(T)$ can be viewed as the set of of events that happen before the stopping time T. With this definition, we can now state and prove the strong Markov property of Brownian motion, which says that a Brownian motion times-hifted by a stopping time is also a Brownian motion and is independent of all the events that happen before the stopping time.

Theorem 1.11 (Strong Markov Property)**.** *For every almost surely finite stopping time T, the process $\{B(T+t) - B(T) : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. Recall that $T_n := (m+1)2^{-n}$ if $m2^{-n} \leq T < (m+1)2^{-n}$. We first show our statement holds for the stopping times T_n which discretely approximate T from above. Define the random processes $\{B_k(t) : t \geq 0\}$ and $\{B_*(t) : t \geq 0\}$ where $B_k(t) := B(t+\frac{k}{2^n}) - B(\frac{k}{2^n})$ and $B_*(t) := B(t+T_n) - B(T_n)$. Suppose that $E \in \mathcal{F}^+(T_n)$. Then for every event $\{B_* \in A\}$, we have

$$\mathbb{P}(\{B_* \in A\} \cap E) = \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\} \cap E \cap \{T_n = k2^{-n})$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\})\mathbb{P}(E \cap \{T_n = k2^{-n}\})$$

using the fact that $\{B_k \in A\}$ is independent of $E \cap \{T_n = k2^{-n} \in \mathcal{F}^+(k2^{-n})$ by Theorem 1.8. Now, by the Markov property, $\mathbb{P}(\{B_k \in A\}) = \mathbb{P}(\{B \in A\})$, where B is a standard BM. Thus we get

$$\sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\})\mathbb{P}(E \cap \{T_n = k2^{-n}\}) = \mathbb{P}(\{B \in A\}) \sum_{k=0}^{\infty} \mathbb{P}(E \cap \{T_n = k2^{-n}\})$$

$$= \mathbb{P}(\{B \in A\})\mathbb{P}(E).$$

Taking E to be the entire probability space, we notice that B_* has the same distributions as B. Hence, B_* is a standard BM. It then follows that $\mathbb{P}(\{B_* \in A\} \cap E) = \mathbb{P}(\{B \in A\})\mathbb{P}(E) = \mathbb{P}(\{B_* \in A\})\mathbb{P}(E)$, which shows that B_* is a BM and independent of E, hence of $\mathcal{F}^+(T_n)$, as claimed.

It remains to generalize this to general stopping times T. As $T_n \searrow T$, we have that $\{B(s + T_n) - B(T_n) : s \geq 0\}$ is a BM independent of $\mathcal{F}^+(T_n) \supset \mathcal{F}^+(T)$. Hence, the increments

$$B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T) - B(t+T_n)$$

of the process $\{B(r+T) - B(T) : r \geq 0\}$ are independent and normally distributed with mean zero and variance s. As the process is clearly almost surely continuous, it is a Brownian motion. Moreover, all increments $B(s+t+T_n) - B(t+T_n)$ are independent of $\mathcal{F}^+(T)$. Therefore, all increments $B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T_n)_B(t+T_n)$ are independent of $\mathcal{F}^+(T)$. Thus, the process $\{B(t+T) - B(T) : t \geq 0\}$ is independent of $\mathcal{F}^+(T)$. □

The primary application of the strong Markov property comes in demonstrating the reflection principle.

Theorem 1.12 (Reflection Principle). *If T is a stopping time and $\{B(t) : t \geq 0\}$ is a standard Brownian motion, then the process $\{B^*(t) : t \geq 0\}$ (called Brownian motion reflected at T) defined by*

$$B^*(t) = \left\{ \begin{array}{ll} B(t) & t \leq T \\ 2B(T) - B(t) & t > T \end{array} \right\}$$

is also a standard Brownian motion.

Proof. If T is finite, the strong Markov property implies that both

$$\{B(t+T) - B(T) : t \geq 0\} \ \text{ and } \ \{-(B(t+T) - B(T)) : t \geq 0\}$$

are Brownian motions independent of the beginning $\{B(t) : t \in [0, T]\}$. Hence, the concatenations of the process up to time T with each of the processes after time T have the same distributions, i.e. the processes defined by

$$B(t)\mathbf{1}_{t\leq T} + (B(t+T) - B(T) + B(T))\mathbf{1}_{t>T}$$

and

$$B(t)\mathbf{1}_{t\leq T} + (-B(t+T) + B(T) + B(T))\mathbf{1}_{t>T}$$

have the same distributions. Because the first process is $\{B(t) : t \geq 0\}$ and the second is $\{B^*(t) : t \geq 0\}$, it follows that $B^*(t)$ is also a standard Brownian motion. $\square$

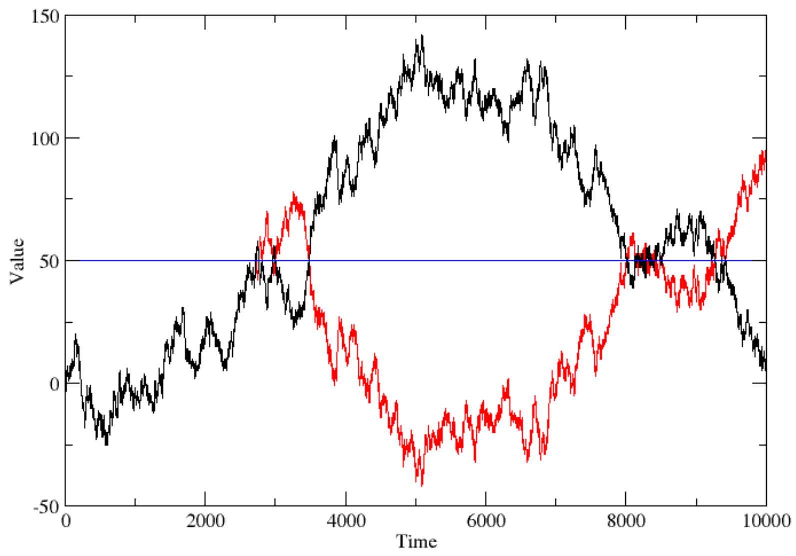

FIGURE 1.5: 1-d Brownian Motion and its reflection upon reaching a crossing point.

Remark 1.13. We define the supremum process of BM as $S_t := \sup_{0 \leq u \leq t} B_u$. The reflection principle implies that:

$$\mathbb{P}(S_t \geq b, B_t \leq a) = \mathbb{P}(B_t \geq 2b - a),$$

which we motivate by stopping at time S_t, and using SMP for BM, even though it is not a stopping time. By getting $a = b$, we get:

$$\mathbb{P}(S_t \geq b) = \mathbb{P}(S_t \geq b, B_t \geq b) + \mathbb{P}(B_t \geq b) = 2\mathbb{P}(B_t \geq b) = \mathbb{P}(|B| \geq b),$$

and conclude that

$$S_t := \sup_{0 \leq u \leq t} B_u \stackrel{d}{=} |B|_t \text{ for each } t \geq 0. \tag{1.1}$$

1.5 Nondifferentiability of Brownian Motion

In spite of the fact that Brownian motion is almost surely continuous, this paragraph will show that BM is almost surely differentiable nowhere. To deliver an sign as to how BM is quite erratic, we present the following proposition:

Proposition 1.14. *Almost surely, for all* $0 < a < b < \infty$*, Brownian motion is not monotone on the interval* $[a, b]$.

Proof. First we fix an interval $[a, b]$. Suppose that the BM $\{B(t) : t \geq 0\}$ is monotone on $[a, b]$. Choose times $a = a_1 < a_2 < ... < a_n < a_{n+1} = b$ in order to divide the interval $[a, b]$ into n sub-intervals. If B is monotone on $[a, b]$, then each increment $B(a_i) - B(a_{i-1})$ must have the same sign. Because the increments are independent and normally distributed, the event that all the increments $B(a_i) - B(a_{i-1})$ have the same sign occurs with probability $2 \cdot 2^{-n}$. Taking $n \to \infty$ shows that $[a, b]$ is an interval of monotonicity with probability 0.

It follows that if we take the countable union over all the intervals $[a, b]$ where a and b are rational endpoints, we see that almost surely BM is not monotone on any interval with rational endpoints. For any other interval $[r, s]$, we can find a sub-interval $[a, b] \subset [r, s]$ where a and b are rational. Because monotonicity on $[r, s]$ implies monotonicity on $[a, b]$, it follows that almost surely BM is not monotone on any interval $[a, b]$. □

Now that we have depict an intuition of how Brownian motion can be unstable, we can begin to show that BM is almost surely nowhere differentiable.

The proof of the next lemma uses the definition of exchangeable events, the Hewitt-Savage 0-1 law and Fatou's lemma which we now recall. A proof of the Hewitt-Savage 0-1 law and Fatou's lemma may be found in Chapter 7 (Stochatsic Processes, p.496) and Chapter 3 (Integration,p.209) respectively of Billingsley's *Probability and Measures*[4].

Definition 1.15. Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a set A of sequences such that $\{\omega \in \Omega | X_n(\omega \in A\} \in \mathcal{F}$. The event $\{X_1, X_2, ... \in A\}$ is called **exchangeable** if

$$\{X_1, X_2, ... \in A\} \subset \{X_{\sigma 1}, X_{\sigma 2}, ... \in A\}$$

for all finite permutations $\sigma : \mathbb{N} \to \mathbb{N}$. Here, finite permutations means that σ is a bijection with $\sigma_n = n$ for all sufficiently large n.

Theorem 1.16 (Hewitt-Savage 0-1 law)**.** *If A is an exchangeable event for an independent, identically distributed sequence, then $\mathbb{P}(A)$ is* 0 *or* 1.

Definition 1.17 (limsup and liminf)**.** Let $\{an\}$ be a sequence. Then the **limit superior** of $\{a_n\}$, denoted by $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

Similarly, the **limit inferior** of $\{a_n\}$, denoted by $\liminf_{n\to\infty} a_n$, is defined by

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

Theorem 1.18 (Fatou's lemma). *For non-negative a_n,*

$$\int \liminf_n a_n d\mu \leq \liminf_n \int a_n d\mu$$

Lemma 1.19. *Almost surely,*

$$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{b}} = +\infty \quad \textit{and} \quad \liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = -\infty$$

Proof. By Fatou's lemma,

$$\mathbb{P}\{B(n) > c\sqrt{n} \text{ infinitely often}\} \geq \limsup_{n\to\infty} \mathbb{P}\{B(n) > c\sqrt{n}\}$$

Using scaling invariance with $a = \frac{1}{\sqrt{n}}$, we have $\limsup_{n\to\infty} \mathbb{P}\{B(n) > c\sqrt{n}\} = \mathbb{P}\{B(1) > c\}$ which is greater than 0 since $B(1)$ has a standard normal distribution.
Let $X_n = B(n) - B(n-1)$ and note that this event

$$\{B(n) > c\sqrt{n} \text{ infinitely often}\} = \left\{\sum_{j=1}^{n} X_j > c\sqrt{n} \text{ infinitely often}\right\}$$

is an exchangeable event. It then follows from the Hewitt-Savage 0-1 law that, almost surely, $B(n) > c\sqrt{n}$ infinitely often. Taking the intersection over all the positive integers c gives that $\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = +\infty$.
The second part of the theorem follows by similar logic. ☐

Definition 1.20. For a function f, we define the **upper** and **lower right derivatives**

$$D^* f(t) := \limsup_{h\to 0} \frac{f(t+h) - f(t)}{h} \quad \text{and} \quad D_* f(t) := \liminf_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

Theorem 1.21. *Fix $t \geq 0$. Then, almost surely, Brownian motion is not differentiable at t. Moreover, $D^* B(t) = +\infty$ and $D_* B(t) = -\infty$.*

Proof. Given a standard BM $\{B(t) : t \geq 0\}$ we construct another standard BM $\{X(t) : t \geq 0\}$ by time inversion as in Theorem 1.4. It then follows that

$$D^* X(0) \geq \limsup_{n\to\infty} \frac{X(\frac{1}{n}) - X(0)}{\frac{1}{n}} \geq \limsup_{n\to\infty} \sqrt{n} X(\frac{1}{n}) = \limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}}$$

which is infinite by Lemma 1.19. By similar way, $D_* X(0) = -\infty$. Hence, standard Brownian motion is not differentiable at 0.
Now let $t > 0$ be arbitary and let $\{B(t) : t \geq 0\}$ be BM. It follows from the time-shift invariance that $Y(s) := B(t+s) - B(t)$ defines a standard BM. We just showed that standard Brownian motion is not differentiable at $s = 0$, which implies that B is not differentiable at t. ☐

Theorem 1.22 (Paley, Wiener and Zygmund, 1933). *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t,*

$$\text{either } D^*B(t) = +\infty \quad \text{or} \quad D_*B(t) = -\infty \quad \text{or both.}$$

A proof of the theorem above can be found in the book *"Brownian Motion"* of Peter Mörters and Yuval Peres [22](Chapter 1,p.34).

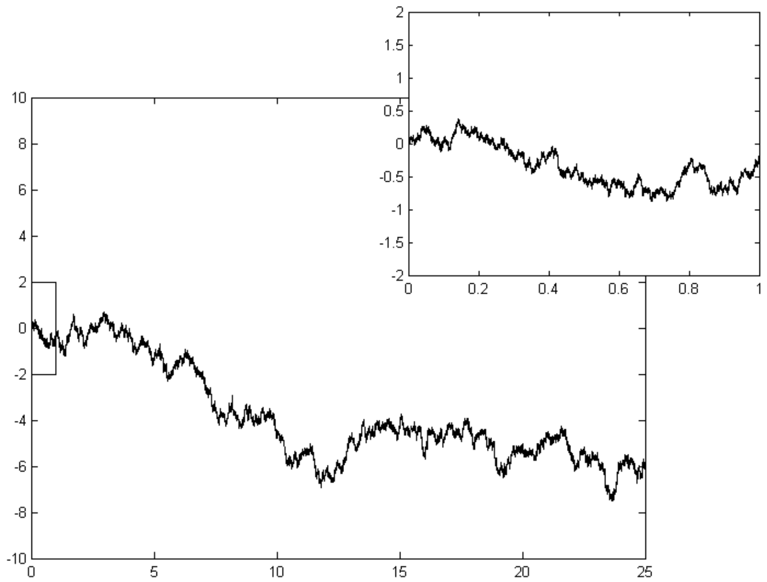

FIGURE 1.6: Brownian motion in zoom

1.6 The martingale property of Brownian motion

In this section we study a different feature of Brownian motion, the martingale property, as a starting point.

Definition 1.23. A real-valued stochastic process $\{X(t) : t \geq 0\}$ is a **martingale** with respect to a filtration $(\mathcal{F}(t) : t \geq 0)$ if it satisfies the following conditions,

(i) $X(t)$ is $\mathcal{F}_t$ measurable for every $t \geq 0$;

(ii) $\mathbb{E}[X(t)] < \infty$ for all $t \geq 0$;

(iii) $\mathbb{E}[X(t)|\mathcal{F}(s)] = X(s)$ for all $s, t \geq 0, s \leq t$

The process is called a **submartingale** if $\geq$ holds, and a **supermartingale** if $\leq$ holds in the display above.

Intuitively, a martingale is a process where the current state $X(t)$ is always the best prediction for its further states. In this sense, martingales describe fair games. If $\{X(t) : t \geq 0\}$ is a martingale, the process $\{|X(t)| : t \geq 0\}$ need not be a martingale, but it still is a submartingale, as a simple application of the triangle inequality shows.
We call a stochastic process $\{X(t) : t \geq 0\}$ *adapted* to the filtration $(\mathcal{F}(t) : t \geq 0)$ if $X(t)$ is for each $t \geq 0$ an $\mathcal{F}(t)$ measurable random variable. Clearly, X is $\mathcal{F}(t)$ adapted if, and only if, $\mathcal{F}(t)^X \subset \mathcal{F}(t)$ for all $t \geq 0$ where $\mathcal{F}(t)^X = \sigma(X(s) : s \geq 0, s \leq t)$ is the natural filtration of X.

Example 1.1. *Let $\{B(t)\}$ be a standard BM. Then*

$$\mathbb{E}[B(t)|\mathcal{F}^+(s)] = \mathbb{E}[B(t) - B(s)|\mathcal{F}^+] + B(s)$$

$$= \mathbb{E}[B(t) - B(s)] + B(s)$$

$$= B(s)$$

by the Markov property. Hence BM is a MG.

Here, we obtain a version of the optional stopping theorem for Brownian motions.

Theorem 1.24 (Optional stopping theorem)**.** *Suppose $\{X(t) : T \geq 0\}$ is a continuous martingale, and $0 \leq S \leq T$ are stopping times. If the process $\{X(\max(T, t) : t \geq 0\}$ is dominated by an integrable random variable, then*

$$\mathbb{E}[X(T)|\mathcal{F}(S)] = X(S),$$

almost surely.

Proof. Here $a \wedge b = \max\{a, b\}$ and $\lfloor a \rfloor$ =the largest integer below a given real number a. We proceed by discretization. Fix N and consider the discrete-time MG

$$X_n = X(T \wedge n2^{-N})$$

and the stopping times

$$S'_N = \lfloor 2^N S \rfloor + 1$$

and

$$T'_N = \lfloor 2^N T \rfloor + 1$$

with respect to the filtration

$$\mathcal{G}_n = \mathcal{F}(n2^{-N}).$$

The discrete-time optional stopping theorem gives

$$\mathbb{E}[X_{T'_N}|\mathcal{G}_{S'_N}] = X_{S'_N},$$

which is equivalent to

$$\mathbb{E}[X(T \wedge 2^{-N}T'_N|\mathcal{F}(2^{-N}S'_N)] = \mathbb{E}[X(T)|\mathcal{F}(2^{-N}S'_N)] = X(T \wedge 2^{-N}S'_N).$$

For $A \in \mathcal{F}(S) \subset \mathcal{F}(2^{-N}S'_N),$, by the definition of the conditional expectation and the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}[X(T); A] &= \lim_N \mathbb{E}[\mathbb{E}[X(T)|\mathcal{F}(2^{-N}S'_N)]; A] \\ &= \mathbb{E}[\lim_N X(T \wedge 2^{-N}S'_N); A] \\ &= \mathbb{E}[X(S); A],\end{aligned}$$

where we used continuity. The first line above is true for each N and therefore for the limit. □

A typical application is Wald's lemma.

Theorem 1.25 (Wald's lemma)**.** *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{F^+(t)\}$ such that either:*

1. $\mathbb{E}[T] < +\infty,$ *or*
2. $\{B(\max\{t, T\}\}$ *is dominated by an integrable random variable.*

Then $\mathbb{E}[B(T)] = 0.$

Proof. The result under the second condition follows immediately from the optional stopping theorem 1.24 with $S = 0$. We show that the first condition implies the second one.
Here $\lceil a \rceil$ is the smallest integer above a given real number a.
Assume $\mathbb{E}[T] < +\infty$. Define

$$M_k = \max_{0 \le t \le 1} |B(t+k) - B(k)|,$$

and

$$M = \sum_{k=1}^{\lceil T \rceil} M_k,$$

and note that $|B(max\{t,T\})| \leq M$.

Then

$$\begin{aligned} \mathbb{E}[M] &= \sum_k \mathbb{E}[\mathbf{1}_{\{T>k-1\}} M_k] \\ &= \sum_k \mathbb{P}[T > k-1]\mathbb{E}[M_k] \\ &= \mathbb{E}[M_0]\mathbb{E}[T+1] < +\infty \end{aligned}$$

where, using Fubini's theorem and the fact that $\mathbb{P}\{M(t) > a\} \leq \frac{\sqrt{2t}}{a\sqrt{\pi}} \exp\left\{-\frac{a^2}{2t}\right\}$,[Mörters and Peres][22](Remark 2.19,p.50), we have

$$\mathbb{E}[M_0] = \int_0^{\infty} \mathbb{P}\left\{\max_{0\leq t\leq 1} |B(t)| > x\right\} dx \leq \int_0^{\infty} \frac{2\sqrt{2}}{x\sqrt{\pi}} \exp\left\{-\frac{x^2}{2t}\right\} dx < \infty.$$

□

To find the second moment of $B(T)$ and thus prove Wald's second lemma, we identify a further martingale derived from Brownian motion.

Lemma 1.26. *Suppose $\{B(t) : t \geq 0\}$ is a linear BM. Then the process*

$$\{B(t)^2 - t : t \geq 0\}$$

is a martingale.

Proof. The process is adapted to the natural filtration $\{\mathcal{F}^+(s) : s > 0\}$ of Brownian motion and for $0 \leq s < t$ we have

$$\mathbb{E}[B(t)^2 - t|\mathcal{F}^+(s)] = \mathbb{E}[(B(t) - B(s))^2|\mathcal{F}^+(s)] + 2\mathbb{E}[B(t)B(s)|\mathcal{F}^+(s)] - B(s)^2 - t$$

$$= (t-s) + 2B(s)^2 - B(s)^2 - t = B(s)^2 - s$$

which completes the proof. □

We state without proof :

Theorem 1.27 (Wald's second lemma)**.** *Let $\{B(t)\}$ be a standard BM and T a stopping time such that $\mathbb{E}[T] < \infty$. Then*

$$\mathbb{E}[B(T)^2] = \mathbb{E}[T].$$

A proof can be found in *"Brownian Motion"* of Mörters and Peres [22], Chapter 2.4 "The martingale property of Brownian motion"(p.60).

Now we will show that the process $\{e^{\lambda B(t) - \frac{\lambda^2}{2}t} : t \geq 0\}$ for $\lambda \in \mathbb{R}$ and $B(t)$ be a standard BM, is a martingale.

For $0 \leq s < t$ it is

$$e^{\lambda B(t) - \frac{\lambda^2}{2}t} = e^{\lambda\{B(t)-B(s)\}} e^{\lambda B(s) - \frac{\lambda^2}{2}t}$$

. The process $e^{\lambda B(s)}$ is countable under $\{\mathcal{F}^+(s) : s \geq 0\}$, the $B(t) - B(s)$ is independent of $\mathcal{F}^+$ and follows the distribution $N(0, t-s)$. So,

$$\mathbb{E}[e^{\lambda B(t) - \frac{\lambda^2}{2}t} | \mathcal{F}^+(s)] = \mathbb{E}[e^{\lambda\{B(t) - B(s)\}}] e^{\lambda B(s) - \frac{\lambda^2}{2}t}$$

$$= e^{\frac{\lambda^2}{2}(t-s) + \lambda B(s) - \frac{\lambda^2}{2}t} = e^{\lambda B(s) - \frac{\lambda^2}{2}s}.$$

The result follows.

1.7 Variation and quadratic variation

Definition 1.28. The **variation** of f on $[a, b]$ over some partition $\Delta = \{a = t_0 < t_1 < ... < t_n = b\}$ is defined by

$$V_\Delta = \sum |f(t_i) - f(t_{i-1})|$$

$$V(f) = \lim_{||\Delta|| \to 0} V_\Delta(f)$$

where $||\Delta||$ is the norm (or mesh) of the partition and equals to the length of the longest subinterval,e.g.

$$||\Delta|| := \max\{|t_i - t_{i-1}| : i = 1, ..., n\}.$$

A function has **bounded variation** if the variation is bounded by some constant.
In other words, to have bounded variation, the sum of the difference between values that the function takes over any arbitrary partition must be finite. One way to think about this is the "wobbliness" of the function on a closed interval. Intuitively, for most smooth functions that one would consider, on a closed interval, it would seem obvious that the variation would be bounded - since we are just measuring the length of a "nice" function on a closed interval, surely this would be finite. However, there are smooth functions that do not have bounded variation. If a function does not have bounded variation, is there any other type of variation that we can use? The answer lies with quadratic variation.

Definition 1.29. The **quadratic variation** of a function f on the interval $[a, b]$ over some partition $\Delta = \{a = t_0 < t_1 < ... < t_n = b\}$ is defined by the following.

$$Q_\Delta(f) = \sum |f(t_i) - f(t_{i-1})|^2$$

$$Q(f) = \lim_{||\Delta|| \to 0} Q_\Delta(f)$$

Recalling what Brownian motion looks like, it is clear that it is extremely "wobbly". And, since we know that Brownian motion is nowhere differentiable with probability 1, we should anticipate that it probably does not have bounded variation.

Theorem 1.30. *The variation of a Brownian motion path does not converge, with probability one; in other words, the following equation diverges:*

$$\lim_{n\to\infty}\sum_{i=1}^{2^n}\left|B\left(\frac{kt}{2^n}\right)-B\left(\frac{(k-1)t}{2^n}\right)\right|$$

Proof. Note that the following inequality is true:

$$\sum_{i=1}^{2^n}\left|B(\frac{kt}{2^n})-B(\frac{(k-1)t}{2^n})\right| \geq \frac{\sum_{i=1}^{2^n}|B(\frac{kt}{2^n})-B(\frac{(k-1)t}{2^n})|^2}{\max_{n=0,1,2,..}|B(\frac{kt}{2^n})-B(\frac{(k-1)t}{2^n})|}$$

Consider the numerator and denominator separately. In the numerator, this is the quadratic variation of Brownian motion: After this, we will show that the quadratic variation of Brownian motion is t. Meanwhile, the denominator will converge to zero, because as the time increments indexing Brownian motion become arbitrarily close, the distance between also converges to zero. So, Brownian motion does not have bounded variation. □

Definition 1.31. The quadratic variation of Brownian motion can be defined using the following:

$$\lim_{n\to\infty}\mathbb{E}\left[\sum_{i=1}^{2^n}\left|B\left(\frac{kt}{2^n}\right)-B\left(\frac{(k-1)t}{2^n}\right)\right|^2\right]=t$$

Proposition 1.32. *The quadratic variation of standard Brownian motion on the interval* $[a,b]$ *is* $a-b$*.*

Proof. To prove this, for the sake of simplicity and the sanity of the reader, we will only consider the case on the interval $[0,t]$, and prove that the quadratic variation L^2 converges to t. In other words, we want to prove that

$$\mathbb{E}[(Q(B)-t)^2]=0 \tag{1.2}$$

Note that for some partition $\{t_1,t_2,...,t_n\}$, we can express t with the following sum:

$$t=\sum_{i=1}^{n}t_i-t_{i-1}$$

We also will use the following notation:

$$B_{t_i}-B_{t_{i-1}}=\Delta B_i$$

By definition of BM, we know that $\Delta B_i \sim N(0, t_i - t_{i-1})$. We have the following inequality:

$$Q(B) - t = \sum_i (\Delta B_i)^2 - (t_i - t_{i-1})$$

We want the expectation of the square of $Q(B) - t$ to converge to zero:

$$\mathbb{E}[(Q(B) - t)^2] = \sum_i \mathbb{E}[((\Delta B_i)^2 - (t_i - t_{i-1}))^2].$$

First, we square the terms inside of the expectation, and simplify. To simplify, separate the sum using linearity of expectation, and the independence of $(\Delta B_i)^2$ and $t_i - t_{i-1}$:

$$= \sum_i \mathbb{E}[((\Delta B_i)^4 - 2(\Delta B_i)^2(t_i - t_{i-1}) + (t_i - t_{i-1})^2]$$

$$= \sum_i \mathbb{E}[(\Delta B_i)^4] - 2\sum_i \mathbb{E}[(\Delta B_i)^2](t_i - t_{i-1}) + \sum_i (t_i - t_{i-1})^2$$

Note that we are able to remove $(t_i - t_{i-1})$ from inside of the expectation, since $(t_i - t_{i-1})$ is not random. To simplify this further, remember that (ΔB_i) is normally distributed. In probability, we can find higher powers of expectation with moment generating functions. In particular, for the normal distribution, where $X \sim N(0, \sigma^2)$, we have that $\mathbb{E}[X^2] = \sigma^2$ and $\mathbb{E}[X^4] = 3\sigma^4$. Using this fact, we can simplify the sum more:

$$= \sum_i 3(t_i - t_{i-1})^2 - 2\sum_i (t_i - t_{i-1})(t_i - t_{i-1}) + \sum_i (t_i - t_{i-1})^2 = 2\sum_i (t_i - t_{i-1})^2$$

Let Π^* be the supremum of distances between points $(t_i - t_{i-1})$ of the partition Π. We have the following inequality (because we are partitioning a finite interval):

$$= 2\sum_i (t_i - t_{i-1})^2 \leq 2\Pi^* \sum_i (t_i - t_{i-1}) = 2\Pi^* t$$

To find the quadratic variation, we must take the limit as the length of the distances between the points in the partitions (for any partition Π) approaches zero. In particular, Π^* will approach zero, meaning that this product also converges to zero:

$$\lim_{|\Pi| \to 0} 2\Pi^* t = 0$$

And clearly this bound means that

$$\mathbb{E}[(Q(B) - t^2)] = 0$$

This shows L^2 convergence. L^2 convergence also implies that the quadratic variation of standard BM converges to t with probability 1; this will directly follow from a variation on Chebyshev's Inequality:

Theorem 1.33 (Chebyshev's Inequality). *For a random variable X, for any $\epsilon > 0$:*

$$P[(X-k) \geq \epsilon] \leq \frac{\mathbb{E}[(X-k)^2]}{\epsilon^2}.$$

Intuitively, Chebyshev's Inequality is a way to use standard deviation to bound the probability that a random variable will deviate beyond a certain radius of the mean.
Applying this, it is clear that

$$P(|Q(B)-t| > \epsilon) \leq \frac{\mathbb{E}[(Q(B)-k)^2]}{\epsilon^2} \to 0$$

This tells us that the quadratic variation of standard BM converges in probability to t.
Comment: Converging in probability is still slightly weaker than saying that it converges with probability 1, which can be proven using the Borel-Cantelli Lemma A.1. This is proven in Lawler[18] pp.65. □

1.8 Itô's Formula

For a function $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative, it is $df(x) = f'(x)dx$ which as integral form means

$$f(b) - f(a) = \int_a^b f'(x)dx$$

for all $a < b$. If we have another function $g : \mathbb{R} \to \mathbb{R}$ with continuous derivative, then the equations above for the $f \circ g$ become $d(f(g(x))) = f'(g(x))g'(x)dx = f'(g(x))dg(x)$ and for $a < b$,

$$f(g(b)) - f(g(a)) = \int_a^b f'(g(x))dg(x).$$

When g is a BM (so it is not differentiable), the chain rule is expressed by Itô's formula. It provides a way to differentiate stochastic processes that involve random variables with respect to time.

Theorem 1.34 (Itô's Lemma). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with continuous second derivative and B a one-dimensional BM. Then with probability 1, we have*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \tag{1.3}$$

for each $t > 0$.

For a detailed proof of the above theorem the reader can refer to Hassler's book (2016) [13]. Itô's formula is widely used in mathematical finance and the modelling of stochastic processes, particularly in the field of quantitative finance, where it plays a crucial role in the pricing and hedging of derivative securities under uncertainty.

1.9 Bougerol's identity

Bougerol's celebrated identity in law has been extensively studied by various authors ever since its initial formulation in 1983. This research has been motivated by two main factors. Firstly, the identity holds significant mathematical interest, and secondly, it has found numerous practical applications, particularly in the field of Finance, such as the pricing of Asian options and others. Nonetheless, there is still a sense that further insights and a deeper understanding of this identity are yet to be uncovered.

Bougerol's remarkable identity [2, 5] states that if we have two independent linear Brownian motions $(B_u, u \geq 0)$ and $(\beta_u, u \geq 0)$ both started from 0, the we have the identity

$$\sinh B_t \overset{(law)}{=} \beta_{A_t(B)}, \quad \text{for every fixed } t \geq 0, \tag{1.4}$$

where

$$A_t(B) = \int_0^t ds \exp\left(2B_s\right)$$

For the proof based on Alili et all(1997)[2] we have:

Proof. On the one hand, applying Itô's formula (1.3) for the $S_t \equiv \sinh\left(B_t\right)$ we have:

$$S_t = \int_0^t \sqrt{1+S_s^2}\,dB_s + \frac{1}{2}\int_0^t S_s ds. \tag{1.5}$$

On the other hand, a time reversal argument for BM yields that for fixed $t \geq 0$,

$$\beta_{A_t(B)} = \int_0^t e^{B_s} d\gamma_s \overset{(law)}{=} e^{B_t}\int_0^t e^{-B_s} d\gamma_s \equiv Q_t, \tag{1.6}$$

where $(\gamma_s, s \geq 0)$ denotes another 1-dim BM, independent from $(B_s, s \geq 0)$. Applying once more Itô's formula to Q_t, we have:

$$dQ_t = \frac{1}{2}Q_t dt + \left(Q_t dB_t + d\gamma_t\right) = \frac{1}{2}Q_t dt + \sqrt{Q_t^2 + 1}\,d\delta_t, \tag{1.7}$$

where δ is another 1-dimensional Brownian motion, depending on B and on γ. From (1.5) and (1.7) we deduce that S and Q satisfy the same Stochastic Differential Equation with Lipschitz coefficients, hence, we obtain (1.4). □

For a great number of applications and other developments of this identity the reader can see Vakeroudis(2012)[29].

Chapter 2

Basic Properties of planar Brownian Motion

2.1 Introduction to planar Brownian Motion

Let start this section with the definition of higher dimensional Brownian motion.

Definition 2.1. If $B_1, ..., B_d$ are independent linear Brownian motions started in $x_1, ..., x_d$ then the stochastic process $\{B(t) : t \geq 0\}$ given by

$$B(t) = (B_1(t), ..., B_d(t))^T$$

is called a **d-dimensional Brownian motion** started in $(x_1, ..., x_d)^T$. The $d-$ dimensional BM started in the origin is also called **standard Brownian motion.**
Two-dimensional BM is called **planar Brownian motion.**

Let $\{Z(t) = X(t) + iY(t) : t \geq 0\}$ denote a standard planar BM, starting from $x_0 + i0$, where $x_0 > 0$ and $\{X(t) : t \geq 0\}, \{Y(t) : t \geq 0\}$ are two independent linear BM, starting respectively from x_0 and 0.
As is widely known, for every fixed $t > 0$, with probability 1, the curve $Z(t)$ carries out an infinite number of windings around the point x_0 without visiting it almost surely (see e.g. Itô-McKean [16]). For this reason, Z may be written in polar coordinates as

$$Z_t = r_t \exp(i\theta_t) \qquad , t \geq 0 \tag{2.1}$$

where $r = |Z|$ is the radial part of Z, and θ_t is the total winding of Z about 0 up to time t. More accurately, $(\theta_t, t \geq 0)$ is the unique continuous process such that $\theta_0 = 0$ and (2.1) holds.

The total winding up to time t may be regarded as in complex analysis as the integral along the path of Z between times 0 and t of the closed differential form on $\mathbb{C} \setminus \{0\}$

$$\operatorname{Im}\left(\frac{dZ}{Z}\right) = \frac{x\ dy - y\ dx}{x^2 + y^2}.$$

Thus if we define

$$\log Z_t = \int_{Z[0,t]} \frac{dZ}{Z}, \tag{2.2}$$

the integral along the path of Z between 0 and t of dZ/Z (see Yor,1997a [32]),then $(\log Z_t, t \geq 0)$ is the unique continuous determination of a logarithm of Z_t starting at $\log Z_0 = 0$, and

$$\log Z_t = \log r_t + i\theta_t.$$

2.2 Skew product representation of planar Brownian Motion

In this section we present a result for the complex BM Z which we study under the law P_a for $a \neq 0$. Since $Z \neq 0$ for all t P_a-a.s., we may choose a continuous determination $\theta_t(\omega)$ of the argument of $Z_t(\omega)$ such that $\theta_0(\omega)$ is a constant and $e^{i\theta_0} = a/|a|$. We then have that $Z_t = r_t \exp(i\theta_t)$ and θ is adapted to the filtration of Z_t.
Before we give a short proof of how we reach at the skew-product representation, we have to present some tools that will be used.
The *Dambis/Dubins-Schwarz Theorem* shows that many processes admit a time change under which they become a Brownian motion.

Theorem 2.2 (Dambis/Dubins-Schwarz). *If $\{M_t, t \geq 0\}$ is a continuous local martingale with respect to filtration $\{\mathcal{F}\}$, such that $\langle M \rangle_\infty = \infty$, then the process:*

$$\beta_u = M_{\tau_u} \quad , \qquad \textit{where } \tau_u = \inf\{t : \langle M \rangle_t \ > u\}$$

is a $\{\mathcal{F}_{\tau_u}\}$ Brownian motion, and M may be represented as:

$$M_t = \beta_{\langle M \rangle_t}, \quad t \geq 0.$$

A proof of the theorem can be found at Revuz,Yor (1999)-p.181 [25].
Moreover, we remind that if W_t is a BM then

1. $\langle dW_s \rangle = ds$
2. $\langle \int_0^t k_s dW_s \rangle = \int_0^t k_s^2 \langle dW_s \rangle = \int_0^t k_s^2 ds$

The processes r_t and θ_t may be analysed in the following way.
We begin from the planar BM $Z_t = X_t + iY_t$ where X, Y are two independent one-dimensional BM. We write Z_t as:

$$Z_t = r_t e^{i\theta_t} = |Z_t| e^{i\theta_t}$$

$$\log Z_t = \log |Z_t| + i\theta_t$$

Moreover, we have from (2.2) that

$$\begin{aligned}
\log Z_t = \int_0^t \frac{dZ_s}{Z_s} &= \int_0^t \frac{dX_s + idY_s}{X_s + iY_s} = \int_0^t \frac{(X_s - iY_s)(dX_s + idY_s)}{X_s^2 + Y_s^2} \\
&= \int_0^t \frac{X_s dX_s + Y_s dY_s}{X_s^2 + Y_s^2} + i \int_0^t \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} \qquad (2.3)
\end{aligned}$$

From Dambis-Dubins-Schwarz theorem (2.2)

$$= \int_0^t \frac{X_s dX_s + Y_s dY_s}{X_s^2 + Y_s^2} = \beta_{H_t} \quad \& \quad \int_0^t \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} = \gamma_{H_t}$$

where

$$H_t = \langle \int_0^t \frac{X_s dX_s + Y_s dY_s}{X_s^2 + Y_s^2} \rangle = \langle \int_0^t \frac{X_s}{X_s^2 + Y_s^2} dX_s + \int_0^t \frac{Y_s}{X_s^2 + Y_s^2} dY_s \rangle =$$

$$= \langle \int_0^t \frac{X_s}{X_s^2 + Y_s^2} dX_s \rangle + \langle \int_0^t \frac{Y_s}{X_s^2 + Y_s^2} dY_s \rangle + \underbrace{\langle \int_0^t \frac{X_s}{X_s^2 + Y_s^2} dX_s \, , \int_0^t \frac{Y_s}{X_s^2 + Y_s^2} dY_s \rangle}_{= 0 \text{ as } X \perp\!\!\!\perp Y}$$

$$\begin{aligned}
&= \int_0^t \frac{X_s^2}{(X_s^2 + Y_s^2)^2} \langle dX_s \rangle + \int_0^t \frac{Y_s^2}{(X_s^2 + Y_s^2)^2} \langle dY_s \rangle + 0 \\
&= \int_0^t \frac{X_s^2 + Y_s^2}{(X_s^2 + Y_s^2)^2} ds = \int_0^t \frac{1}{X_s^2 + Y_s^2} ds \\
&= \int_0^t \frac{1}{|Z_s|^2} ds
\end{aligned}$$

as $|Z_t| = \sqrt{X_t^2 + Y_t^2}$.
Thus,

$$\int_0^t \frac{X_s dX_s + Y_s dY_s}{X_s^2 + Y_s^2} = \beta_{H_t}, \quad \text{where } H_t = \int_0^t \frac{ds}{|Z_s|^2}. \qquad (2.4)$$

Similarly,

$$\int_0^t \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} = \gamma_{H_t}, \quad \text{where } H_t = \int_0^t \frac{ds}{|Z_s|^2}. \qquad (2.5)$$

So, we conclude that

$$\log Z_t = \log |Z_t| + i\theta_t \overset{(2.3),(2.4),(2.5)}{=} \beta_{H_t} + i\gamma_{H_t} \implies$$

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t} \tag{2.6}$$

A more compact way to depict the skew product representation is

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u)|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}} \tag{2.7}$$

As we saw above, $(\beta_u + i\theta_u, u \geq 0)$ is another planar BM from $\log 1 + i0 = 0$. The process $H_t = \int_0^t \frac{ds}{|Z_s|^2}$ is called *Bessel clock* [35] since, on the one hand, it involves only the Bessel process $(|Z_t|, t \geq 0)$ and, on the other hand, $(H_t, t \geq 0)$ is the time-change, or 'clock' and plays important role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$(see,e.g.Yor[33]).

From this illustration we can perceive that, as one might expect, the smaller the modulus of Z, the more rapidly the argument of Z varies. Furthermore, as θ_t is a time-changed BM, it is easy to see that

$$\liminf_{t\to\infty} \theta_t = -\infty, \qquad \limsup_{t\to\infty} \theta_t = +\infty \ \text{ a.s.}$$

Thus, the planar BM winds itself arbitrarily large numbers of times around 0, then unwinds itself and does this infinitely often.

2.3 Exit times for a planar Brownian Motion

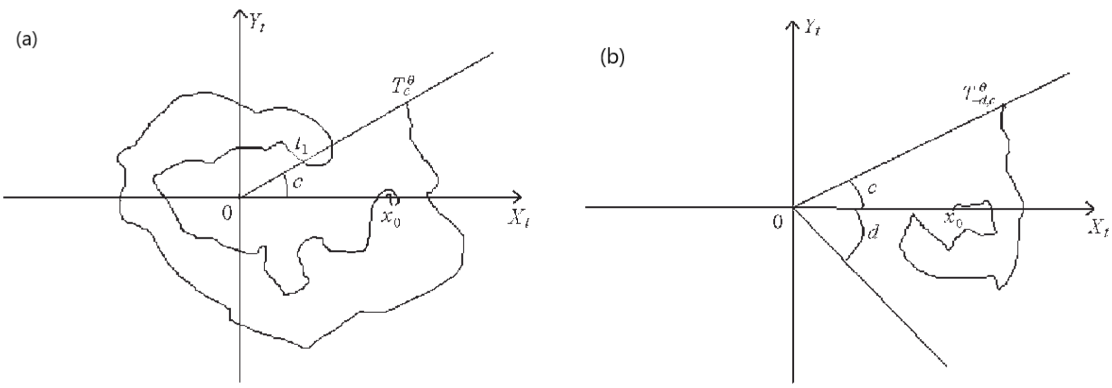

FIGURE 2.1: Exit times for a planar BM.

This figure presents the exit times (a) T_c^θ (t_1 doesn't matter because the angle is negative) and (b) $T_{-d,c}^\theta$ for a planar BM starting from $x_0 + i0$.

In this paragraph we are going to study the exit times for a planar BM. Going back to Spitzer [27], we define the process $\theta_R(t)$, the angular part of the plane Brownian motion. The subscript R denotes the radial initial condition, i.e. $|X(0) + iY(0)| = R > 0$. Then, $\theta_R(t)$ is the winding number of the continuous path $Z_\tau = X_\tau + iY_\tau$, where $0 \leq \tau \leq t$, about the origin. P.Lévy [19] deduced that the distribution of $\theta_R(t)$ must have infinite variance from the fact that $\theta_R(t)$

tends to assume very large values, when $R(t)$ is small.
Now, we define the first hitting time $T^{\theta}_{-d,c}$ is the time when the path of the BM Z_t comes out from angle c above the X_t-axis and angle d below the X_t-axis, as:

$$T^{\theta}_{-d,c} \equiv \inf\{t : \theta_t \notin (-d, c)\}, \quad (c, d > 0) \tag{2.8}$$

Especially, in Spitzer [27], Burkholder [6] and Revuz-Yor [25] can the reader find the bellow criterion:

$$\mathbb{E}[(T^{\theta}_{-d,c})^p] \text{ if and only if } p < \frac{\pi}{2(c+d)}. \tag{2.9}$$

which says that the moment of order p has finite expected value if and only if $p < \frac{\pi}{2(c+d)}$.
Moreover, Spitzer's asymptotic theorem (see e.g. Spitzer [27]) states that:

$$\frac{2\theta_t}{\log t} \xrightarrow[t\to\infty]{(law)} C_1 \overset{(law)}{=} \gamma_{T_1^{\beta}} \tag{2.10}$$

2.4 On the Laplace transform of the distribution of the hitting time $T^{\theta}_c \equiv T^{\theta}_{-\infty,c}$

Here, we use the skew-product representation (2.6) to access the distribution of the hitting time T^{θ}_c (see Figure 2.1a). It is defined $T^{\gamma}_c \equiv \inf\{t : \gamma_t \notin (-\infty, c)\}$ as the hitting time connected with the BM $(\gamma_t, t \geq 0)$. From 2.6 we have:
$H_{T^{\theta}_c} = T^{\gamma}_c$ hence $T^{\theta}_c = H^{-1}_{u=T^{\gamma}_c}$ where,

$$H^{-1}_u \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2\beta_s) := A_u \tag{2.11}$$

So, we have acquired

$$T^{\theta}_c = A_{T^{\gamma}_c} \tag{2.12}$$

where $(A_u, u \geq 0)$ and T^{γ}_c are independent, since β and γ are independent.

Remark 2.3. **In the sequel, we will state most of the results for $\mathbf{T^{\theta}_c}$, but from (2.12), it is obvious that they will also hold for $\mathbf{A_{T^{\gamma}_c}}$.**

Moreover, β_s can be written as $\beta_s = \log x_0 + \beta^{(0)}_s$, with $(\beta^{(0)}_s, s \geq 0)$ a standard one-dimensional Brownian motion starting from 0. Thus, we obtain from (2.12) that

$$T^{\theta}_c = x_0^2 \left(\int_0^{T^{\gamma}_c} ds \exp(2\beta^{(0)}_s) \right) \tag{2.13}$$

From now on, for simplicity, we shall take $x_0 = 1$, but this is really no restriction as the dependency in x_0 is noticed from (2.13) is really plane.

From (1.4) and (2.13), and as it is well known [25], the law of $\beta_{T_c^\gamma}$ is the Cauchy law with parameter c, i.e., with density:

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)},$$

we deduce that:

Proposition 2.4. *For fixed $c > 0$, there is the following identity in law:*

$$\sinh(C_c) \overset{(law)}{=} \hat{\beta}_{(T_c^\theta)}, \tag{2.14}$$

where, on the left hand side, $(C_c, c \geq 0)$ denotes a standard Cauchy process and on the right hand side, $(\hat{\beta}_u, u \geq 0)$ is one- dimensional BM, independent from T_c^θ.

From the Proposition above we have that $C_c \sim Cauchy(1, 0)$ and

$$h_c(x) = \frac{1}{\pi(1 + x^2)}$$

Let $Y = \sinh C_c$ be the transformed random variable, where C_c has the density function $h_c(x)$. First, we find the inverse function of $\sinh(x)$. Since $\sinh(x)$ is not a one-to-one function, we consider its principal branch:

$$y = \sinh x$$

$$x = \sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$$

Next, we apply the transformation rule for probability density functions:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y))$$

where $f_X(x)$ is the density function of X, and $g^{-1}(y)$ is the inverse function of the transformation.

In our case, the density function of $\sinh(C_c)$ would be

$$f_Y(y) = f_X(\ln(y + \sqrt{y^2 + 1})) \cdot \frac{d}{dy}(\ln(y + \sqrt{y^2 + 1})).$$

Substituting the density function $h_c(x) = \frac{1}{\pi(1+x^2)}$ and evaluating the derivative, we have:

$$f_Y(y) = \frac{1}{\pi(1 + \ln^2(y + \sqrt{y^2 + 1}))} \frac{1}{\sqrt{y^2 + 1}}$$

Therefore, the density function of $\sinh(C_c)$ is:

$$f_Y(y) = \frac{1}{\pi(1+\ln^2(y+\sqrt{y^2+1}))}\frac{1}{\sqrt{y^2+1}}$$

For simplicity, we write:

$$f_Y(y) = \frac{1}{\sqrt{y^2+1}}h_c(\sinh^{-1}(y)) = \frac{1}{\sqrt{y^2+1}}h_c(a(y))$$

where $a(y) = \arg\sinh(y) = \sinh^{-1}(y)$.
On the right hand side we have:

$$\mathbb{E}\left[\frac{1}{\sqrt{2\pi T_c^{\theta}}}\exp(-\frac{x^2}{2T_c^{\theta}})\right].$$

Thus, we have secured the following:

Proposition 2.5. *The distribution of T_c^{θ} may be characterized by:*

$$\mathbb{E}\left[\frac{1}{\sqrt{2\pi T_c^{\theta}}}\exp(-\frac{x^2}{2T_c^{\theta}})\right] = \frac{1}{\pi(1+\ln^2(x+\sqrt{x^2+1}))}\frac{1}{\sqrt{x^2+1}}, \quad x \geq 0 \tag{2.15}$$

The proof of Proposition 2.5 follows by making the change of variable $y^2 = x$ at the $a(y) = \sinh^{-1}(y) \equiv \log(y+\sqrt{1+y^2})$. Now, it is the time to define the probability:

$$Q_c = \sqrt{\frac{\pi c^2}{2T_c^{\theta}}} \cdot P.$$

From (2.15) for $x = 0$ comes that Q_c is a probability. Thus we obtain that $c\mathbb{E}[\sqrt{\pi/2T_c^{\theta}}] = 1$, and we may write:

$$\mathbb{E}_{Q_c}[\exp(-\frac{x}{2T_c^{\theta}})] = \frac{1}{\sqrt{1+x}}\frac{1}{1+\frac{1}{c^2}\log^2(\sqrt{x}+\sqrt{1+x})}, \quad \forall x \geq 0, \tag{2.16}$$

which yields the Laplace transform of $1/T_c^{\theta}$ under Q_c.
Now we are going to investigate what happens if we make $c \to \infty$. If we denote by $T_1^{\beta} \equiv \inf\{t : \beta_t = 1\}$ the first hitting time of level 1 for a standard BM β and by N a standard Gaussian variable $\mathcal{N}(0,1)$, from equation (2.16), we obtain:

$$\lim_{c\to\infty}\mathbb{E}_{Q_c}\left[e^{-\frac{x}{2T_c^{\theta}}}\right] = \mathbb{E}\left[e^{-\frac{xN^2}{2}}\right] = \mathbb{E}\left[e^{-\frac{x}{2T_1^{\beta}}}\right], \tag{2.17}$$

which means that $T_c^{\theta} \xrightarrow[c\to\infty]{\text{(law)}} T_1^{\beta}$.
Note: *At this point, one may wonder whether there is some kind of convergence in law involving*

$(\theta_u, u \geq 0)$, *under* Q_c, *as* $c \to \infty$, *but, we shall not touch this point.*

From Proposition (2.5) we deduce the following:

Corollary 2.6. *Let* $\phi(x)$ *denote the Laplace transform (2.16), that is the Laplace transform of* $1/2T_c^\theta$ *under* Q_c. *Then, the Laplace transform of* $1/2T_c^\theta$ *under* P *is:*

$$\mathbb{E}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \int_x^\infty \frac{dw}{\sqrt{w-x}}\phi(w). \tag{2.18}$$

Proof. From Fubini's theorem, we deduce from (2.16) that:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] &= \int_0^\infty \frac{dy}{\sqrt{y}}\mathbb{E}\left[\frac{1}{\sqrt{2\pi T_c^\theta}}\exp\left(-\frac{x+y}{2T_c^\theta}\right)\right] \\
&= \int_0^\infty \frac{dy}{\sqrt{y}}\phi(x+y) \\
&\overset{y=xt}{=} \sqrt{x}\int_0^\infty \frac{dt}{\sqrt{t}}\phi(x(1+t)) \\
&\overset{u=1+t}{=} \sqrt{x}\int_1^\infty \frac{du}{\sqrt{u-1}}\phi(xu) \\
&\overset{w=xw}{=} \int_x^\infty \frac{dw}{\sqrt{w-x}}\phi(w),
\end{aligned}$$

which is the equation (2.18). □

2.5 Further properties of planar Brownian Motion

This section is strongly associated with the publication of D.Dufresne and M.Yor(2011)[10]. Here we examine in a different way the combination of Bougerol's identity (1.4) and the skew-product representation (2.7) leading to the next stunning identities in law:

Proposition 2.7. *Let* $(\delta_u, u \geq 0)$ *be a one-dimensional Brownian motion independent of the planar Brownian motion* $(Z_u, u \geq 0)$, *starting from* $1 + i0$. *Then, for any* $b \geq 0$, *the following identities in law hold:*

$$(i)\ H_{T_b^\delta} \overset{(law)}{=} T_{a(b)}^\beta \quad (ii)\ \theta_{T_b^\delta} \overset{(law)}{=} C_{a(b)} \quad (iii)\ \bar{\theta}_{T_b^\delta} \overset{(law)}{=} |C_{a(b)}|,$$

where C_A *is a Cauchy variable with parameter* A *and* $\bar{\theta} = \sup_{s \leq u} \theta_s$.

Proof. From the symmetry principle, Bougerol's identity may be equivalently stated as:

$$\sinh(\bar{\beta}_u) \overset{(law)}{=} \bar{\delta}_{A_u(b)}. \tag{2.19}$$

Consequently, the law of the first hitting times of a fixed level b by the processes on each side of (2.19) are identical that is

$$H_t = \int_0^t \frac{1}{|Z|^2} ds \text{ and } T_a^\beta = \inf t \geq 0 : \beta_t = a$$

where from skew-product representation we have that

$$T_{a(b)}^\beta \stackrel{\text{(law)}}{=} H_{T_b^\delta},$$

which is (i).
(ii) follows from (i) since:

$$\theta_u \stackrel{\text{(law)}}{=} \gamma_{H_u},$$

where $(\gamma_s : s \geq 0)$ is a 1-dimensional BM independent of $(H_u : u \geq 0)$ which follows Cauchy distribution with parameter A. Also, $(C_u : u \geq 0)$ may be represented as $(\gamma_{T_u^\beta} : u \geq 0)$.
(iii) We have from symmetry principle:

$$\bar{\theta}_{T_b^\delta} = \sup_{s<T_b^\delta} \theta_s = |C_{a(b)}|.$$

□

2.6 Williams' "pinching method"

For the proof of Spitzer's theorem that is discussed at the next section we need to explain Williams'"pinching method". We consider, on a given probability space $(\Omega, \mathcal{F}, P)$, an increasing family of positive, finite, random variables $(S_r : r \geq 0)$ such that

$$\frac{\log(S_r)}{\phi(r)} \xrightarrow[r\to\infty]{a.s.} 1,$$

with ϕ a continuous, strictly increasing function, such that $\phi(r) \xrightarrow{r\to\infty} \infty$. We define ψ as the inverse of ϕ. The letter u will denote a regularly varying function, with exponent $\rho > 0$, that is, u may be written in the form

$$u(x) = a(x)x^\rho \exp \int_1^x \frac{\epsilon(y)}{y} dy,$$

with

$$a(x) \xrightarrow{x\to\infty} c < \infty \ , \quad \epsilon(y) \xrightarrow{y\to\infty} 0.$$

The main property of u is that

$$\frac{u(tx)}{u(t)} \xrightarrow{t\to\infty} x^\rho.$$

From Williams[31] we have the following general statement.

Theorem 2.8. *Let X be a real-valued random variable (defined on some probability space), and let $\mathbf{\Phi} : (t, \omega) \to \mathbf{\Phi}_t(\omega)$ be a right-continuous process such that*

$$\frac{\mathbf{\Phi}_{S(r)}}{u(\phi(r))} \xrightarrow[r\to\infty]{d} X. \tag{2.20}$$

Under the hypothesis that the family of distributions of

$$\left\{ \frac{1}{u[\phi(r_2) - \phi(r_1)]} \sup_{S(r_1)\leq s,t\leq S(r_2)} |\mathbf{\Phi_t} - \mathbf{\Phi_s}| : 0 < r_1 < r_2 < \infty \right\} \text{ is tight,} \tag{2.21}$$

we have

$$\frac{\mathbf{\Phi}_t}{u[\log t]} \xrightarrow[t\to\infty]{d} X. \tag{2.22}$$

A family of distributions is said to be *tight* if its probability measures do not "spread out" too much as the parameters vary. More precisely, a family of distributions is tight if, for any given $\epsilon > 0$, there exists a compact set K such that the probability measures of all distributions in the family have their mass concentrated inside K with a total probability of at least $1 - \epsilon$.
In simpler terms, tightness ensures that the probability mass of the distributions in the family does not escape to infinity as the parameters change. Instead, the mass remains concentrated within a bounded region of the probability space.
We have to notice the follow about the theorem above:

1. In order to ensure that $\sup\{|\mathbf{\Phi_t} - \mathbf{\Phi_s}| : S(r_1) \leq s, t \leq S(r_2)\}$ is measurable, we assume $(\mathbf{\Phi_t})$ to be right-continuous.

2. Since u is regularly varying, it is easy to prove that $\frac{u(\log S_r)}{u(\phi(r))} \xrightarrow[r\to\infty]{d} 1$,and, therefore, that (2.20) is equivalent to
$$\frac{\mathbf{\Phi}_{s(r)}}{u(\log S_r)} \xrightarrow[r\to\infty]{d} X,$$
which makes conclusion (2.22) really intuitive.

3. If, in (2.20), the convergence in distribution is replaced by convergence in probability, then the same change should be made in (2.22).

In the sequel we apply Theorem 2.8 under P_{z_0}, the law of BM, with

$$S(r) = T(r) \equiv \inf\{t : |Z_t| > r\}, \quad \phi(r) = 2\log r$$

(it is well known that $\frac{\log T(r)}{2\log r} \xrightarrow[r\to\infty]{a.s.} 1$), and $u(x) = x^\gamma$ $(\gamma > 0)$.

2.7 Spitzer's theorem

Spitzer's theorem is associated with the winding number of a planar BM $\{Z_t = X_t + iY_t, t \geq 0\}$. The winding of a two-dimensional process refers to the number of times the process wraps around a fixed point in the plane. For planar Brownian motion, which is a stochastic process that models the random movement of a particle, the winding number measures how many times the particle loops around the origin.

Spitzer [27] showed that if $\{\theta_t(\omega), t \geq 0\}$ denotes the (continuous) total winding number of the trajectory $\{Z_u(\omega), 0 \leq u \leq t\}$ around 0, then

$$\frac{2}{\log t}\theta_t \xrightarrow[t\to\infty]{(law)} C_1 \tag{2.23}$$

with C_1 denoting a standard Cauchy variable.

Williams[31] applies the pinching method with $\gamma = 1$ to $\mathbf{\Phi}_t = \theta_t$ $(t \geq 0)$. It is convenient to assume that $z_0 = 1$. To give a simpler proof of (2.23), we remark that, since $\theta_t = \gamma_{C_t}$ with

$$C_t = \int_0^t \rho_s^{-2} ds \qquad \rho_s \equiv |Z_s| \tag{2.24}$$

and (γ_t) a real-valued BM starting from 0 independent from ρ, it suggices, in order to prove (2.23), to show that

$$\left(\frac{2}{\log t}\right)^2 C_t \xrightarrow[t\to\infty]{d} \sigma_1 \ , \tag{2.25}$$

where $\sigma_1 = \inf\{u|\beta_u = 1\}$, and (β_u) is a one-dimensional BM beginning from 0.

Now, one has $\log \rho_t = \beta_{C_t}$, with (β_t) so the identity (ii) in Proposition 2.7 is wistful of Williams' remark (see[21],[31]), that

$$H_{T_r^R} \overset{(law)}{=} T^\delta_{\log r} \tag{2.26}$$

where in this occasion R starts from 1 and δ starts from 0 (in fact, this is a sequence of (2.6)). Someone can find a great number of variants of (2.26) at [20]. We note that in (ii), T_b^δ is independent of the process $(\theta_u, u \geq 0)$ while in 2.26 T_r^R depends on $(\theta_u, u \geq 0)$. The idea is employed in a systematic manner to derive, without the need for explicit calculations, the limiting behaviour (in distribution), as $t \to \infty$ of

$$\int_0^t \{h(Z_s)dX_s + k(Z_s)dY_s\}$$

for certain functions $h, k : \mathbb{C} \to \mathbb{R}$.

Proposition 2.9 (A new proof of Spitzer's theorem). *As* $t \to \infty$, $\theta_{T^\delta_{\sqrt{t}}} - \theta_t$ *converges in law, which implies that:*

$$\frac{1}{\log t}\left(\theta_{T^\delta_{\sqrt{t}}} - \theta_t\right) \xrightarrow[t\to\infty]{(P)} 0, \tag{2.27}$$

and, in turn, implies Spitzer's theorem (see formula (2.23)):

$$\frac{2}{\log t}\theta_t \xrightarrow[t\to\infty]{(law)} C_1.$$

A new proof of the Proposition above based on Vakeroudis (2011) [28] is presented bellow.

Proof. From equation (ii) of Proposition 2.7 we note that:

$$\frac{1}{\log b}\theta_{T_b^\delta} \overset{(law)}{=} \frac{C_{a(b)}}{\log b} \xrightarrow[b\to\infty]{(law)} C_1$$

which for $b = \sqrt{t}$ becomes:

$$\frac{2}{\log t}\theta_{T_{\sqrt{t}}^\delta} \xrightarrow[t\to\infty]{(law)} C_1.$$

On the other hand, by applying Williams' "pinching method", we note that:

$$\frac{1}{\log t}\left(\theta_{T_{\sqrt{t}}^\delta} - \theta_t\right) \xrightarrow[t\to\infty]{(law)} 0,$$

since $Z_u = x_0 + Z_u^{(0)}$. Moreover, if we change variables $u = tv$ and by the use of scaling property of BM, we get:

$$\theta_{T_{\sqrt{t}}^\delta} - \theta_t \equiv \mathrm{Im}\left(\int_t^{T_{\sqrt{t}}^\delta} \frac{dZ_u}{Z_u}\right) \xrightarrow[t\to\infty]{(law)} \mathrm{Im}\left(\int_1^{T_1^\delta} \frac{sZ_v^{(0)}}{Z_v^{(0)}}\right).$$

□

Chapter 3

Windings of planar Brownian Motion in Financial Mathematics

Financial mathematics is a dynamic and interdisciplinary field that combines mathematics, statistics, and economic theory to tackle the intricate challenges of modern finance. Through the development and refinement of sophisticated mathematical models, such as option pricing models, portfolio optimization techniques, and risk management frameworks, financial mathematics plays a pivotal role in shaping investment strategies, asset pricing, and risk assessment. In this thesis we are going to limit and deepen to studying options.

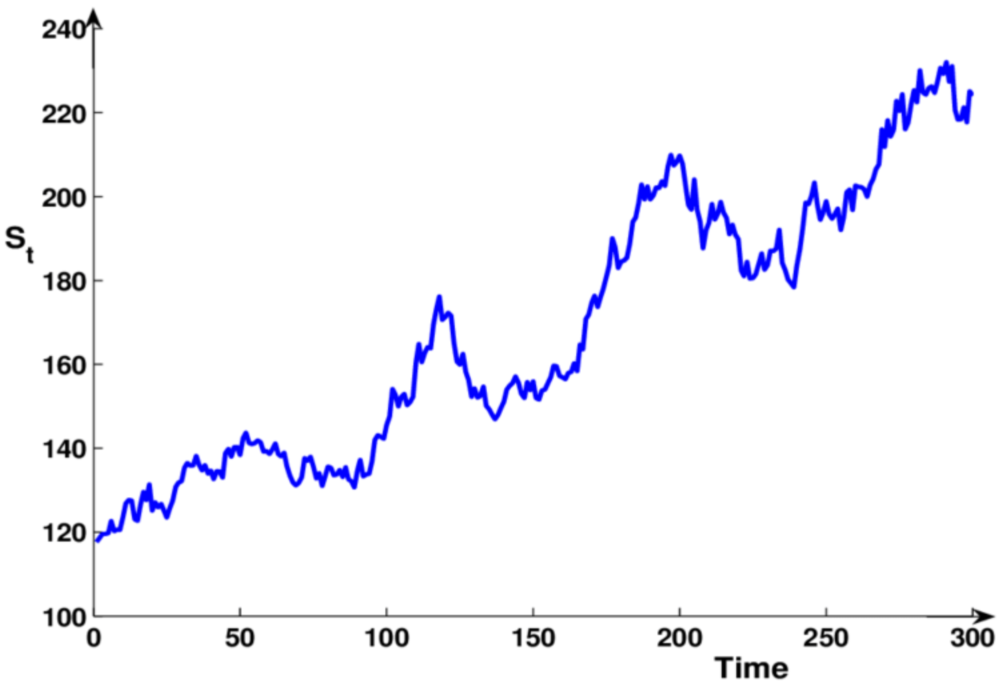

FIGURE 3.1: The evolution of a stock price in a Brownian motion.[24]

3.1 Options

Option is a financial derivative contract with two types. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A long call can be used to speculate on the price of the underlying rising, since it has unlimited upside potential but the maximum loss is the premium (price) paid for the option.
On the other hand, a *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. A long put is a short position in the underlying security, since the put gains value as the underlying's price falls. Protective puts can be purchased as a sort of insurance, providing a price floor for investors to hedge their positions.
Buying an option offers the right, but not the obligation, to purchase or sell the underlying asset. The price in the contract is known as the *exercise price* or *strike price* and the date in the contract is known as the *expiration date* or *maturity*[15].
In the field of finance, the *style* or *family* of an option is determined by the class to which it belongs, typically determined by the specific exercise dates allowed for the option. The majority of options fall into two main categories: **European** or **American** options. These options, along with others that have a similar calculation for their payoff, are commonly known as "vanilla options". On the contrary, options with a different method of calculating the payoff are classified as "exotic options".
The basic difference between American and European option is associated with the date when the option is exercised:

- A **European option** can only be exercised at the expiration date of the contract. This means that the option holder can only exercise their right to buy or sell the underlying asset on the predetermined expiration date.
- An **American option** provide more flexibility as they can be exercised at any time before the expiration date. This means that the option holder has the freedom to exercise the option and buy or sell the underlying asset at any point during the life of the contract.

When the payoff occurs its the same for both and is calculated as:

- $\max\{(S-K),0\}$, for a call option
- $\max\{(K-S,0)\}$, for a put option

where K is the strike price and S is the spot price of the underlying asset.
As regards exotic options, their payoff may be base on multiple underlying assets, have non-linear relationships, or incorporate complex pricing formulas. There are various types of exotic options, including but not limited to:

- **Barrier Options**: These options come with predefined price barriers. The option's payoff depends on whether the underlying asset price breaches or stays within the specified barrier levels during the option's life.

- **Asian Options**: Asian options have payoffs based on the average price of the underlying asset over a specific period. They aim to reduce the impact of short-term price fluctuations.

- **Binary Options**: Binary options have a fixed payout if a specific condition is met at expiration. They typically pay either a predetermined amount or nothing at all.

- **Lookback Options**: Lookback options have payoffs based on the extreme (maximum or minimum) price of the underlying asset over the option's life. They allow the holder to benefit from favorable price movements.

Exotic options can pose significant challenges in terms of valuation and hedging strategies. Their complex features and non-standard payoffs require advanced mathematical models and specialized techniques for pricing and risk management. Thus, the content of the previous chapters can be applied to the Asian options which we will examine in detail below.

3.2 Asian Options

Windings of 2-dimensional processes, and especially of planar Brownian motion have several applications in Financial Mathematics for instance, where the exponential functionals of Brownian motion are of special interest. A fundamental example is the pricing of Asian options.
During the 1980s, Mark Standish was employed at Bankers Trust in London, where he focused on fixed income derivatives and proprietary arbitrage trading. David Spaughton, on the other hand, worked as a systems analyst in the financial markets at Bankers Trust starting from 1984, when the Bank of England granted licenses to banks for foreign exchange options in the London market. While conducting business in Tokyo in 1987, Standish and Spaughton collaborated and devised the first pricing formula for options linked to the average price of crude oil, which went on to be used commercially. They called this exotic option the Asian option because they were in Asia.
More specifically, an **Asian option** is a type of financial derivative contract that derives its value based on the average price of the underlying asset over a specified period of time. It is also known as an average option.
In a typical option contract, the payoff is determined by the price of the underlying asset at a specific point in time (e.g., at the expiration date). However, with an Asian option, the payoff is based on the average price of the underlying asset over a predetermined period. Asian options

are often used by investors and traders who are interested in taking positions based on the average price of an asset over time, rather than the price at a specific point in time. They can be useful in hedging against the volatility of the underlying asset or for investors who have a long-term perspective on the market. They can be utilized for various purposes, including risk management, speculation, and taking advantage of potential trends or patterns in the average price movement. Also, the higher the number of observations, the lower the volatility and the lower the option price. Because of this lower volatility, asian options are generally cheaper than vanilla options (with the same characteristics) and also are characterized as fair options both for the seller and the buyer.[23]

It's important to note that the valuation and pricing of Asian options can be more complex compared to standard options due to the averaging feature. Various mathematical models and numerical methods are used to calculate the fair value of Asian options, taking into account factors such as the average calculation method, volatility, interest rates, and the specific terms of the option contract.

The payout of an Asian call option is given by:

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t ds \exp(\beta_s + \nu s) - K\right)^+\right],$$

where $(\beta_u : u \geq 0)$ is real BM, $\nu \in \mathbb{R}$ and the non-negative number K is the strike price.

For a put Asian option we have:

$$\mathbb{E}\left[\left(K - \frac{1}{t}\int_0^t ds \exp(\beta_s + \nu s)\right)^+\right],$$

and the same study stands for this type of option. In our study we will examine the call option. It is easy to show (for more details, see Geman and Yor(1993)[11]) that the computation of this expectation simplifies to the computation of

$$\mathbb{E}\left[\left(\int_0^t ds \exp(\beta_s + \nu s) - K\right)^+\right],$$

which implies studying the quantity

$$\mathbb{E}\left[\int_0^t ds \exp(\beta_s + \nu s)\right].$$

In particular, in Yor's article (1993) [34] can be found more information about the distribution of the exponential functional

$$A_t^{(\nu)} := \int_0^t ds \exp(\beta_s + \nu s)$$

taken up to a random time T_λ which follows the exponential distribution with parameter $\lambda > 0$ and is independent from β.
We suppose here for simplicity that $\nu = 0$ and applying the scaling property of BM (1.3) we can invoke Bougerol's identity (1.4) and as a result we can study exponential functionals in terms of planar Brownian motion, i.e. taken up to an independent random time, that is

$$\int_0^{T_c^\gamma} \exp(2\beta_u) du$$

where $(\gamma_s, s \geq 0)$ is another real Brownian motion independent from β and the exit time T^γ is given by (2.8).

3.3 Asian options and exponential functionals of Brownian Motion

In this subsection, we want to characterize the distribution of

$$A_t^Z = \int_0^t \exp(2\beta_u) du,$$

in order to compute $\mathbb{E}\left[(\frac{1}{t} A_t^Z - K)^+ \right]$. The aim here is to study the distribution of A_t.

Proposition 3.1. *The following convergence in law holds*

$$\frac{1}{t} \log A_{t^2}^Z \xrightarrow[t\to\infty]{(law)} 2|\beta|_{T_1^\gamma} \overset{(law)}{=} 2|C|_1,$$

where C_1 is a standard Cauchy random variable.

For the proof we apply Williams'"pinching" method ([21, 31]), which was introduced in Subsection 2.6.

Proof. First, we notice that

$$\log \left(\frac{A_{T_t^\gamma}^Z}{A_{t^2}^Z} \right) = \log \left(\frac{\int_0^{T_t^\gamma} \exp(2\beta_u) du}{\int_0^{t^2} \exp(2\beta_u du)} \right),$$

which is a random variable that exists (and which seems to be of no other interest here). Renormalising by t (which is obviously an increasing function in $t > 0$), by WPM we get

$$\frac{1}{t} \left(\log A_{T_t^\gamma}^Z - \log A_{t^2}^Z \right) = \frac{1}{t} \log \left(\frac{A_{T_t^\gamma}^Z}{A_{t^2}^Z} \right) \xrightarrow[t\to\infty]{(law)} 0.$$

As a result, studying asymptotically $t^{-1}\log A^Z_{T^\gamma_t}$, as $t\to\infty$, would imply similar results for $t^{-1}\lg A^Z_{t^2}$, according to Jedidi and Vakeroudis (2018)[17].

Remark 3.2. Here we will shortly prove that

$$T^\gamma_t \overset{(law)}{=} t^2 T^\gamma_1, \quad \text{for fixed } t.$$

We have

$$T^\gamma_t = \inf\{s\geq 0 : \gamma_s = t\} = \inf\{s\geq 0 : \frac{1}{t}\gamma_s = 1\}$$

applying scaling property (1.3)

$$\overset{(law)}{=} \inf\{s\geq 0 : \gamma_{\frac{s}{t^2}} = 1\} \overset{u=\frac{s}{t^2}}{=} \inf\{t^2 u : \gamma_u = 1\}$$

$$= t^2 \inf\{u : \gamma_u = 1\} = t^2 T^\gamma_1$$

which implies that $T^\gamma_t \overset{(law)}{=} t^2 T^\gamma_1$, for fixed t.

Following some generalisations of Vakeroudis and Yor (2011)[30], applying the scaling property of BM (1.3) and making a change of variables, we have that

$$A^Z_{T^\gamma_t} = \int_0^{T^\gamma_t} e^{2\beta_u} du \overset{(law)}{=} \int_0^{t^2 T^\gamma_1} e^{2\beta_u} du \tag{3.1}$$

We make now the following change of variables $u = t^2 v$ and we get:

$$u = t^2 v \to du = t^2 dv$$

and the edges

$$u = t^2 T^\gamma_1 \to v = T^\gamma_1$$

$$u = 0 \to v = 0.$$

Then (3.1) becomes

$$A^Z_{T^\gamma_t} = \int_0^{T^\gamma_1} e^{2\beta_{t^2 v}} t^2 dv \overset{\substack{\text{(law)}\\ \text{scaling}}}{=} t^2 \int_0^{T^\gamma_1} e^{2\sqrt{t^2}\beta_v} dv$$

$$= t^2 \int_0^{T^\gamma_1} e^{2t\beta_v} dv = t^2 \int_0^{T^\gamma_1} e^{2t\beta_u} du$$

so that, for all $t\geq 0$, we have that

$$\frac{1}{t}\log A^Z_{T^\gamma_t} \overset{(law)}{=} \frac{1}{t}\log\left(t^2 \int_0^{T^\gamma_1} e^{2t\beta_u} du\right) = \frac{2\log t}{t} + \log\left(\int_0^{T^\gamma_1} e^{2t\beta_u} du\right)^{1/t}.$$

Using the fact that the $p-$norm converges to the ∞-norm when $p \to \infty$ (see Berberian[3]), the latter converges for $t \to \infty$ towards $2\sup_{0\le u\le T_1^\gamma}\beta_u$. From Theorem 1.12 (Reflection Principle), and more specifically from 1.1—Remark 1.13— we have

$$\sup_{0\le u\le T_1^\gamma}\beta_u \overset{(law)}{=} |\beta|_{T_1^\gamma} \overset{(law)}{=} |C_1|$$

and we deduce that

$$\frac{1}{t}\log A^Z_{T_t^\gamma} \xrightarrow[t\to\infty]{(law)} 2|C_1|.$$

The results for A_t^Z follows immediately from Remark 3.2. □

Chapter 4

Simulations

In the world of finance, options are powerful instruments that allow investors to hedge risks, speculate on price movements, and diversify their portfolios. Asian options, in particular, have gained significant attention due to their unique payoff structure and potential for mitigating volatility. To address this challenge, this chapter delves into the concept of using Brownian motion and simulations to calculate Asian options based on real data. More precisely, we will now proceed to simulations (involving real data), in order to verify that the previously presented novel approach for the pricing of Asian Options through the windings of planar BM (see e.g. Section 3) yelds correct results.

The motivation behind this chapter stems from the need to develop robust pricing models for Asian options that capture the complexities of real-world data. While closed-form solutions exist for certain types of options, Asian options require a more nuanced approach due to their dependence on average prices over a period. Simulating real data using Brownian motion offers a promising solution by providing a more realistic representation of asset price dynamics.

By simulating asset price movements using the principles of Brownian motion, we aim to capture the inherent randomness and volatility observed in financial markets. Additionally, we will discuss practical considerations, such as parameter estimation, model validation, and the role of historical data in the simulation process.

The simulations of this chapter are based on real data coming from the site investing.com. This is a site which gives you the chance to download or examine historical data from a various of commodities or stocks.

In this thesis were used the global prices of:

1. Natural Gas
2. Gold
3. Apple's stocks

In this part, it is important to refer that an option contract is usually signed for short time periods such us 2, 3 or 6 months.

4.1 Use of Brownian motion to price Asian options

The aim of this Section is to invoke BM in order to compute the fair Asian call option price as it was presented in Section 3.2. The equation for the payout of an Asian call option is:

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t ds\exp(\beta_s+\nu s)-K\right)^+\right],$$

where $(\beta_u : u \geq 0)$ is real BM, $\nu \in \mathbb{R}$ and the non-negative number K is the strike price. Here, as in the previous sections, for simplicity, we assume that $\nu = 0$. As a result our model will be:

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t ds\exp(\beta_s)-K\right)^+\right] \tag{4.1}$$

The aim here is to simulate some Brownian motions, which will take the place of β_s in (4.1) and then to calculate the price of the Asian option for various values of the Strike price.
As it is shown in the Section 3.2, in order to calculate the price of the Asian option we have to estimate the mean of the exponential values of (β_s) minus the strike price K. The idea for the simulation is that the real data can be modelled as a BM, so we want to determine the price of the Asian option by producing a great number of random walks which will give us a reliable guess when they are imported to our model (4.1).
The simulation of the BM is based in its construction by involving a Random Walk and then taking an infinitesimally small time step. The input of the Algorithm 1 is the underlying asset price (S), the Strike price (K), the time to maturity (t), the number of time steps (N) and the number of simulations we want to carry out. Then the option value is returned .The algorithm can be found in the Appendix 1 where the reader can find more details.

The values of N (number of steps),S (initial price) and t (time to maturity) can be set according to the situation of the data we have. More specifically, depending on the period of the option we want to investigate, we can change the number of steps or the time increment in order to make the simulation more accurate.
For the estimation of the Asian option we have to calculate the arithmetic mean of the exponential values of BM (β_s), as they were simulated in steps 1-9 of Algorithm 1, minus the strike price K, as it referred in the model (4.1). It is calculated in the steps 11-16 of the Algorithm 1. We subtract the Strike price K because we want to determine the call option, otherwise we would add it.
Then we will compare the price found with the actual price of each commodity or stock, and

the accuracy of the Option depending on the period we decided to study.
Moreover, because we discuss about purchases and sales, it is obvious that the price of a stock or a commodity can change during a day because demand and availability change in a short period of time. As a result, the data we examine contain the date, the highest price achieved on this date, the lowest price, the initial price at which a financial instrument starts trading at the beginning of the day and the price, which is the last price of the day. For our study we used only the date and the price of the financial item we examine. Furthermore, we have to underline the fact that dates do not include weekend and international holidays. For example, for the Natural Gas we have 66 observations for a period of 3 months.

4.2 Natural Gas

Natural gas is a fossil fuel primarily composed of methane (CH4) along with smaller amounts of other hydrocarbon gases like ethane, propane, and butane. It is formed from the remains of ancient plants and animals that were buried and subjected to heat and pressure over millions of years. It is commonly located in underground reservoirs or found alongside oil deposits.
As regards economy, natural gas is a commodity of the Energy category. It is generally measured in one of two ways – by volume or by energy content. Because the energy content (or heating value) of natural gas can vary as the quality of each source vary, the most accurate way of measuring the ultimate value of gas is to use units based on energy content. In our data, we have the price in USD per 1 MMBtu (MMBtu is a million of Btu). Btu stands for British Thermal Units. To convert volumetric units to units based on energy content, you must know the heating value of that specific gas. The heating value tells you how many MMBtus are contained in a thousand of cubic feet (Mcf).
The Contract size of NG is 10,000 MMBtu. It refers to the amount or quantity of an underlying security represented by a derivatives contract. The size is often standarlized and as a result makes the trading process more streamlined and clearly sets out the traders' obligations.
Now, let construct the coming scenario. It is 01/01/2019 and an oil company wants to sign an Asian call option contract with a natural gas company that will expire in 3 months. Our aim is to calculate the fair price of the Asian Option that the oil company has to pay in order to acquire the right to buy gas from the natural gas company.
After 1000 simulations for various number of strike prices (K) based on Algorithm 1 we have the following table with the prices of simulated prices of Asian option. Also, we have use the prices we know from the data in order to calculate the actual price of the Asian option. The data we used are depicted in the figure 4.1 which will help us to understand the results.

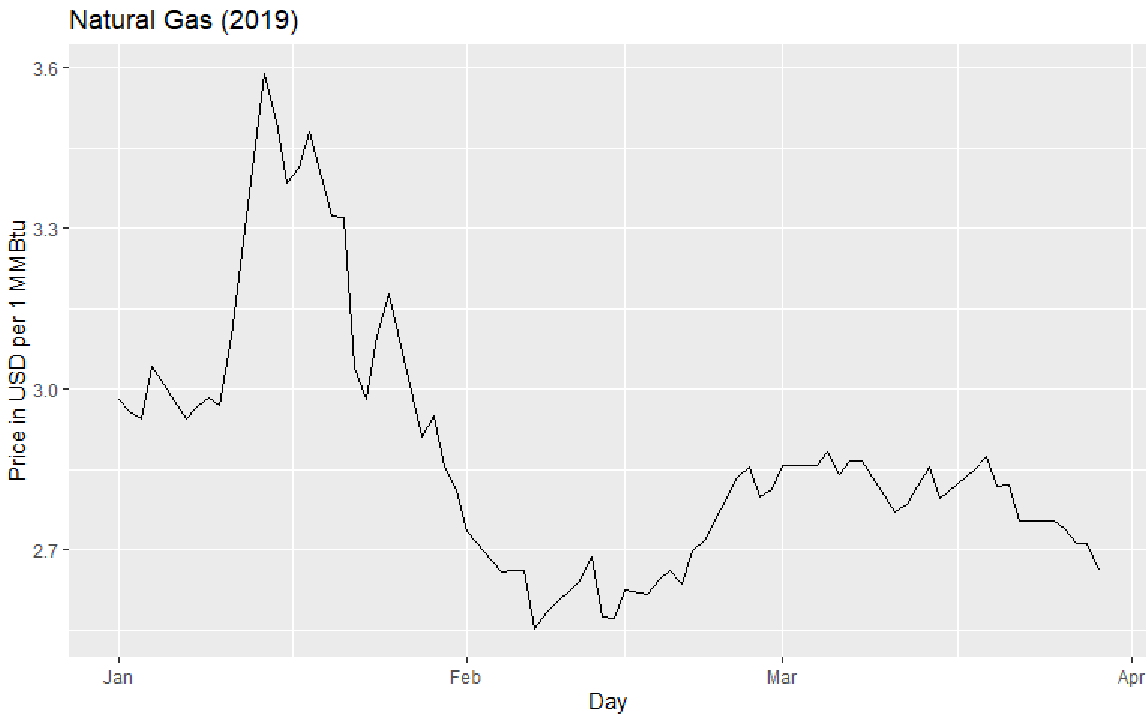

FIGURE 4.1: Prices of natural gas for 3 months (2019)

We replaced the simulated values of the BM with the real prices we had from the data. For 3 months we had 66 observations so $t \leftarrow 66$ and the underlying asset price was 2.98 USD in December 1, 2019 ($S \leftarrow 2.98$). Last, but not least, the number of time steps for each simulation was 100000, $N \leftarrow 100000$.

Strike price (K)	Simulated	Actual	Difference
2.6	0.38698	0.27671	0.11027
2.7	0.27854	0.17671	0.10183
2.8	0.18249	0.07671	0.10578
2.9	0.10261	0	0.10261
3	0.03894	0	0.03894
3.1	0.00987	0	0.00987
3.2	0.00163	0	0.00163
3.3	0.00013	0	0.00013

TABLE 4.1: Simulated and actual Asian options for natural gas.

First of all, someone who has studied Table 4.1 can easily understand that the option price (Simulated or Actual) tends to zero as the Strike price (K) becomes greater. That is obvious from the equation 4.1 as we get the $\max\{\frac{1}{t}\int_0^t ds \exp(\beta_s) - K, 0\}$. Also, in combination with the fact that the price of NG decreases after January 15, 2019 (see figure 4.1), the reader realizes

that the mean will also fall off. Therefore the average price is much lower than the strike so the buyer will not pay for the right he wants to acquire. That observation is cleaner in the Actual price of the option but the difference from the Simulated is really small.
Finally, we can see from the table 4.1 that when the Strike price is low, it is more risky for the buyer, in our example the oil company, to pay more for the option. On the other hand, the reader can see that the difference is small between the Simulated and the Actual price and according to the volume he wants to buy, he can decide whether it is worth to risk or not.

4.3 Gold

In this subsection, we are going to study an agreement concerning gold. It is a chemical symbol with the symbol Au. Gold plays a significant role in the global economy and has been valued for centuries for its beauty and scarcity. More specifically, it has traditionally been considered a store of value and a safe haven asset. During times of economic uncertainty or market volatility, investors often turn to gold as a hedge against inflation and currency fluctuations. It has historically retained its value over the long term. Furthermore, gold is traded on various exchanges worldwide, and investors can buy and sell gold in the form of bars, coins, or through financial instruments such as futures contracts.
In economics, gold is a commodity in the group of precious metals and is measured by troy ounce and by gram. In our data we have the price of gold per 100 troy ounces. Troy weight is a system of units of mass that originated in 15th-century England [12]. The troy weight units are the grain, the pennyweight (24 grains), the troy ounce (20 pennyweights), and the troy pound (12 troy ounces). One troy ounce (oz t) equals exactly 31.1034768 grams. According to the international trade association LBMA and ISO 4217 the troy ounce is internationally recognized and standardized as 31.1035 grams. The contract size is 100 tr. ounces.
Our simulation is taking place for the calculation of a call Asian option which has 6 months duration. The date we want to sign the contract is July 1, 2022 and the price of Gold is 1801.5 USD per tr. ounce. To implement Algorithm 1 we have $S \leftarrow 1801.5$. From the real data we have 132 observations for 6 months so $t \leftarrow 132$. The algorithm ran 1000 times $(numSimulations \leftarrow 1000)$ and with 100000 steps $(N \leftarrow 100000)$ per simulation. The real data are depicted in the figure bellow.

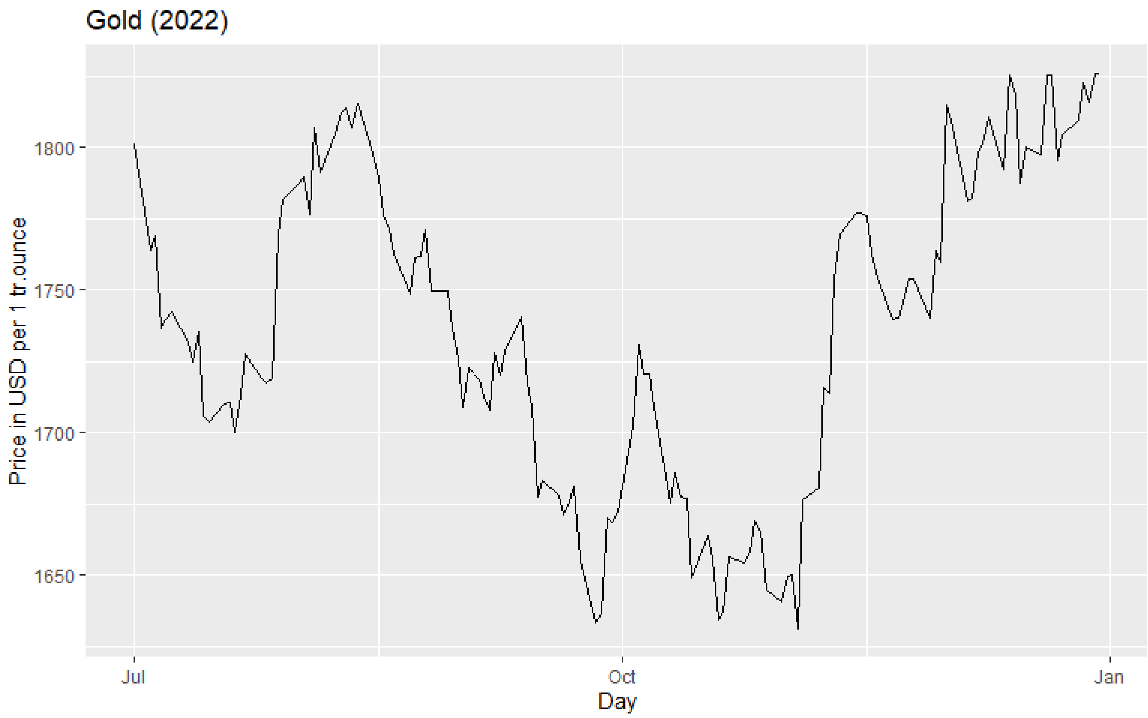

FIGURE 4.2: Prices of gold for 6 months (2022)

The table bellow shows the values resulting from the simulations for various strike prices (K) compared to the actual prices as calculated by substituting the simulated BM values for the true gold prices for the period we want to study.

Strike price (K)	Simulated	Actual	Difference
1650	151.4917	86.6178	64.8739
1675	126.4740	61.6178	64.8562
1700	101.4742	36.6178	64.8564
1725	76.4907	11.6178	64.8729
1750	51.4816	0	51.4816
1775	26.4744	0	26.4744
1800	1.4875	0	1.4875
1825	0	0	0

TABLE 4.2: Simulated and actual Asian options for gold

As it was mentioned in the subsection 4.2 the reader can understand that the price of the Asian option tends to zero as the Strike price gets higher. Here, we can realise that the simulated prices get to 0 with less rate than the actual. Moreover, the difference between simulated and actual prices is stable for the first 4 Strike prices. That happens because the price reduction for the first semester follows similar path with the increase of the second semester as it can be detected in figure 4.2. Finally, we have to mention that the difference which is observed is due to the increase in the duration of the simulations and the high price of gold.

4.4 Apple Inc. stock

Apple Inc. is an American multinational technology company founded on April 1, 1976, by Steve Jobs, Steve Wozniak, and Ronald Wayne. The company is renowned for its consumer electronics and software products, such as Mac computers, iPhones, iPads, iPods, and the Apple Watch. Their mobile operating system, iOS, and desktop operating system, macOS, are widely used globally. Apple's products are known for their sleek design and user-friendly interfaces. The company has a history of innovation and has played a pivotal role in shaping the technology industry. The launch of the iPhone in 2007 revolutionized the smartphone market, while the iPad introduced a new era for tablet computing in 2010.

As regards profits, Apple is the world's largest technology company with 394.3 billion USD in 2022 revenue [1]. It is also the world's biggest company by market capitalization, as of March 2023 according to companiesmarketcap.com. Market capitalization refers to the total dollar market value of a company's outstanding shares of stock and it is used to determine a company's size instead of sales or total asset figures. The stock price of Apple has experienced significant growth over the years, driven by strong sales and investor confidence in the company's ability to innovate. Because of that we studied the Asian options related to the share price of Apple (AAPL).

Here, we studied the value of Asian options for Apple's stock. Our beginning date is August 2, 2021 when hypothetically a contract is about to sign. In this day, the price of a stock is $145.52\$$ which is the initial value for our BM we want to simulate via Algorithm 1. The expiration date is September 30, 2021 -two months later- and form real data we have 43 observations, so in Algorithm 1 $t \leftarrow 43$. Similarly, with the previous simulations the algorithm ran 1000 times ($numSimulations \leftarrow 1000$) and with 100000 steps ($N \leftarrow 100000$) per simulation.

After that, the real data, which are depicted in figure 4.3, took the place of the simulated BM in order to calculate the actual price of the Asian option.

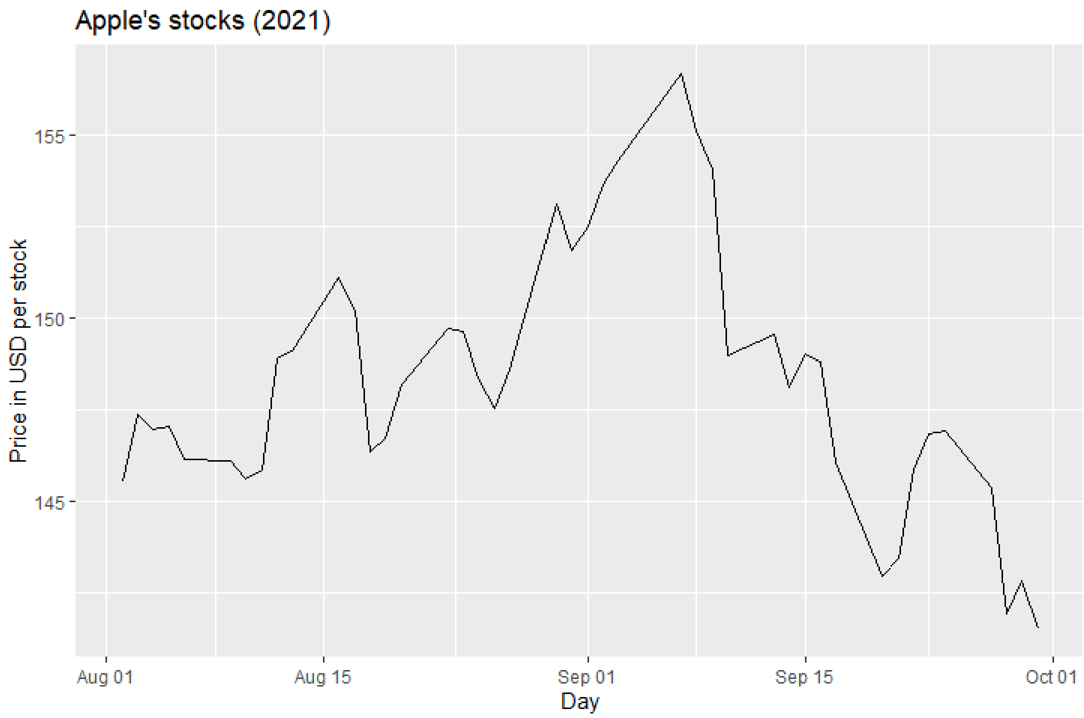

FIGURE 4.3: Prices of Apple Inc. stock for 2 months (2021)

The results are shown in the table bellow which includes the simulated and the real prices of the Asian options for various values of the Strike price (K).

Strike price (K)	Simulated	Actual	Difference
140	5.521720	8.240465	-2.718745
142	3.515956	6.240465	-2.724509
144	1.516011	4.240465	-2.724454
146	0	2.240465	-2.240465
148	0	0.240465	-0.240465
150	0	0	0
152	0	0	0

TABLE 4.3: Simulated and actual Asian options for Apple Inc. stock

In this example of simulations, we face the fact that the price coming from the simulated data is lower than the value of option resulting from the real data. Moreover, it is noticed that the difference is between simulated and actual values is almost constant for Strike price less or equal to 146$. That can maybe explained from the fact that most prices of the stock are bellow of 150$. For that obvious reason, the reader can understand why the simulated and the real value of the Asian options tend to zero for strike prices over 148$.

4.5 Conclusions

In conclusion, this study has explored and analyzed windings of planar Brownian motion in-depth, shedding light on various aspects and providing valuable insights into the subject matter. Through a comprehensive literature review and rigorous data analysis, we have addressed the research questions and objectives set forth at the beginning of this investigation. The findings presented in this study have not only confirmed existing theories but have also revealed novel perspectives for more investigation. Moreover, the methodologies employed in this research have proven to be effective in capturing the intricate nuances of the phenomenon under study, enhancing the validity and reliability of the results obtained.

First of all, as regards the simulations, we have noticed the smallest difference between the values of the Asian option of simulated and actual data in the example with the Natural Gas. All of them, for each Strike price, were about $0.1$$ or less. That happened because the evolution of the price, excluding a pick in the middle of January, was fluctuated between $2.55 - 2.85$$. That means it had a small variation and seems like the simulation responded successfully.

On the other hand, the greatest difference was observed in the simulation which regards gold. This situation can be explained by two reasons. Firstly, because it was selected to be a 6 month period for maturity and as a result uncertainty increases. In addition, the price of the gold for the period it was examined had great discrepancies. For instance, the highest price in this period was about 1825 USD, when the lowest was 1625 USD. In this point we have to underline the fact that although the difference in the price of gold for a 6 month period was about 200 $, the difference between simulated and actual price of the option reached only 64$.

Finally, regarding the simulation for Apple Inc. stocks, we found that the simulated price of an Asian option was less than actual one for about 2.7$. It is logical that some times the values of the simulated prices are higher than the actual and the opposite because we just want to study the randomness and the uncertainty. In addition, it is obvious for the reader to understand that as the strike price gets higher such the value of the option tends to zero. This implies from the equation (4.1).

While this study has made substantial progress in addressing the research questions, it is essential to acknowledge some limitations that may have impacted the results. For example, if the number of steps per simulation was greater than 100,000 or the simulations were more than 1,000, we might have produced better and more qualitative results, but this would increase the running time of the algorithm and would require more specialized hardware. In addition, the selection of the strike price happened voluntarily and not through an elegant way for tuning this parameter. In light of these limitations, future research in this area should focus on overcoming these challenges and delving deeper into certain aspects that warrant further exploration.

In conclusion, this study has made valuable contributions to the field of Stochastic Analysis, advancing our knowledge and providing practical insights that can be utilized by various stakeholders. It is our hope that this work will serve as a catalyst for further research and foster positive developments in economics. Ultimately, we believe that the findings presented here will contribute to a more informed and enlightened society, paving the way for progress and positive change.

Appendix A

Appendix

A.1 Some theoretical observations

A useful tool for our investigations is the following lemma:

Lemma A.1 (Borel-Cantelli Lemma). *If* $\sum_n \mathbb{P}(A_n)$ *converges, then* $\mathbb{P}(\limsup_n A_n) = 0$.

Proof. From $\limsup_n A_n \subset_{k=m}^{\infty} A_k$ follows $\mathbb{P}(\limsup_n A_n) \leq \mathbb{P}(\bigcup_{k=m}^{\infty} A_k) \leq \sum_{k=m}^{\infty} \mathbb{P}(A_k)$, and this sum tends to 0 as $m \to \infty$, if $\sum_n \mathbb{P}(A_n)$ converges. □

Theorem A.2. 1. *For each sequence* $\{A_n\}$

$$\mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n)$$

$$\leq \limsup_n \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_n A_n)$$

2. *If* $A_n \to A$ *then* $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

For the proof see Billingsley, "Probability and Measure"[4].
Combining the Lemma and the theorem above, $\mathbb{P}(A_n) \to 0$ implies that $\mathbb{P}(\liminf_n A_n) = 0$.

Here is a proof for the existence theorem (1.2) based on Mörters and Peres[22].
Brownian motion is constructed as a uniform limit of continuous functions in order to ensure that it has continuous paths. We need only to construct a standard BM $\{B(t) : t \geq 0\}$, as $X(t) = x + B(t)$ is a BM started from point x. The proof exploits properties of Gaussian random vectors, which are the higher-dimensional analogue of the normal distribution.

Definition A.3. A random vector $(X_1, ...X_n))$ is called a Gaussian random vector if there exists an $n \times m$ matrix A, and an n-dimensional vector b such that $X^T = AY + b$, where Y is an m-dimensional vector with independent standard normal entries.

Proof. We will construct the BM in $[0,1]$ as a random variable with values in $\mathbb{C}[0,1]$ of continuous functions on $[0.1]$ and the extension to $[0,\infty)$ will be immediate.
The point is to construct the right joint distribution of Brownian motion step by step on the finite sets

$$\mathcal{D}_n := \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\}$$

of dyadic points. We then interpolate the values on $\mathcal{D}_n$ linearly and check that the uniform limit of these continuous functions exists and is a Brownian motion.
To do this let

$$\mathcal{D} := \bigcup_{n=0}^{\infty} \mathcal{D}_n$$

the set of parties in $[0,1]$ and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent, standard normally distributed random variables can be defined.
We set $B(0) := 0$ and $B(1) := Z_1$. For each $n \in \mathbb{N}$ we define the random variables $B(d), d \in \mathcal{D}_n$ such that:

1. for all $r < s < t$ in $\mathcal{D}_n$ the random variable $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$, and is independent of $B(s) - B(r)$,
2. the vectors $(B(d) : d \in \mathcal{D}_n)$ and $(Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n)$ are independent.

Note that we have already done this for $\mathcal{D}_0 = \{0, 1\}$. Proceeding inductively we may assume that we have succeeded in doing it for some $n-1$. Now we determine $B(d)$ for $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ by

$$B(d) = \frac{B(d-2^{-n}) + B(d+2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}} \tag{A.1}$$

Note that the first sum is the linear interpolation of the values of B at the neighbouring points of d in $\mathcal{D}_{n-1}$. Therefore $B(d)$ is independent of $(Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n)$ and the second property is fulfilled.
Furthermore, as $\frac{1}{2}[B(d+2^{-n}) - B(d-2^{-n})]$ depends only on $(Z_t : t \in \mathcal{D}_{n-1})$, it is independent of $Z_d/2^{(n+1)/2}$. Both terms are normally distributed with mean zero and variance $2^{-(n+1)}$. Hence their sum $B(d) - B(d-2^{-n})$ and their difference $B(d+2^{-n}) - B(d)$ are independent and normally distributed with mean zero and variance 2^{-n} as it is known by the Corollary 3.4 of Mörters and Peres[22] which is writen bellow:

Theorem A.4 (Corollary 3.4). *Let X_1 and X_2 be be independent and normally distributed with expectation 0 and variance $\sigma^2 > 0$. Then $X_1 + X_2$ and $X_1 - X_2$ are independent and normally distributed with expectation 0 and variance $2\sigma^2$.*

Actually, all increments $B(d) - B(d-2^{-n})$, for $d \in \mathcal{D}_n \setminus \{0\}$, are independent. To see this it suffices to show that they are pairwise independent, as the vector of these increments is

Gaussian. In the previous paragraph we show that the pairs $B(d) - B(d - 2^{-n})$ and $B(d + 2^{-n}) - B(d)$ with $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ are independent. The other possibility is that the increments are over intervals separated by some $d \in \mathcal{D}_{n-1}$. Choose $d \in \mathcal{D}_j$ with this propert and minimal j, so that the two intervals are contained in $[d - 2^{-j}, d]$, respectively $[d, d + 2^{-j}]$. By induction the increments over these two intervals of length 2^{-j} are independent, and the increments over the intervals of length 2^{-n} are constructed from the independent increments $B(d) - B(d - 2^{-j})$, respectively $B(d + 2^{-j}) - B(d)$, using a disjoint set of variables $(Z_t : t \in \mathcal{D}_n)$. Hence they are independent and this implies the first property, and completes the induction step.

Having thus chosen the values of the process on all dyadic points, we interpolate between them. Formally, define the continuous functions $F_n : [0, 1] \to \mathbb{R}, n \in \mathbb{N}$ as

$$F_0(t) = \left\{ \begin{array}{ll} Z_1, & t = 1 \\ 0, & t = 0 \\ \text{linear}, & t \in (0, 1) \end{array} \right\}$$

and for $n \geq 0$

$$F_n(t) = \left\{ \begin{array}{ll} 2^{-(n+1)/2} Z_t, & t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1} \\ 0, & t \in \mathcal{D}_{n-1} \\ \text{linear} & \text{between consecutive points in } \mathcal{D}_n \end{array} \right\}$$

For $d \in \mathcal{D}_n$ we have

$$B(d) = \sum_{k=0}^{n} F_k(d) = \sum_{k=0}^{\infty} F_k(d) \tag{A.2}$$

This can be seen by induction. It holds for $n = 0$. Suppose that it holds for $n - 1$. Let $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Since for $0 \leq k \leq n - 1$ the function F_k is linear on $[d - 2^{-n}, d + 2^{-n}]$, we get

$$\sum_{k=0}^{n-1} F_k(d) = \sum_{k=1}^{n-1} \frac{F_k(d - 2^{-n}) + F_k(d + 2^{-n})}{2} = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2}$$

Since $F_n(d) = 2^{-(n+1)/2} Z_d$ which gives A.2. Now we will use a useful estimate for standard normal random variables, which is quite precise for large x.

Lemma A.5. *Suppose X is a standard normally distributed. Then, for all $x > 0$,*

$$\frac{x}{x^2 + 1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \mathbb{P}(X > x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

By the definition of Z_d and by LemmaA.5, for $c > 0$ and large n we have

$$\mathbb{P}(|Z_d| \geq c\sqrt{n}) \leq \frac{1}{c\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-c^2 n/2} \leq \exp\left(\frac{-c^2 n}{2}\right)$$

so that the series

$$\sum_{n=0}^{\infty} \mathbb{P}(\text{there exists } d \in \mathcal{D}_n \text{ with } |Z_d| \geq c\sqrt{n}) \leq \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} \mathbb{P}(|Z_d| \geq c\sqrt{n})$$

$$\leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2 n}{2}\right)$$

We use the ratio test in order to test the convergence.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2^{n+1}+1)\exp\left(\frac{-c^2(n+1)}{2}\right)}{(2^n+1)\exp\left(\frac{-c^2 n}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{(2^{n+1}+1)}{2^n+1} \exp\left(\frac{-c^2 n - c^2 + c^2 n}{2}\right) = \lim_{n\to\infty} \frac{2^{n+1}+1}{2^n+1} \exp\left(\frac{-c^2}{2}\right) =$$

$$\lim_{n\to\infty} \frac{2+\frac{1}{2^n}}{1+\frac{1}{2^n}} \exp\left(\frac{-c^2}{2}\right) = 2\exp\left(\frac{-c^2}{2}\right)$$

The series converges if the limit is less than 1, so

$$2\exp\left(\frac{-c^2}{2}\right) < 1 \Leftrightarrow \exp\left(\frac{-c^2}{2}\right) < \frac{1}{2}$$

$$\frac{-c^2}{2} < \ln\frac{1}{2} \Leftrightarrow -c^2 < 2\ln\frac{1}{2} \Leftrightarrow c^2 > -2\ln\frac{1}{2} = 2\ln 2 \Leftrightarrow$$

$$c < -\sqrt{2\ln 2} \quad \text{or} \quad c > \sqrt{2\ln 2}$$

converges as soon as $c > \sqrt{2\ln 2}$ because we have assume $c > 0$. Fix such a c. By the Borel-Cantelli lemma A.1 [9] there exists a random (but almost surely finite) N such that for all $n \geq N$ and $d \in \mathcal{D}_n$ we have $|Z_d| < c\sqrt{n}$. Hence, for every $n \geq N$,

$$||F_n||_\infty < c\sqrt{n}2^{-n/2}.$$

This upper bound means that, almost surely, the series

$$B(t) = \sum_{n=0}^{\infty} F_n(t)$$

converges uniformly on $[0,1]$. The continuous limit is denoted as $\{B(t) : t \in [0,1]\}$.
It remains to check that the increments of this process have the right marginal distributions. This follows directly from the properties of B on the dense set $\mathcal{D} \subset [0,1]$ and the continuity of the paths. Indeed, suppose that $t_1 < t_2 < ... < t_n$ are in $[0,1]$. We find $t_{1,k} < t_{2,k} < ... < t_{n,k}$

in $\mathcal{D}$ with $\lim_{k\to\infty} t_{i,k} - t_i$ and infer from the continuity of B that, for $1 \le i \le n-1$,

$$B(t_{i+1}) - B(t_i) = \lim_{k\to\infty} (t_{i+1,k}) - B(t_{i,k}).$$

As

$$\lim_{k\to\infty} \mathbb{E}[B(t_{i,k}) - B(t_{i,k})] = 0$$

and

$$\lim_{k\to\infty} \operatorname{Cov}(B(t_{i+1,k}) - B(t_{i,k}), B(t_{j+1,k}) - B(t_{j,k}))$$
$$= \lim_{k\to\infty} \mathbf{1}_{i=j}(t_{i+1,l} - t_{i,k}) = \mathbf{1}_{i=j}(t_{i+1} - t_i)$$

the increments $B(t_{i+1}) - B(t_i)$ are independent Gaussian random variables with mean 0 and variance $t_{i+1} - t_i$, as required by the below propositionA.6.

Proposition A.6 (Proposition 3.7 of [9]). *Suppose* $\{X_n : n \in \mathbb{N}\}$ *is a sequence of Gaussian random vectors and* $\lim_n X_n = X$*, almost surely. If* $b := \lim_{n\to\infty} \mathbb{E}X_n$ *and* $C := \lim_{n\to\infty} \operatorname{Cov}X_n$ *exists, then* X *is Gaussian with mean* b *and covariance matrix* C*.*

We have constructed a continuous process $B : [0,1] \to \mathbb{R}$ with the same marginal distributions as Brownian motion. Take a sequence $B_1, B_2, \ldots$ of independent $\mathbb{C}[0,1]$-valued random variables with the distribution of this process, and define $\{B(t) : t \ge 0\}$ by gluing together the parts, more precisely by

$$B(t) = B_{\lfloor t \rfloor}{}^{1}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1), \text{ for all } \ t \ge 0$$

This defines a continuous random function $B : [0,\infty) \to \mathbb{R}$ and one can see easily from what we have shown so far that the requirements of a standard Brownian motion are fulfilled. □

[1] $\lfloor t \rfloor$: the largest integer smaller or equal to t

A.2 Some codes

Simple Brownian Motion Code in R for Two Dimensions based on the idea of Chapter 20 of Schilling and Partzsch (2012)[26]

```
N=10000
xdis = rnorm(N, 0 ,1);
ydis = rnorm(N, 0 ,1);
xdis = cumsum(xdis);
ydis = cumsum(ydis);
plot(xdis, ydis, type='l', main ='Brownian Motion in Two Dimension',
xlab='x displacement', ylab = 'y displacement')
```

The graph of five sampled Brownian motions was constructed with the following code:

```
n = 1000
t = 100
No.Ex = 5
steps = seq(0,t,length=n+1)
BM = replicate(No.Ex, -bm ¡- c(0,cumsum(rnorm(n,0,sqrt(t/n)))))

matplot(BM, type = "l", lty = 1,ylab="Bt",xlab="time")
```

Here is the code used for the simulation of an Asian Option using Brownian motion.
Some Brownian motions were simulated in order to take the place of β_s in (4.1). They were simulated by a creation of a Random Walk and taking an infinitesimally small time step. Then used in order to calculate the value of the Asian option. The actual value of the Asian option was estimated by changing the path of the simulated prices with the real data we mined from the site investing.com.
The algorithm follows in pseudo-code.

Algorithm 1 Simulation of an Asian Option using Brownian Motion

Require: Underlying asset price (S), Strike price (K), Time to maturity (t), Number of time steps (N), Number of simulations (numSimulations)

Ensure: Option value

function SIMULATEASIANOPTION(S, K, t, N, numSimulations)
    $dt \leftarrow t/N$ ▷ Time step
    $optionPayoffs \leftarrow$ empty array ▷ Array to store option payoffs
    for sim in 1 to numSimulations **do**
        $pricePath \leftarrow$ empty array ▷ Array to store price path
        $pricePath[1] \leftarrow S$ ▷ Initial price
        for i in 2 to N **do**
            $dW \leftarrow rnorm(1, 0, dt)$random number from $N(0, dt)$ ▷ Increment of Brownian motion
            $pricePath[i] \leftarrow pricePath[i-1] + dW$
        end for
        $averagePrice \leftarrow$ mean($pricePath$) ▷ Average price over the specified period
        if $averagePrice > K$ **then**
            $optionPayoffs$[sim] $\leftarrow averagePrice - K$ ▷ Option payoff
        else
            $optionPayoffs$[sim] $\leftarrow 0$ ▷ Option payoff
        end if
    end for
    $optionValue \leftarrow$ mean($optionPayoffs$) ▷ Option value
    return $optionValue$
end function

The actual code used for the simulations in R is presented bellow:

```
##############################################################
### Function to simulate Asian option using Brownian motion ###
##############################################################
simulateAsianOption ¡- function(S, K, t, N, numSimulations) –
    dt ¡- t/N  # Time step
    optionPayoffs ¡- numeric(numSimulations)  # Array to store option payoffs
    for (sim in 1:numSimulations) –
        pricePath ¡- numeric(N+1)  # Array to store price path
        pricePath[1] ¡- S  # Initial price
        for (i in 2:N) –
            dW ¡- rnorm(1,0,dt)  # Increment of Brownian motion
            pricePath[i] ¡- pricePath[i-1]+ dW


    averagePrice ¡- mean(pricePath)  # Average price over the specified period
```

```
    optionPayoffs[sim] ¡- ifelse(averagePrice ¿ K, averagePrice - K, 0)  # Option payoff


  optionValue ¡- mean(optionPayoffs)  # Option value
  return(optionValue)




##############################
## import data for Nat. Gas ##
##############################
library(readr)
NatGas¡- readcsv("Natural Gas Futures Historical Data.csv")
View(NatGas)
head(NatGas)
#prepare date in appropriate format
md¡-c()
for(i in 1:length(NatGas$Date))-
  md¡-c(md,substr(NatGas$Date[i],1,6))

y¡-c()
for(i in 1:length(NatGas$Date))-
  y¡-c(y,substr(NatGas$Date[i],9,10))

days¡-paste(md,y)
days¡-gsub(" ", "", paste(md,y))
day¡-as.Date(days,format="%m/%d/%y")

#new data frame with the days and the prices
day¡-day[1:66]
price¡-NatGas$Price[1:66]
NatGas1¡-data.frame(day,price)

#plot prices
library(ggplot2)
ggplot(NatGas1,aes(x=day,y=price))+
  geomline()+
  labs(x="Day",y="Price in USD per 1 MMBtu",title="Natural Gas (2019)")


##############################
#### For Natural Gas ########
####    3 months    ########
############################
set.seed(123)
S ¡-NatGas1$price[1]   # Underlying asset price
K ¡- c(2.6,2.7,2.8,2.9,3,3.1,3.2,3.3)  # Strike price
t ¡- 66  # Time to maturity
N ¡- 100000  # Number of time steps
numSimulations ¡- 1000  # Number of simulations
#Options for each K
optionValue ¡-c()
for(i in 1:length(K))-
  optionValue¡-c(optionValue,simulateAsianOption(S, K[i], t, N, numSimulations))
```

```
print(paste(”For K=”,K,”Asian option value:”, round(optionValue,6)))


#Actual options for each K
averageActPrice¡-mean(NatGas1$price)
optionActPayoffs¡-c()
for(i in 1:length(K))-
optionActPayoffs¡-c(optionActPayoffs,ifelse(averageActPrice ¿ K[i], averageActPrice- K[i], 0))

print(paste(”For K=”,K,”actual Asian option:”,optionActPayoffs))

###########################
## import data for Gold ##
##########################
Gold¡- readcsv(”Gold Futures Historical Data.csv”)
View(Gold)
head(Gold)
#prepare the date in appropriate format
md¡-c()
for(i in 1:length(Gold$Date))-
  md¡-c(md,substr(Gold$Date[i],1,6))

y¡-c()
for(i in 1:length(Gold$Date))-
  y¡-c(y,substr(Gold$Date[i],9,10))

days¡-paste(md,y)
days¡-gsub(” ”, ””, paste(md,y))
day¡-as.Date(days,format=”%m/%d/%y”)
day¡-day[1:132]
#new data frame with the days and the prices
price¡-Gold$Price[1:132]
Gold1¡-data.frame(day,price)
#plot prices
library(ggplot2)
ggplot(Gold1,aes(x=day,y=price))+
  geomline()+
  labs(x=”Day”,y=”Price in USD per 1 tr.ounce”,title=”Gold (2022)”)


####################
####   For Gold  ####
####  6 months   ####
###################
set.seed(123)
S ¡-Gold$Price[1]   # Underlying asset price
K ¡- c(1650,1675,1700,1725,1750,1775,1800,1825)  # Strike price
t ¡- 132  # Time to maturity
N ¡- 100000  # Number of time steps
numSimulations ¡- 1000  # Number of simulations
#Options for each K
optionValue ¡-c()
for(i in 1:length(K))-
  optionValue¡-c(optionValue,simulateAsianOption(S, K[i], t, N, numSimulations))
```

```
print(paste("For K=",K,"Asian option value:", round(optionValue,4)))


#Actual options for each K
averageActPrice¡-mean(Gold1$price)
optionActPayoffs¡-c()
for(i in 1:length(K))-
  optionActPayoffs¡-c(optionActPayoffs,ifelse(averageActPrice ¿ K[i], averageActPrice- K[i], 0))

print(paste("For K=",K,"actual Asian option:",optionActPayoffs))


####################################
## import data for APPLE's stocks ##
####################################

Appl¡- readcsv("AAPL Historical Data.csv")
View(Appl)
head(Appl)
#prepare date in appropriate format
md¡-c()
for(i in 1:length(Appl$Date))-
  md¡-c(md,substr(Appl$Date[i],1,6))

y¡-c()
for(i in 1:length(Appl$Date))-
  y¡-c(y,substr(Appl$Date[i],9,10))

days¡-paste(md,y)
days¡-gsub(" ", "", paste(md,y))
day¡-as.Date(days,format="%m/%d/%y")

#new data frame with the days and the prices
day¡-day[64:106]
price¡-Appl$Price[64:106]
Appl1¡-data.frame(day,price)

#plot prices
ggplot(Appl1,aes(x=day,y=price))+
  geomline()+
  labs(x="Day",y="Price in USD per stock",title="Apple's stocks (2021)")


##############################
#### For Apple's stocks  #####
####    1 month         #####
############################
set.seed(123)
S ¡-Appl1$price[1]   # Underlying asset price
K ¡- c(140,142,144,146,148,150,152)  # Strike price
t ¡- 43  # Time to maturity
N ¡- 100000 # Number of time steps
numSimulations ¡- 1000  # Number of simulations
#Options for each K
optionValue ¡-c()
```

```
for(i in 1:length(K))-
  optionValue¡-c(optionValue,simulateAsianOption(S, K[i], t, N, numSimulations))

print(paste(”For K=”,K,”Asian option value:”, round(optionValue,6)))


#Actual options for each K
averageActPrice¡-mean(Appl1$price)
optionActPayoffs¡-c()
for(i in 1:length(K))-
  optionActPayoffs¡-c(optionActPayoffs,ifelse(averageActPrice ¿ K[i], averageActPrice- K[i], 0))

print(paste(”For K=”,K,”actual Asian option:”,optionActPayoffs))
```

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