Warming up RBC (McKay) Analytics

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1 Problem

Assume that preferences are given by:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$

while

$$Y_t = Z_t K_{t-1}^{\alpha}\bar{L}^{1-\alpha} \tag{2}$$

Production Function

$$Y_t = C_t + I_t \tag{3}$$

Resource Constraint

$$K_t = (1-\delta)K_{t-1} + I_t \tag{4}$$

Law of Motion of Capital

$$\log Z_t = \rho \log Z_{t-1} + \epsilon_t \tag{5}$$

Productivity Shock process

Combining (3) and (4) we get:

$$Y_t = K_t - (1-\delta)K_{t-1} + C_t \tag{6}$$

So, now the problem states as [1]:

$$L = \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\frac{C_t^{1-\gamma}}{1-\gamma} + \lambda_t[Z_t K_{t-1}^{\alpha}\bar{L}^{1-\alpha} + (1-\delta)K_{t-1} - K_t - C_t]\right\} \tag{7}$$

[1] we could omit L (as we do next) since labor supply is exogenously given

<u>FOC</u>

C_t:

$$\beta^t(C_t^{-\gamma} - \lambda_t) = 0 \to \lambda_t = C_t^{-\gamma} \tag{8}$$

K_t:

$$\begin{aligned} \beta^t(-\lambda_t) + \beta^{t+1}\mathbb{E}\{\lambda_{t+1}(\alpha Z_{t+1}K_t^{\alpha-1}\bar{L}^{1-\alpha} + 1 - \delta)\} = 0 \\ \lambda_t = \beta\mathbb{E}\{\lambda_{t+1}(\alpha Z_{t+1}K_t^{\alpha-1}\bar{L}^{\alpha-1} + 1 - \delta)\} \end{aligned} \tag{9}$$

Combining (8) and (9) we get:

$$C_t^{-\gamma} = \beta\mathbb{E}\{C_{t+1}^{-\gamma}(\alpha Z_{t+1}K_t^{\alpha-1}\bar{L}^{1-\alpha} + 1 - \delta)\} \tag{10}$$

Denoting the gross real interest rate as $R_{t+1} = \alpha Z_{t+1}K_t^{\alpha-1}\bar{L}^{1-\alpha} + 1 - \delta$, we get:

$$\boxed{C_t^{-\gamma} = \beta\mathbb{E}[R_{t+1}C_{t+1}^{-\gamma}]} \tag{11}$$

Euler for Consumption

Equilibrium

We have a system of 5 equations on 5 unknowns $\{\mathrm{C}_t,\ R_t,\ K_t,\ Y_t,\ Z_t\ \}$

$$C_t^{-\gamma} = \beta \mathbb{E}(R_{t+1} C_{t+1}^{-\gamma}) \tag{12}$$

$$R_t = \alpha Z_t K_{t-1}^{\alpha-1} \bar{L}^{1-\alpha} + 1 - \delta \tag{13}$$

$$K_t = (1-\delta) K_{t-1} + Y_t - C_t \tag{14}$$

$$Y_t = Z_t K_{t-1}^{\alpha} \bar{L}^{1-\alpha} \tag{15}$$

$$\log Z_t = \rho \log Z_{t-1} + \epsilon_t \tag{16}$$

Steady State

Equation (12) becomes:

$$C^{-\gamma} = \beta R C^{-\gamma} \tag{17}$$
$$R = \frac{1}{\beta}$$

Equation (13) becomes:

$$R = \alpha Z K^{\alpha-1} \bar{L}^{1-\alpha} + (1-\delta) \tag{18}$$
$$K = \left[\frac{R-1+\delta}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

Equation (14) becomes:

$$C = Y - \delta K \tag{19}$$

Equation (15) becomes:

$$Y = K^{\alpha} \tag{20}$$

Equation (16) becomes:

$$Z = 1 \tag{21}$$

Log-Linearization

Let's start by loglinearizng equation (12):

$$C_t^{-\gamma} = \beta \mathbb{E}[R_{t+1} C_{t+1}^{-\gamma}]$$

Taking logs on both sides

$$-\gamma log(C_t) = log(\beta) + log(R_{t+1}) - \gamma log(C_{t+1})$$

Taking a First Order Taylor Expansion (FOTE):

$$-\gamma log(C) - \gamma \tfrac{1}{C}(C_t - C) =$$
$$log(\beta) + log(R) + \tfrac{1}{R}(R_{t+1} - R) - \gamma log(C) - \tfrac{\gamma}{C}(C_{t+1} - C)$$

We know that in SS:

$$log(R) = -log(\beta)$$

so, we have that:

$$\frac{-\gamma}{C}(C_t - C) = \frac{1}{R}(R_{t+1} - R) - \frac{\gamma}{C}(C_{t+1} - C)$$
$$-\gamma \tilde{c_t} = \tilde{r}_{t+1} - \gamma \tilde{c}_{t+1}$$
$$\boxed{\tilde{c}_{t+1} - \tilde{c_t} = \frac{1}{\gamma}\tilde{r}_{t+1}} \quad (22)$$

This is the log-linearized Euler for Consumption.

Now let's log-linearize (15)

$$Y_t = Z_t K_{t-1}^{\alpha} \bar{L}^{1-\alpha}$$

Following the same steps as before we get:

$$log(Y) + \tfrac{1}{Y}(Y_t - Y) = log(Z) + \tfrac{1}{Z}(Z_t - Z) + \alpha log(K) + \tfrac{\alpha}{K}(K_{t-1} - K)$$
$$\tfrac{1}{Y}(Y_t - Y) = \tfrac{1}{Z}(Z_t - Z) + \tfrac{\alpha}{K}(K_{t-1} - K)$$

$$\boxed{\tilde{y_t} = \zeta_t + \alpha \tilde{k}_{t-1}} \quad (23)$$

This is the log-linearized Production Function.

Note that ζ_t is the $log(Z_t)$ since $\log(\frac{Z_t}{Z}) = log(Z_t)$ because $log(Z) = 0$

Let's log-linearize (14)

$$K_t = (1-\delta)K_{t-1} + Y_t - C_t$$

$$log(K) + \frac{1}{K}(K_t - K) = log[(1-\delta)K + Y - C] + \frac{1(1-\delta)}{(1-\delta)K+Y-C}(K_{t-1} - K) + \frac{1}{(1-\delta)K+Y-C}(Y_t - Y) - \frac{1}{(1-\delta)K+Y-C}(C_t - C)$$

$$\frac{1}{K}(K_t - K) = \frac{1-\delta}{(1-\delta)K+Y-C}(K_{t-1} - K) + \frac{1}{(1-\delta)K+Y-C}(Y_t - Y) - \frac{1}{(1-\delta)K+Y-C}(C_t - C)$$

$$\frac{1}{K}(K_t - K) = \frac{1-\delta}{K}(K_{t-1} - K) + \frac{1}{K}(Y_t - Y) - \frac{1}{K}(C_t - C)$$

where we have used that in SS: $K = (1-\delta)K + Y - C$

Now, we multiply and divide by Y,C where appropriate

$$\tilde{k}_t = (1-\delta)\tilde{k}_{t-1} + \frac{Y}{K}\frac{(Y_t - Y)}{Y} - \frac{C}{K}\frac{(C_t - C)}{C}$$

$$\tilde{k}_t = (1-\delta)\tilde{k}_{t-1} + \frac{Y}{K}\tilde{y}_t - \frac{C}{K}\tilde{c}_t$$

If you check the equlibrium conditions: $\frac{Y}{K} = \frac{R-1+\delta}{\alpha}$ and $\frac{C}{K} = \frac{R-1+\delta(1-\alpha)}{\alpha}$

so, we get

$$\boxed{\tilde{k}_t = (1-\delta)\tilde{k}_{t-1} + \left(\frac{R-1+\delta}{\alpha}\right)\tilde{y}_t - \left(\frac{R-1+\delta(1-\alpha)}{\alpha}\right)\tilde{c}_t} \tag{24}$$

This is the log-linearized capital accumulation equation.

Let's loglinearize (13)

$$R_t = \alpha Z_t K_{t-1}^{\alpha-1} + 1 - \delta \longrightarrow R_t = \alpha \tfrac{Y_t}{K_{t-1}} + 1 - \delta$$

$$log(R) + \tfrac{1}{R}(R_t - R) = log\left(\alpha \tfrac{Y}{K} + 1 - \delta\right) + \tfrac{\frac{\alpha}{K}}{\alpha\frac{Y}{K}+1-\delta}(Y_t - Y) - \tfrac{\alpha(\frac{Y}{K^2})}{\alpha\frac{Y}{K}+1-\delta}(K_{t-1} - K)$$

We multiply and divide by Y,K when appropriate

$$log(R) + \tfrac{1}{R}(R_t - R) = log\left(\alpha \tfrac{Y}{K} + 1 - \delta\right) + \tfrac{\frac{\alpha}{K}}{\alpha\frac{Y}{K}+1-\delta} Y \tfrac{(Y_t - Y)}{Y} - \tfrac{\alpha(\frac{Y}{K^2})}{\alpha\frac{Y}{K}+1-\delta} K \tfrac{(K_{t-1} - K)}{K}$$

$$log(R) + \tfrac{1}{R}(R_t - R) = log\left(\alpha \tfrac{Y}{K} + 1 - \delta\right) + \tfrac{\alpha\frac{Y}{K}}{\alpha\frac{Y}{K}+1-\delta}\tilde{y}_t - \tfrac{\alpha\frac{Y}{K}}{\alpha\frac{Y}{K}+1-\delta}\tilde{k}_{t-1}$$

From the SS conditions we can see that

$$\alpha \tfrac{Y}{K} = R - 1 + \delta$$

Also, in SS

$$log(R) = \left(\alpha \tfrac{Y}{K} + 1 - \delta\right)$$

so

$$\tfrac{1}{R}(R_t - R) = \tfrac{R-1+\delta}{R}(\tilde{y}_t - \tilde{k}_{t-1})$$

$$\boxed{\tilde{r_t} = \frac{R - 1 + \delta}{R}(\tilde{y}_t - \tilde{k}_{t-1})} \tag{25}$$

Now we are just left with the shock process, which takes the form:

$$\boxed{\zeta_t = \rho_z \zeta_{t-1} + \epsilon_t} \tag{26}$$

Let's rewrite the log-linearized equations for clarity:

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\gamma}\tilde{r}_{t+1} \tag{27}$$

$$\tilde{y_t} = \zeta_t + \alpha\tilde{k}_{t-1} \tag{28}$$

$$\tilde{k}_t = (1-\delta)\tilde{k}_{t-1} + \left(\frac{R-1+\delta}{\alpha}\right)\tilde{y_t} - \left(\frac{R-1+\delta(1-\alpha)}{\alpha}\right)\tilde{c}_t \tag{29}$$

$$\tilde{r_t} = \frac{R-1+\delta}{R}(\tilde{y_t} - \tilde{k}_{t-1}) \tag{30}$$

$$\zeta_t = \rho_z\zeta_{t-1} + \epsilon_t \tag{31}$$

System Reduction

Substituting (30) into (27)

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\tilde{y}_{t+1} - \frac{1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\tilde{k}_t \tag{32}$$

So, we have eliminated the equation for the real interest rate (30)

Now, substituting (28) into (32), and (28) into (29)

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\zeta_{t+1} + \frac{\alpha-1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\tilde{k}_t \tag{33}$$

and

$$\tilde{k}_t = R\tilde{k}_{t-1} + \left(\frac{R-1+\delta}{\alpha}\right)\zeta_t - \left(\frac{R-1+\delta(1-\alpha}{\alpha}\right)\tilde{c}_t \tag{34}$$

Thus, we have eliminated the equation for the production function (28)

In the (unlikely) event that we have no mistakes, our final system consists of:

$$\boxed{\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\zeta_{t+1} + \frac{\alpha-1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\tilde{k}_t} \tag{35}$$

$$\boxed{\tilde{k}_t = R\tilde{k}_{t-1} + \left(\frac{R-1+\delta}{\alpha}\right)\zeta_t - \left(\frac{R-1+\delta(1-\alpha)}{\alpha}\right)\tilde{c}_t} \tag{36}$$

$$\boxed{\zeta_t = \rho_z\zeta_{t-1} + \epsilon_t} \tag{37}$$

Bringing everything on the LHS:

$$\boxed{\tilde{c}_{t+1} - \tilde{c}_t - \frac{1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\zeta_{t+1} - \frac{\alpha-1}{\gamma}\left(\frac{R-1+\delta}{R}\right)\tilde{k}_t = 0} \tag{38}$$

$$\boxed{\tilde{k}_t - R\tilde{k}_{t-1} - \left(\frac{R-1+\delta}{\alpha}\right)\zeta_t + \left(\frac{R-1+\delta(1-\alpha)}{\alpha}\right)\tilde{c}_t = 0} \tag{39}$$

$$\boxed{\zeta_t - \rho_z\zeta_{t-1} + \epsilon_t = 0} \tag{40}$$

Now, we are asked to express the system in the form

$$A\mathbb{E}X_{t+1} + BX_t + CX_{t-1} + \varepsilon\epsilon_t = 0$$

So, we have that:

$$A = \begin{bmatrix} 1 & 0 & -\left(\frac{1}{\gamma}\right)\left(\frac{R-1+\delta}{R}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{41}$$

$$X_{t+1} = \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \zeta_{t+1} \end{bmatrix} \tag{42}$$

$$B = \begin{bmatrix} -1 & -\left(\frac{\alpha-1}{\gamma}\right)\left(\frac{R-1+\delta}{R}\right) & 0 \\ \left(\frac{R-1+\delta(1-\alpha)}{\alpha}\right) & 1 & -\left(\frac{R-1+\delta}{\alpha}\right) \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

$$X_t = \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \zeta_t \end{bmatrix} \tag{44}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -R & 0 \\ 0 & 0 & -\rho_z \end{bmatrix} \tag{45}$$

$$X_{t-1} = \begin{bmatrix} \tilde{c}_{t-1} \\ \tilde{k}_{t-1} \\ \zeta_{t-1} \end{bmatrix} \tag{46}$$

$$\varepsilon = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{47}$$

ATTENTION: capital is an (endogenous) state variable so its coefficients should not appear in the A matrix, which contains only the coefficients of the forward-looking (control variables). Instead they will appear in X_{t-1} which contains the coefficients of the backward-looking (state) variables (here capital and prod.).