Model Elliptical copula

Data structure

Assume we get data set with size n

$$\begin{bmatrix} x_{11}, x_{12}, x_{13} \\ x_{21}, x_{22}, x_{23} \\ \cdots\cdots \\ x_{n1}, x_{n2}, x_{n3} \end{bmatrix}$$

After probability integral transformation

$$\begin{bmatrix} u_{11}, u_{12}, u_{13} \\ u_{21}, u_{22}, u_{23} \\ \cdots\cdots \\ u_{n1}, u_{n2}, u_{n3} \end{bmatrix}$$

The likelihood is

$$\begin{aligned} L(\alpha, \beta, \mathbf{w}, A; \mathbf{U}) &= \prod_{i=1}^{n} c(u_{i1}, u_{i2}) \\ &= \prod_{i=1}^{n} |A|^{-1} (r_i^2)^{(1-p)/2} h((r_i^2)^{1/2}) \\ &= \prod_{i=1}^{n} |A|^{-1} (r_i)^{(1-p)} h(r_i) \\ &= \prod_{i=1}^{n} |A|^{-1} (r_i)^{(1-p)} \prod_{j=1}^{k} (w_j f(r_i | \alpha_j, \beta_j)^{z_{ij}}) \end{aligned}$$

where

$$
\begin{aligned}
r_i^2 &= (\mathbf{x})'\Omega^{-1}(\mathbf{x}) \\
&= (Q_{EC}(u_{i1}), Q_{EC}(u_{i2}))\Omega^{-1}(Q_{EC}(u_{i1}), Q_{EC}(u_{i2}))'
\end{aligned}
$$

$$
X = (Q_{EC}(u_{i1}), Q_{EC}(u_{i2}))
$$

In this likelihood, **both h function and quantile function Q_{EC} contain parameters α, β as well as weights w**.

Conditional likelihood

- $L(\alpha, \beta, w|\Omega, U, X, R)$
- $L(\Omega|\alpha, \beta, w, U, X, R)$

These two likelihood above need the whole pdf involved.

α's are not mutually independent. Every time we update α_j, R has to be updated to make sure next operations are conditioned on the current estimate of α_j. This requirement also applies on the estimation of β as well as weights.

Log likelihood

$$
\begin{aligned}
&L(\alpha, \beta, \mathbf{w}, A; \mathbf{U}) \\
&= \prod_{i=1}^{n} c(u_{i1}, u_{i2}) \\
&= \prod_{i=1}^{n} |A|^{-1} (r_i^2)^{(1-p)/2} h((r_i^2)^{1/2}) \\
&= \prod_{i=1}^{n} |A|^{-1} (r_i)^{(1-p)} h(r_i) \\
&= \prod_{i=1}^{n} \{|A|^{-1} (r_i)^{(1-p)} \prod_{j=1}^{k} (w_j f(r_i|\alpha_j, \beta_j))^{z_{ij}}\} \\
&= -nlog|A| + (1-p) \sum_{i=1}^{n} log(r_i) + \sum_{i=1}^{n} log\Big(\prod_{j=1}^{k} (w_j f(r_i|\alpha_j, \beta_j))^{z_{ij}}\Big) \\
&= -nlog|A| + (1-p) \sum_{i=1}^{n} log(r_i) + \sum_{i=1}^{n} \sum_{j=1}^{k} log\Big(w_j f(r_i|\alpha_j, \beta_j)\Big)^{z_{ij}} \\
&= -nlog|A| + (1-p) \sum_{i=1}^{n} log(r_i) + \sum_{i=1}^{n} \sum_{j=1}^{k} z_{ij} \Big(log(w_j) + log\big(f(r_i|\alpha_j, \beta_j)\big)\Big) \\
&= -nlog|A| + (1-p) \sum_{i=1}^{n} log(r_i) + \sum_{i=1}^{n} \sum_{j=1}^{k} \Big(z_{ij} log(w_j) + z_{ij} log\big(f(r_i|\alpha_j, \beta_j)\big)\Big) \\
&= -nlog|A| + (1-p) \sum_{i=1}^{n} log(r_i) + \sum_{i=1}^{n} \sum_{j=1}^{k} \Big(z_{ij} log(w_j)\Big) + \\
&\sum_{i=1}^{n} \sum_{j=1}^{k} \Big(z_{ij} log\big(f(r_i|\alpha_j, \beta_j)\big)\Big)
\end{aligned}
$$

Priors

For α's

$$\frac{c}{\alpha_j^{c+1}}$$

For β's

$$\frac{b^a}{\Gamma(a)}\beta_j^{a-1}e^{-b\beta_j}$$

For v's

$$\gamma(1-v_j)^{\gamma-1}$$

For γ

$$\gamma^{\eta_1-1}e^{-\eta_2\gamma}$$

For ρ's

$$\mathbf{1}_{(-1\leq\rho\leq1)}$$

Log priors

$$\begin{aligned}
&\sum_{j=1}^{k}\big(log(c)-(c+1)log(\alpha_j)\big)\\
&+\sum_{j=1}^{k}\big(alog(b)+(a-1)log(\beta_j)-b\beta_j-logamma(a)\big)\\
&+\sum_{j=1}^{k-1}\big(log(\gamma)+(\gamma-1)log(1-v_j)\big)\\
&+(\eta_1-1)log(\gamma)-\eta_2\gamma
\end{aligned}$$

Conditional log posteriors for MCMC process

α from different components are not mutually independent any more. and β's and weights are also like this. This cause sampling much more difficult and time consuming.

- for α_j

$$(1-p)\sum_{i=1}^{n}log(r_i)+\sum_{i=1}^{n}\sum_{j=1}^{k}\Big(z_{ij}log\big(f(r_i|\alpha_j,\beta_j)\big)\Big)+\sum_{j=1}^{k}(c+1)log(\alpha_j)$$

- for β_j

$$(1-p)\sum_{i=1}^{n}log(r_i)+\sum_{i=1}^{n}\sum_{j=1}^{k}\Big(z_{ij}log\big(f(r_i|\alpha_j,\beta_j)\big)\Big)+ \\ \sum_{j=1}^{k}\big((a-1)log(\beta_j)-b\beta_j\big)$$

- for v_j

$$(1-p)\sum_{i=1}^{n}log(r_i)+\sum_{i=1}^{n}\sum_{j=1}^{k}\Big(z_{ij}log(w_j)\Big)+ \\ \sum_{i=1}^{n}\sum_{j=1}^{k}\Big(z_{ij}log\big(f(r_i|\alpha_j,\beta_j)\big)\Big)+\sum_{j=1}^{k-1}\big((\gamma-1)log(1-v_j)\big)$$

- for θ

$$-nlog|A|+(1-p)\sum_{i=1}^{n}log(r_i)+\sum_{i=1}^{n}\sum_{j=1}^{k}\Big(z_{ij}log\big(f(r_i|\alpha_j,\beta_j)\big)\Big)$$

Using Metropolis or HMC to draw samples from these log posterior pdfs.

Conditional posterior distributions for latent variable

- for latent variables Z

$$P(\mathbf{Z}|\alpha, \beta, \mathbf{V}, \gamma, \rho, \mathbf{R}) = \frac{P(\mathbf{R}|\alpha, \beta, \mathbf{V}, \gamma, \rho, \mathbf{Z}) \times P(\mathbf{Z})}{P(\mathbf{R}|\alpha, \beta, \mathbf{V}, \gamma, \rho)}$$

$$\begin{aligned} r_i^2 &= (\mathbf{x})'\Omega^{-1}(\mathbf{x}) \\ &= (Q_{EC}(u_{i1}), Q_{EC}(u_{i2}))\Omega^{-1}(Q_{EC}(u_{i1}), Q_{EC}(u_{i2}))' \\ &= Q_{EC}(\mathbf{U}) \end{aligned}$$

In this case, Z_i's are independent, of course, since they only depend on the corresponding X_i, thus the joint posterior distribution of $\mathbf{Z}$ is actually a set of independent univariate multinomial distributions.

In our case, the latent variable is defined as

$$Z_{ji} = 1, \text{if } r_i \text{is from component } j$$

we will have $30 \times n$ matrix

$$\begin{bmatrix} 101 \cdots 0 \\ 010 \cdots 0 \\ \vdots \ddots \vdots \\ 000 \cdots 1 \end{bmatrix}$$

which only has a single 1 in its each column. Each column is a multinomial distribution.

$$P(Z_{ji} = 1|\alpha, \beta, \mathbf{V}, \gamma, \rho, \mathbf{X}) = \frac{P(\mathbf{X}|\alpha, \beta, \mathbf{V}, \gamma, \rho, Z_{ji} = 1) \times P(Z_{ji} = 1)}{P(\mathbf{X}|\alpha, \beta, \mathbf{V}, \gamma, \rho)}$$