FEEG6005 Assignment 1

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"I am aware of the requirements of good academic practice and the potential penalties for any breaches. I confirm that this assignment is all my own work"

QUESTION 1

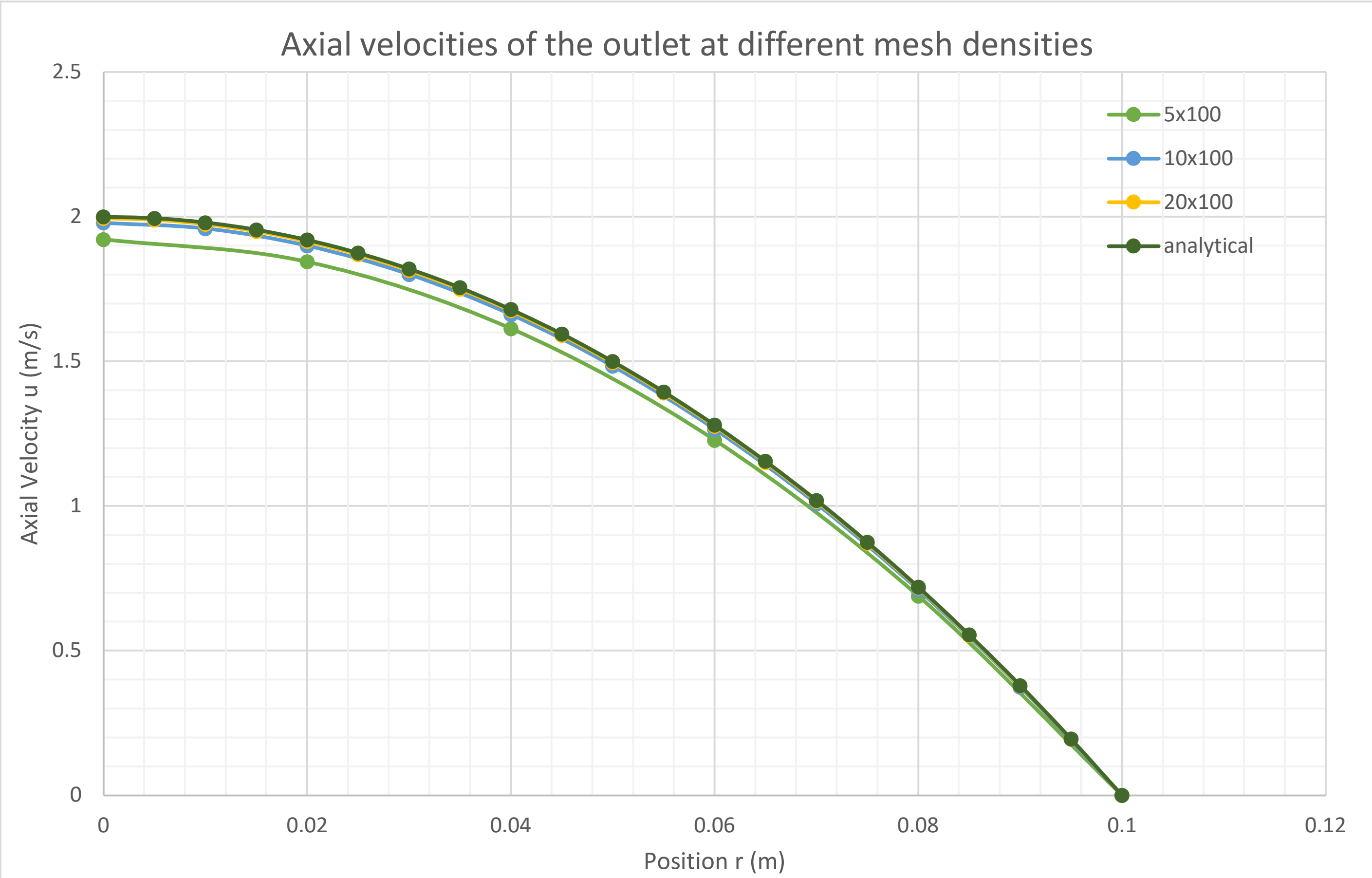

Figure 1. Axial velocities of the outlet at different mesh densities

As seen in Figure 1, all velocity profiles lie reasonably inside the same shape, expected from a no slip pipe condition. As the wall of the pipe is approached, values start to converge to the no slip condition where u=0, but they start from different velocities at the centreline depending on the mesh densities, being the 5x100 the smallest compared to the analytical values, and this difference decreasing proportionally with the number of cells.

To be more concise, the highest velocity for the 5x100 mesh is 1.9216 m/s; for 10x100 mesh is 1.97936 m/s; and for the 20x100 is 1.99462 m/s. The theoretical value 2 m/s.

Therefore, results are independent on the mesh as we approach the pipe wall, but mesh density dictates the maximum velocity of the flow. Velocity profiles overall lie on the same trendline, with the error being directly proportional to the mesh density used.

QUESTION 2

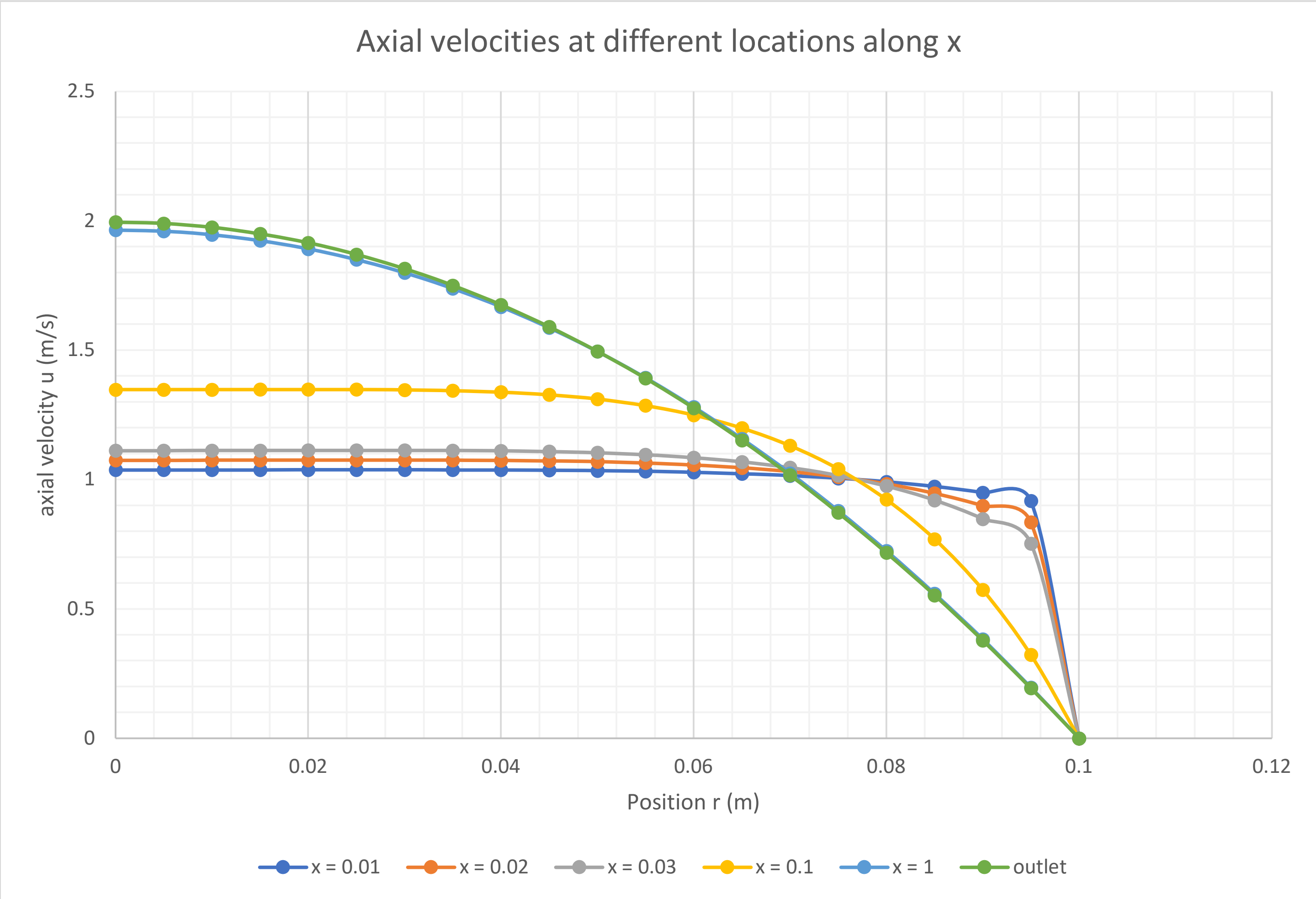

Figure 2. Axial velocities at different locations along x

For the locations, I have picked 3 immediately close to the inlet, one slightly further away and one at 1m from the outlet, and compared them to the fully developed velocity.

All flows are considered to have two main regions: The entrance and the fully developed. It can be clearly seen that in the locations x=0.01,0.02,0.03 and 0.1 flow is not fully developed as it is very different to the outlet velocity profile. Furthermore, the maximum velocity located at the centreline, grows from 1 to 2 m/s as we travel along the pipe. Looking at the profile at x=1, it looks almost exactly as the outlet, with a slight difference at the maximum velocity; flow is, therefore, almost fully developed at this region.

Plotting the axial velocity in the centreline of the pipe along with pipe X coordinate, and using a 0.05% tolerance, flow can be said to be fully developed from x=2.08 m, point from which velocity at centreline does not vary more than 0.001 m/s as we increase X.

QUESTION 3

i)

Uniform mesh	τw (Fluent)	τw (analytical)	% error
100 x 5	0.0769231	0.08	3.846
100 x 10	0.0792082	0.08	0.990
100 x 20	0.0798987	0.08	0.127

Table 1. Wall shear stresses and errors for different mesh densities

Values for the shear stress were taken at x=4.64, to ensure full flow development. As expected, the finer the mesh, the more precise values of shear stress can be achievable.

ii)

Using $\Delta\tau = \tau_x - \tau_{x-\Delta x}$ as a tolerance method, the start of the fully developed region can be estimated. For the finer mesh, with a tolerance of 1% ($\Delta\tau = 0.01$), the start of the region is located at x=0.72m. For a tolerance of 0.01% instead, it is located at 1.84. Fluent has a limited number of decimals, to the order of $1e^{-6}$, so the point at which the tolerance goes to 0% ($\Delta\tau = 1\mathrm{e}-6$) is at x=2.4m.

In terms of Re and D, the start of the fully developed region is $x_{dev} = \frac{D*Re}{10}$.

iii)

The greatest wall shear stress is shown at x=0. This is the point at which the flow, travelling at 1m/s, enters a no slip pipe (u = 0m/s). The clash between the velocity difference produces a reaction force that is traduced into wall shear stress, as the wall tries to decelerate the flow. This is later balanced as we travel along the pipe, and therefore wall shear is reduced as velocity is decreased close to the pipe wall. This can be clearly seen in the velocity profiles in Figure 2, where velocity is almost the same along the whole axial region at the start and then it progressively transforms into the fully developed profile.

The condition u = 0 at the pipe wall, forces the centreline velocity to 2 in order to maintain mass conservation.

QUESTION 4

i)

Uniform mesh	Radial Step Δr	u_c (2nd order upwind)	u_c (1st order upwind)	u exact	Error % (ε) (2nd order)	Error % (ε) (1st order)
100 x 5	0.02	1.9216	1.9218	2	3.92	3.91
100 x 10	0.01	1.97937	1.97947	2	1.0315	1.0265

Table 2. Centreline velocities and errors using different schemes for calculations

Results do not vary much from the errors obtained by the shear stress calculations in question 3. It can be appreciated that as we increase the mesh density, the error decreases. Between the numerical schemes, there is not much error other than a very slight decrease when using first order upwind.

ii)

second order upwind

$(u_c)_{\Delta r} - u_{exact}$	$(u_c)_{\Delta r/2} - u_{exact}$	p (2nd Order)
0.0784	0.02063	1.9261
0.02063	0.00531	1.9579

Table 3. Second order upwind p prediction

first order upwind

$(u_c)_{\Delta r} - u_{exact}$	$(u_c)_{\Delta r/2} - u_{exact}$	p (1st Order)
0.0782	0.02053	1.9294
0.02053	0.00527	1.9618

Table 4. First order upwind p prediction

First noticeable similarities are that p tends to 2 as we increase the mesh density. Comparing the values for the same Δr, the difference between 2nd and 1st order is less than 1% of their values.

QUESTION 5

Shear stresses were acquired at x = 6 m to ensure full flow development.

i) First order upwind. $Re_{allocated}$ = 159.5

	Numerical			Analytical		
	$Re_{allocated}$	Re = 3190	Re = 4785	Re	20*Re	30*Re
Shear	0.050141	0.003086	0.00223135	0.050157	0.002508	0.001672
	Errors					
Absolute	1.57E-05	0.000578	0.000559459			
Relative %	0.031381	23.06422	33.46262188			

Table 5. First order upwind wall shear stresses and errors at different Reynolds Number

ii) Second order upwind. $Re_{allocated}$ = 159.5

	Numerical			Analytical		
	$Re_{allocated}$	Re = 3190	Re = 4785	Re	20*Re	30*Re
Shear	0.05013	0.002994	0.00215411	0.050157	0.002508	0.001672
	Errors					
Absolute	2.7E-05	0.000486	0.000482219			
Relative %	0.053911	19.39492	28.84270438			

Table 6. Second order upwind wall shear stresses and errors at different Reynolds Number

In the last few tables relative errors can be appreciated for different Reynolds numbers at different schemes. Comparing the errors for the normal Reynolds number, it can be appreciated that, as little as it is, the second order relative error is almost twice the one appreciated in the first order scheme. On the other hand, as we start increasing the Reynolds number drastically, both errors start to go very high, but there is a slight difference between the two schemes, as the first order upwind appreciates more error. In conclusion, it can be confirmed that the use of different schemes is dependent, along with other critical factors, on the Reynolds number, and therefore the kind of flow we are trying to solve: laminar, in which first order scheme is recomended(up to Re = 2000); or turbulent, in which second order scheme is recommended (from Re = 4000).

iii)

According to the equation:

$$Re = \frac{\rho u_{in} D}{\mu}$$

A way to increase the Reynolds number up to 20*$Re_{allocated}$ is to reduce the viscosity μ by a factor of 20. So in order to get a Reynolds number of 159.5 (allocated), the equation above is derived to get $\mu = 0{,}01254$.

Target Re	Viscosity (μ)	Inlet velocity (m/s)	1
159.5	0.001254	**Density (kg/m³)**	1
3190	6.27E-05	**Diameter (m)**	0.1
4785	4.18E-05		

Table 7. Crucial numerical settings for Q5

This way, diameter, velocity and density are preserved. A previous test was executed by increasing velocity instead, but it produced undeveloped flow even after the whole pipe length. Therefore a viscosity change was used to modify the Reynolds number.

QUESTION 6

1. What mesh density was used during simulation?
   As seen in question 3, mesh density is very close related to the amount of error observed during simulations.

2. Have you considered the no slip condition during experimental data might not be perfect?
   This can be a source of error mostly when dealing with wall shear stress, since we are trying to simulate a perfect condition in real life.

3. Is the flow laminar or turbulent?
   This will determine whether to use first or second order upwind scheme for momentum, as seen in question 4 and 5.

4. Is the velocity used too largo or too small?
   Very critical values might cause the values to differ too much

5. Is convergence limit set very low?
   It might cause calculations to finish earlier that it should, leaving them incomplete and faulty.

QUESTION 7

Analysis of Numerical methods

Assignment 1. 1

$\boxed{\text{Question 7}}$

for eqn. 3 if $\Gamma = 0$ then:

$$\frac{d\phi}{dt} + u\frac{d\phi}{dx} = 0$$

we know: from Taylor expansion around node j,n:

$$\phi_{j,n+1} = \phi_{jn} + \Delta t\,\dot{\phi}_{j,n} + \frac{1}{2}\Delta t^2\ddot{\phi}_{j,n}\dots$$

$$\phi_{j-1,n} = \phi_{j,n} - \Delta x\,\phi'_{j,n} + \frac{1}{2}\Delta x^2\phi''_{j,n}\dots$$

$$\phi_{j+1,n} = \phi_{jn} + \Delta x\,\phi'_{jn} + \frac{1}{2}\Delta x^2\phi''_{j,n}\dots$$

-> Substituting into equation 4.

$$\cancel{\phi_{jn}} + \Delta t\,\dot{\phi}_{jn}\dots = \cancel{\phi_{jn}} - \frac{p}{2}\left(\cancel{\phi_{jn}} + \Delta x\phi'_{jn}\dots - \cancel{\phi_{j,n}} + \Delta x\phi'\right)$$

$$+ \frac{p^2}{2}\left(\cancel{\phi_{jn}} + \cancel{\Delta x\phi'_{jn}}\dots - \cancel{2\phi_{jn}} + \cancel{\phi_{jn}} - \cancel{\Delta x\phi'_{jn}}\dots\right)$$

gives

$$\Delta t\,\dot{\phi}_{jn} + \frac{1}{2}\Delta t^2\ddot{\phi}_{jn} = \frac{-p}{2}\left(2\Delta x\,\phi'_{jn}\right) + \frac{p^2}{2}\left(\cancel{\tfrac{2}{2}}\Delta x^2\phi''_{jn}\right)$$

$p = \frac{u\Delta t}{\Delta x}$

$$\Delta t\,\dot{\phi}_{jn} + \frac{1}{2}\Delta t^2\ddot{\phi}_{jn} = \frac{-u\Delta t}{2\Delta x}\left(2\Delta x\,\phi'_{jn}\right) + \frac{-u^2\Delta t^2}{2\Delta x^2}\left(\Delta x^2\phi''_{jn}\right)$$

↳

then:

$$\Delta t\, \dot{\phi}_{j,n} + \frac{1}{2}\Delta t^2 \ddot{\phi}_{j,n} = -u\Delta t\, \phi'_{j,n} + \frac{u^2\Delta t^2}{2}\phi''_{j,n}$$

dividing by Δt:

$$\text{FDE: } \dot{\phi}_{j,n} + u\phi'_{j,n} = \frac{u^2\Delta t}{2}\phi''_{j,n} - \frac{1}{2}\Delta t\, \ddot{\phi}_{j,n}$$

For consistency:

$$\text{PDE: } \frac{d\phi}{dt} + \frac{u\, d\phi}{dx} = 0$$

$$\lim_{\Delta x, \Delta t \to 0} \text{FDE} - \underset{\downarrow \atop 0}{\text{PDE}} = 0$$

$$\text{FDE} = 0 \to \lim_{\Delta x, \Delta t \to 0} \frac{u^2\Delta t}{2}\phi''_{j,n} - \frac{1}{2}\Delta t\, \ddot{\phi}_{j,n} = \boxed{0}$$

The scheme is consistent

The truncation error from the taylor serie is given by the very small terms in the expansion. Starting from Δx^3 and Δt^3 which were assumed 0 since the time and space steps Δt and Δx are normally very small.

QUESTION 8

Question 8

Von Neumann stability:

$$\phi_{j,n} = A^n e^{i\alpha j}$$

Replacing in eq. S:

$$A^{n+1} e^{i\alpha j} = A^{n-1} e^{i\alpha j} - p\left(A^n e^{i\alpha(j+1)} - A^n e^{i\alpha(j-1)}\right)$$

dividing by $A^n e^{i\alpha j}$ we have

$$A = \frac{1}{A} - p\left(e^{i\alpha} - e^{-i\alpha}\right)$$

recall $e^{i\alpha} = \cos\alpha + i\sin\alpha$ and $e^{-i\alpha} = \cos\alpha - i\sin\alpha$

so $A = \frac{1}{A} - p\left(\cancel{\cos\alpha} + i\sin\alpha - \cancel{\cos\alpha} + i\sin\alpha\right)$

$A = \frac{1}{A} - 2pi\sin\alpha \rightarrow A - \frac{1}{A} = -2pi\sin\alpha$

$$A - \frac{1}{A} = -2pi\sin\alpha$$

$$A^2 - 1 = -2A\,pi\sin\alpha$$

$$A^2 - 2A\,pi\sin\alpha - 1 = 0$$

$$A = \frac{-2pi\sin\alpha \pm \sqrt{(-2pi\sin\alpha)^2 + 4}}{2} = \frac{-2pi\sin\alpha \pm \sqrt{(-4)p^2\sin\alpha + 4}}{2}$$

for stability $|A| \leq 1$

$$\frac{-2\,pi\sin\alpha \pm \sqrt{-4p^2\sin^2\alpha + 4}}{2} \leq 1$$

$$\frac{-\cancel{2}\,pi\sin\alpha \pm \sqrt{\cancel{4}(-p^2\sin\alpha + 1)}}{\cancel{2}} \leq 1$$

$$-pi\sin\alpha \pm \sqrt{p^2\sin^2\alpha + 1} \leq 1$$

$$\left(\pm\sqrt{p^2\sin^2\alpha + 1}\right)^2 \leq (1 + pi\sin\alpha)^2$$

$$\cancel{p^2\sin^2\alpha}\ \cancel{+1} \leq -\cancel{p^2\sin^2}\alpha + 2pi\sin\alpha\ \cancel{+1}$$

$$0 \leq (2pi\sin\alpha)^2$$

$$0 \leq -4p^2\sin^2\alpha$$

$-4p^2\sin^2\alpha \geqslant 0$

$p^2\sin^2\alpha \leq 0$ $\qquad p > 0$

so $\sin^2\alpha \leq 0$

$\frac{1}{2}(1-\cos 2\alpha) \leq 0$

$1-\cos 2\alpha \leq 0$

$+\cos 2\alpha \leq 1$

true for $\boxed{\alpha = \frac{\pi n}{2} \quad \text{for } n \in \mathbb{Z}}$

QUESTION 9

i)

$$\frac{\partial \phi}{\partial t} + u\frac{d\phi}{dx} = \frac{\Gamma}{\rho}\frac{d^2\phi}{dx^2}$$

Question 9 i) 4

$\frac{\partial \phi}{\partial t} = 0$

$\phi_0 = 1$

$\phi_L = 0$ @ $L = 1 = x$

u Γ ρ = const.

$$\downarrow$$

$$u\frac{d\phi}{dx} = \frac{\Gamma}{\rho}\frac{d^2\phi}{dx^2}$$

using backward (upwind)

$$\frac{d\phi_j}{dx} = \frac{\phi_j - \phi_{j-1}}{\Delta x}$$

using central difference

$$\frac{d^2\phi}{dx^2} = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}$$

$$u\frac{\phi_j - \phi_{j-1}}{\Delta x} = \frac{\Gamma}{\rho}\frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}$$

$$\frac{u\phi_j}{\Delta x} - \frac{u}{\Delta x}\phi_{j-1} = \frac{\Gamma}{\rho\Delta x^2}(\phi_{j+1} - 2\phi_j + \phi_{j-1})$$

FDE:

$$\left(\frac{u}{\Delta x} + \frac{2\Gamma}{\rho\Delta x^2}\right)\phi_j - \left(\frac{u}{\Delta x} + \frac{\Gamma}{\rho\Delta x^2}\right)\phi_{j-1} - \frac{\Gamma}{\rho\Delta x^2}\phi_{j+1} = 0$$

for $j=1$

$$\frac{u}{\Delta x} + \frac{2\Gamma}{\rho \Delta x^2} = A \qquad \frac{\Gamma}{\rho \Delta x^2} = C$$

$$\frac{u}{\Delta x} + \frac{\Gamma}{\rho \Delta x^2} = B \qquad \begin{array}{l} \phi_0 = 1 \\ \phi_5 = 0 \end{array}$$

$$A\phi_1 - B\phi_0 - C\phi_2 = 0$$

for $j=2$ $\quad A\phi_2 - B\phi_1 - C\phi_3$

for $j=3$ $\quad A\phi_3 - B\phi_2 - C\phi_4$

for $j=4$ $\quad A\phi_4 - B\phi_3 - C\phi_5$

$$A\phi_1 - B - C\phi_2 = 0$$

$$A\phi_2 - Bd_1 - C\phi_3 = 0 \longrightarrow \begin{bmatrix} A & -C & 0 & 0 \\ -B & A & -C & 0 \\ 0 & -B & A & -C \\ 0 & 0 & -B & A \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}$$

$$A\phi_3 - B\phi_2 - C\phi_4 = 0$$

$$A\phi_4 - B\phi_3 = 0$$

6

In matrix form:

$$\begin{bmatrix} A & -C & 0 & 0 \\ -B & A & -C & 0 \\ 0 & -B & A & -C \\ 0 & 0 & -B & A \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} +B \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

This last equation was solved using excel to give the values for $\phi_{1,2,3,4}$.

j	**x**	**Φ(exact)**	**Φ(analytical)**	**error**	**%**
0	0	1	1	0	0
1	0.2	1	0.99935691	-0.00064308	0.06430848
2	0.4	0.99999969	0.99549839	-0.0045013	0.45013032
3	0.6	0.9999546	0.97234727	-0.02760733	2.76085866
4	0.8	0.99326205	0.83344051	-0.15982154	16.0905713
5	1	0	0	0	0

Table 8. FDE solutions and errors for the exact and the analytical method

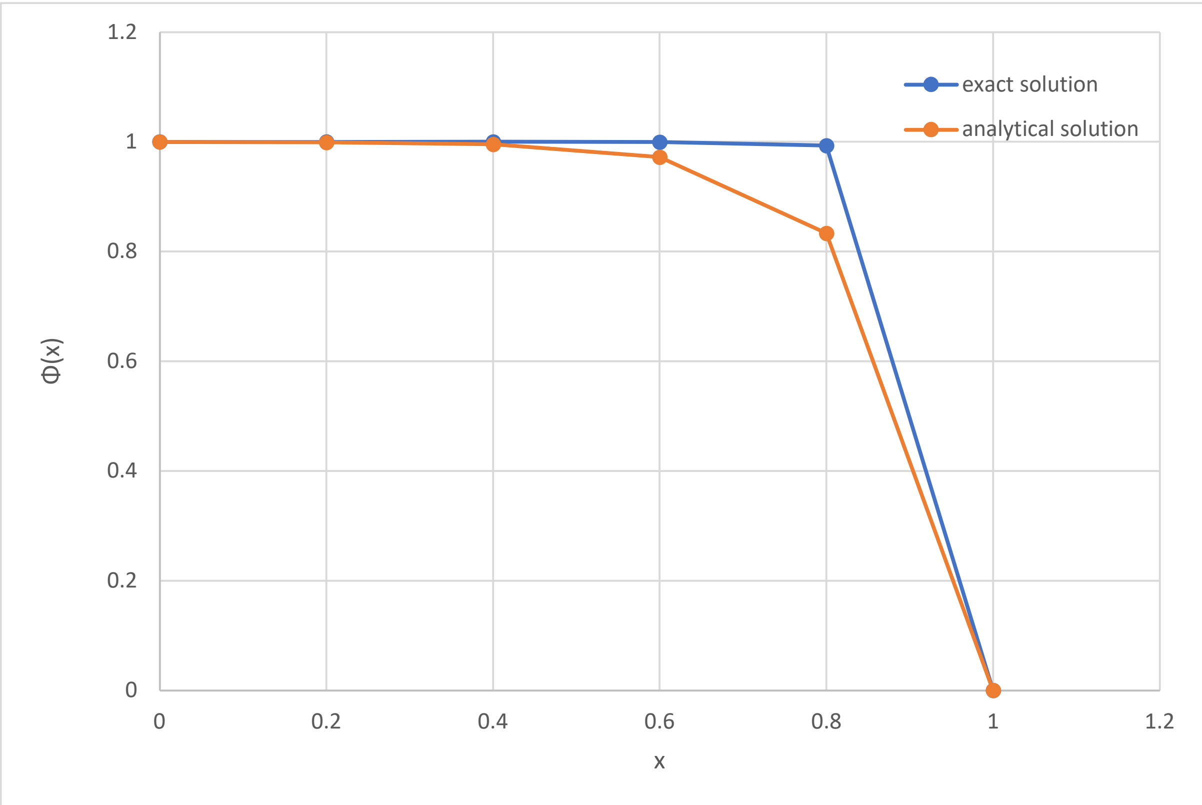

Figure 3. Plot of the exact and analytical solutions of the FDE

Tendency of the analytical solution follows the same shape as the exact, but values start to differ as we increase x.

ii)

ii) $\boxed{\text{Question } 9}$ ii) 7

$$\frac{d\phi}{dt} + u\frac{d\phi}{dx} = \frac{\Gamma}{\rho}\frac{d^2\phi}{d^2x^2} \qquad \frac{d\phi}{dt} = 0$$

$$\phi_0 = 1$$

$$u\frac{d\phi}{dx} = \frac{\Gamma}{\rho}\frac{d^2\phi}{dx^2} \qquad \phi_L = \phi_5 = 0$$

$$\frac{d\phi}{dx} = \xrightarrow{\text{central diff.}} \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}$$

$$\frac{d^2\phi}{dx^2} = \xrightarrow{\text{central diff}} \frac{\phi_{j+1} - 2\phi_j + \phi_{j+1}}{\Delta x^2}$$

$$\frac{u}{2\Delta x}(\phi_{j+1} - \phi_{j-1}) = \frac{\Gamma}{\rho\Delta x^2}\left(\phi_{j+1} - 2\phi_j + \phi_{j+1}\right)$$

FDE:

$$\underbrace{\frac{2\Gamma}{\rho\Delta x^2}}_{D}\phi_j + \underbrace{\left(\frac{u}{2\Delta x} - \frac{\Gamma}{\rho\Delta x^2}\right)}_{E}\phi_{j+1} + \underbrace{\left(\frac{-u}{2\Delta x} - \frac{\Gamma}{\rho\Delta x^2}\right)}_{F}\phi_{j+1} = 0$$

for $j=1$ $\quad A\phi_1 + E\phi_2 + F\phi_0 \xrightarrow{=0} A\phi_1 + E = -F$

for $j=2$ $\quad A\phi_2 + E\phi_3 + F\phi_1 = 0$

for $j=3$ $\quad A\phi_3 + E\phi_4 + F\phi_2 = 0$

for $j=4$ $\quad A\phi_4 + E\cancel{\phi_5}^{0} + F\phi_3 = 0$

In matrix form:

$$\begin{bmatrix} D & E & 0 & 0 \\ F & D & E & 0 \\ 0 & F & D & E \\ 0 & 0 & F & D \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} -F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This last equation was solved using excel to give the values for $\phi_{1,2,3,4}$.

j	**x**	**Φ(exact)**	**Φ(analytical)**	**error**	**%**
0	0	1	1	0	0.00
1	0.2	1	0.952492669	-0.04750733	4.75
2	0.4	0.99999969	1.063343109	0.06334341	6.33
3	0.6	0.9999546	0.804692082	-0.19526252	19.53
4	0.8	0.99326205	1.408211144	0.41494909	41.78
5	1	0	0	0	0

Table 9. FDE solutions and errors for the second analytical method

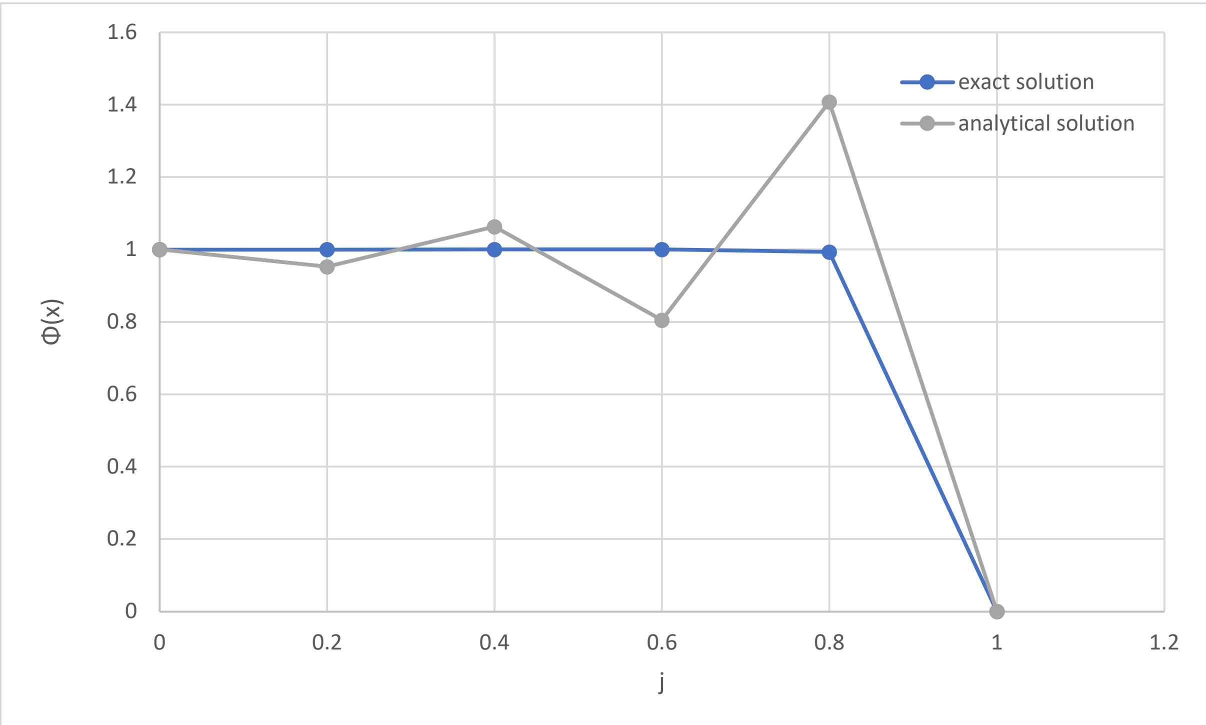

Figure 4. Plot of the exact and analyitical solution

The biggest difference appreciated between methods i) and ii), are the stability. We can clearly see in the graphs that method 2 fluctuates up and down from the exact value, incrementing the error with every step, while method 1 is limited by the boundary conditions.

If we were to reduce the Δx step, we will find that method 1 is much more precise than method 2.