The Gauss circle problem for arithmetic progressions

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Chapter 1

Introduction

1.1 The Gauss circle problem

The Gauss circle problem is the problem of counting the number of integers u, v such that $u^2 + v^2 \leq x$ for some predetermined x. Mathematically this means estimating the function $G(x)$ defined as,

$$G(x) = \sum_{n \leq x} r(n), \tag{1.1}$$

where

$$r(n) = \{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\}. \tag{1.2}$$

The Gauss circle problem also has a geometric interpretation. Notice that $\sqrt{a^2 + b^2}$ is the distance of the point (a, b) from the origin in 2-dimensional Euclidean space. So

$$G(x) = \#\{\text{integer points inside a circle of radius } \sqrt{x}, \text{ centered at the origin}\},$$

see Figure 1.1 for an example. This geometric interpretation suggests that for large x, the main term of $G(x)$ is πx, the area of a circle with radius equal to $\sqrt{x}$. Let $R(x)$ be defined by the relation

$$G(x) = \pi x + R(x). \tag{1.3}$$

The problem lies in estimating the smallest θ such that

$$R(x) \ll O(x^{\theta}). \tag{1.4}$$

Gauss (1834) proved the bound for $\theta = 1/2$. The bound was then improved to $\theta = 1/3$ by Voronoi (1903) and Sierpinski (1906) independently. Then it was lowered to $\theta = 27/82 + \epsilon$ by Van der Corput (1928). The current best known bound is $\theta = \theta^* + \epsilon$ where $\theta^* = 0.314483\ldots$ and is defined to be the negative of the unique solution to the equation

$$-\frac{8}{25}x - \frac{1}{200}\sqrt{2(1 - 14x)} - 5\sqrt{-1 - 8x}^2 + \frac{51}{200} = -x$$

on the interval $[-0.35, -0.3]$. This was proved by Li and Yang [12].

gcp_images_for_thesis/original_gcp_radius_10_visual.png

Figure 1.1: visual for $G(100)$

Later in this chapter we will prove (1.4) for $\theta = 27/82$.

Theorem 1. *Let $r(n)$ be defined as in (1.2). Then*

$$r(n) = 4 \sum_{d|n} \chi_4(d), \tag{1.5}$$

where χ_4 is the unique non-trivial Dirichlet character of modulus 4 defined by

$$\chi_4(d) = \begin{cases} 1, & d \equiv 1(4), \\ -1, & d \equiv 3(4). \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We follow the argument from [2]. Let

$$\delta(n) = \sum_{d|n} \chi_4(d) = d_1(n) - d_3(n),$$

where d_1 and d_3 count the number of divisors of n of the form $4m+1$ and $4m+3$ respectively. Say

$$n = 2^a \prod_{j=1}^{k} {p_j}^{b_j} \prod_{j=1}^{l} {q_j}^{c_j},$$

where p_j are primes of the form $4m+1$ and q_j are primes of the form $4m+3$. If we restrict our attention to the case $n = \prod_{j=1}^{l} {q_j}^{c_j}$, we see that $\delta(n) = \prod_{j=1}^{l}(1+(-1)^{c_j})/2$. Now for general n, we note that multiplying a divisor d of n by a power of p_j leaves the congruency class of d modulo 4 unchanged because $p_j \equiv 1(4)$. We also note that even divisors give no contribution to δ, so for general n we have

$$\delta(n) = \prod_{j=1}^{k}(b_j+1) \prod_{j=1}^{l} \left(\frac{1+(-1)^{c_j}}{2} \right).$$

Now say $n = A^2 + B^2 = (A+Bi)(A-Bi)$. In $\mathbb{Z}[i]$ we have

$$A + Bi = i^r (1+i)^{a_1} (1-i)^{a_2} \prod_{j=1}^{k} (x_j + iy_j)^{b_{j,1}} (x_j - iy_j)^{b_{j,2}} \prod_{j=1}^{l} q^{\gamma_j},$$

where $a_1 + a_2 = a$, $b_{j,1} + b_{j,2} = b_j$ and $p_j = (x_j + iy_j)(x_j - iy_j)$ is the unique factorization of a prime of the form $4m+1$ in $\mathbb{Z}[i]$. If we now consider the factorization of $A - Bi$, we must have that the exponents of conjugate factors between $A+Bi$ and $A-Bi$ are the same, hence we get

$$A - Bi = i^{-r} (1+i)^{a_2} (1-i)^{a_1} \prod_{j=1}^{k} (x_j + iy_j)^{b_{j,2}} (x_j - iy_j)^{b_{j,1}} \prod_{j=1}^{l} q^{\gamma_j}.$$

Now note that $(1+i)/(1-i) = i$ is a unit so the choice of a_1, a_2 causes no variation in A and B beyond the choice of r. We finally note that each unique b_{j_1} will correspond to a unique $A+Bi$ and that each c_j must be even because each $\gamma_j = c_j/2$. So the total number of representations of n as a sum of two squares is

$$4 \prod_{j=1}^{k}(b_j+1) \prod_{j=1}^{l} \left(\frac{1+(-1)^{c_j}}{2} \right) = 4\delta(n).$$

□

1.2 Bounding exponential sums

Let

$$e(x) = e^{2\pi i x}, \tag{1.6}$$

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n|,$$

$$\psi(x) = x - [x] - 1/2, \tag{1.7}$$

where $[x]$ is the integer part of x. Exponential sums naturally appear in the Gauss circle problem when we consider the Fourier series of ψ. This section is devoted to referencing some results from [9] that will enable us to bound the exponential sums that appear.

1.2.1 Van der Corput estimates

Let $I = (a, b]$ where $a, b \in \mathbb{Z}$ for this section.

Theorem 2. *If f is continuously differentiable, f' is monotonic, and $\|f'\| \geq \lambda > 0$ on I then*

$$\sum_{n \in I} e(f(n)) \ll \lambda^{-1}.$$

Theorem 3. *Suppose that f is a real valued function with two continuous derivatives on I. Suppose also there is some $\alpha \geq 1$ and $\lambda > 0$ such that*

$$\lambda \leq |f''(x)| \leq \alpha\lambda$$

on I. Then

$$\sum_{n \in I} e(f(n)) \ll \alpha|I|\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. Case 1: $\lambda \geq 1/4$. We note that $1/2 \leq \alpha\lambda^{1/2}$ so by the trivial estimate

$$\left|\sum_{n \in I} e(f(n))\right| \leq 2\alpha|I|\lambda^{1/2}.$$

Case 2: $\lambda < 1/4$. We will try to split I into intervals based on the behaviour of $f'(x)$ so that we may apply Theorem 2 and the trivial estimate to prove the theorem. By the conditions of the theorem we have that f' is monotonic, and due to the formula

$$\left|\sum_{n \in I} e(f(n))\right| = \left|\sum_{n \in I} e(-f(n))\right|,$$

we may assume f' is increasing. The conditions then tell us the range of f' on I is $(k, k + |I|D]$ where $D \in [\lambda, \alpha\lambda]$ and k is some real number.

Let $0 < \delta < 1/2$. We now wish to split I into two collections of intervals which we will denote X_1 and X_2. The intervals in X_1 will be such that $\|f'\| \geq \delta$, and the remaining intervals will have $\|f'\| < \delta$ and be placed in X_2. Due to the range of f' we have that

$$|X_1| \leq \alpha|I|\lambda + 2.$$

Note that we add two here to account for the fact that k may not be an integer and $|I|D$ may not be an integer. Similarly we have

$$|X_2| \leq \alpha|I|\lambda + 2.$$

Now for each interval in X_1, we can apply Theorem 2. We have at most $\alpha|I|\lambda + 2$ intervals and so

$$\sum_{J \in X_1} \left|\sum_{n \in J} e(f(n))\right| \ll (\alpha|I|\lambda + 2)\delta^{-1}.$$

For each interval $J \in X_2$, we have that f' is within δ range of an integer in J. Since we also know $\lambda \leq f''(x)$, we have $|J| \leq 2\delta/\lambda$. For each interval in X_2, we apply the trivial estimate using that the number of integer points in an interval J is bounded by $|J|+1$. We have at most $\alpha|I|\lambda+2$ intervals and so

$$\sum_{J\in X_2}\left|\sum_{n\in J} e(f(n))\right| \leq (\alpha|I|\lambda+2)(2\delta\lambda^{-1}+1).$$

By triangle inequality we then have

$$\sum_{n\in I} e(f(n)) \ll (\alpha|I|\lambda+1)(\delta^{-1}+\delta\lambda^{-1}+1).$$

Let $\delta=\lambda^{1/2}$ and observe that since $\lambda<1/4$, 1 is dominated by $\lambda^{-1/2}$ which gives us

$$\sum_{n\in I} e(f(n)) \ll \alpha|I|\lambda^{1/2}+\lambda^{-1/2}.$$

□

1.2.2 The method of exponent pairs.

The Van der Corput estimates do not provide a strong enough bound for later results in the Gauss circle problem. In the 1930s a method by Phillips emerged to produce a stronger bound for certain exponential sums.

Definition. *Let N, y, s and ϵ be positive real numbers with $\epsilon<1/2$ and let P be a non-negative integer. Define $F(N,P,s,y,\epsilon)$ to be the family of functions f satisfying:*

i) f is defined and has P continuous derivatives in some interval $[a,b]$, with $[a,b]\in[N,2N]$.

ii) If $0\leq p\leq P-1$ and $a\leq x\leq b$ then

$$\left|f^{(p+1)}(x)-(-1)^p(s)_p y x^{-s-p}\right| < \epsilon(s)_p y x^{-s-p}.$$

where $(s)_p$ is defined recursively by $(s)_0=1$ and $(s)_{k+1}=(s+k)(s)_k$. Note that if $p\geq 1$, then $(s)_p=s(s+1)\cdots(s+p-1)$.

Definition. *Let k and l be real numbers such that $0\leq k\leq 1/2\leq l\leq 1$. Suppose that for every $s>0$, there is some $P=P(k,l,s)$ and some $\epsilon=\epsilon(k,l,s)$ such that for every $N>0$, every $y>0$, and every $f\in F(N,P,s,y,\epsilon)$, the estimate*

$$\sum_{n\in I} e(f(n)) \ll (yN^{-s})^k N^l + y^{-1}N^s$$

holds. Here it is also assumed that f is defined on $[a,b]$ and that the implied constant depends only on k,l,s. We then say that (k,l) is an exponent pair.

Theorem 4. *Suppose $(k.l)$ is an exponent pair, and that $P=P(k,l)$, $\epsilon=\epsilon(k,l)$ are the corresponding parameters guaranteed by the definition of exponent pairs. If $f\in F(N,P,s,y,\epsilon)$ and f is defined on $[a,b]$, then*

$$\sum_{n\in I}\psi(f(n)) \ll y^{k/(k+1)}N^{((1-s)k+l)/(k+1)}+y^{-1}N^s,$$

where the implicit constant depends only on k,l,s.

Note that $(0,1)$ is clearly an exponent pair. We now describe two processes that generate new exponent pairs.

Theorem 5 (The A-process)**.** *If (k,l) is an exponent pair, then so is*

$$A(k,l)=\left(\frac{k}{2k+2},\frac{k+l+1}{2k+2}\right).$$

Theorem 6 (The B-process)**.** *If (k,l) is an exponent pair, then so is*

$$B(k,l)=(l-1/2,k+1/2).$$

For all the results we obtain using exponent pairs in this paper, we will use the exponent pair

$$BA^3B(0,1)=BA^3(1/2,1/2)=BA^2(1/6,2/3)=\cdots=(11/30,16/30).$$

1.3 Bounding the error term for the Gauss circle problem

In this section we will prove that

$$R(x)\ll x^{27/82+\epsilon}$$

by following pages 42 to 44 from [9]. The key to this proof will be using the theory of exponent pairs. We begin by substituting (1.5) into (1.1) and then applying the Dirichlet's hyperbola method to get,

$$\begin{aligned}\sum_{n\le x}\sum_{d|n}\chi_4(d)&=\sum_{md\le x}\chi_4(d)\\&=\sum_{d\le x^{1/2}}\chi_4(d)\sum_{m\le x/d}1+\sum_{m\le x^{1/2}}\sum_{d\le x/m}\chi_4(d)-\sum_{m\le x^{1/2}}\sum_{d\le x^{1/2}}\chi_4(d)\\&=\sum\nolimits_1+\sum\nolimits_2-\sum\nolimits_3.\end{aligned}\tag{1.8}$$

To proceed, we need more information about sums involving χ_4.

Theorem 7. *Let $S(y)$ be defined by*

$$S(y)=\sum_{m\le y}\chi_4(m).$$

Then

$$S(y)=\frac{1}{2}-\psi\left(\frac{y-1}{4}\right)+\psi\left(\frac{y-3}{4}\right),$$

and

$$\sum_{d\le y}\frac{\chi_4(d)}{d}=\frac{\pi}{4}+\frac{S(y)-1/2}{y}+O\left(\frac{1}{y^2}\right).\tag{1.9}$$

Proof. To prove the first statement, we note that

$$S(y)=\sum_{\substack{1\le n\le y\\ n\equiv1(4)}}1-\sum_{\substack{1\le n\le y\\ n\equiv3(4)}}1=\left\lfloor\frac{y-1}{4}\right\rfloor-\left\lfloor\frac{y-3}{4}\right\rfloor.\tag{1.10}$$

By using the definition of ψ from (1.7), we obtain

$$S(y) = \left(\frac{y-1}{4} - \psi\left(\frac{y-1}{4}\right) - \frac{1}{2}\right) - \left(\frac{y-3}{4} - \psi\left(\frac{y-3}{4}\right) - \frac{1}{2}\right) = \frac{1}{2} - \psi\left(\frac{y-1}{4}\right) + \psi\left(\frac{y-3}{4}\right)$$

which proves the result. To prove the second statement, we observe that

$$\sum_{d \le y} \frac{\chi_4(d)}{d} = \frac{\pi}{4} - \sum_{d>y} \frac{\chi_4(d)}{d}. \tag{1.11}$$

Summation by parts followed by substituting (1.10) gives us,

$$\begin{aligned}\sum_{d>y} \frac{\chi_4(d)}{d} &= \left[\frac{S(t)}{t}\right]_y^\infty + \int_y^\infty \frac{1}{2} - \psi\left(\frac{t-1}{4}\right) + \psi\left(\frac{t-3}{4}\right) \frac{dt}{t^2} \\ &= -\frac{S(y)-1/2}{y} + \int_y^\infty \psi\left(\frac{t-3}{4}\right) - \psi\left(\frac{t-1}{4}\right) \frac{dt}{t^2}.\end{aligned}$$

We now focus on

$$\int_y^\infty \psi\left(\frac{t-a}{4}\right) \frac{dt}{t^2}, \tag{1.12}$$

where $a = 1$ or $a = 3$. Let

$$\psi_1(t) = \int_0^t \psi(w) dw. \tag{1.13}$$

Observe that $\psi_1(u)$ is bounded because $\psi_1(0) = \psi_1(1) = 0$ and ψ has period 1. We now perform integration by parts on (1.12), making use of the boundedness of ψ_1, to give us

$$\begin{aligned}\int_y^\infty \psi\left(\frac{t-a}{4}\right) \frac{dt}{t^2} &= \left[4\psi_1\left(\frac{t-a}{4}\right)\frac{1}{t^2}\right]_y^\infty + 8\int_y^\infty \psi_1\left(\frac{t-a}{4}\right) \frac{dt}{t^3} \\ &= -4\psi_1\left(\frac{y-a}{4}\right)\frac{1}{y^2} + O\left(\int_y^\infty \frac{dt}{t^3}\right) \\ &\ll y^{-2}.\end{aligned}$$

Substituting this bound into (1.12), combined with (1.11), gives us (1.9). $\square$

Theorem 8. *Let $R(x)$ be defined by (1.3), then*

$$R(x) = 4\left(\sum_{d \le x^{1/2}/4} \left\{\psi\left(\frac{x}{4d+1}\right) - \psi\left(\frac{x}{4d+3}\right)\right\} + \sum_{d \le x^{1/2}} \left\{\psi\left(\frac{x}{4d} - \frac{3}{4}\right) - \psi\left(\frac{x}{4d} - \frac{1}{4}\right)\right\}\right) + O(1). \tag{1.14}$$

If (k,l) is an exponent pair such that $(k,l) \ne \left(\frac{1}{2}, \frac{1}{2}\right)$, then

$$R(x) \ll x^{(k+l)/(2k+2)} \log x. \tag{1.15}$$

In particular, $R(x) \ll x^{27/82+\epsilon}$.

Proof. Let $\sum_1$, $\sum_2$, $\sum_3$ be as defined in (1.8). Using the definition of ψ and Theorem 7, we obtain

$$\begin{aligned}\sum\nolimits_1 &= \sum_{d\le x^{1/2}} \chi_4(d)\left[\frac{x}{d}\right] \\ &= x\sum_{d\le x^{1/2}} \frac{\chi_4(d)}{d} - \frac{1}{2}S(x^{1/2}) - \sum_{d\le x^{1/2}} \chi_4(d)\psi\left(\frac{x}{d}\right) \\ &= \frac{\pi}{4}x + \left(S(x^{1/2}) - \frac{1}{2}\right)x^{1/2} - \sum_{d\le x^{1/2}} \chi_4(d)\psi\left(\frac{x}{d}\right) + O(1).\end{aligned}$$

Another application of Theorem 7 yields,

$$\begin{aligned}\sum\nolimits_2 &= \sum_{m\le x^{1/2}} S\left(\frac{x}{m}\right) \\ &= \frac{1}{2}x^{1/2} + \sum_{m\le x^{1/2}} \psi\left(\frac{x/m-3}{4}\right) - \psi\left(\frac{x/m-1}{4}\right) + O(1).\end{aligned}$$

Lastly we have,

$$\sum\nolimits_3 = \sum_{m\le x^{1/2}} S(x^{1/2}) = x^{1/2}S(x^{1/2}) + O(1).$$

Combining the estimates for $\sum_1$, $\sum_2$, and $\sum_3$ gives (1.14).

We will bound $R(x)$ by bounding

$$\sum_{d\le x^{1/2}/4} \psi\left(\frac{x}{4d+a}\right),$$

and then

$$\sum_{d\le x^{1/2}} \psi\left(\frac{x}{4d} - \frac{a}{4}\right),$$

where $a = 1$ or $a = 3$.

We begin by performing a dyadic decomposition on the first sum to give us,

$$\sum_{d\le x^{1/2}/4} \psi\left(\frac{x}{4d+a}\right) = \sum_{1\le j\le J}\sum_{n\in I_j} \psi\left(\frac{x}{4n+a}\right),$$

where

$$I_j = \{n : 2^{-j-2}x^{1/2} < n \le 2^{-j-1}x^{1/2}\},$$

and J represents how far we have to go along the number line before I_j contains no integers, so $J = O(\log x)$. We note that ψ is an odd function so

$$\left|\sum_{n\in I_j} \psi\left(\frac{x}{4n+a}\right)\right| = \left|\sum_{n\in I_j} \psi\left(\frac{-x}{4n+a}\right)\right|.$$

Now let $g(u) = -x/(4u+a)$, we choose to work with this function (rather than $-g$) because the derivative of g is increasing so we can apply exponent pairs. We will show that $g \in F(N_j, P, 2, x/4, \epsilon)$, where $P = P(11/30, 16/30, 2)$, $\epsilon = \epsilon(11/30, 16/30, 2)$, and $N_j = 2^{-j-2}x^{1/2}$, the size of the interval $[N_j, 2N_j]$, satisfies

$$1 - \left(1 - \frac{1}{N_j+1}\right)^{P+1} < \epsilon,$$

or equivalently $N_j \geq K$ for some K fixed by ϵ and P. To do this we have to show two things. First we show that $g(u)$ has P continuous derivatives in the interval $[N_j, 2N_j]$. Observe that

$$g^{(p+1)}(u) = (-1)^p x 4^{p+1}(p+1)!(4u+a)^{-(p+2)}, \tag{1.16}$$

so $g(u)$ is infinitely differentiable and defined in the interval $[N_j, 2N_j]$ for any positive N_j.

Second, we aim to show that for $0 \leq p \leq P-1$, and $u \in [N_j, 2N_j]$, we have

$$\left| g^{(p+1)}(u) - (-1)^p x(p+1)! u^{-(p+2)}/4 \right| < \epsilon(p+1)! x u^{-(p+2)}/4.$$

We substitute (1.16) into this inequality, so it suffices to show

$$\left| \left(\frac{4}{4u+a}\right)^{p+2} - \left(\frac{1}{u}\right)^{p+2} \right| < \epsilon \left(\frac{1}{u}\right)^{p+2}.$$

We have,

$$\begin{aligned}
\left| \left(\frac{4}{4u+a}\right)^{p+2} - \left(\frac{1}{u}\right)^{p+2} \right| &= \left(\frac{1}{u}\right)^{p+2} \left(1 - \left(1 - \frac{a}{4u+a}\right)^{p+2}\right) \\
&< \left(\frac{1}{u}\right)^{p+2} \left(1 - \left(1 - \frac{1}{N_j+1}\right)^{P+1}\right) \\
&< \epsilon \left(\frac{1}{u}\right)^{p+2},
\end{aligned}$$

where we have used that

$$\frac{a}{4u+a} < \frac{4}{4N_j+4}.$$

So $g \in F(N_j, P, 2, x/4, \epsilon)$ for $N_j \geq K$. Now observe that $I_j \subset [N_j, 2N_j]$, so now we substitute $y = x/4$, $s = 2$, and $N = 2^{-j-2}x^{1/2}$ into Theorem 4 which gives us

$$\sum_{n \in I_j} \psi\left(g(n)\right) \ll \left(\frac{x}{4}\right)^{k/(k+1)} \left(2^{-j-2}x^{1/2}\right)^{(-k+l)/(k+1)} + \left(\frac{4}{x}\right)\left(2^{-j-2}x^{1/2}\right)^2,$$

for any j satisfying $N_j \geq K$. We note that $\sum_{j=1}^{\infty} 2^{-2j}$ converges, and provided $(k,l) \neq (1/2, 1/2)$, then $\sum_{j=1}^{\infty} 2^{(k-l)j/(k+1)}$ also converges. We also know that $J = O(\log x)$, therefore

$$\begin{aligned}
\sum_{\substack{1 \leq j \leq J \\ N_j \geq K}} \sum_{n \in I_j} \psi\left(g(n)\right) &\ll \sum_{\substack{1 \leq j \leq J \\ N_j \geq K}} \left(x^{(k+l)/(2k+2)} + 1\right) \\
&\ll x^{(k+l)/(2k+2)} \log x.
\end{aligned}$$

Note that we have not dealt with bounding in the case $N_j < K$. This case is simple to deal with though because K is fixed, so

$$\sum_{\substack{1\le j\le J\\ N_j<K}} \sum_{n\in I_j} \psi\left(g(n)\right) = O(1).$$

So altogether we obtain

$$\sum_{d\le x^{1/2}/4} \psi\left(\frac{x}{4d+a}\right) \ll x^{(k+l)/(2k+2)} \log x. \tag{1.17}$$

By a similar argument, it can be shown that

$$\sum_{d\le x^{1/2}} \psi\left(\frac{x}{4d} - \frac{a}{4}\right) \ll x^{(k+l)/(2k+2)} \log x. \tag{1.18}$$

Then (1.15) follows from (1.17) and (1.18) applied to (1.14), and in particular, if we let $(k,l) = BA^3B(0,1) = (11/30, 16/30)$, then

$$R(x) \ll x^{27/82} \log x,$$

hence $R(x) \ll x^{27/82+\epsilon}$. □

1.4 The Gauss circle problem for arithmetic progressions.

We now look at a variant of the Gauss circle problem. We begin by considering the sum

$$S_{q,a}(x) = \sum_{\substack{n\le x\\ n\equiv a(q)}} r(n). \tag{1.19}$$

This sum is the same as $G(x)$, defined in (1.1), but with an extra congruence condition. We wish to approximate this sum. We first define an important arithmetic function that will help us do so. Let

$$\eta_a(q) = \#\{(x,y) \in \left(\mathbb{Z}/q\mathbb{Z}\right)^2 : x^2+y^2 \equiv a \pmod q\}, \tag{1.20}$$

then for large x compared to q, (1.19) can be approximated well by

$$\pi\frac{\eta_a(q)}{q^2}x. \tag{1.21}$$

As we did for the original Gauss circle problem, we will motivate this approximation geometrically. Consider the congruence

$$x^2+y^2 \equiv a \pmod q. \tag{1.22}$$

Note that the solutions to this congruence are doubly-periodic. So we can visualize each solution in the Euclidean plane and will have that the solutions tile the Euclidean plane with $q\times q$ square tiles, each copies of each other. We now consider a tiling where one of the tiles is centered at the origin. Each tile contains $\eta_a(q)$ points (if q is even, there may be points on the boundary of the tile so the definition of a tile containing a point has to chosen slightly more carefully).

gcp_images_for_thesis/lattice_for_points_1_mod_7_radius_30.png

Figure 1.2: Visual for $S_{7,1}(900)$.

So for large x compared to q, we see $S_{q,a}(x)$ is approximately the number of integer points, weighted by $\eta_a(q)$, inside the circle of radius $\sqrt{x}/q$. See Figure 1.2 and Figure 1.3 for an example of this tiling with $a = 1, q = 7, x = 900$. We now define the remainder term

$$R_{q,a}(x) = S_{q,a}(x) - \pi \frac{\eta_a(q)}{q^2} x. \tag{1.23}$$

In 1968 Smith [10] showed that if $q = O(x^{2/3})$, then

$$R_{q,a}(x) \ll x^{\frac{2}{3}+\xi} q^{-\frac{1}{2}(1+3\xi)} (q, a)^{1/2} \tau(q)$$

for any $0 < \xi < 1/3$. Here $\tau(q)$ is the divisor function and (q, a) means the highest common factor of a and q.

In 2011 Tolev [11] proved that

$$R_{q,a}(x) \ll (q^{\frac{1}{2}} + x^{\frac{1}{3}})(a, q)^{\frac{1}{2}} \tau^4(q) \log^4(x). \tag{1.24}$$

This bound is stronger than Smith's if $q = O(x^{2/3})$. In the last two chapters of this paper, we will prove the following theorem:

Theorem 9. *For $R_{q,a}(x)$ defined by (1.23) we have*

$$R_{q,a}(x) \ll \left(x^{27/82} q^{115/82} + q^2 \right) \tau((a, q)) \log x.$$

This bound is stronger than Tolev's for fixed q or q that grows very slowly compared to x but is weaker otherwise.

gcp_images_for_thesis/tiled_lattice_for_points_1_mod_7_radius_30.png

Figure 1.3: Tiled visual for $S_{7,1}(900)$.

Chapter 2

The main term for the Gauss circle problem for arithmetic progressions

Recall (1.21), in order to calculate this main term of $S_{q,a}(x)$, we must first calculate $\eta_a(q)$. We will prove an explicit formula for $\eta_a(q)$ by first calculating $\eta_a(q)$ for prime q, then using this result to prove the case where q is a prime power, and then we will show that $\eta_a(q)$ is multiplicative to get a result for q in general. First we let

$$U_{a,q} = \{(x,y) \in (\mathbb{Z}/q\mathbb{Z})^2 : x^2 + y^2 \equiv a \pmod{q}\}, \tag{2.1}$$

so $\eta_a(q) = |U_{a,q}|$. We will use $U_{a,q}$ repeatedly throughout this chapter. We now prove the following theorem which gives an explicit formula for $\eta_a(q)$ if q is an odd prime:

Theorem 10. *Let p be an odd prime.*
If $p \equiv 1 \ (4)$, then

$$\eta_a(p) = \begin{cases} 2p-1, & p \mid a, \\ p-1, & p \nmid a. \end{cases}$$

If $p \equiv 3 \ (4)$, then

$$\eta_a(p) = \begin{cases} 1, & p \mid a, \\ p+1, & p \nmid a. \end{cases}$$

Proof. We will be counting the number of solutions to (1.22) in $(\mathbb{Z}/p\mathbb{Z})^2$.

Case 1: $a = 0$. If $y = 0$, then (1.22) has only one solution. Suppose now that we fix some $y \not\equiv 0 \pmod{p}$. In this case we have to count the number of solutions to

$$(x\overline{y})^2 \equiv -1 \pmod{p}. \tag{2.2}$$

If $p \equiv 3 \pmod 4$, then by the First Nebensatz this gives no solutions, so we have one solution when $p \equiv 3 \pmod 4$ in total. If $p \equiv 1 \pmod 4$, then by the First Nebensatz, (2.2) has solutions only when $x\overline{y} \equiv \pm k \pmod{p}$, for some k dependent on p. For any non-zero y, there will be two x values that solve this equation, so we have $2(p-1) + 1 = 2p - 1$ solutions when $p \equiv 1 \ (4)$ in total.

Case 2: $a \neq 0$. For this case we follow a slightly generalized argument from [8]. We consider first the case that a is a quadratic residue (QR) mod p. Suppose that $x \neq 0$ and $y \neq 0$. Then $a = b^2$ (for some b) and (1.22) can be rewritten as $1 \equiv (b\overline{y} - x\overline{y})(b\overline{y} + x\overline{y}) \pmod{p}$. Now let $m = b\overline{y}$ and $n = x\overline{y}$. Then $m + n = k$, for some k, and $m - n = k^{-1}$. So $m = (k + k^{-1})/2$ and $n = (k - k^{-1})/2$ hence k determines m and n, which determines y and then x because $x, y \neq 0$. We require $m \neq 0$ and $n \neq 0$ so: If $p \equiv 1 \pmod{4}$ then we get $p - 5$ possibilities for k and if $p \equiv 3 \pmod{4}$ then we get $p - 3$ possibilities for k (This comes from the First Nebensatz). We now have to consider the solutions $(0, 1), (1, 0), (-1, 0), (0, -1)$ for when $x = 0$ or $y = 0$. Hence we've proven the theorem for a a quadratic residue.

Now suppose a is a quadratic non-residue (QNR). By the Primitive Root Theorem, there exists a primitive root g of p and any QNR is an odd power of g. Now consider the function

$$f : U_{a,p} \to U_{ag^{2k},p}, \qquad \text{with} \qquad f(x, y) = (xg^k, yg^k).$$

Since $(g, p) = 1$, this function gives a bijection between $U_{a,p}$ and $U_{ag^{2k},p}$ for every $k \in \mathbb{Z}$ (even negative k), so $\eta_a(p) = \eta_{ag^{2k}}(p)$. Combining our calculations for the case $a = 0$ or a is a QR with the fact that there are $(p - 1)/2$ QRs and $(p - 1)/2$ QNRs, we obtain:

If $p \equiv 1 \pmod{4}$: There are

$$\left(p^2 - \frac{p-1}{2}(p-1) - (2p-1)\right) / \left(\frac{p-1}{2}\right) = p - 1$$

solutions for a a QNR.

If $p \equiv 3 \pmod{4}$: There are

$$\left(p^2 - \frac{p-1}{2}(p+1) - 1\right) / \left(\frac{p-1}{2}\right) = p + 1$$

solutions for a a QNR. □

Theorem 11. *The function $\eta_a(q)$ is multiplicative with respect to q.*

We first prove the following lemma.

Lemma 1. *Let $m, n \in \mathbb{N}$ such that $(m, n) = 1$. Then there exists $c \in \mathbb{N}$ such that*

$$c^2(m^2 + n^2) \equiv 1 \pmod{mn}.$$

Proof. Let $c = \overline{m + n}$. Observe that $(mn, m + n) = 1$ so c exists. Then $c^2(m^2 + n^2) \equiv c^2(m + n)^2 \equiv 1 \pmod{mn}$. □

Proof of Theorem 11. We will show that there exists an injective function from $U_{a,m} \times U_{a,n}$ to $U_{a,mn}$ and an injective function from $U_{a,mn}$ to $U_{a,m} \times U_{a,n}$. We use c from Lemma 1 and define

$$f : U_{a,m} \times U_{a,n} \to U_{a,mn}, \qquad \text{with} \qquad f((x_1, y_1), (x_2, y_2)) = (cnx_1 + cmx_2, cny_1 + cmy_2). \tag{2.3}$$

By simple modular arithmetic we see

$$\begin{aligned}(cnx_1 + cmx_2)^2 + (cny_1 + cmy_2)^2 &\equiv c^2n^2a + c^2m^2a \pmod{mn} \\ &\equiv a \pmod{mn},\end{aligned}$$

therefore this function does map elements from $U_{a,m} \times U_{a,n}$ to $U_{a,mn}$. Next we show that f is injective. Let $(x_1, y_1), (u_1, v_1) \in U_{a,m}$ and $(x_2, y_2), (u_2, v_2) \in U_{a,n}$, that is

$$x_1^2 + y_1^2 \equiv a \pmod{m},$$
$$u_1^2 + v_1^2 \equiv a \pmod{m},$$
$$x_2^2 + y_2^2 \equiv a \pmod{n},$$
$$u_2^2 + v_2^2 \equiv a \pmod{n},$$

and suppose

$$cnx_1 + cmx_2 \equiv cnu_1 + cmu_2 \pmod{mn},$$
$$cny_1 + cmy_2 \equiv cnv_1 + cmv_2 \pmod{mn}.$$

Then we have

$$n(x_1 - u_1) - m(u_2 - x_2) \equiv 0 \pmod{mn},$$
$$n(x_1 - u_1) - m(u_2 - x_2) \equiv 0 \pmod{mn}.$$

Since $(m, n) = 1$, it follows from the last two displayed equations that $x_1 \equiv u_1 \ (m)$, and $x_2 \equiv u_2 \ (n)$, and $y_1 \equiv v_1 \ (m)$, and $y_2 \equiv v_2 \ (n)$. So f is an injective function.

We now describe an injective function from $U_{a,mn}$ to $U_{a,m} \times U_{a,n}$. Define the function

$$g : U_{a,mn} \to U_{a,m} \times U_{a,n} \qquad \text{with} \qquad g((x, y)) := ((x, y), (x, y)).$$

This function clearly maps elements from $U_{a,mn}$ to $U_{a,m} \times U_{a,n}$ and is also injective by Chinese remainder theorem, which can be used since $(m, n) = 1$. $\square$

Theorem 12. *Let $k \in \mathbb{N}_{\geq 2}$. Then*

$$\eta_a(2^k) = \begin{cases} 2^{k+1}, & a \equiv 1 \quad (4), \\ 0, & a \equiv 3 \quad (4), \\ 2\eta_{a/2}(2^{k-1}), & a \equiv 0 \quad (2). \end{cases}$$

Proof. Case 1: $a \equiv 1$ (4). We will prove this by showing $2\eta_a(2^k) = \eta_a(2^{k+1})$ and then calculating a base case.

We assume first that $k \geq 2$. Now let $R_{a,q,r}$ be the set of solutions to (1.22) such that $0 \leq x, y < r$ (and so $R_{a,q,q} = U_{a,q}$). Observe now that $(x, y) \in U_{a,2^{k+1}}$ is equivalent to

$$(2^{k+1} - x, y) \in U_{a,2^{k+1}}, \qquad \text{or} \qquad (x, 2^{k+1} - y) \in U_{a,2^{k+1}}, \qquad \text{or} \qquad (2^{k+1} - x, 2^{k+1} - y) \in U_{a,2^{k+1}},$$

hence $\eta_a(2^{k+1}) = 4\left|R_{a,2^{k+1},2^k}\right|$. So we want to show $\eta_a(2^k) = 2\left|R_{a,2^{k+1},2^k}\right|$. Observe that the identity map from $U_{a,2^k}$ to $R_{a,2^{k+1},2^k} \cup R_{a+2^k,2^{k+1},2^k}$ is a bijection, because for every $(x, y) \in U_{a,2^k}$ we must have

$$x^2 + y^2 \equiv a \pmod{2^{k+1}}, \qquad \text{or} \qquad x^2 + y^2 \equiv a + 2^k \pmod{2^{k+1}},$$

and the identity map is clearly injective and surjective, so $\eta_a(2^k) = \left|R_{a,2^{k+1},2^k}\right| + \left|R_{a+2^k,2^{k+1},2^k}\right|$. Furthermore, the function

$$f : R_{a,2^{k+1},2^k} \to R_{a+2^k,2^{k+1},2^k} \qquad \text{with} \qquad f((x, y)) = \begin{cases} (x - 2^{k-1}, y), & x \text{ odd}, \\ (x, y - 2^{k-1}), & x \text{ even}, \end{cases}$$

also gives a bijection. To see this, first note that for $(x, y) \in U_{a,2^k}$, one of x, y must be odd and the other even because we are assuming $a \equiv 1$ (4). Suppose that x is odd, then $f((x, y)) = (x - 2^{k-1}, y)$ and

$$\begin{aligned}(x - 2^{k-1})^2 + y^2 &\equiv x^2 + y^2 - 2^k x \quad (2^{k+1}) \\ &\equiv a + 2^k \quad (2^{k+1}).\end{aligned}$$

If, instead, y is odd, then $f(x, y)$ again maps to the correct space. This function can then simply be shown to be injective and surjective. We therefore have that $\eta_a(2^k) = 2\left|R_{a,2^{k+1},2^k}\right|$ which is what we wanted to show, hence $2\eta_a(2^k) = \eta_a(2^{k+1})$. Now we evaluate the base case $k = 2$, which gives us $\eta_1(2^2) = 8$, and so we we get the desired result.

Case 2: $a \equiv 3$ (4). If $k \geq 2$, then by reducing (1.22) modulo 4, we see $x^2 + y^2 \equiv 3$ (4). This congruence has no solutions so we get $U_{a,2^k} = \emptyset$.

Case 3: $a \equiv 0$ (2). We will first show $\eta_a(2^k) = 4\eta_{a/4}(2^{k-2})$ and then calculate two base cases (we do it this way because 4 is a square number).

Assume $k \geq 2$ and consider the function

$$f : U_{a,2^k} \to R_{4a,2^{k+2},2^{k+1}}, \qquad \text{with} \qquad f(x, y) = (2x, 2y).$$

Note that $0 \leq 2x, 2y \leq 2^{k+1}$ and $(2x)^2 + (2y)^2 \equiv 4a$ (2^{k+2}) so this function is indeed well-defined. Now let $(x_1, y_1), (x_2, y_2) \in U_{a,2^k}$. If $(2x_1, 2y_1) = (2x_2, 2y_2)$ in $R_{a,2^{k+2},2^{k+1}}$, then $x_1 \equiv x_2$ (2^k) and $y_1 \equiv y_2$ (2^k) so f is injective. It is also surjective since $x^2 + y^2 \equiv 4a$ (2^{k+2}) implies x and y are even, so we have

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 \equiv a \pmod{2^k},$$

therefore $f(x/2, y/2) = (x, y)$.

So f is bijective which means we get the desired recurrence relation. We now deal with the base cases $k = 1$ and $k = 2$ by hand for which we see $\eta_a(2^2) = 4$ and $\eta_{a/2}(2) = 2$ hence $\eta_a(2^k) = 2\eta_{a/2}(2^{k-1})$ as required. ☐

Note that there's one very simple case the above theorem doesn't deal with, that is $d = 2$. For this case we can just calculate $\eta_a(2) = 2$ by hand.

Now we prove the following theorem, allowing us to calculate $\eta_a(q)$ recursively for q an odd prime power:

Theorem 13. *Let p be an odd prime and let $k \in \mathbb{N}$.*
If $p \equiv 1$ mod 4, then

$$\eta_a(p^k) = \begin{cases} 2p^{k-1}(p-1) + p^2\eta_{a/p^2}(p^{k-2}), & p^2 \mid a, \\ 2p^{k-1}(p-1), & p \mid a, p^2 \nmid a, \\ p^{k-1}(p-1), & p \nmid a. \end{cases}$$

If $p \equiv 3$ mod 4, then

$$\eta_a(p^k) = \begin{cases} p^2\eta_{a/p^2}(p^{k-2}), & p^2 \mid a, \\ 0, & p \mid a, p^2 \nmid a, \\ p^{k-1}(p+1), & p \nmid a. \end{cases}$$

Proof. *Case 1:* $p^2 \mid a$. Let $a = p^n c$ where $p \nmid c$.

Looking at (1.22), we see that if $p \mid y$ then $p \mid x$ and so if we let $x = px_1$ and $y = py_1$, we can reduce (1.22) to $x_1^2 + y_1^2 \equiv p^{n-2}c \ (p^{k-2})$. Note that x_1 and y_1 were defined in $(\mathbb{Z}/p^{k-1}\mathbb{Z})^2$, so when counting solutions (x_1, y_1) in the space $(\mathbb{Z}/p^{k-2}\mathbb{Z})^2$ we get a factor of p^2. Hence there are $p^2\eta_{a/p^2}(p^{k-2})$ solutions.

If $p \nmid y$ then $p \nmid x$. First consider $p \equiv 3 \ (4)$. Then we can reduce (1.22) to $(x\overline{y})^2 \equiv -1 \ (p)$ but this has no solutions (follows from the First Nebensatz), so in this case we get no extra solutions which proves the result for $p \equiv 3 \ (4)$. Now consider $p \equiv 1 \ (4)$ and fix a value y, coprime to p. Let

$$f(x) = x^2 + y^2 - p^n c.$$

We know there exists an x_0 such that $(x_0)^2 \equiv -y^2 \ (p)$ (this follows from the First Nebensatz). So, $f(x_0) \equiv 0 \ (p)$ and we can see $f'(x_0) \not\equiv 0 \ (p)$, hence we can use Hensel's Lemma to get a root of f modulo any power of p. Denote the solution that we obtain modulo p^k by z. Now suppose we have another solution w. Note that $p \nmid z$ and $p \nmid w$ since $p \nmid y$. Then we have $z^2 + y^2 \equiv p^n c \ (p^k)$ and $w^2 + y^2 \equiv p^n c \ (p^k)$ which implies $(w - z)(w + z) \equiv 0 \ (p^k)$. This means $w \equiv \pm z \ (p^k)$ because $p \nmid 2z$. We see that $w = \pm z$ are both solutions and must be the only solutions for a fixed y. There are $p^{k-1}(p-1)$ possible y values that are coprime to p, and so we have $2p^{k-1}(p-1)$ extra solutions in this case, which proves the result for $p \equiv 1 \ (4)$.

Case 2: $p \mid a$, $p^2 \nmid a$. Let $a = pc$ where $p \nmid c$. Looking at (1.22), we see that if $p \mid y$ then $p \mid x$. Reducing (1.22) modulo p^2 then gives $0 \equiv pc \ (p^2)$ so we get no solutions if $p \mid y$. We now consider $p \nmid x$. To deal with this case, we consider the function

$$f(x) = x^2 + y^2 - pc,$$

for a fixed y coprime to p, and use the same reasoning as in Case 1 by applying Hensel's Lemma to get the desired results.

Case 3: $p \nmid a$. We will prove this in a similar way to how we did for Case 1 of Theorem 12 by relating the solutions to (1.22) for p^k and for p^{k+1}, and then applying induction. We now show $\eta_a(p^k) = p\eta_a(p^{k-1})$.

Let $T_{a,q,m}$ be the set of pairs $(x, y) \in (\mathbb{Z}/(mq)\mathbb{Z})^2$ that are solutions to (1.22). We have

$$m^2\eta_a(q) = |T_{a,q,m}|.$$

Now observe that the identity map gives a bijection from

$$T_{a,p^k,p} \to \bigcup_{n=0}^{p-1} U_{a+np^k,p^{k+1}}.$$

So

$$p^2\eta_a(p^k) = \sum_{n=0}^{p-1} \eta_{a+np^k}(p^{k+1}). \tag{2.4}$$

We will now show that $\eta_{a+np^k}(p^{k+1})$ is the same value for each n by constructing bjiections. Consider the collection of maps $f_{\alpha,\beta} : U_{a+\alpha p^k,p^{k+1}} \to U_{a+\beta p^k,p^{k+1}}$, with $\alpha, \beta \in \{1, \ldots, p-1\}$ and $\alpha \neq \beta$, described by

$$f_{\alpha,\beta}(x,y) := \begin{cases} (x + (\beta-\alpha)\overline{2x}p^k, y), & p \nmid x, \\ (x, y + (\beta-\alpha)\overline{2y}p^k), & p \mid x. \end{cases}$$

where $\overline{2x}$ is the inverse of $2x$ modulo p^{k+1}. Note that we make use of the fact $p \nmid a$ in this definition since $p \nmid a$ implies $p \nmid x$ or $p \nmid y$, hence the required inverses exist.

As an example to see that this set of functions maps to the correct space, choose $\alpha = 0, \beta = 1$ and take some $(x,y) \in U_{a,p^{k+1}}$ such that $p \nmid x$. Then, $f_{0,1}(x,y) = (x + \overline{2x}p^k, y)$. Observe that

$$(x + \overline{2x}p^k)^2 + y^2 \equiv a + 2x\overline{2x}p^k \equiv a + p^k \quad (p^{k+1})$$

hence $f_{0,1}(x,y)$ maps from $U_{a,p^{k+1}}$ to $U_{a+p^k,p^{k+1}}$ as required, and this remains true for α, β, x, y in general.

We now show that each $f_{\alpha,\beta}$ is injective. Let $(x_1,y_1),(x_2,y_2) \in U_{a+\alpha p^k,p^{k+1}}$ such that $f_{\alpha,\beta}(x_1,y_1) = f_{\alpha,\beta}(x_2,y_2)$. Then we want to show $x_1 \equiv x_2 \ (p^{k+1})$ and $y_1 \equiv y_2 \ (p^{k+1})$

Suppose $p \mid x_1, p \mid x_2$. Then we have $x_1 \equiv x_2 \ (p^{k+1})$ and

$$y_1 + (\beta-\alpha)\overline{2y_1}p^k \equiv y_2 + (\beta-\alpha)\overline{2y_2}p^k \quad (p^{k+1}). \tag{2.5}$$

This means that $y_1 \equiv y_2 \ (p^k)$ so $\overline{y_1} \equiv \overline{y_2} \ (p^k)$. So then (2.5) shows that $y_1 \equiv y_2 \ (p^{k+1})$. So we have shown $x_1 \equiv x_2 \ (p^{k+1})$ and $y_1 \equiv y_2 \ (p^{k+1})$ hence $f_{\alpha,\beta}$ is injective in this case. By a similar argument, we see that $f_{\alpha,\beta}$ is also injective in the case $p \nmid x_1, p \nmid x_2$.

Now suppose that $p \mid x_1, p \nmid x_2$. Then $x_1 \equiv x_2 + (\beta-\alpha)\overline{2x_2}p^k \ (p^{k+1})$. As $k \geq 1$, we reach a contradiction by reducing this congruence modulo p. By a similar argument, we reach a contradiction in the case $p \nmid x_1, p \mid x_2$.

Hence we have found an injective function from $U_{a+\alpha p^k,p^{k+1}}$ to $U_{a+\beta p^k,p^{k+1}}$ for all $\alpha, \beta \in \{1, \ldots, p-1\}$, with $\alpha \neq \beta$. So

$$\eta_{a+\alpha p^k}(p^{k+1}) = \eta_{a+\beta p^k}(p^{k+1}).$$

Therefore

$$\sum_{n=0}^{p-1} \eta_{a+np}(p^{k+1}) = p\eta_a(p^{k+1}),$$

so by (2.4), we have

$$\eta_a(p^{k+1}) = p\eta_a(p^k).$$

We now simply refer to Theorem 10 to get our base cases. If $p \equiv 1 \pmod 4$: $\eta_a(p) = p-1$ so $\eta_a(p^k) = p^{k-1}(p-1)$. If $p \equiv 3 \pmod 4$: $\eta_a(p) = p+1$ so $\eta_a(p^k) = p^{k-1}(p+1)$. $\square$

We are now able to calculate $\eta_a(q)$ for every $a \in \mathbb{Z}$, and $q \in \mathbb{N}$. Next we state an equivalent form of Theorem 12 and then an equivalent form of Theorem 13 which will be easier for us to work with.

Theorem 14. *Let $r \geq 2$ and $a = 2^s A$ where A is odd, and say $A = 0$ if $r \leq s$, then*

$$\eta_a(2^r) = 2^r(1 + \mathbf{1}_{\{r-s\geq 2\}}\chi_4(A)),$$

where $\mathbf{1}_{\{r-s\geq 2\}} = 1$, if $r - s \geq 2$, and is zero otherwise.

Proof. We split the proof into two cases based on whether $r - s \geq 2$ or not.

If $r - s \geq 2$, then we must show

$$\eta_a(2^r) = 2^r(1 + \chi_4(A)).$$

We will prove this by induction on r. The statement can be verified for the base case $r = 2$. Now assume the statement holds for all $r \leq n$ for some n. Consider $r = n + 1$.

If $2 \nmid a$, then by Theorem 12 we see $\eta_a(2^{n+1}) = 2^{n+1}(1 + \chi_4(A))$. If $2 \mid a$, then by Theorem 12 we get $\eta_a(2^{n+1}) = 2\eta_{a/2}(2^n)$ and then by the inductive hypothesis we have $\eta_a(2^{n+1}) = 2(2^n(1 + \chi_4(A))$. So the result holds for $n + 1$.

If $r - s < 2$, then we must show

$$\eta_a(2^r) = 2^r. \tag{2.6}$$

Since $r - s < 2$, we must have

$$\eta_a(2^r) = 2^{r-1}\eta_k(2),$$

for some $k \in \mathbb{Z}$ by Theorem 12. Since $\eta_0(2) = \eta_1(2) = 2$, we then get (2.6). □

Theorem 15. *Let $k \in \mathbb{N}$, $a \in \mathbb{Z}$, p be an odd prime, and χ_4 be the non-trivial Dirichlet character modulo 4.*
If $v_p(a) < k$, then

$$\eta_a(p^k) = p^{k-1}(p - \chi_4(p)) \sum_{0 \leq b \leq v_p(a)} \chi_4(p^b). \tag{2.7}$$

If $v_p(a) \geq k$, then

$$\eta_a(p^k) = p^k\chi_4(p^k) + p^{k-1}(p - \chi_4(p)) \sum_{0 \leq b \leq k-1} \chi_4(p^b). \tag{2.8}$$

Proof. We prove (2.7) first. First consider the case where $p \equiv 1$ (4). In this case, proving (2.7) is equivalent to showing

$$\eta_a(p^k) = p^{k-1}(p - 1)(v_p(a) + 1), \tag{2.9}$$

for $v_p(a) < k$. We will prove two base cases and then do induction on $v_p(a)$. If $v_p(a) = 0$ or $v_p(a) = 1$, then the result follows from Theorem 13. Suppose now the result holds for all $v_p(a)$ up to an integer $n \geq 1$. We now consider $v_p(a) = n + 2$. By Theorem 13 and the inductive hypothesis, we have

$$\begin{aligned}
\eta_a(p^k) &= 2p^{k-1}(p - 1) + p^2\eta_{a/p^2}(p^{k-2}) \\
&= 2p^{k-1}(p - 1) + p^2(p^{k-3}(p - 1)(n + 1)) \\
&= p^{k-1}(p - 1)(n + 3),
\end{aligned}$$

as required.

The case $p \equiv 3$ (4) for (2.7) can be proved in a very similar way, by proving the equivalent statement

$$\eta_a(p^k) = \begin{cases} p^{k-1}(p+1), & v_p(a) \text{ even}, \\ 0, & v_p(a) \text{ odd}, \end{cases} \tag{2.10}$$

for $v_p(a) < k$, by performing induction on $v_p(a)$.

Now we prove (2.8). First consider the case where $p \equiv 1$ (4). In this case, proving (2.8) is equivalent to showing

$$\eta_a(p^k) = p^k + p^{k-1}(p-1)k, \tag{2.11}$$

for $v_p(a) \geq k$. We will prove two base cases and then do induction on k. If $k = 0$, the result is trivial, and if $k = 1$, then the result follows from Theorem 10. Suppose now the result holds for all k up to an integer $n \geq 1$. We now consider $k = n + 2$. By Theorem 13 and the inductive hypothesis, we have

$$\begin{aligned} \eta_a(p^k) &= 2p^{k-1}(p-1) + p^2\eta_{a/p^2}(p^{k-2}) \\ &= 2p^{n+1}(p-1) + p^2(p^n + p^{n-1}(p-1)(n)) \\ &= p^{n+2} + p^{n+1}(p-1)(n+2), \end{aligned}$$

as required.

The case $p \equiv 3$ (4) for (2.8) can be proved in a very similar way, by proving the equivalent statement

$$\eta_a(p^k) = \begin{cases} p^k, & k \text{ even}, \\ p^{k-1}, & k \text{ odd}, \end{cases} \tag{2.12}$$

for $v_p(a) \geq k$, by performing induction on k. $\square$

Corollary 1. $\eta_a(q) \ll q\tau(q)$

Proof. Let $k \in \mathbb{N}$. By Theorem 14 we see $\eta_a(2^k) \leq 2^k(k+1)$, and by looking at each of (2.9), (2.10), (2.11), and (2.12), we see that $\eta_a(p^k) \leq p^k(k+1)$. Now let $q = \prod_{i=1}^{n} {p_i}^{k_i}$. Then by multiplicativity (Theorem 11), we obtain

$$\eta_a(q) = \prod_{i=1}^{n} \eta_a({p_i}^{k_i}) \leq \prod_{i=1}^{n} p^{k_i}(k_i + 1) = q\tau(q),$$

hence $\eta_a(q) \ll q\tau(q)$. $\square$

Chapter 3

Tolev's bound

3.1 Introduction.

It follows from (1.5) that $r(n) \ll \tau(n)$. We now aim to bound $\tau(n)$ using the following result by Chen.

Theorem 16 ([1], Theorem 1.6.)**.** *For any $\epsilon > 0$ we have*

$$\tau(n) \ll_\epsilon n^\epsilon.$$

Proof. Let $n = \prod_{j=1}^{r} {p_j}^{u_j}$. By multiplicativity of $\tau(n)$, it follows that

$$\frac{\tau(n)}{n^\epsilon} = \prod_{j=1}^{r} \frac{(u_j+1)}{{p_j}^{\epsilon u_j}}.$$

Clearly $\tau(n) \leq n$ so we only need to prove the case $\epsilon < 1$. If $2 \leq p_j < 2^{1/\epsilon}$ then we have

$$p^{\epsilon u_j} \geq 2^{\epsilon u_j} = e^{\epsilon u_j \log 2} > 1 + \epsilon u_j \log 2 > (1+u_j)\epsilon \log 2,$$

where we have used that $e^x > 1 + x$ for positive x. It then follows that

$$\frac{(u_j+1)}{p^{\epsilon u_j}} < \frac{1}{\epsilon \log 2}.$$

On the other hand, if $p_j \geq 2^{1/\epsilon}$, then

$$\frac{(u_j+1)}{{p_j}^{\epsilon u_j}} \geq \frac{u_j+1}{2^{u_j}}.$$

So we obtain

$$\frac{\tau(n)}{n^\epsilon} < \prod_{p<2^{1/\epsilon}} \frac{1}{\epsilon \log 2},$$

a positive constant depending only on ϵ. □

Using Theorem 16 we get $r(n) \ll_\epsilon n^\epsilon$. Hence

$$\begin{aligned} S_{q,a}(x) &\ll_\epsilon x^\epsilon \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} 1 \\ &\ll_\epsilon x^{1+\epsilon} q^{-1}. \end{aligned} \tag{3.1}$$

For $q > x^{2/3}$, (1.24) follows from (3.1), and (1.23) combined with Corollary 1. So we are left to deal with the case $q \leq x^{2/3}$ and this is the case Tolev deals with in his paper. For the remainder of this chapter we describe his proof.

We first observe that $S_{q,a}(x)$ simply counts the number of pairs of integers u, v such that

$$u^2 + v^2 \leq x, \qquad u^2 + v^2 \equiv a \pmod{q}. \tag{3.2}$$

So if we let S' be the number of pairs of positive integers u, v satisfying (3.2), and S'' be the number of positive integers u satisfying

$$u^2 \leq x, \qquad u^2 \equiv a \pmod{q},$$

then

$$S_{q,a}(x) = 4S' + 4S'' + O(1). \tag{3.3}$$

We will show why this is the case with an example figure:

gcp_images_for_thesis/gcpfap_20_5_s_p_s_d_p_visual.png

Figure 3.1: Visual for the points that S' and S'' count for $q = 5$, $a = 0$, $x = 400$.

As this figure illustrates, the factor of 4 for S' and S'' in (3.3) comes from considering all quadrants, and $O(1)$ comes from consideration of the origin. For the remainder of Chapter 3, by u and v we mean natural numbers only.

Clearly

$$S'' = \frac{\omega_a(q)}{q}\sqrt{x} + O(\omega_a(q)), \tag{3.4}$$

where $\omega_a(q) = \#\{1 \leq \alpha \leq q : \alpha^2 \equiv a \ (q)\}$.

To understand the asymptotic behaviour of S'', it therefore suffices to understand the asymptotic behaviour of $\omega_a(q)$.

Theorem 17. *For $\omega_a(q)$ as defined above, we have*

$$\omega_a(q) \ll (q,a)^{\frac{1}{2}} \tau(q).$$

Proof. Let $q = \prod_{i=1}^{r} {p_i}^{k_i}$. Consider the function

$$f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/{p_1}^{k_1}\mathbb{Z} \times ... \times \mathbb{Z}/{p_r}^{k_r}\mathbb{Z} \qquad \text{with} \qquad f(\alpha) = (\alpha, \ldots, \alpha).$$

Observe that this function is injective by the Chinese remainder theorem and that $\alpha^2 \equiv a \pmod q$ implies $\alpha^2 \equiv a \pmod{{p_i}^{k_i}}$, hence $\omega_a(q) \leq \prod_{i=1}^{r} \omega_a({p_i}^{k_i})$. We now focus on bounding $\omega_a(p^k)$ for p a prime and $k \in \mathbb{N}$.

Case 1: p is odd. Let $a = bp^{2d}$ where $p^2 \nmid b$ and let $k = 2d + e$. Then $\alpha^2 \equiv bp^{2d} \pmod{p^{2d+e}}$ so $(\alpha/p^d)^2 \equiv b \pmod{p^e}$. Now we will bound the number of solutions (over $\mathbb{Z}/p^e\mathbb{Z}$) to the congruence $\beta^2 \equiv b \pmod{p^e}$. Note that if $e = 0$ or $e = 1$ then this congruence has at most two solutions, so now assume $e \geq 2$. Suppose we have two solutions to this congruence, m and n. Then $(m-n)(m+n) \equiv 0 \pmod{p^e}$ and $(m+n) - (m-n) = 2n$. Since $e \geq 2$, we can not have $p \mid n$ otherwise $p^2 \mid b$ which contradicts the definition of b. Since p is an odd prime, we then see $p \nmid 2n$ so p must divide only one of the numbers $(m-n)$ and $(m+n)$, so we must have that $m \equiv n \pmod{p^e}$ or $m \equiv -n \pmod{p^e}$. Hence there are at most two solutions to $\beta^2 \equiv b \pmod{p^e}$.

Using this bound, we now bound the number of solutions to the congruence $(\alpha/p^d)^2 \equiv b \pmod{p^e}$. We must consider the subtle point that α was defined in the space $\mathbb{Z}/(p^{2d+e})\mathbb{Z}$, hence α/p^d is defined in the space $\mathbb{Z}/(p^{d+e})\mathbb{Z}$. But for our congruence we are considering α/p^d in the space $\mathbb{Z}/p^e\mathbb{Z}$ hence for every β, there are p^d values of α which map to that β. So in total this congruence actually has at most $2p^d$ solutions. And note that clearly $p^{2d} \mid (a, p^k)$ so $p^d \leq (a, p^k)^{\frac{1}{2}}$ and so $\omega_a(p^k) \leq 2(a, p^k)^{\frac{1}{2}}$.

Case 2: $p = 2$. We approach this case in a very similar way but end up with a slightly worse bound. Let $a = b2^{2d}$ where $2^2 \nmid b$. Now let $k = 2d + e$, then $\alpha^2 \equiv b2^{2d} \pmod{2^{2d+e}}$ so $(\alpha/2^d)^2 \equiv b \pmod{2^e}$. Now we will bound the number of solutions to the congruence $\beta^2 \equiv b \pmod{2^e}$. Note that if $e = 0$ or $e = 1$ then this congruence has at most one solution so now assume $e \geq 2$. Suppose we have two solutions to this congruence, m and n. Then $(m-n)(m+n) \equiv 0 \pmod{2^e}$. Now note that $(m+n) - (m-n) = 2n$. Since $e \geq 2$, we can not have $2 \mid n$ otherwise $2^2 \mid b$ which contradicts the definition of b. So $2 \nmid n$ and so $2 \mid 2n$ but $2^2 \nmid 2n$, so we must have that 2 divides both $(m-n)$ and $(m+n)$ but 4 only divides one of these numbers. This means we have two cases.
The first case, $(m-n) \equiv 0 \pmod 2$ and $(m+n) \equiv 0 \pmod{2^{e-1}}$. This implies $m \equiv -n \pmod{2^e}$ or $m \equiv -n + 2^{e-1} \pmod{2^e}$.
The second case, $(m+n) \equiv 0 \pmod 2$ and $(m-n) \equiv 0 \pmod{2^{e-1}}$. This implies $m \equiv n \pmod{2^e}$ or $m \equiv n + 2^{e-1} \pmod{2^e}$.
So altogether this gives at most 4 solutions to $\beta^2 \equiv b \pmod{2^e}$.

For the same reason as in Case 1, we have to include a factor of 2^d due to the spaces our variables are defined in. We note that $2^{2d} \mid (a, 2^k)$ implies $2^d \leq (a, 2^k)^{\frac{1}{2}}$ and so $\omega_a(2^k) \leq 4(a, 2^k)^{\frac{1}{2}}$.

Now that we have dealt with both cases, we get

$$\omega_a(q) \leq \prod_{i=1}^{r} \omega_a({p_i}^{k_i}) \leq 2 \prod_{i=1}^{r} 2(a, {p_i}^{k_i})^{\frac{1}{2}}.$$

This factor of 2 outside the product comes from whether q is odd or even because we may have to use the slightly greater bound that we calculated in Case 2 above. Then

$$2\prod_{i=1}^{r} 2(a, {p_i}^{k_i})^{\frac{1}{2}} = 2(a,q)^{\frac{1}{2}} \prod_{i=1}^{r} 2 \leq 2(a,q)^{\frac{1}{2}} \tau(q),$$

where we used a multiplicative property of $(-,-)$ and that $\prod_{i=1}^{r} 2 \leq \tau(q)$. Hence $\omega_a(q) \ll (q,a)^{\frac{1}{2}} \tau(q)$ as desired. $\square$

We now focus on S'. We will count points on S' in a way similar to Dirichlet's hyperbola method, but under a quarter-circle instead. The following plot shows specifically how we will count the points:

gcp_images_for_thesis/S_prime_split_figure.png

Figure 3.2: Counting S' for $x = 400$. Points in the green and red domains are contained in the circle.

To calculate S', we will sum over the points in S' that are in the green and red domain but notice that we will then be double counting the points that lie in both a green and red domain. We now formalise this idea algebraically.

We have that

$$S' = 2S_1 - S_2, \tag{3.5}$$

where S_1 is the number of pairs of natural numbers u, v such that

$$u^2 + v^2 \equiv a(q), \qquad u \leq \sqrt{x/2}, \qquad v \leq \sqrt{x - u^2}, \tag{3.6}$$

and S_2 is the number of pairs of natural numbers u, v such that

$$u^2 + v^2 \equiv a(q), \qquad u \leq \sqrt{x/2}, \qquad v \leq \sqrt{x/2}. \tag{3.7}$$

We now divide S_1 and S_2 into parts based on the congruence classes of u, v modulo q. For simplicity we will henceforth write

$$\sum_{\alpha,\beta} \qquad \text{for} \qquad \sum_{\substack{1\le\alpha,\beta\le q \\ \alpha^2+\beta^2\equiv a(q)}} .$$

Using this notation we have

$$S_1 = \sum_{\alpha,\beta} G_1(\alpha,\beta), \qquad S_2 = \sum_{\alpha,\beta} G_2(\alpha,\beta),$$

where $G_1(\alpha,\beta)$ is the number of pairs u, v satisfying (3.6) and

$$u \equiv a(q), \qquad v \equiv b(q), \tag{3.8}$$

and $G_2(\alpha,\beta)$ is the number of pairs u, v satisfying (3.7) and (3.8). So,

$$G_1(\alpha,\beta) = \sum_{\substack{u\le\sqrt{x/2} \\ u\equiv\alpha(q)}} \sum_{\substack{v\le\sqrt{x-u^2} \\ v\equiv\beta(q)}} 1, \tag{3.9}$$

and

$$G_2(\alpha,\beta) = \sum_{\substack{u\le\sqrt{x/2} \\ u\equiv\alpha(q)}} \sum_{\substack{v\le\sqrt{x/2} \\ v\equiv\beta(q)}} 1. \tag{3.10}$$

We will now try to evaluate $G_1(\alpha,\beta)$. Let

$$\rho(y) = -\psi(y),$$

where ψ is defined in (1.7). We now prove a theorem that will allow us to express G_1 in terms of ρ.

Theorem 18. *For $y \ge 0$, we have*

$$\sum_{\substack{1\le u\le y \\ u\equiv\gamma(q)}} 1 = \frac{y}{q} + \rho\left(\frac{y-\gamma}{q}\right) - \rho\left(\frac{-\gamma}{q}\right).$$

Proof. We will equivalently show

$$\sum_{\substack{1\le u\le y \\ u\equiv\gamma(q)}} 1 = \left[\frac{y-\gamma}{q}\right] - \left[\frac{-\gamma}{q}\right]. \tag{3.11}$$

Clearly It suffices to prove (3.11) for $1 \le y, \gamma \le q$. We then consider the two cases, $y \ge \gamma$ and $y < \gamma$ for which we see (3.11) holds. $\square$

So by applying Theorem 18 to the inner sum of (3.9) we have that

$$G_1(\alpha,\beta) = \frac{1}{q} \sum_{\substack{u\le\sqrt{x/2} \\ u\equiv\alpha(q)}} \sqrt{x-u^2} + \sum_{\substack{u\le\sqrt{x/2} \\ u\equiv\alpha(q)}} \rho\left(\frac{\sqrt{x-u^2}-\beta}{q}\right) - \sum_{\substack{u\le\sqrt{x/2} \\ u\equiv\alpha(q)}} \rho\left(\frac{-\beta}{q}\right). \tag{3.12}$$

We substitute (3.12) into S_1 and we get

$$S_1 = \frac{1}{q}S_1^{(0)} + S_1^{(1)} - S_1^{(2)}, \tag{3.13}$$

where

$$S_1^{(0)} = \sum_{\alpha,\beta} \sum_{\substack{u \le \sqrt{x/2} \\ u \equiv \alpha(q)}} \sqrt{x-u^2},$$

$$S_1^{(1)} = \sum_{\alpha,\beta} \sum_{\substack{u \le \sqrt{x/2} \\ u \equiv \alpha(q)}} \rho\left(\frac{\sqrt{x-u^2}-\beta}{q}\right),$$

$$S_1^{(2)} = \sum_{\alpha,\beta} \sum_{\substack{u \le \sqrt{x/2} \\ u \equiv \alpha(q)}} \rho\left(\frac{-\beta}{q}\right).$$

Similarly, we evaluate G_2 by applying Theorem 18 to the inner sum of (3.10) and we get

$$S_2 = \frac{\sqrt{x/2}}{q}S_2^{(0)} + S_2^{(1)} - S_1^{(2)}, \tag{3.14}$$

where $S_1^{(2)}$ is defined as above and

$$S_2^{(0)} = \sum_{\alpha,\beta} \sum_{\substack{u \le \sqrt{x/2} \\ u \equiv \alpha(q)}} 1,$$

$$S_2^{(1)} = \sum_{\alpha,\beta} \sum_{\substack{u \le \sqrt{x/2} \\ u \equiv \alpha(q)}} \rho\left(\frac{\sqrt{x/2}-\beta}{q}\right).$$

We now return to $S_{q,a}(x)$ which we reexpress using (3.3), (3.4), Theorem 17, (3.5), (3.13) and (3.14). This gives us,

$$S_{q,a}(x) = \frac{8}{q}S_1^{(0)} + 8S_1^{(1)} - 4S_1^{(2)} - 4\frac{\sqrt{x/2}}{q}S_2^{(0)} - 4S_2^{(1)} + 4\frac{\omega_a(q)}{q}\sqrt{x} + O\left((q,a)^{1/2}\tau(q)\right). \tag{3.15}$$

We will now evaluate the sums $S_1^{(0)}$, $S_1^{(1)}$, $S_1^{(2)}$, $S_2^{(0)}$, $S_2^{(1)}$ one by one and once we have bounded each of these sums sufficiently well, we will have proved (1.24). We start with $S_1^{(1)}$, the most difficult of these 5 sums to bound sufficiently well.

3.2 Evaluation of $S_1^{(1)}$.

A large portion of Tolev's paper deals with bounding $S_1^{(1)}$ because as we will see at the end of this chapter, $S_1^{(1)}$ gives the largest error term from the RHS of (3.15). For any integer $M \geq 2$, we have

$$\rho(y) = \sum_{1 \le |n| \le M} \frac{e(ny)}{2\pi i n} + O\left(\min\left(1, \frac{1}{M\|y\|}\right)\right), \tag{3.16}$$

where $e(ny)$ is defined as in (1.6), and we assume $\min\left(1, \frac{1}{0}\right) = 1$. We also have

$$\min\left(1, \frac{1}{M\|y\|}\right) = \sum_{n\in\mathbb{Z}} c_n e(ny), \tag{3.17}$$

where

$$c_n \ll \begin{cases} M^{-1}\log M, & \text{for all } n, \\ Mn^{-2}, & \text{for } n \neq 0. \end{cases} \tag{3.18}$$

The proofs of (3.16), (3.17) and (3.18) are contained in [6].

Now take an integer $M \geq 2$, that will be chosen later. Applying (3.16) to $S_1^{(1)}$, we obtain

$$S_1^{(1)} = \sum_{\alpha,\beta} \sum_{\substack{u\leq\sqrt{x/2} \\ u\equiv\alpha(q)}} \sum_{1\leq|n|\leq M} \frac{1}{2\pi i n} e\left(\frac{\sqrt{x-u^2}-\beta}{q} n\right) + O(\Delta), \tag{3.19}$$

where,

$$\Delta = \sum_{\alpha,\beta} \sum_{\substack{u\leq\sqrt{x/2} \\ u\equiv\alpha(q)}} \min\left(1, M^{-1}\left\|\frac{\sqrt{x-u^2}-\beta}{q}\right\|^{-1}\right).$$

Now we apply (3.17) to get

$$\Delta = \sum_{\alpha,\beta} \sum_{\substack{u\leq\sqrt{x/2} \\ u\equiv\alpha(q)}} \sum_{n\in\mathbb{Z}} c_n e\left(\frac{\sqrt{x-u^2}-\beta}{q} n\right) = \sum_{n\in\mathbb{Z}} c_n \mathcal{F}_n,$$

where,

$$\mathcal{F}_n = \sum_{\alpha,\beta} \sum_{\substack{u\leq\sqrt{x/2} \\ u\equiv\alpha(q)}} e\left(\frac{\sqrt{x-u^2}-\beta}{q} n\right). \tag{3.20}$$

Next we apply (3.18) to get

$$\Delta \ll \frac{\log M}{M}|\mathcal{F}_0| + \log M \sum_{1\leq|n|\leq M} M^{-1}|\mathcal{F}_n| + M \sum_{|n|>M} n^{-2}|\mathcal{F}_n|. \tag{3.21}$$

Using (3.19) we see

$$S_1^{(1)} \ll \sum_{1\leq|n|\leq M} |n|^{-1}|\mathcal{F}_n| + O(\Delta),$$

so combined with (3.21) we get

$$S_1^{(1)} \ll \frac{\log M}{M}|\mathcal{F}_0| + \log M \sum_{1\leq|n|\leq M} |n|^{-1}|\mathcal{F}_n| + M \sum_{|n|>M} n^{-2}|\mathcal{F}_n|. \tag{3.22}$$

To proceed we must manipulate $\mathcal{F}_n$. We begin by using the elementary identity

$$\sum_{h(q)} e\left(\frac{hm}{q}\right) = \begin{cases} q, & \text{for } q \mid m, \\ 0, & \text{otherwise,} \end{cases} \tag{3.23}$$

where the summation is taken over all residue classes h modulo q. This identity will serve us as an 'indicator function' that gets rid of the annoying congruence condition in (3.20). We will use this idea again later in this chapter.

Using this identity we have

$$\begin{aligned} \mathcal{F}_n &= \sum_{\alpha,\beta} \sum_{u \le \sqrt{x/2}} e\left(\frac{\sqrt{x-u^2}-\beta}{q} n\right) \frac{1}{q} \sum_{h(q)} e\left(\frac{h(u-\alpha)}{q}\right) \\ &= \frac{1}{q} \sum_{h(q)} \mathcal{H}_{h,n} \mathcal{T}_{h,n}, \end{aligned}$$

where

$$\mathcal{H}_{h,n} = \mathcal{H}_{h,n}(q,a) = \sum_{\alpha,\beta} e\left(\frac{-\alpha h - \beta n}{q}\right),$$

$$\mathcal{T}_{h,n} = \mathcal{T}_{h,n}(q) = \sum_{u \le \sqrt{x/2}} e(f(u)),$$

$$f(u) = \left(n\sqrt{x-u^2} + hu\right) q^{-1}.$$

Observe that $\mathcal{H}_{h,n}$ and $\mathcal{T}_{h,n}$ both have period q in the h variable, so in combination with the triangle inequality we get

$$|\mathcal{F}_n| \le \frac{1}{q} \sum_{|h| \le q/2} |\mathcal{H}_{h,n}||\mathcal{T}_{h,n}|. \tag{3.24}$$

So in order to bound $|\mathcal{F}_n|$, we must first bound $|\mathcal{H}_{h,n}|$ and $|\mathcal{T}_{h,n}|$.

3.3 Estimation of $\mathcal{H}_{h,n}$

In this section we will show that

$$|\mathcal{H}_{h,n}| \le 4q^{\frac{1}{2}} \tau^2(q)(q,h,n)^{\frac{1}{2}}(q,a,h^2+n^2)^{\frac{1}{2}}. \tag{3.25}$$

We will do so by proving (3.25) in the case where q is odd and then in the case where q is a power of two. Then we will use a multiplicative-like property of $\mathcal{H}_{h,n}$ to prove (3.25) for general q. To deal with these two cases, we must first manipulate $\mathcal{H}_{h,n}$.

To begin, we make the simple substitutions $\alpha \to q - \alpha$ and $\beta \to q - \beta$ to give us

$$\mathcal{H}_{h,n}(q,a) = \sum_{\alpha,\beta} e\left(\frac{\alpha h + \beta n}{q}\right).$$

Now we define the Gauss sum,

$$S(q;k,m) = \sum_{\alpha(q)} e\left(\frac{k\alpha^2 + m\alpha}{q}\right).$$

We now use (3.23) again which gives us the indicator function

$$\frac{1}{q}\sum_{k(q)} e\left(\frac{k(\alpha^2+\beta^2-a)}{q}\right) = \begin{cases} 1, & \text{for } \alpha^2+\beta^2 \equiv a \pmod q, \\ 0, & \text{otherwise.} \end{cases}$$

So we obtain

$$\begin{aligned} \mathcal{H}_{h,n}(q,a) &= \sum_{\alpha(q)}\sum_{\beta(q)} e\left(\frac{\alpha h + \beta n}{q}\right)\frac{1}{q}\sum_{k(q)} e\left(\frac{k(\alpha^2+\beta^2-a)}{q}\right) \\ &= \frac{1}{q}\sum_{k(q)} e\left(\frac{-\alpha k}{q}\right) S(q;k,h)S(q;k,n). \end{aligned}$$

We now reference a theorem that will allow us to manipulate the Gauss sums appearing in this sum.

Theorem 19. *Let $d=(k,q)$, then*

$$S(q;k,n) = \begin{cases} dS(q/d;k/d,n/d), & \text{for } d \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This result is proved in [4, Lemma 7]. □

Note that

$$\sum_{d|q}\sum_{\substack{k(q) \\ (k,q)=\frac{q}{d}}} = \sum_{k(q)},$$

so we have

$$\mathcal{H}_{h,n} = \frac{1}{q}\sum_{d|q}\sum_{\substack{k(q) \\ (k,q)=\frac{q}{d}}} e\left(\frac{-\alpha k}{q}\right) S(q;k,h)S(q;k,n).$$

Now we apply Theorem 19 with $q/d=(k,q)$ to give us

$$\mathcal{H}_{h,n} = \frac{1}{q}\sum_{\substack{d|q \\ \frac{q}{d}|(h,n)}}\sum_{\substack{k(q) \\ (k,q)=\frac{q}{d}}} e\left(\frac{-\alpha k}{q}\right)\left(\frac{q}{d}S(d;kd/q,hd/q)\right)\left(\frac{q}{d}S(d;kd/q,nd/q)\right).$$

Now note that for each d we iterate over, requiring $(k,q)=\frac{q}{d}$ for any k we then iterate over means that $k=ql/d$ for some $(l,d)=1$, hence we can iterate over l instead of k and we obtain

$$\mathcal{H}_{h,n} = q\sum_{\substack{d|q \\ \frac{q}{d}|(h,n)}} d^{-2}B(d;a,hd/q,nd/q), \tag{3.26}$$

where

$$B(d;a,m,t) = \sum_{l(d)}^{*} e\left(\frac{-al}{d}\right) S(d;l,m)S(d;l,t). \tag{3.27}$$

Here

$$\sum_{l(d)}^{*} = \sum_{\substack{l(d)\\(l,d)=1}} . \tag{3.28}$$

Now define the Kloosterman sum

$$K(q;k,n) = \sum_{\alpha(q)}^{*} e\left(\frac{k\alpha + n\overline{\alpha}}{q}\right), \tag{3.29}$$

where $\overline{\alpha}$ is the inverse of α modulo q. We now reference a result by A.Weil which is proved in [5, Chapter 11] which states

$$|K(q;k,n)| \le q^{\frac{1}{2}}\tau(q)(q,k,n)^{\frac{1}{2}}. \tag{3.30}$$

3.3.1 The case $2 \nmid q$.

In this subsection we prove that

$$|\mathcal{H}_{h,n}| \le q^{\frac{1}{2}}\tau^2(q)(q,h,n)^{\frac{1}{2}}(q,a,h^2+n^2)^{\frac{1}{2}} \qquad \text{for} \qquad 2 \nmid q. \tag{3.31}$$

Having in mind (3.26), we will bound $\mathcal{H}_{h,n}$ by bounding $B(d;a,m,t)$ for each $d \mid q$. We will bound $B(d;a,m,t)$ by relating it to Kloosterman sums and Gauss sums. To proceed we first prove two theorems about Gauss sums. First let $S(q;1) = S(q;1,0)$ and let $\left(\frac{k}{q}\right)$ be the Jacobi symbol.

Theorem 20. *For $(q,2k)=1$ we have,*

$$S(q;k) = \left(\frac{k}{q}\right) S(q,1).$$

Proof. A more general version of this theorem is proved in [4, Lemma 9]. □

Next we prove a result that is very similar to [4, Lemma 3]:

Theorem 21. *For $(q,2k)=1$, we have*

$$S(q;k,m) = e\left(\frac{-\overline{(4k)}m^2}{q}\right)\left(\frac{k}{q}\right) S(q;1),$$

where $\overline{(4k)}$ is the inverse of $4k$ modulo q.

Proof. Since $(q,2k)=1$, we know $\overline{4k}$ exists modulo q and so

$$\begin{aligned} S(q;k,m) &= \sum_{\alpha(q)} e\left(\frac{k\alpha^2 + m\alpha + \overline{(4k)}m^2 - \overline{(4k)}m^2}{q}\right) \\ &= e\left(\frac{-\overline{(4k)}m^2}{q}\right) \sum_{\alpha(q)} e\left(\frac{k\big(\alpha + \overline{2k}m\big)^2}{q}\right) \\ &= e\left(\frac{-\overline{(4k)}m^2}{q}\right) S(q;k). \end{aligned}$$

Then we apply Theorem 20 to finish the proof. □

Since q is odd, if $d \mid q$ and $(d,l)=1$ then $(d,2l)=1$ also. So we may apply Theorem 21 twice to get

$$\begin{aligned} B(d;a,m,t) &= \sum_{l(d)}^{*} e\left(\frac{-al}{d}\right)\left(e\left(\frac{-\overline{(4l)}m^2}{d}\right)\left(\frac{l}{d}\right)S(d;1)\right)\left(e\left(\frac{-\overline{(4l)}t^2}{d}\right)\left(\frac{l}{d}\right)S(d;1)\right) \\ &= S^2(d;1)K(d,a,\overline{4}(m^2+t^2)), \end{aligned}$$

where we have used the substitution $l \to d-l$ to remove the negative sign appearing in the exponential sum, and where $\overline{4}$ is the inverse of 4 modulo d. We then use the result

$$|S(d;1)|^2 = d \qquad \text{for} \qquad 2 \nmid d,$$

which is proved in [7, Chapter 7], combined with (3.30) to get

$$|B(d;a,m,t)| \le d^{\frac{3}{2}}\tau(d)(d,a,m^2+t^2)^{\frac{1}{2}}. \tag{3.32}$$

We now substitute this bound for B into (3.26), with $m = hd/q$ and $t = nd/q$. We use that $\tau(d) \le \tau(q)$ and that $(d,a,(d/q)^2(h^2+n^2)) \le (q,a,h^2+n^2)$ because d divides q and q/d divides (h,n), to obtain

$$|\mathcal{H}_{h,n}| \le q\tau(q)(q,a,h^2+n^2)^{\frac{1}{2}}\mathfrak{X}, \qquad \text{with} \qquad \mathfrak{X} = \sum_{\substack{d \mid q \\ \frac{q}{d} \mid (h,n)}} d^{-\frac{1}{2}}.$$

We then see by iterating over q/d instead of d that

$$\mathfrak{X} = \sum_{\substack{\frac{q}{d} \mid q \\ d \mid (h,n)}} \left(\frac{q}{d}\right)^{-\frac{1}{2}} = \sum_{d \mid (q,h,n)} \left(\frac{q}{d}\right)^{-\frac{1}{2}} \le q^{-\frac{1}{2}}\tau(q)(q,h,n)^{\frac{1}{2}},$$

hence we obtain (3.31).

3.3.2 The case $q = 2^\theta$.

In this subsection we prove that

$$|\mathcal{H}_{h,n}| \le 4(2^\theta)^{\frac{1}{2}}\tau^2(2^\theta)(2^\theta,n,h)^{\frac{1}{2}}(2^\theta,a,h^2+n^2)^{\frac{1}{2}}. \tag{3.33}$$

We again obtain a bound on $\mathcal{H}_{h,n}$ by first studying the sum B. We will show

$$|B(2^\nu,a,m,t)| \le 4(2^\nu)^{\frac{3}{2}}\tau(2^\nu)(2^\nu,a,m^2+t^2)^{\frac{1}{2}}. \tag{3.34}$$

This inequality is obvious if $\nu < 2$ (because $|B(2,a,m,t)| \le 4$), so we may assume $\nu \ge 2$.

Theorem 22. *If $2 \nmid r$ and $\nu \ge 2$, then*

$$S(2^\nu,r,t) = \begin{cases} e\left(\frac{-\overline{r}(t/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}\frac{1+i^r}{\sqrt{2}}, & \text{for } 2 \mid t,\ 2 \mid \nu, \\ e\left(\frac{-\overline{r}(t/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}e\left(\frac{r}{8}\right), & \text{for } 2 \mid t,\ 2 \nmid \nu, \\ 0, & \text{for } 2 \nmid t. \end{cases}$$

Proof. This result is proved in [4, Section 6] and [7, Chapter 7]. □

From (3.27) and Theorem 22, we see that if $2 \nmid m$ or $2 \nmid t$ then $B = 0$, hence (3.34) is satisfied. So assume now that $2 \mid m$ and $2 \mid t$. If we also have that $2 \mid \nu$ then we use Theorem 22 to get

$$\begin{aligned} B(2^\nu, a, m, t) &= \sum_{l(2^\nu)}^{*} e\left(\frac{-al}{2^\nu}\right)\left(e\left(\frac{-\bar{l}(m/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}\frac{1+i^l}{\sqrt{2}}\right)\left(e\left(\frac{-\bar{l}(t/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}\frac{1+i^l}{\sqrt{2}}\right) \\ &= 2^{\nu+1}\sum_{l(2^\nu)}^{*} e\left(\frac{-la-\bar{l}\left((m/2)^2+(t/2)^2\right)}{2^\nu}\right)\left(\frac{1+i^l}{\sqrt{2}}\right)^2. \end{aligned} \tag{3.35}$$

Observe that in this sum we only consider odd l. For odd l we have

$$\left(\frac{1+i^l}{\sqrt{2}}\right)^2 = i(-1)^{\frac{l-1}{2}} = e\left(\frac{2^{\nu-2}l}{2^\nu}\right).$$

Substituting this into (3.35), while keeping (3.29) in mind, gives us

$$B(2^\nu, a, m, t) = 2^{\nu+1}K\left(2^\nu; 2^{\nu-2}-a, -\frac{m^2+t^2}{4}\right).$$

Now we apply (3.30) and obtain

$$B(2^\nu, a, m, t) \le 2(2^\nu)^{\frac{3}{2}}\tau(2^\nu)\left(2^\nu, 2^{\nu-2}-a, -\frac{m^2+t^2}{4}\right)^{\frac{1}{2}}.$$

Next observe that

$$\left(2^\nu, 2^{\nu-2}-a, -\frac{m^2+t^2}{4}\right) \Big| \; 4\left(2^\nu, a, m^2+t^2\right),$$

so (3.34) follows.

If $2 \nmid \nu$, we begin again by applying Theorem 22 to obtain

$$\begin{aligned} B(2^\nu, a, m, t) &= \sum_{l(2^\nu)}^{*} e\left(\frac{-al}{2^\nu}\right)\left(e\left(\frac{-\bar{l}(m/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}e\left(\frac{l}{8}\right)\right)\left(e\left(\frac{-\bar{l}(t/2)^2}{2^\nu}\right)2^{\frac{\nu+1}{2}}e\left(\frac{l}{8}\right)\right) \\ &= 2^{\nu+1}K\left(2^\nu; 2^{\nu-2}-a, -\frac{m^2+t^2}{4}\right). \end{aligned}$$

This is the same expression we obtained for $B(2^\nu, a, m, t)$ in the case $2 \nmid \nu$ and so (3.34) follows. Hence we have proved (3.34) for all $\nu \ge 0$. Note that (3.34) is the same as (3.32) apart from a constant factor so (3.33) follows from what we have already done at the end of Subsection 3.3.1.

3.3.3 The estimate for $\mathcal{H}_{h,n}$.

Theorem 23. *For $(q_1, q_2) = 1$ we have,*

$$\mathcal{H}_{h,n}(q_1q_2, a) = \mathcal{H}_{h\overline{q_2},n\overline{q_2}}(q_1, a)\mathcal{H}_{h\overline{q_1},n\overline{q_1}}(q_2, a),$$

where $\overline{q_2}$ and $\overline{q_1}$ denote the inverses of q_2 and q_1 modulo q_1 and q_2 respectively.

Proof. By definition we have

$$\mathcal{H}_{h,n}(q_1q_2,a)=\sum_{\substack{1\le x,y\le q_1q_2\\ x^2+y^2\equiv a(q_1q_2)}} e\left(\frac{-xh-yn}{q_1q_2}\right).$$

Now recall (2.3) which defined a bijective function between $U_{a,mn}$ and $U_{a,m}\times U_{a,n}$ provided that $(m,n)=1$, where $U_{a,q}$ was defined in (2.1). Let $c=\overline{q_1+q_2}$ where the inverse is taken modulo q_1q_2, then

$$\begin{aligned}\mathcal{H}_{h,n}(q_1q_2,a)&=\sum_{\substack{1\le\alpha,\beta\le q_1\\ \alpha^2+\beta^2\equiv a(q_1)}}\sum_{\substack{1\le\gamma,\delta\le q_2\\ \gamma^2+\delta^2\equiv a(q_2)}} e\left(\frac{-(cq_2\alpha+cq_1\gamma)h-(cq_2\beta+cq_1\delta)n}{q_1q_2}\right)\\ &=\sum_{\substack{1\le\alpha,\beta\le q_1\\ \alpha^2+\beta^2\equiv a(q_1)}} e\left(\frac{-c\alpha h-c\beta n}{q_1}\right)\sum_{\substack{1\le\gamma,\delta\le q_2\\ \gamma^2+\delta^2\equiv a(q_2)}} e\left(\frac{-c\gamma h-c\delta n}{q_2}\right).\end{aligned}$$

To finish the proof, we observe that $c\equiv\overline{q_2}\ (q_1)$ and similarly $c\equiv\overline{q_1}\ (q_2)$, where $\overline{q_2}$ and $\overline{q_1}$ denote the inverses of q_2 and q_1 modulo q_1 and q_2 respectively. □

Now using (3.33), (3.31) and Theorem 23 we bound $\mathcal{H}_{h,n}$ in the general case. Let $q=q_1q_2$ where q_1 is odd and q_2 is a power of two. Then we get

$$\mathcal{H}_{h,n}(q_1q_2,a)=\mathcal{H}_{h\overline{q_2},n\overline{q_2}}(q_1,a)\mathcal{H}_{h\overline{q_1},n\overline{q_1}}(q_2,a)\le 4q^{\frac12}\tau^2(q)\alpha\beta\gamma\delta$$

where,

$$\begin{aligned}\alpha&=(q_1,h\overline{q_2},n\overline{q_2}),\\ \beta&=(q_2,h\overline{q_1},n\overline{q_1}),\\ \gamma&=(q_1,a,h^2\overline{q_2}^2+n^2\overline{q_2}^2),\\ \delta&=(q_2,a,h^2\overline{q_1}^2+n^2\overline{q_1}^2).\end{aligned}$$

To prove (3.25) it clearly suffices to show $\alpha\beta\mid(q,h,n)$ and $\gamma\delta\mid(q,a,h^2+n^2)$. Furthermore, since $(q_1,q_2)=1$, it follows that $(\alpha,\beta)=1$ and $(\gamma,\delta)=1$, so it suffices to show $\alpha,\beta\mid(q,h,n)$ and $\gamma,\delta\mid(q,a,h^2+n^2)$ which is trivial.

So we've proved (3.25). Note that in order to prove (1.24) we will only use the simple consequence of (3.25):

$$|\mathcal{H}_{h,n}|\le 4q^{\frac12}\tau^2(q)(q,h,n)^{\frac12}(q,a)^{\frac12}. \tag{3.36}$$

Recalling (3.24), we now move onto bounding $\mathcal{T}_{h,n}$.

3.4 Estimation of $\mathcal{T}_{h,n}$ and $\mathcal{F}_n$.

3.4.1 The sum $\mathcal{T}_{h,n}$

We assume $|h|\le q/2$ throughout Subsection 3.4.1. We will show that

$$\mathcal{T}_{h,n}\ll\min(\sqrt{x},q|h|^{-1}),\qquad\text{if}\qquad n=0\qquad\text{or}\qquad 0<2|n|\le|h|, \tag{3.37}$$

and

$$\mathcal{T}_{h,n} \ll x^{\frac{1}{4}}(|n|^{\frac{1}{2}}q^{-\frac{1}{2}} + |n|^{-\frac{1}{2}}q^{\frac{1}{2}}) \qquad \text{for} \qquad n \neq 0. \tag{3.38}$$

Case 1: $n = 0$. We have

$$\mathcal{T}_{h,0}(q) = \sum_{u \leq \sqrt{\frac{x}{2}}} e\left(\frac{hu}{q}\right).$$

We can bound this in 2 ways. First we trivially have $|\mathcal{T}_{h,0}| \leq \sqrt{x/2}$ by the triangle inequality. The second way uses Theorem 2. To use this theorem, we note that hu/q is continuously differentiable, $f'(u) = h/q$ is monotonic, and $\|h/q\| = |h|/q$, so we have $\mathcal{T}_{h,0} \ll q|h|^{-1}$. Altogether this gives

$$\mathcal{T}_{h,0} \ll \min(\sqrt{x}, q|h|^{-1}).$$

Case 2: $n \neq 0$. We have by differentiation

$$f(u) = (n\sqrt{x-u^2} + hu)q^{-1},$$

$$f'(u) = \left(-nu(x-u^2)^{-1/2} + h\right)q^{-1},$$

$$f''(u) = -nx(x-u^2)^{-3/2}q^{-1}.$$

We will now complete the proof of (3.37) by bounding $|f'|$ so that we may apply Theorem 2. Suppose $0 < 2|n| \leq |h|$. By the definition of $\mathcal{T}_{h,n}$, we only consider $0 \leq u \leq \sqrt{x/2}$ so we get

$$\left|nu(x-u^2)^{-1/2}q^{-1}\right| \leq \left|nux^{-1/2}q^{-1}\right| \leq \frac{|h|}{2}\sqrt{\frac{x}{2}}\frac{1}{\sqrt{x}}q^{-1} \leq \frac{|h|}{2q}.$$

So

$$\frac{|h|}{q} - \frac{|h|}{2q} \leq \left|\frac{-nu(x-u^2)^{-\frac{1}{2}} + h}{q}\right| \leq \frac{|h|}{q} + \frac{|h|}{2q},$$

by using the triangle inequality and inverse triangle inequality. Hence,

$$\frac{|h|}{2q} \leq |f'(u)| \leq \frac{3|h|}{2q}.$$

Now we use our assumption that $|h| \leq q/2$ to obtain

$$\frac{|h|}{2q} \leq 1 - \frac{3|h|}{2q}.$$

So we can now apply Theorem 2 using that $f'(u)$ is continuously differentiable, $f'(u)$ is monotonic for $u \in [0, \sqrt{x/2}]$ because $f''(u) = 0$ is strictly negative or strictly positive for $u \in [0, \sqrt{x/2}]$, and that

$$\|f'(u)\| \leq \frac{|h|}{2q},$$

to get $\mathcal{T}_{h,n} \ll q|h|^{-1}$. We also have the trivial bound $\mathcal{T}_{h,n} \leq \sqrt{x/2}$ so altogether we get that

$$\mathcal{T}_{h,0} \ll \min(\sqrt{x}, q|h|^{-1})$$

for $0 < 2|n| \le |h|$, so we have proved (3.37).

Now we prove (3.38). Recall that

$$|f''(u)| = \left|nx(x-u^2)^{-3/2}q^{-1}\right| = q^{-1}|n|x^{-1/2}\left|\left(\frac{x}{x-u^2}\right)^{3/2}\right|.$$

Observe

$$1 \le \left|\left(\frac{x}{x-u^2}\right)^{3/2}\right| \le 2\sqrt{2},$$

where the left and right bounds come from substituting $u = 0$ and $u = \sqrt{x/2}$ respectively. Therefore

$$q^{-1}|n|x^{-1/2} \le |f''(u)| \le 2\sqrt{2}q^{-1}|n|x^{-1/2}.$$

We may now use Theorem 3, using the above inequality, that f is continuously differentiable, and that the length of the interval for u we're considering is $\sqrt{x/2}$, to obtain

$$\mathcal{T}_{h,n} \ll 2\sqrt{2}\sqrt{x/2}(q^{-1}|n|x^{-\frac{1}{2}})^{\frac{1}{2}} + (q^{-1}|n|x^{-\frac{1}{2}})^{-\frac{1}{2}}.$$

Since the two terms on the RHS are positive we can ignore constants which gives us

$$\mathcal{T}_{h,n} \ll \sqrt{x}(q^{-1}|n|x^{-\frac{1}{2}})^{\frac{1}{2}} + (q^{-1}|n|x^{-\frac{1}{2}})^{-\frac{1}{2}}.$$

Expanding this gives (3.38), as desired.

3.4.2 The sum $\mathcal{F}_n$.

We have now bounded $\mathcal{H}_{h,n}$ and $\mathcal{T}_{h,n}$ so recalling (3.24), we can now bound $\mathcal{F}_n$. We bound $\mathcal{F}_n$ in two cases based on whether n is zero or not.

Case 1: $n = 0$. We apply our bound for $\mathcal{H}_{h,0}$, (3.36), and our bound for $\mathcal{T}_{h,0}$, (3.37), to get

$$\begin{aligned}\mathcal{F}_0 &\ll q^{-1}\sum_{|h|\le\frac{q}{2}}\left(q^{\frac{1}{2}}\tau^2(q)(q,h)^{\frac{1}{2}}(q,a)^{\frac{1}{2}}\right)\min(\sqrt{x}, q|h|^{-1})\\ &= q^{-\frac{1}{2}}\tau^2(q)(q,a)^{\frac{1}{2}}\sum_{|h|\le\frac{q}{2}}(q,h)^{\frac{1}{2}}\min(\sqrt{x}, q|h|^{-1}).\end{aligned}$$

Now we separate the term corresponding to $h = 0$ from the sum, and then use that $\min(\sqrt{x}, q|h|^{-1}) \le q|h|^{-1}$ as well as that $(q,h)^{\frac{1}{2}}|h|^{-1}$ is an even function to obtain

$$\begin{aligned}\mathcal{F}_0 &\le \tau^2(q)(q,a)^{\frac{1}{2}}\left(\sqrt{x} + q^{\frac{1}{2}}\sum_{1\le|h|\le\frac{q}{2}}(q,h)^{\frac{1}{2}}|h|^{-1}\right)\\ &\ll \tau^2(q)(q,a)^{\frac{1}{2}}\left(\sqrt{x} + q^{\frac{1}{2}}\sum_{1\le h\le q}\frac{(q,h)^{\frac{1}{2}}}{h}\right).\end{aligned} \tag{3.39}$$

Notice that

$$\sum_{1\le n\le y}\frac{(q,n)^{\frac12}}{n}\le\sum_{1\le n\le y}\sum_{\substack{\delta|q\\ \delta|n}}\frac{\delta^{\frac12}}{n}=\sum_{\delta|q}\delta^{\frac12}\sum_{\substack{n\le y\\ \delta|n}}\frac{1}{n}=\sum_{\delta|q}\delta^{-\frac12}\sum_{n\le\frac{y}{\delta}}\frac{1}{n},$$

so if $y\ge 2$, then

$$\sum_{1\le n\le y}\frac{(q,n)^{\frac12}}{n}\ll\tau(q)\log(y). \tag{3.40}$$

We now substitute this bound into (3.39) to get

$$\begin{aligned}\mathcal{F}_0&\ll\tau^2(q)(q,a)^{\frac12}\left(\sqrt{x}+q^{\frac12}\tau(q)\log(q)\right)\\ &\ll\tau^3(q)(q,a)^{\frac12}\log(q)\left(\sqrt{x}+q^{\frac12}\right).\end{aligned}$$

We assumed in Section 3.1 that throughout this entire chapter $q\le x^{2/3}$, so

$$\mathcal{F}_0\ll x^{\frac12}\tau^3(q)(q,a)^{\frac12}\log(x). \tag{3.41}$$

Case 2: $n\neq 0$. We substitute our bound for $\mathcal{H}_{h,n}$, (3.36), into (3.24), which gives us

$$\mathcal{F}_n\ll q^{-\frac12}\tau^2(q)(q,a)^{\frac12}(q,n)^{\frac12}\mathfrak{T}_n, \tag{3.42}$$

where

$$\mathfrak{T}_n=\sum_{|h|\le\frac{q}{2}}|\mathcal{T}_{h,n}|.$$

To bound $\mathcal{F}_n$, we must first bound $\mathfrak{T}_n$. We use our bounds for $\mathcal{T}_{h,n}$, (3.37) and (3.38), to get

$$\mathfrak{T}_n\ll\sum_{2|n|\le|h|\le\frac{q}{2}}q|h|^{-1}+\sum_{|h|\le\min(2|n|,\frac{q}{2})}x^{\frac14}\left(|n|^{\frac12}q^{-\frac12}+|n|^{-\frac12}q^{\frac12}\right).$$

Observe then that

$$\sum_{2|n|\le|h|\le\frac{q}{2}}q|h|^{-1}\ll q\log q\ll q\log x,$$

and

$$\begin{aligned}\sum_{|h|\le\min(2|n|,\frac{q}{2})}x^{\frac14}\left(|n|^{\frac12}q^{-\frac12}+|n|^{-\frac12}q^{\frac12}\right)&\le x^{\frac14}\left(\sum_{|h|\le\frac{q}{2}}|n|^{\frac12}q^{-\frac12}+\sum_{|h|\le 2|n|}|n|^{-\frac12}q^{\frac12}\right)\\ &\ll x^{\frac14}|n|^{\frac12}q^{\frac12}.\end{aligned}$$

Hence

$$\mathfrak{T}_n\ll q\log x+x^{\frac14}|n|^{\frac12}q^{\frac12}.$$

So now we use this bound for $\mathfrak{T}_n$ and substitute it into (3.42) to get

$$\mathcal{F}_n\ll\left(q^{\frac12}+x^{\frac14}|n|^{\frac12}\right)\tau^2(q)(q,a)^{\frac12}(q,n)^{\frac12}\log x\qquad\text{for}\qquad n\neq 0.$$

3.5 Estimation of $S_1^{(1)}$.

We may now substitute our bounds for $\mathcal{F}_n$, namely (3.41) and (3.42), into (3.22) to get

$$S_1^{(1)} \ll \left(x^{\frac{1}{2}} M^{-1} + \left(q^{\frac{1}{2}} + x^{\frac{1}{4}} M^{\frac{1}{2}}\right) \sum_{1\le n\le M} \frac{(q,n)^{\frac{1}{2}}}{n} + M \sum_{n>M} \frac{(q,n)^{\frac{1}{2}}}{n^2} \left(q^{\frac{1}{2}} + x^{\frac{1}{4}} n^{\frac{1}{2}}\right)\right) \tag{3.43}$$
$$\times \tau^3(q)(q,a)^{\frac{1}{2}} \log x \log M.$$

Next we bound the sums appearing on the RHS. For any $y \ge 1$ we have

$$\begin{aligned}\sum_{n>y} \frac{(q,n)^{\frac{1}{2}}}{n^2} &\le \sum_{\delta|q} \delta^{\frac{1}{2}} \sum_{\substack{n>y\\ n\equiv 0(\delta)}} \frac{1}{n^2}\\ &\le \sum_{\delta|q} \delta^{-\frac{3}{2}} \sum_{n>\frac{y}{\delta}} \frac{1}{n^2}\\ &\le \sum_{\substack{\delta|q\\ \delta\le y}} \delta^{-\frac{3}{2}} \sum_{n>\frac{y}{\delta}} \frac{1}{n^2} + \sum_{\substack{\delta|q\\ \delta>y}} \delta^{-\frac{3}{2}} \sum_{n>\frac{y}{\delta}} \frac{1}{n^2}.\end{aligned}$$

We now bound both of these sums. By Monotone comparison theorem we get

$$\sum_{n>\frac{y}{\delta}} \frac{1}{n^2} = \sum_{n>1} \frac{1}{n+(\frac{y}{\delta}-1)^2} \le \int_1^\infty \frac{1}{t+(\frac{y}{\delta}-1)^2} dt + O\left(\frac{\delta^2}{y^2}\right) = \frac{\delta}{y} + O\left(\frac{\delta^2}{y^2}\right) = O\left(\frac{\delta}{y}\right).$$

Hence

$$\sum_{\substack{\delta|q\\ \delta\le y}} \delta^{-\frac{3}{2}} \sum_{n>\frac{y}{\delta}} \frac{1}{n^2} \ll \sum_{\substack{\delta|q\\ \delta\le y}} \delta^{-\frac{1}{2}} y^{-1} \ll \tau(q) y^{-1}.$$

Now we bound the second sum,

$$\sum_{\substack{\delta|q\\ \delta>y}} \delta^{-\frac{3}{2}} \sum_{n>\frac{y}{\delta}} \frac{1}{n^2} \ll \sum_{\substack{\delta|q\\ \delta>y}} \delta^{-\frac{3}{2}} \ll y^{-1} \sum_{\substack{\delta|q\\ \delta>y}} \delta^{-\frac{1}{2}} \ll \tau(q) y^{-1}.$$

Hence altogether,

$$\sum_{n>y} \frac{(q,n)^{\frac{1}{2}}}{n^2} \ll \tau(q) y^{-1}. \tag{3.44}$$

Following a very similar method, we find

$$\sum_{n>y} \frac{(q,n)^{\frac{1}{2}}}{n^{\frac{3}{2}}} \ll \tau(q) y^{-\frac{1}{2}}. \tag{3.45}$$

So by substituting our bounds (3.40), (3.44) and (3.45) into (3.43), we obtain

$$S_1^{(1)} \ll \left(x^{\frac{1}{2}} M^{-1} + q^{\frac{1}{2}} + x^{\frac{1}{4}} M^{\frac{1}{2}}\right) \tau^4(q)(q,a)^{\frac{1}{2}} \log x \log^2 M.$$

Now we choose $M = [x^{\frac{1}{6}}]$ and we get

$$S_1^{(1)} \ll \left(q^{\frac{1}{2}} + x^{\frac{1}{3}}\right) \tau^4(q)(q,a)^{\frac{1}{2}} \log^4 x. \tag{3.46}$$

3.6 Evaluation of the sums $S_1^{(0)}$, $S_1^{(2)}$, $S_2^{(0)}$, $S_2^{(1)}$.

3.6.1 The sum $S_1^{(0)}$.

We first manipulate $S_1^{(0)}$ to express it as the following single sum,

$$S_1^{(0)} = \sum_{u \leq \sqrt{x/2}} b_u \sqrt{x-u^2}, \qquad b_u = \omega_{a-u^2}(q).$$

Now we apply summation by parts to get

$$S_1^{(0)} = \sqrt{x/2} \sum_{u \leq \sqrt{x/2}} b_u - \int_0^{\sqrt{x/2}} \left(\sum_{u \leq t} b_u \right) \frac{d}{dt} \sqrt{x-t^2}\, dt. \tag{3.47}$$

Using Theorem 18, we see that

$$\begin{aligned} \sum_{u \leq t} b_u &= \sum_{\substack{u \leq t \\ 1 \leq \beta \leq q \\ \beta^2 \equiv a-u^2 (q)}} 1 \\ &= \sum_{\alpha,\beta} \sum_{\substack{u \leq t \\ u \equiv \alpha (q)}} 1 \\ &= \sum_{\alpha,\beta} \left(\frac{t}{q} + \rho\left(\frac{t-\alpha}{q}\right) - \rho\left(\frac{-\alpha}{q}\right) \right). \end{aligned} \tag{3.48}$$

So by substituting (3.48) into (3.47) we get,

$$S_1^{(0)} = \frac{\eta_a(q)}{2q} x + \sqrt{x/2}\, \mathfrak{N} - \sqrt{x/2}\, \mathfrak{N}_0 + \sum_{\alpha,\beta} \Gamma_\alpha + \int_0^{\sqrt{x/2}} \left(\sum_{\alpha,\beta} \frac{t}{q} - \rho\left(\frac{-\alpha}{q}\right) \right) \frac{t}{\sqrt{x-t^2}} dt, \tag{3.49}$$

where

$$\mathfrak{N} = \sum_{\alpha,\beta} \rho\left(\frac{\sqrt{x/2}-\alpha}{q} \right), \qquad \mathfrak{N}_0 = \sum_{\alpha,\beta} \rho\left(\frac{-\alpha}{q}\right)$$

and

$$\Gamma_\alpha = \int_0^{\sqrt{x/2}} \rho\left(\frac{t-\alpha}{q}\right) \frac{t\ dt}{\sqrt{x-t^2}}. \tag{3.50}$$

We first evaluate

$$\begin{aligned} \int_0^{\sqrt{x/2}} \sum_{\alpha,\beta} \frac{t}{q} \frac{t}{\sqrt{x-t^2}} dt &= \frac{\eta_a(q)}{q} \int_0^{\sqrt{x/2}} \frac{t^2}{\sqrt{x-t^2}} dt \\ &= \left(\frac{\pi}{8} - \frac{1}{4} \right) \frac{\eta_a(q)}{q} x, \end{aligned} \tag{3.51}$$

where $\eta_a(q)$ is as defined in (1.20).

By a similar process of exchanging a sum with an integral and then evaluating the integral, we have

$$\begin{aligned}\int_0^{\sqrt{x/2}} \sum_{\alpha,\beta} \rho\left(\frac{-\alpha}{q}\right) \frac{t}{\sqrt{x-t^2}} dt &= \sum_{\alpha,\beta} \rho\left(\frac{-\alpha}{q}\right) \left(\sqrt{x} - \sqrt{x/2}\right) \\ &= \left(\sqrt{x} - \sqrt{x/2}\right) \mathfrak{N}_0. \end{aligned} \tag{3.52}$$

So by substituting (3.51) and (3.52) into (3.49) we have,

$$S_1^{(0)} = \left(\frac{\pi}{8} + \frac{1}{4}\right) \frac{\eta_a(q)}{q} x + \sqrt{x/2}\, \mathfrak{N} - \sqrt{x}\, \mathfrak{N}_0 + \sum_{\alpha,\beta} \Gamma_\alpha. \tag{3.53}$$

We will proceed by evaluating $\mathfrak{N}_0$ and bounding $\sum_{\alpha,\beta} \Gamma_\alpha$ so that we may bound $S_1^{(0)}$. Note that all terms in the final expression for $S_{q,a}(x)$ involving $\mathfrak{N}$ will cancel out so there is no need to bound $\mathfrak{N}$.

We will now show that

$$\mathfrak{N}_0 = \frac{1}{2} \omega_a(q). \tag{3.54}$$

We have

$$\mathfrak{N}_0 = \sum_{\beta(q)} \sum_{\substack{\alpha(q) \\ \alpha^2 \equiv a - \beta^2 (q)}} \rho\left(\frac{-\alpha}{q}\right) = \sum_{\beta(q)} \mathcal{Y}_\beta,$$

say. We will now evaluate $\mathcal{Y}_\beta$ by considering two cases.

Case 1: $\beta^2 \not\equiv a(q)$. In this case there is no term corresponding to $\alpha = q$, and the term corresponding to $\alpha = q/2$ (if it even exists) will be zero. The remaining terms can be divided into couples $\rho\left(\frac{-\alpha}{q}\right) + \rho\left(\frac{\alpha - q}{q}\right)$, where $1 \le \alpha \le q/2$ and the sum of the terms of each such couple is zero because ρ is an odd function, hence $\mathcal{Y}_\beta = 0$.

Case 2: $\beta^2 \equiv a(q)$. This case can argued very similarly. The contribution to $\mathcal{Y}_\beta$ from the terms corresponding to $1 \le \alpha < q$ again vanishes. In this case we now get a corresponding term for $\alpha = q$ that is equal to $1/2$.

So

$$\mathfrak{N}_0 = \sum_{\substack{\beta(q) \\ \beta^2 \equiv a(q)}} \frac{1}{2}$$

which proves (3.54). Similar arguments also imply

$$\left| \sum_{\alpha,\beta} \rho\left(\frac{-\alpha}{q}\right) \xi_\beta \right| \le \omega_a(q) \qquad \text{for} \qquad |\xi_\beta| \le 1. \tag{3.55}$$

and

$$\sum_{\alpha,\beta} \sin\left(\frac{2\pi n \alpha}{q}\right) = 0, \tag{3.56}$$

Note that

$$\rho(y) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n y)}{\pi n} \qquad \text{for} \qquad y \notin \mathbb{Z}.$$

We may now substitute this expression for ρ into (3.50) to give us

$$\Gamma_\alpha = \sum_{n=1}^{\infty} \frac{1}{\pi n} \int_0^{\sqrt{x/2}} \sin\left(2\pi n \left(\frac{t-\alpha}{q}\right)\right) \frac{t\ dt}{\sqrt{x-t^2}},$$

where we may use the dominated convergence theorem to justify exchanging the sum and the integral.

We now use the sum angle formula for sine and (3.56) to give us

$$\sum_{\alpha,\beta} \Gamma_\alpha = \sum_{n=1}^{\infty} \frac{1}{\pi n} \mathcal{D}_n \mathcal{E}_n, \tag{3.57}$$

where

$$\mathcal{D}_n = \sum_{\alpha,\beta} \cos\left(\frac{2\pi n \alpha}{q}\right), \qquad \mathcal{E}_n = \int_0^{\sqrt{x/2}} \sin\left(\frac{2\pi n t}{q}\right) \frac{t\ dt}{\sqrt{x-t^2}}.$$

Observe that because of (3.56), we have

$$\mathcal{H}_{0,n} = \mathcal{D}_n.$$

So we may now use our bound for $\mathcal{H}_{h,n}$, (3.36), to get

$$\mathcal{D}_n \ll q^{\frac{1}{2}} \tau^2(q) (q,n)^{\frac{1}{2}} (q,a)^{\frac{1}{2}}.$$

Next we bound $\mathcal{E}_n$ using integration by parts which gives us

$$\begin{aligned}
\mathcal{E}_n &= \left[-\frac{q}{2\pi n} \cos\left(\frac{2\pi n t}{q}\right) \frac{t}{\sqrt{x-t^2}}\right]_{t=0}^{t=\sqrt{x/2}} + \int_0^{\sqrt{x/2}} \frac{q}{2\pi n} \cos\left(\frac{2\pi n t}{q}\right) \left(\frac{1}{\sqrt{x-t^2}} + t^2 (x-t^2)^{-3/2}\right) dt \\
&\ll \frac{q}{n} + \frac{q}{n} \int_0^{\sqrt{x/2}} x^{-1/2} dt \\
&\ll \frac{q}{n}.
\end{aligned}$$

We substitute these bounds for $\mathcal{D}_n$ and $\mathcal{E}_n$ into (3.57), and then use (3.44) with $y = 1$, to get

$$\begin{aligned}
\sum_{\alpha,\beta} \Gamma_\alpha &\ll q^{\frac{3}{2}} \tau^2(q) (q,a)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(q,n)^{\frac{1}{2}}}{n^2} \\
&\ll q^{\frac{3}{2}} \tau^3(q) (q,a)^{\frac{1}{2}}.
\end{aligned} \tag{3.58}$$

So now we substitute (3.54) and (3.58) into (3.53) which gives us,

$$S_1^{(0)} = \left(\frac{\pi}{8} + \frac{1}{4}\right) \frac{\eta_a(q)}{q} x + \sqrt{x/2}\ \mathfrak{N} - \frac{\sqrt{x}}{2}\ \omega_a(q) + O\left(q^{\frac{3}{2}} \tau^3(q) (q,a)^{\frac{1}{2}}\right). \tag{3.59}$$

3.6.2 The sums $S_1^{(2)}, S_2^{(0)}, S_2^{(1)}$.

We can bound $S_1^{(2)}$ quite easily now using results we have already shown. We use Theorem 18, (3.54) and (3.55) to give us,

$$\begin{aligned} S_1^{(2)} &= \sum_{\alpha,\beta} \rho\left(\frac{-\beta}{q}\right)\left(\frac{\sqrt{x/2}}{q} + \rho\left(\frac{\sqrt{x/2}-\alpha}{q}\right) - \rho\left(\frac{-\alpha}{q}\right)\right) \\ &= \frac{\omega_a(q)}{2q}\sqrt{x/2} + O\left(\omega_a(q)\right). \end{aligned}$$

Next we use Theorem 17, hence

$$S_1^{(2)} = \frac{\omega_a(q)}{2q}\sqrt{x/2} + O\left((q,a)^{\frac{1}{2}}\tau(q)\right). \tag{3.60}$$

We bound $S_1^{(2)}$ now by using Theorem 18 and (3.54) again to give us,

$$\begin{aligned} S_2^{(0)} &= \sum_{\alpha,\beta}\left(\frac{\sqrt{x/2}}{q} + \rho\left(\frac{\sqrt{x/2}-\alpha}{q}\right) - \rho\left(\frac{-\alpha}{q}\right)\right) \\ &= \frac{\eta_a(q)}{q}\sqrt{x/2} + \mathfrak{N} - \frac{1}{2}\omega_a(q). \end{aligned} \tag{3.61}$$

To bound $S_1^{(2)}$ we use Theorem 18 and (3.54) again, as well as (3.55), to give us

$$\begin{aligned} S_2^{(1)} &= \sum_{\alpha,\beta} \rho\left(\frac{\sqrt{x/2}-\beta}{q}\right)\left(\frac{\sqrt{x/2}}{q} + \rho\left(\frac{\sqrt{x/2}-\alpha}{q}\right) - \rho\left(\frac{-\alpha}{q}\right)\right) \\ &= \frac{\sqrt{x/2}}{q}\mathfrak{N} + \mathcal{D} + O\left((q,a)^{\frac{1}{2}}\tau(q)\right), \end{aligned} \tag{3.62}$$

where

$$\mathcal{D} = \sum_{\alpha,\beta} \rho\left(\frac{\sqrt{x/2}-\alpha}{q}\right)\rho\left(\frac{\sqrt{x/2}-\beta}{q}\right).$$

We now apply (3.16) with $M = M_1$ to get

$$\mathcal{D} = \sum_{\alpha,\beta} \rho\left(\frac{\sqrt{x/2}-\alpha}{q}\right) \sum_{1\leq|n|\leq M_1} \frac{1}{2\pi i n} e\left(\frac{\sqrt{x/2}-\beta}{q} n\right) + O\left(\Delta_1\right),$$

where

$$\Delta_1 = \sum_{\alpha,\beta} \min\left(1, {M_1}^{-1}\left\|\frac{\sqrt{x/2}-\beta}{q}\right\|^{-1}\right),$$

by (3.17). Next we apply (3.16) again, with $M = M_1$ again, to get

$$\begin{aligned} \mathcal{D} &= \sum_{\alpha,\beta}\sum_{1\leq|m|\leq M_1} \frac{1}{2\pi i m} e\left(\frac{\sqrt{x/2}-\alpha}{q} m\right) \sum_{1\leq|n|\leq M_1} \frac{1}{2\pi i n} e\left(\frac{\sqrt{x/2}-\beta}{q} n\right) + O\left(\Delta_1 \log M_1\right) \\ &= \sum_{1\leq|m|,|n|\leq M_1} \frac{e\left((m+n)q^{-1}\sqrt{x/2}\right)}{(2\pi i)^2 mn}\mathcal{H}_{m,n} + O\left(\Delta_1 \log M_1\right). \end{aligned}$$

Next we use (3.17) to get

$$\Delta_1 = \sum_{\alpha,\beta}\sum_{n\in\mathbb{Z}} c_n e\left(\frac{\sqrt{x/2}-\beta}{q}n\right) = \sum_{n\in\mathbb{Z}} c_n e\left(\frac{\sqrt{x/2}}{q}n\right)\mathcal{H}_{0,n}.$$

Now we use (3.18) and choose the bound ${M_1}^{-1}\log M_1$ if $|n| < M_1$ or the bound $M_1 n^{-2}$ otherwise to get

$$\mathcal{D} \ll \sum_{1\le|m|,|n|\le M_1} \frac{|\mathcal{H}_{m,n}|}{|mn|} + \log^2 M_1\left(\frac{\eta_a(q)}{M_1} + \sum_{1\le|n|\le M_1}\frac{|\mathcal{H}_{0,n}|}{|n|} + M_1\sum_{|n|>M_1}\frac{|\mathcal{H}_{0,n}|}{n^2}\right),$$

where we have used that $\mathcal{H}_{0,0} = \eta_a(q)$. Next we use (3.36) to get

$$\mathcal{D} \ll q^{\frac{1}{2}}\tau^2(q)(q,a)^{\frac{1}{2}}\left(\sum_{1\le m,n\le M_1}\frac{(q,m,n)^{\frac{1}{2}}}{mn} + \sum_{1\le n\le M_1}\frac{(q,n)^{\frac{1}{2}}}{n} + M_1\sum_{n>M_1}\frac{(q,n)^{\frac{1}{2}}}{n^2}\right)\log^2 M_1 + \frac{\log^2 M_1}{M_1}\eta_a(q).$$

We note that

$$\sum_{1\le m,n\le M_1}\frac{(q,m,n)^{\frac{1}{2}}}{mn} \le \sum_{1\le m\le M_1}\frac{1}{m}\sum_{1\le n\le M_1}\frac{(q,n)^{\frac{1}{2}}}{n} \ll \log M_1 \sum_{1\le n\le M_1}\frac{(q,n)^{\frac{1}{2}}}{n},$$

and then use (3.40) as well as (3.44) to get

$$\mathcal{D} \ll q^{\frac{1}{2}}\tau^3(q)(q,a)^{\frac{1}{2}}\log^4 M_1 + \frac{\log^2 M_1}{M_1}\eta_a(q).$$

We now choose $M_1 = q^2$ and use Corollary 1 to get

$$\mathcal{D} \ll q^{\frac{1}{2}}\tau^3(q)(q,a)^{\frac{1}{2}}\log^4 x. \tag{3.63}$$

From (3.62) and (3.63) we obtain

$$S_2^{(1)} = \frac{\sqrt{x/2}}{q}\mathfrak{N} + O\left(q^{\frac{1}{2}}\tau^3(q)(q,a)^{\frac{1}{2}}\log^4 x\right). \tag{3.64}$$

3.7 The end of the proof.

It remains to substitute our bounds (3.46), (3.59), (3.60), (3.61), and (3.64) into (3.15). This gives us,

$$\begin{aligned}
S_{q,a}(x) =& \frac{8}{q}\left(\left(\frac{\pi}{8}+\frac{1}{4}\right)\frac{\eta_a(q)}{q}x + \sqrt{x/2}\,\mathfrak{N} - \frac{\sqrt{x}}{2}\,\omega_a(q) + O\left(q^{\frac{3}{2}}\tau^3(q)(q,a)^{\frac{1}{2}}\right)\right)\\
&+ O\left(\left(q^{\frac{1}{2}} + x^{\frac{1}{3}}\right)\tau^4(q)(q,a)^{\frac{1}{2}}\log^4 x\right)\\
&- 4\left(\frac{\omega_a(q)}{2q}\sqrt{x/2} + O\left((q,a)^{\frac{1}{2}}\tau(q)\right)\right) - 4\frac{\sqrt{x/2}}{q}\left(\frac{\eta_a(q)}{q}\sqrt{x/2} + \mathfrak{N} - \frac{1}{2}\omega_a(q)\right)\\
&- 4\left(\frac{\sqrt{x/2}}{q}\mathfrak{N} + O\left(q^{\frac{1}{2}}\tau^3(q)(q,a)^{\frac{1}{2}}\log^4 x\right)\right) + 4\frac{\omega_a(q)}{q}\sqrt{x} + O\left((q,a)^{1/2}\tau(q)\right).
\end{aligned}$$

We can see here that the terms involving $\mathfrak{N}$ cancel out and the terms involving $\omega_a(q)$ also cancel out. This leaves us with

$$S_{q,a}(x) = \pi\frac{\eta_a(q)}{q^2}x + O\left((q^{\frac{1}{2}} + x^{\frac{1}{3}})(a,q)^{\frac{1}{2}}\tau^4(q)\log^4 x\right),$$

thus proving (1.24).

Chapter 4

A new estimate for the error term in arithmetic progressions for $4 \nmid q$.

4.1 Initial comments

Over the next two chapters we prove Theorem 9. Chapter 4 deals with the case $4 \nmid q$, and then Chapter 5 deals with the case $4 \mid q$ by relating it to the case $4 \nmid q$. I chose to split the cases in this way because the behaviour of the Dirichlet characters we encounter becomes more problematic if $4 \mid q$ (see Theorem 27), and so splitting the two cases made the proof easier to manage.

For the remainder of this paper, we use the notation

$$\mathbf{1}_{\{\text{statement}\}} = \begin{cases} 1, & \text{statement is TRUE,} \\ 0, & \text{statement is FALSE.} \end{cases}$$

Also, for the remainder of the paper χ_0 will always refer to a trivial Dirichlet character with modulus specified by the context.

The outline of my method closely follows the method in Section 1.3, but ends up being longer.

We assume throughout the remainder of the paper that $x \geq q$. If this were not the case, then Theorem 9 would follow trivially.

4.2 Rewriting $S_{q,a}(x)$

We look at

$$\begin{aligned} S_{q,a}(x) &= \sum_{\substack{n \leq x \\ n \equiv a(q)}} r(n) \\ &= 4 \sum_{\substack{md \leq x \\ md \equiv a(q)}} \chi_4(d). \\ &= 4 \sum_{md \leq x} \chi_4(d) \mathbf{1}_{\{md \equiv a(q)\}}. \end{aligned}$$

We now manipulate the above sum so we may sum m, d over a fixed residue class of q. We then have,

$$\sum_{md\leq x}\chi_4(d)\mathbf{1}_{\{md\equiv a(q)\}} = \sum_{\alpha(q)}\sum_{\beta(q)}\sum_{\substack{md\leq x\\ m\equiv\alpha(q)\\ d\equiv\beta(q)}}\chi_4(d)\mathbf{1}_{\{md\equiv a(q)\}}$$

$$= \sum_{\alpha(q)}\sum_{\beta(q)}\mathbf{1}_{\{\alpha\beta\equiv a(q)\}}\sum_{md\leq x}\chi_4(d)\mathbf{1}_{\{m\equiv\alpha(q)\}}\mathbf{1}_{\{d\equiv\beta(q)\}}. \tag{4.1}$$

Throughout this entire paper we will repeatedly use the following notation,

$$(\alpha, q) = k_1, \qquad (\beta, q) = k_2, \tag{4.2}$$

$$\alpha = \alpha_1 k_1, \qquad \beta = \beta_2 k_2 \tag{4.3}$$

$$q = k_1 q_1, \qquad q = k_2 q_2. \tag{4.4}$$

In order for

$$\sum_{md\leq x}\chi_4(d)\mathbf{1}_{\{m\equiv\alpha(q)\}}\mathbf{1}_{\{d\equiv\beta(q)\}}$$

to be non-zero, we require k_1 and k_2 to divide m and d respectively. Let $m = k_1 m_1$ and $d = k_2 d_2$, then

$$\sum_{md\leq x}\chi_4(d)\mathbf{1}_{\{m\equiv\alpha(q)\}}\mathbf{1}_{\{d\equiv\beta(q)\}} = \sum_{md\leq x}\chi_4(d)\mathbf{1}_{\{m_1\equiv\alpha_1(q_1)\}}\mathbf{1}_{\{d_2\equiv\beta_2(q_2)\}}\mathbf{1}_{\{k_1|m\}}\mathbf{1}_{\{k_2|d\}}$$

$$= \sum_{MD\leq X}\chi_4(Dk_2)\mathbf{1}_{\{M\equiv\alpha_1(q_1)\}}\mathbf{1}_{\{D\equiv\beta_2(q_2)\}}, \tag{4.5}$$

where

$$X = \frac{x}{k_1 k_2}. \tag{4.6}$$

Now we introduce the following indicator function:

For any $(n, q) = 1$ and any $k \in \mathbb{Z}$, we have

$$\mathbf{1}_{\{k\equiv n(q)\}} = \frac{1}{\varphi(q)}\sum_{\chi(q)}\chi(k)\overline{\chi}(n), \tag{4.7}$$

where $\sum_{\chi(q)}$ sums over all Dirichlet characters modulo q. We will very frequently use this identity during Chapter 4 and Chapter 5 to re-express a congruence condition appearing in a sum as a character sum.

Using the above indicator function, we have that

$$\sum_{MD\leq X}\chi_4(Dk_2)\mathbf{1}_{\{M\equiv\alpha_1(q_1)\}}\mathbf{1}_{\{D\equiv\beta_2(q_2)\}} = \frac{\chi_4(k_2)}{\varphi(q_1)\varphi(q_2)}\sum_{MD\leq X}\sum_{\chi_i(q_1)}\chi_i(M)\overline{\chi_i}(\alpha_1)\sum_{\chi_j(q_2)}\chi_j(D)\overline{\chi_j}(\beta_2)\chi_4(D)$$

$$= \frac{\chi_4(k_2)}{\varphi(q_1)\varphi(q_2)}\sum_{\chi_i(q_1)}\overline{\chi_i}(\alpha_1)\sum_{\chi_j(q_2)}\overline{\chi_j}(\beta_2)\sum_{MD\leq X}\chi_i(M)\chi_j(D)\chi_4(D).$$

For shorthand, we let

$$\mathfrak{L}_{i,j}(X;q_1,q_2)=\sum_{md\le X}\chi_i(m)\chi_j(d)\chi_4(d),$$

where q_1,q_2 as function parameters mean that χ_i is a Dirichlet character modulo q_1, and χ_j is a Dirichlet character modulo q_2. Also let

$$\mathcal{K}(X;q,\alpha,\beta)=\sum_{\chi_i(q_1)}\overline{\chi_i}(\alpha_1)\sum_{\chi_j(q_2)}\overline{\chi_j}(\beta_2)\mathfrak{L}_{i,j}(X;q_1,q_2),$$

so altogether we have,

$$S_{q,a}(x)=4\sum_{\alpha(q)}\sum_{\beta(q)}\mathbf{1}_{\{\alpha\beta\equiv a(q)\}}\frac{\chi_4(k_2)}{\varphi(q_1)\varphi(q_2)}\mathcal{K}(X;q,\alpha,\beta). \tag{4.8}$$

4.3 Re-expressing $\mathfrak{L}_{i,j}(X;q_1,q_2)$.

By Dirichlet's hyperbola method, we have

$$\mathfrak{L}_{i,j}(X;q_1,q_2)=\sum\nolimits_1^{(i,j,q_1,q_2)}+\sum\nolimits_2^{(i,j,q_1,q_2)}-\sum\nolimits_3^{(i,j,q_1,q_2)}, \tag{4.9}$$

where

$$\sum\nolimits_1^{(i,j,q_1,q_2)}=\sum_{d\le X^{1/2}}\chi_4(d)\chi_j(d)\sum_{m\le X/d}\chi_i(m),$$

$$\sum\nolimits_2^{(i,j,q_1,q_2)}=\sum_{m\le X^{1/2}}\chi_i(m)\sum_{d\le X/m}\chi_4(d)\chi_j(d),$$

$$\sum\nolimits_3^{(i,j,q_1,q_2)}=\sum_{m\le X^{1/2}}\chi_i(m)\sum_{d\le X^{1/2}}\chi_4(d)\chi_j(d).$$

For simplicity, we henceforth write

$$\sum\nolimits_k^{(i,j)}\qquad\text{for}\qquad\sum\nolimits_k^{(i,j,q_1,q_2)}.$$

Now we prove some theorems on character sums. Let

$$S(\chi,y)=\sum_{1\le n\le y}\chi(n),$$

where χ is a Dirichlet character. For the remainder of the paper, we will more specifically mean

$$\sum_{\gamma(q)}=\sum_{1\le\gamma\le q}.$$

Theorem 24. *Let χ be a non-trivial Dirichlet character modulo q, then*

$$S(\chi, y) = \sum_{\gamma(q)}^{*} \chi(\gamma) \left(\frac{-\gamma}{q} - \psi\left(\frac{y-\gamma}{q} \right) \right),$$

where $\sum^*$ is as defined in (3.28).

Proof. We sum over the residue classes modulo q to get

$$S(\chi, y) = \sum_{\gamma(q)}^{*} \chi(\gamma) \sum_{\substack{1 \le n \le y \\ n \equiv \gamma(q)}} 1.$$

If we choose $1 \le \gamma \le q$ to be the representatives for the residue classes and then use (3.11), we get

$$S(\chi, y) = \sum_{\gamma(q)}^{*} \chi(\gamma) \left(\left[\frac{y-\gamma}{q} \right] + 1 \right). \tag{4.10}$$

Lastly we use that $[w] = w - \psi(w) - 1/2$ and then use that

$$\sum_{\gamma(q)}^{*} \chi(\gamma) = 0$$

to get the result. $\square$

Theorem 25. *Let χ_0 be the trivial Dirichlet character modulo q. Then,*

$$S(\chi_0, y) = \varphi(q) \left(\frac{y}{q} + \frac{1}{2} \right) - \sum_{\gamma(q)}^{*} \left\{ \frac{\gamma}{q} + \psi\left(\frac{y-\gamma}{q} \right) \right\}.$$

Proof. We can prove this in a very similar way to how to we proved Theorem 24. Using (4.10), and that we are now working with a trivial Dirichlet character, we obtain

$$S(\chi_0, y) = \sum_{\gamma(q)}^{*} \left(\left[\frac{y-\gamma}{q} \right] + 1 \right).$$

Lastly we use that $[w] = w - \psi(w) - 1/2$ which proves the theorem. $\square$

Next we prove a result that is very similar to (1.15).

Theorem 26. *Let $\gamma \in \mathbb{R}$, $0 \le \delta \le q$, $L \ge 2$, χ be a Dirichlet character modulo q, and*

$$g(u) = \frac{X}{qu + \delta} - \gamma.$$

Then,

$$\sum_{n \le L} \psi\left(g(n)\right) \ll \left(X^{11/41} q^{-11/41} L^{5/41} + qL^2 X^{-1} \right) \log L, \tag{4.11}$$

where the implicit constant has no dependence on any variables.

Proof. The following proof will be very similar to the proof of (1.15). We will instead prove (4.11) for $-g(u)$ (so that the derivative is increasing), which is clearly sufficient since ψ is an odd function. We now perform a dyadic decomposition to obtain

$$\sum_{n\leq L}\psi\left(-g(n)\right)=\sum_{1\leq j\leq J}\sum_{n\in I_j}\psi\left(-g(n)\right),$$

where

$$I_j=\{n:2^{-j}L<n\leq 2^{-j+1}L\},$$

and J represents how far we have to go along the number line before I_j contains no integers. So $J=O(\log x)$. We will now show that $-g\in F(N_j,P,2,X/q,\epsilon)$, where $P=P(11/30,16/30,2)$, $\epsilon=\epsilon(11/30,16/30,2)$, and $N_j=2^{-j}L$, the size of the interval $[N_j,2N_j]$, satisfies

$$1-\left(1-\frac{1}{N_j+1}\right)^{P+1}<\epsilon,$$

or equivalently $N_j\geq K$ for some K fixed by ϵ and P. To do this we have to show two things. First we show that $-g(u)$ has P continuous derivatives in the interval $[N_j,2N_j]$. Observe that

$$-g^{(p+1)}(u)=(-1)^pXq^{p+1}(p+1)!(qu+\delta)^{-(p+2)}.$$

So $-g(u)$ is infinitely differentiable and defined in the interval $[N_j,2N_j]$ for any $N_j>0$.

Second, we aim to show that for $0\leq p\leq P-1$, and $u\in[N_j,2N_j]$, we have

$$\left|-g^{(p+1)}(u)-(-1)^p(p+1)!Xq^{-1}u^{-(p+2)}\right|<\epsilon(p+1)!Xq^{-1}u^{-(p+2)}.$$

By substituting our formula for $-g^{(p+1)}(u)$ into the above inequality, it suffices to show

$$\left|\left(\frac{q}{qu+\delta}\right)^{p+2}-\left(\frac{1}{u}\right)^{p+2}\right|<\epsilon\left(\frac{1}{u}\right)^{p+2}.$$

Now observe

$$\begin{aligned}\left|\left(\frac{q}{qu+\delta}\right)^{p+2}-\left(\frac{1}{u}\right)^{p+2}\right|&=\left(\frac{1}{u}\right)^{p+2}\left(1-\left(1-\frac{\delta}{qu+\delta}\right)^{p+2}\right)\\&\leq\left(\frac{1}{u}\right)^{p+2}\left(1-\left(1-\frac{1}{N_j+1}\right)^{P+1}\right)\\&<\epsilon\left(\frac{1}{u}\right)^{p+2},\end{aligned}$$

where we have used that

$$\frac{\delta}{qu+\delta}\leq\frac{q}{qN_j+q}.$$

So $-g\in F(N_j,P,2,X/q,\epsilon)$ for $N_j\geq K$. Next observe that $I_j\subset[N_j,2N_j]$, so now we substitute $y=X/q$, $s=2$ and $N=2^{-j}L$ into Theorem 4 which gives us

$$\sum_{n\in I_j}\psi\left(-g(n)\right)\ll\left(\frac{X}{q}\right)^{k/(k+1)}(2^{-j}L)^{(-k+l)/(k+1)}+\left(\frac{q}{X}\right)(2^{-j}L)^2,$$

for any j satisfying $N_j \geq K$ and where the implicit constant here depends only on k and l since s has been fixed. We note that $\sum_{j=1}^{\infty} 2^{-2j}$ converges, and provided $(k,l) \neq (1/2, 1/2)$, then $\sum_{j=1}^{\infty} 2^{(k-l)j/(k+1)}$ also converges. We also know that $J = O(\log x)$, therefore

$$\begin{aligned}\sum_{\substack{1\leq j\leq J\\ N_j\geq K}} \sum_{n\in I_j} \psi\left(-g(n)\right) &\ll \sum_{\substack{1\leq j\leq J\\ N_j\geq K}} \left(\frac{X}{q}\right)^{k/(k+1)} L^{(-k+l)/(k+1)} + qL^2X^{-1}\\ &\ll \left(\left(\frac{X}{q}\right)^{k/(k+1)} L^{(-k+l)/(k+1)} + qL^2X^{-1}\right)\log L.\end{aligned}$$

Note that we have not dealt with bounding in the case $N_j < K$. This case is simple to deal with though because K is fixed, so

$$\sum_{\substack{1\leq j\leq J\\ N_j< K}} \sum_{n\in I_j} \psi\left(-g(n)\right) = O(1).$$

So altogether we obtain

$$\sum_{1\leq j\leq J} \sum_{n\in I_j} \psi\left(-g(n)\right) \ll \left(\left(\frac{X}{q}\right)^{k/(k+1)} L^{(-k+l)/(k+1)} + qL^2X^{-1}\right)\log L.$$

Now we use the exponent pair $(k,l) = BA^3B(0,1) = (11/30, 16/30)$ to obtain

$$\sum_{1\leq j\leq J} \sum_{n\in I_j} \psi\left(-g(n)\right) \ll \left(X^{11/41}q^{-11/41}L^{5/41} + qL^2X^{-1}\right)\log L.$$

Note that since k and l have also been fixed now, the implicit constant here has no dependence on any variables. $\square$

Theorem 27. *Let χ be a Dirichlet character modulo q. Then,*

$$\chi_4\chi = \chi_0,$$

where χ_0 is the trivial Dirichlet character modulo $\operatorname{lcm}(q,4)$, has no solutions if $4 \nmid q$, and exactly one solution when $4 \mid q$. This one solution is described by

$$\chi(n) = {\chi_4}^{-1}(n),$$

for all n coprime to q, and is 0 otherwise.

Proof. If $(q,4) = 1$, then $\chi_4(2q+1)\chi(2q+1) = -1$.

If $(q,4) = 2$, then $\chi_4(q+1)\chi(q+1) = -1$.

If $(q,4) = 4$, then there can be at most one character satisfying $\chi_4\chi = \chi_0$ since there is a requirement for each value of the character, specifically $\chi(n) = {\chi_4}^{-1}(n)$ for all $(n,q) = 1$ and $\chi(n) = 0$ otherwise. This character then exists since it satisfies the conditions of being a Dirichlet character modulo q. $\square$

We now deal with re-expressing $\mathfrak{L}_{i,j}(X; q_1, q_2)$. To do this we will break down each $\sum_{k}^{(i,j)}$ in (4.9), and then reassemble $\mathfrak{L}_{i,j}(X; q_1, q_2)$ as a sum of different sums.

4.3.1 Evaluating $\sum_1^{(i,j)}$.

We have four cases to deal with as $\sum_1^{(i,j)}$ varies based on whether χ_i, χ_j is trivial or not.

If $i, j \neq 0$ we apply Theorem 24 to get

$$\begin{aligned}\sum\nolimits_1^{(i,j)} &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d) \sum_{m \leq X/d} \chi_i(m) \\ &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d) \sum_{\gamma(q_1)}^{*} \chi_i(\gamma) \left(\frac{-\gamma}{q_1} - \psi\left(\frac{X/d - \gamma}{q_1} \right) \right).\end{aligned}$$

If $i \neq 0$ and $j = 0$ we similarly deduce that

$$\begin{aligned}\sum\nolimits_1^{(i,0)} &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_0(d) \sum_{m \leq X/d} \chi_i(m) \\ &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_0(d) \sum_{\gamma(q_1)}^{*} \chi_i(\gamma) \left(\frac{-\gamma}{q_1} - \psi\left(\frac{X/d - \gamma}{q_1} \right) \right).\end{aligned}$$

If $i = 0$ and $j \neq 0$ we instead apply Theorem 25 to get

$$\begin{aligned}\sum\nolimits_1^{(0,j)} &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d) \sum_{m \leq X/d} \chi_0(m) \\ &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d) \left(\varphi(q_1)\left(\frac{X}{dq_1} + \frac{1}{2} \right) - \sum_{\gamma(q_1)}^{*} \left\{ \frac{\gamma}{q_1} + \psi\left(\frac{X/d - \gamma}{q_1} \right) \right\} \right).\end{aligned}$$

If $i, j = 0$ we similarly deduce that

$$\begin{aligned}\sum\nolimits_1^{(0,0)} &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_0(d) \sum_{m \leq X/d} \chi_0(m) \\ &= \sum_{d \leq X^{1/2}} \chi_4(d)\chi_0(d) \left(\varphi(q_1)\left(\frac{X}{dq_1} + \frac{1}{2} \right) - \sum_{\gamma(q_1)}^{*} \left\{ \frac{\gamma}{q_1} + \psi\left(\frac{X/d - \gamma}{q_1} \right) \right\} \right).\end{aligned}$$

Altogether this gives us

$$\sum\nolimits_1^{(i,j)} = \mathcal{E}_1^{(i,j)} + \mathcal{E}_3^{(i,j)} + \mathcal{E}_5^{(i,j)} + \mathcal{E}_6^{(i,j)},$$

where

$$\mathcal{E}_1^{(i,j)} = -\sum_{\gamma(q_1)}^{*} \chi_i(\gamma) \sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d)\psi\left(\frac{X/d - \gamma}{q_1} \right),$$

$$\mathcal{E}_3^{(i,j)} = -\sum_{d \leq X^{1/2}} \chi_4(d)\chi_j(d) \sum_{\gamma(q_1)}^{*} \frac{\gamma\chi_i(\gamma)}{q_1},$$

$$\mathcal{E}_5^{(i,j)} = \delta_{i0} \frac{\varphi(q_1)}{2} \sum_{d \le X^{1/2}} \chi_4(d)\chi_j(d),$$

$$\mathcal{E}_6^{(i,j)} = \delta_{i0} \frac{\varphi(q_1)X}{q_1} \sum_{d \le X^{1/2}} \frac{\chi_4(d)\chi_j(d)}{d},$$

and δ_{ij} denotes the Kronecker delta function.

4.3.2 Evaluating $\sum_2^{(i,j)}$.

Now let

$$l_2 = \operatorname{lcm}(q_2, 4).$$

For the remainder of this paper we will frequently encounter l_2. Note that $l_2 \le 4q_2$. Then for $i, j \ne 0$, we note that $\chi_4\chi_j$ is a Dirichlet character modulo l_2 and is necessarily non-trivial by Theorem 27. We then apply Theorem 24 to obtain

$$\begin{aligned}\sum\nolimits_2^{(i,j)} &= \sum_{m \le X^{1/2}} \chi_i(m) \sum_{d \le X/m} \chi_4(d)\chi_j(d) \\ &= \sum_{m \le X^{1/2}} \chi_i(m) \sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma) \left(\frac{-\gamma}{l_2} - \psi\left(\frac{X/m - \gamma}{l_2} \right) \right).\end{aligned}$$

For $i \ne 0$ and $j = 0$ we again apply Theorem 24, using that $\chi_4\chi_0$ is non-trivial, to get

$$\begin{aligned}\sum\nolimits_2^{(i,0)} &= \sum_{m \le X^{1/2}} \chi_i(m) \sum_{d \le X/m} \chi_4(d)\chi_0(d) \\ &= \sum_{m \le X^{1/2}} \chi_i(m) \sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma) \left(\frac{-\gamma}{l_2} - \psi\left(\frac{X/m - \gamma}{l_2} \right) \right).\end{aligned}$$

For $i = 0$ and $j \ne 0$ we may again apply Theorem 24 to get

$$\begin{aligned}\sum\nolimits_2^{(0,j)} &= \sum_{m \le X^{1/2}} \chi_0(m) \sum_{d \le X/m} \chi_4(d)\chi_j(d) \\ &= \sum_{m \le X^{1/2}} \chi_0(m) \sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma) \left(\frac{-\gamma}{l_2} - \psi\left(\frac{X/m - \gamma}{l_2} \right) \right).\end{aligned}$$

For $i, j = 0$ we similarly deduce that

$$\begin{aligned}\sum\nolimits_2^{(0,0)} &= \sum_{m \le X^{1/2}} \chi_0(m) \sum_{d \le X/m} \chi_4(d)\chi_0(d) \\ &= \sum_{m \le X^{1/2}} \chi_0(m) \sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma) \left(\frac{-\gamma}{l_2} - \psi\left(\frac{X/m - \gamma}{l_2} \right) \right).\end{aligned}$$

Then

$$\sum\nolimits_2^{(i,j)} = \mathcal{E}_2^{(i,j)} + \mathcal{E}_7^{(i,j)},$$

where,

$$\mathcal{E}_2^{(i,j)} = -\sum_{m\le X^{1/2}} \chi_i(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X/m-\gamma}{l_2}\right),$$

$$\mathcal{E}_7^{(i,j)} = -\sum_{m\le X^{1/2}} \chi_i(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\chi_j(\gamma)\frac{\gamma}{l_2}.$$

The $\mathcal{E}_7^{(i,j)}$ term will cancel out in the next section.

4.3.3 Evaluating $\sum_3^{(i,j)}$.

For $i,j\neq 0$, we again note that $\chi_4\chi_j$ is a non-trivial Dirichlet character modulo l_2, and we apply Theorem 24 to get

$$\begin{aligned}\sum\nolimits_3^{(i,j)} &= \sum_{m\le X^{1/2}} \chi_i(m) \sum_{d\le X^{1/2}} \chi_4(d)\chi_j(d)\\ &= \sum_{m\le X^{1/2}} \chi_i(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\chi_j(\gamma)\left(\frac{-\gamma}{l_2} - \psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right).\end{aligned}$$

For $i,j=0$ we apply Theorem 24 again, using that $\chi_4\chi_0$ is non-trivial, to obtain

$$\begin{aligned}\sum\nolimits_3^{(0,0)} &= \sum_{m\le X^{1/2}} \chi_0(m) \sum_{d\le X^{1/2}} \chi_4(d)\chi_0(d)\\ &= \sum_{m\le X^{1/2}} \chi_0(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\left(\frac{-\gamma}{l_2} - \psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right).\end{aligned}$$

For $i=0$ and $j\neq 0$ we again apply Theorem 24 to get

$$\begin{aligned}\sum\nolimits_3^{(0,j)} &= \sum_{m\le X^{1/2}} \chi_0(m) \sum_{d\le X^{1/2}} \chi_4(d)\chi_j(d)\\ &= \sum_{m\le X^{1/2}} \chi_0(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\chi_j(\gamma)\left(\frac{-\gamma}{l_2} - \psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right).\end{aligned}$$

For $i\neq 0$ and $j=0$ we similarly deduce that

$$\begin{aligned}\sum\nolimits_k^{(i,0)} &= \sum_{m\le X^{1/2}} \chi_i(m) \sum_{d\le X^{1/2}} \chi_4(d)\chi_0(d)\\ &= \sum_{m\le X^{1/2}} \chi_i(m) \sum_{\gamma(l_2)}^{*} \chi_4(\gamma)\left(\frac{-\gamma}{l_2} - \psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right).\end{aligned}$$

Then

$$\sum\nolimits_3^{(i,j)} = -\mathcal{E}_4^{(i,j)} + \mathcal{E}_7^{(i,j)},$$

where,

$$\mathcal{E}_4^{(i,j)} = \sum_{m \le X^{1/2}} \chi_i(m) \sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right),$$

and $\mathcal{E}_7^{(i,j)}$ is defined as in the previous subsection.

4.3.4 Recollecting terms.

We now gather the sums we have calculated in the previous three subsections to give us

$$\mathfrak{L}_{i,j}(X; q_1, q_2) = \sum_{n=1}^{6} \mathcal{E}_n^{(i,j)},$$

Looking back at (4.8), we now write

$$\sum_{\alpha,\beta,i,j} \qquad \text{for} \qquad \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{\varphi(q_1)\varphi(q_2)} \sum_{\chi_i(q_1)} \overline{\chi_i}(\alpha_1) \sum_{\chi_j(q_2)} \overline{\chi_j}(\beta_2),$$

and have that

$$S_{q,a}(x) = 4\sum_{n=1}^{6} \mathcal{M}_n, \tag{4.12}$$

where

$$\mathcal{M}_n = \sum_{\alpha,\beta,i,j} \mathcal{E}_n^{(i,j)}. \tag{4.13}$$

4.4 Bounding $\mathcal{M}_1, \mathcal{M}_2$.

We begin with bounding $\mathcal{M}_1$. By using the equation for the indicator function $\mathbf{1}_{\{\gamma\equiv\alpha_1(q_1)\}}$, (4.7), we get

$$\mathcal{M}_1 = -\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{\varphi(q_2)} \sum_{\chi_j(q_2)} \overline{\chi_j}(\beta_2) \sum_{d\le X^{1/2}} \chi_4(d)\chi_j(d)\psi\left(\frac{X/d-\alpha_1}{q_1}\right).$$

We now use the method of exponent pairs to deal with the sum

$$\sum_{d\le X^{1/2}} \chi_4(d)\chi_j(d)\psi\left(\frac{X/d-\alpha_1}{q_1}\right).$$

We first re-express this sum as

$$\sideset{}{^*}\sum_{\delta(l_2)} \chi_4(\delta)\chi_j(\delta) \sum_{0\le d\le (X^{1/2}-\delta)/l_2} \psi\left(\frac{X/(l_2 d+\delta)-\alpha_1}{q_1}\right).$$

Now we use Theorem 26 to get,

$$\begin{aligned}\sum_{0\le d\le (X^{1/2}-\delta)/l_2} \psi\left(\frac{X/(l_2 d+\delta)-\alpha_1}{q_1}\right) &\ll \left(X^{27/82}{q_1}^{-11/41}{l_2}^{-16/41} + q_1{l_2}^{-1}\right)\log X \\ &\ll \left(x^{27/82}q^{-11/41}q^{-27/82} + q_1{q_2}^{-1}\right)\log x,\end{aligned}$$

where we have used that $q_2 \le l_2$. Hence

$$\begin{aligned}\mathcal{M}_1 &\ll \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{1}{\varphi(q_2)} \sum_{\chi_j(q_2)} \sum_{\delta(l_2)}^{*} \left(x^{27/82}q^{-49/82} + q_1{q_2}^{-1}\right)\log x \\ &= \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \varphi(l_2)\left(x^{27/82}q^{-49/82} + q_1{q_2}^{-1}\right)\log x \\ &\ll \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \left(x^{27/82}q^{-49/82}q_2 + q_1\right)\log x,\end{aligned}$$

where we have used the bound $\varphi(l_2) \le l_2 \le 4q_2$. Next we simply bound q_1 and q_2 by q to give us

$$\mathcal{M}_1 \ll \left(x^{27/82}q^{33/82} + q\right)\log x \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}}. \tag{4.14}$$

Next we use the following theorem.

Theorem 28. *We have,*

$$\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \ll q\tau((a,q)).$$

Proof. We have

$$\begin{aligned}\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} &= \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha_1 k_1\beta\equiv a(q_1k_1)\}} \\ &= \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\beta\equiv a\overline{\alpha_1}/k_1(q_1)\}}\mathbf{1}_{\{k_1|a\}} \\ &= \sum_{\alpha(q)} k_1\mathbf{1}_{\{k_1|a\}}.\end{aligned}$$

We now manipulate this sum to iterate over divisors of q so we get

$$\begin{aligned}\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} &= \sum_{d|q} d\mathbf{1}_{\{d|a\}} \sum_{\substack{\alpha(q)\\ k_1=d}} 1 \\ &= \sum_{d|q} d\mathbf{1}_{\{d|a\}}\varphi(q_1).\end{aligned}$$

Then we use that $\varphi(q_1) \le q_1 = q/d$ to get the result. □

Applying this theorem to (4.14), we get

$$\mathcal{M}_1 \ll \left(x^{27/82}q^{115/82} + q^2\right)\tau((a,q))\log x. \tag{4.15}$$

Next we bound $\mathcal{M}_2$, which will be a very similar process to how we bounded $\mathcal{M}_1$. By using the equation for the indicator function $\mathbf{1}_{\{\gamma\equiv\beta_2(q_2)\}}$, we have

$$\mathcal{M}_2 = -\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{\varphi(q_1)} \sum_{\chi_i(q_1)} \overline{\chi_i}(\alpha_1) \sum_{\substack{\gamma(l_2)\\ \gamma\equiv\beta_2(q_2)}}^{*} \chi_4(\gamma) \sum_{m\le X^{1/2}} \chi_i(m)\psi\left(\frac{X/m-\gamma}{l_2}\right)$$

We now use the method of exponent pairs to deal with the sum

$$\sum_{m \le X^{1/2}} \chi_i(m) \psi\left(\frac{X/m - \gamma}{l_2}\right).$$

We first re-express this sum as

$$\sideset{}{^*}\sum_{\delta(q_1)} \chi_i(\delta) \sum_{m \le (X^{1/2} - \delta)/q_1} \psi\left(\frac{X/(q_1 m + \delta) - \gamma}{l_2}\right).$$

We now proceed in a very similar manner to bounding $\mathcal{M}_1$. We use Theorem 26 to get,

$$\begin{aligned}\sum_{m \le (X^{1/2} - \delta)/q_1} \psi\left(\frac{X/(q_1 m + \delta) - \gamma}{l_2}\right) &\ll \left(X^{27/82} {q_1}^{-16/41} {l_2}^{-11/41} + {q_1}^{-1} l_2\right) \log X \\ &\ll \left(x^{27/82} q^{-27/82} q^{-11/41} + {q_1}^{-1} q_2\right) \log x,\end{aligned}$$

where we have used that $q_2 \le l_2 \le 4q_2$. Hence

$$\mathcal{M}_2 \ll \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{1}{\varphi(q_1)} \sum_{\chi_i(q_1)} \sideset{}{^*}\sum_{\delta(q_1)} \sideset{}{^*}\sum_{\substack{\gamma(l_2) \\ \gamma \equiv \beta_2(q_2)}} \left(x^{27/82} q^{-49/82} + {q_1}^{-1} q_2\right) \log x.$$

We note that

$$\sideset{}{^*}\sum_{\substack{\gamma(l_2) \\ \gamma \equiv \beta_2(q_2)}}$$

is a sum of at most four terms, so by a very similar process to how we bounded $\mathcal{M}_1$, we obtain

$$\mathcal{M}_2 \ll \left(x^{27/82} q^{115/82} + q^2\right) \tau((a, q)). \tag{4.16}$$

4.5 Bounding $\mathcal{M}_3, \mathcal{M}_5$.

Theorem 29. *If $4 \nmid q_2$ then,*

$$\sum_{\substack{d \le y \\ d \equiv \beta_2(q_2)}} \chi_4(d) = O(1),$$

and if y is a multiple of l_2 then,

$$\sum_{\substack{d \le y \\ d \equiv \beta_2(q_2)}} \chi_4(d) = 0,$$

Proof. We will prove this by showing that due to cancellation, the modulus of this sum is bounded by 4 for any y, and that if $l_2 \mid y$, then the sum completely cancels.

If $(4, q_2) = 1$, then $\chi_4(\beta_2) + \chi_4(q_2 + \beta_2) + \chi_4(2q_2 + \beta_2) + \chi_4(3q_2 + \beta_2) = 0$.

If $(4, q_2) = 2$, then $\chi_4(\beta_2) + \chi_4(q_2 + \beta_2) = 0$. $\square$

By using the equation for the indicator function $\mathbf{1}_{\{\gamma\equiv\alpha_1(q_1)\}}$ we have,

$$\mathcal{M}_3 = -\frac{1}{q}\sum_{\alpha(q)}\sum_{\beta(q)}\mathbf{1}_{\{\alpha\beta\equiv a(q)\}}\alpha\frac{\chi_4(k_2)}{\varphi(q_2)}\sum_{\chi_j(q_2)}\overline{\chi_j}(\beta_2)\sum_{d\le X^{1/2}}\chi_4(d)\chi_j(d).$$

Now we use the equation for the indicator function $\mathbf{1}_{\{d\equiv\beta_2(q_2)\}}$, and then that $\sum_{d\le X^{1/2}}\chi_4(d)=O(1)$ to get

$$\mathcal{M}_3 \ll \frac{1}{q}\sum_{\alpha(q)}\sum_{\beta(q)}\mathbf{1}_{\{\alpha\beta\equiv a(q)\}}\alpha.$$

We now use the bound $\alpha\le q$ and Theorem 28 to get

$$\mathcal{M}_3 \ll q\tau((a,q)). \tag{4.17}$$

Next we bound $\mathcal{M}_5$. We use the equation for the indicator function $\mathbf{1}_{\{d\equiv\beta_2(q_2)\}}$ to get

$$\mathcal{M}_5 = \frac{1}{2}\sum_{\alpha(q)}\sum_{\beta(q)}\mathbf{1}_{\{\alpha\beta\equiv a(q)\}}\chi_4(k_2)\sum_{\substack{d\le X^{1/2}\\ d\equiv\beta_2(q_2)}}\chi_4(d).$$

The innermost sum is $O(1)$ by Theorem 29. We then apply Theorem 28 to get the bound

$$\mathcal{M}_5 \ll q\tau((a,q)). \tag{4.18}$$

It remains to evaluate $\mathcal{M}_4$ and $\mathcal{M}_6$. We will deal with these terms together as they experience cancellation.

4.6 Evaluating $\mathcal{M}_4+\mathcal{M}_6$.

Recalling (4.13), we will first evaluate $\mathcal{E}_4^{(i,j)}$ and $\mathcal{E}_6^{(i,j)}$, and then re-express $\mathcal{M}_4+\mathcal{M}_6$ as a different sum involving Dirichlet L-functions.

We begin by evaluating $\mathcal{E}_4^{(0,j)}$. Using Theorem 25, we get

$$\mathcal{E}_4^{(0,j)} = \left(\varphi(q_1)\left(\frac{X^{1/2}}{q_1}+\frac{1}{2}\right)-\sum_{\delta(q_1)}^{*}\frac{\delta}{q_1}+\psi\left(\frac{X^{1/2}-\delta}{q_1}\right)\right)\left(\sum_{\gamma(l_2)}^{*}\chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right). \tag{4.19}$$

Next we evaluate $\mathcal{E}_4^{(i,j)}$ for $i\neq 0$. For this we use Theorem 24 to get

$$\mathcal{E}_4^{(i,j)} = \left(-\sum_{\delta(q_1)}^{*}\chi_i(\delta)\left(\frac{\delta}{q_1}+\psi\left(\frac{X^{1/2}-\delta}{q_1}\right)\right)\right)\left(\sum_{\gamma(l_2)}^{*}\chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right). \tag{4.20}$$

We now evaluate $\mathcal{E}_6^{(0,j)}$. We will do so by relating it to a Dirichlet L-function that comes with an error term. The following theorem will allow us to bound the error term that appears:

Theorem 30. *For $y\ge 1$ we have,*

$$\int_y^\infty \psi\left(\frac{t-\gamma}{q}\right)\frac{dt}{t^2} \ll qy^{-2}.$$

Proof. Recall the definition of ψ_1 given in (1.13). Integration by parts gives us,

$$\int_y^\infty \psi\left(\frac{t-\gamma}{q}\right)\frac{dt}{t^2} = \left[q\psi_1\left(\frac{t-\gamma}{q}\right)t^{-2}\right]_y^\infty + 2q\int_y^\infty \psi_1\left(\frac{t-\gamma}{q}\right)\frac{dt}{t^3}.$$

The result then follows because ψ_1 is $O(1)$. □

The next theorem allows us to relate a Dirichlet L-function to a truncated Dirichlet L-function that will appear in $\mathcal{E}_6^{(0,j)}$:

Theorem 31. *Let χ be a non-trivial Dirichlet character modulo q. Then,*

$$\sum_{d\le y}\frac{\chi(d)}{d} = L(1,\chi) - \frac{1}{y}\sideset{}{^*}\sum_{\gamma(q)}\chi(\gamma)\psi\left(\frac{y-\gamma}{q}\right) + \sideset{}{^*}\sum_{\gamma(q)}\chi(\gamma)\int_y^\infty \psi\left(\frac{t-\gamma}{q}\right)\frac{dt}{t^2}.$$

Proof. We have,

$$\begin{aligned}\sum_{d\le y}\frac{\chi(d)}{d} &= \sum_{d=1}^\infty\frac{\chi(d)}{d} - \sum_{d>y}^\infty\frac{\chi(d)}{d}\\ &= L(1,\chi) - \sum_{d\ge y}^\infty\frac{\chi(d)}{d}. \end{aligned} \tag{4.21}$$

We have used here that χ is non-trivial so that $L(1,\chi)$ converges. Then we use summation by parts to get

$$\sum_{d\ge y}^\infty\frac{\chi(d)}{d} = -\frac{S(\chi,y)}{y} + \int_y^\infty\frac{S(\chi,t)}{t^2}dt. \tag{4.22}$$

Applying Theorem 24 we get

$$\frac{S(\chi,y)}{y} = \sideset{}{^*}\sum_{\gamma(q)}\chi(\gamma)\left(\frac{-\gamma}{q} - \psi\left(\frac{y-\gamma}{q}\right)\right)y^{-1}, \tag{4.23}$$

and,

$$\int_y^\infty\frac{S(\chi,t)}{t^2}dt = -\sideset{}{^*}\sum_{\gamma(q)}\chi(\gamma)\frac{\gamma}{q}y^{-1} - \sideset{}{^*}\sum_{\gamma(q)}\chi(\gamma)\int_y^\infty\psi\left(\frac{t-\gamma}{q}\right)\frac{dt}{t^2}. \tag{4.24}$$

Combining (4.21), (4.22), (4.23), and (4.24) proves the theorem. □

We know that $\chi_4\chi_j$ is non-trivial by Theorem 27, so we may use Theorem 31 to get

$$\mathcal{E}_6^{(0,j)} = \frac{\varphi(q_1)}{q_1}\left(L(\chi_4\chi_j,1)X - X^{1/2}\sideset{}{^*}\sum_{\gamma(l_2)}\chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right) + \sideset{}{^*}\sum_{\gamma(l_2)}\chi_4(\gamma)\chi_j(\gamma)\int_{X^{1/2}}^\infty\psi\left(\frac{t-\gamma}{l_2}\right)\frac{dt}{t^2}\right). \tag{4.25}$$

We now recollect the terms (4.19), (4.20), and (4.25), noting that there is cancellation between $\mathcal{E}_4^{(0,j)}$ and $\mathcal{E}_6^{(0,j)}$, to get

$$\mathcal{M}_4 + \mathcal{M}_6 = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)}, \tag{4.26}$$

where

$$\mathcal{M}^{(0)} = X \sum_{\alpha,\beta,0,j} \frac{\varphi(q_1)}{q_1} L(\chi_4\chi_j, 1),$$

$$\mathcal{M}^{(1)} = \frac{1}{2} \sum_{\alpha,\beta,0,j} \varphi(q_1) \left(\sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right),$$

$$\mathcal{M}^{(2)} = -\sum_{\alpha,\beta,i,j} \sideset{}{^*}\sum_{\delta(q_1)} \chi_i(\delta)\left(\frac{\delta}{q_1} + \psi\left(\frac{X^{1/2}-\delta}{q_1}\right)\right)\left(\sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right),$$

$$\mathcal{M}^{(3)} = \sum_{\alpha,\beta,0,j} \sideset{}{^*}\sum_{\gamma(l_2)} \frac{\varphi(q_1)}{q_1}\chi_4(\gamma)\chi_j(\gamma)\int_{X^{1/2}}^{\infty} \psi\left(\frac{t-\gamma}{l_2}\right)\frac{dt}{t^2}.$$

We will see that $\mathcal{M}^{(0)}$ provides the main term of $S_{q,a}(x)$, while the other $\mathcal{M}^{(i)}$ can be bounded by $O(q^2\tau((a,q)))$.

4.7 Bounding $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \mathcal{M}^{(3)}$.

We first bound $\mathcal{M}^{(1)}$. We expand out $\sum_{\alpha,\beta,0,j}$, then use the indicator function for $\gamma \equiv \beta_2(q_2)$ and then apply Theorem 28 to get

$$\begin{aligned}
\mathcal{M}^{(1)} &= \frac{1}{2}\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{\varphi(q_2)} \sum_{\chi_j(q_2)} \overline{\chi_j}(\beta_2)\left(\sideset{}{^*}\sum_{\gamma(l_2)} \chi_4(\gamma)\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right)\\
&= \frac{1}{2}\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \chi_4(k_2)\left(\sideset{}{^*}\sum_{\substack{\gamma(l_2)\\ \gamma\equiv\beta_2(q_2)}} \chi_4(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right)\\
&= \sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} O(1)\\
&\ll q\tau((a,q)).
\end{aligned} \tag{4.27}$$

Now we bound $\mathcal{M}^{(2)}$ in a very similar way. We expand out $\sum_{\alpha,\beta,i,j}$, then use the indicator functions for $\gamma \equiv \beta_2(q_2)$ and $\delta \equiv \alpha_1(q_1)$, and then apply Theorem 28 to get

$$\begin{aligned}
\mathcal{M}^{(2)} &= -\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \chi_4(k_2)\left(\frac{\alpha_1}{q_1} + \psi\left(\frac{X^{1/2}-\alpha_1}{q_1}\right)\right)\left(\sideset{}{^*}\sum_{\substack{\gamma(l_2)\\ \gamma\equiv\beta_2(q_2)}} \chi_4(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{l_2}\right)\right)\\
&= -\sum_{\alpha(q)}\sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} O(1)\\
&\ll q\tau((a,q)).
\end{aligned} \tag{4.28}$$

Finally we bound $\mathcal{M}^{(3)}$. We use the indicator function for $\gamma \equiv \beta_2(q_2)$ to get

$$\mathcal{M}^{(3)} = \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{\chi_4(k_2)}{q_1} \sideset{}{^*}\sum_{\substack{\gamma(l_2) \\ \gamma \equiv \beta_2(q_2)}} \chi_4(\gamma) \int_{X^{1/2}}^{\infty} \psi\left(\frac{t-\gamma}{l_2}\right) \frac{dt}{t^2}.$$

Now we use Theorem 30 to get

$$\begin{aligned}
\mathcal{M}^{(3)} &= \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{\chi_4(k_2)}{q_1} \sideset{}{^*}\sum_{\substack{\gamma(l_2) \\ \gamma \equiv \beta_2(q_2)}} \chi_4(\gamma) \left(O(l_2 X^{-1})\right) \\
&= \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{\chi_4(k_2)}{q_1} O(l_2 X^{-1}) \\
&\ll x^{-1} \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} {k_1}^2 \\
&\ll \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} k_1. \qquad (4.29)
\end{aligned}$$

We bound k_1 by q and then apply Theorem 28 to get

$$\mathcal{M}^{(3)} \ll q^2 \tau((a,q)).$$

It remains now to evaluate $\mathcal{M}^{(0)}$, which we will see gives us exactly the main term for $S_{q,a}(x)$.

4.8 Relating $\mathcal{M}^{(0)}$ to the main term.

We have the following expression

$$\mathcal{M}^{(0)} = \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{\chi_4(k_2)}{q_1 \varphi(q_2)} \sum_{\chi_j(q_2)} \overline{\chi_j}(\beta_2) \sum_{D=1}^{\infty} \frac{\chi_4(D)\chi_j(D)}{D} X.$$

We wish to exchange sums to evaluate this expression. We justify this by using the following theorem proved in [3].

Theorem 32. *Let f be some two variable function, I be non-empty and finite, and suppose that for each $x \in I$ the series $\sum_{y=0}^{\infty} f(x,y)$ converges. Then the series $\sum_{y=0}^{\infty} \sum_{x \in I} f(x,y)$ also converges, and we have*

$$\sum_{x \in I} \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} \sum_{x \in I} f(x,y)$$

Proof. We have,

$$\begin{aligned}
\sum_{x \in I} \sum_{y=0}^{\infty} f(x,y) &= \sum_{x \in I} \left(\lim_{n \to \infty} \sum_{y=0}^{n} f(x,y) \right) = \lim_{n \to \infty} \left(\sum_{x \in I} \sum_{y=0}^{n} f(x,y) \right) \\
&= \lim_{n \to \infty} \left(\sum_{y=0}^{n} \sum_{x \in I} f(x,y) \right) = \sum_{y=0}^{\infty} \sum_{x \in I} f(x,y).
\end{aligned}$$

□

So by using the indicator function for $d \equiv \beta_2(q_2)$, and (4.6), and Theorem 32, we have that

$$\mathcal{M}^{(0)} = \frac{x}{q} \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{k_2} \sum_{\substack{D=1 \\ D\equiv\beta_2(q_2)}}^{\infty} \frac{\chi_4(D)}{D}.$$

Remark. *The innermost sum of this expression converges.*

Now let

$$\mathcal{M}^{(0)} = \frac{x}{q} \mathcal{Z}_a(q), \tag{4.30}$$

where

$$\mathcal{Z}_a(q) = \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\chi_4(k_2)}{k_2} \sum_{\substack{D=1 \\ D\equiv\beta_2(q_2)}}^{\infty} \frac{\chi_4(D)}{D}.$$

We will first manipulate $\mathcal{Z}_a(q)$ into a more useful expression, which will allow us to calculate $\mathcal{Z}_a(q)$ explicitly in terms of $\eta_a(q)$, defined in (1.20). Observe that

$$\begin{aligned} \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} &= \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\equiv a\overline{\beta_2}/k_2(q_2)\}} \mathbf{1}_{\{k_2|a\}} \\ &= k_2 \mathbf{1}_{\{k_2|a\}}, \end{aligned} \tag{4.31}$$

so

$$\begin{aligned} \mathcal{Z}_a(q) &= \sum_{\beta(q)} \mathbf{1}_{\{k_2|a\}} \chi_4(k_2) \sum_{\substack{D=1 \\ D\equiv\beta_2(q_2)}}^{\infty} \frac{\chi_4(D)}{D} \\ &= \sum_{d|q} \mathbf{1}_{\{d|a\}} \chi_4(d) \sum_{\substack{\beta(q) \\ k_2=d}} \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \mathbf{1}_{\{D\equiv\beta_2(q_2)\}}. \end{aligned}$$

Observe that

$$\sum_{\substack{\beta(q) \\ k_2=d}} \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \mathbf{1}_{\{D\equiv\beta_2(q_2)\}} = \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \sum_{\substack{\beta(q) \\ k_2=d}} \mathbf{1}_{\{D\equiv\beta_2(q_2)\}}. \tag{4.32}$$

We may justify this exchange of sums using Theorem 32 and by our earlier remark that this innermost infinite sum converges. So we now focus on the finite inner sum of (4.32). Each term is non-zero unless d divides β, so each β is of the form Kd for some K giving us,

$$\begin{aligned} \sum_{\substack{\beta(q) \\ (\beta,q)=d}} \mathbf{1}_{\{D\equiv\beta_2(q_2)\}} &= \sum_{K(q_2)}^{*} \mathbf{1}_{\{D\equiv K(q_2)\}} \\ &= \mathbf{1}_{\{(D,q_2)=1\}}. \end{aligned}$$

And so,

$$\mathcal{Z}_a(q) = \sum_{d|q} \mathbf{1}_{\{d|a\}} \chi_4(d) \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \mathbf{1}_{\{(D,q_2)=1\}}. \tag{4.33}$$

Note that $\mathbf{1}_{\{(D,q_2)=1\}} = \chi_0(D)$ where χ_0 is the trivial Dirichlet character modulo q_2, and that we have now removed k_2 from the sum and q_2 is now equal to q/d.

We will prove the following theorem which will allow us to get the main term for $S_{q,a}(x)$:

Theorem 33. *Let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $4 \nmid q$. Then*

$$\frac{4}{\pi}\mathcal{Z}_a(q) = \frac{\eta_a(q)}{q}.$$

To prove Theorem 33, we will first show that the LHS is multiplicative, and then prove the result for q an odd prime power or $q = 2$.

Theorem 34. *The function $4\mathcal{Z}_a(q)/\pi$ is multiplicative in q.*

Proof. Now let $q = mn$, where $(m,n) = 1$, and let d be a divisor of q that we iterate through in (4.33) such that $d = d_1 d_2$ where $d_1 \mid m$ and $d_2 \mid n$, and $m = d_1 m_1$, $n = d_2 n_2$ (so $q_2 = m_1 n_2$). We have

$$\frac{4}{\pi}\mathcal{Z}_a(q) = \frac{4}{\pi}\sum_{d|q}\mathbf{1}_{\{d|a\}}\chi_4(d)\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,q_2)=1\}},$$

by (4.33). Since $(m,n) = 1$, the sum over the divisors of q can be rewritten as the sum over the divisors of m and the divisors of n. Furthermore, we have $\mathbf{1}_{\{d|a\}}$ and $\chi_4(d)$ are both multiplicative functions in d, and we may manipulate the following indicator function

$$\mathbf{1}_{\{(D,q_2)=1\}} = \mathbf{1}_{\{(D,m_1)=1\}}\mathbf{1}_{\{(D,n_2)=1\}}.$$

Hence,

$$\frac{4}{\pi}\mathcal{Z}_a(mn) = \sum_{d_1|m}\mathbf{1}_{\{d_1|a\}}\chi_4(d_1)\sum_{d_2|n}\mathbf{1}_{\{d_2|a\}}\chi_4(d_2)\left(\frac{4}{\pi}\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,m_1)=1\}}\mathbf{1}_{\{(D,n_2)=1\}}\right). \tag{4.34}$$

Remark. *The function $\chi_4(D)\mathbf{1}_{\{(D,k)=1\}}$ is a non-trivial Dirichlet character modulo $\operatorname{lcm}(4,k)$ and $\chi_4(D)\mathbf{1}_{\{(D,k_1)=1\}}\mathbf{1}_{\{(D,k_2)=1\}}$ is a non-trivial Dirichlet character modulo $\operatorname{lcm}(4,k_1,k_2)$.*

We also have by (4.33) that

$$\left(\frac{4}{\pi}\right)^2\mathcal{Z}_a(m)\mathcal{Z}_a(m) = \left(\frac{4}{\pi}\right)^2\left(\sum_{d_1|m}\mathbf{1}_{\{d_1|a\}}\chi_4(d_1)\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,m_1)=1\}}\right)$$
$$\times\left(\sum_{d_2|n}\mathbf{1}_{\{d_2|a\}}\chi_4(d_2)\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,n_2)=1\}}\right). \tag{4.35}$$

By equating (4.34) and (4.35), to show multiplicativity, it therefore suffices to show

$$\frac{\pi}{4}\left(\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,m_1)=1\}}\mathbf{1}_{\{(D,n_2)=1\}}\right) = \left(\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,m_1)=1\}}\right)\left(\sum_{D=1}^{\infty}\frac{\chi_4(D)}{D}\mathbf{1}_{\{(D,n_2)=1\}}\right). \tag{4.36}$$

For simplicity, we use the notation

$$\mathbf{1}_n(D) = \mathbf{1}_{\{(D,n)=1\}}.$$

Lemma 2. *Let* $(m_1, n_2) = 1$*, then*

$$(\chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}) * \chi_4(D) = (\chi_4 \mathbf{1}_{m_1}) * (\chi_4 \mathbf{1}_{n_2})(D).$$

Proof. Since the LHS and RHS are both just convolutions of products of Dirichlet characters, both sides must be multiplicative and so it suffices to prove this result for prime powers. Next observe that $(\chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}) * \chi_4 = \chi_4(\mathbf{1}_{m_1} \mathbf{1}_{n_2} * 1)$ and $(\chi_4 \mathbf{1}_{m_1}) * (\chi_4 \mathbf{1}_{n_2}) = \chi_4(\mathbf{1}_{m_1} * \mathbf{1}_{n_2})$ because χ_4 is completely multiplicative, so we will show $(\mathbf{1}_{m_1} \mathbf{1}_{n_2}) * 1 = \mathbf{1}_{m_1} * \mathbf{1}_{n_2}$.

Let $D = p^k$ where p is a prime number. Then we have

$$(\mathbf{1}_{m_1} \mathbf{1}_{n_2}) * 1 = \sum_{d|p^k} \mathbf{1}_{m_1}(d) \mathbf{1}_{n_2}(d) = 1 + k \mathbf{1}_{m_1}(p) \mathbf{1}_{n_2}(p),$$

and

$$\mathbf{1}_{m_1} * \mathbf{1}_{n_2} = \sum_{d|p^k} \mathbf{1}_{m_1}(d) \mathbf{1}_{n_2}\left(\frac{p^k}{d}\right) = \mathbf{1}_{m_1}(p) + (k-1)\mathbf{1}_{m_1}(p)\mathbf{1}_{n_2}(p) + \mathbf{1}_{n_2}(p).$$

Hence it suffices to show

$$1 + \mathbf{1}_{m_1}(p)\mathbf{1}_{n_2}(p) = \mathbf{1}_{m_1}(p) + \mathbf{1}_{n_2}(p).$$

Note that since $(m_1, n_2) = 1$, we can not have that $\mathbf{1}_{m_1}(p) = \mathbf{1}_{n_2}(p) = 0$. For every other $\mathbf{1}_{m_1}(p), \mathbf{1}_{n_2}(p) \in \{0, 1\}$, we see that the above equation holds. □

We now note the equation

$$L(s, \chi) = s \int_1^\infty S(\chi, t) t^{-s-1} dt$$

for a non-trivial Dirichlet character modulo q, which can be proved by summation by parts on the partial sums of $L(s, \chi)$ and then taking a limit. The integral here converges absolutely for $\mathrm{Re}(s) > 0$, since $|S(\chi, t)| \le q$, and so we have that $\sigma_\chi \le 1/2$, where σ_χ denotes the abscissa of convergence of $L(s, \chi)$.

Since χ_4, $\chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}$, $\chi_4 \mathbf{1}_{m_1}$ and $\chi_4 \mathbf{1}_{n_2}$ are all non-trivial Dirichlet characters, we have $\sigma_{\chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}}$, σ_{χ_4}, $\sigma_{\chi_4 \mathbf{1}_{m_1}}$, $\sigma_{\chi_4 \mathbf{1}_{n_2}} \le 1/2$. So by Lemma 2, we have

$$\begin{aligned} L(s, \chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}) L(s, \chi_4) &= L(s, \chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2} * \chi_4) \\ &= L(s, \chi_4 \mathbf{1}_{m_1} * \chi_4 \mathbf{1}_{n_2}) \\ &= L(s, \chi_4 \mathbf{1}_{m_1}) L(s, \chi_4 \mathbf{1}_{n_2}), \end{aligned}$$

for $\mathrm{Re}(s) > 1/2$, and in particular $L(1, \chi_4 \mathbf{1}_{m_1} \mathbf{1}_{n_2}) L(1, \chi_4) = L(1, \chi_4 \mathbf{1}_{m_1}) L(1, \chi_4 \mathbf{1}_{n_2})$. So (4.36) follows. □

We will first prove two theorems which will allow us to calculate $\mathcal{Z}_a(q)$ for q an odd prime power or $q = 2$.

Theorem 35. *Let* q *be a prime power. If* χ_0 *is the trivial character modulo* q *and* $\chi_0(d)\chi_4(d) = g * \chi_4(d)$*, then*

$$g(d) = \begin{cases} \chi_4(d)(\chi_0(d) - 1), & d \text{ prime}, \\ 1, & d = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. First we show that g is multiplicative. Then it suffices to prove the result for d a prime power.

Let $m, n \in \mathbb{Z}^{\geq 2}$ such that $(m, n) = 1$. If m or n are composite then $g(mn) = g(m)g(n) = 0$, hence we assume now that m and n are prime. Then mn is composite so $g(mn) = 0$ hence it suffices to show that $g(m)$ or $g(n)$ is zero. Since m and n are both prime and relatively prime, $\chi_0(m) = 1$ or $\chi_0(n) = 1$, therefore $g(m)$ or $g(n)$ equal zero. So g is multiplicative.

Now we show that $\chi_0(d)\chi_4(d) = g * \chi_4(d)$ for $d = p^k$ where p is prime and $k \in \mathbb{N}$.

We have, $g * \chi_4(d) = g(1)\chi_4(p^k) + g(p)\chi_4(p^{k-1}) = \chi_4(p^k)\chi_0(p) = \chi_4(p^k)\chi_0(p^k) = \chi_4(d)\chi_0(d)$, as desired. ☐

Theorem 36. *Suppose $q = p^k$, where p is prime and $k \in \mathbb{N}$. Let χ_0 be the trivial Dirichlet character modulo q, then*

$$\sum_{d=1}^{\infty} \frac{\chi_0(d)\chi_4(d)}{d} = \frac{\pi\eta_1(q)}{4q},$$

if p is an odd prime, or if $q = 2$.

Proof. Taking the function g from Theorem 35, we have

$$\begin{aligned} L(s, g) &= 1 + \sum_{d=2}^{\infty} \frac{g(d)}{d^s} \\ &= 1 - \sum_{\substack{p \mid q \\ p \text{ prime}}} \frac{\chi_4(p)}{p^s}. \end{aligned} \tag{4.37}$$

This is just a finite sum so $\sigma_g = -\infty$. We know that if χ is non-trivial, then $\sigma_\chi \leq 1/2$, so we must have

$$L(s, \chi_0\chi_4) = L(s, g)L(s, \chi_4),$$

for $\mathrm{Re}(s) > \max(\sigma_g, \sigma_\chi) = 1/2$, and in particular $L(1, \chi_0\chi_4) = L(1, g)L(1, \chi_4)$. Hence

$$\begin{aligned} \sum_{d=1}^{\infty} \frac{\chi_0(d)\chi_4(d)}{d} &= L(1, \chi_4)L(1, g) \\ &= \frac{\pi}{4} L(1, g). \end{aligned} \tag{4.38}$$

We then use Theorem 10 and that $\eta_1(2) = 1$ to obtain

$$1 - \sum_{\substack{p \mid q \\ p \text{ prime}}} \frac{\chi_4(p)}{p} = 1 - \frac{\chi_4(p)}{p} = \frac{\eta_1(q)}{q}. \tag{4.39}$$

Both equalities here use our assumption that $q = p^k$ where p is an odd prime, or $q = 2$, as equality does not hold for general q. The result then follows by (4.37), (4.38), and (4.39). ☐

Theorem 37. *Suppose $q = p^k$, where p is prime and $k \in \mathbb{N}$, then*

$$\mathcal{Z}_a(q) = \frac{\pi}{4q}\eta_a(q),$$

if p is an odd prime, or if $q = 2$.

Proof. We have

$$\mathcal{Z}_a(q) = \sum_{d|(a,q)} \chi_4(d) \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \mathbf{1}_{\{(D,q_2)=1\}},$$

by (4.33). Let $m = \min(v_p(a), k)$. Since q is a power of p, we have

$$\sum_{d|(a,q)} \chi_4(d) = \sum_{0 \le b \le m} \chi_4(p^b),$$

and now we have that $q_2 = q/p^b$. Hence

$$\mathcal{Z}_a(q) = \sum_{0 \le b \le m} \chi_4(p^b) \sum_{D=1}^{\infty} \frac{\chi_4(D)\chi_0(D)}{D},$$

where χ_0 is the trivial Dirichlet character modulo q_2. Since q_2 is a prime power and $4 \nmid q_2$, we may use Theorem 36 to give us

$$\mathcal{Z}_a(q) = \sum_{0 \le b \le m} \chi_4(p^b) \frac{\pi \eta_1(q_2)}{4q_2}.$$

We now consider two cases based on whether m is equal to k or not.

If $m < k$, then since q_2 is a prime power and $4 \nmid q$, we use Theorem 10 and Theorem 13, and that $\eta_1(2) = 2$, to get

$$\frac{\eta_1(q_2)}{q_2} = \frac{\eta_1(p)}{p} = \frac{p - \chi_4(p)}{p}, \tag{4.40}$$

for each $0 \le b \le m$. Since $m < k$, then $m = v_p(a)$. By (2.7), we therefore get

$$\begin{aligned} \mathcal{Z}_a(p^k) &= \frac{\pi}{4} \left(\frac{p - \chi_4(p)}{p} \right) \sum_{0 \le b \le m} \chi_4(p^b) \\ &= \frac{\pi}{4} \frac{\eta_a(p^k)}{p^k}, \end{aligned}$$

which proves the theorem in the case $m < k$.

If $m = k$, then $q/p^m = 1$, so

$$\begin{aligned} \mathcal{Z}_a(q) &= \frac{\pi}{4} \sum_{0 \le b \le m} \chi_4(p^b) \frac{\eta_1(q_2)}{q_2} \\ &= \frac{\pi}{4} \left(\chi_4(p^m) + \sum_{0 \le b \le m-1} \chi_4(p^b) \frac{\eta_1(q_2)}{q_2} \right). \end{aligned}$$

All the b values in this sum are now strictly less than m, so we can now use (4.40) because $q_2 \neq 1$ and so q_2 is a prime power. We then use (2.8) to obtain

$$\begin{aligned} \mathcal{Z}_a(q) &= \frac{\pi}{4} \left(\chi_4(p^m) + \left(\frac{p - \chi_4(p)}{p} \right) \sum_{0 \le b \le m-1} \chi_4(p^b) \right) \\ &= \frac{\pi}{4} \frac{\eta_a(p^k)}{p^k}, \end{aligned}$$

which proves the theorem now for the case $m = k$. $\square$

Now we are able to prove Theorem 33.

Proof of Theorem 33. Since $4 \nmid q$, we can decompose q into a product of odd prime powers and a power of two with exponent at most one. By multiplicativity (Theorem 34), it suffices to prove that

$$\frac{4}{\pi}\mathcal{Z}_a(q) = \frac{\eta_a(q)}{q},$$

for q an odd prime power or $q = 2$, which we proved in Theorem 37. $\square$

Recalling (4.30), we use Theorem 33 to get

$$\mathcal{M}^{(0)} = \frac{\pi\eta_a(q)}{4q^2}x. \tag{4.41}$$

It now remains to sum together the many sums we have bounded or explicitly calculated to acquire our bound for $R_{q,a}(x)$. We substitute (4.27), (4.28), (4.29), and our explicit calculation for $\mathcal{M}^{(0)}$, (4.41), into (4.26). We then substitute this term for (4.26), along with (4.15), (4.16), (4.17), and (4.18), into (4.12), which gives us

$$S_{q,a}(x) = \pi\frac{\eta_a(q)}{q^2}x + O\left(\left(x^{27/82}q^{115/82} + q^2\right)\tau((a,q))\log x\right),$$

hence

$$R_{q,a}(x) \ll \left(x^{27/82}q^{115/82} + q^2\right)\tau((a,q))\log x,$$

which proves Theorem 9 in the case $4 \nmid q$.

Chapter 5

A new estimate for the error term in arithmetic progressions for $4 \mid q$.

5.1 Rewriting $S_{q,a}(x)$.

In this chapter we will prove Theorem 9 in the case $4 \mid q$. For this entire chapter we assume $4 \mid q$. We go back to (4.1) and use that $4 \mid q$ to get $\chi_4(d) = \chi_4(\beta)$, hence

$$S_{q,a}(x) = 4 \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \sum_{\substack{md\leq x \\ m\equiv\alpha(q) \\ d\equiv\beta(q)}} 1.$$

We will use the same variables, (4.2), (4.3), (4.4), (4.6) as defined in the previous chapter. Then in a very similar way to how we got (4.5), we obtain

$$\sum_{\substack{md\leq x \\ m\equiv\alpha(q) \\ d\equiv\beta(q)}} 1 = \sum_{\substack{MD\leq X \\ M\equiv\alpha_1(q_1) \\ D\equiv\beta_2(q_2)}} 1.$$

Now let

$$\mathfrak{L}(X;\alpha_1,q_1,\beta_2,q_2) = \mathfrak{L}(X) = \sum_{\substack{md\leq X \\ m\equiv\alpha_1(q_1) \\ d\equiv\beta_2(q_2)}} 1.$$

Then we have

$$S_{q,a}(x) = 4 \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \mathfrak{L}(X). \tag{5.1}$$

By Dirichlet's hyperbola method, we obtain

$$\mathfrak{L}(X) = \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3,$$

where,

$$\sum\nolimits_1 = \sum_{\substack{d\leq X^{1/2} \\ d\equiv\beta_2(q_2)}} \sum_{\substack{m\leq X/d \\ m\equiv\alpha_1(q_1)}} 1,$$

$$\sum\nolimits_2 = \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \sum_{\substack{d \le X/m \\ d \equiv \beta_2 (q_2)}} 1,$$

$$\sum\nolimits_3 = \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \sum_{\substack{d \le X^{1/2} \\ d \equiv \beta_2 (q_2)}} 1.$$

5.2 Evaluating $\mathfrak{L}(X)$.

We apply Theorem 18, using that $\rho = -\psi$, to each of the above three sums and we obtain

$$\sum\nolimits_1 = \sum_{\substack{d \le X^{1/2} \\ d \equiv \beta_2 (q_2)}} \left(\frac{X}{dq_1} - \psi\left(\frac{X/d - \alpha_1}{q_1}\right) + \psi\left(\frac{-\alpha_1}{q_1}\right)\right),$$

$$\sum\nolimits_2 = \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \left(\frac{X}{mq_2} - \psi\left(\frac{X/m - \beta_2}{q_2}\right) + \psi\left(\frac{-\beta_2}{q_2}\right)\right),$$

$$\sum\nolimits_3 = \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \left(\frac{X^{1/2}}{q_2} - \psi\left(\frac{X^{1/2} - \beta_2}{q_2}\right) + \psi\left(\frac{-\beta_2}{q_2}\right)\right).$$

Expanding further, and using Theorem 18 again, we then have that

$$\sum\nolimits_1 = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3,$$

where

$$\mathcal{S}_1 = \frac{X}{q_1} \sum_{\substack{d \le X^{1/2} \\ d \equiv \beta_2 (q_2)}} \frac{1}{d},$$

$$\mathcal{S}_2 = -\sum_{\substack{d \le X^{1/2} \\ d \equiv \beta_2 (q_2)}} \psi\left(\frac{X/d - \alpha_1}{q_1}\right),$$

$$\mathcal{S}_3 = \psi\left(\frac{-\alpha_1}{q_1}\right)\left(\frac{X^{1/2}}{q_2} - \psi\left(\frac{X^{1/2} - \beta_2}{q_2}\right) + \psi\left(\frac{-\beta_2}{q_2}\right)\right), \tag{5.2}$$

and

$$\sum\nolimits_2 - \sum\nolimits_3 = \mathcal{S}_4 + \mathcal{S}_5 + \mathcal{S}_6,$$

where

$$\mathcal{S}_4 = \frac{X}{q_2} \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \frac{1}{m},$$

$$\mathcal{S}_5 = - \sum_{\substack{m \le X^{1/2} \\ m \equiv \alpha_1 (q_1)}} \psi\left(\frac{X/m - \beta_2}{q_2}\right),$$

$$\mathcal{S}_6 = \left(-\frac{X^{1/2}}{q_2} + \psi\left(\frac{X^{1/2} - \beta_2}{q_2}\right)\right)\left(\frac{X^{1/2}}{q_1} - \psi\left(\frac{X^{1/2} - \alpha_1}{q_1}\right) + \psi\left(\frac{-\alpha_1}{q_1}\right)\right). \tag{5.3}$$

Looking back at (5.1), we now introduce new notation for the remainder of this paper. We will write

$$\sum_{\alpha,\beta} \quad \text{for} \quad \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}}. \tag{5.4}$$

Note that $\sum_{\alpha,\beta}$ means something different than in Chapter 3. Using this notation we have

$$S_{q,a}(x) = 4 \sum_{i=1}^{6} \mathcal{N}_i, \tag{5.5}$$

where

$$\mathcal{N}_i = \sum_{\alpha,\beta} \mathcal{S}_i. \tag{5.6}$$

5.3 Evaluating $\mathcal{N}_2, \mathcal{N}_5$.

This subsection is very similar to Subsection 4.4. We bound $\mathcal{N}_2$ in a very similar way to how we bounded $\mathcal{M}_1$. We substitute the character sum indicator function, (4.7), for $\mathbf{1}_{\{d \equiv \beta_2 (q_2)\}}$, to get

$$\mathcal{N}_2 = - \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{1}{\varphi(q_2)} \sum_{\chi_j(q_2)} \overline{\chi_j}(\beta_2) \sum_{d \le X^{1/2}} \psi\left(\frac{X/d - \alpha_1}{q_1}\right) \chi_j(d).$$

We re-express the innermost sum as

$$\sum_{\delta(q_2)}^{*} \chi_j(\delta) \sum_{d \le (X^{1/2} - \delta)/q_2} \psi\left(\frac{X/(q_2 d + \delta) - \alpha_1}{q_1}\right),$$

and then apply Theorem 26 to get

$$\mathcal{N}_2 \ll \left(x^{27/82} q^{115/82} + q^2\right) \tau((a,q)) \log x. \tag{5.7}$$

We bound $\mathcal{N}_5$ in a very similar way to how we bounded $\mathcal{M}_2$, which gives us

$$\mathcal{N}_5 \ll \left(x^{27/82} q^{115/82} + q^2\right) \tau((a,q)) \log x. \tag{5.8}$$

5.4 Evaluating $\mathcal{N}_1 + \mathcal{N}_3 + \mathcal{N}_4 + \mathcal{N}_6$.

5.4.1 Evaluating $\mathcal{S}_3$ and $\mathcal{S}_6$.

We can see from (5.2) that

$$\mathcal{S}_3 = \psi\left(\frac{-\alpha_1}{q_1}\right)\frac{X^{1/2}}{q_2} + O(1).$$

Expanding out (5.3), we obtain

$$\mathcal{S}_6 = -\frac{X}{q_1 q_2} + X^{1/2}\left(\frac{1}{q_1}\psi\left(\frac{X^{1/2}-\beta_2}{q_2}\right) + \frac{1}{q_2}\psi\left(\frac{X^{1/2}-\alpha_1}{q_1}\right) - \frac{1}{q_2}\psi\left(\frac{-\alpha_1}{q_1}\right)\right) + O(1).$$

So,

$$\mathcal{S}_3 + \mathcal{S}_6 = -\frac{X}{q_1 q_2} + X^{1/2}\left(\frac{1}{q_1}\psi\left(\frac{X^{1/2}-\beta_2}{q_2}\right) + \frac{1}{q_2}\psi\left(\frac{X^{1/2}-\alpha_1}{q_1}\right)\right) + O(1). \tag{5.9}$$

We will see that the X term here offers no contribution within $\sum_{\alpha,\beta}$, and that the $X^{1/2}$ term here will cancel out with the $X^{1/2}$ term from $\mathcal{S}_1 + \mathcal{S}_4$.

5.4.2 Evaluating $\mathcal{S}_1$ and $\mathcal{S}_4$.

Using our indicator functions we have,

$$\mathcal{S}_1 = \frac{X}{q_1\varphi(q_2)}\sum_{\chi_j(q_2)}\overline{\chi_j}(\beta_2)\sum_{d\le X^{1/2}}\frac{\chi_j(d)}{d},$$

and,

$$\mathcal{S}_4 = \frac{X}{q_2\varphi(q_1)}\sum_{\chi_i(q_1)}\overline{\chi_i}(\alpha_1)\sum_{m\le X^{1/2}}\frac{\chi_i(m)}{m}.$$

We would like to get an expression involving Dirichlet L-functions as we did in the case where $4 \nmid q$. This is only problematic now in the case where χ is a trivial Dirichlet character, as $L(1,\chi_0)$ doesn't converge. To deal with this problem, we will separate the term corresponding to the trivial character from both $\mathcal{S}_1$ and $\mathcal{S}_4$ and deal with this term separately. So we get,

$$\mathcal{S}_1 = \frac{X}{q_1\varphi(q_2)}\sum_{\substack{\chi_j(q_2)\\ j\neq 0}}\overline{\chi_j}(\beta_2)\sum_{d\le X^{1/2}}\frac{\chi_j(d)}{d} + \frac{X}{q_1\varphi(q_2)}\sum_{d\le X^{1/2}}\frac{1}{d},$$

and,

$$\mathcal{S}_4 = \frac{X}{q_2\varphi(q_1)}\sum_{\substack{\chi_i(q_1)\\ i\neq 0}}\overline{\chi_i}(\alpha_1)\sum_{m\le X^{1/2}}\frac{\chi_i(m)}{m} + \frac{X}{q_2\varphi(q_1)}\sum_{m\le X^{1/2}}\frac{1}{m}.$$

We will see that the term corresponding to the trivial character for both $\mathcal{S}_1$ and $\mathcal{S}_4$ in fact offers no contribution to $S_{q,a}(x)$.

So now we use Theorem 31 to get,

$$\begin{aligned}\mathcal{S}_1 =&\frac{1}{q_1\varphi(q_2)}\sum_{\substack{\chi_j(q_2)\\ j\neq 0}}\overline{\chi_j}(\beta_2)\left(L(1,\chi_j)X - X^{1/2}\sideset{}{^*}\sum_{\gamma(q_2)}\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{q_2}\right) + X\sideset{}{^*}\sum_{\gamma(q_2)}\chi_j(\gamma)\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2}\right)\\ &+\frac{X}{q_1\varphi(q_2)}\sum_{d\le X^{1/2}}\frac{1}{d},\end{aligned} \tag{5.10}$$

and similarly,

$$\begin{aligned}\mathcal{S}_4 =&\frac{1}{q_2\varphi(q_1)}\sum_{\substack{\chi_i(q_1)\\ i\neq 0}}\overline{\chi_i}(\alpha_1)\left(L(1,\chi_i)X - X^{1/2}\sideset{}{^*}\sum_{\gamma(q_1)}\chi_i(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{q_1}\right) + X\sideset{}{^*}\sum_{\gamma(q_1)}\chi_i(\gamma)\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_1}\right)\frac{dt}{t^2}\right)\\ &+\frac{X}{q_2\varphi(q_1)}\sum_{m\le X^{1/2}}\frac{1}{m}.\end{aligned} \tag{5.11}$$

Corollary 2. *If $q\in\mathbb{N}$ and $(n,q)=1$ then,*

$$\mathbf{1}_{\{k\equiv n(q)\}} - \mathbf{1}_{\{(k,q)=1\}} = \frac{1}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi\ \textit{non-trivial}}}\chi(k)\overline{\chi}(n).$$

Proof. Follows from (4.7). □

We handle the terms from $\mathcal{S}_1$ first. Since $(\beta_2,q_2)=1$ and $(\alpha_1,q_1)=1$ by definition, it follows from Corollary 2 that

$$\begin{aligned}-\frac{1}{\varphi(q_2)}&\sum_{\substack{\chi_j(q_2)\\ j\neq 0}}\overline{\chi_j}(\beta_2)X^{1/2}\sideset{}{^*}\sum_{\gamma(q_2)}\chi_j(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{q_2}\right)\\ &= -X^{1/2}\psi\left(\frac{X^{1/2}-\beta_2}{q_2}\right) + X^{1/2}\sideset{}{^*}\sum_{\gamma(q_2)}\psi\left(\frac{X^{1/2}-\gamma}{q_2}\right).\end{aligned} \tag{5.12}$$

We look at another term from $\mathcal{S}_1$ now. We add and subtract a term in order to include the trivial character back into the character sum, and then use the indicator function for $\beta_2\equiv\gamma(q_2)$ to obtain

$$\begin{aligned}&\frac{1}{\varphi(q_2)}\sum_{\substack{\chi_j(q_2)\\ j\neq 0}}\overline{\chi_j}(\beta_2)\sideset{}{^*}\sum_{\gamma(q_2)}\chi_j(\gamma)\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2}\\ &\quad=\frac{1}{\varphi(q_2)}\sum_{\chi_j(q_2)}\overline{\chi_j}(\beta_2)\sideset{}{^*}\sum_{\gamma(q_2)}\chi_j(\gamma)\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2} - \frac{1}{\varphi(q_2)}\sideset{}{^*}\sum_{\gamma(q_2)}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2}\\ &\quad=\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\beta_2}{q_2}\right)\frac{dt}{t^2} - \frac{1}{\varphi(q_2)}\sideset{}{^*}\sum_{\gamma(q_2)}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2}.\end{aligned} \tag{5.13}$$

Now we handle the terms from $\mathcal{S}_4$. Similarly to how we got (5.12), we obtain

$$
\begin{aligned}
-\frac{1}{\varphi(q_1)} &\sum_{\substack{\chi_i(q_1)\\ i\neq 0}} \overline{\chi_i}(\alpha_1) X^{1/2} \sideset{}{^*}\sum_{\gamma(q_1)} \chi_i(\gamma)\psi\left(\frac{X^{1/2}-\gamma}{q_1}\right) \\
&= -X^{1/2}\psi\left(\frac{X^{1/2}-\alpha_1}{q_1}\right) + X^{1/2}\sideset{}{^*}\sum_{\gamma(q_1)}\psi\left(\frac{X^{1/2}-\gamma}{q_1}\right), \qquad (5.14)
\end{aligned}
$$

and similarly to how we got (5.13), we obtain

$$
\begin{aligned}
\frac{1}{\varphi(q_1)} &\sum_{\substack{\chi_i(q_1)\\ i\neq 0}} \overline{\chi_i}(\alpha_1) \sideset{}{^*}\sum_{\gamma(q_1)} \chi_i(\gamma) \int_{X^{1/2}}^{\infty} \psi\left(\frac{t-\gamma}{q_1}\right)\frac{dt}{t^2} \\
&= \int_{X^{1/2}}^{\infty} \psi\left(\frac{t-\alpha_1}{q_1}\right)\frac{dt}{t^2} - \frac{1}{\varphi(q_1)}\sideset{}{^*}\sum_{\gamma(q_1)}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_1}\right)\frac{dt}{t^2}. \qquad (5.15)
\end{aligned}
$$

Hence by combining (5.9), (5.10), (5.11), (5.12), (5.13), (5.14), (5.15), and recalling (4.6), we obtain

$$
\mathcal{S}_1 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_6 = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \mathcal{S}^{(3)} + O(1), \qquad (5.16)
$$

where

$$
\mathcal{S}^{(0)} = \frac{x}{qk_2\varphi(q_2)} \sum_{\substack{\chi_j(q_2)\\ j\neq 0}} \overline{\chi_j}(\beta_2) L(1,\chi_j),
$$

$$
\mathcal{S}^{(1)} = \frac{x}{qk_1\varphi(q_1)} \sum_{\substack{\chi_i(q_1)\\ i\neq 0}} \overline{\chi_i}(\alpha_1) L(1,\chi_i),
$$

$$
\mathcal{S}^{(2)} = \frac{X}{q_1}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\beta_2}{q_2}\right)\frac{dt}{t^2} + \frac{X}{q_2}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\alpha_1}{q_1}\right)\frac{dt}{t^2},
$$

$$
\begin{aligned}
\mathcal{S}^{(3)} = &-\frac{X}{q_1q_2} + \frac{X}{q_1\varphi(q_2)}\sum_{d\le X^{1/2}}\frac{1}{d} + \frac{X}{q_2\varphi(q_1)}\sum_{m\le X^{1/2}}\frac{1}{m} + \frac{1}{q_1}X^{1/2}\sideset{}{^*}\sum_{\gamma(q_2)}\psi\left(\frac{X^{1/2}-\gamma}{q_2}\right) \\
&+ \frac{1}{q_2}X^{1/2}\sideset{}{^*}\sum_{\gamma(q_1)}\psi\left(\frac{X^{1/2}-\gamma}{q_1}\right) - \frac{X}{q_1\varphi(q_2)}\sideset{}{^*}\sum_{\gamma(q_2)}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_2}\right)\frac{dt}{t^2} - \frac{X}{q_2\varphi(q_1)}\sideset{}{^*}\sum_{\gamma(q_1)}\int_{X^{1/2}}^{\infty}\psi\left(\frac{t-\gamma}{q_1}\right)\frac{dt}{t^2},
\end{aligned}
$$

where $\mathcal{S}^{(3)}$ has been formed to contain all terms from $\mathcal{S}_1+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_6$ that have no dependence on α_1 or β_2. The sums have been organized so that $\mathcal{S}^{(0)}$ and $\mathcal{S}^{(1)}$ give the main terms of $S_{q,a}(x)$, $\mathcal{S}^{(2)}$ gives an error term, and $\mathcal{S}^{(3)}$ gives zero contribution to $S_{q,a}(x)$. We have re-expressed $\mathcal{S}_1+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_6$ so recalling (5.6), we now let

$$
\mathcal{N}^{(i)} = \sum_{\alpha,\beta}\mathcal{S}^{(i)},
$$

where $\sum_{\alpha,\beta}$ is as defined in (5.4), in order to relate our new $\mathcal{S}^{(i)}$ values to $S_{q,a}(x)$. So by (5.16), we have

$$\mathcal{N}_1 + \mathcal{N}_3 + \mathcal{N}_4 + \mathcal{N}_6 = \mathcal{N}^{(0)} + \mathcal{N}^{(1)} + \mathcal{N}^{(2)} + \mathcal{N}^{(3)} + O\left(\sum_{\alpha,\beta} 1\right). \tag{5.17}$$

The sum $\mathcal{S}^{(3)}$ looks quite complicated but we will now prove the following theorem which will very easily show us that $\mathcal{N}^{(3)}$ offers no contribution to $S_{q,a}(x)$:

Theorem 38. *If $4 \mid q$, then for any two variable function f which is independent of α_1, β_2, we have*

$$\sum_{\alpha,\beta} f(\alpha,\beta) = 0,$$

where $\sum_{\alpha,\beta}$ is as defined in (5.4).

Proof. Since $4 \mid q$ and f is independent of α_1 and β_2, we have that

$$\chi_4(\beta)\mathbf{1}_{\{\alpha\beta\equiv a(q)\}} f(\alpha,\beta) = -\chi_4(q-\beta)\mathbf{1}_{\{(q-\alpha)(q-\beta)\equiv a(q)\}} f(q-\alpha, q-\beta).$$

This equation shows that any term corresponding to an (α,β) pair which we iterate over, is necessarily the negative of the term corresponding to the $(q-\alpha, q-\beta)$ pair. If (α,β) is distinct from $(q-\alpha, q-\beta)$, we get from the above equation that the sum of their corresponding terms is zero. If (α,β) is not distinct from $(q-\alpha, q-\beta)$ then β is even (because $4 \mid q$), and so the above equation shows that the term corresponding to (α,β) is zero. Hence the sum over all (α,β) pairs is zero. ☐

We chose $S^{(3)}$ to be completely independent of α_1 and β_2 so by the above theorem, we have that

$$\mathcal{N}^{(3)} = \sum_{\alpha,\beta} \mathcal{S}^{(3)} = 0, \tag{5.18}$$

thus $\mathcal{S}^{(3)}$ offers no contribution to $S_{q,a}(x)$ due to (5.5), (5.6) and (5.17).

Next we look at $\mathcal{S}^{(2)}$. To bound this term, we use the triangle inequality and then Theorem 30 to obtain

$$\mathcal{S}^{(2)} \ll \frac{q_2}{q_1} + \frac{q_1}{q_2}.$$

Then

$$\begin{aligned}\mathcal{N}^{(2)} &\ll \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right) \\ &\ll q \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}}.\end{aligned}$$

Now we use Theorem 28 to get

$$\mathcal{N}^{(2)} \ll q^2 \tau((a,q)). \tag{5.19}$$

Lastly for this subsection, we also bound the contribution from the $O\left(\sum_{\alpha,\beta} 1\right)$ term of (5.17). We simply apply Theorem 28 to get

$$\sum_{\alpha,\beta} \ll \sum_{\alpha(q)} \sum_{\beta(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \ll q\tau((a,q)). \tag{5.20}$$

5.4.3 Evaluating $\mathcal{N}^{(0)}, \mathcal{N}^{(1)}$.

Now we look at $\mathcal{N}^{(0)}$. We apply Theorem 32 multiple times to pull the infinite summation outside which gives us

$$\mathcal{N}^{(0)} = \frac{x}{q} \sum_{D=1}^{\infty} \sum_{\alpha,\beta} \frac{1}{k_2 \varphi(q_2)} \sum_{\substack{\chi_j(q_2) \\ j \neq 0}} \overline{\chi_j}(\beta_2) \frac{\chi_j(D)}{D}.$$

Now we use Corollary 2, (4.31), and substitute the definition of $\sum_{\alpha,\beta}$, to get

$$\begin{aligned}
\mathcal{N}^{(0)} &= \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta \equiv a(q)\}} \frac{\mathbf{1}_{\{D \equiv \beta_2(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}}{k_2} \\
&= \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{\substack{\beta(q) \\ k_2 | a}} \chi_4(\beta) (\mathbf{1}_{\{D \equiv \beta_2(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}) \\
&= \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{d|q} \mathbf{1}_{\{d|a\}} \sum_{\substack{\beta(q) \\ k_2 = d}} \chi_4(\beta) (\mathbf{1}_{\{D \equiv \beta_2(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}).
\end{aligned}$$

We simplify this innermost sum to get

$$\sum_{\substack{\beta(q) \\ k_2 = d}} \chi_4(\beta) (\mathbf{1}_{\{D \equiv \beta_2(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}) = \chi_4(d) \sum_{K(q_2)}^{*} \chi_4(K) (\mathbf{1}_{\{D \equiv K(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}),$$

hence

$$\mathcal{N}^{(0)} = \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{d|q} \chi_4(d) \mathbf{1}_{\{d|a\}} \sum_{K(q_2)}^{*} \chi_4(K) (\mathbf{1}_{\{D \equiv K(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}).$$

Now let

$$q = 2^r Q, \tag{5.21}$$

where Q is odd (note that $r \geq 2$). Then,

$$\sum_{d|q} = \sum_{d_1 | 2^r} \sum_{d_2 | Q}.$$

So

$$\mathcal{N}^{(0)} = \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{d_1 | 2^r} \sum_{d_2 | Q} \chi_4(d_1 d_2) \mathbf{1}_{\{d_1 d_2 | a\}} \sum_{K(q_2)}^{*} \chi_4(K) (\mathbf{1}_{\{D \equiv K(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}).$$

If d is even, then $\chi_4(d_1 d_2) = 0$, so we only get a contribution if $d_1 = 1$, thus

$$\mathcal{N}^{(0)} = \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{d_2 | Q} \chi_4(d_2) \mathbf{1}_{\{d_2 | a\}} \sum_{K(q_2)}^{*} \chi_4(K) (\mathbf{1}_{\{D \equiv K(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}).$$

Note that since $q_2 = q/d_2$, we have $4 \mid q_2$. So

$$\sideset{}{^*}\sum_{K(q_2)} \chi_4(K)(\mathbf{1}_{\{D\equiv K(q_2)\}} - \mathbf{1}_{\{(D,q_2)=1\}}) = \chi_4(D)\mathbf{1}_{\{(D,q_2)=1\}} - \mathbf{1}_{\{(D,q_2)=1\}} \sideset{}{^*}\sum_{K(q_2)} \chi_4(K).$$

Observe that

$$\sideset{}{^*}\sum_{K(q_2)} \chi_4(K) = 0,$$

because $4 \mid q_2$ and so $\chi_4(K) = -\chi_4(q_2 - K)$, hence

$$\begin{aligned}\mathcal{N}^{(0)} &= \frac{x}{q} \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \sum_{d_2 \mid Q} \chi_4(d_2)\mathbf{1}_{\{d_2 \mid a\}}\mathbf{1}_{\{(D,q_2)=1\}} \\ &= \frac{x}{q}\mathcal{Z}_a(Q),\end{aligned} \tag{5.22}$$

where we have used (4.33).

Now we focus on $\mathcal{N}^{(1)}$. We apply Theorem 32 multiple times to get,

$$\mathcal{N}^{(1)} = \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{\alpha,\beta} \frac{1}{k_1\varphi(q_1)} \sum_{\substack{\chi_i(q_1) \\ i\neq 0}} \overline{\chi_i}(\alpha_1)\chi_i(D).$$

We use Corollary 2 and write out $\sum_{\alpha,\beta}$ to get,

$$\mathcal{N}^{(1)} = \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{\beta(q)} \chi_4(\beta) \sum_{\alpha(q)} \mathbf{1}_{\{\alpha\beta\equiv a(q)\}} \frac{\mathbf{1}_{\{D\equiv\alpha_1(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}}{k_1}.$$

We prove the following theorem that will allow us to proceed with manipulating $\mathcal{S}^{(1)}$:

Theorem 39. *We have,*

$$\sum_{\beta(q)} \chi_4(\beta)\mathbf{1}_{\{\alpha\beta\equiv a(q)\}} = \mathbf{1}_{\{k_1 \mid a\}}\mathbf{1}_{\{4 \mid q_1\}} k_1 \chi_4\left(\frac{a\overline{\alpha_1}}{k_1}\right).$$

Proof. We have,

$$\sum_{\beta(q)} \chi_4(\beta)\mathbf{1}_{\{\alpha\beta\equiv a(q)\}} = \mathbf{1}_{\{k_1 \mid a\}} \sum_{\beta(q)} \chi_4(\beta)\mathbf{1}_{\{\beta\equiv a\overline{\alpha_1}/k_1(q_1)\}}.$$

Now we deal with the sum on the RHS by cases.

If $v_2(q_1) \geq 2$, then χ_4 is the same for each β satisfying the condition $\beta \equiv a\overline{\alpha_1}/k_1(q_1)$, so in this case we get

$$\sum_{\beta(q)} \chi_4(\beta)\mathbf{1}_{\{\beta\equiv a\overline{\alpha_1}/k_1(q_1)\}} = k_1\chi_4\left(\frac{a\overline{\alpha_1}}{k_1}\right).$$

If $v_2(q_1) = 1$, then $\chi_4(n) + \chi_4(q_1+n) = 0$ for any integer n. There are k_1 terms that are summed over, and since $v_2(q_1) = 1$, we have $v_2(k_1) \geq 1$ (because $4 \mid q$). Hence, the sum is zero in this case.

If $v_2(q_1) = 0$, then $\chi_4(n) + \chi_4(q_1+n) + \chi_4(2q_1+n) + \chi_4(3q_1+n) = 0$ for any integer n. There are k_1 terms that are summed over, and since $v_2(q_1) = 0$, we have $v_2(k_1) \geq 2$ (because $4 \mid q$). Hence the sum is zero in this case. $\square$

Using this theorem, we get

$$\begin{aligned}
\mathcal{N}^{(1)} &= \frac{x}{q}\sum_{D=1}^{\infty}\frac{1}{D}\sum_{\alpha(q)}\left(\mathbf{1}_{\{D\equiv\alpha_1(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}\right)\mathbf{1}_{\{k_1|a\}}\mathbf{1}_{\{4|q_1\}}\chi_4\left(\frac{a\overline{\alpha_1}}{k_1}\right) \\
&= \frac{x}{q}\sum_{D=1}^{\infty}\frac{1}{D}\sum_{d|q}\mathbf{1}_{\{d|a\}}\sum_{\substack{\alpha(q)\\ k_1=d}}\left(\mathbf{1}_{\{D\equiv\alpha_1(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}\right)\mathbf{1}_{\{4|q_1\}}\chi_4\left(\frac{a\overline{\alpha_1}}{d}\right) \\
&= \frac{x}{q}\sum_{D=1}^{\infty}\frac{1}{D}\sum_{d|q}\mathbf{1}_{\{d|a\}}\mathbf{1}_{\{4|q_1\}}\sideset{}{^*}\sum_{K(q_1)}\left(\mathbf{1}_{\{D\equiv K(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}\right)\chi_4\left(\frac{a\overline{K}}{d}\right). \qquad (5.23)
\end{aligned}$$

where $\overline{K}$ is the inverse of K modulo q_1, and where we have removed α from the sum and now $q_1 = q/d$. We recall (5.21) and therefore have

$$\sum_{d|q} = \sum_{d_1|2^r}\sum_{d_2|Q}.$$

Now we look at the expression

$$\mathbf{1}_{\{4|q_1\}}\chi_4\left(\frac{a\overline{K}}{d}\right).$$

The condition $4 \mid q_1$ means that q_1 is even and so any K we iterate over in (5.23) is odd. For $\chi_4\left(\frac{a\overline{K}}{d}\right)$ to be non-zero, we must have that $v_2(a\overline{K}) = v_2(d) = v_2(d_1)$. Hence we only get a non-zero contribution when $d_1 = 2^{v_2(a)}$. Now let

$$a = 2^s A,$$

where A is odd. Recalling (5.21), the condition $4 \mid q_1$ can therefore be restated as $r - s \geq 2$. So

$$\mathcal{N}^{(1)} = \mathbf{1}_{\{r-s\geq 2\}}\frac{x}{q}\sum_{D=1}^{\infty}\frac{1}{D}\sum_{d|Q}\mathbf{1}_{\{2^s d|a\}}\sideset{}{^*}\sum_{K(q_1)}\left(\mathbf{1}_{\{D\equiv K(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}\right)\chi_4\left(\frac{a\overline{K}}{2^s d}\right),$$

where now $q_1 = q/(2^s d)$. Then

$$\begin{aligned}
\sideset{}{^*}\sum_{K(q_1)}\left(\mathbf{1}_{\{D\equiv K(q_1)\}} - \mathbf{1}_{\{(D,q_1)=1\}}\right)\chi_4\left(\frac{a\overline{K}}{2^s d}\right) &= \mathbf{1}_{\{(D,q_1)=1\}}\chi_4\left(\frac{a\overline{D}}{2^s d}\right) - \mathbf{1}_{\{(D,q_1)=1\}}\chi_4\left(\frac{a}{2^s d}\right)\sideset{}{^*}\sum_{K(q_1)}\chi_4(\overline{K}) \\
&= \mathbf{1}_{\{(D,q_1)=1\}}\chi_4\left(\frac{a\overline{D}}{2^s d}\right),
\end{aligned}$$

because $4 \mid q_1$.

Observe that since $4 \mid q_1$, we have $\chi_4(\overline{D}) = \chi_4(D)$, and so

$$\mathcal{N}^{(1)} = \mathbf{1}_{\{r-s\geq 2\}} \frac{x}{q} \sum_{D=1}^{\infty} \frac{1}{D} \sum_{d|Q} \mathbf{1}_{\{2^s d|a\}} \mathbf{1}_{\{(D,q_1)=1\}} \chi_4\left(\frac{a}{2^s d}\right) \chi_4(D).$$

We now substitute in the definition of q_1 and use that $\mathbf{1}_{\{2^s d|a\}} = \mathbf{1}_{\{d|a\}}$, because $2^s \mid a$ by definition of s, and $(d, 2^s) = 1$. We also note that for d dividing Q, we have $\chi_4(D)\mathbf{1}_{\{(D,Q/d)=1\}} = \chi_4(D)\mathbf{1}_{\{(D,q/(2^s d))=1\}}$, so we get

$$\begin{aligned}
\mathcal{N}^{(1)} &= \mathbf{1}_{\{r-s\geq 2\}} \frac{x\chi_4\left(\frac{a}{2^s}\right)}{q} \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \sum_{d|Q} \mathbf{1}_{\{d|a\}} \mathbf{1}_{\{(D,q/(2^s d))=1\}} \frac{1}{\chi_4(d)} \\
&= \mathbf{1}_{\{r-s\geq 2\}} \frac{x\chi_4\left(A\right)}{q} \sum_{d|Q} \mathbf{1}_{\{d|a\}} \chi_4(d) \sum_{D=1}^{\infty} \frac{\chi_4(D)}{D} \mathbf{1}_{\{(D,Q/d)=1\}} \\
&= \mathbf{1}_{\{r-s\geq 2\}} \frac{x\chi_4(A)}{q} \mathcal{Z}_a(Q), \qquad (5.24)
\end{aligned}$$

where we have used (4.33).

5.4.4 Collecting sums.

Substituting (5.18), (5.19), (5.20), (5.22), (5.24), into (5.17), we get

$$\begin{aligned}
\mathcal{N}_1 + \mathcal{N}_3 + \mathcal{N}_4 + \mathcal{N}_6 &= \frac{x}{q} \mathcal{Z}_a(Q) + \mathbf{1}_{\{r-s\geq 2\}} \frac{\chi_4(A)x}{q} \mathcal{Z}_a(Q) + O\left(q^2 \tau((a,q))\right) \\
&= \frac{\pi x}{4qQ} \left((1 + \mathbf{1}_{\{r-s\geq 2\}} \chi_4(A))\eta_a(Q)\right) + O\left(q^2 \tau((a,q))\right),
\end{aligned}$$

where we have used Theorem 33. Then by multiplicativity, we have $\eta_a(2^r Q) = \eta_a(2^r)\eta_a(Q)$. We now use Theorem 14 to get

$$\begin{aligned}
\mathcal{N}_1 + \mathcal{N}_3 + \mathcal{N}_4 + \mathcal{N}_6 &= \frac{\pi x}{4qQ} \left(\frac{\eta_a(q)}{2^r}\right) + O\left(q^2 \tau((a,q))\right) \\
&= \frac{\pi x}{4q^2} \left(\eta_a(q)\right) + O\left(q^2 \tau((a,q))\right). \qquad (5.25)
\end{aligned}$$

It now remains to sum together the many sums we have bounded or explicitly calculated to acquire our bound for $R_{q,a}(x)$. We substitute (5.7), (5.8), and (5.25) into (5.5) to get

$$S_{q,a}(x) = \pi \frac{\eta_a(q)}{q^2} x + O\left(\left(x^{27/82} q^{115/82} + q^2\right) \tau((a,q)) \log x\right),$$

hence

$$R_{q,a}(x) \ll \left(x^{27/82} q^{115/82} + q^2\right) \tau((a,q)) \log x,$$

which proves Theorem 9 in the case $4 \mid q$.

5.5 Conclusion.

We studied the Gauss circle problem for arithmetic progressions and found an improvement to the bound produced by Tolev in the x aspect at the expense of the new bound being worse in the q aspect.

In Chapter 4 we in fact showed

$$R_{q,a}(x) = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}^{(3)} + O\left(q\tau(a,q)\right) \qquad \text{for} \qquad 4 \nmid q,$$

and in Chapter 5 we in fact showed

$$R_{q,a}(x) = \mathcal{N}_2 + \mathcal{N}_5 + \mathcal{N}^{(2)} + O\left(q\tau(a,q)\right) \qquad \text{for} \qquad 4 \mid q,$$

hence an improvement to the bound for $R_{q,a}(x)$ can be made if $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}^{(3)}, \mathcal{N}_2, \mathcal{N}_5, \mathcal{N}^{(2)}$ can be bounded better. The sums $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_2$ and $\mathcal{N}_5$ are all similar which leaves hope that an improvement to the bound on any one of these sums can be extended to the others. The sums $\mathcal{M}^{(3)}$ and $\mathcal{N}^{(2)}$ also look similar. These two sums could potentially be decomposed into a sum of terms that provide cancellation with one of the other aforementioned sums but I have not managed to find any such cancellation.

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