Module - 6

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- Random variable :- A random variable is defined as a variable which takes numerical values determined by the outcomes of a random experiment.
There are 2 types of Random variable —

1 Discrete Random Variable — A RV which can take only some isolated values is known as a discrete RV.

eg :- No. of childrens in a family, no of misprints in a page, No. of tails obtained by 3 tosses.

2 Continuous RV — A RV which takes any value in some interval of real nos. It takes infinite no. of possible values.
Eg:- height, weight, etc.

- Probability distribution of a Random Variable:

The probability describes how the total probability is distributed over the possible values of a random variable. It tells us what the possible values of a RV. X are and how the probabilities assigned to those values.

eg:- obtain the probability distribution of a RV, X, the no. of heads obtained in 3 tosses of a fair coin.

$X = 0, 1, 2, 3$

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

So, the probability distribution is

x	0	1	2	3
$P(X=x)$	1/8	3/8	3/8	1/8

Q Consider the experiment of throwing two fair dice starting from the S of this experiment. Obtain the probability distribution of a RV.

"No. of sixes observed", ~~daft~~ defined on this S.

Ans. S = (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (6,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

PMF → Probability Mass Function
PDF → Probability Density Functions.

Let the RV be X ;

$X = 0, 1, 2$

x	0	1	2
$P(X=x)$	25/36	10/36	1/36

The probability distribution of a RV. of a ~~that~~ discrete RV is known as a Discrete Probability Distribution / PMF

Whereas the probability distribution of a continuous RV is known as a Continuous Probability Distribution / PDF

• Probability Mass Function (PMF)

For a discrete random variable, X, there exists a function $f(x)$ such that $f(x) = P(X=x)$, x being the all possible values of X. A function of this type satisfying the conditions :-

i) $f(x) \geq 0$, for all x

ii) $\sum_x f(x) = 1$

is called the PMF of X

Q For what value of 'a', $f(x)$ will be the PMF of a discrete RV, X whose Probability distribution is given below.

x	0	1	2	3	4
$f(x)$	a	$2a$	$7a^2$	$2a$	a

Write down the entire probability distribution of X.

i) $P(X<2)$

ii) $P(X \geq 2)$

iii) $P(X \geq 1)$

iv) $P(X<3 \mid X \geq 1)$

Ans $\Sigma f(x) = 1$

$a+2a+7a^2+2a+a = 1$

$\Rightarrow 6a+7a^2 = 1$

$\Rightarrow 7a^2+6a-1 = 0$

$\Rightarrow 7a^2+(7-1)a-1 = 0$

$\Rightarrow 7a^2+7a-a-1 = 0$

$\Rightarrow 7a(a+1) - 1(a+1) = 0$

$\Rightarrow (a+1)(7a-1) = 0$

So, $a = -1$ & $\frac{1}{7}$

-1 → neglected

So,

x	0	1	2	3	4
$f(x)$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$

$P(X<2) = \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$

$P(X \geq 2) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}$

$P(x<3 / x \geq 1)$

$$= \frac{P(x<3 \cap x \geq 1)}{P(x \geq 1)}$$

$$= \frac{\frac{2}{7} + \frac{1}{7}}{\frac{2}{7} + \frac{1}{7} + \frac{2}{7} + \frac{1}{7}} = \frac{3}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{6}\,2} = \frac{1}{2}$$

Q For what value of c the value of $f(x)$ will be a PMF. if

$$f(x) = \begin{cases} cx & ; \text{ for } x = 1, 2, 3, 4, 5 \\ 0 & ; \text{ otherwise} \end{cases}$$

Obtain i) $P(X < 3)$

ii) $P(X \leq 3)$

iii) $P(X \leq 5)$

iv) $P(2 < X < 4)$

v) $P(X > 2 \mid X \leq 4)$

Ans $\sum_{x=1}^{x=5} f(x) = 1$

$\sum_{x=1}^{x=5} cx = 1$

$c[1 + 2 + 3 + 4 + 5] = 1$

$c \cdot 15 = 1$

$c = \frac{1}{15}$.

$x =$	1	2	3	4	5
$f(x) =$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$P(X<3) = \frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

$P(X \leq 3) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} = \frac{\cancel{6}^{2}}{\cancel{15}_{5}} = \frac{2}{5}$

$P(X \leq 5) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = \frac{15}{15} = 1$

$P(2<X<4) = \frac{\cancel{3}^{1}}{\cancel{15}_{5}} = \frac{1}{5}$

$P(X>2 \mid x \leq 4) = \frac{P(X>2 \cap X \leq 4)}{P(x \leq 4)}$

$= \frac{\frac{3}{15} + \frac{4}{15}}{\frac{10}{15}}$

$= \frac{7}{\cancel{15}} \times \frac{\cancel{15}}{10}$

$= \frac{7}{10}$

- **Mean & Varience of a Discrete random variable :-**

$X \rightarrow DRV$

$E(X)$

$V(X) = E(x^2) - \{E(X)\}^2$

$E(X) = \sum x \cdot P(X=x)$

$E(x^2) = \sum x^2 \, P(X=x)$

Mean or expectation of a DRV, X is defined as $E(X) = \sum_x x \cdot P(X=x)$

$$= \sum_x x \, f(x)$$

The variance of a discrete random variable

$$V = E(X^2) - \{E(X)\}^2$$

where $E(X^2) = \sum x^2 \; P(X=x)$

$$= \sum x^2 f(x)$$

Q Obtain the mean & variance of RV X, the no. of heads obtained in tossing a coin twice.

$x^2 \cdot f(x)$	0	$\frac{1}{2}$	1
x	0	1	2
$P(X=x)$ ⇓ $f(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$
$x \cdot f(x)$	0	$\frac{1}{2}$	$\frac{1}{2}$

$E(x) = 1$

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Probability Density Function :- (P.d.f)

- For a continuous random variable X, there may exist a function $f(x)$ such that the following conditions are satisfied:

i) $f(x) \geq 0 \quad \forall x$

ii) $\int_x f(x)\, dx = 1$

Q ~~Let~~ Let the random variable X.

$$f(x) = \begin{cases} cx(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

i) Find 'c'

ii) Find probability $(x > \frac{1}{2})$

iii) Find probability $(\frac{1}{2} \leq x \leq 1)$.

Ans

$$\cancel{\sum_{x=0}^{x=2} c/x = 1}$$

$$\cancel{c[0+1+2] = 1}$$

i) $\int_0^2 cx(2-x)\, dx = 1$

$\Rightarrow c\left[\int_0^2 2x\, dx - \int_0^2 x^2\, dx\right] = 1$

$$\Rightarrow C\left[2\int_0^2 x\,dx - \int_0^2 x^2\,dx\right] = 1$$

$$\Rightarrow C\left[2\left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2\right] = 1$$

$$\Rightarrow C\left[2\times\left(\frac{4}{2}-0\right) - \left(\frac{8}{3}-0\right)\right] = 1$$

$$\Rightarrow C\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow C\left[\frac{12-8}{3}\right] = 1$$

$$\Rightarrow C\left[\frac{4}{3}\right] = 1$$

$$\Rightarrow C = \frac{3}{4}$$

So,

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

ii) $P\left(X > \frac{1}{2}\right) = \int_{1/2}^{2} \frac{3}{4} x (2-x)\, dx$

$= \frac{3}{4}\left[\int_{1/2}^{2} 2x\, dx - \int_{1/2}^{2} x^2\, dx\right]$

$= \frac{3}{4}\left[2\left(\frac{x^2}{2}\right)_{1/2}^{2} - \left(\frac{x^3}{3}\right)_{1/2}^{2}\right]$

$\cancel{= \frac{3}{4}\left[2\left(\frac{4}{2}\right)\cdot\left(\frac{2}{4}\right) - \frac{8}{3} - \frac{3}{8}\right]}$

$= \frac{3}{4}\left[2\left\{\left(\frac{4}{2}\right) - \left(\frac{1}{8}\right)\right\} - \left\{\left(\frac{8}{3}\right) - \frac{1}{24}\right\}\right]$

$= \frac{3}{4}\left[2\left(\frac{16-1}{8}\right) - \left(\frac{64-1}{24}\right)\right]$

$$= \frac{3}{4}\left[\cancel{8}\left(\frac{15}{\cancel{4}} - \frac{63}{24}\right)\right]$$

$$= \frac{3}{4}\left[\cancel{8}\ \frac{90-63}{\cancel{8}24}\right]$$

$$= \frac{\cancel{3}}{4} \times \frac{27}{\cancel{24}8}$$

$$= \frac{27}{32}$$

iii) $P\left(\frac{1}{2} \leq x \leq 1\right)$

$$= \int_{1/2}^{1} \frac{3}{4} x (2-x)\, dx$$

$$= \frac{3}{4}\left[\int_{1/2}^{1} 2x\, dx - \int_{1/2}^{1} x^2\, dx\right]$$

$$= \frac{3}{4}\left[\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_{1/2}^{1} - \left(\frac{x^3}{3}\right)_{1/2}^{1}\right]$$

$$= \frac{3}{4}\left[\left(1 - \frac{1}{4}\right) - \left(\frac{1}{3} - \frac{1}{24}\right)\right]$$

$$= \frac{3}{4}\left[\left(\frac{4-1}{4}\right) - \left(\frac{8-1}{24}\right)\right]$$

$$= \frac{3}{4}\left[\frac{3}{4} - \frac{7}{24}\right]$$

$$= \frac{3}{4}\left[\frac{18-7}{24}\right]$$

$$= \frac{\cancel{3}}{4}\left[\frac{11}{\cancel{24}_{8}}\right]$$

$$= \frac{11}{32}$$

Q2 $f(x) = \begin{cases} Kx(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

i) Find 'K'

ii) Find $E(x)$

Ans i) $\int_0^1 Kx(1-x)\,dx = 1$

$\Rightarrow K\left[\int_0^1 x(1-x)\,dx\right] = 1$

$\Rightarrow K\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right] = 1$

$\Rightarrow K\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right] = 1$

$\Rightarrow K\left[\left(\frac{1}{2} - 0\right) - \left(\frac{1}{3} - 0\right)\right] = 1$

$\Rightarrow K\left[\left(\frac{1}{2} - \frac{1}{3}\right)\right] = 1$

$\Rightarrow K\left(\frac{3-2}{6}\right) = 1$

$\Rightarrow K\left(\frac{1}{6}\right) = 1$

$\Rightarrow \underline{K = 6}$

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So, $f(x) = \begin{cases} 6x(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

ii) $E(x)$

$= \int_0^1 x \, f(x) \, dx$

$= \int_0^1 x \, 6x(1-x) \, dx$

$= 6 \int_0^1 x^2 (1-x) \, dx$

$= 6 \left[\int_0^1 x^2 \, dx - \int_0^1 x^3 \, dx \right]$

$= 6 \left[\frac{x^3}{3} \Big|_0^1 - \frac{x^4}{4} \Big|_0^1 \right]$

$= 6 \left[\frac{1}{3} - \frac{1}{4} \right]$

$= 6 \left(\frac{4-3}{12} \right)$

$= 6 \left(\frac{1}{12} \right) = \frac{\cancel{6}\,1}{\cancel{12}\,2} = \frac{1}{2}$

Remark →

1) If $Y = ax$

$\therefore E(Y) = aE(x)$

& $V(Y) = a^2 var(X)$

2) If $Y = a \pm bx$

$E(Y) = a \pm bE(X)$

$V(Y) = +b^2 var(X)$

3) If X and Y independent

$E(XY) = E(X) \cdot E(Y)$

Q1 For a RV. $X, E(X-1)^2 = 10$ and $E(X-2)^2 = 6$

Find $E(X)$ & $var(X)$.

Ans $E(X-1)^2 = 10$

$\Rightarrow E[X^2 - 2X + 1] = 10$

$\Rightarrow E(X^2) - 2E(X) + 1 = 10$

$\Rightarrow E(X^2) - 2E(X) = 9$ —— (1)

$E(X-2)^2 = 6$

$\Rightarrow E(X^2 - 4X + 4) = 6$

$\Rightarrow E(X^2) - 4E(X) + 4 = 6$

$\Rightarrow E(X^2) - 4E(X) = 2$ ——— (2)

Solving (1) & (2)

$$\begin{array}{r} \cancel{E(X^2)} - 2E(X) = 9 \\ -\cancel{E(X^2)} \mp 4E(X) = -2 \\ \hline 2E(X) = 7 \end{array}$$

$E(X) = \frac{7}{2}$

Putting in (1)

$E(X^2) - \left(\cancel{2} \times \frac{7}{\cancel{2}}\right) = 9$

$\Rightarrow E(X^2) = 9 + 7$

$\Rightarrow E(X^2) = 16$

$V(X) = E(X^2) - \{E(X)\}^2$

$= 16 - \left(\frac{7}{2} \times \frac{7}{2}\right)$

$= 16 - \frac{49}{4}$

$$= \frac{64-49}{4}$$

$$= \frac{15}{4}$$

Q2 If a variable X assume only two values -2 & $+1$, such that

$2P(X=-2) = P(X=1)$

find var(X)

Ans $2P(X=-2) = P(X=1) = p$ (let).

then, $2P(X=-2) = p$ | $P(X=1) = p$

$P(X=-2) = \frac{p}{2}$ |

$$\frac{p}{2} + p = 1$$

$$\Rightarrow \frac{p+2p}{2} = 1$$

$$\Rightarrow \frac{3p}{2} = 1$$

$$\Rightarrow p = \frac{2}{3}$$

So, $P(X=-2) = \frac{p}{2} = \frac{1}{3}$ | $P(X=1) = p = \frac{2}{3}$

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$$X = \begin{cases} -2 & \text{with prob. } \frac{1}{3} \\ 1 & \text{" } \frac{2}{3} \end{cases}$$

$$E(X) = (-2) \times \frac{1}{3} + \left(1 \times \frac{2}{3}\right)$$

$$= \frac{-2}{3} + \frac{2}{3}$$

$$= 0$$

$$E(X^2) = (-2)^2 \times \frac{1}{3} + 1^2 \times \frac{2}{3}$$

$$= \frac{4}{3} + \frac{2}{3}$$

$$= \frac{\cancel{6}^{2}}{\cancel{3}} = 2 = Var(X).$$

- Cumulative distribution function (CDF)

For any real no. of x the function

$$\boxed{F(x) = P(X \le x)}$$

is known as the CDF.

Q Find the CDF of the R.V. X, the no. of heads obtained is tossing a coin thrice.

x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$F(x)$ of X for $x = 0, 1, 2, 3$ are as follows →

$F(0) = P(X \leq 0)$

$= f(0)$

$= \frac{1}{8}$

$F(1) = P(X \leq 1)$

$= f(0) + f(1)$

$= \frac{4}{8} = \frac{1}{2}$

$F(2) = P(X \leq 2)$

$= f(0) + f(1) + f(2)$

$= \frac{7}{8}$

$F(3) = P(X \leq 3)$

$= f(0) + f(1) + f(2)$

$= \frac{7}{8}$