11.4 Characteristic Equation and Characteristic roots of a Recurrence Relation

A recurrence relation of order k typically has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \tag{11.2}$$

where $c_1, c_2, \ldots, c_k$ are constants.
Convert the recurrence relation into its characteristic (or auxiliary) equation by replacing each term $a_n, a_{n-1}, \ldots, a_{n-k}$ with powers of a variable r:

$$a_n \to r^n, \quad a_{n-1} \to r^{n-1}, \quad a_{n-2} \to r^{n-2}, \quad \cdots, a_{n-k} \to r^{n-k}$$

Then the relation (11.2) becomes:

$$\begin{aligned} r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\ \Rightarrow r^k &= c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k \text{ [Divide by } r^{n-k} \text{ to remove } n\text{-dependence]} \end{aligned}$$

Now solve the characteristic (or auxiliary) equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

Let $r_1, r_2, \ldots, r_k$ be distinct roots of the characteristic equation.

11.4.1 Construct the General Solution

The general solution of the recurrence relation depends on the nature of the roots:

(i) **If all roots are distinct:** The solution is given by:

$$a_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n,$$

where $A_1, A_2, \ldots, A_k$ are constants determined by initial conditions.

(ii) **If there are repeated roots:** For a root r_i of multiplicity p_i, the contribution to the solution is:

$$(B_0 + B_1 n + B_2 n^2 + \cdots + B_{p_i-1} n^{p_i-1}) r_i^n,$$

where $B_0, B_1, \ldots, B_{p_i-1}$ are constants.

(iii) **Combine contributions from all roots:** The general solution is:

$$a_n = \sum_{i=1}^{m} \left(\sum_{j=0}^{p_i-1} B_{ij} n^j \right) r_i^n,$$

where B_{ij} are constants determined by initial conditions.

11.4.2 Illustrative Examples

Example 11.6 The characteristic equation of $5a_n - 4a_{n-1} - 6a_{n-2} = 6^n$ is $5r^2 - 4r - 6 = 0$.
Example 11.7 Consider the recurrence relation:

$$a_n = 3a_{n-1} - 4a_{n-2}.$$

- The characteristic equation is:

$$r^2 - 3r + 4 = 0.$$

- Solving $r^2 - 3r + 4 = 0$, we find the roots $r_1 = 2$ and $r_2 = 1$.
- Since the roots are distinct, the solution is:

$$a_n = A_1 \cdot 2^n + A_2 \cdot 1^n,$$

where A_1 and A_2 are determined by the initial conditions.

11.5 Worked Out Exercises

Example 11.8 Find the characteristic roots of the recurrence relation $a_n - 3a_{n-1} - 4a_{n-2} = 0$.

Solution *Its characteristic equation is*

$$\lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow \lambda^2 - 4\lambda + \lambda - 4 = 0$$
$$\Rightarrow (\lambda + 1)(\lambda - 4) = 0$$

This gives $\lambda = -1, 4$. $\therefore$ *the characteristic roots are -1 and 4 .*

Exercise 11.4 Find the characteristic equation and characteristic roots of the relation $a_{n+2} - 5a_{n+1} + 6a_n = 2$.

Solution *Its characteristic equation is* $\lambda^2 - 5\lambda + 6 = 0$ *or,* $(\lambda - 3)(\lambda - 2) = 0$

$\therefore$ *the characteristic roots are 3 and 2 .*

Exercise 11.5 Solve the following recurrence relation: $a_n - a_{n-1} - 12a_{n-2} = 0$, $a_0 = 0$, $a_1 = 1$.

Solution *We are solving the recurrence relation:*

$$a_n - a_{n-1} - 12a_{n-2} = 0, \quad a_0 = 0, \quad a_1 = 1.$$

Assume a solution of the form $a_n = r^n$. *Substituting this into the recurrence relation gives:*

$$r^n - r^{n-1} - 12r^{n-2} = 0$$
$$\Rightarrow r^2 - r - 12 = 0 \text{ [Divide by } r^{n-2} \text{ (for } r \neq 0)]$$
$$\Rightarrow (r - 4)(r + 3) = 0$$
$$\Rightarrow r_1 = 4, \quad r_2 = -3.$$

The general solution of the given recurrence relation is:

$$a_n = C_1(4^n) + C_2(-3^n),$$

where C_1 *and* C_2 *are constants to be determined from the initial conditions.*

Using $a_0 = 0$*:*

$$a_0 = C_1(4^0) + C_2(-3^0) = C_1 + C_2 = 0. \tag{1}$$

Using $a_1 = 1$*:*

$$a_1 = C_1(4^1) + C_2(-3^1) = 4C_1 - 3C_2 = 1. \tag{2}$$

From equation (1)*:*

$$C_2 = -C_1.$$

Substitute $C_2 = -C_1$ *into equation* (2)*:*

$$4C_1 - 3(-C_1) = 1,$$

$$4C_1 + 3C_1 = 1,$$

$$7C_1 = 1 \quad \Longrightarrow \quad C_1 = \frac{1}{7}.$$

Using $C_2 = -C_1$:

$$C_2 = -\frac{1}{7}.$$

Substitute C_1 and C_2 into the general solution:

$$a_n = \frac{1}{7}(4^n) - \frac{1}{7}(-3^n).$$

Simplify:

$$a_n = \frac{1}{7}(4^n + 3^n).$$