6.1 Worked Out Exercises

✍ **Exercise 6.1** Find the value of x^3 in the group of integers under the operation $+$ (arithmetic addition).

Solution *In the group of integers under addition, $x^n = x + x + \cdots + x$ (n times).*
So $x^3 = x + x + x$.

✍ **Exercise 6.2** How many binary operations are needed for a set to satisfy the ring axioms?

Solution *A ring requires two binary operations: one for addition (denoted $+$) and one for multiplication (denoted $\cdot$). So, the set needs two binary operations to satisfy the ring axioms.*

✍ **Exercise 6.3** Show that if a group has two identity elements e and f, then $e = f$.

Solution *Let e and f be two identity elements of a group G. By definition, for any element $g \in G$, we have:*

$$e \cdot g = g \quad \textit{and} \quad f \cdot g = g.$$

Also, since e is an identity element, we have $e \cdot f = f$. Substituting f for g, we get:

$$e \cdot f = f \quad \textit{and} \quad f \cdot e = e.$$

Thus, $e = f$, showing that there can only be one identity element in a group.

✍ **Exercise 6.4** Determine the number of binary operations required for a set to form a field.

Solution *A field requires two binary operations: one for addition (denoted $+$) and one for multiplication (denoted $\cdot$). Therefore, two binary operations are needed for a set to form a field.*

✍ **Exercise 6.5** In any group, how many trivial subgroups can be identified?

Solution *In any group, there are two trivial subgroups: the subgroup containing only the identity element, $\{e\}$, and the group itself, G. These are the only trivial subgroups.*

✍ **Exercise 6.6** Verify that the set of roots of $x^4 - 1 = 0$ satisfies the group axioms under the operation of multiplication.

Solution *The equation $x^4 - 1 = 0$ has the solutions $x = 1, -1, i, -i$, where i is the imaginary unit. The set of roots is $\{1, -1, i, -i\}$.*

1. ***Closure:*** *The product of any two elements in $\{1, -1, i, -i\}$ is another element in $\{1, -1, i, -i\}$.*
2. ***Associativity:*** *Multiplication is associative for complex numbers.*
3. ***Identity element:*** *The identity element is 1, since multiplying any element by 1 gives the element itself.*
4. ***Inverse elements:*** *Each element has an inverse: $1^{-1} = 1$, $(-1)^{-1} = -1$, $i^{-1} = -i$, and $(-i)^{-1} = i$.*

Thus, the set $\{1, -1, i, -i\}$ satisfies the group axioms under multiplication.

✍ **Exercise 6.7** Prove that $(G, *)$ is abelian if and only if $(a * b)^2 = a^2 * b^2$ holds for all $a, b \in G$.

Solution *First, assume that $(G, *)$ is abelian. This means that $a * b = b * a$ for all $a, b \in G$. Now, compute:*

$$(a * b)^2 = (a * b) * (a * b) = a * (b * a) * b = a * (a * b) * b = a^2 * b^2.$$

Thus, if $(G, *)$ is abelian, then $(a * b)^2 = a^2 * b^2$.

Conversely, assume that $(a * b)^2 = a^2 * b^2$ *holds for all* $a, b \in G$*. Expanding both sides:*

$$(a * b)^2 = a * (b * a) * b = a^2 * b^2.$$

Since this holds for all $a, b \in G$*, we conclude that* $a * b = b * a$*, and therefore* $(G, *)$ *is abelian.*

✍ **Exercise 6.8** Check whether $(\mathbb{Z}, +, \cdot)$ forms a commutative ring with unity under addition and multiplication.

Solution *The set of integers* $\mathbb{Z}$ *with addition and multiplication forms a commutative ring with unity. The axioms of a commutative ring with unity are satisfied: - Addition and multiplication are commutative and associative. - The distributive property holds. - The additive identity is* 0*, and the multiplicative identity is* 1*. Therefore,* $(\mathbb{Z}, +, \cdot)$ *is a commutative ring with unity.*

✍ **Exercise 6.9** Prove that the set $\{1, \omega, \omega^2\}$ is an abelian group with respect to multiplication.

Solution *This was covered in an earlier solution for the set of roots of* $x^3 - 1 = 0$*, which is* $\{1, \omega, \omega^2\}$*, and is indeed an abelian group under multiplication.*