Chapter 2 Matrix Representation of Graph

2.1 Introduction

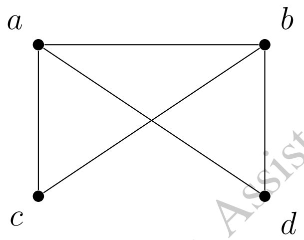
Diagrammatic or pictorial representation of a graph is very important and very convenient for a visual study. It is commonly used to represent graphs for computer processing. Many known results of matrix algebra can be readily applied to study the structural properties of graphs. In this chapter, we shall discuss three important matrix representations of undirected and directed graphs: adjacency matrix, incidence matrix, and circuit matrix.

2.1.1 Adjacency Matrix Representation for Undirected Graphs

Suppose that $G = (V, E)$ is a simple graph with n vertices. The adjacency matrix $A(G)$ of the simple graph is the $n \times n$ zero-one matrix. In other words, if the adjacency matrix is $A(G) = (a_{ij})_{n\times n}$, then

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \text{ is an edge in } G \\ 0, & \text{otherwise} \end{cases}$$

Example 2.1 This example illustrates the adjacency matrix $A(G)$ for an undirected graph G.



G

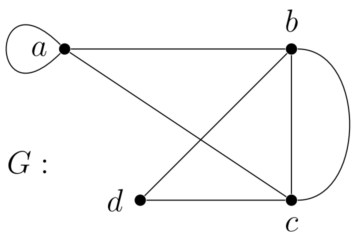
$$A(G) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 1 & 0 & 0 \end{array}$$

Note *A pseudograph can also be represented by an adjacency matrix. It is defined as:*

$$a_{ij} = \begin{cases} 1, & \textit{if } (v_i, v_j) \textit{ is an edge in } G \\ n, & \textit{if } (v_i, v_j) \textit{ is connected by } n \textit{ edges in } G \\ 0, & \textit{otherwise} \end{cases}$$

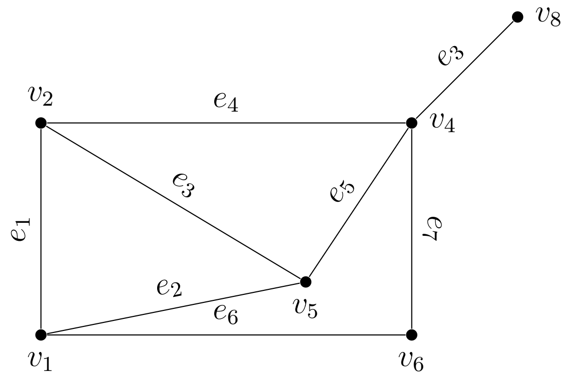
Example 2.2 In this example $A(G)$ is the adjacency matrix of the graph G.

G :



$$A(G) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Exercise 2.1 Find the adjacency matrix of the graph given below:



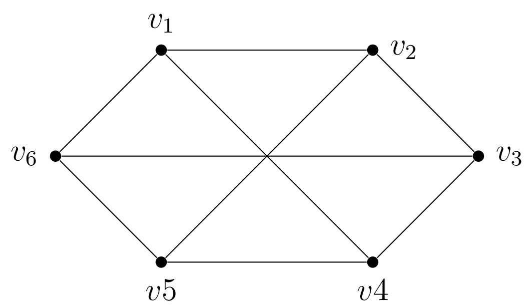
Solution *The graph has* 6 *vertices. Therefore the adjacent matrix of order* 6×6 *is*

$$A(G) = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

✍ **Exercise 2.2** Draw the graph of the following adjacency matrix:

$$A(G) = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution *The required graph is*

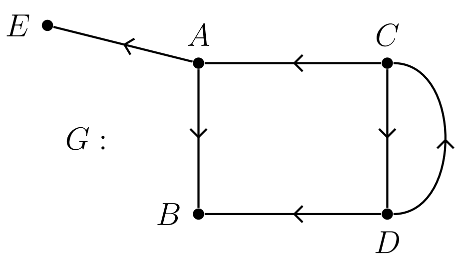


2.2 Adjacency Matrix for Directed Graph

Suppose that $G = (V, E)$ is a directed graph (di-graph) with n vertices. The adjacency matrix $A(G)$ of the di-graph G is defined as an $n \times n$ matrix. Specifically, if the adjacency matrix is represented as $A(G) = (a_{ij})_{n\times n}$, then the elements of the matrix are given by:

$$a_{ij} = \begin{cases} 1, & \text{if } G \text{ has a directed edge from } i \text{ to } j \\ 0, & \text{otherwise} \end{cases}$$

Example 2.3 In this example $A(G)$ is the adjacency matrix of the graph G.

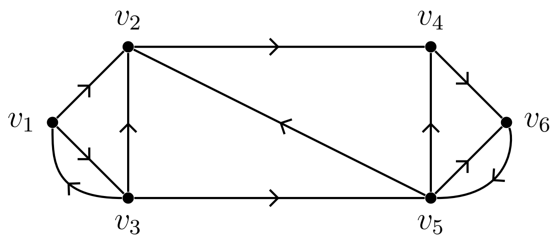


$$A(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

✍ **Exercise 2.3** Design the graph of the following adjacency matrix:

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution *The graph is*

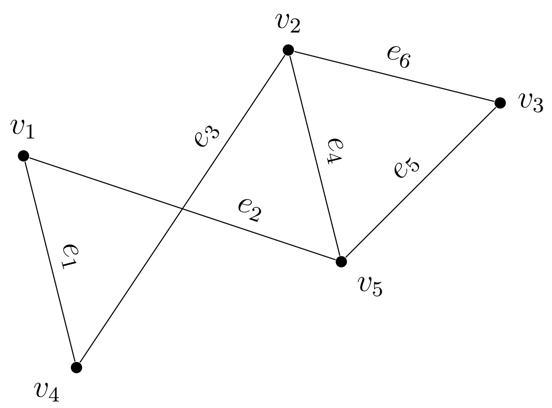


2.3 Incidence Matrix for Undirected Graph

Suppose that $G = (V, E)$ is an undirected graph with n vertices and m edges. The incidence matrix $I(G) = (m_{ij})_{n\times m}$ for the graph G is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with vertex} v_i \\ 0, & \text{otherwise} \end{cases}$$

Example 2.4 Consider the following graph:



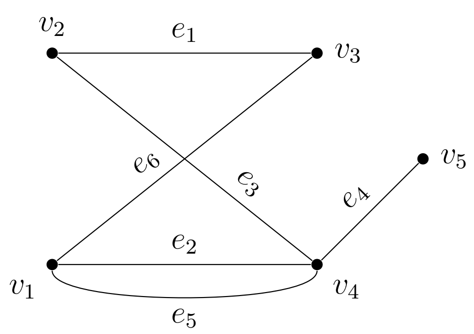
The incidence matrix of the above graph G is

$$I(G) = \begin{array}{cc} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example 2.5 Let us consider the incidence matrix $I(G)$:

$$A(G) = \begin{array}{cc} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The graph of the above incidence matrix $I(G)$ is given below:

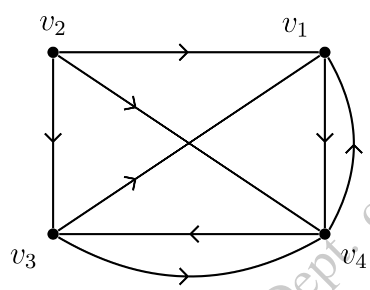


2.4 Incidence Matrix for Directed Graph (or Di-graph)

Suppose that $G = (V, E)$ is a di-graph with n vertices. The incidence matrix $I(G) = (a_{ij})_{n\times m}$ for the di-graph G is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if edge } e_j \text{ is incident out of the vertex } v_i \\ -1, & \text{if edge } e_j \text{ is incident into the vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

Example 2.6 Write the incidence matrix of the di-graph.



The incidence matrix is :

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Chapter 2 Exercise

1. Find the graph that have the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

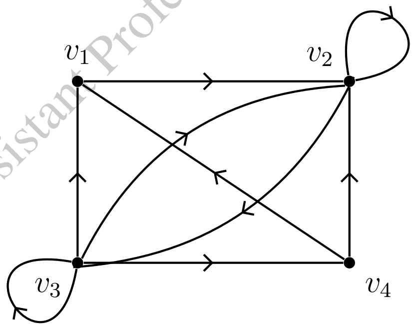
2. Draw the graph of the following adjacency matrix: (6)

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

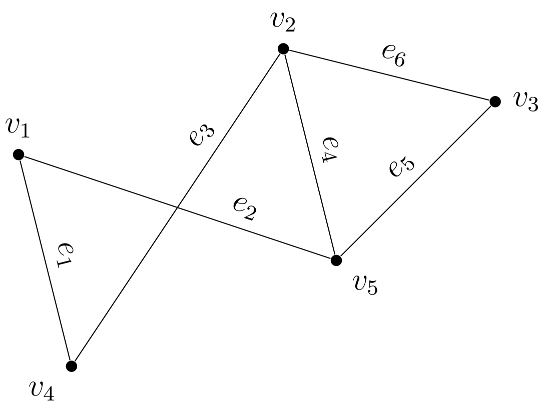
3. Draw the graph of the following adjacency matrix:

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

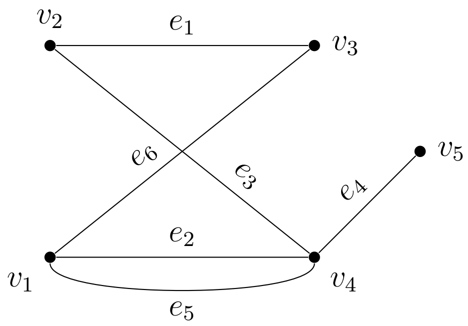
4. Find the adjacency matrix of the following graph:



5. Construct the incidence matrix of the graph G



6. The incidence matrix of the graph G is:



7. The incidence matrix of the graph G is:

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$