4.1 Worked Out Exercises

✍ **Exercise 4.1** Determine whether the relation $R = \{(1,3),(3,5),(5,3),(5,7)\}$ on the set $\{1,3,5,7\}$ is reflexive, symmetric, or transitive.

Solution *Let $R = \{(1,3),(3,5),(5,3),(5,7)\}$ be a relation on the set $\{1,3,5,7\}$.*

We need to check whether the relation is reflexive, symmetric, and transitive.

1. ***Reflexive:*** *A relation R on a set is reflexive if for every element x in the set, the pair (x,x) is in R.*
   The set is $\{1,3,5,7\}$, so we need to check if $(1,1),(3,3),(5,5),(7,7)$ are all in R.
   *None of these pairs are in R, so the relation is **not reflexive**.*
2. ***Symmetric:*** *A relation R is symmetric if for every pair (x,y) in R, the pair (y,x) must also be in R.*
   Checking the pairs:
   $$(1,3) \in R \quad \Rightarrow \quad (3,1) \notin R$$
   $$(3,5) \in R \quad \Rightarrow \quad (5,3) \in R$$
   $$(5,3) \in R \quad \Rightarrow \quad (3,5) \in R$$
   $$(5,7) \in R \quad \Rightarrow \quad (7,5) \notin R$$
   *Since $(1,3)$ and $(5,7)$ do not have their symmetric pairs in R, the relation is **not symmetric**.*
3. ***Transitive:*** *A relation R is transitive if whenever $(x,y) \in R$ and $(y,z) \in R$, we have $(x,z) \in R$.*
   Checking the pairs:
   $$(1,3) \in R \text{ and } (3,5) \in R \quad \Rightarrow \quad (1,5) \notin R$$
   $$(3,5) \in R \text{ and } (5,3) \in R \quad \Rightarrow \quad (3,3) \notin R$$
   $$(5,3) \in R \text{ and } (3,5) \in R \quad \Rightarrow \quad (5,5) \notin R$$
   $$(5,7) \in R \text{ and } (7,x) \notin R \quad \Rightarrow \quad \text{No further pair to check.}$$
   *Since $(1,5)$, $(3,3)$, and $(5,5)$ are not in R, the relation is **not transitive**.*

Hence, the relation $R = \{(1,3),(3,5),(5,3),(5,7)\}$ is neither reflexive, symmetric, nor transitive.

✍ **Exercise 4.2** On the set $\mathbb{Z}$ of integers, define a binary relation R by aRb if and only if $a-b$ is divisible by 7. Show that R is an equivalence relation.

Solution *To show that R is an equivalence relation, we need to check three properties: reflexivity, symmetry, and transitivity.*

(i) ***Reflexive:*** *A relation R is reflexive if for all $a \in \mathbb{Z}$, aRa.*
For any integer a, we have:
$$a - a = 0 \quad \text{and} \quad 0 \text{ is divisible by } 7.$$
Therefore, aRa for all $a \in \mathbb{Z}$, and R is reflexive.

(ii) ***Symmetric:*** *A relation R is symmetric if whenever aRb, we also have bRa.*

Suppose aRb, i.e., $a - b$ is divisible by 7, i.e., $a - b = 7k$ for some integer k. Then:

$$b - a = -(a - b) = -7k = 7(-k).$$

Since $-k$ is an integer, $b - a$ is divisible by 7, so bRa. Therefore, R is symmetric.

(iii) ***Transitive:*** *A relation R is transitive if whenever aRb and bRc, we also have aRc.*
Suppose aRb and bRc, i.e., $a - b = 7k_1$ and $b - c = 7k_2$ for some integers k_1 and k_2. Then:

$$a - c = (a - b) + (b - c) = 7k_1 + 7k_2 = 7(k_1 + k_2).$$

Since $k_1 + k_2$ is an integer, $a - c$ is divisible by 7, so aRc. Therefore, R is transitive.

Since R is reflexive, symmetric, and transitive, we conclude that R is an equivalence relation on $\mathbb{Z}$.

✍ **Exercise 4.3** A relation ρ on the set of integers $\mathbb{Z}$ is defined by $\rho = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } |a - b| \leq 5\}$. Is the relation reflexive, symmetric, and transitive?

Solution *We are given a relation ρ on $\mathbb{Z}$ defined by:*

$$\rho = \{(a, b) \mid a, b \in \mathbb{Z} \textit{ and } |a - b| \leq 5\}.$$

We need to determine whether the relation is reflexive, symmetric, and transitive.

(i) ***Reflexive:*** *A relation ρ is reflexive if for all $a \in \mathbb{Z}$, $(a, a) \in \rho$.*
For any integer a, we have:

$$|a - a| = 0 \quad \textit{and} \quad 0 \leq 5.$$

Since $|a - a| = 0$, the pair (a, a) is always in ρ. Thus, the relation is ***reflexive****.*

(ii) ***Symmetric:*** *A relation ρ is symmetric if whenever $(a, b) \in \rho$, we also have $(b, a) \in \rho$.*
Suppose $(a, b) \in \rho$, which means $|a - b| \leq 5$. We also have:

$$|b - a| = |a - b| \quad \textit{(since absolute value is symmetric)}.$$

Therefore, if $(a, b) \in \rho$, then $(b, a) \in \rho$ as well. Thus, the relation is ***symmetric****.*

(iii) ***Transitive:*** *A relation ρ is transitive if whenever $(a, b) \in \rho$ and $(b, c) \in \rho$, we also have $(a, c) \in \rho$.*
Suppose $(a, b) \in \rho$ and $(b, c) \in \rho$. This means:

$$|a - b| \leq 5 \quad \textit{and} \quad |b - c| \leq 5.$$

To check if the relation is transitive, we need to determine if $(a, c) \in \rho$, i.e., if $|a - c| \leq 5$.
By the triangle inequality:

$$|a - c| \leq |a - b| + |b - c| \leq 5 + 5 = 10.$$

This shows that $|a - c|$ can be at most 10, but for (a, c) to be in ρ, we need $|a - c| \leq 5$. Therefore, the relation is ***not transitive****, as the inequality $|a - c| \leq 5$ is not guaranteed.*

Thus, the relation ρ is reflexive and symmetric, but it is not transitive.

✍ **Exercise 4.4** The relation $R = \{(x, y) : x, y \in \mathbb{Z},\ x \neq y\}$ is defined on $\mathbb{Z}$. What properties does the relation R have?

Solution *We are given a relation R on $\mathbb{Z}$, defined by:*

$$R = \{(x, y) : x, y \in \mathbb{Z},\ x \neq y\}.$$

That is, R consists of all pairs of integers (x, y) where $x \neq y$.

We need to determine what properties this relation has, namely, whether it is reflexive, symmetric, and transitive.

(i) ***Reflexive:*** *A relation R is reflexive if for all $x \in \mathbb{Z}$, the pair $(x, x) \in R$.*
In this case, R consists of pairs where $x \neq y$. Since (x, x) would require $x = x$, which contradicts the condition $x \neq y$, no pair of the form (x, x) can be in R.
*Therefore, the relation R is **not reflexive**.*

(ii) ***Symmetric:*** *A relation R is symmetric if whenever $(x, y) \in R$, we also have $(y, x) \in R$.*
Suppose $(x, y) \in R$, which means $x \neq y$. We want to check if (y, x) is also in R. Since $y \neq x$ is the same as $x \neq y$, it follows that $(y, x) \in R$ whenever $(x, y) \in R$.
*Therefore, the relation R is **symmetric**.*

(iii) ***Transitive:*** *A relation R is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$.*
Suppose $(x, y) \in R$ and $(y, z) \in R$. This means that $x \neq y$ and $y \neq z$. To check if $(x, z) \in R$, we need to determine if $x \neq z$.
However, $x \neq y$ and $y \neq z$ do not guarantee that $x \neq z$. For example, if $x = 1$, $y = 2$, and $z = 1$, then $x \neq y$ and $y \neq z$, but $x = z$, meaning $(x, z) \notin R$.
*Therefore, the relation R is **not transitive**.*

Thus, the relation R is symmetric but not reflexive or transitive.

✍ **Exercise 4.5** If $B = \{-3, -2, -1, 0, 1\}$ and $g : B \to \mathbb{R}$ is defined as $g(x) = x^2 - 1$, then $g(B) =$

Solution *The function $g : B \to \mathbb{R}$ is defined by:*

$$g(x) = x^2 - 1 \quad \textit{for all} \quad x \in B.$$

We are tasked with finding the set $g(B)$, which is the image of the set B under the function g. That is, we need to compute the set of values $g(x)$ for each element $x \in B$.

We compute $g(x)$ for each element of $B = \{-3, -2, -1, 0, 1\}$:

$$\begin{gathered} g(-3) = (-3)^2 - 1 = 9 - 1 = 8, \\ g(-2) = (-2)^2 - 1 = 4 - 1 = 3, \\ g(-1) = (-1)^2 - 1 = 1 - 1 = 0, \\ g(0) = 0^2 - 1 = 0 - 1 = -1, \\ g(1) = 1^2 - 1 = 1 - 1 = 0. \end{gathered}$$

Therefore, the image of B under g is:

$$g(B) = \{8, 3, 0, -1\}.$$

✍ **Exercise 4.6** Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 9$, for $x \in \mathbb{R}$, is bijective.

Solution *To show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 5x + 9$, is bijective, we need to prove that it is both **injective** and **surjective**.*

(i) ***Injective:*** *A function f is injective (or one-to-one) if for all $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.*

Assume that $f(x_1) = f(x_2)$. Then:

$$5x_1 + 9 = 5x_2 + 9.$$

Subtracting 9 from both sides gives:

$$5x_1 = 5x_2.$$

Dividing both sides by 5:

$$x_1 = x_2.$$

Therefore, f is injective.

(ii) ***Surjective:*** *A function f is surjective (or onto) if for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$.*
Let $y \in \mathbb{R}$. We need to find $x \in \mathbb{R}$ such that $f(x) = y$, i.e., $5x + 9 = y$. Solving for x:

$$5x = y - 9,$$
$$x = \frac{y-9}{5}.$$

Since y is any real number, we can always find such an $x \in \mathbb{R}$. Therefore, f is surjective.

Since f is both injective and surjective, we conclude that f is bijective.

Exercise 4.7 If $f, g : \mathbb{R} \to \mathbb{R}$ where $f(x) = ax + b$, $g(x) = 1 - x + x^2$, and $(g \circ f)(x) = 9x^2 - 9x + 3$, find the values of a and b.

Solution *We are given the functions:*

$$f(x) = ax + b, \quad g(x) = 1 - x + x^2,$$

and the composition of the functions:

$$(g \circ f)(x) = 9x^2 - 9x + 3.$$

We need to find the values of a and b.

The composition $(g \circ f)(x)$ is defined as:

$$(g \circ f)(x) = g(f(x)).$$

Substituting the expression for $f(x) = ax + b$ into $g(x)$, we get:

$$g(f(x)) = g(ax + b) = 1 - (ax + b) + (ax + b)^2.$$

Now, simplify the expression for $g(ax + b)$:

$$g(ax + b) = 1 - (ax + b) + (ax + b)^2.$$

Expanding the terms:

$$1 - ax - b + (ax + b)^2 = 1 - ax - b + (a^2x^2 + 2abx + b^2).$$

So, we have:

$$g(ax + b) = a^2x^2 + 2abx + b^2 - ax - b + 1.$$

Simplifying further:

$$g(ax + b) = a^2x^2 + (2ab - a)x + (b^2 - b + 1).$$

We are given that:

$$g(f(x)) = 9x^2 - 9x + 3.$$

Now, compare the two expressions for $g(f(x))$*:*

$$a^2x^2 + (2ab - a)x + (b^2 - b + 1) = 9x^2 - 9x + 3.$$

By comparing the coefficients of like powers of x*, we get the following system of equations: - For the* x^2*-term:* $a^2 = 9$*, - For the* x*-term:* $2ab - a = -9$*, - For the constant term:* $b^2 - b + 1 = 3$*.*

Solve each equation: (1). From $a^2 = 9$*, we get two possible values for* a*:*

$$a = 3 \quad \textit{or} \quad a = -3.$$

(2). From $b^2 - b + 1 = 3$*, simplify to:*

$$b^2 - b - 2 = 0.$$

Factoring the quadratic equation:

$$(b - 2)(b + 1) = 0.$$

Thus, $b = 2$ *or* $b = -1$*.*

(3). Now, substitute these values of a *and* b *into the equation* $2ab - a = -9$ *and check which pair satisfies the equation.*

- For $a = 3$ *and* $b = 2$*:*

$$2(3)(2) - 3 = 12 - 3 = 9 \quad (\textit{not } -9).$$

- For $a = 3$ *and* $b = -1$*:*

$$2(3)(-1) - 3 = -6 - 3 = -9 \quad (\textit{this is correct}).$$

- For $a = -3$ *and* $b = 2$*:*

$$2(-3)(2) - (-3) = -12 + 3 = -9 \quad (\textit{this is correct}).$$

- For $a = -3$ *and* $b = -1$*:*

$$2(-3)(-1) - (-3) = 6 + 3 = 9 \quad (\textit{not } -9).$$

Thus, the possible pairs (a, b) *are* $(3, -1)$ *and* $(-3, 2)$*.*