INSTRUMENTATION AND CONTROL

COEG 304

LECTURE 2

KATHMANDU UNIVERSITY

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Mathematical Models

The process of representation of the dynamics of any physical system with mathematical equations that approximately relates the physical system quantities to the system components is mathematical modeling.

In order to understand the behavior of systems, mathematical models are required.

Mathematical models are equations which describe the relationship between the input and output of a system.

The basis for any mathematical model is provided by the fundamental physical laws that govern the behavior of the system.

Mathematical Model

Most common mathematical models to represent any system are :

- Differential equations
- Transfer function
- State equations

Mechanical System Building Blocks

Basic building block: **spring**, **dashpots**, and **masses**.

Springs: The reaction force ‘***f***k’ developed due to compression or elongation of the spring is equal to stiffness ’k’ and the amount of deformation of the spring. It represent the stiffness of a system.

Dashpots represent the forces opposing motion, for example frictional or damping effects. A **dashpot** is a mechanical device, a damper which resists motion via viscous friction.

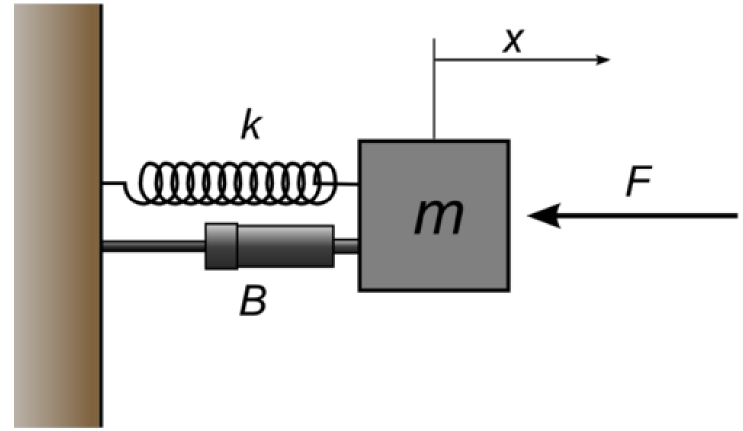
Masses represent the inertia or resistance to acceleration.

Mechanical systems does not have to be really made up of springs, dashpots, and masses but have the properties of stiffness, damping, and inertia.

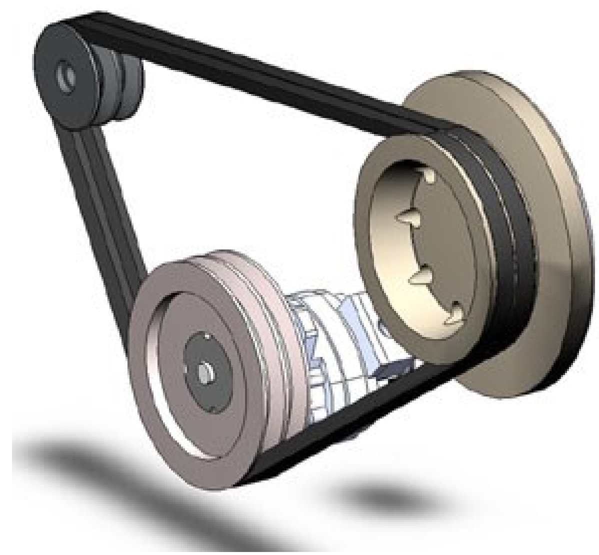
All these building blocks may be considered to have a force as an input and displacement as an output.

Basic Types of Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion



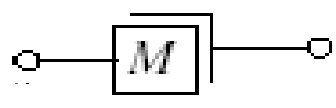
Basic Elements of Translational Mechanical Systems

Translational Spring

i)

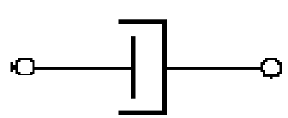
Translational Mass

ii)



Translational Damper

iii)

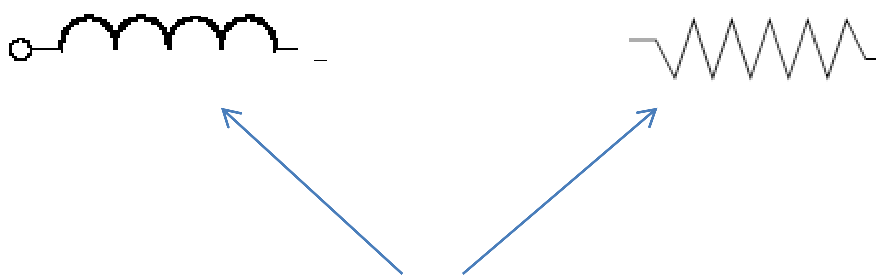


I. Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)

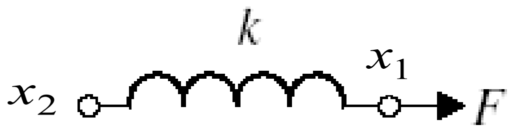


Circuit Symbols

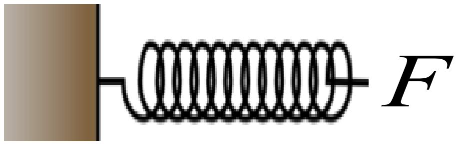
Translational Spring

I. Translational Spring

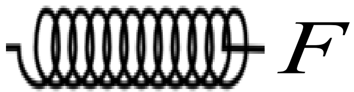
- If *F* is the applied force



- Then x_1 is the deformation if $x_2 = 0$



- Or $(x_1 - x_2)$ is the deformation.



- The equation of motion is given as

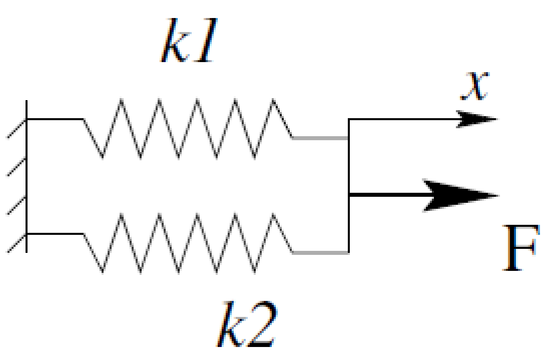
$$F = k(x_1 - x_2)$$

- Where k is stiffness of spring expressed in N/m

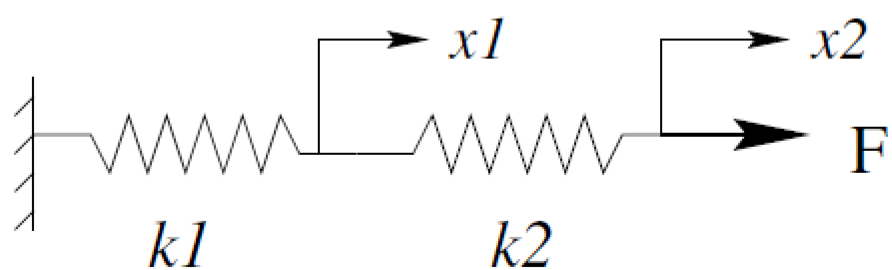
I. Translational Spring

- Given two springs with spring constant k_1 and k_2, obtain the equivalent spring constant k_{eq} for the two springs connected in:

(1) Parallel



(2) Series



I. Translational Spring

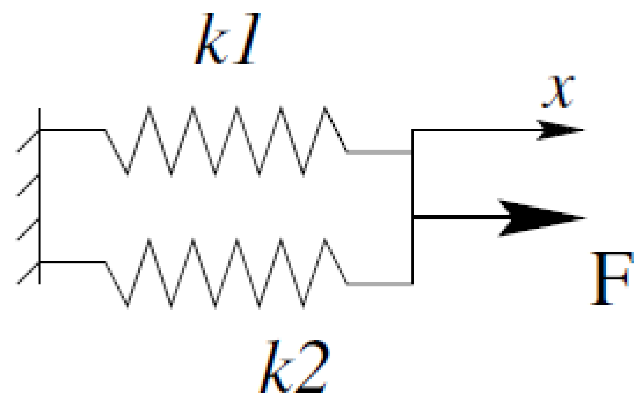
- The two springs have same displacement therefore:

$$k_1 x + k_2 x = F$$

$$(k_1 + k_2)x = F$$

$$k_{eq} x = F$$

$$\boxed{k_{eq} = k_1 + k_2}$$



(1) Parallel

- If ***n*** springs are connected in parallel then:

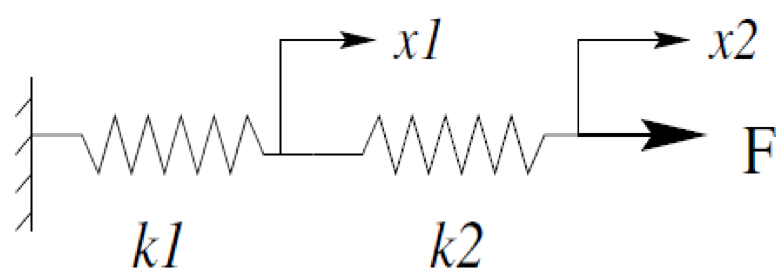
$$\boxed{k_{eq} = k_1 + k_2 + \cdots + k_n}$$

I. Translational Spring

- The forces on two springs are same, *F*, however displacements are different therefore:

$$k_1 x_1 = F \text{ and } k_2(x_2 - x_1) = F$$

(2) Series



$$k_2\left(x_2 - \frac{F}{k_1}\right) = F$$

$$\text{or, } k_2 x_2 = F + \frac{k_2}{k_1}F = \frac{k_1 + k_2}{k_1}F$$

- The equivalent spring constant k_{eq} for this case is then found as

$$\boxed{k_{eq} = \frac{F}{x_2} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}}$$

I. Translational Spring

$$\boxed{k_{eq} = \frac{F}{x_2} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}}$$

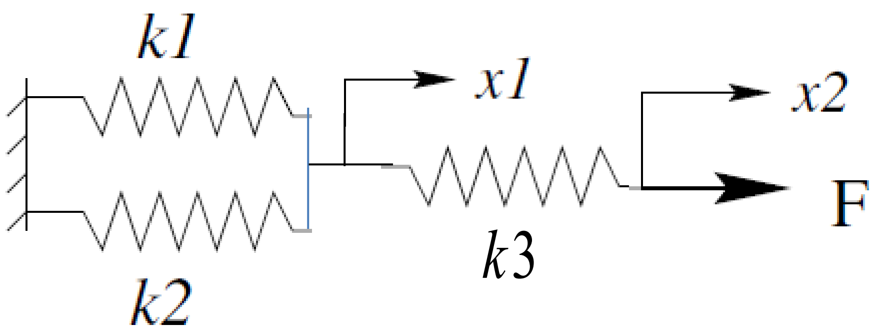
- If ***n*** springs are connected in series then:

$$\boxed{k_{eq} = \frac{k_1 k_2 \cdots k_n}{k_1 + k_2 + \cdots + k_n}}$$

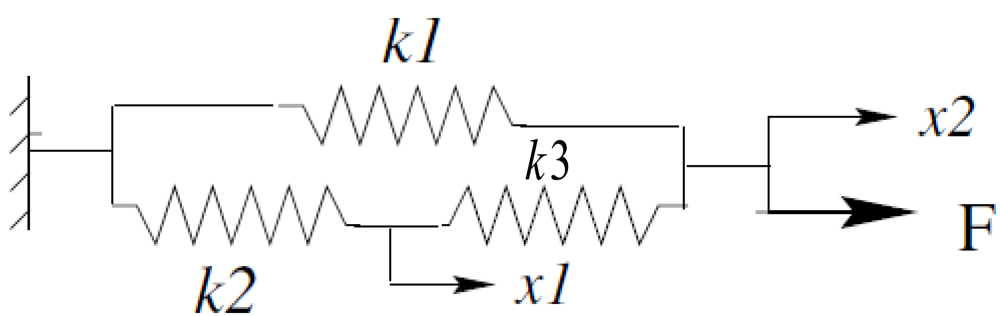
I. Translational Spring

- **Exercise:** Obtain the equivalent stiffness for the following spring networks.

i)



ii)



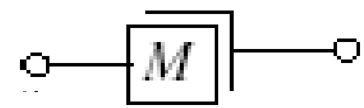
II. Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation between force and displacements is given by Newton's law.

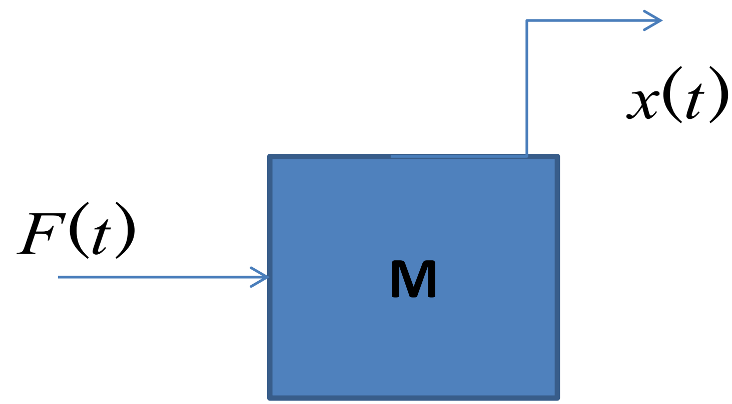
$$\boxed{F = M\ddot{x}}$$

Translational Mass

ii)



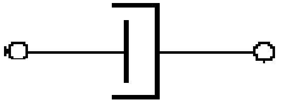
constant



III. Translational Damper

- When the viscosity or drag is not negligible in a system, we often model them with the damping force.
- All the materials exhibit the property of damping to some extent.
- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

iii) **Translational Damper**



Common Uses of Dashpots

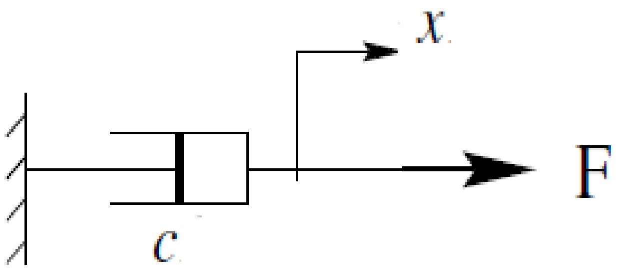
Door Stoppers

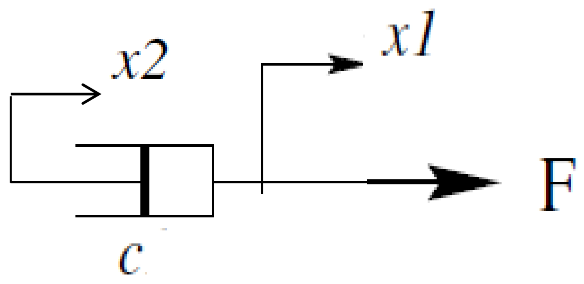
Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper





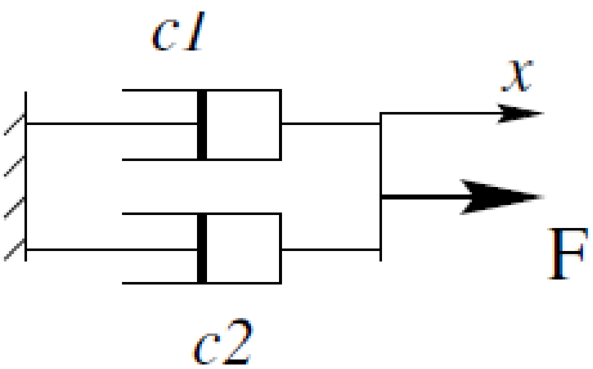
$$\boxed{F = C\dot{x}}$$

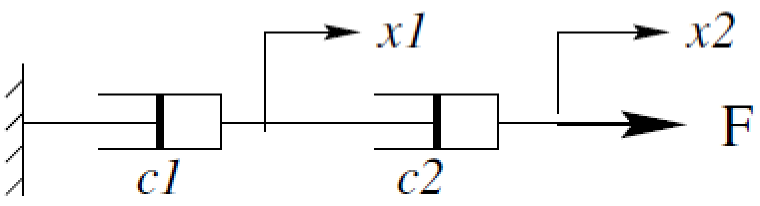
$$\boxed{F = C(\dot{x}_1 - \dot{x}_2)}$$

- Where C is damping coefficient (***N/ms^{-1}***).

Translational Damper

- Translational Dampers in series and parallel.



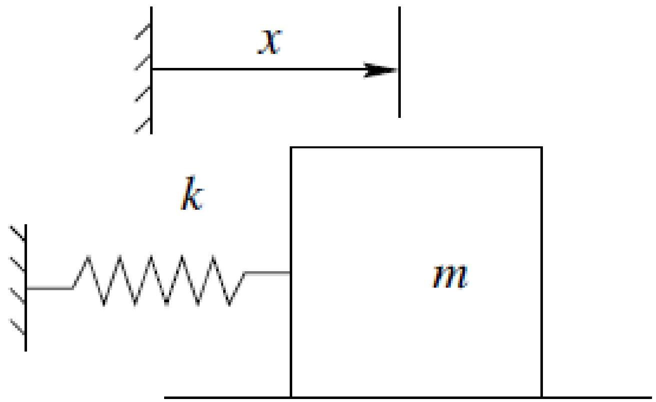


$$\boxed{C_{eq} = C_1 + C_2}$$

$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}}$$

Modelling a simple Translational System

- **Example-1**: Consider a simple horizontal spring-mass system on a frictionless surface, as shown in figure below.



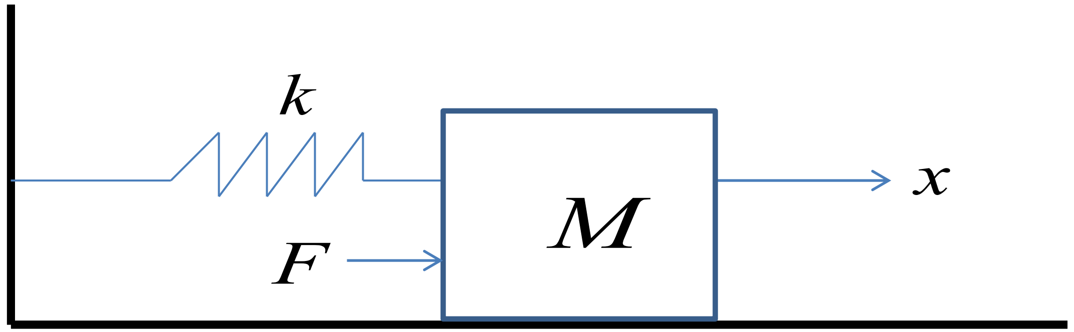
$$m\ddot{x} = -kx$$

or

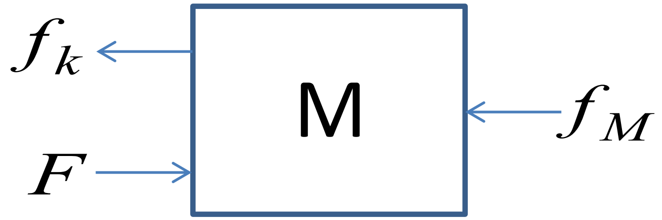
$$m\ddot{x} + kx = 0$$

Example-2

- Consider the following system (friction is negligible)

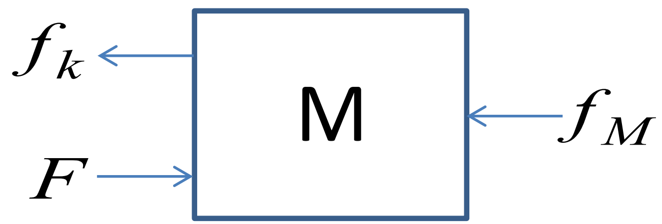


- Free Body Diagram



- Where f_k and f_M are force applied by the spring and inertial force respectively.

Example-2



$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

$$F(s) = Ms^2 X(s) + kX(s)$$

Example-2

$$F(s) = Ms^2X(s) + kX(s)$$

- The transfer function of the system is

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}}$$

- if

$$M = 1000kg$$

$$k = 2000\,Nm^{-1}$$

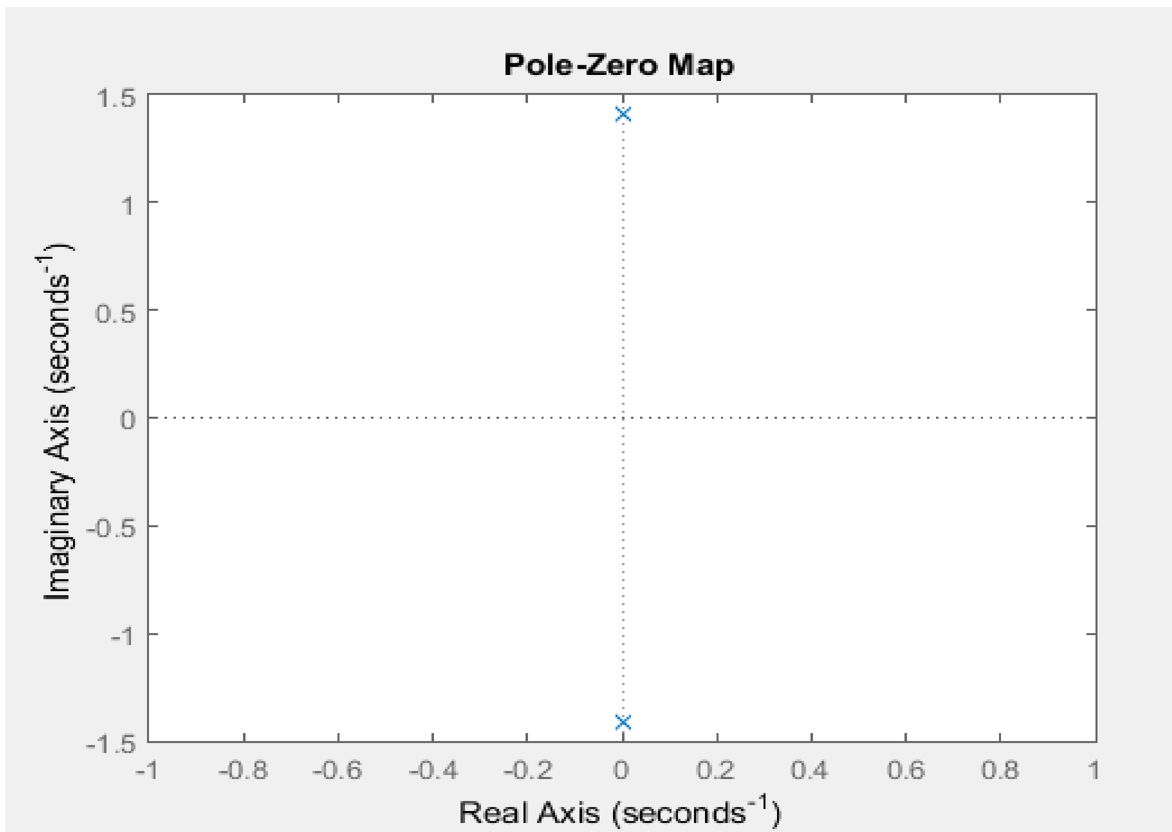
$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

Example-2

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

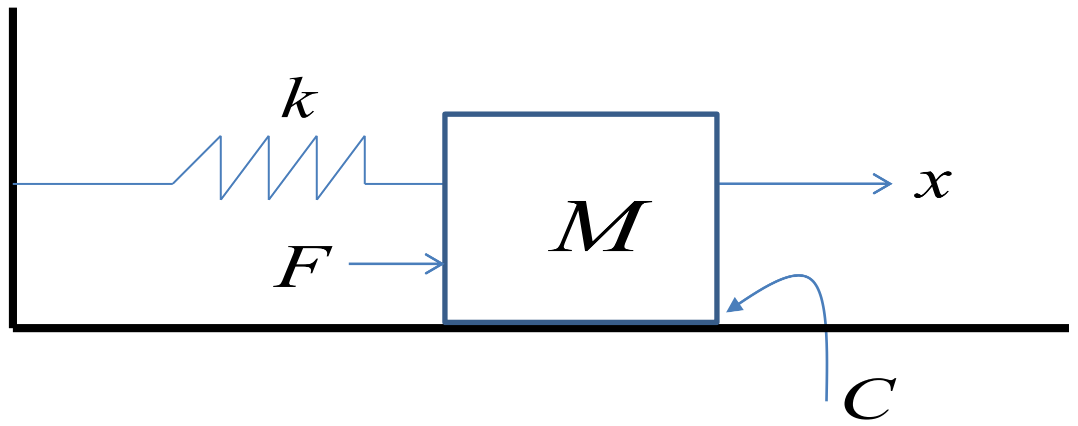
```
p =

   0.0000 + 1.4142i
   0.0000 - 1.4142i
```

- The pole-zero map of the system is

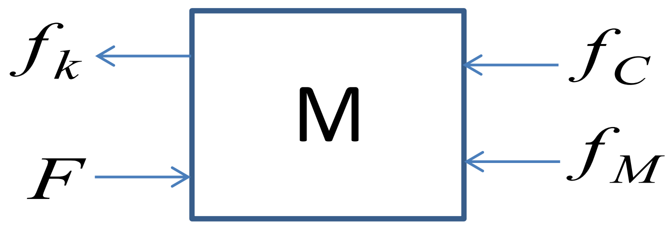


Example-3

- Consider the following system



- Free Body Diagram



$$F = f_k + f_M + f_C$$

Example-3

Differential equation of the system is:

$$F = M\ddot{x} + C\dot{x} + kx$$

Taking the Laplace Transform of both sides and ignoring Initial conditions we get

$$F(s) = Ms^2 X(s) + CsX(s) + kX(s)$$

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + k}}$$

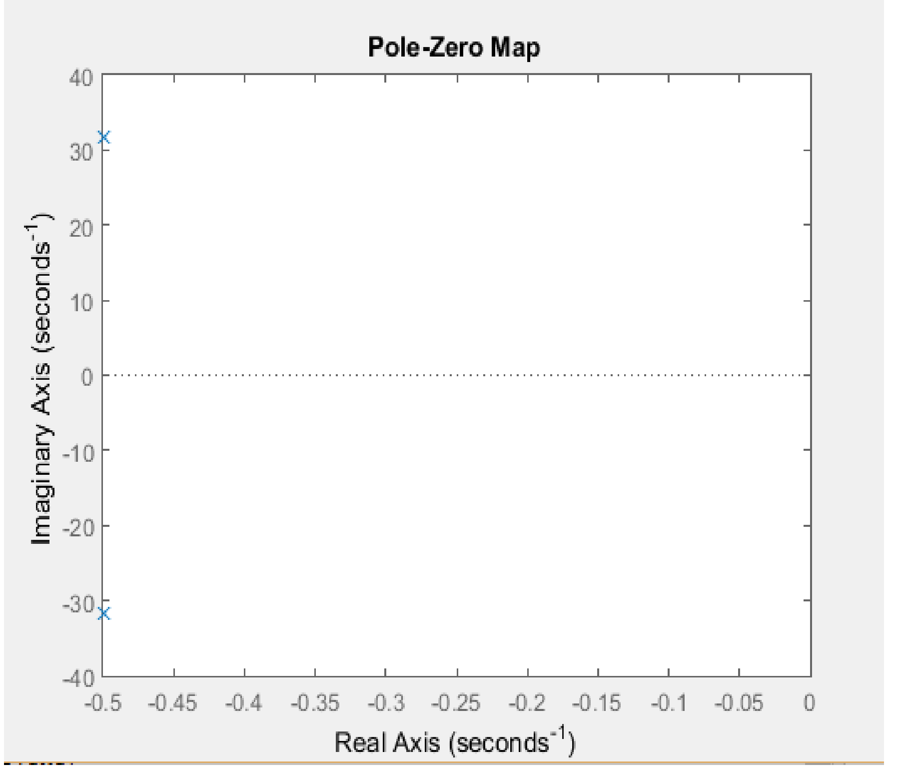
Example-3

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + k}}$$

```
p =

  -0.5000 +31.6188i
  -0.5000 -31.6188i
```

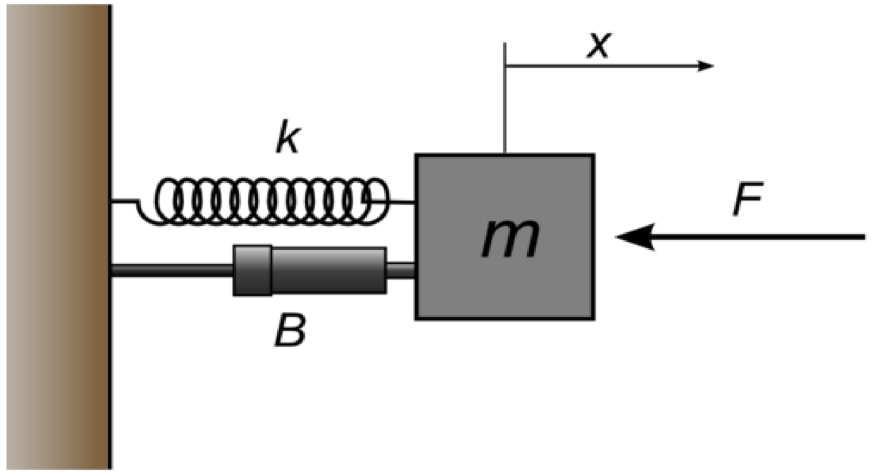
- if

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + s + 1000}$$

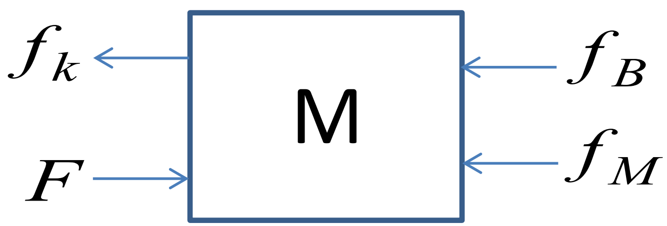


Example-4

- Consider the following system



- Free Body Diagram (same as example-3)



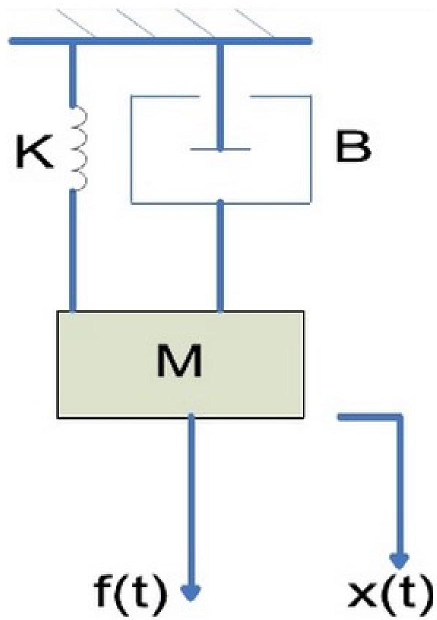
$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}}$$

$$F = f_k + f_M + f_B$$

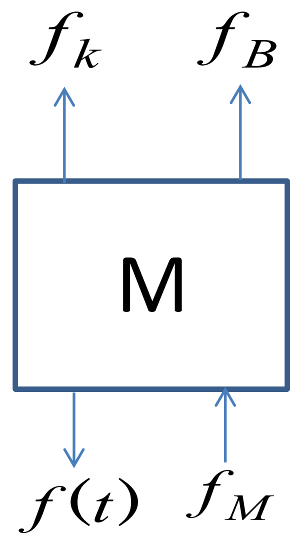
Example-8

- Find the transfer function of the mechanical translational system given in Figure-1.

Figure-1



Free Body Diagram

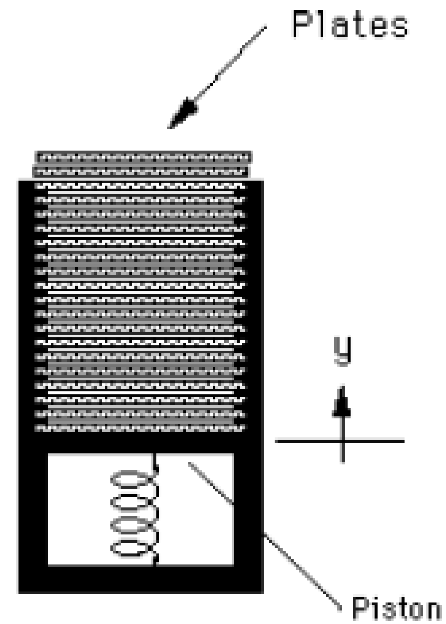


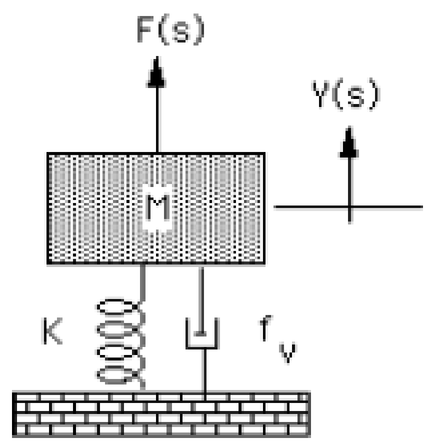
$$f(t) = f_k + f_M + f_B$$

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}}$$

Example-9

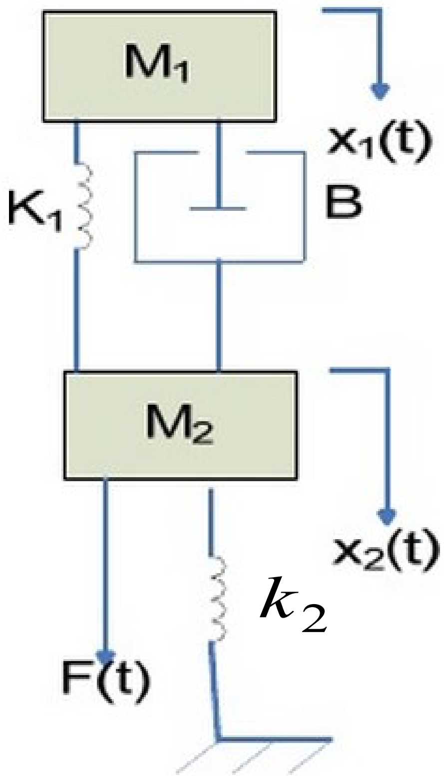
- Restaurant plate dispenser



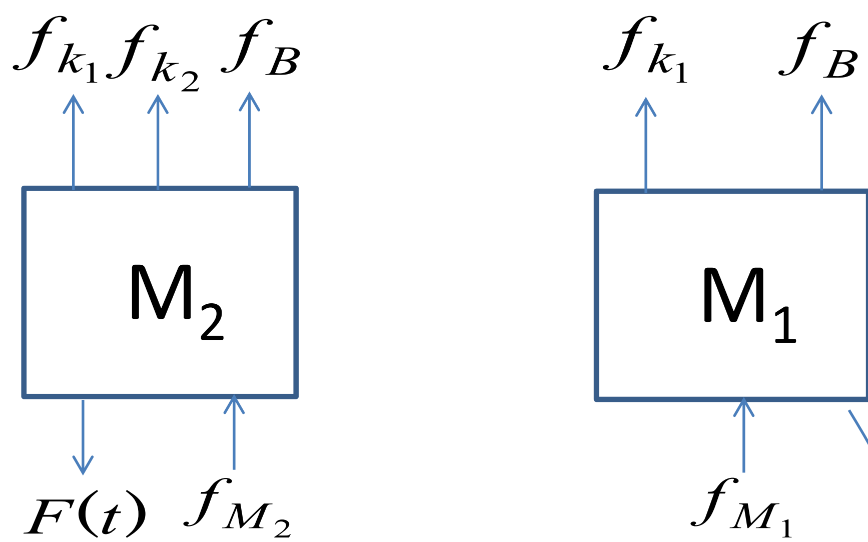


Example-10

- Find the transfer function $X_2(s)/F(s)$ of the following system.



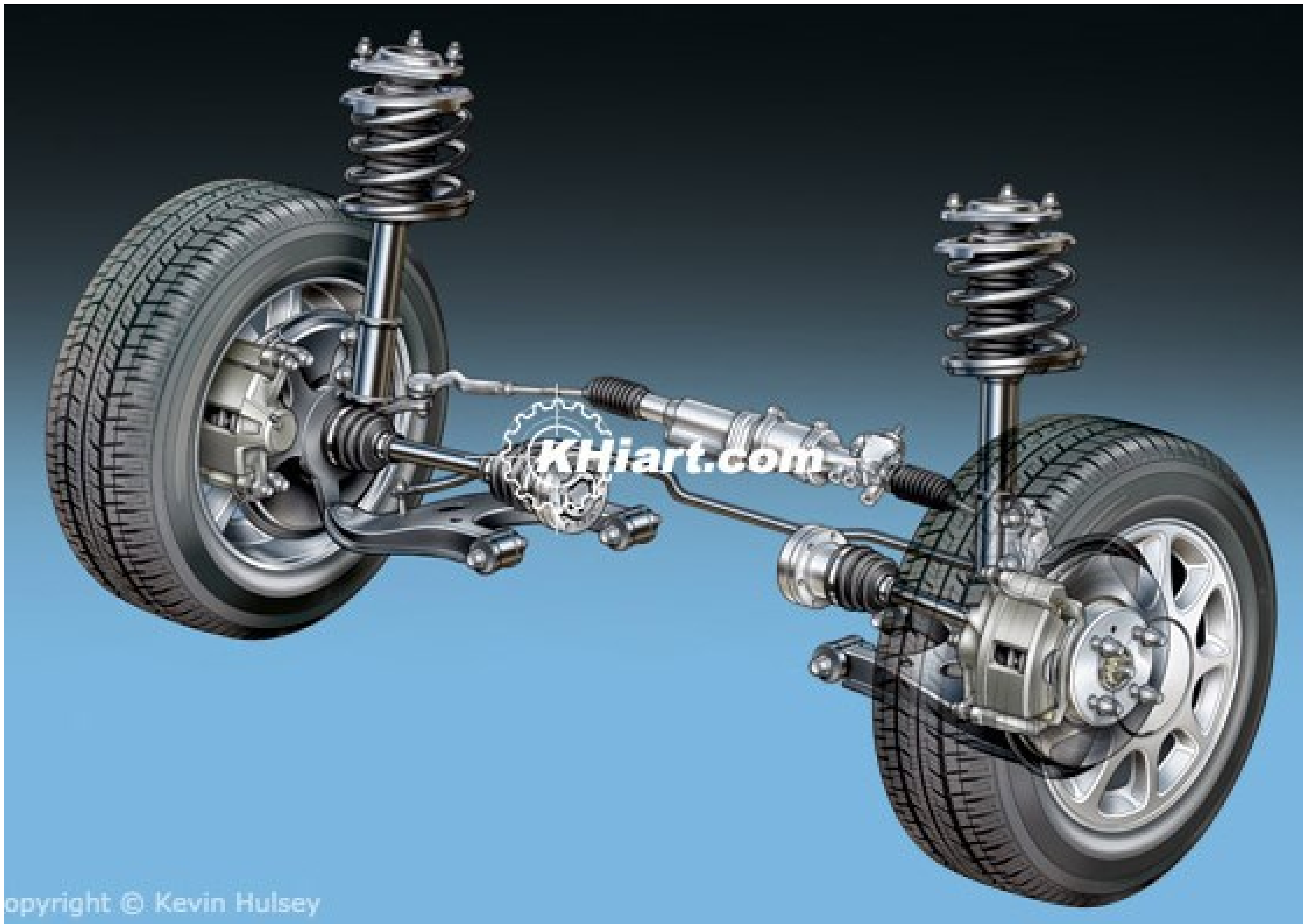
Free Body Diagram



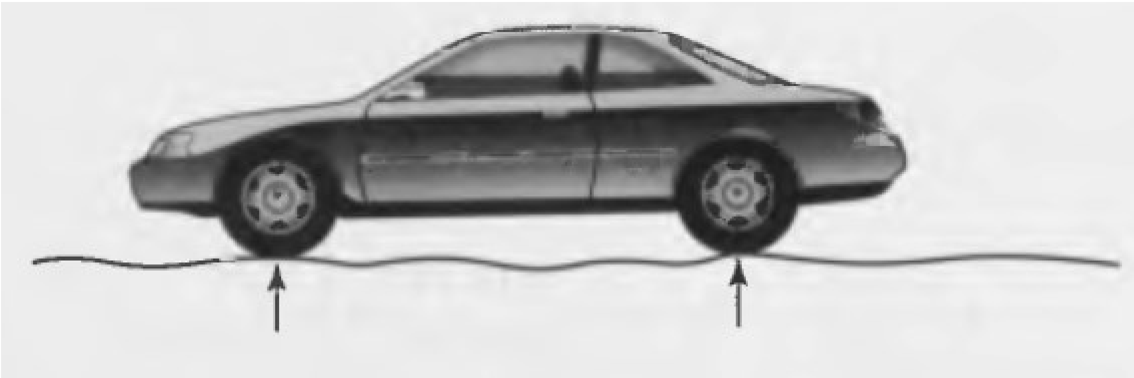
$$F(t) = f_{k_1} + f_{k_2} + f_{M_2} + f_B$$

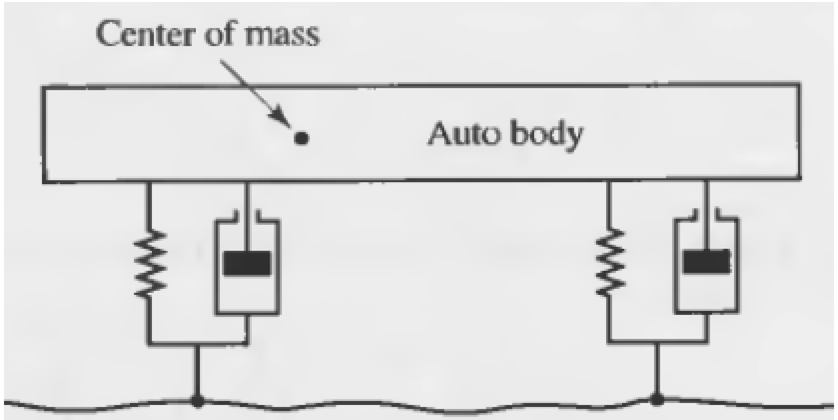
$$0 = f_{k_1} + f_{M_1} + f_B$$

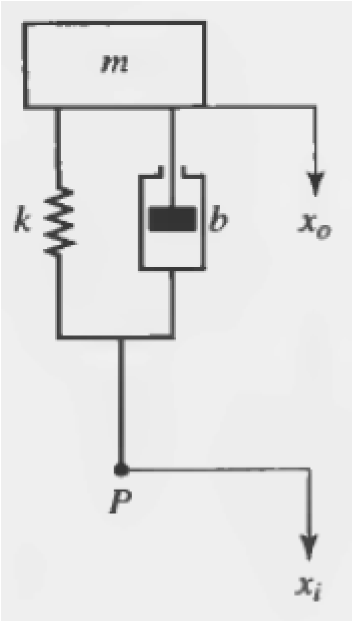
Example-11: Automobile Suspension



Automobile Suspension



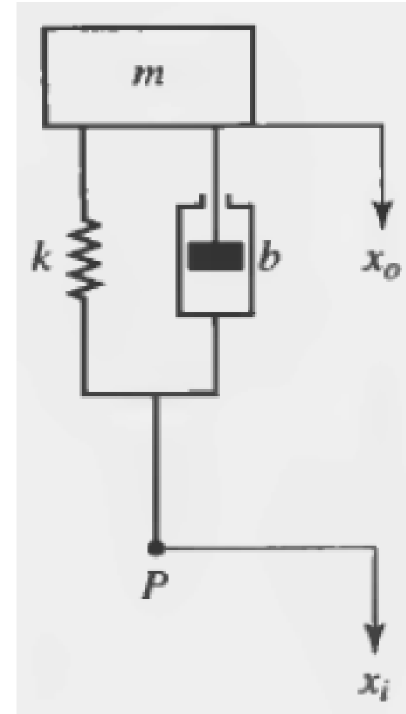




Automobile Suspension

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \qquad \text{(eq .1)}$$

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \qquad \text{eq. 2}$$



Taking Laplace Transform of the equation (2)

$$ms^2 X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Electrical System Building Blocks

The basic building blocks of electrical systems are resistance, inductance and capacitance.

$$\text{Resistor}: \; v = iR; \; P = i^2 R$$

$$\text{Inductor}: i = \frac{1}{L}\int v\,dt; \; E = \frac{1}{2} L i^2$$

$$\text{Capacitor}: i = C\frac{dv}{dt}; \; E = \frac{1}{2} C v^2$$

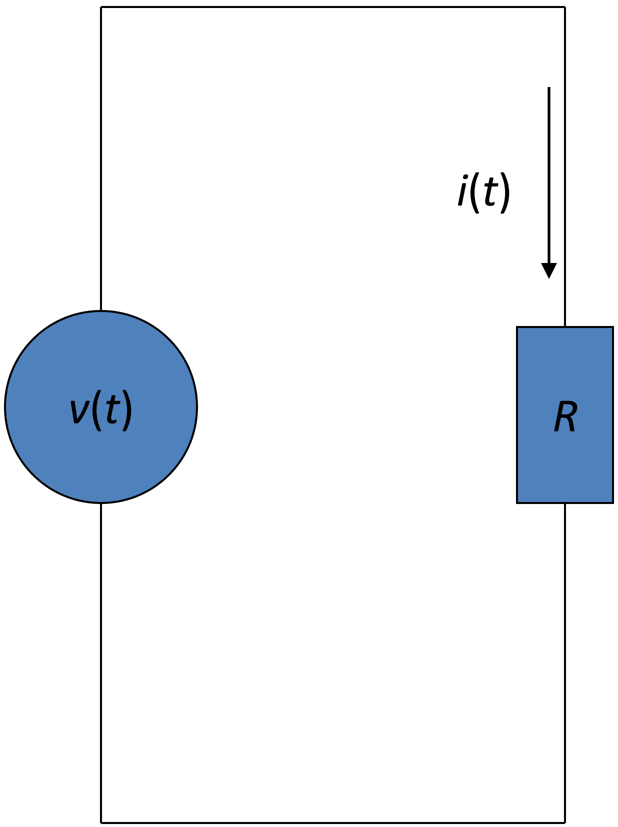
Resistance, *R* (ohm)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = Ri(t)$$

$$i(t) = \frac{1}{R}v(t)$$



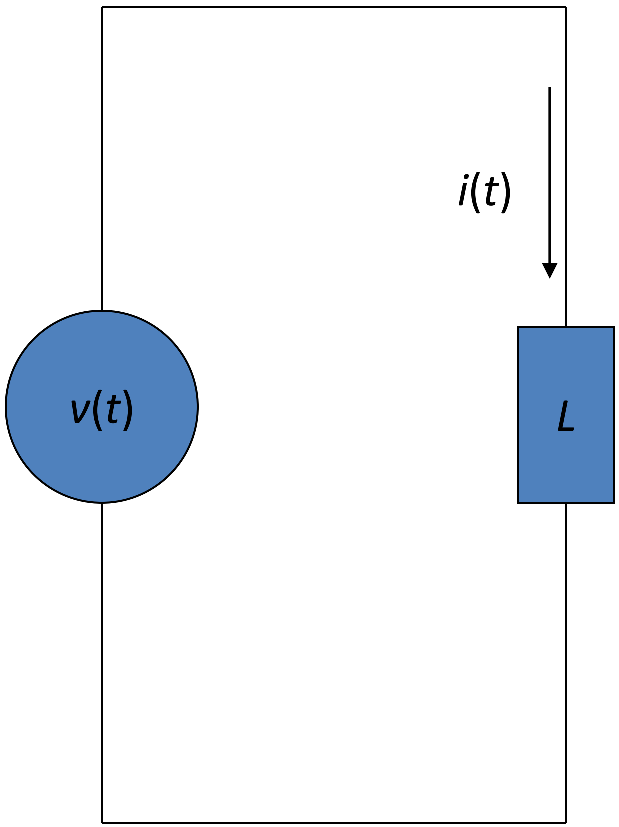
Inductance, *L* (H)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)dt$$



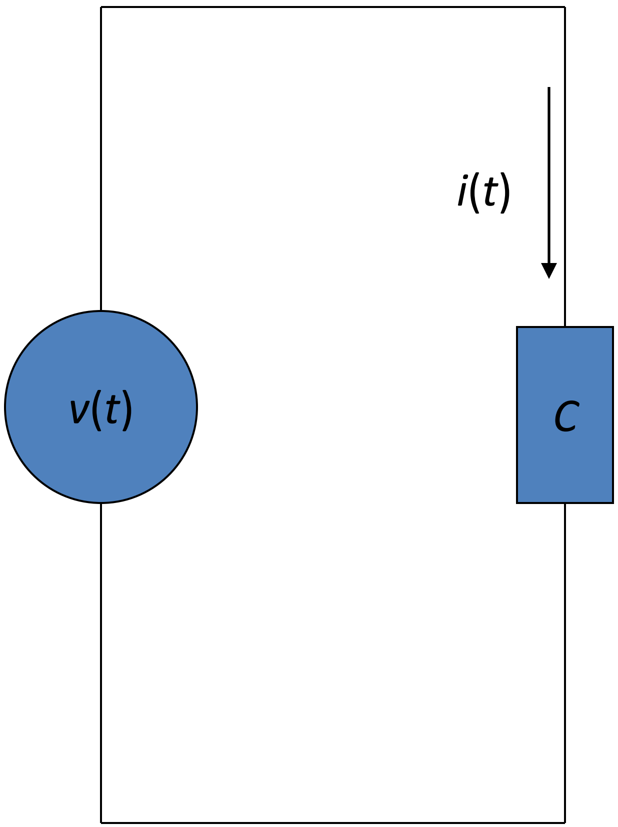
Capacitance, *C* (F)

Appied voltage $v(t)$

Current $i(t)$

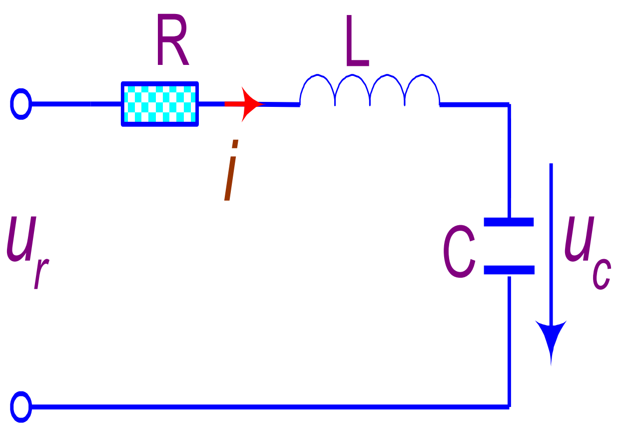
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$i(t) = C\frac{dv(t)}{dt}$$



Example 13

A passive circuit



***define**: input → u_r output → u_c。*
***we have**：*

$$Ri + L\frac{di}{dt} + u_c = u_r \qquad i = C\frac{du_c}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2u_c}{dt^2} + RC\frac{du_c}{dt} + u_c = u_r$$

$$make: \quad RC = T_1 \quad \frac{L}{R} = T_2 \quad \Rightarrow \quad T_1T_2\frac{d^2u_c}{dt^2} + T_1\frac{du_c}{dt} + u_c = u_r$$

Modelling of Electric Systems:

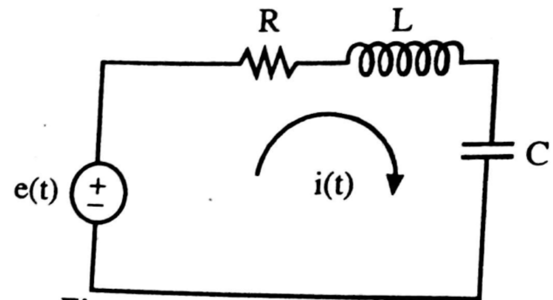


Figure 3.1 A simple RLC circuit

The differential equations describing the dynamics of the circuit are

$$e(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i(t)d(t)$$

Taking Laplace transform of the above equation with all initial conditions set to zero, we get

$$E(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s)$$

The transfer function of the given system is given by the ratio of current $I(s)$ in the circuit to the emf $E(s)$ in the circuit.

$$\therefore \quad \frac{I(s)}{E(s)} = \frac{sC}{s^2LC + sRC + 1}$$

Phase Lag Network

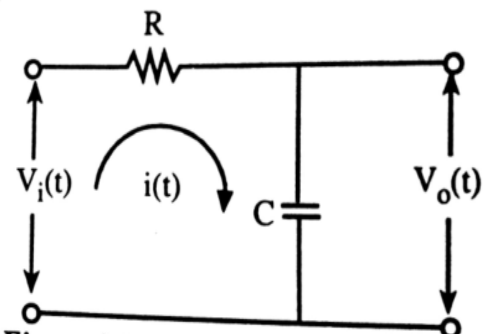


Figure 3.2 A simple phase lag network

Applying KVL across input, we get

$$V_i(t) = Ri(t) + \frac{1}{C}\int i(t)d(t)$$

Taking Laplace Transform of the above equation, we get

$$V_i(s) = \frac{1}{sC}I(s) + RI(s) \tag{3.5}$$

Applying KVL across output, we get

$$V_0(t) = \frac{1}{C}\int i(t)dt$$

Taking Laplace Transform of the above equation, we get

$$V_0(s) = \frac{1}{sC}I(s) \tag{3.6}$$

Thus, the transfer function of the phase lag network is the ratio of output voltage $V_0(s)$ to the input voltage $V_i(s)$

$$\therefore \quad \frac{V_0(s)}{V_i(s)} = \frac{1}{1+sCR}$$

Phase Lead Network

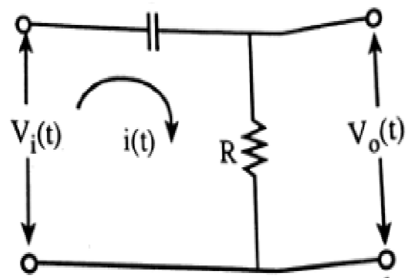


Figure 3.3 *A phase lead network*

Applying KVL across input, we get

$$V_i(t) = \frac{1}{C}\int i(t).dt + R.i(t)$$

Taking Laplace transform of the above equation, we get

$$V_i(s) = \frac{1}{sC} I(s) + RI(s)$$

Applying KVL across output, we get

$$V_0(t) = Ri(t)$$

Taking Laplace Transform of the above equation, we get

$$V_0(s) = RI(s) \tag{3.8}$$

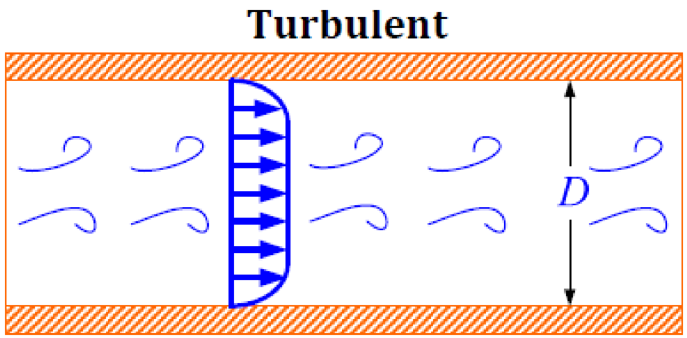
Thus, the transfer function of the phase lead network is the ratio of output voltage $V_o(s)$ to the input voltage $V_i(s)$

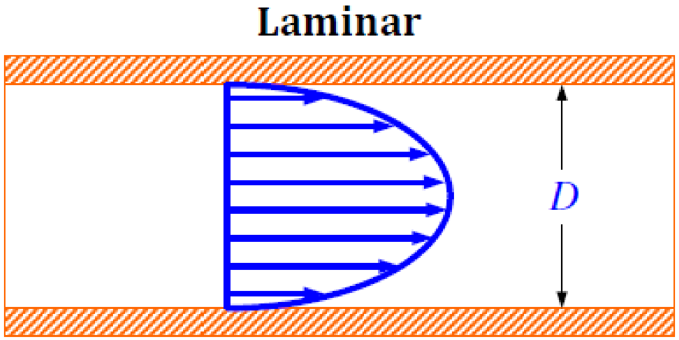
$$\therefore \quad \frac{V_0(s)}{V_i(s)} = \frac{sCR}{1+sCR}$$

Analogy between Mechanical and Electrical System

SN	Mechanical Translational System	Electrical System	
		Voltage Analogy	Current Analogy
1.	Force, $F(t)$	Voltage, $e(t)$	Current, $i(t)$
2.	Mass, M	Inductance, L	Capacitance, C
3.	Stiffness, K	Reciprocal of capacitance, $1/C$	Reciprocal of inductance, $1/L$
4.	Damping Coefficient, B	Resistance, R	Reciprocal of resistance, $1/R$
5.	Displacement, $x(t)$	Charge, $q(t)$	Flux Linkage, $\varphi(t)$

Laminar *vs* Turbulent Flow





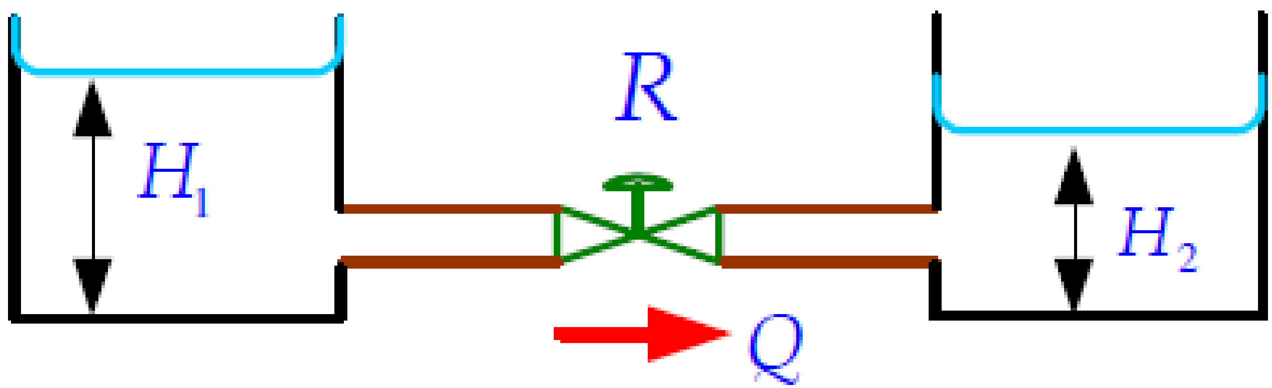
- Laminar Flow

-Laminar flow or streamline flow occurs when a fluid flows in parallel layers, with no disruption between the layers. At low velocities, the fluid tends to flow without lateral mixing.

- Turbulent Flow
  - When inertia forces dominate, the flow is called turbulent flow and is characterized by an irregular motion of the fluid.

Resistance of Liquid-Level Systems

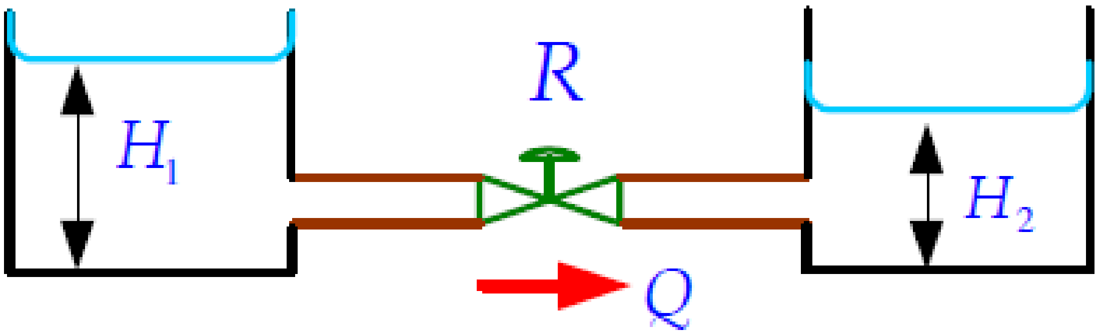
- Consider the flow through a short pipe connecting two tanks as shown in Figure.



- Where H_1 is the height (or level) of first tank, H_2 is the height of second tank, R is the resistance in flow of liquid and Q is the flow rate.

Resistance of Liquid-Level Systems

- The resistance for liquid flow in such a pipe is defined as the change in the level difference necessary to cause a unit change inflow rate.



$$Resistance = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate} = \frac{m}{m^3/s}$$

$$\boxed{R = \frac{\Delta(H_1 - H_2)}{\Delta Q} = \frac{m}{m^3/s}}$$

Resistance in Laminar Flow

- For laminar flow, the relationship between the steady-state flow rate and steady state height at the restriction is given by:

$$Q = k_l H$$

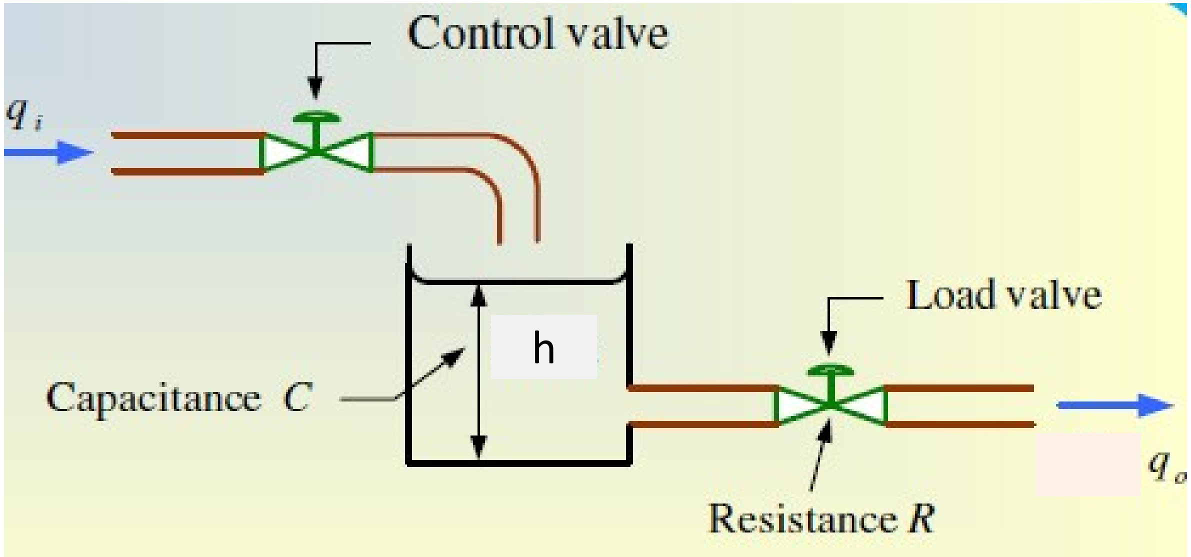
- Where Q = steady-state liquid flow rate in m^3/s
- *K_l = constant in m^2/s*
- *and H = steady-state* height in *m*.
- The resistance R_ℓ is

$$R_l = \frac{dH}{dQ} \qquad R = \frac{\text{change in level difference, m}}{\text{change in flow rate, m}^3/\text{sec}}$$

The laminar flow resistance is constant and is analogous to the electrical resistance

Capacitance of Liquid-Level Systems

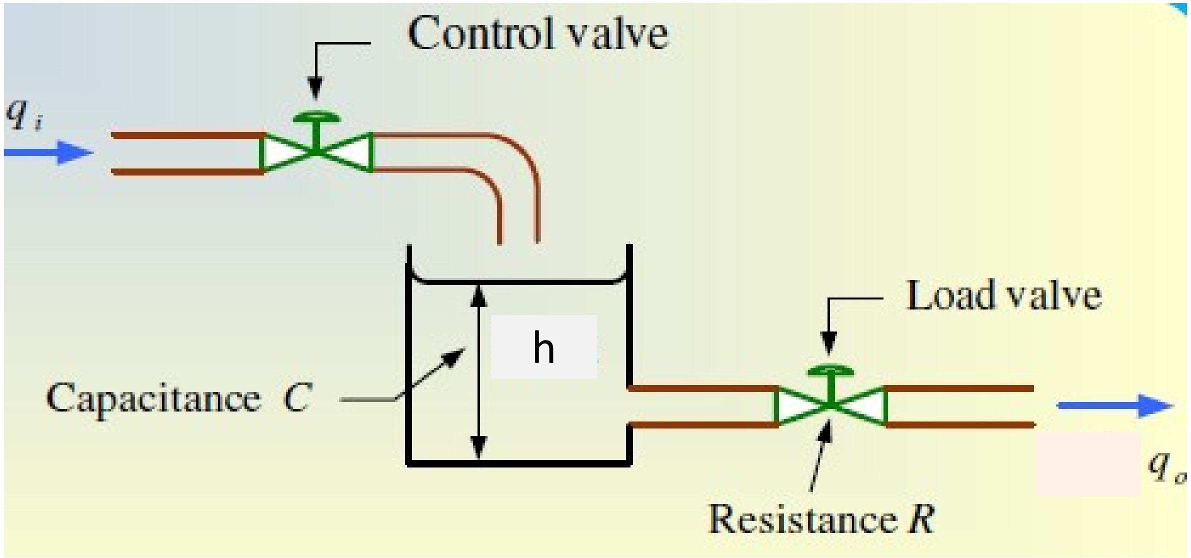
- The capacitance of a tank is defined to be the change in quantity of stored liquid necessary to cause a unity change in the height.



$$Capacitance = \frac{change\ in\ liquid\ stored}{change\ in\ height} = \frac{m^3}{m}\ or\ m^2$$

- Capacitance (C) is cross sectional area (A) of the tank.

Capacitance of Liquid-Level Systems

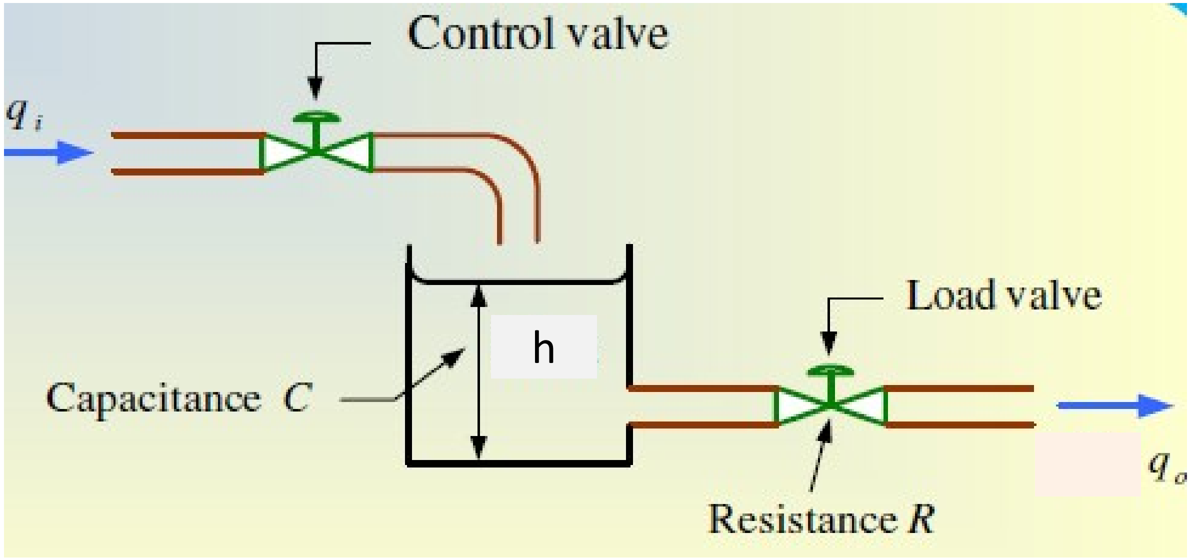


$$Rate\ of\ \ change\ \ of\ \ fluid\ volume\ in\ the\ tank = flow\,in - flow\,out$$

$$\frac{dV}{dt} = q_i - q_o$$

$$\frac{d(A \times h)}{dt} = q_i - q_o$$

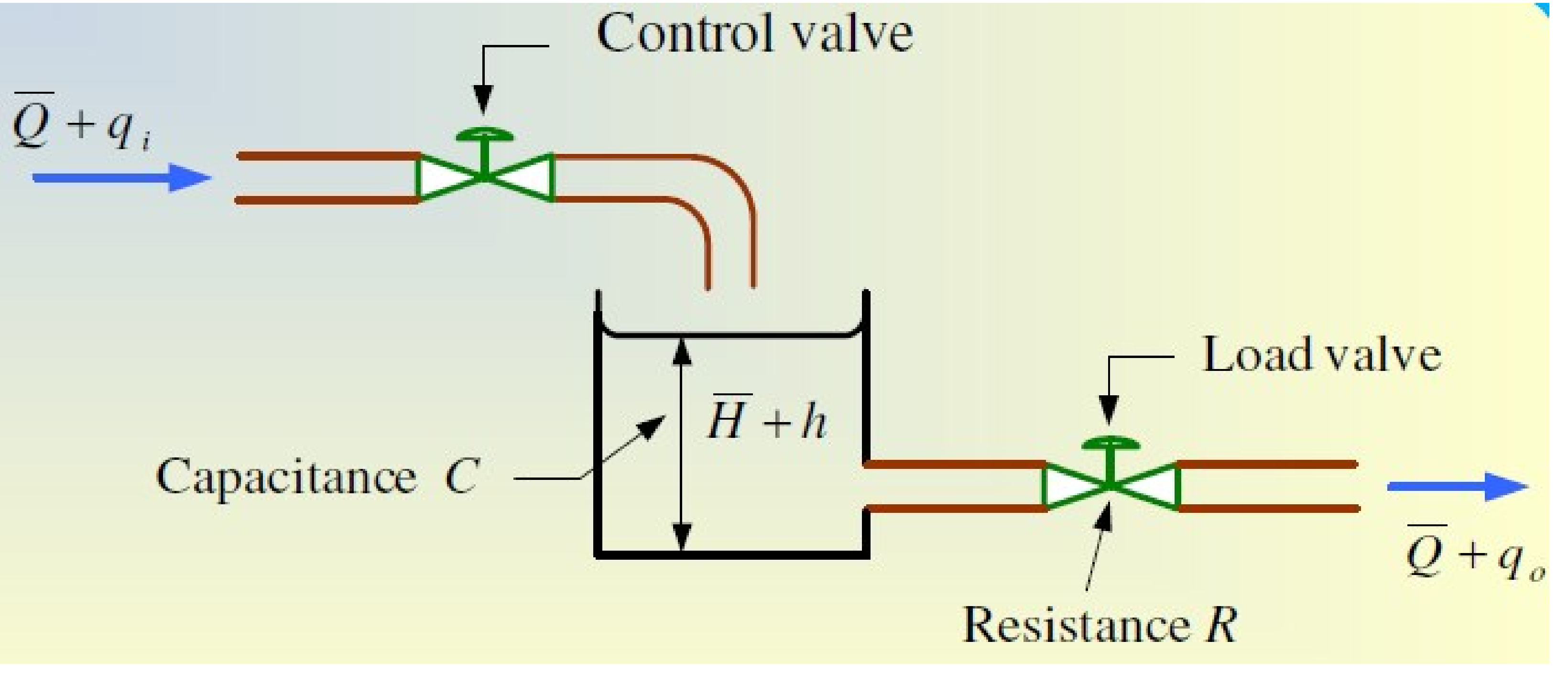
Capacitance of Liquid-Level Systems



$$A\frac{dh}{dt} = q_i - q_o$$

$$\boxed{C\frac{dh}{dt} = q_i - q_o}$$

Modelling Example#1



$\bar{H} =$ steady-state head (before any change has occurred), m.
$h =$ small deviation of head from its steady-state value, m.
$\bar{Q} =$ steady-state flow rate (before any change has occurred), m^3/s.
$q_i =$ small deviation of inflow rate from its steady-state value, m^3/s.
$q_o =$ small deviation of outflow rate from its steady-state value, m^3/s.

Modelling Example#1

- The rate of change in liquid stored in the tank is equal to the flow in minus flow out.

$$C\frac{dh}{dt} = q_i - q_o \longrightarrow \quad (1)$$

- The resistance *R* may be written as

$$R = \frac{dH}{dQ} = \frac{h}{q_0} \longrightarrow \quad (2)$$

- Rearranging equation (2)

$$q_0 = \frac{h}{R} \longrightarrow \quad (3)$$

Modelling Example#1

$$C\frac{dh}{dt} = q_i - q_o \longrightarrow \quad (1) \qquad q_0 = \frac{h}{R} \longrightarrow \quad (3)$$

- Substitute q_o in equation (3) to equation (1)

$$C\frac{dh}{dt} = q_i - \frac{h}{R}$$

- After simplifying above equation

$$RC\frac{dh}{dt} + h = Rq_i$$

- Taking Laplace transform considering initial conditions to zero

$$RCsH(s) + H(s) = RQ_i(s)$$

Modelling Example#1

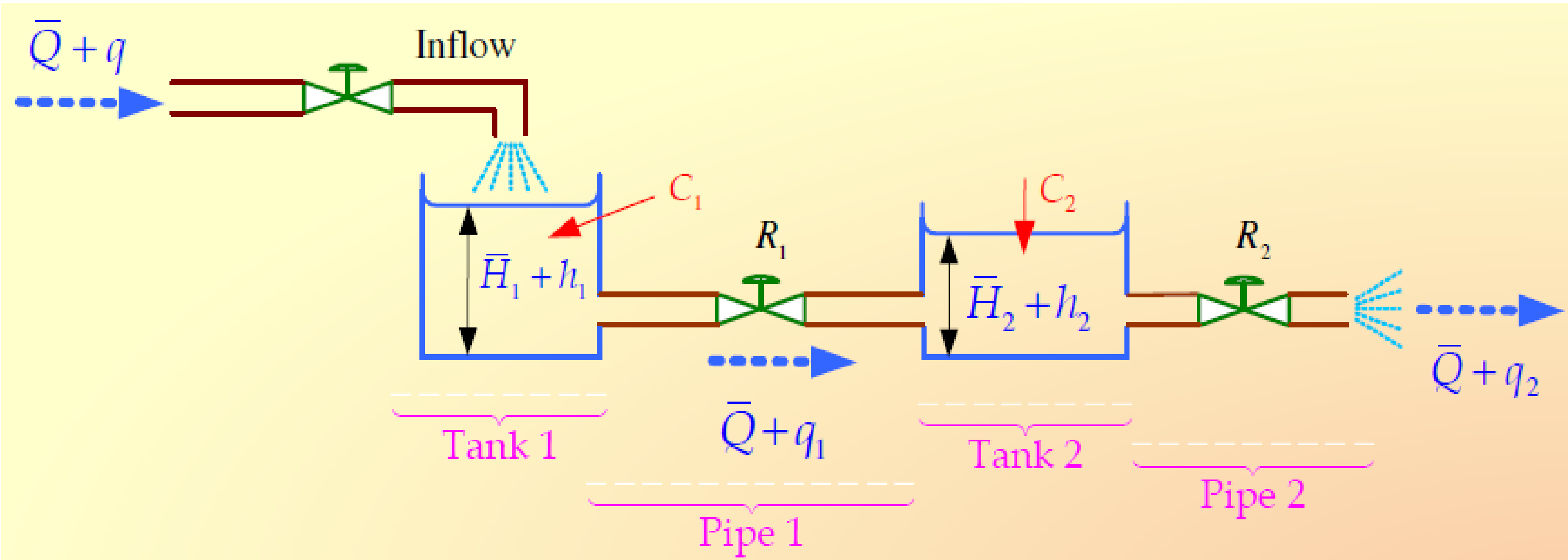
$$RCsH(s) + H(s) = RQ_i(s)$$

- The transfer function can be obtained as

$$\boxed{\frac{H(s)}{Q_i(s)} = \frac{R}{(RCs + 1)}}$$

Modelling Example#2

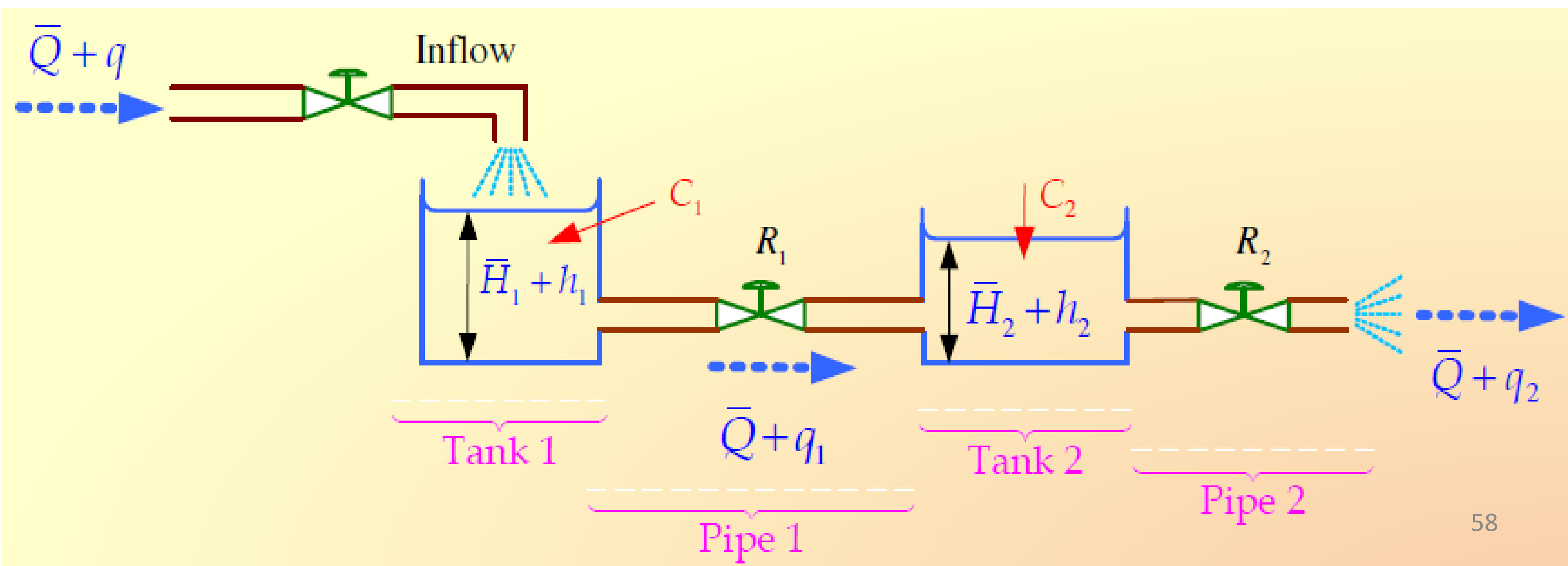
- Consider the liquid level system shown in following Figure. In this system, two tanks interact. Find transfer function $Q_2(s)/Q(s)$.



Modelling Example#2

- Tank 1 $$C_1 \frac{dh_1}{dt} = q - q_1$$ Pipe 1 $$R_1 = \frac{h_1 - h_2}{q_1}$$

- Tank 2 $$C_2 \frac{dh_2}{dt} = q_1 - q_2$$ Pipe 2 $$R_2 = \frac{h_2}{q_2}$$



Modelling Example#2

- Tank 1 $$C_1 \frac{dh_1}{dt} = q - \frac{h_1 - h_2}{R_1}$$ Pipe 1 $$q_1 = \frac{h_1 - h_2}{R_1}$$

- Tank 2 $$C_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$ Pipe 2 $$q_2 = \frac{h_2}{R_2}$$

- Re-arranging above equation

$$C_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = q + \frac{h_2}{R_1}$$

$$C_2 \frac{dh_2}{dt} + \frac{h_2}{R_1} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

Modelling Example#2

$$C_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = q + \frac{h_2}{R_1} \qquad C_2 \frac{dh_2}{dt} + \frac{h_2}{R_1} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

- Taking LT of both equations considering initial conditions to zero [i.e. $h_1(0)=h_2(0)=0$].

$$\left(C_1 s + \frac{1}{R_1}\right) H_1(s) = Q(s) + \frac{1}{R_1} H_2(s) \tag{1}$$

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1} H_1(s) \tag{2}$$

Modelling Example#2

$$\left(C_1 s + \frac{1}{R_1}\right)H_1(s) = Q(s) + \frac{1}{R_1}H_2(s) \quad (1)$$

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right)H_2(s) = \frac{1}{R_1}H_1(s) \quad (2)$$

- From Equation (1)

$$H_1(s) = \frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}$$

- Substitute the expression of $H_1(s)$ into Equation (2), we get

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right)H_2(s) = \frac{1}{R_1}\left(\frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}\right)$$

Modelling Example#2

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1}\left(\frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}\right)$$

- Using $H_2(s) = R_2 Q_2(s)$ in the above equation

$$\left[(R_2 C_2 s + 1)(R_1 C_1 s + 1) + R_2 C_1 s\right] Q_2(s) = Q(s)$$

$$\boxed{\frac{Q_2(s)}{Q(s)} = \frac{1}{R_2 C_1 R_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1}}$$

Assignment Modelling Example#3

- Write down the system differential equations.