Advanced Calculus I

- single variable real analysis

Real numbers ($\mathbb{R}$)

- $(\mathbb{R}, +, \cdot)$ is a field.
- $(\mathbb{R}, \geq)$ is a partially ordered set.

Absolute value

- $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- $|x-a|$ is the distance between x and a.

Properties of the absolute value

1. $|x| \geq 0$
2. $|x| = 0 \Leftrightarrow x = 0$
3. $|xy| = |x| \cdot |y|$
   $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \neq 0$
4. $|x+y| \leq |x| + |y|$
   $|x| - |y| \leq |x-y|$
5. If $a > 0$ then
   $|x| < a \Leftrightarrow -a < x < a$
   $|x| > a \Leftrightarrow x > a$ or $x < -a$

Other consequences

1. $|-x| = |x|$
2. If $a > 0$ then
   $|x-b| < a \Leftrightarrow b - a < x < b + a$
3. $|x-y| = 0 \Leftrightarrow x = y$
4. $\big||x| - |y|\big| \leq |x-y|$

Supremum and Infimum

Definition. Let $S \subseteq \mathbb{R}$ and $u, v \in \mathbb{R}$.

1. u is an upper bound of S if for all $s \in S$, $s \leq u$
2. v is a lower bound of S if for all $s \in S$, $s \geq v$

Definition.

1. If S has an upper [lower] bound, then S is said to be bounded above [below].
2. If S is bounded above and below, then S is said to be bounded.

Remark.

1. S is bounded above if
   $(\exists u \in \mathbb{R})(\forall s \in S)(s \leq u)$
   S is bounded below if
   $(\exists v \in \mathbb{R})(\forall s \in S)(s \geq v)$
2. S is bounded
   $\Leftrightarrow (\exists u, v \in \mathbb{R})(\forall s \in S)(v \leq s \leq u)$
   $\Leftrightarrow (\exists M > 0)(\forall s \in S)(|s| \leq M)$

Definition. Let $S \subseteq \mathbb{R}$ and $u, v \in \mathbb{R}$.

1. u is the supremum (or least upper bound) of S if:
   (a) u is an upper bound of S
   (b) and for all upper bounds d of S, $u \leq d$.
2. v is the infimum (or greatest lower bound) of S if:
   (a) v is a lower bound of S
   (b) and for all lower bounds b of S, $v \geq b$.

Remark.

1. Notation:
   $\sup S = u$
   $\inf S = v$
2. The supremum and infimum of S are not necessarily in S.
3. Since $\varnothing$ is bounded above and below by any $a \in \mathbb{R}$, $\varnothing$ has neither a supremum nor an infimum.
4. S is not bounded above implies that S has no supremum.
   S is not bounded below implies that S has no infimum.

Theorem.
Let $S \subseteq \mathbb{R}$. If a supremum [infimum] exists, then it is unique.
Proof.
Suppose that u and v are suprema of S. For the sake of contradiction, assume that $u \neq v$. Without loss of generality, assume that $u < v$. By definition of supremum, u is an upper bound of S. Also by definition of supremum, $v \leq d$ for any upper bound d of S. Taking $d = u$, we get $v \leq u$. Then $u < v \leq u$, which is absurd. Thus, $u = v$. $\square$

Theorem.
Let u be an upper bound of a non-empty set $S \subseteq \mathbb{R}$. Then the following are equivalent:

1. $\sup S = u$
2. $(\forall x \in \mathbb{R})(x < u \Rightarrow (\exists s \in S)(x < s))$
3. $(\forall \varepsilon > 0)(\exists s \in S)(u - \varepsilon < s)$

Proof.
[(1) $\Rightarrow$ (2)] Suppose that $\sup S = u$. Let $x < u$. Since u is the least upper bound then any number less than u is not an upper bound. In particular, x is not an upper bound. Thus, there exists $s \in S$ such that $x < s$.
[(2) $\Rightarrow$ (3)] Suppose that $(\forall x \in \mathbb{R})(x < u \Rightarrow (\exists s \in S)(x < s))$. Let $\varepsilon > 0$. Take $x = u - \varepsilon < u$. Thus,

there exists $s \in S$ such that $x < s$ so that $u - \varepsilon < s$.
[(3) $\Rightarrow$ (2)] Suppose that $(\forall \varepsilon > 0)(\exists s \in S)(u - \varepsilon < s)$. For the sake of contradiction, suppose that u is not the supremum of S. Thus, there exists w such that w is an upper bound of S and $w < u$. Take $\varepsilon = u - w > 0$. Then, there exists s such that $u - \varepsilon < s$ so that $w < s$. Thus, w is not an upper bound, which is a contradiction. $\square$

Theorem.
Let v be a lower bound of a non-empty set $S \subseteq \mathbb{R}$. Then the following are equivalent:

1. $\inf S = v$
2. $(\forall x \in \mathbb{R})(x > v \Rightarrow (\exists s \in S)(x > s)$
3. $(\forall \varepsilon > 0)(\exists s \in S)(v + \varepsilon > s)$

Proof.
Left as an exercise. $\square$

Definition.
Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$.

1. $a + S = \{a + s \mid s \in S\}$
2. $-S = \{-s \mid s \in S\}$

Theorem.
Let $S \subseteq \mathbb{R}$ and $a \in \mathbb{R}$.

1. If S is bounded above then $\sup(a + S) = a + \sup S$.
2. If S is bounded below then $\inf(a+S) = a + \inf S$.
3. If S is counded then
   (a) $\inf(-S) = -\sup S$
   (b) $\sup(-S) = -\inf S$

Proof.
Let $S \subseteq \mathbb{R}$ and $a \in \mathbb{R}$.

1. Let $u = \sup S$ and $v = \sup(a + S)$. Since $u = \sup S$, we know that u is an upper bound of S. Hence, for all $s \in S$, $s \leq u$. Thus, for all $s \in S$, $a + s \leq u + a$. Then, $u + a$ is an upper bound of S. Therefore, since v is the least upper bound of S, $v \leq u + a$. Since $v = \sup(a + S)$, we know that v is an upper bound of $a + S$. Hence, for all $s \in S$, $a + s \leq v$ so that $s \leq v - a$ for all $s \in S$. Thus, v is an upper bound of S. Since u is the least upper bound of S, we know that $u \leq v - a$ so that $u + a \leq v$.
2. Exercise.
3. Exercise. $\square$

The Completeness Axiom of $\mathbb{R}$

- every non-empty subset of $\mathbb{R}$ that has an upper [lower] bound has a supremum [infimum].

Theorem. Archimedean Property.

1. For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.
2. For every $y > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.

Proof.

1. Suppose, for the sake of contradiction, that there exists $x \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $x \geq n$. Thus, x is an upper bound of $\mathbb{N}$. Also, $\mathbb{N}$ is clearly non-empty. Thus, by the completeness axiom of $\mathbb{R}$, there exists $u \in \mathbb{R}$ such that $u = \sup \mathbb{N}$. Since $1 > 0$, by a previous theorem, we know that there exists $n \in \mathbb{N}$ such that $u - 1 < n$ so that $u < n + 1 \in \mathbb{N}$. Thus, u is not an upper bound of $\mathbb{N}$ so that u is not a supremum of $\mathbb{N}$, which is a contradiction.
2. Let $y > 0$. Take $x = 1/y$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{y} = x < n$ so that $\frac{1}{n} < y$. $\square$

Corollary.
For any $y > 0$, there exists $n \in \mathbb{N}$ such that $n - 1 \leq y < n$.
Proof.
Let $y > 0$. Consider the set $S = \{n \in \mathbb{N} \mid y < n\}$. By the Archimedean Property, $S \neq \varnothing$. By the well-ordering principle, there exists $n \in S$ such that $n = \min S$ so that for all $m \in S$, $m \geq n$. Since $n \in S$, we know that $y < n$. From the fact that n is the least element of S, we get that $m < n$ implies that $m \notin S$ for any $m \in \mathbb{R}$. In particular, $n - 1 < n$ so that $n - 1 \notin S$. Thus, $n - 1 \leq y$. $\square$

Theorem. Density Theorem.
For every $x, y \in \mathbb{R}$ such that $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.
Proof.
Since $y - x > 0$, then by the Archimedean Property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. Thus, $1 < |ny - nx|$. This means that the distance of ny from nx is greater than 1 so that there exists $m \in \mathbb{Z}$ such that $nx < m < ny$ and thus, $x < \frac{m}{n} < y$. Taking $r = \frac{m}{n}$, we are done. $\square$

Corollary.
For every $x, y \in \mathbb{R}$ such that $x < y$, there exists $r' \in \mathbb{Q}^{\mathsf{c}}$ such that $x < r' < y$.
Proof.
Exercise. $\square$

Nested Interval Property.

Definition.
A set $S \subseteq \mathbb{R}$ that has at least two elements is an interval if for every $s, r \in S$ such that $r < s$, we have that $\{x \in \mathbb{R} \mid r < x < s\} \subseteq S$.

Definition.
A collection $(I_n)_{n=1}^{\infty}$ of intervals in $\mathbb{R}$ is said to be nested if $I_{j+1} \subset I_j$ for all $j \in \mathbb{N}$.

Theorem. Nested Interval Property.
Let the collection $([a_n, b_n])_{n=1}^{\infty}$ be nested. Then the following are true:

1. $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \varnothing$
2. $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$ implies $\exists! x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{x\}$

Proof.
Observe that since $([a_n, b_n])$ is nested, then for all $i, j \in \mathbb{N}$ such that $i \leq j$, we have that $a_i \leq a_j$ and $b_i \geq b_j$. And since each $[a_n, b_n]$ is an interval, we know that $a_n < b_n$ for all $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$, we have that $1 \leq n$ so that $b_n \leq b_1$ and thus, $a_n < b_1$.

1. Let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is non-empty and bounded above by b_1, then by the completeness axiom of $\mathbb{R}$, there exists $x \in \mathbb{R}$ such that $\sup A = x$ so that x is an upper bound of A. Thus, $a_n \leq x$ for any $n \in \mathbb{N}$. Let $m, n \in \mathbb{R}$. If $m \leq n$ then $a_m \leq a_n < b_n$. If $m > n$ then $a_m < b_m \leq b_n$. Thus, b_n is an upper bound of A. Since x is the least upper bound of A, we have that $x \leq b_n$.
2. Let $B = \{b_n \mid n \in \mathbb{N}\}$. Similar arguments show that if $y = \inf B$ then for any $m, n \in \mathbb{N}$, $a_m \leq y \leq b_n$. Assume $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$. Note that if $z \in \bigcap_{n=1}^{\infty}[a_n, b_n]$, then $x \leq z \leq y$. Otherwise, if for example, $z > y$, then there exists $n \in \mathbb{N}$ such that $b_n < z$ so that $z \notin [a_n, b_n]$. This implies that $\bigcap_{n=1}^{\infty}[a_n, b_n] \subseteq [x, y]$. Note that for all $n \in \mathbb{N}$, $b_n - a_n \geq y - x$ since $b_n \geq y$ and $a_n \leq x$. Thus, $y - x$ is a lower bound of $\{b_n - a_n \mid n \in \mathbb{N}\}$. Since $0 = \inf\{b_n - a_n \mid n \in \mathbb{N}\}$, we have that $y - x \leq 0$ so that $y \leq x$. Since $x \leq y$, then $x = y$. $\square$

Topology.

- study of properties of a space that is preserved under continuous deformations

Definition.
Let $\varepsilon > 0$ and $x \in \mathbb{R}$. The ε-neighborhood of x, denoted by $N_\varepsilon(x)$, is the set $N_\varepsilon(x) = \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$, also called the ball about x of radius ε.

Definition.

1. A set $G \subseteq \mathbb{R}$ is open if for all $x \in G$, there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq G$.
2. A set $F \subseteq \mathbb{R}$ is closed if F^{c} is open.

Theorem.

1. The arbitrary union of open sets and finite intersections of open sets are open.
2. The arbitrary intersection of closed sets and finite unions of closed sets are closed.

Proof.

1. Let G_α be open for all $\alpha \in \mathbb{R}$. Define $G = \bigcup_{\alpha \in \mathbb{R}} G_\alpha$. Let $x \in G$. Then, there exists $\alpha \in \mathbb{R}$ such that $x \in G_\alpha$. Since G_α is open, we know that there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq G_\alpha$. Thus, $N_\varepsilon(x) \subseteq G_\alpha \subseteq G$.
   Let $G_1, G_2, \ldots, G_n$ be open sets. Define $G = \bigcap_{i=1}^{n} G_i$. Let $x \in G$. Then, for $i = 1, 2, \ldots, n$, $x \in G_i$. Since G_i is open, there exists $\varepsilon_i > 0$ such that $N_{\varepsilon_i}(x) \subseteq G_i$. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. Then, for $i = 1, 2, \ldots, n$, we have that $N_\varepsilon(x) \subseteq N_{\varepsilon_i}(x)$ so that $N_\varepsilon(x) \subseteq G_i$. Thus, $N_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} G_i = G$.
2. Exercise. $\square$

Definition.
Let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. We say that x is a cluster point of A if for any $\varepsilon > 0$, we have that $(A \setminus \{x\}) \cap N_\varepsilon(x) \neq \varnothing$. We denote by A' the set of all cluster points of A.

Example. Proof is an exercise.

A	A'
$[a,b]$, (a,b)	$[a,b]$
finite	$\varnothing$
$\mathbb{N}$	$\varnothing$
$\mathbb{Q}$, $\mathbb{Q}^{\mathsf{c}}$	$\mathbb{R}$

Trivia.
G is dense in F if $G' = F$.

Proof of $\mathbb{Q}' = \mathbb{R}$.
Let $x \in \mathbb{R}$. Let $\varepsilon > 0$. Since $x - \varepsilon < x$, we know by the density theorem that there exists $y \in \mathbb{Q}$ such that $x - \varepsilon < y < x$ so that $y \neq x$. Thus, $x - y < \varepsilon$ so that $|x - y| < \varepsilon$. □

Theorem.
A set $F \subseteq \mathbb{R}$ is closed if and only if it contains all of its cluster points.
Proof.
Let $F \subseteq \mathbb{R}$ be closed. Let $x \in F'$. Suppose $x \notin F$. Then, $x \in F^{\mathsf{c}}$. Note that F^{c} is open. Hence, there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq F^{\mathsf{c}}$. Thus, no element of F is in $N_\varepsilon(x)$ so that x is not a cluster point, which is a contradiction.
Let F contain all of its cluster points. Let $x \in F^{\mathsf{c}}$. Then, x is not a cluster point of F. Thus, there exists $\varepsilon > 0$ such that $(F \setminus \{x\}) \cap N_\varepsilon(x) = \varnothing$. Since $x \notin F$, we get that $F \setminus \{x\} = F$ so that $F \cap N_\varepsilon(x) = \varnothing$. Therefore, $N_\varepsilon(x) \subseteq F^{\mathsf{c}}$. □

Definition.
The closure of A, denoted by $\bar{A}$, is the smallest closed set containing A. We state the following facts without proof:

1. $\bar{A} = A \cup A'$
2. A is closed if and only if $\bar{A} = A$

Compact Sets.

Definition.
Let $A \subseteq \mathbb{R}$. An open cover of A is a collection $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{R}}$ of open setssuch that $A \subseteq \bigcup_{\alpha \in \mathbb{R}} G_\alpha$.
A subcover of $\mathcal{G}$ is a subcollection $\{G'_\alpha\}_{\alpha \in \mathbb{R}}$ such that $A \subseteq \bigcup_{\alpha \in \mathbb{R}} G'_\alpha$.

Definition.
A set $K \subseteq \mathbb{R}$ is compact if every open cover of K has a finite subcover.

Lemma.
Let $n \in \mathbb{N}$. Then $n \leq 2^n$.
Proof.
Let $n \in \mathbb{N}$. If $n = 1$, then $n = 1 < 2 = 2^1 = 2^n$. Suppose that $n \leq 2^n$. Then $n + 1 \leq n + n = 2n \leq 2(2^n) = 2^{n+1}$. Thus, by the principle of mathematical induction, $n \leq 2^n$ for any $n \in \mathbb{N}$.

Theorem.
$[a,b]$ is compact.
Proof.
Suppose, for the sake of contradiction, that $\mathcal{G} = \{G_\alpha\}$ is an open cover of $[a,b]$ that does not have a finite subcover.
Consider the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. Since $[a,b]$ does not have a finite subcover, then one of the two is also not finitely covered, say I_1.
Bisecting I_1, we again can conclude that one of these two subintervals will not be finitely covered, say I_2.
Continuing this, we obtain a sequence of intervals $\{I_n\}_{n=1}^\infty$ such that
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$
Clearly, the sequence is a nested set of closed and bounded intervals.
Suppose that $I_n = [a_n, b_n]$. If $n = 1$, then $b_1 - a_1 = \frac{b-a}{2}$. If $b_n - a_n = \frac{b-a}{2^n}$ then $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$. So by the principle of mathematical induction, $b_n - a_n = \frac{b-a}{2^n}$ for any $n \in \mathbb{N}$.
Consider the set $A = \{b_n - a_n \mid n \in \mathbb{N}\} = \{\frac{b-a}{2^n} \mid n \in \mathbb{N}\}$. Note that $\frac{b-a}{2^n} > 0$ for any $n \in \mathbb{N}$ so that 0 is a lower bound of A. Let $\varepsilon > 0$. By the archimedean property, there exists $r \in \mathbb{N}$ such that $\frac{1}{r} < \varepsilon$. Also by the archimedean property, there exists $s \in \mathbb{N}$ such that $r(b-a) < s$ so that $\frac{b-a}{s} < \frac{1}{r}$. Taking $n = s$, we get $\frac{b-a}{2^n} \leq \frac{b-a}{s} < \frac{1}{r} < \varepsilon$. Therefore, $\inf A = 0$.
By the nested interval property, $\exists! x \in \mathbb{R}$ such that $\bigcap_{n=1}^\infty I_n = \{x\}$. Since $x \in [a,b]$, there exists α such that $x \in G_\alpha$. Since G_α is open, there exists ε such that $N_\varepsilon(x) \subseteq G$. Choose N large enough such that $\frac{b-a}{2^N} < \varepsilon$. Then, $I_N \subseteq N_\varepsilon(x)$ so that I_N is finitely

covered, which is a contradiction. □

Theorem. Heine Borel Theorem.
A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.
Proof.
Suppose K is compact.
Consider $\mathcal{G} = \{G_n\}$ where $G_n = (-n, n)$. Clearly, $\mathcal{G}$ covers $\mathbb{R}$. In particular, it covers K. Since K is compact, there exists a finite subcover $\mathcal{G}' = \{G_{m_1}, G_{m_2}, \cdots, G_{m_n}\}$. Take $M = \max\{m_1, m_2, \cdots, m_n\}$. Thus, $\bigcup_{i=1}^{n} G_{m_i} = (-M, M) \supseteq K$, so that K is bounded.
Let $x \in K^{\mathsf{c}}$. Take $\mathcal{G} = \{G_n\}$, where $G_n = \{y \mid |x - y| > \frac{1}{n}\}$. Then, $\bigcup_{n=1}^{\infty} G_n = \mathbb{R} \setminus \{x\} \supseteq K$. Since K is compact, there exists a finite subcover $\mathcal{G}' = \{G_{m_1}, G_{m_2}, \cdots, G_{m_n}\}$. Take $M = \max\{m_1, m_2, \cdots, m_n\}$. Thus, $\bigcup_{i=1}^{n} G_{m_i} = G_M \supseteq K$. Take $\varepsilon = \frac{1}{M}$ so that $N_\varepsilon(x) \subseteq G_M^{\mathsf{c}}$ and thus, $N_\varepsilon(x) \subseteq K^{\mathsf{c}}$.
Suppose K is closed and bounded. Let $\mathcal{G} = \{G_\alpha\}$ be an open cover for K. Since K is bounded, then there exists $M > 0$ such that $K \subseteq [-M, M]$. Since K is closed, K^{c} is open. Note that $\bigcup G_\alpha \supseteq K$ so that $\bigcup G_\alpha \cup K^{\mathsf{c}} \supseteq K \cup K^{\mathsf{c}} = \mathbb{R}$. Thus, $\mathcal{G} \cup \{K^{\mathsf{c}}\}$ is an open cover for $\mathbb{R}$ and in particular, $[-M, M]$. Since $[-M, M]$ is compact, there exists a finite subcover of $\mathcal{G} \cup \{K^{\mathsf{c}}\}$ so that there exists $\mathcal{G}' = \{G_{m_1}, G_{m_2}, \cdots, G_{m_n}\}$ such that $\bigcup_{i=1}^{n} G_{m_i} \cup K^{\mathsf{c}} \supseteq [-M, M] \supseteq K$. Thus, $\bigcup_{i=1}^{n} G_{m_i} \supseteq K$. □

Sequences in $\mathbb{R}$

Definition.
A sequence in $\mathbb{R}$ is a function defined on the set $\mathbb{N}$ and whose range is contained in $\mathbb{R}$.

Definition.
Let (x_n) be a sequence of real numbers. We say that (x_n) approaches a real number x, denoted by $\lim x_n = x$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \in N$ and $n \geq N$, $|x_n - x| < \varepsilon$. In this case, we say that (x_n) is convergent. Otherwise, we say that (x_n) is divergent.

Remark.
1. $\lim x_n = x$
   $\Leftrightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then
   $$|x_n - x| < \varepsilon$$
   $\Leftrightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,
   $$|x_n - x| < \varepsilon$$
2. (x_n) is convergent $\Leftrightarrow \exists x \in \mathbb{R}$ such that $\lim x_n = x$.
   (x_n) is divergent $\Leftrightarrow \forall x \in \mathbb{R}$, $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $|x_n - x| \geq \varepsilon$.
3. $\lim x_n = x$ implies that there are only a finite number of terms outside of $N_\varepsilon(x)$.
4. If N satisfies our definition, then any $N_0 > N$ will also satisfy our definition.

Example.
1. $\lim c = c$.
   Proof.
   Let $\varepsilon > 0$. Choose $N = 1$. Thus if $n \geq N$ then $|x_n - x| = |c - c| = 0 < \varepsilon$.
2. $\lim \frac{1}{n} = 0$.
   Proof.
   Let $\varepsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Thus if $n \geq N$ then $|x_n - x| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.
3. $\lim \frac{n^2-5}{n^2+5} = 1$.
   Proof.
   Let $\varepsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{10}$. Thus if $n \geq N$ then $|x_n - x| = |\frac{n^2-5}{n^2+5} - 1| = |\frac{-10}{n^2+5}| = \frac{10}{n^2+5} < \frac{10}{n^2} \leq \frac{10}{n} \leq \frac{10}{N} < \varepsilon$.
4. Let $\varepsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{10}{\varepsilon} + 5 < N$. Thus if $n \geq N$ then $|x_n - x| = |\frac{n^2+5}{n^2-5} - 1| = |\frac{10}{n^2-5}| = \frac{10}{n^2-5} \leq \frac{10}{N-5} < \varepsilon$.

Theorem.
If for all $\varepsilon > 0$, $0 \leq x < \varepsilon$, then $x = 0$.
Proof.
Let $\varepsilon > 0$. Suppose that $0 \leq x < \varepsilon$. Then $0 \leq x$. Suppose, for the sake of contradiction, that $x \neq 0$. Then $0 < x$. Take $\varepsilon = x$. Then $0 \leq x < x$ so that $x < x$, which is a contradiction. Thus, $x = 0$. □

Corollary.

If $|x-y| < \varepsilon$ for all $\varepsilon > 0$, then $x = y$.
Proof.
Exercise. $\square$

Theorem.
If $\lim x_n$ exists, then it is unique.
Proof.
Suppose that x and y are limits of x_n. Let $\varepsilon > 0$. Since $\lim x_n = x$, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. Since $\lim x_n = y$, there exists $N_2 \in \mathbb{N}$ such that $|x_n - y| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Then $|x_N - x| < \frac{\varepsilon}{2}$ so that $-\frac{\varepsilon}{2} < x_N - x < \frac{\varepsilon}{2}$. Also, $|y - x_N| = |x_N - y| < \frac{\varepsilon}{2}$ so that $-\frac{\varepsilon}{2} < y - x_N < \frac{\varepsilon}{2}$. Hence, $-\varepsilon < x_N - x + y - x_N < \varepsilon$ so that $|y - x| < \varepsilon$. Thus, by the previous corollary, $x = y$. $\square$

Theorem.
If (x_n) is convergent then $\{x_n \mid n \in \mathbb{N}\}$ is bounded.
Proof.
Let $\lim x_n = x$. Then there exists $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq N$. Then $|x_n| - |x| \leq |x_n - x| < 1$ so that $|x_n| < 1 + |x|$ for all $n \geq N$. Take $M = \max\{1+|x|, |x_1|, |x_2|, \ldots, |x_{N-1}|\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$, so that $\{x_n \mid n \in \mathbb{N}\}$ is bounded. $\square$

Definition.
Let (x_n) and (y_n) be sequences.
1. $(x_n) + (y_n) := (x_n + y_n)$
2. $(x_n)(y_n) := (x_n y_n)$
3. $\frac{(x_n)}{(y_n)} := (\frac{x_n}{y_n})$, $y_n \neq 0$

Theorem.
Let (x_n) and (y_n) be convergent sequences with $\lim x_n = x$ and $\lim y_n = y$. Then the following are true.
1. $\lim x_n + y_n = x + y$
2. $\lim x_n y_n = xy$
3. $\lim \frac{1}{x_n} = \frac{1}{x}$, provided that $x \neq 0$ and $x_n \neq 0$ $\forall n \in \mathbb{N}$

Proof.
1. Let $\varepsilon > 0$. Since $\lim x_n = x$, there exists $N_1 \in \mathbb{N}$ such $|x_n - x| < \frac{\varepsilon}{2}$ for any $n \geq N_1$. Since $\lim y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2}$ for any $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Thus if $n \geq N$, then $|(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - 2)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
2. Let $\varepsilon > 0$. Since (y_n) is convergent, there exists $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{N}$. Take $K = \max\{M, |x|\}$. Since $\lim x_n = x$, then there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2K}$ for any $n \geq N_1$. Since $\lim y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2K}$ for any $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Thus if $n \geq N$, then $|x_n y_n - xy| = |(x_n y_n - x y_n) + (x y_n - xy)| \leq |y_n(x_n - x)| + |x(y_n - y)| = |y_n| \cdot |x_n - x| + |x| \cdot |y_n - y| < K|x_n - x| + K|y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
3. Let $\varepsilon > 0$. Since $\lim x_n = x$ and $x \neq 0$, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{|x|}{2}$ for any $n \geq N_1$. Hence, $|x| - |x_n| < \frac{|x|}{2}$ so that $\frac{|x|}{2} < |x_n|$. And since $\lim x_n = x$, there exists $N_2 \in \mathbb{N}$ such that $|x_n - x| < \frac{|x|^2}{2}$ for any $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Thus if $n \geq N$, then $|\frac{1}{x_n} - \frac{1}{x}| = |\frac{x - x_n}{x_n x}| = \frac{|x_n - x|}{|x_n| \cdot |x|} < \frac{\frac{x}{2}^2 \cdot \varepsilon}{\frac{|x|}{2} \cdot |x|} = \varepsilon$. $\square$

Theorem.
If (x_n) diverges and (y_n) converges then $(x_n + y_n)$ diverges.
Proof.
Suppose, for the sake of contradiction, that $(x_n + y_n)$ converges. Consider $(z_n) := (x_n + y_n) + (-1)(y_n)$, then this sequence converges by the previous theorem. Since $(z_n) = (y_n)$, we have a contradiction. $\square$

Theorem.
Let (x_n) and (y_n) be convergent sequences. Then the following are true.
1. If $x_n \geq 0$ for any $n \in \mathbb{N}$, then $\lim x_n \geq 0$.
2. If $x_n \geq y_n$ for any $n \in \mathbb{N}$, then $\lim x_n \geq \lim y_n$.

Proof.
1. Let $\lim x_n = x$. Suppose, for the sake of contradiction, that $x < 0$. Then $-x > 0$, so that there exists $N \in \mathbb{N}$ such that $|x_n - x| < -x$ for any $n \geq N$. In particular, $|x_N - x| < -x$. But $|x_N - x| = x_N - x$, so that $x_N - x < -x$. Hence, $x_N < 0$, which is a contradiction. Thus, $x \geq 0$.
2. Consider $(z_n) = (x_n) + (-1)(y_n)$. By a previous theorem, (z_n) is convergent. Also, $z_n =$

$x_n - y_n \geq 0$ for any $n \in \mathbb{N}$ so that $\lim z_n \geq 0$. But $\lim z_n = \lim x_n + -1 \cdot \lim y_n = \lim x_n - \lim y_n$. Hence, $\lim x_n - \lim y_n \geq 0$ so that $\lim x_n \geq \lim y_n$. $\square$

Theorem. Squeeze Theorem for Limits.
If (x_n), (y_n), and (z_n) are sequences satisfying

1. $x_n \leq y_n \leq z_n$ for any $n \in \mathbb{N}$ and
2. (x_n) and (z_n) are convergent with $\lim x_n = x$ and $\lim z_n = x$,

then $\lim y_n = x$.
Proof.
Let $\varepsilon > 0$. Since $\lim x_n = x$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$,

$$|x_n - x| < \varepsilon$$
$$\Leftrightarrow -\varepsilon < x_n - x < \varepsilon.$$

Since $\lim z_n = x$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$,

$$|z_n - x| < \varepsilon$$
$$\Leftrightarrow -\varepsilon < z_n - x < \varepsilon.$$

Take $N = \max\{N_1, N_2\}$. Thus if $n \geq N$, then $-\varepsilon < x_n - x \leq y_n - x \leq z_n - x < \varepsilon$ so that $|y_n - x| < \varepsilon$. $\square$

Remark.
In general, we can assume that $\exists N \in \mathbb{N}$ such that $x_n \leq y_n \leq z_n$ $\forall n \geq N$.

Theorem. Monotone Subsequence Theorem.
Every sequence has a monotone subsequence.
Proof.
Let (x_n) be a subsequence. Let S be the set of peaks of (x_n), $S := \{x_m \mid m \leq n \Rightarrow x_m \geq x_n\}$.
Suppose S is infinite. Since for all m_n such that $x_{m_n} \in S$, then $x_{m_{n_1}} \geq x_{m_{n_2}}$ if $m_{n_1} \geq m_{n_2}$. Thus the subsequence (x_{m_n}) is decreasing.
Now, suppose that S is finite. Thus, there exists $N_1 \in \mathbb{N}$ such that $N_1 > m$ for any $x_m \in S$. So x_{N_1} is not a peak, so that there exists $N_2 > N_1$ such that $x_{N_1} < x_{N_2}$. So x_{N_2} is not a peak so that there exists N_3 such that $x_{N_2} < x_{N_3}$. Continuing this, we obtain a strictly increasing subseqence (x_{N_k}). $\square$

Theorem. Bolzano-Weierstrass Theorem.
Every bounded sequence has a convergent subsequence.
Proof.
Let (x_n) be a bounded sequence. By the monotone subsequence theorem, there exists (x_{n_k}) monotone subsequence of (x_n). Sinc (x_n) is bounded, then (x_{n_k}) is also bounded. By the monotone convergence theorem, (x_{n_k}) is convergent. $\square$

Theorem.
Let (x_n) be a bounded sequence. If every convergent subsequence of (x_n) converges to x, then (x_n) converges to x.
Proof.
Suppose, for the sake of contradiction, that (x_n) does not converge to x. Then there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - x| \geq \varepsilon$.

Claim.
There exists a subsequence (x_{n_k}) such that

$$|x_{n_k} - x| \geq \varepsilon.$$

Proof.
Indeed, if $N = 1$, there exists x_n such that

$$|x_{n_1} - x| \geq \varepsilon.$$

Also, if $N = n_1 + 1$, there exists x_{n_2} such that

$$|x_{n_2} - x| \geq \varepsilon.$$

Continuing this for $N = n_k + 1$, there exists $X_{n_{k+1}}$ such that

$$|x_{k+1} - x| \geq \varepsilon.$$

And we are done.

Since (x_n) is bounded, then (x_{n_k}) is bounded. Thus by the Bolzano-Weierstrass theorem, there is a convergent subsequence of (x_{n_k}), say (x'_{n_k}). By assumption, $\lim x'_{n_k} = x$. Thus there exists $N \in \mathbb{N}$ such that if $n_k \geq N$ then $|x'_{n_k} - x| < \varepsilon$. This is a contradiction since $|x'_{n_k} - x| \geq \varepsilon$ by construction of (x_{n_k}). $\square$

Definition.
Let (x_n) be a sequence. We say that y is a subsequential limit of (x_n) if there exists a subsequence (x_{n_k}) that converges to y.

Remark.

1. The sequence (x_n) will diverge if it has at least two distinct subsequential limits.
2. Sometimes called a limit point, but definitely not

a cluster point.

3. y is a subsequential limit of (x_n) if and only if for any $\varepsilon > 0$, $N_\varepsilon(y)$ has infinitely many terms of (x_n).

Definition.
Let (x_n) be a bounded sequence. Let $a_n = \inf\{x_m \mid m \geq n\}$ and $b_n = \sup\{x_m \mid m \geq n\}$.

1. The limit inferior of (x_n), denoted by $\liminf$, is given by $\lim a_n$.
2. The limit superior of (x_n), denoted by $\limsup$, is given by $\lim b_n$.

Remark.

1. The sequences (a_n) and (b_n) are monotone. In particular, (a_n) is increasing and (b_n) is decreasing.
2. If (x_n) is bounded, the $\liminf$ and $\limsup$ will always exist.

Remark.

1. The limit inferior and superior are subsequential limit points of (x_n).
2. Every subsequential limit y of (x_n) satisfies
$$\liminf x_n \leq y \leq \limsup x_n.$$
3. (x_n) converges to x if and only if $\liminf x_n = \limsup x_n = x$.

Cauchy Sequences.

Definition.
A sequence (x_n) is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then
$$|x_n - x_m| < \varepsilon.$$

Example.
$(\frac{1}{n})$ is Cauchy.
Proof.
Let $\varepsilon > 0$. Choose $N > \frac{2}{\varepsilon}$. If $n, m \geq N$, then
$$\begin{aligned} |x_n - x_m| &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} \\ &< \varepsilon. \ \square \end{aligned}$$

Theorem.
A Cauchy sequence is bounded.
Proof.
Let (x_n) be Cauchy. Then there exists $N \in \mathbb{N}$ such that if $n, m \in\geq N$,
$$|x_n - x_m| < 1.$$
In particular, for all $m \geq N$,
$$|x_N - x_m| < 1.$$
Hence $|x_m| < 1 + |x_N|$ for any $m \geq N$. Take $M = \max\{1+|x_n|, |x_1|, |x_2|, \ldots, |x_{N-1}|\}$. Thus for all $m \in \mathbb{N}$,
$$|x_m| \leq M$$
and thus (x_n) is bounded. $\square$

Theorem. Cauchy Criterion.
A sequence (x_n) converges if and only if it is Cauchy.
Proof.
Suppose that (x_n) converges. Let $\varepsilon > 0$. Since (x_n) converges, there exists $x \in \mathbb{R}$ such that $\lim x_n = x$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$,
$$|x_n - x| < \frac{\varepsilon}{2}.$$
Thus if $n, m \geq N$,
$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
Now, suppose that (x_n) is Cauchy. Then (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a subsequence (x_{n_k}) such that
$$\lim x_{n_k} = x$$
for some $x \in \mathbb{R}$. Let $\varepsilon > 0$. Since (x_n) is Cauchy,

there exists $N_1 \in \mathbb{N}$ such that if n, $m \geq N$,
$$|x_n - x_m| < \frac{\varepsilon}{2}.$$
Since $\lim x_{n_k} = x$, there exists $N_2 \in \mathbb{N}$ such that if $k \geq N_2$,
$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$
Let $k = \max\{N_1, N_2\}$. Thus if $n \geq N_1$,
$$\begin{aligned}|x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \square\end{aligned}$$

Example.
Let (x_n) be a sequence such that $|x_{n+1} - x_n| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Then (x_n) converges.
Proof.
Let $\varepsilon > 0$. Choose N such that $1 + log_2 \frac{1}{\varepsilon} < N$.
Thus if $n > m \geq N$, then
$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \cdots + x_{m+1} - x_m| \\ &< \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} \\ &< \frac{1}{2^m} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots \\ &= \frac{1}{2^{m-1}}. \ \square\end{aligned}$$

Theorem.
A monotone sequence is properly divergent if it is unbounded. In addition, if (x_n) is increasing then $\lim x_n = +\infty$ and if (x_n) is decreasing then $\lim x_n = -\infty$.
Proof.
Suppose that (x_n) is an increasing unbounded sequence.
Let $\alpha > 0$. Since (x_m) is unbounded, there exists $N \in \mathbb{N}$ such that
$$0 < \alpha < x_N.$$
Since (x_n) is increasing, if $n \geq N$ then $\alpha < x_N \leq x_n$.
The left is an exercise. $\square$

Theorem.
Let (x_n) and (y_n) be sequences such that for all $n \in \mathbb{N}$,
$$x_n \leq y_n.$$

1. If $\lim x_n = +\infty$ then $\lim y_n = +\infty$.
2. If $\lim y_n = -\infty$ then $\lim x_n = -\infty$.

Proof.

1. Let $\alpha > 0$. Since $\lim x_n = +\infty$, then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\alpha < x_n$. But $x_n \leq y_n$ for all $n \in \mathbb{N}$, in particular, for $n \geq N$. Hence, $\alpha < y_n$. Therefore, $\lim y_n = +\infty$.
2. Exercise. $\square$

Remark.
We can further generalize by only assuming that $x_m < y_n$ for all $n \leq K$ for some $K \in \mathbb{N}$.

Theorem.
Let (x_n) and (y_n) be positive sequences such that $\lim \frac{x_n}{y_n} = L$ for some $L > 0$. Then $\lim x_n = +\infty$ if and only if $\lim y_n = +\infty$.
Proof.
Since $\lim \frac{x_n}{y_n} = L$, there exists $K \in \mathbb{N}$ such that if $n \geq K$, then
$$\begin{aligned}&|\frac{x_n}{y_n} - L| < \frac{L}{2} \\ \Leftrightarrow\ &\frac{L}{2} < \frac{x_n}{y_n} < \frac{3L}{2} \\ \Leftrightarrow\ &\frac{L}{2} y_n < x_n < \frac{3L}{2} y_n.\end{aligned}$$
Let $x_n := \frac{L}{2} y_n$ and $z_n := \frac{3L}{2} y_n$. Note that it can be proven that
$$\begin{aligned}&\lim y_n = +\infty \\ \Leftrightarrow\ &\lim w_n = +\infty \\ \Leftrightarrow\ &\lim z_n = +\infty.\end{aligned}$$
By our previous theorem (with remark),
$$\begin{aligned}&\lim w_n = +\infty \\ \Rightarrow\ &\lim x_n = +\infty\end{aligned}$$
and
$$\begin{aligned}&\lim x_n = +\infty \\ \Rightarrow\ &\lim z_n = +\infty. \ \square\end{aligned}$$

Remark.
If (x_n) is properly divergent, then for any $\alpha > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$,
$$\alpha < |x_n|.$$

Exercise.
If (y_n) is properly divergent and (x_n) is a bounded sequence, then $\lim \frac{x_n}{y_n} = 0$.