Math 123.1
Homework 1
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1. Let $u \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ such that $S \neq \varnothing$. Suppose that for every $n \in \mathbb{N}$, $u + \frac{1}{n}$ is an upper bound of S and $u - \frac{1}{n}$ is an upper bound of S.
   Suppose that u is not an upper bound of S. Then $u < x$ for some $x \in S$. Let $\varepsilon = x - u > 0$. Then, by the archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. So $u + \frac{1}{n} < u + \varepsilon = x$, which is absurd since $u + \frac{1}{n}$ is an upper bound of S. Thus, u is an upper bound of S.
   Suppose that u is not the supremum of S. Then, there exists some upper bound u' of S such that $u' < u$. Let $\varepsilon' = u - u' > 0$. Then by the archimedean property, there exists $n' \in \mathbb{N}$ such that $\frac{1}{n'} < \varepsilon'$. Hence, $\frac{1}{n'} < u - u'$ so that $u' < u - \frac{1}{n'}$. But since $u - \frac{1}{n'}$ is not an upper bound of S, there exists $x' \in S$ such that $u - \frac{1}{n'} < x'$. Then $u' < x'$, which contradicts our assumption that u' is an upper bound of S. Therefore, u is the supremum of S, as required.
2. Lemma 1.
   Let R, S be sets. Suppose that R is closed and $S \subseteq R$. Then $\overline{S} \subseteq R$.
   Proof.
   Let R, S be sets. Suppose that R is closed and $S \subseteq R$. Let
   $$T := \{U \mid S \subseteq U, U \text{ is closed}\}.$$
   Then $\overline{S} = \bigcap T$ and $R \in T$. Hence, $\overline{S} \cap R = \overline{S}$ so that $\overline{S} \subseteq R$, as required.
   Lemma 2.
   $S \subseteq \overline{S}$ for any set S.
   Proof.
   Let S be a set. Let
   $$T := \{U \mid S \subseteq U, U \text{ is closed}\}.$$
   Suppose that $S \nsubseteq \overline{S}$. Then for some $x \in S$, $x \notin \overline{S}$. Since $S = \bigcap T$, then there exists $X \in T$ such that $x \notin X$. But $S \subseteq X$ and $x \in S$ so that $x \in X$. Clearly, this is a contradiction. Hence, $S \subseteq \overline{S}$, as required.

   Let A, B be sets. by Lemma 2, we have that $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Hence, $A \cup B \subseteq \overline{A} \cup \overline{B}$. Recall that the finite union and arbitrary intersection of closed sets are closed sets. Hence, both $\overline{A} = \bigcap\{U \mid A \subseteq U, U \text{ is closed}\}$ and $\overline{B} = \bigcap\{U \mid B \subseteq U, U \text{ is closed}\}$ are closed. So $\overline{A} \cup \overline{B}$ is a union of two closed sets and is thus also closed. Hence, by Lemma 1, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Note that $\overline{A \cup B} = \bigcap\{U \mid A \cup B \subseteq U, U \text{ is closed}\}$ is also an intersection of closed sets and is thus closed. Also, by Lemma 2, $A \cup B \subseteq \overline{A \cup B}$. But $A \subseteq A \cup B$ and $B \subseteq A \cup B$ so that $A \subseteq \overline{A \cup B}$ and $B \subseteq \overline{A \cup B}$. Hence, by Lemma 1, $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. Thus, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.
   Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, as required.
3. (a) Let $n \in \mathbb{N}$. So $4x_n(1 - x_{n+1}) > 1$ and $0 < x_n$. Hence, $1 - x_{n+1} > \frac{1}{4x_n}$ so that $x_{n+1} < 1 - \frac{1}{4x_n}$.

   Claim.
   $1 - \frac{1}{4y} \leq y$ for any $y > 0$.
   Proof.
   Let $y > 0$. So $\frac{1}{2y} - 1 \in \mathbb{R}$ so that by the trivial inequality, $(\frac{1}{2y} - 1)^2 \geq 0$. Hence, $\frac{1}{4y^2} - \frac{1}{y} + 1 \geq 0$. So $1 \geq \frac{1}{y} - \frac{1}{4y^2}$. Multiplying by y, we get $y \geq 1 - \frac{1}{4y}$, as required.

   In particular, $1 - \frac{1}{4x_n} \leq x_n$ so that $x_{n+1} < x_n$. Thus, (x_n) is decreasing. Therefore, (x_n) is monotone, as required.

   (b) Note that (x_n) is bounded above by 1 and bounded below by 0. Hence, (x_n) is bounded. So by the monotone convergence theorem, (x_n) converges. Then there exists $x \in \mathbb{R}$ such that $x = \lim x_n$. Then by a previous theorem, $x = \lim x_{n+1}$. Consider the sequence
   $$\begin{aligned}(z_n) &:= (4x_n(1 + (-1) \cdot x_{n+1})) \\ &= (4x_n(1 - x_{n+1})).\end{aligned}$$
   Note that (4), (x_n), (1), (-1), and (x_{n+1}) are all convergent sequences so that by a previous theorem, (z_n) converges and
   $$\begin{aligned}\lim z_n &= 4(\lim x_n)(1 + (-1) \cdot (\lim x_{n+1}) \\ &= 4x(1 + (-1) \cdot x) \\ &= 4x - 4x^2.\end{aligned}$$
   But $z_n = 4x_n(1 - x_{n+1}) > 1$ for all $n \in \mathbb{N}$ so that $\lim z_n \geq \lim 1$. Hence, $4x - 4x^2 \geq 1$. So $0 \geq 4x^2 - 4x + 1 = (2x - 1)^2$. But $2x - 1 \in \mathbb{R}$ so that by the trivial inequality, $0 \leq (2x - 1)^2$. Thus, $(2x - 1)^2 = 0$. Hence, $2x - 1 = 0$ so that $x = \frac{1}{2}$. Therefore, the limit of (x_n) is $\frac{1}{2}$.