1. Show that 1001 is divisible by 7, by 11, and by 13.

2. Find the quotient and remainder in the division algorithm when the following numbers are divided by 17.

   (a) 100

   (b) 289

   (c) -100

3. Prove the following:

   (a) If $a|b$ and $b|a$, then $a = \pm b$.

   (b) If $a|b$ and $c|d$, then $ac|bd$.

   (c) If $a|b$, then $a^k|b^k$ for all positive integers k.

4. Show that if a is an integer, then 3 divides $a^3 - a$.

5. Show that the fourth power of any integer is either of the form $5k$ or $5k+1$.

6. Verify that $2|a(a+1)$ and $3|a(a+1)(a+2)$ for any integer a.

7. Show that if a is odd, then $32|(a^2+3)(a^2+7)$.

8. Prove that if a is an integer not divisible by 2 or 3, then $24|(a^2+23)$.

9. Prove that the sum of the squares of two odd integers cannot be a perfect square.

Solutions.

1. Note that $1001 = 7 \cdot 143 = 11 \cdot 91 = 13 \cdot 77$ where $143, 91, 77 \in \mathbb{Z}$ so that 1001 is divisible by 7, by 11, and by 13.

2. (a) $100 = 17 \cdot 5 + 15$ so that the quotient and remainder when 100 is divided by 17 are 5 and 15, respectively.

   (b) $289 = 17 \cdot 17 + 0$ so that the quotient and remainder when 289 is divided by 17 are 17 and 0, respectively.

   (c) $-100 = 17 \cdot -6 + 2$ so that the quotient and remainder when -100 is divided by 17 are -6 and 2, respectively.

3. (a) Suppose that $a|b$ and $b|a$. Then, by definition, there exists $m, n \in \mathbb{Z}$ such that $b = am$ and $a = bn$. If $b = 0$, then $a = 0n = 0$. Note that $0 = \pm 0$ so that $a = \pm b$. Otherwise, assume that $b \neq 0$. Then, $b = (bn)m = b(nm)$ so that $1 = nm$. Since $m \in \mathbb{Z}$, then $n|1$ so that $n = \pm 1$. Thus, $a = b \cdot \pm 1 = \pm b$.

   (b) Suppose that $a|b$ and $c|d$. Then, by definition, there exists $m, n \in \mathbb{Z}$ such that $b = am$ and $d = cn$. Hence, $bd = (am)(cn) = ac(mn)$ where $mn \in \mathbb{Z}$. Thus, by definition, $ac|bd$.

   (c) Suppose that $a|b$. Then, by definition, there exists $m \in \mathbb{Z}$ such that $b = am$. Hence, $b^1 = a^1 m$ so that $a^1|b^1$. Let k be a positive integer. Suppose that $a^k|b^k$. Then, by definition, there exists $n \in \mathbb{Z}$ such that $b^k = a^k n$. Thus, $(b^k)(b) = (a^k n)(am)$. Hence, $b^{k+1} = a^{k+1}(mn)$ where $mn \in \mathbb{Z}$. Thus, by definition, $a^{k+1}|b^{k+1}$. By the principle of mathematical induction, $a^k|b^k$ for all positive integers k.

4. By the division algorithm, any integer a can be written in one of the following forms:

$$a = 3q,$$

$$a = 3q + 1, \text{ and}$$

$$a = 3q + 2$$

   where $q \in \mathbb{Z}$. If $a = 3q$ then $a^3 - a = 27q^3 - 3q = 3(9q^3 - q)$, where $9q^3 - q \in \mathbb{Z}$ so that $3|a^3 - a$. If $a = 3q+1$ then $a^3 - a = 27q^3 + 27q^2 + 9q + 1 - (3q+1) = 27q^3 + 27q^2 + 6q = 3(9q^3 + 9q^2 + 2q)$, where $9q^3 + 9q^2 + 2 \in \mathbb{Z}$ so that $3|a^3 - a$. If $a = 3q+2$ then $a^3 - a = 27q^3 + 54q^2 + 36q + 8 - (3q+2) = 27q^3 + 54q^2 + 33q + 6 = 3(9q^3 + 18q^2 + 11q + 2)$, where $9q^3 + 18q^2 + 11q + 2 \in \mathbb{Z}$ so that $3|a^3 - a$. Thus, $3|a^3 - a$ for any integer a.

5. Let n be an integer. By the division algorithm, n can be written in one of the following forms:

$$n = 5q,$$

$$n = 5q + 1,$$

$$n = 5q + 2,$$

$$n = 5q + 3, \text{ and}$$

$$n = 5q + 4$$

where $q \in \mathbb{Z}$. If $n = 5q$ then $n^4 = 625q^4 = 5(125q^4)$. Taking $k = 125q^4 \in \mathbb{Z}$, we can write n^4 as $5k$. If $n = 5q + 1$ then $n^4 = 625q^4 + 500q^3 + 150q^2 + 20q + 1 = 5(125q^4 + 100q^3 + 30q^2 + 4q) + 1$. Taking $k = 125q^4 + 100q^3 + 30q^2 + 4q \in \mathbb{Z}$, we can write n^4 as $5k+1$. If $n = 5q+2$ then $n^4 = 625q^4 + 1000q^3 + 600q^2 + 160q + 16 = 5(125q^4 + 200q^3 + 120q^2 + 32q + 3) + 1$. Taking $k = 125q^4 + 200q^3 + 120q^2 + 32q + 3 \in \mathbb{Z}$, we can write n^4 as $5k+1$. If $n = 5q+3$ then $n^4 = 625q^4 + 1500q^3 + 1350q^2 + 540q + 81) = 5(125q^4 + 300q^3 + 270q^2 + 108q + 16) + 1$. Taking $k = 125q^4 + 300q^3 + 270q^2 + 108q + 16 \in \mathbb{Z}$, we can write n^4 as $5k + 1$. If $n = 5q + 4$ then $n^4 = 625q^4 + 2000q^3 + 2400q^2 + 1280q + 256 = 5(125q^4 + 400q^3 + 480q^2 + 256q + 51) + 1$. Taking $k = 125q^4 + 400q^3 + 480q^2 + 256q + 51) \in \mathbb{Z}$, we can write n^4 as $5k + 1$. Thus, the fourth power of any integer is either of the form $5k$ or $5k + 1$.

6. Let a be an integer. Then $a(a + 1) \in \mathbb{Z}$ so that $3|3a(a + 1)$. We know that $3|a^3 - a = a(a + 1)(a - 1)$. Thus, $3|a(a + 1)(a - 1) + 3a(a + 1) = a(a + 1)(a - 1 + 3) = a(a + 1)(a + 2)$. By the division algorithm, either $a = 2q$ or $a = 2q+1$ for some $q \in \mathbb{Z}$. If $a = 2q$ then $a(a+1) = 2q(2q+1) = 2(q(2q + 1))$, where $q(2q + 1) \in \mathbb{Z}$ so that $2|a(a + 1)$. If $a = 2q + 1$ then $a(a+1) = (2q+1)(2q+1+1) = 2((q+1)(2q+1))$, where $(q+1)(2q+1) \in \mathbb{Z}$ so that $2|a(a + 1)$.

7. Suppose that a is odd. Thus, $a = 2q + 1$ for some $q \in \mathbb{Z}$. Hence, $a^2 + 3 = 4q^2 + 4q + 1 + 3 = 4(q^2 + q + 1)$ and $a^2 + 7 = 4q^2 + 4q + 1 + 7 = 4(q^2 + q + 2)$. We know that $2|q(q + 1)$ and $2|2$ so that $2|q(q + 1) + 2 = q^2 + q + 2$. Thus, by definition, $q^2 + q + 2 = 2k$ for some $k \in \mathbb{Z}$. Hence, $(a^2 + 3)(a^2 + 7) = (4(q^2 + q + 1))(4(2k)) = 32k(q^2 + q + 1) = 32(k(q^2 + q + 1))$, where $k(q^2 + q + 1) \in \mathbb{Z}$ so that $32|(a^2 + 3)(a^2 + 7)$.

8. Suppose that a is an integer not divisible by 2 or 3. By the division algorithm, any integer a can be written in one of the following forms:

$$a = 6q,$$

$$a = 6q + 1,$$

$$a = 6q + 2,$$

$$a = 6q + 3,$$

$$a = 6q + 4, \text{ and}$$

$$a = 6q + 5$$

where $q \in \mathbb{Z}$. Since a is not divisible by 3, we get that $a \neq 6q = 3(2q)$ and $a \neq 6q+3 = 3(2q+1)$. Since a is not divisible by 2, we additionally get that $a \neq 6q + 2 = 2(3q + 1)$ and $a \neq 6q + 4 = 2(3q + 2)$. Thus, either $a = 6q + 1$

or $a = 6q + 5$. If $a = 6q + 1$ then $a^2 + 23 = 36q^2 + 12q + 1 + 23 = 24(q^2 + 1) + 12(q^2 + q)$. We know that $2|q^2 + q$ so that for some $k \in \mathbb{Z}$, $q^2 + q = 2k$. Thus, $a^2 + 23 = 24(q^2 + 1) + 12(2k) = 24(q^2 + 1 + k)$, where $q^2 + 1 + k \in \mathbb{Z}$ so that $24|a^2 + 23$. If $a = 6q + 5$ then $a^2 + 23 = 36q^2 + 60q + 25 + 23 = 24(q^2 + 2q + 2) + 12(q^2 + q) = 24(q^2 + 2q + 2) + 12(2k) = 24(q^2 + 2q + 2 + k)$, where $q^2 + 2q + 2 + k \in \mathbb{Z}$ so that $24|a^2 + 23$.

9. Let a and b be odd integers. Thus, $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Assume, for the sake of contradiction, that the sum of their squares is a perfect square. Thus, for some $c \in \mathbb{Z}$, $a^2 + b^2 = c^2$. Note that $a^2 = 4m^2 + 4m + 1$ and $b^2 = 4n^2 + 4n + 1$ so that $c^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + m + n^2 + n) + 2$, where $m^2 + m + n^2 + n \in \mathbb{Z}$ and $0 \leq 2 < 4$ so that the remainder when c^2 is divided by 4 is 2. By the division algorithm, either $c = 2k$ or $c = 2k + 1$ for some $k \in \mathbb{Z}$. Thus, either $c^2 = 4k^2$ or $c^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, where $k^2, k^2 + k \in \mathbb{Z}$ and $0 \leq 0, 1 < 4$ so that the remainder when c^2 is divided by 4 is either 0 or 1. Thus, we get that 2 is either 0 or 1, which is absurd. Thus, the sum of the squares of two odd integers cannot be a perfect square.