Advanced Calculus I

- single variable real analysis

Real numbers ($\mathbb{R}$)

- $(\mathbb{R}, +, \cdot)$ is a field.
- $\mathbb{R}$ with $\geq$ is a partially ordered set.

Absolute value

- $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- $|x - a|$ is the distance between x and a.

Properties of the absolute value

1. $|x| \geq 0$
2. $|x| = 0 \Leftrightarrow x = 0$
3. $|xy| = |x| \cdot |y|$
   $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$
4. $|x + y| \leq |x| + |y|$
   $|x| - |y| \leq |x - y|$
5. If $a > 0$ then
   $|x| < a \Leftrightarrow -a < x < a$
   $|x| > a \Leftrightarrow x > a$ or $x < -a$

Other consequences

1. $|-x| = |x|$
2. If $a > 0$ then
   $|x - b| < a \Leftrightarrow b - a < x < b + a$
3. $|x - y| = 0 \Leftrightarrow x = y$
4. $\big||x| - |y|\big| \leq |x - y|$

Supremum and Infimum
Definition. Let $S \subseteq \mathbb{R}$ and $u, v \in \mathbb{R}$.

1. u is an upper bound of S if for all $s \in S$, $s \leq u$
2. v is a lower bound of S if for all $s \in S$, $s \geq v$

Definition.

1. If S has an upper [lower] bound, then S is said to be bounded above [below].

2. If S is bounded above and below, then S is said to be bounded.

Remark.

1. S is bounded above if
   $(\exists u \in \mathbb{R})(\forall s \in S)(s \leq u)$
   S is bounded below if
   $(\exists v \in \mathbb{R})(\forall s \in S)(s \geq v)$

2. S is bounded
   $\Leftrightarrow (\exists u, v \in \mathbb{R})(\forall s \in S)(v \leq s \leq u)$
   $\Leftrightarrow (\exists M > 0)(\forall s \in S)(|s| \leq M)$

Definition. Let $S \subseteq \mathbb{R}$ and $u, v \in \mathbb{R}$.

1. u is the supremum (or least upper bound) of S if:
   (a) u is an upper bound of S
   (b) and for all upper bounds d of S, $u \leq d$.

2. v is the infimum (or greatest lower bound) of S if:
   (a) v is a lower bound of S
   (b) and for all lower bounds b of S, $v \geq b$.

Remark.

1. Notation:
   $\sup S = u$
   $\inf S = v$

2. The supremum and infimum of S are not necessarily in S.

3. Since $\varnothing$ is bounded above and below by any $a \in \mathbb{R}$, $\varnothing$ has neither a supremum nor an infimum.

4. S is not bounded above implies that S has no supremum.
   S is not bounded below implies that S has no infimum.

Theorem. Let $S \subseteq \mathbb{R}$. If a supremum [infimum] exists, then it is unique.
Proof. Suppose that u and v are suprema of S. For the sake of contradiction, assume that $u \neq v$. Without loss of generality, assume that $u < v$. By definition of supremum, u is an upper bound of S. Also by definition of supremum, $v \leq d$ for any upper bound d of S. Taking $d = u$, we get $v \leq u$. Then $u < v \leq u$, which is absurd. Thus, $u = v$.
Theorem. Let u be an upper bound of a non-empty set $S \subseteq \mathbb{R}$. Then the following are equivalent:

1. $\sup S = u$
2. $(\forall x \in \mathbb{R})(x < u \Rightarrow (\exists s \in S)(x < s))$
3. $(\forall \varepsilon > 0)(\exists s \in S)(u - \varepsilon < s)$