



**University of
Zurich**^{UZH}

University of Zurich
Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 327

Shrinkage Estimation of Large Covariance Matrices: Keep it Simple, Statistician?

Olivier Ledoit and Michael Wolf

First version: July 2019
This version: February 2020

Shrinkage Estimation of Large Covariance Matrices: Keep It Simple, Statistician?

Olivier Ledoit*

Department of Economics
University of Zurich
CH-8032 Zurich, Switzerland
olivier.ledoit@econ.uzh.ch

Michael Wolf

Department of Economics
University of Zurich
CH-8032 Zurich, Switzerland
michael.wolf@econ.uzh.ch

First version: July 2019

This version: February 2020

Abstract

Under rotation-equivariant decision theory, sample covariance matrix eigenvalues can be optimally shrunk by recombining sample eigenvectors with a (potentially nonlinear) function of the unobservable population covariance matrix. The optimal shape of this function reflects the loss/risk that is to be minimized. We solve the problem of optimal covariance matrix estimation under a variety of loss functions motivated by statistical precedent, probability theory, and differential geometry. A key ingredient of our *nonlinear shrinkage* methodology is a new estimator of the angle between sample and population eigenvectors, without making strong assumptions on the population eigenvalues. We also introduce a broad family of covariance matrix estimators that can handle all regular functional transformations of the population covariance matrix under large-dimensional asymptotics. In addition, we compare via Monte Carlo simulations our methodology to two simpler ones from the literature, linear shrinkage and shrinkage based on the spiked covariance model.

KEY WORDS: Large-dimensional asymptotics, random matrix theory, rotation equivariance.

JEL CLASSIFICATION NOS: C13.

*Second affiliation: AlphaCrest Capital, 120 W 45th St, New York NY 10036, USA.

1 Introduction

Ever since [Stein \(1956\)](#) proved that the usual estimator of the mean is inadmissible in dimensions greater than three, decision theory has taken the edge over likelihood maximization in multivariate statistics. This leaves open the question of which loss function to minimize in a practical application. In this respect, the more loss functions available the better, as different researchers may pursue different goals. Regarding the second moments, that is, covariance matrix estimation, six loss functions have been investigated so far within the framework of large-dimensional asymptotics by [Ledoit and Wolf \(2018\)](#) and [Engle et al. \(2019\)](#), yielding a grand total of three different optimal nonlinear shrinkage formulas.

This paper delivers the technology to double the number of loss functions that can be handled from six to 12, without making strict assumptions. The six new loss functions considered are potentially attractive to applied researchers, as they have been promoted before by statisticians for decision-theoretical estimation of the covariance matrix. In order to achieve this degree of generality, we identify a formula from random matrix theory (RMT) that enables us to develop a new estimator of the angle of any sample eigenvector with any population eigenvector, in the large-dimensional limit. Using this new technique opens the door to addressing a large set of loss functions that were previously unattainable within the framework of large-dimensional asymptotics with the techniques of [Ledoit and Wolf \(2018\)](#): In addition to the six specific new loss functions considered, we can also handle two infinite general families of loss functions based on all regular transformations of the population covariance matrix.

Before starting to develop our methodology, it will be useful to give a brief review of the relevant literature. Likelihood maximization has done wonders for statistics in general; however, in the particular context of multivariate statistics when the number of parameters to be estimated is large, it tends to overfit in-sample data, at the expense of good out-of-sample performance. In reaction to that, decision theory favors estimators that perform well out-of-sample with respect to some given loss function. These estimators critically depend on the loss function selected by the end-user.

For covariance matrix estimation, we place ourselves firmly within the paradigm pioneered by [Stein \(1975, 1986\)](#): (i) no assumption on the eigenvalues of the population covariance matrix apart from positive definiteness; (ii) equivariance with respect to rotation of the original orthonormal basis of variables; and (iii) full flexibility to modify the eigenvalues of the sample covariance matrix as deemed necessary.

This is a tall order, and even Stein’s finite-sample mathematical prowesses achieved limited progress. It was only after cross-pollination from RMT, a field originated by Nobel Prize-winning physicist Eugene [Wigner \(1955\)](#), and specifically the notion of large-dimensional asymptotics, that conclusive strides forward could be made. Charles Stein himself was well aware, as early as 1969, of the potential of large-dimensional asymptotics to unlock the multivariate application problems that preoccupied him ([Stein, 1969](#), pp. 79–81). However, he left some work on the table for his intellectual successors in this respect.

There are currently three ‘simplified’ large-dimensional asymptotic strands of literature that fall short of Stein’s ambitious program in one way or another. Sparsity (Bickel and Levina, 2008) violates point (ii) because it assumes *a priori* knowledge of a specific orthonormal basis where (unlike for most other bases) the proportion of covariances equal to zero approaches 100%. Linear shrinkage (Ledoit and Wolf, 2004) violates point (iii) because it can only modify the eigenvalues of the sample covariance matrix through a linear transformation. The spiked covariance model of Johnstone (2001) violates point (i) because it assumes that all population eigenvalues are equal to each other, except for a vanishingly small proportion of them (called ‘spikes’).

By contrast, the present paper inscribes itself in a strand of literature called *nonlinear shrinkage* (Ledoit and Wolf, 2012, 2015, 2018) which does not compromise on any of these three points, and so remains in line with Stein’s original ambitious paradigm. A key ingredient is consistent estimation of the eigenvalues of the population covariance matrix. This was not even deemed possible until El Karoui (2008) proved otherwise. Since then, it has been more of a discussion of which estimation scheme to use, such as El Karoui’s own numerical procedure or a more modern approach based on supersymmetry (Jun, 2017); in this paper, we use the QuEST function of Ledoit and Wolf (2015).

In recent related work, the spiked covariance model of Johnstone (2001) has been used by Donoho et al. (2018) to derive shrinkage covariance matrix estimators for a *ménagerie* of 26 different loss functions. They promote the spiked model because, as they state in their Section 10,

the simple shrinkage rules we propose here may be more likely to be applied correctly in practice, and to work as expected, even in relatively small sample sizes.

It is, therefore, of interest to study whether our ‘more complicated’ nonlinear shrinkage rules actually lead to improved performance or whether applied researchers are just as well served by the rules of Donoho et al. (2018) according to their implicitly alluded to KISS (*Keep it simple, statistician!*) principle.

The remainder of this paper is organized as follows. Section 2 presents an intuitively understandable analysis in finite samples. Section 3 defines the large-dimensional asymptotics under which our results are derived. Section 4 investigates a wide variety of loss functions and, for each one, finds a *bona fide* covariance matrix estimator that is asymptotically optimal. Section 5 extends the analysis to the challenging yet empirically relevant case when the dimension exceeds the sample size. Section 6 presents Monte Carlo simulations. Section 7 concludes. An appendix collects various mathematical results to keep the presentation of the main paper compact.

2 Analysis in Finite Samples

2.1 Basic Setup

Assumption 1. Y is an $n \times p$ matrix of n independent and identically distributed (i.i.d.) observations on a system of $p < n$ random variables with mean zero and positive definite covariance matrix Σ with eigenvalues (τ_1, \dots, τ_p) , sorted in nondecreasing order without loss of generality (w.l.o.g.), and corresponding eigenvectors (v_1, \dots, v_p) .

(The case $p > n$ is treated in Section 5.)

The sample covariance matrix is $S := Y'Y/n$. Its spectral decomposition is $S = U\Lambda U'$, where Λ is a diagonal matrix and U is orthogonal. Let $\Lambda = \text{Diag}(\boldsymbol{\lambda})$ where $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_p)'$, with the eigenvalues again sorted in nondecreasing order w.l.o.g. The i th sample eigenvector is u_i , the i th column vector of U , so that $S = \sum_{i=1}^p \lambda_i \cdot u_i u_i'$. Note that it holds similarly $\Sigma = \sum_{i=1}^p \tau_i \cdot v_i v_i'$.

Definition 2.1. We consider rotation-equivariant covariance matrix estimators of the type $\tilde{S} := U\tilde{D}U'$, where \tilde{D} is a diagonal matrix: $\tilde{D} := \text{Diag}(\tilde{d}_1, \dots, \tilde{d}_p)$.

This class assumes no *a priori* information about the orientation of the orthonormal basis of (unobservable) population covariance matrix eigenvectors; this is different from the sparsity literature, which requires *a priori* knowledge of an orthonormal basis in which most covariances are zero. For many loss functions, there exists a finite-sample optimal (FSOPT) ‘estimator’ in this class of the form

$$\tilde{S} := \sum_{i=1}^p \tilde{d}_i \cdot u_i u_i', \quad \text{with} \quad \tilde{d}_i := \gamma^{-1} [u_i' \gamma(\Sigma) u_i] \quad \forall i = 1, \dots, p, \quad (2.1)$$

where γ denotes some smooth invertible function mapping of $(0, +\infty)$ onto \mathbb{R} . As is standard, applying a univariate function γ to a diagonalizable matrix means preserving its eigenvectors and applying γ to each eigenvalue individually; for example, $\log(\Sigma) := \sum_{i=1}^p \log(\tau_i) \cdot v_i v_i'$.

Remark 2.1. \tilde{S} in Equation (2.1) is obviously not feasible in practice, hence the single quotation marks around the word ‘estimator’. ■

Remark 2.2. To simplify the notation, we assume that all variables have mean zero. In many applications, variables do not have mean zero, or at least it is not known whether they do. In such a setting, it is more common to base the sample covariance matrix on the demeaned data instead: $S_n := \tilde{Y}_n' \tilde{Y}_n / (n - 1)$, where \tilde{Y}_n is obtained from Y_n by the operation of columnwise demeaning. In this case, n needs to be replaced everywhere with the ‘effective’ sample size $n - 1$. As shown at the beginning of Section 3 of [Silverstein and Bai \(1995\)](#), demeaning is a rank-one perturbation which in turn, thanks to Lemma 2.5a of the same paper, implies that it has no impact on large-dimensional asymptotic convergence results. ■

2.2 A Brief Summary of Known Results on Nonlinear Shrinkage

So far, only six loss functions have been solved in the very general rotation-equivariant framework of Assumption 1 and Definition 2.1. In the second column of Table 2.1, the loss functions are streamlined for readability; the actual ones could be squared and have various constants added or multiplied in ways that are irrelevant to estimator optimality. The way to read the fourth column is that the i th sample eigenvalue $\lambda_i = u_i' S u_i$ ($i = 1, \dots, p$) should be replaced by the quantity in the fourth column, optimally with respect to the same-row loss function, in finite samples: so it is the optimally ‘shrunk’ eigenvalue. We use the standard notation for the Frobenius norm of M , a square matrix: $\|M\|_F := \sqrt{\text{Tr}[MM']}$.

Name	Stylized Loss Function	Reference	Shrinkage
Frobenius	$\ \tilde{S} - \Sigma\ _F$	Leung and Muirhead (1987)	$u_i' \Sigma u_i$
Inverse Stein	$\text{Tr}[\tilde{S}^{-1} \Sigma] - \log[\det(\tilde{S}^{-1} \Sigma)]$	Ghosh and Sinha (1987)	$u_i' \Sigma u_i$
Minimum Variance	$\text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}] / (\text{Tr}[\tilde{S}^{-1}])^2$	Engle et al. (2019)	$u_i' \Sigma u_i$
Stein	$\text{Tr}[\tilde{S} \Sigma^{-1}] - \log[\det(\tilde{S} \Sigma^{-1})]$	James and Stein (1961)	$\frac{1}{u_i' \Sigma^{-1} u_i}$
Inverse Frobenius	$\ \tilde{S}^{-1} - \Sigma^{-1}\ _F$	Haff (1979a)	$\frac{1}{u_i' \Sigma^{-1} u_i}$
Symmetrized Stein	$\text{Tr}[\tilde{S} \Sigma^{-1} + \tilde{S}^{-1} \Sigma]$	Kubokawa and Konno (1990)	$\sqrt{\frac{u_i' \Sigma u_i}{u_i' \Sigma^{-1} u_i}}$

Table 2.1: Existing set of finite-sample optimal (FSOPT) nonlinear shrinkage formulas.

Table 2.1 shows that the six loss functions really only yield three different nonlinear shrinkage formulas. The first two are of the type (2.1), with $\gamma(x) = x$ and $\gamma(x) = 1/x$, respectively, and the third one is simply their geometric mean.

2.3 Additional Loss Functions

The easiest way to start this investigation is to look for different loss functions that give rise to the same nonlinear shrinkage formulas as the ones in Section 2.2. Table 2.2 presents two of them.

Name	Stylized Loss Function	Reference	Shrinkage
Weighted Frobenius	$\text{Tr}[(\tilde{S} - \Sigma)^2 \Sigma^{-1}]$	Sharma and Krishnamoorthy (1985)	$\frac{1}{u_i' \Sigma^{-1} u_i}$
Disutility	$\text{Tr}[(\tilde{S}^{-1} - \Sigma^{-1})^2 \Sigma]$	Appendix A	$u_i' \Sigma u_i$

Table 2.2: Two more loss functions leading to existing nonlinear shrinkage formulas.

The second loss function is new. It is derived from the Sharma and Krishnamoorthy (1985) loss in the same way that the Inverse Frobenius loss is derived from the Frobenius loss, or that

the Inverse Stein’s loss of [Ghosh and Sinha \(1987\)](#) is derived from the original Stein’s loss: by substituting the covariance matrix with its inverse, the precision matrix. At the same time, it has a more interesting justification as minus the quadratic utility function of [Markowitz \(1952\)](#) in large dimensions, as argued in [Appendix A](#) (hence the name *disutility*). It is a close cousin of the Minimum Variance loss function, with a tighter grip on the scale of the estimator. Reassuringly, both of them give rise to the same optimal nonlinear shrinkage formula.

There are three interlocking reasons for bringing up these loss functions, even though they fall back on the known estimators of [Section 2.2](#). First, to avoid the well-known ‘file-drawer problem’ (also called publication bias), whereby results that are deemed less interesting remain unpublished. Second, some applied researcher may well look at one of these three loss functions and recognize that it suits his or her objective perfectly, in which case it does not matter whether the shrinkage formula is old or new. Third, in the end, the choice of estimator is a choice of shrinkage formula, and the best way to know what a specific shrinkage really means is to list as many loss functions as possible that lead to it.

2.4 New Shrinkage Formulas

The main point of the paper is to go beyond the two cases $\gamma(x) = x^{\pm 1}$ and thereby to study other functions of the population covariance matrix (through the prism of sample eigenvectors). We introduce four more: \sqrt{x} , $\log(x)$, x^2 , and $1/x^2$. Hence, we triple the number of functions that can be utilized for this purpose, from two to six. We could have introduced as many new functions as we wanted, but this should be enough to make the point. Nor is this frivolous or arbitrary: these four functional transformations arise naturally in the study of four well-regarded loss functions that have remained as open problems. In what follows, the symbol \mathbb{I} denotes a conformable identity matrix.

Type of Loss	Loss Function	Reference	Shrinkage
Log-Euclidian	$\ \log(\tilde{S}) - \log(\Sigma)\ _F$	Arsigny et al. (2006)	$\exp[u_i' \log(\Sigma) u_i]$
Fréchet	$\ \tilde{S}^{1/2} - \Sigma^{1/2}\ _F$	Dowson and Landau (1982)	$(u_i' \Sigma^{1/2} u_i)^2$
Quadratic	$\ \Sigma^{-1} \tilde{S} - \mathbb{I}\ _F$	$L^{F,3}$ in Donoho et al. (2018)	$\frac{u_i' \Sigma^{-1} u_i}{u_i' \Sigma^{-2} u_i}$
Inverse Quadratic	$\ \tilde{S}^{-1} \Sigma - \mathbb{I}\ _F$	$L^{F,4}$ in Donoho et al. (2018)	$\frac{u_i' \Sigma^2 u_i}{u_i' \Sigma u_i}$

Table 2.3: New set of finite-sample optimal (FSOPT) nonlinear shrinkage formulas.

Log-Euclidian It is defined as the Euclidian distance on the logarithm of the manifold of symmetric positive-definite matrices, hence the name. It is a close cousin of the geodesic distance on the smooth Riemannian manifold of positive-definite matrices. It has essentially the same properties, but is much more tractable for statistical applications. In particular, it is invariant with respect to matrix inversion, so eigenvalues close to zero are

treated like eigenvalues close to infinity.

Fréchet The Fréchet discrepancy, named after the French mathematician Maurice Fréchet (1878–1973), is originally a measure of distance between two probability distributions. In the multivariate normal case, it directly implies a notion of distance between any two symmetric positive-definite matrices. Intuitively, we should think of it as a measure of ‘how far apart’ are the distributions that these two covariance matrices generate.

Quadratic This is a recent variant of the quadratic-type loss function that can be traced back to pioneers in the field such as Selliah (1964, Section 2.2.4) and Haff (1979b, loss function L_2). Its signature is that it promotes accuracy in the direction of the smallest principal components of the population covariance matrix.

Inverse Quadratic Same as above, but with the inverse sample covariance matrix. Mechanically, it promotes accuracy in the direction of the largest principal components of the population covariance matrix.

The logarithm and the square root are directly embedded into the first two shrinkage formulas (Log-Euclidian and Fréchet), but the square and inverse-square functions only appear in the last two loss formulas as part of combinations, echoing what happened with the Symmetrized Stein’s loss. Proof that the loss functions in the second column of the tables give rise to the FSOPT formulas in the fourth column can be found in Appendix B.

These seven nonlinear shrinkage formulas given rather different results. Researchers may wonder how they compare to each other. One interesting mathematical observation is that they do not cross, but one is always above (or below) the other across the whole spectrum — with the sole exception of Symmetrized Stein vs. Log-Euclidian shrinkage. The following proposition reveals the ordering.

Proposition 2.1. *Under Assumption 1, with probability one, for all $i = 1, \dots, p$,*

$$\frac{u_i' \Sigma^2 u_i}{u_i' \Sigma u_i} > u_i' \Sigma u_i > \left(u_i' \sqrt{\Sigma} u_i\right)^2 > \exp \left[u_i' \log(\Sigma) u_i\right] > \frac{1}{u_i' \Sigma^{-1} u_i} > \frac{u_i' \Sigma^{-1} u_i}{u_i' \Sigma^{-2} u_i} \quad (2.2)$$

$$\left(u_i' \sqrt{\Sigma} u_i\right)^2 > \sqrt{\frac{u_i' \Sigma u_i}{u_i' \Sigma^{-1} u_i}} > \frac{1}{u_i' \Sigma^{-1} u_i} \quad (2.3)$$

Proof. Follows from Jensen’s inequality and the Cauchy-Schwarz inequality once we remark that $u_i' \gamma(\Sigma) u_i = \sum_{j=1}^p \gamma(\tau_i) \cdot (u_i' v_j)^2$ for $\gamma(x) = x, 1/x, x^2, 1/x^2, \sqrt{x}$, or $\log(x)$, and that $\sum_{j=1}^p (u_i' v_j)^2 = 1$, for every $i = 1, \dots, p$. ■

2.5 Preview of General Result

FSOPT ‘estimators’ of the form (2.1) cannot be used directly because they depend on the population covariance matrix Σ , which is unobservable. So it stands to reason to ask: How is it even possible that this approach leads anywhere? First of all, note that we do not need to

estimate all $p(p+1)/2$ entries of the symmetric matrix Σ , we only need p quantities: $u_i' \gamma(\Sigma) u_i$, for $i = 1, \dots, p$, which is much more manageable. When the matrix dimension p is large, it is possible to approximate these quantities by the general formula

$$u_i' \gamma(\Sigma) u_i \approx \frac{1}{p} \sum_{j=1}^p \gamma(\hat{\tau}_j) \cdot \left\{ \frac{\frac{p}{n} \lambda_i \hat{\tau}_j}{|\hat{\tau}_j [1 - \frac{p}{n} - \frac{p}{n} \lambda_i \check{m}_{n,p}^{\hat{\tau}}(\lambda_i)] - \lambda_i|^2} \right\}, \quad (2.4)$$

where $\hat{\tau} := (\hat{\tau}_1, \dots, \hat{\tau}_p)'$ is an estimator of the population eigenvalues, and $\check{m}_{n,p}^{\hat{\tau}}(\cdot)$ is the complex-valued function of real argument due to Ledoit and Wolf (2015, Section 2). Formula (2.4) generates *bona fide* covariance matrix estimators of the type (2.1) for all the loss functions in Table 2.3 by setting $\gamma(x)$ equal to $\log(x)$, \sqrt{x} , x^{-2} , or x^2 . Given that $u_i' \gamma(\Sigma) u_i = \frac{1}{p} \sum_{j=1}^p \gamma(\tau_j) \cdot \{p(u_i' v_j)^2\}$, the term between curly brackets in (2.4) is simply an estimator of the dimension-normalized squared dot product of the i th sample eigenvector with the j th population eigenvector.

3 Large-Dimensional Asymptotic Framework

We now move on to formally establishing that plugging the approximation (2.4) into the generic nonlinear shrinkage formula (2.1) yields optimal rotation-equivariant covariance matrix estimators under large-dimensional asymptotics with respect to the loss functions listed. First of all, to make the paper self-contained, we need to restate some sets of assumptions that have been used a number of times before. We shall do so in a condensed fashion; any unfamiliar reader interested in getting more background information should refer to some earlier paper such as, for example, Ledoit and Wolf (2018, Section 3.1), and the references therein.

In a nutshell: The dimension p goes to infinity along with the sample size n , their ratio p/n converges to some limit $c \in (0, 1)$, and we seek to asymptotically optimize the way to nonlinearly shrink sample eigenvalues. Also, from now on, all dimension-dependent objects are subscripted by the sample size n .

3.1 Large-Dimensional Asymptotic Framework

Assumption 2 (Dimension). *Let n denote the sample size and $p := p(n)$ the number of variables. It is assumed that the ratio p/n converges, as $n \rightarrow \infty$, to a limit $c \in (0, 1)$ called the limiting concentration (ratio). Furthermore, there exists a compact interval included in $(0, 1)$ that contains p/n for all n large enough.*

Assumption 3 (Population Covariance Matrix).

- a. *The $p \times p$ population covariance matrix Σ_n is nonrandom symmetric positive-definite.*
- b. *Let $\tau_n := (\tau_{n,1}, \dots, \tau_{n,p})'$ denote a system of eigenvalues of Σ_n , and H_n their empirical distribution function (e.d.f.): $H_n(x) := \sum_{i=1}^p \mathbb{1}_{[\tau_{n,i}, +\infty)}(x)/p$, where $\mathbb{1}$ denotes the indicator function of a set. It is assumed that H_n converges weakly to some limit law H , called the limiting spectral distribution (function).*

- c. $\text{Supp}(H)$, the support of H , is the union of a finite number of closed intervals in $(0, +\infty)$.
- d. There exists a compact interval $[\underline{h}, \bar{h}] \subset (0, \infty)$ that contains $\{\tau_{n,1}, \dots, \tau_{n,p}\}$ for all n large enough.

Note that this includes [Johnstone's \(2001\)](#) spiked covariance model as a special case where the limiting population spectral distribution H is a step function with a single step.

Assumption 4 (Data Generating Process). X_n is an $n \times p$ matrix of i.i.d. random variables with mean zero, variance one, and finite 12th moment. The matrix of observations is $Y_n := X_n \sqrt{\Sigma_n}$. Neither $\sqrt{\Sigma_n}$ nor X_n are observed on their own: only Y_n is observed.

Note that this includes [Johnstone's \(2001\)](#) spiked covariance model as a special case where the variates are assumed to be normal.

Remark 3.1 (Moment Condition). The existence of a finite 12th moment is assumed to prove certain mathematical results using the methodology of [Ledoit and P  ch   \(2011\)](#). However, Monte Carlo studies in [Ledoit and Wolf \(2012, 2015\)](#) indicate that this assumption is not needed in practice and can be replaced with the existence of a finite fourth moment. This is a generic requirement that does not depend on any particular loss function. ■

The sample covariance matrix is defined as $S_n := n^{-1} Y_n' Y_n = n^{-1} \sqrt{\Sigma_n} X_n' X_n \sqrt{\Sigma_n}$. It admits a spectral decomposition $S_n = U_n \Lambda_n U_n'$, where Λ_n is a diagonal matrix, and U_n is an orthogonal matrix: $U_n U_n' = U_n' U_n = \mathbb{I}_n$, where \mathbb{I}_n (in slight abuse of notation) denotes the identity matrix of dimension $p \times p$. Let $\Lambda_n := \text{Diag}(\boldsymbol{\lambda}_n)$ where $\boldsymbol{\lambda}_n := (\lambda_{n,1}, \dots, \lambda_{n,p})'$. We can assume w.l.o.g. that the sample eigenvalues are sorted in increasing order: $\lambda_{n,1} \leq \lambda_{n,2} \leq \dots \leq \lambda_{n,p}$. Correspondingly, the i th sample eigenvector is $u_{n,i}$, the i th column vector of U_n . Under Assumptions 2–4, the e.d.f. of sample eigenvalues $F_n(x) := \sum_{i=1}^p \mathbb{1}_{[\lambda_{n,i}, +\infty)}(x)/p$ converges almost surely to a nondeterministic cumulative distribution function F that depends only on H and c :

$$F_n(x) \xrightarrow{\text{a.s.}} F(x) \quad \forall x \in (0, +\infty) .$$

How to go from (H, c) to F is determined by the following equation, due to [Silverstein \(1995\)](#): for all z in \mathbb{C}^+ , the half-plane of complex numbers with strictly positive imaginary part, $m := m_F(z)$ is the unique solution in the set $\{m \in \mathbb{C} : -\frac{1-c}{z} + cm \in \mathbb{C}^+\}$ to the equation

$$m = \int \frac{1}{\tau[1 - c - czm] - z} dH(\tau) , \tag{3.1}$$

where m_F denotes the [Stieltjes \(1894\)](#) transform of F , whose standard definition is:

$$\forall z \in \mathbb{C}^+ \quad m_F(z) := \int \frac{1}{\lambda - z} dF(\lambda) .$$

The Stieltjes transform admits a well-known inversion formula:

$$G(b) - G(a) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_a^b \text{Im}[m_G(\xi + i\eta)] d\xi$$

if G is continuous at both a and b . Although the Stieltjes transform of F , m_F , is a function whose domain is the upper half of the complex plane, it admits an extension to the real line, since [Silverstein and Choi \(1995\)](#) show that: $\forall x \in (0, +\infty)$, $\lim_{z \in \mathbb{C}^+ \rightarrow x} m_F(z) =: \check{m}_F(x)$ exists and is continuous. The imaginary part of \check{m}_F is the derivative of F , up to rescaling by π ; therefore, (3.1) enables us to pin down the location of the sample eigenvalues, a fact exploited by the QuEST function; see Section 3.2. Furthermore, the support of the limiting distribution of the sample eigenvalue $\text{Supp}(F)$ is the union of a finite number $\kappa \geq 1$ of compact intervals: $\text{Supp}(F) = \bigcup_{k=1}^{\kappa} [a_k, b_k]$, where $0 < a_1 < b_1 < \dots < a_{\kappa} < b_{\kappa} < \infty$.

Definition 3.1 (Rotation-Equivariant Estimators). *We consider covariance matrix estimators of the type $\tilde{S}_n := U_n \tilde{D}_n U_n'$, where \tilde{D}_n is a diagonal matrix: $\tilde{D}_n := \text{Diag}(\tilde{\varphi}_n(\lambda_{n,1}) \dots, \tilde{\varphi}_n(\lambda_{n,p}))$, and $\tilde{\varphi}_n$ is a (possibly random) real univariate function which can depend on S_n .*

Assumption 5 (Nonlinear Shrinkage Function). *We assume that there exists a nonrandom real univariate function $\tilde{\varphi}$ defined on $\text{Supp}(F)$ and continuously differentiable on $\bigcup_{k=1}^{\kappa} [a_k, b_k]$ such that $\tilde{\varphi}_n(x) \xrightarrow{\text{a.s.}} \tilde{\varphi}(x)$ for all $x \in \text{Supp}(F)$. Furthermore, this convergence is uniform over $x \in \bigcup_{k=1}^{\kappa} [a_k + \eta, b_k - \eta]$, for any small $\eta > 0$. Finally, for any small $\eta > 0$, there exists a finite nonrandom constant \tilde{K} such that almost surely, over the set $x \in \bigcup_{k=1}^{\kappa} [a_k - \eta, b_k + \eta]$, $|\tilde{\varphi}_n(x)|$ is uniformly bounded by \tilde{K} , for all n large enough.*

3.2 The QuEST Function

Once again, to make the paper self-contained, we need to restate the definition of a key mathematical object called the QuEST (*quantized eigenvalues sampling transform*) function. We shall do so in condensed fashion; the interested reader is referred to [Ledoit and Wolf \(2015, 2017\)](#) for full background information.

In a nutshell: QuEST is a multivariate deterministic function mapping population eigenvalues into sample eigenvalues, valid asymptotically as p and n go to infinity together.

Definition 3.2 (QuEST). *For any given n and p , $Q_{n,p}$ maps $\mathbf{t} := (t_1, \dots, t_p) \in [0, +\infty)^p$ into*

$$Q_{n,p}(\mathbf{t}) := (q_{n,p}^1(\mathbf{t}), \dots, q_{n,p}^p(\mathbf{t})) \quad \text{where} \quad q_{n,p}^i(\mathbf{t}) := p \int_{(i-1)/p}^{i/p} (F_{n,p}^{\mathbf{t}})^{-1}(u) du ,$$

$$(F_{n,p}^{\mathbf{t}})^{-1} \text{ is the inverse function of } F_{n,p}^{\mathbf{t}}(v) := \frac{1}{\pi} \int_{-\infty}^v \text{Im} [\check{m}_{n,p}^{\mathbf{t}}(x)] dx ,$$

and, for all x in \mathbb{R} , $\check{m}_{n,p}^{\mathbf{t}}(x)$ is the unique solution $m \in \mathbb{C}^+$ to the fundamental equation:

$$m = \frac{1}{p} \sum_{j=1}^p \frac{1}{t_j \left(1 - \frac{p}{n} - \frac{p}{n} x m\right) - x} . \quad (3.2)$$

Theorem 3.1 ([Ledoit and Wolf \(2015\)](#)). *Suppose Assumptions 2–4 are satisfied. Define*

$$\hat{\tau}_n := \underset{\mathbf{t} \in (0, +\infty)^p}{\text{argmin}} \frac{1}{p} \sum_{i=1}^p [q_{n,p}^i(\mathbf{t}) - \lambda_{n,i}]^2 , \quad (3.3)$$

The Radon-Nikodym derivative $\theta(\lambda_{n,i}, \tau_{n,j})$ is ‘essentially like’ the squared dot product $p(u'_{n,i}v_{n,j})^2$ for large p and n . In order to operationalize Equation (3.6), we need *bona fide* estimators for its ingredients, and they are provided by Section 3.2’s QuEST function:

$$\hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j}) := \frac{\frac{p}{n} \lambda_{n,i} \hat{\tau}_{n,j}}{\left| \hat{\tau}_{n,j} \left[1 - \frac{p}{n} - \frac{p}{n} \lambda_{n,i} \check{m}_{n,p}^{\hat{\tau}_{n,j}}(\lambda_{n,i}) \right] - \lambda_{n,i} \right|^2} . \quad (3.7)$$

Although the expression may seem a bit unusual, it is just what comes out of RMT, and we should count ourselves lucky to have any closed-form solution at all. This ‘luck’ is first and foremost due to the pioneering efforts of probabilists who came before. If Equations (3.1), (3.2), (3.6), and (3.7) appear to be descendents from each other, *it is because they are*. A graphical illustration in the case where the population eigenvalues are evenly spread in the interval $[1, 5]$, with concentration ratio $p/n = 0.5$, is given by Figure 3.1.

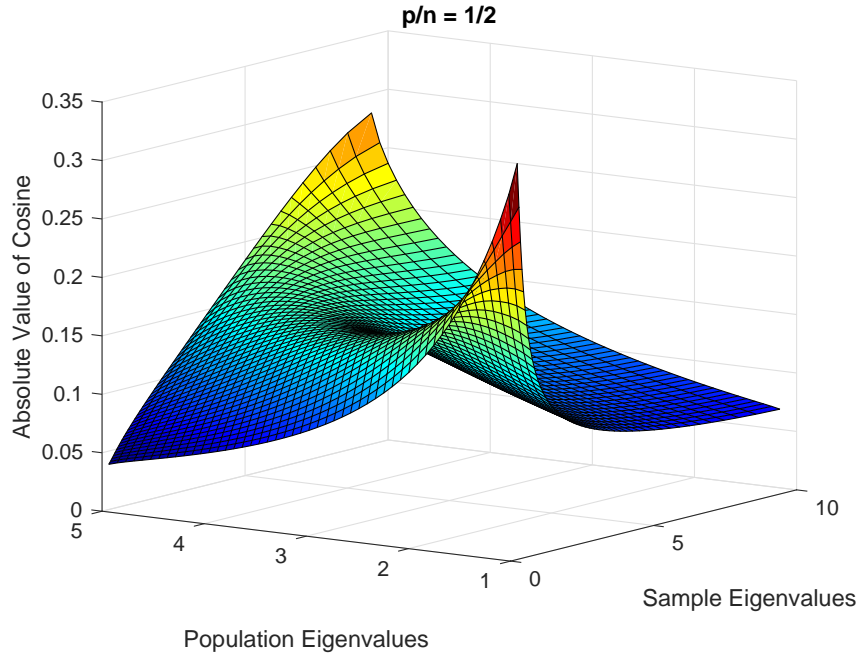


Figure 3.1: Absolute value of the cosine of the angle between population and sample eigenvectors. On the horizontal axes, eigenvectors are indexed by their respective eigenvalues.

One can see that the spread of sample eigenvalues is much wider: from 0.2 to 10.2. Top-ranked sample eigenvectors are more aligned with top-ranked population eigenvectors, and bottom-ranked sample eigenvectors are more aligned with bottom-ranked population eigenvectors. The overall pattern is complicated and can only be captured by the function θ of Theorem 3.2.

4 Asymptotically Optimal Nonlinear Shrinkage Estimators

The two loss functions from Table 2.2 are easy to handle using the techniques of Ledoit and Wolf (2018): The nonlinear shrinkage estimator that they call \hat{S}_n^* is optimal with respect to the

Weighted Frobenius loss under large-dimensional asymptotics; and the estimator that they call \widehat{S}_n° is optimal with respect to the Disutility loss. These results are stated without proof, as they are just minor extensions of the arguments put forward by [Ledoit and Wolf \(2018\)](#).

Regarding the loss functions of Table 2.3, they are vastly more challenging, and cannot be handled with existing techniques. Instead, they can only be handled by using the new technique of angle estimation introduced in Section 3.3 above, as we shall now proceed to demonstrate.

4.1 Four Specific Loss Functions

All remaining theorems are proven in Appendix C. We start with asymptotically optimal *bona fide* estimators based on Table 2.3.

Theorem 4.1 (Log-Euclidian). *For any estimator \widetilde{S}_n in Definition 3.1, the Log-Euclidian loss*

$$\mathcal{L}_n^{LE}(\widetilde{S}_n, \Sigma_n) := \frac{1}{p} \text{Tr} \left[\left\{ \log(\widetilde{S}_n) - \log(\Sigma_n) \right\}^2 \right], \quad (4.1)$$

converges under Assumptions 2–5 almost surely to a deterministic limit that depends only on H , c , and $\widetilde{\varphi}$. This limit is minimized if $\widetilde{\varphi}_n(\lambda_{n,i})$ is equal to

$$\widehat{\varphi}_n^{LE}(\lambda_{n,i}) := \exp \left(\frac{1}{p} \sum_{j=1}^p \log(\widehat{\tau}_{n,j}) \cdot \widehat{\theta}_n(\lambda_{n,i}, \widehat{\tau}_{n,j}) \right), \quad (4.2)$$

where $\widehat{\tau}_n = (\widehat{\tau}_{n,j})_{j=1,\dots,p}$ denotes the estimator of population covariance matrix eigenvalues in Theorem 3.1, and $\widehat{\theta}_n(\lambda_{n,i}, \widehat{\tau}_{n,j})$ is the estimator of the (dimension-normalized) squared dot product of the i th sample eigenvector with the j th population eigenvector in Equation (3.7). The resulting covariance matrix estimator is $\widehat{S}_n^{LE} := \sum_{i=1}^p \widehat{\varphi}_n^{LE}(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

Theorem 4.2 (Fréchet). *The Fréchet loss $\mathcal{L}_n^{FRÉ}(\widetilde{S}_n, \Sigma_n) := \|\widetilde{S}_n^{1/2} - \Sigma_n^{1/2}\|_F^2/p$ converges almost surely to a deterministic limit that is minimized if $\widetilde{\varphi}_n(\lambda_{n,i})$ is equal to*

$$\widehat{\varphi}_n^{FRÉ}(\lambda_{n,i}) := \left(\frac{1}{p} \sum_{j=1}^p \sqrt{\widehat{\tau}_{n,j}} \cdot \widehat{\theta}_n(\lambda_{n,i}, \widehat{\tau}_{n,j}) \right)^2. \quad (4.3)$$

The resulting covariance matrix estimator is $\widehat{S}_n^{FRÉ} := \sum_{i=1}^p \widehat{\varphi}_n^{FRÉ}(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

Theorem 4.3 (Quadratic). *The Quadratic loss $\mathcal{L}_n^Q(\widetilde{S}_n, \Sigma_n) := \|\Sigma_n^{-1} \widetilde{S}_n - \mathbb{I}_n\|_F^2/p$ converges almost surely to a deterministic limit that is minimized if $\widetilde{\varphi}_n(\lambda_{n,i})$ is equal to*

$$\widehat{\varphi}_n^Q(\lambda_{n,i}) := \frac{\frac{1}{p} \sum_{j=1}^p \frac{1}{\widehat{\tau}_{n,j}} \cdot \widehat{\theta}_n(\lambda_{n,i}, \widehat{\tau}_{n,j})}{\frac{1}{p} \sum_{j=1}^p \frac{1}{\widehat{\tau}_{n,j}^2} \cdot \widehat{\theta}_n(\lambda_{n,i}, \widehat{\tau}_{n,j})}. \quad (4.4)$$

The resulting covariance matrix estimator is $\widehat{S}_n^Q := \sum_{i=1}^p \widehat{\varphi}_n^Q(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

Theorem 4.4 (Inverse Quadratic). *The Inverse Quadratic loss function, which is defined as $\mathcal{L}^{QINV}(\tilde{S}_n, \Sigma_n) := \|\tilde{S}_n^{-1}\Sigma_n - \mathbb{I}_n\|_F^2/p$, converges almost surely to a deterministic limit that is minimized if $\tilde{\varphi}_n(\lambda_{n,i})$ is equal to*

$$\hat{\varphi}_n^{QINV}(\lambda_{n,i}) := \frac{\frac{1}{p} \sum_{j=1}^p \hat{\tau}_{n,j}^2 \cdot \hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j})}{\frac{1}{p} \sum_{j=1}^p \hat{\tau}_{n,j} \cdot \hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j})} . \quad (4.5)$$

The resulting covariance matrix estimator is $\hat{S}_n^{QINV} := \sum_{i=1}^p \hat{\varphi}_n^{QINV}(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

4.2 Two Infinite Families of Loss Functions

We have so far covered 12 loss functions, including many of the classic ones, from which we have derived a total of 7 different optimal nonlinear shrinkage formulas (as there are some commonalities). It is tedious to keep adding more by hand. Most applied researchers should have already been able to find ‘the shoe that fits’ in this rather extensive list by now.

If not, the only systematic method is to study an (*uncountably*) *infinite* number of loss functions, and to find the nonlinear shrinkage formula exactly optimized with respect to each of them. To the best of our knowledge, an ambitious project on this scale has never been envisioned before. In doing so, we will meet again some old acquaintances: 6 of the 12 loss functions already analyzed manually are special cases of the two general theorems presented below. The first infinite family of loss functions is what we call Generalized Frobenius.

Theorem 4.5 (Generalized Frobenius). *For any invertible and continuously differentiable function γ defined on $(0, +\infty)$, the Generalized Frobenius loss $\mathcal{L}_n^{\gamma, F}(\tilde{S}_n, \Sigma_n) := \left\| \gamma(\tilde{S}_n) - \gamma(\Sigma_n) \right\|_F^2/p$ converges almost surely to a deterministic limit that is minimized if $\tilde{\varphi}_n(\lambda_{n,i})$ is equal to*

$$\hat{\varphi}_n^\gamma(\lambda_{n,i}) := \gamma^{-1} \left(\frac{1}{p} \sum_{j=1}^p \gamma(\hat{\tau}_{n,j}) \cdot \hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j}) \right) . \quad (4.6)$$

The resulting covariance matrix estimator is $\hat{S}_n^\gamma := \sum_{i=1}^p \hat{\varphi}_n^\gamma(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

The Frobenius, Inverse Frobenius, Log-Euclidian, and Fréchet losses are special cases of the General Frobenius family, corresponding, respectively, to $\gamma(x)$ equal to x , $1/x$, $\log(x)$, and \sqrt{x} .

A second infinite family of loss functions is based on the [Kullback and Leibler \(1951\)](#) divergence. Given two multivariate normal distributions $\mathcal{N}(0, A_i)$ with zero mean and covariance matrix A_i , for $i \in \{1, 2\}$, their dimension-normalized Kullback-Leibler divergence is:

$$D_{KL}(\mathcal{N}(0, A_1) \parallel \mathcal{N}(0, A_2)) := \frac{1}{2p} \{ \text{Tr}[A_2^{-1}A_1] - \log [\det(A_2^{-1}A_1)] - p \} . \quad (4.7)$$

Stein’s loss and the Inverse Stein loss are special cases of the Generalized Kullback-Leibler family defined below, obtained by setting $\gamma(x)$ equal to $1/x$ and x , respectively.

Theorem 4.6 (Generalized Kullback-Leibler). *For any invertible and continuously differentiable function γ defined on $(0, +\infty)$, the Generalized Kullback-Leibler loss function*

$$\mathcal{L}_n^{\gamma, KL}(\tilde{S}_n, \Sigma_n) := \frac{1}{2p} \left\{ \text{Tr} \left[\gamma(\tilde{S}_n)^{-1} \gamma(\Sigma_n) \right] - \log \det \left[\gamma(\tilde{S}_n)^{-1} \gamma(\Sigma_n) \right] - p \right\} \quad (4.8)$$

converges almost surely to a deterministic limit that is minimized if $\tilde{\varphi}_n(\lambda_{n,i})$ is equal to the quantity $\hat{\varphi}_n^{\gamma}(\lambda_{n,i})$ defined in Equation (4.6) (for $i = 1, \dots, p$).

Both infinite families of loss functions confirm the asymptotic optimality of the *same* infinite family of nonlinear shrinkage estimators \hat{S}_n^{γ} . The Frobenius norm is important because it is just the Euclidian distance on the space of matrices, and the Kullback-Leibler divergence is important in a completely different field: information theory. Two justifications coming from such different perspectives combine to give strong backing to the covariance matrix estimator \hat{S}_n^{γ} , no matter which function γ the end-user is interested in.

Remark 4.1. The three other nonlinear shrinkage formulas that do not fit into the mold of Equation (4.6) are just elementary combinations of $\hat{\varphi}_n^{\gamma}(\cdot)$ for two different γ functions. ■

Remark 4.2. It should be pointed out that, apart from the two special cases $\gamma(x) = x^{\pm 1}$, these two infinite families of loss functions can also only be handled by using the new technique of angle estimation introduced in Section 3.3 above. ■

5 Singular Case: $p > n$

This is a case of great practical importance. When it happens, the sample covariance matrix is singular: It has $p - n$ eigenvalues equal to zero; therefore, it is only positive *semi*-definite. There then exist some linear combinations of the original variables that falsely appear to have zero variance when one only looks in-sample. In a sense, the sample covariance matrix, with its $p(p + 1)/2$ degrees of freedom, ‘overfits’ the data set of dimension $n \times p$.

5.1 Finite-Sample Analysis

With respect to the loss functions studied in this paper, the optimal nonlinear shrinkage formula applied to the n non-zero sample eigenvalues remains the same as in the case $p < n$, so no need to revisit. The only item to be determined is how to shrink the $p - n$ null sample eigenvalues. Recall that we sort the sample eigenvalues in nondecreasing order w.l.o.g., so the null eigenvalues are the first $p - n$ ones. To build intuition, we start as before with the finite-sample case: Table 5.1 presents a counterpart to Tables 2.1–2.3, listing how to optimally shrink null sample eigenvalues.

The pattern is clear: compute how all the eigenvectors in the null space of the sample covariance matrix relate to (a function of) the population covariance matrix, and take the average. There is a rotational indeterminacy in this null space of dimension $p - n$, but the formulas in the last column are invariant to rotation of the basis of null eigenvectors, so it does not matter.

Type of Loss	Stylized Loss Function	Null Shrinkage
Frobenius	$\ \tilde{S} - \Sigma\ _F$	$\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i$
Inverse Stein	$\text{Tr}[\tilde{S}^{-1} \Sigma] - \log[\det(\tilde{S}^{-1} \Sigma)]$	$\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i$
Minimum Variance	$\text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}] / (\text{Tr}[\tilde{S}^{-1}])^2$	$\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i$
Stein	$\text{Tr}[\tilde{S} \Sigma^{-1}] - \log[\det(\tilde{S} \Sigma^{-1})]$	$\left(\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-1} u_i\right)^{-1}$
Inverse Frobenius	$\ \tilde{S}^{-1} - \Sigma^{-1}\ _F$	$\left(\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-1} u_i\right)^{-1}$
Symmetrized Stein	$\text{Tr}[\tilde{S} \Sigma^{-1} + \tilde{S}^{-1} \Sigma]$	$\sqrt{\frac{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i}{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-1} u_i}}$
Weighted Frobenius	$\text{Tr}[(\tilde{S} - \Sigma)^2 \Sigma^{-1}]$	$\left(\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-1} u_i\right)^{-1}$
Disutility	$\text{Tr}[(\tilde{S}^{-1} - \Sigma^{-1})^2 \Sigma]$	$\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i$
Log-Euclidian	$\ \log(\tilde{S}) - \log(\Sigma)\ _F$	$\exp\left[\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \log(\Sigma) u_i\right]$
Fréchet	$\ \tilde{S}^{1/2} - \Sigma^{1/2}\ _F$	$\left(\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{1/2} u_i\right)^2$
Quadratic	$\ \Sigma^{-1} \tilde{S} - \mathbb{I}\ _F$	$\frac{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-1} u_i}{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^{-2} u_i}$
Inverse Quadratic	$\ \tilde{S}^{-1} \Sigma - \mathbb{I}\ _F$	$\frac{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma^2 u_i}{\frac{1}{p-n} \sum_{i=1}^{p-n} u_i' \Sigma u_i}$

Table 5.1: Formulas for shrinking null eigenvalues.

5.2 Large-Dimensional Asymptotic Framework

Assumption 6 (Singular). *The ratio p/n converges, as $n \rightarrow \infty$, to a finite limit $c > 1$. Furthermore, there exists a compact interval included in $(1, +\infty)$ that contains p/n for all n large enough.*

Given that the first $p - n$ sample eigenvalues are devoid of informational content, it is judicious to focus on the e.d.f of the n other ones: $\forall x \in \mathbb{R} \quad \underline{F}_n(x) := \frac{1}{n} \sum_{i=p-n+1}^p \mathbb{1}_{[\lambda_{n,i}, +\infty)}(x)$. Under Assumptions 3–6, it admits a nonrandom limit:

$$\forall x \in \mathbb{R} \quad \underline{F}_n(x) \xrightarrow{\text{a.s.}} \underline{F}(x) := (1 - c)\mathbb{1}_{[0, +\infty)}(x) + cF(x) . \quad (5.1)$$

Of particular interest will be its Stieltjes transform: $\forall z \in \mathbb{C}^+ \quad m_{\underline{F}}(z) := \int \frac{1}{\lambda - z} d\underline{F}(\lambda)$, which admits a continuous extension onto the real line: $\forall x \in \mathbb{R} \quad \check{m}_{\underline{F}}(x) := \lim_{z \in \mathbb{C}^+ \rightarrow x} m_{\underline{F}}(z)$.

5.3 Optimal Shrinkage of Null Sample Eigenvalues

At this stage, what we need is an equivalent of Equation (2.4) that pertains to the shrinkage of the null sample eigenvalues. It comes from Theorem 9 of [Ledoit and P  ch   \(2011\)](#):

$$\frac{1}{p-n} \sum_{i=1}^{p-n+1} u'_{n,i} \gamma(\Sigma_n) u_{n,i} \approx \frac{1}{p} \sum_{j=1}^p \gamma(\widehat{\tau}_{n,j}) \cdot \frac{1}{\left(1 - \frac{n}{p}\right) \left[1 + \check{m}_{n,p}^{\widehat{\tau}_n}(0) \widehat{\tau}_{n,j}\right]}, \quad (5.2)$$

where $\widehat{\tau}_n := \{\widehat{\tau}_{n,j}\}_{j=1}^p$ is, as before, the estimator of population eigenvalues obtained by numerically inverting the QuEST function, and $\check{m}_{n,p}^{\widehat{\tau}_n}(0)$ is a strongly consistent estimator of $\check{m}_{\underline{F}}(0)$ which is another by-product of the QuEST function (when $p > n$). As per [Ledoit and Wolf \(2015, Section 3.2.2\)](#), $\check{m}_{n,p}^{\widehat{\tau}_n}(0)$ is the unique solution $m \in (0, \infty)$ to the equation

$$\frac{1}{m} = \frac{1}{n} \sum_{j=1}^p \frac{\hat{\tau}_{n,j}}{1 + \hat{\tau}_{n,j} m} . \quad (5.3)$$

Equation (5.2) enables us to extend the squared-dot-product function $\theta(x, t)$ presented in Section 3.3 to handle $x = 0$. The next figure graphs

$$\theta(0, t) := \frac{1}{\left(1 - \frac{1}{c}\right) \left[1 + \check{m}_F(0) t\right]} \quad (5.4)$$

as a function of t for various values of the concentration ratio p/n . We use the same baseline scenario as in Figure 3.1: the population eigenvalues are evenly spread in the interval $[1, 5]$.

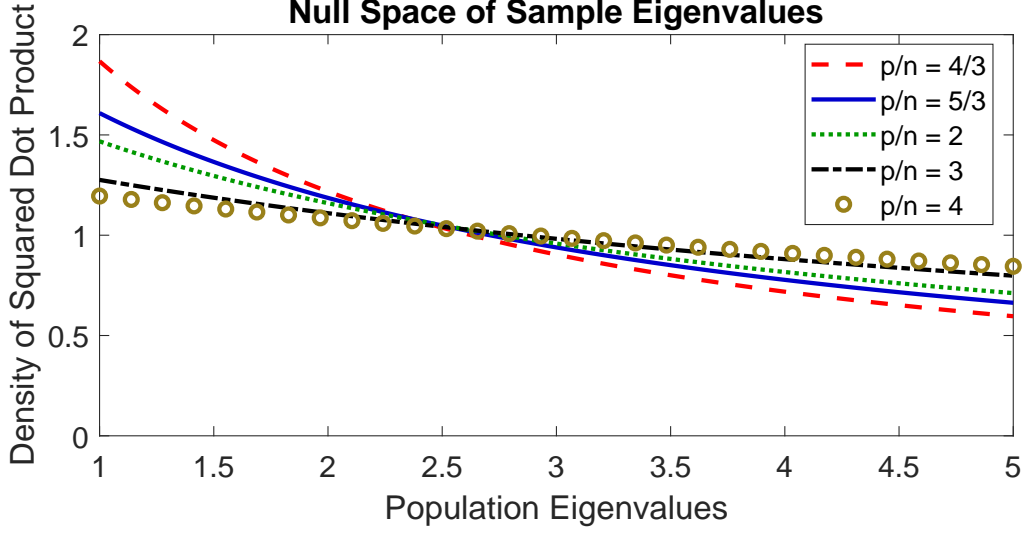


Figure 5.1: The Radon-Nykodym derivative $\theta(0, t)$ as a function of the population eigenvalues. This plot shows how aligned the null-space sample eigenvectors are with the population eigenvectors.

Eigenvectors in the null space of the sample covariance matrix tend to be more (less) aligned with population eigenvectors corresponding to small (large) population eigenvalues, which makes intuitive sense. The degree of preferential alignment is inversely related to the concentration ratio, as a high ratio p/n disorients the sample eigenvectors. The overall pattern is highly nonlinear, and could only be pinned down through Equations (5.3)–(5.4) from RMT. Note that, by construction, the dimension-normalized density of the squared dot-product averages to 1, so it is deviations from the baseline number of 1 that are informative.

5.4 Covariance Matrix Estimation in the Singular Case

Theorems 4.1–4.6 remain valid when $c > 1$, with the understanding that the estimator of the squared dot-product in the null space of the sample covariance matrix ($i = 1, \dots, p - n$) is

$$\forall j = 1, \dots, p \quad \hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j}) = \hat{\theta}_n(0, \hat{\tau}_{n,j}) := \frac{1}{\left(1 - \frac{n}{p}\right) \left[1 + \underline{m}_{n,p}^{\hat{\tau}}(0) \hat{\tau}_{n,j}\right]}. \quad (5.5)$$

In order to show how this works, we need only state and prove the singular-case counterpart of Theorem 4.5, as the other theorems are adapted from $p < n$ to the $p > n$ case in similar fashion.

Theorem 5.1. *Under Assumptions 3–6, the Generalized Frobenius loss admits an almost sure (deterministic) limit, which is minimized by the nonlinear shrinkage formula*

$$\hat{\varphi}_n^\gamma(\lambda_{n,i}) := \gamma^{-1} \left(\frac{1}{p} \sum_{j=1}^p \gamma(\hat{\tau}_{n,j}) \cdot \hat{\theta}_n(\lambda_{n,i}, \hat{\tau}_{n,j}) \right), \quad (5.6)$$

where the bivariate function $\hat{\theta}_n(x, t)$ is given by Equation (3.7) for $x > 0$, and $\hat{\theta}_n(0, t)$ is given by Equation (5.5). The resulting covariance matrix estimator is $\hat{S}_n^\gamma := \sum_{i=1}^p \hat{\varphi}_n^\gamma(\lambda_{n,i}) \cdot u_{n,i} u_{n,i}'$.

6 Monte Carlo Simulations

The goal of this section is to illustrate on simulated data that there is generally great benefit in using the shrinkage estimator that is tailored to the loss function one has selected.

6.1 General Setup

The population eigenvalues are distributed as follows: 20% are equal to 1, 40% are equal to 3, and 40% are equal to 10. This is a challenging problem originally introduced by [Bai and Silverstein \(1998\)](#). We use the 12 loss functions from Tables 2.1–2.3. For each one, we compute the FSOPT ‘estimator’ specific to the particular loss function, as well as all 7 *bona fide* shrinkage estimators presented in the paper. We use the same notation as [Ledoit and Wolf \(2018\)](#): \hat{S}_n° is the estimator optimal with respect to Frobenius, Inverse Stein and Minimum Variance losses; \hat{S}_n^* is the one optimal with respect to Stein and Inverse Frobenius losses; and $\hat{S}_n^{*\circ}$ the one optimal with respect to the Symmetrized Stein’s loss. In addition, the identity matrix (rescaled to have same trace as the sample covariance matrix), the sample covariance matrix, and the linear shrinkage estimator of [Ledoit and Wolf \(2004\)](#) are also computed for reference purposes. The results are averaged over 1,000 simulations.

6.2 Nonsingular Case

To produce the results of Table 6.1, the matrix dimension is $p = 100$ and the sample size is $n = 200$.

In each row, the performance of the best *bona fide* estimator is printed in bold. One can see that the winner is always the estimator tailor-made for the loss function of the given row. Sometimes the difference with the other estimators is quite stark. Obviously, the FSOPT always dominates, but usually the excess loss of the best *bona fide* estimator is quite small. This finding reinforces the message that the asymptotically optimal estimators listed in the present paper perform as well as they ought to, even in finite samples.

Regarding the other (reference) estimators, linear shrinkage does better than the two ingredients that it interpolates, the scaled identity matrix and the sample covariance matrix, with respect to all but one of the 12 loss functions. This is good news because in theory its shrinkage intensity is optimized with respect to the Frobenius loss only. Linear shrinkage performs honorably across the board for such a simple estimator: it even manages to beat some nonlinear shrinkage estimators in almost every row, typically a couple of them. Needless to say, linear shrinkage never beats the nonlinear shrinkage formula optimized to the loss function in the given row, which shows that it ‘leaves some money on the table’ and that shrinking

Loss Function	FSOPT	Identity	Sample	Linear	\hat{S}_n°	\hat{S}_n^*	\hat{S}_n^\circledast	\hat{S}_n^{LE}	$\hat{S}_n^{\text{FRÉ}}$	\hat{S}_n^{Q}	\hat{S}_n^{QINV}
Frobenius	5.755	14.644	14.771	7.382	5.925	7.747	6.441	6.297	6.016	16.094	8.226
Inverse Stein	0.152	0.326	0.710	0.184	0.157	0.216	0.171	0.174	0.161	0.464	0.226
Minimum Variance	1.095	2.721	2.757	1.370	1.138	1.162	1.144	1.163	1.148	1.172	1.359
Stein	0.150	0.690	0.310	0.289	0.213	0.154	0.168	0.168	0.186	0.222	0.513
Inverse Frobenius	0.048	0.144	0.852	0.098	0.069	0.051	0.055	0.054	0.060	0.069	0.126
Symmetrized Stein	0.329	1.016	1.020	0.473	0.370	0.371	0.339	0.342	0.347	0.686	0.739
Weighted Frobenius	0.228	1.016	0.504	0.377	0.317	0.233	0.251	0.256	0.281	0.336	0.743
Disutility	0.290	0.504	5.257	0.342	0.298	0.442	0.329	0.343	0.311	0.919	0.405
Log-Euclidian	0.291	0.859	0.756	0.427	0.329	0.324	0.301	0.300	0.307	0.598	0.637
Fréchet	0.286	0.772	0.585	0.367	0.300	0.347	0.302	0.299	0.294	0.703	0.504
Quadratic	0.292	4.212	1.013	1.289	0.978	0.462	0.647	0.668	0.803	0.298	2.927
Inverse Quadratic	0.260	0.503	9.490	0.376	0.449	1.104	0.685	0.737	0.576	2.642	0.264

Table 6.1: Average losses computed for various estimators when $p = 100$ and $n = 200$. Best numbers are in **bold face**.

nonlinearly (in the appropriate way) delivers yet another round of improvement over and above linear shrinkage.

6.3 Comparison of Shrinkage Formulas

Confirming the ordering of Proposition 2.1, Figure 6.1 gives further insight into the loss functions by showing how the 7 estimators shrink the sample eigenvalues in this case.

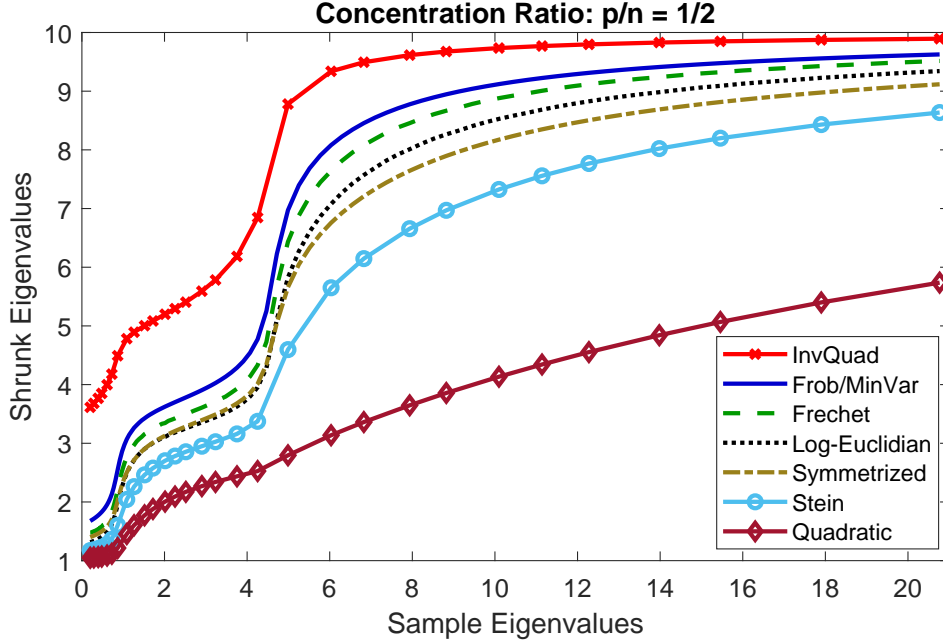


Figure 6.1: Comparison of 7 nonlinear shrinkage formulas.

The Quadratic and the Inverse Quadratic shrinkage formulas stand out as ‘outliers’, as shown by Proposition 2.1. In Table 6.1, the estimators \hat{S}_n^Q and \hat{S}_n^{QINV} display erratic performances when measured against other loss functions than their own. The other estimators are better able to deliver respectable performance across foreign loss functions. The estimators \hat{S}_n° and \hat{S}_n^* have strong backing, from the Minimum-Variance and Stein’s loss, respectively; the Log-Euclidian estimator \hat{S}_n^{LE} represents an excellent ‘neutral’ compromise that has strong foundations in the differential geometry of the manifold of tensors (a.k.a. positive definite matrices).

6.4 Singular Case

Table 6.2 presents further results when $p = 200$ and $n = 100$. Once again, the pattern is confirmed overall, except for one violation: \hat{S}_n^{LE} beats \hat{S}_n° both ‘home’ and ‘away’: with respect to the Log-Euclidian loss *and*, unexpectedly, with respect to the Symmetrized Stein’s loss also. (In other simulations not reported here, we double-checked that \hat{S}_n° does beat \hat{S}_n^{LE} with respect to the Symmetrized Stein’s loss when dimension is high enough, as implied by large-dimensional asymptotic theory.) Both of these estimators plow the same narrow but interesting field of

Loss Function	FSOPT	Identity	Linear	\hat{S}_n°	\hat{S}_n^*	\hat{S}_n^\oplus	\hat{S}_n^{LE}	$\hat{S}_n^{\text{FRÉ}}$	\hat{S}_n^{Q}	\hat{S}_n^{QINV}
Frobenius	11.250	14.644	11.774	11.360	15.343	12.590	12.559	11.688	22.044	17.560
Inverse Stein	0.274	0.326	0.280	0.275	0.418	0.308	0.315	0.285	0.729	0.358
Minimum Variance	2.221	2.721	2.271	2.232	2.255	2.239	2.253	2.241	2.301	2.362
Stein	0.290	0.690	0.510	0.496	0.299	0.356	0.347	0.407	0.339	1.071
Inverse Frobenius	0.091	0.144	0.128	0.126	0.094	0.107	0.104	0.114	0.102	0.163
Symmetrized Stein	0.656	1.016	0.789	0.772	0.716	0.665	0.662	0.693	1.068	1.428
Weighted Frobenius	0.397	1.015	0.707	0.697	0.406	0.475	0.470	0.557	0.463	1.933
Disutility	0.453	0.504	0.459	0.455	0.691	0.501	0.516	0.469	1.292	0.530
Log-Euclidian	0.587	0.859	0.687	0.672	0.636	0.595	0.592	0.614	0.914	1.123
Fréchet	0.572	0.772	0.610	0.595	0.697	0.592	0.593	0.577	1.032	1.012
Quadratic	0.395	4.210	2.718	2.648	0.803	1.422	1.367	1.895	0.490	8.418
Inverse Quadratic	0.321	0.503	0.517	0.525	1.851	0.947	1.033	0.726	4.148	0.322

Table 6.2: Average losses computed for various estimators when $p = 200$ and $n = 100$. Best numbers are in **bold face**.

estimators that are equivariant with respect to matrix inversion, so it is not completely surprising that the estimator that beats \hat{S}_n^{\otimes} on its home turf shares the same desirable property.

Remarks regarding the two simple estimators (scaled identity and linear shrinkage) essentially go in the same direction as in Section 6.2. We excluded the sample covariance matrix because it is not invertible, so most of the loss functions return $+\infty$.

6.5 Comparison with Simpler Alternatives: The Matrix as a Whole

We examine two alternative approaches that make compromises in order to obtain formulas that are simpler than the ones developed in this paper. Linear shrinkage (Ledoit and Wolf, 2004) compromises by forcing all eigenvalues to be shrunk towards the same target with the same shrinkage intensity, and by considering only the Frobenius loss. The spiked-model approach (Donoho et al., 2018) compromises by assuming that the *bulk* of the population eigenvalues (meaning: all of them except for a vanishing fraction of them, called spikes) are equal.

In this Monte Carlo simulation, we take both the simpler linear shrinkage and the simpler spiked model ‘outside of their comfort zone’ by considering 8 different loss functions, and by considering specifications where the bulk of the population eigenvalues can be different from each other. Most applied researchers will be interested to know how robust the simplified formulas are against violations of the framework under which they have been derived.

The 8 loss functions that we consider are all of the ones in the intersection of the 12 that we consider in the present paper with the 18 for which Donoho et al. (2018) deliver closed-form spike shrinkage: 1) Frobenius, 2) Stein, 3) Inverse Frobenius, 4) Inverse Stein, 5) Symmetrized Stein, 6) Fréchet, 7) Quadratic, and 8) Inverse Quadratic. The code for spike shrinkage was taken directly from Donoho et al. (2016). As far as the population eigenvalues are concerned, the initial specification is to have a single spike at 10, and the $p - 1$ bulk eigenvalues equal to 1. From this base, we will allow for heterogeneity in the bulk by keeping half of the bulk equal to 1, while setting the other half equal to $\bar{\tau} \in [1, 5]$. It is only fair to allow bulk eigenvalues to not all be equal: after all, this is the generic case, and the special case where all bulk eigenvalues are equal is a measure-zero subset of the set of all possible eigenvalue combinations, so it is not necessarily representative of real-world applications. We put 100 eigenvalues in the bulk, plus (as mentioned above) a single spike, for a total of $p = 101$ eigenvalues. We take the (limiting) concentration ratio to be $c = 1/2$, which implies $n = 202$.

Figure 6.2 displays the Percentage Relative Improvement in Average Loss (PRIAL):

$$\text{PRIAL}(\mathcal{L}_n^i, \tilde{S}_n) := \frac{\mathbb{E}[\mathcal{L}_n^i(S_n, \Sigma_n)] - \mathbb{E}[\mathcal{L}_n^i(\tilde{S}_n, \Sigma_n)]}{\mathbb{E}[\mathcal{L}_n^i(S_n, \Sigma_n)] - \mathbb{E}[\mathcal{L}_n^i(\hat{S}_n^{*,i}, \Sigma_n)]} \times 100\%, \quad (6.1)$$

where \mathcal{L}_n^i denotes one of the eight loss functions listed above, $\hat{S}_n^{*,i}$ denotes the FSOPT ‘estimator’ tailored to each specific loss function as per Tables 2.1–2.3, \tilde{S}_n denotes the estimator under consideration (whether linear shrinkage, spike shrinkage, or nonlinear shrinkage), and the expectation is approximated by the average of 1,000 Monte Carlo simulations. By construction,

the PRIAL of the sample covariance matrix is 0% whereas the PRIAL of the FSOPT ‘estimator’ is 100%. The PRIAL measures how much of the potential for improvement relative to the sample covariance matrix is attained by a given estimator \tilde{S}_n .

One can see that, even though the dimension is not overly large ($p \approx 100$), nonlinear shrinkage captures nearly 100% of the potential improvement with respect to all loss functions, regardless of how spread out are the bulk population eigenvalues. Linear shrinkage has more of a mixed performance, but still manages to capture at least 50% of the potential improvement most of the time. It beats the sample covariance matrix with respect to all 8 loss functions, which shows that its attractiveness extends far beyond the Frobenius loss under which it was originally derived. It beats spike shrinkage as long as $\bar{\tau} \geq 2.5$ in all cases but one (the Quadratic loss, where they are essentially identical). Also worth noting is that linear shrinkage is the only estimator that keeps the same formula in all 8 subplots of Figure 6.2, so it is ‘fighting with one hand tied behind the back’ when it has to compete against the other two shrinkage estimators under the 7 loss functions different from Frobenius loss.

As expected, the performance of spike shrinkage is near-perfect when its specification matches reality ($\bar{\tau} = 1$: all bulk eigenvalues are equal), but it monotonically degrades as soon as the bulk population eigenvalues become heterogeneous. This drop in performance is not so pronounced with the Inverse Frobenius, Inverse Stein and Inverse Quadratic losses, but it is very pronounced with the 5 other loss functions. There is even a case ($\bar{\tau} = 5$ and Fréchet loss) where spike shrinkage underperforms the sample covariance matrix, which results in a *negative* PRIAL. This is a result that should be expected purely from theory: Unlike linear and nonlinear shrinkage, spike shrinkage can actually be *worse* than the sample covariance matrix even in the large-dimensional asymptotic limit.

Compared to the two simpler alternatives, nonlinear shrinkage does better across the board. In particular, there are scenarios where the optimal nonlinear shrinkage formula is (nearly) linear; and, even then, nonlinear shrinkage performs just as well as linear shrinkage, for all practical purposes. Similarly, there are scenarios where the spiked covariance model holds perfectly true (bulk maximum equal to one); and, even then, nonlinear shrinkage performs just as well as spike shrinkage, for all practical purposes.

The overall conclusion is that, among the simpler formulas, linear shrinkage can ‘leave some money on the table’ when the optimal shrinkage is highly nonlinear whereas spike shrinkage is vulnerable to the risk that its stringent specification of bulk-eigenvalue equality is violated by reality. Only the full-blown nonlinear shrinkage formulas derived in this paper avoid both pitfalls and deliver state-of-the-art enhancement of the sample covariance matrix across the board.

6.6 Comparison with Simpler Alternatives: Focus on the Spike

There might be a perception that nonlinear shrinkage does better on the bulk, whereas spike shrinkage does better on the spike. This is hard to justify formally, as the loss functions used in the spike literature pertain to the whole covariance matrix, placing no special over-weight on

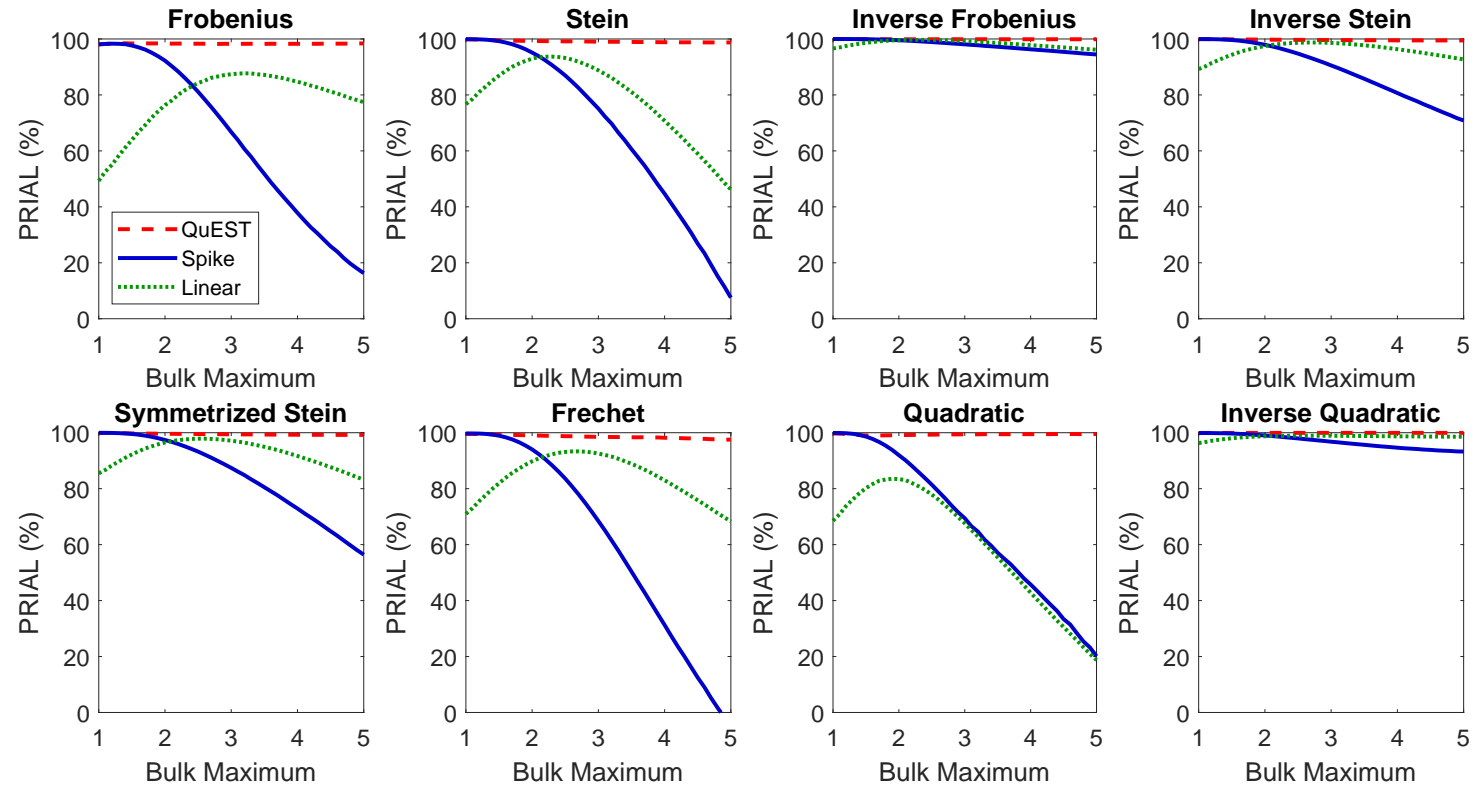


Figure 6.2: PRIAL as a function of the spread of the bulk of population eigenvalues for 4 *bona fide* estimators under 8 loss functions.

the spike. Nonetheless, in Table B.1 we isolate the spike’s contribution to overall risk, for the 8 loss functions from Section 6.5. This is easy to do because every one of these loss functions can be decomposed as a sum of contributions attributable to each of the p eigenvalues. Figure 6.3 displays the Monte Carlo simulation results.

The performance of the sample covariance matrix can be so erratic as to not even be on the same scale as the other estimators in many scenarios. The risk contribution of the FSOPT is zero by construction. The pattern is that spike shrinkage and nonlinear shrinkage estimate the spike equally well when the spike model’s assumption of equal bulk eigenvalues is satisfied, or nearly satisfied. However, as the bulk spreads out, nonlinear shrinkage estimates the spike more accurately than spike shrinkage for all 8 loss functions where methods overlap.

7 Conclusion

In this paper, we have

- developed a new estimator of the angle between *any* sample eigenvector and *any* population eigenvector by exploiting a sophisticated equation from random matrix theory (RMT);
- doubled the number of loss functions that can be handled from 6 to 12 (compared to related earlier work), which can only be achieved by the new technique of angle estimation;
- proposed a classification of loss functions by their finite-sample optimal shrinkage formulas;
- increased the number of asymptotically optimal nonlinear shrinkage formulas from 3 to 7 (compared to related earlier work);
- established an ordering of the nonlinear shrinkage formulas (from largest to smallest);
- delivered two infinite families of loss functions and their (correspondingly infinite family of) optimal nonlinear shrinkage formulas, which can only be achieved by the new technique of angle estimation;
- and introduced a new loss function founded on the economic concept of utility maximization.

As a simpler alternative approach, Donoho et al. (2018) consider a *ménagerie* of 26 loss functions under the spiked covariance model of Johnstone (2001). The key distinction in this model is between the bulk, which is comprised by eigenvalues packed shoulder-to-shoulder like sardines, and the spikes, which are a few select eigenvalues large enough to separate from the bulk. Donoho et al. (2018) treat the spikes carefully, but they just collapse the bulk. This approach is perfectly legitimate under the assumption that they make, namely, that all bulk population eigenvalues are equal. However, in many applications, such an assumption is unrealistic or may not be known to hold. In the general case, bulk population eigenvalues are nonequal, so valuable information can be gleaned from the angle between sample and population eigenvectors, and from applying differentiated shrinkage inside the bulk. Monte Carlo simulations show that the resulting nonlinear shrinkage performs, for all practical purposes, just as well as spike shrinkage

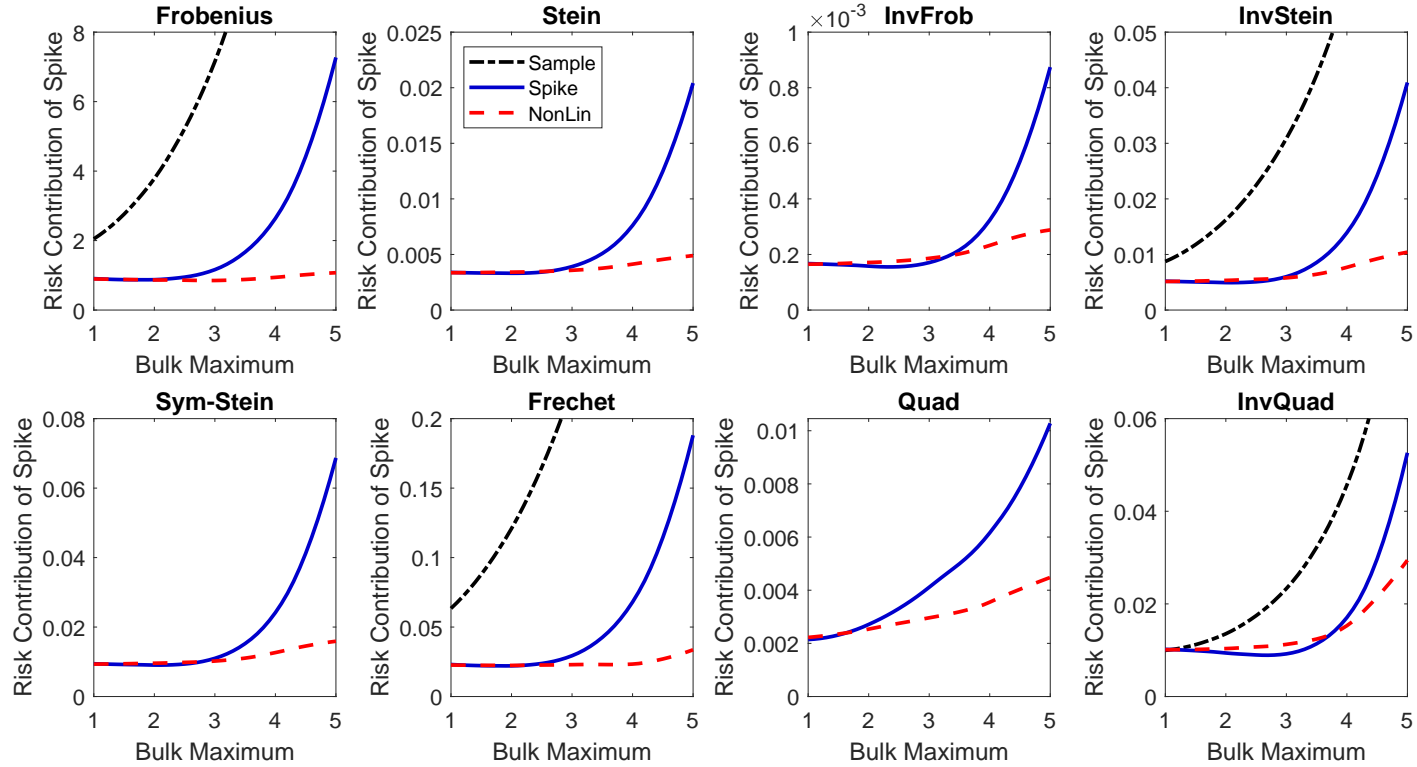


Figure 6.3: Normalized contribution of the spike to the overall risk of the covariance matrix for 8 loss functions, as a function of the spread of the bulk of population eigenvalues. The setup is the same as in Section 6.5, except that we run 10,000 simulations.

when bulk population eigenvalues are equal, but is often much better when they are unequal. Therefore, the KISS (*Keep it simple, statistician!*) principle does not seem to benefit applied researchers: by upgrading instead from spike shrinkage to full-blown nonlinear shrinkage, they have, basically, nothing to lose but much to gain. (In addition, at least currently, spike shrinkage is only available for the case where the dimension is smaller than the sample size, which limits practical applications.)

Having said this, [Donoho et al. \(2018\)](#) roll out a clever technology that convincingly documents three closely interrelated facts that have not garnered sufficient attention in this field:

1. The choice of loss function has a profound effect on optimal estimation.
2. Eigenvalue inconsistency: The sample eigenvalues are spread, biased, and shifted away from their theoretical (population) counterparts by an asymptotically predictable amount.
3. Eigenvector inconsistency: The angles between the sample eigenvectors and the corresponding population eigenvectors have nonzero asymptotic limits.

Such fundamental truths need to be hammered in again and again, in every possible way.

Finally, we may say a word about the choice of loss function. 12 of them have been solved already, yielding 7 different nonlinear shrinkage formulas, in addition to the infinite families, which should be more than enough to satisfy any reasonable need. By definition it is the duty of the end-user to pick the loss function, but perhaps some light-touch guidance can help orient readers through a forest with so many trees. For anyone interested in using a covariance matrix estimator to minimize variance, risk, or noise in any sense, then certainly the Minimum Variance loss function is the appropriate one. An additional advantage is that a new technology has arisen for this purpose that is no more complex than kernel density estimation, and so is extremely fast and scalable to ultra-high dimensions ([Ledoit and Wolf, 2020](#)). For researchers concerned with the decision-theoretic aspects of the problem, a loss function based on the Kullback-Leibler divergence (also called relative entropy), such as Stein’s loss, is the natural candidate. For other applications, such as fMRI tensors, where it is important to regard eigenvalues close to zero as being ‘as distant’ as eigenvalues close to infinity, then the Log-Euclidian loss function is well suited: It appears a good compromise because it produces shrunken eigenvalues that lie in between the ones from the Minimum-Variance loss and the ones from Stein’s loss.

References

- Arsigny, V., Fillard, P., Pennec, X., and Ayache, N. (2006). Log-euclidean metrics for fast and simple calculus on diffusion tensors. *Magnetic Resonance in Medicine*, 56(2):411–421.
- Bai, Z. D. and Silverstein, J. W. (1998). No eigenvalues outside the support of the limiting spectral distribution of large-dimensional random matrices. *Annals of Probability*, 26(1):316–345.
- Bickel, P. J. and Levina, E. (2008). Regularized estimation of large covariance matrices. *Annals of Statistics*, 36(1):199–227.
- Donoho, D. L., Gavish, M., and Johnstone, I. M. (2016). Code supplement to “optimal shrinkage of eigenvalues in the spiked covariance model”. Available online at <http://purl.stanford.edu/xy031gt1574>.
- Donoho, D. L., Gavish, M., and Johnstone, I. M. (2018). Optimal shrinkage of eigenvalues in the spiked covariance model. *Annals of Statistics*, 46(4):1742–1778.
- Dowson, D. and Landau, B. (1982). The Fréchet distance between multivariate normal distributions. *Journal of Multivariate Analysis*, 12(3):450–455.
- El Karoui, N. (2008). Spectrum estimation for large dimensional covariance matrices using random matrix theory. *Annals of Statistics*, 36(6):2757–2790.
- Engle, R. F., Ledoit, O., and Wolf, M. (2019). Large dynamic covariance matrices. *Journal of Business & Economic Statistics*, 37(2):363–375.
- Ghosh, M. and Sinha, B. (1987). Inadmissibility of the best equivariant estimators of the variance-covariance matrix, the precision matrix, and the generalized variance under entropy loss. *Statistics & Decisions*, 5(3-4):201–228.
- Haff, L. (1979a). Estimation of the inverse covariance matrix: Random mixtures of the inverse wishart matrix and the identity. *Annals of Statistics*, pages 1264–1276.
- Haff, L. (1979b). An identity for the Wishart distribution with applications. *Journal of Multivariate Analysis*, 9(4):531–544.
- James, W. and Stein, C. (1961). Estimation with quadratic loss. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* **1**, pages 361–380.
- Johnstone, I. M. (2001). On the distribution of the largest eigenvalue in principal component analysis. *Annals of Statistics*, 29(2):295–327.
- Jun, W. (2017). *On High-Dimensional Covariance Matrices Estimation*. PhD thesis, National University of Singapore, Department of Statistics and Applied Probability. Available online at <http://scholarbank.nus.edu.sg/handle/10635/135450>.

- Kubokawa, T. and Konno, Y. (1990). Estimating the covariance matrix and the generalized variance under a symmetric loss. *Annals of the Institute of Statistical Mathematics*, 42(2):331–343.
- Kullback, S. and Leibler, R. A. (1951). On information and sufficiency. *Annals of Mathematical Statistics*, 22(1):79–86.
- Ledoit, O. and Péché, S. (2011). Eigenvectors of some large sample covariance matrix ensembles. *Probability Theory and Related Fields*, 150(1–2):233–264.
- Ledoit, O. and Wolf, M. (2004). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, 88(2):365–411.
- Ledoit, O. and Wolf, M. (2012). Nonlinear shrinkage estimation of large-dimensional covariance matrices. *Annals of Statistics*, 40(2):1024–1060.
- Ledoit, O. and Wolf, M. (2015). Spectrum estimation: A unified framework for covariance matrix estimation and PCA in large dimensions. *Journal of Multivariate Analysis*, 139(2):360–384.
- Ledoit, O. and Wolf, M. (2017). Numerical implementation of the QuEST function. *Computational Statistics & Data Analysis*, 115:199–223.
- Ledoit, O. and Wolf, M. (2018). Optimal estimation of a large-dimensional covariance matrix under Stein’s loss. *Bernoulli*, 24(4B). 3791–3832.
- Ledoit, O. and Wolf, M. (2020). Analytical nonlinear shrinkage of large-dimensional covariance matrices. *Annals of Statistics*. Forthcoming.
- Leung, P. L. and Muirhead, R. J. (1987). Estimation of parameter matrices and eigenvalues in manova and canonical correlation analysis. *Annals of Statistics*, 15(4):1651–1666.
- Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, 7:77–91.
- Selliah, J. B. (1964). *Estimation and Testing Problems in a Wishart Distribution*. PhD thesis, Stanford University, Department of Statistics.
- Sharma, D. and Krishnamoorthy, K. (1985). Empirical Bayes estimators of normal covariance matrix. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 247–254.
- Silverstein, J. W. (1995). Strong convergence of the empirical distribution of eigenvalues of large-dimensional random matrices. *Journal of Multivariate Analysis*, 55:331–339.
- Silverstein, J. W. and Bai, Z. D. (1995). On the empirical distribution of eigenvalues of a class of large-dimensional random matrices. *Journal of Multivariate Analysis*, 54:175–192.
- Silverstein, J. W. and Choi, S. I. (1995). Analysis of the limiting spectral distribution of large-dimensional random matrices. *Journal of Multivariate Analysis*, 54:295–309.

- Stein, C. (1956). Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, pages 197–206. University of California Press.
- Stein, C. (1975). Estimation of a covariance matrix. Rietz lecture, 39th Annual Meeting IMS. Atlanta, Georgia.
- Stein, C. (1986). Lectures on the theory of estimation of many parameters. *Journal of Mathematical Sciences*, 34(1):1373–1403.
- Stein, C. M. (1969). Multivariate Analysis I. Technical Report No. 42, Department of Statistics, Stanford University. (Notes prepared by Morris L. Eaton.).
- Stieltjes, T. J. (1894). Recherches sur les fractions continues. *Annales de la Faculté des Sciences de Toulouse 1^{re} Série*, 8(4):J1–J122.
- Wigner, E. P. (1955). Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics*, 62(3):548–564.

A Portfolio Selection and the Disutility Loss

Here we explain how the Weighted Frobenius loss of [Sharma and Krishnamoorthy \(1985\)](#) applied to the precision matrix can be interpreted as quadratic *disutility*. Consider the standard mean-variance optimization problem with quadratic utility function:

$$\max_w w' \mu - \frac{1}{2} \rho w' \Sigma w, \quad (\text{A.1})$$

where μ denotes some vector of expected return selected by the end-user, and $\rho > 0$ the risk aversion parameter (cf. [Markowitz \(1952\)](#)). The first-order condition is $\mu - \rho \Sigma w = 0$, and the solution is $w = \Sigma^{-1} \mu / \rho$. In practice, we only observe an estimator \tilde{S} of the unobservable population covariance matrix Σ , so the plug-in estimator for the optimal weight vector is $\tilde{w} = \tilde{S}^{-1} \mu / \rho$. The quadratic utility associated with this vector is

$$\tilde{w}' \mu - \frac{1}{2} \rho \tilde{w}' \Sigma \tilde{w} = \frac{1}{\rho} \mu' \tilde{S}^{-1} \mu - \frac{1}{2 \rho} \mu' \tilde{S}^{-1} \Sigma \tilde{S}^{-1} \mu. \quad (\text{A.2})$$

At this point, the risk aversion coefficient ρ becomes irrelevant because, regardless of ρ , all investors want to find a covariance matrix estimator \tilde{S} that maximizes

$$\mu' \tilde{S}^{-1} \mu - \frac{1}{2} \mu' \tilde{S}^{-1} \Sigma \tilde{S}^{-1} \mu. \quad (\text{A.3})$$

As argued in further detail in [Engle et al. \(2019, Section 4.3\)](#), under large-dimensional asymptotics in conjunction with RMT, there is a key approximation:

$$\mu' A \mu \approx \|\mu\|^2 \text{Tr}[A]. \quad (\text{A.4})$$

From this we can streamline the objective function so as to make it equally adept at fitting the needs of all users who may have different views on the choice of vector μ :

$$\|\mu\|^2 \text{Tr}[\tilde{S}^{-1}] - \frac{1}{2} \|\mu\|^2 \text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}]. \quad (\text{A.5})$$

The squared Euclidian norm of the linear constraint vector μ becomes irrelevant to the estimation process, so we are left with just maximizing $\text{Tr}[\tilde{S}^{-1}] - \frac{1}{2} \text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}]$.¹ This is obviously equivalent to *minimizing* with respect to the rotation-equivariant estimator \tilde{S} the shifted loss function

$$- \text{Tr}[\tilde{S}^{-1}] + \frac{1}{2} \text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}] + \frac{1}{2} \text{Tr}[\Sigma^{-1}] = \frac{1}{2} \text{Tr}[(\tilde{S}^{-1} - \Sigma^{-1})^2 \Sigma]. \quad (\text{A.6})$$

We recognize immediately the Weighted Frobenius loss function applied to the precision matrix

$$\mathcal{L}^D(\tilde{S}, \Sigma) := \frac{\text{Tr}[(\tilde{S}^{-1} - \Sigma^{-1})^2 \Sigma]}{\text{Tr}[\Sigma]}, \quad (\text{A.7})$$

up to some multiplicative renormalizations. This approach nicely dovetails with the Minimum Variance loss function of [Engle et al. \(2019\)](#), as it gives the same optimal nonlinear shrinkage formula, but pins down the scaling factor internally rather than by appealing to the external argument of trace preservation.

¹Note the close connection with the Minimum Variance loss function, which was essentially based on $\text{Tr}[\tilde{S}^{-1} \Sigma \tilde{S}^{-1}] / (\text{Tr}[\tilde{S}^{-1}])^2$. So, instead of dividing, we are subtracting here. Given that both of them are based on mean-variance portfolio optimization, it is reassuring to observe that they do not contradict each other.

B Finite-Sample Optimal Estimators for Various Losses

In this section, all loss functions are normalized by dimension so they admit an almost sure limit under large-dimensional asymptotics. Given that the objective is to optimize over the rotation-equivariant covariance matrix estimator \tilde{S} , we call ‘constant’ any quantity that does not depend on \tilde{S} . In addition, for the eight loss functions that overlap with Table 2 of [Donoho et al. \(2018\)](#), we indicate how to partial out the incremental contribution to loss attributable to imperfect shrinkage of the largest eigenvalue, in order to be able to focus on the spike in some of the simulations in [Section 6](#).

B.1 Frobenius

$$\mathcal{L}^F(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr} [(\tilde{S} - \Sigma)^2] = \frac{1}{p} \text{Tr} [(\Sigma - \tilde{S})^2] \quad (\text{B.1})$$

$$= \frac{1}{p} \text{Tr} [(\Sigma - U\tilde{D}U')(\Sigma - U\tilde{D}U')] \quad (\text{B.2})$$

$$= \frac{1}{p} \text{Tr} [U'(\Sigma - U\tilde{D}U')UU'(\Sigma - U\tilde{D}U')U] \quad (\text{B.3})$$

$$= \frac{1}{p} \text{Tr} [(U'\Sigma U - \tilde{D})^2] = \frac{1}{p} \sum_{i=1}^p (u'_i \Sigma u_i - \tilde{d}_i)^2 + \text{constant} , \quad (\text{B.4})$$

which is clearly minimized when $\tilde{d}_i = u'_i \Sigma u_i$ for all $i = 1, \dots, p$. It is clear from Equation (B.4) that the incremental contribution to Frobenius loss due to imperfect shrinkage of the largest eigenvalue is $(u'_p \Sigma u_p - \tilde{d}_p)^2$. By the adjective ‘imperfect’, we mean worse than the unattainable FSOPT. Note that here, and for subsequent loss functions, we do not divide by p because there is only a single eigenvalue (the largest one) under consideration.

B.2 Inverse Stein

$$\mathcal{L}^{\text{SINV}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr}[\tilde{S}^{-1}\Sigma] - \frac{1}{p} \log [\det(\tilde{S}^{-1}\Sigma)] - 1 \quad (\text{B.5})$$

$$= \frac{1}{p} \text{Tr}[\Sigma\tilde{S}^{-1}] - \frac{1}{p} \log [\det(\Sigma\tilde{S}^{-1})] - 1 \quad (\text{B.6})$$

$$= \frac{1}{p} \text{Tr}[\Sigma U \tilde{D}^{-1} U'] - \frac{1}{p} \log \left[\frac{\det(\Sigma)}{\det(\tilde{S})} \right] - 1 \quad (\text{B.7})$$

$$= \frac{1}{p} \text{Tr}[U' \Sigma U \tilde{D}^{-1}] - \frac{1}{p} \log [\det(\Sigma)] + \frac{1}{p} \log [\det(\tilde{D})] - 1 \quad (\text{B.8})$$

$$= \frac{1}{p} \sum_{i=1}^p \left[\frac{u'_i \Sigma u_i}{\tilde{d}_i} - \log \left(\frac{u'_i \Sigma u_i}{\tilde{d}_i} \right) - 1 \right] + \text{constant} \quad (\text{B.9})$$

$$\frac{\partial \mathcal{L}^{\text{SINV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = -\frac{u'_i \Sigma u_i}{p \tilde{d}_i^2} + \frac{1}{p \tilde{d}_i} \quad (\forall i = 1, \dots, p) \quad (\text{B.10})$$

$$\frac{\partial \mathcal{L}^{\text{SINV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = u'_i \Sigma u_i . \quad (\text{B.11})$$

From Equation (B.9), we can see that the incremental contribution to Inverse Stein loss due to imperfect shrinkage of the largest eigenvalue is equal to

$$\frac{u'_p \Sigma u_p}{\tilde{d}_p} - \log \left(\frac{u'_p \Sigma u_p}{\tilde{d}_p} \right) - 1 . \quad (\text{B.12})$$

B.3 Minimum Variance

$$\mathcal{L}^{\text{MV}}(\tilde{S}, \Sigma) := \frac{\text{Tr}[\tilde{S}^{-1}\Sigma\tilde{S}^{-1}]/p}{\left(\text{Tr}[\tilde{S}^{-1}]/p\right)^2} - \frac{1}{\text{Tr}[\Sigma^{-1}]/p} \quad (\text{B.13})$$

$$= p \frac{\text{Tr}[U\tilde{D}^{-1}U'\Sigma U\tilde{D}^{-1}U']}{\left(\text{Tr}[U\tilde{D}^{-1}U']\right)^2} + \text{constant} \quad (\text{B.14})$$

$$= p \frac{\text{Tr}[\tilde{D}^{-2}U'\Sigma U]}{\left(\text{Tr}[\tilde{D}^{-1}]\right)^2} + \text{constant} \quad (\text{B.15})$$

$$= p \frac{\sum_{i=1}^p \tilde{d}_i^{-2} u_i' \Sigma u_i}{\left(\sum_{i=1}^p \tilde{d}_i^{-1}\right)^2} + \text{constant} \quad (\text{B.16})$$

$$\frac{\partial \mathcal{L}^{\text{MV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = p \frac{-2\tilde{d}_i^{-3} u_i' \Sigma u_i \left(\sum_{j=1}^p \tilde{d}_j^{-1}\right)^2 + 2\tilde{d}_i^{-2} \left(\sum_{j=1}^p \tilde{d}_j^{-1}\right) \left(\sum_{j=1}^p \tilde{d}_j^{-2} u_j' \Sigma u_j\right)}{\left(\sum_{j=1}^p \tilde{d}_j^{-1}\right)^4} \quad (\text{B.17})$$

$$\frac{\partial \mathcal{L}^{\text{MV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i^{-3} u_i' \Sigma u_i \left(\sum_{j=1}^p \tilde{d}_j^{-1}\right)^2 = \tilde{d}_i^{-2} \left(\sum_{j=1}^p \tilde{d}_j^{-1}\right) \left(\sum_{j=1}^p \tilde{d}_j^{-2} u_j' \Sigma u_j\right) \quad (\text{B.18})$$

$$\Leftrightarrow \tilde{d}_i = \text{scalar} \cdot u_i' \Sigma u_i, \quad (\text{B.19})$$

where the scalar is independent of $i = 1, \dots, p$. Any cursory inspection of the Minimum Variance loss function $\mathcal{L}^{\text{MV}}(\tilde{S}, \Sigma)$ immediately reveals that the scalar cannot be determined internally, because multiplying the estimator \tilde{S} by any strictly positive scalar just washes out. Therefore, we have to invoke other arguments to make a choice. By preservation of the trace, the scalar should be set equal to one.

B.4 Stein

$$\mathcal{L}^S(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr}[\tilde{S}\Sigma^{-1}] - \frac{1}{p} \log \left[\det(\tilde{S}\Sigma^{-1}) \right] - 1 \quad (\text{B.20})$$

$$= \frac{1}{p} \text{Tr}[\Sigma^{-1}\tilde{S}] - \frac{1}{p} \log \left[\det(\Sigma^{-1}\tilde{S}) \right] - 1 \quad (\text{B.21})$$

$$= \frac{1}{p} \text{Tr}[\Sigma^{-1}U\tilde{D}U'] - \frac{1}{p} \log \left[\frac{\det(\tilde{S})}{\det(\Sigma)} \right] - 1 \quad (\text{B.22})$$

$$= \frac{1}{p} \text{Tr}[U'\Sigma^{-1}U\tilde{D}] - \frac{1}{p} \log[\det(\tilde{S})] + \frac{1}{p} \log[\det(\Sigma)] - 1 \quad (\text{B.23})$$

$$= \frac{1}{p} \sum_{i=1}^p \left[u_i' \Sigma^{-1} u_i \tilde{d}_i - \log(\tilde{d}_i) \right] + \text{constant} \quad (\text{B.24})$$

$$\frac{\partial \mathcal{L}^S(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = \frac{1}{p} u_i' \Sigma^{-1} u_i - \frac{1}{p \tilde{d}_i} \quad (\forall i = 1, \dots, p) \quad (\text{B.25})$$

$$\frac{\partial \mathcal{L}^S(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = \frac{1}{u_i' \Sigma^{-1} u_i}. \quad (\text{B.26})$$

From Equation (B.24), we can see that the incremental contribution to Stein's loss due to imperfect shrinkage of the largest eigenvalue is equal to $u_p' \Sigma^{-1} u_p \cdot \tilde{d}_p - \log(u_p' \Sigma^{-1} u_p \cdot \tilde{d}_p) - 1$.

B.5 Inverse Frobenius

$$\mathcal{L}^{\text{FINV}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr} \left[(\tilde{S}^{-1} - \Sigma^{-1})^2 \right] = \frac{1}{p} \text{Tr} \left[(\Sigma^{-1} - \tilde{S}^{-1})^2 \right] \quad (\text{B.27})$$

$$= \frac{1}{p} \text{Tr} \left[(\Sigma^{-1} - U\tilde{D}^{-1}U')(\Sigma^{-1} - U\tilde{D}^{-1}U') \right] \quad (\text{B.28})$$

$$= \frac{1}{p} \text{Tr} \left[U'(\Sigma^{-1} - U\tilde{D}^{-1}U')UU'(\Sigma^{-1} - U\tilde{D}^{-1}U')U \right] \quad (\text{B.29})$$

$$= \frac{1}{p} \text{Tr} \left[(U'\Sigma^{-1}U - \tilde{D}^{-1})^2 \right] = \frac{1}{p} \sum_{i=1}^p \left(u_i' \Sigma^{-1} u_i - \frac{1}{\tilde{d}_i} \right)^2 + \text{constant}, \quad (\text{B.30})$$

which is clearly minimized when $\tilde{d}_i = (u_i' \Sigma^{-1} u_i)^{-1}$ for all $i = 1, \dots, p$. From Equation (B.30), we can see that the incremental contribution to Inverse Frobenius loss due to imperfect shrinkage of the largest eigenvalue is equal to

$$\left(u_p' \Sigma^{-1} u_p - \frac{1}{\tilde{d}_p} \right)^2. \quad (\text{B.31})$$

B.6 Symmetrized Stein

$$\mathcal{L}^{\text{SSYM}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr}[\tilde{S}\Sigma^{-1} + \tilde{S}^{-1}\Sigma] - 2 = \frac{1}{p} \text{Tr}[\Sigma^{-1}\tilde{S} + \Sigma\tilde{S}^{-1}] - 2 \quad (\text{B.32})$$

$$= \frac{1}{p} \text{Tr}[\Sigma^{-1}U\tilde{D}U' + \Sigma U\tilde{D}^{-1}U'] - 2 \quad (\text{B.33})$$

$$= \frac{1}{p} \text{Tr}[U'\Sigma^{-1}U\tilde{D} + U'\Sigma U\tilde{D}^{-1}] - 2 \quad (\text{B.34})$$

$$= \frac{1}{p} \sum_{i=1}^p \left(u_i' \Sigma^{-1} u_i \tilde{d}_i + u_i' \Sigma u_i \tilde{d}_i^{-1} \right) - 2 \quad (\text{B.35})$$

$$\frac{\partial \mathcal{L}^{\text{SSYM}}(\Sigma, \tilde{S})}{\partial \tilde{d}_i} = \frac{1}{p} u_i' \Sigma^{-1} u_i - \frac{u_i' \Sigma u_i}{p \tilde{d}_i^{-2}} \quad (\text{B.36})$$

$$\frac{\partial \mathcal{L}^{\text{SSYM}}(\Sigma, \tilde{S})}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = \sqrt{\frac{u_i' \Sigma u_i}{u_i' \Sigma^{-1} u_i}}. \quad (\text{B.37})$$

From Equation (B.34), we can see that the incremental contribution to Symmetrized Stein loss due to imperfect shrinkage of the largest eigenvalue is equal to

$$u_p' \Sigma^{-1} u_p \cdot \tilde{d}_p + \frac{u_p' \Sigma u_p}{\tilde{d}_p} - 2 \sqrt{u_p' \Sigma u_p \cdot u_p' \Sigma^{-1} u_p}. \quad (\text{B.38})$$

Note that the last term, which is a constant in the sense that it does not depend on the shrunken eigenvalue \tilde{d}_p , has been artificially added so that (B.38) is equal to zero when Equation (B.37) is satisfied, and strictly positive otherwise.

B.7 Weighted Frobenius

$$\mathcal{L}^{\text{FW}}(\tilde{S}, \Sigma) := \frac{\text{Tr}[(\tilde{S} - \Sigma)^2 \Sigma^{-1}]}{\text{Tr}[\Sigma]} = \frac{\text{Tr}[(U\tilde{D}U' - \Sigma)^2 \Sigma^{-1}]}{\text{Tr}[\Sigma]} \quad (\text{B.39})$$

$$= \frac{\text{Tr}[U\tilde{D}^2U'\Sigma^{-1} - 2U\tilde{D}U' + \Sigma]}{\text{Tr}[\Sigma]} = \frac{\text{Tr}[\tilde{D}^2U'\Sigma^{-1}U' - 2\tilde{D} + \Sigma]}{\text{Tr}[\Sigma]} \quad (\text{B.40})$$

$$= \frac{\sum_{i=1}^p \left(\tilde{d}_i^2 u_i' \Sigma^{-1} u_i - 2\tilde{d}_i + \tau_i \right)}{\text{Tr}[\Sigma]} \quad (\text{B.41})$$

$$\frac{\partial \mathcal{L}^{\text{FW}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = \frac{2\tilde{d}_i u_i' \Sigma^{-1} u_i - 2}{\text{Tr}[\Sigma]} \quad (\text{B.42})$$

$$\frac{\partial \mathcal{L}^{\text{FW}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = \frac{1}{u_i' \Sigma^{-1} u_i} \quad (\text{B.43})$$

B.8 Disutility

$$\mathcal{L}^D(\tilde{S}, \Sigma) := \frac{\text{Tr}[(\tilde{S}^{-1} - \Sigma^{-1})^2 \Sigma]}{\text{Tr}[\Sigma^{-1}]} = \frac{\text{Tr}[(U\tilde{D}^{-1}U' - \Sigma^{-1})^2 \Sigma]}{\text{Tr}[\Sigma^{-1}]} \quad (\text{B.44})$$

$$= \frac{\text{Tr}[U\tilde{D}^{-2}U'\Sigma - 2U\tilde{D}^{-1}U' + \Sigma^{-1}]}{\text{Tr}[\Sigma^{-1}]} \quad (\text{B.45})$$

$$= \frac{\text{Tr}[\tilde{D}^{-2}U'\Sigma U' - 2\tilde{D}^{-1} + \Sigma^{-1}]}{\text{Tr}[\Sigma^{-1}]} \quad (\text{B.46})$$

$$= \frac{\sum_{i=1}^p (\tilde{d}_i^{-2} u'_i \Sigma u_i - 2\tilde{d}_i^{-1} + \tau_i^{-1})}{\text{Tr}[\Sigma^{-1}]} \quad (\text{B.47})$$

$$\frac{\partial \mathcal{L}^D(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = \frac{-2\tilde{d}_i^{-3} u'_i \Sigma u_i + 2\tilde{d}_i^{-2}}{\text{Tr}[\Sigma]} \quad (\text{B.48})$$

$$\frac{\partial \mathcal{L}^D(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = u'_i \Sigma u_i \quad (\text{B.49})$$

B.9 Log-Euclidian

$$\mathcal{L}^{\text{LE}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr}[\{\log(\tilde{S}) - \log(\Sigma)\}^2] \quad (\text{B.50})$$

$$= \frac{1}{p} \text{Tr}[\{\log(\Sigma) - \log(\tilde{S})\}^2] = \frac{1}{p} \text{Tr}[\{\log(\Sigma) - \log(U\tilde{D}U')\}^2] \quad (\text{B.51})$$

$$= \frac{1}{p} \text{Tr}[\{\log(\Sigma) - U \log(\tilde{D}) U'\}^2] = \frac{1}{p} \text{Tr}[\{U' \log(\Sigma) U - \log(\tilde{D})\}^2] \quad (\text{B.52})$$

$$= \frac{1}{p} \sum_{i=1}^p \{u'_i \log(\Sigma) u_i - \log(\tilde{d}_i)\}^2 + \text{constant} , \quad (\text{B.53})$$

which is clearly minimized when $\tilde{d}_i = \exp[u'_i \log(\Sigma) u_i]$ for all $i = 1, \dots, p$.

B.10 Fréchet

$$\mathcal{L}^{\text{Fré}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr}[(\tilde{S}^{1/2} - \Sigma^{1/2})^2] = \frac{1}{p} \text{Tr}[(\Sigma^{1/2} - \tilde{S}^{1/2})^2] \quad (\text{B.54})$$

$$= \frac{1}{p} \text{Tr}[(\Sigma^{1/2} - U\tilde{D}^{1/2}U')(\Sigma^{1/2} - U\tilde{D}^{1/2}U')] \quad (\text{B.55})$$

$$= \frac{1}{p} \text{Tr}[U'(\Sigma^{1/2} - U\tilde{D}^{1/2}U')UU'(\Sigma^{1/2} - U\tilde{D}^{1/2}U')U] \quad (\text{B.56})$$

$$= \frac{1}{p} \text{Tr}[(U'\Sigma^{1/2}U - \tilde{D}^{1/2})^2] = \frac{1}{p} \sum_{i=1}^p \left(u'_i \Sigma^{1/2} u_i - \sqrt{\tilde{d}_i}\right)^2 + \text{constant} , \quad (\text{B.57})$$

which is clearly minimized when $\tilde{d}_i = (u_i' \Sigma^{1/2} u_i)^2$ for all $i = 1, \dots, p$. From Equation (B.57), we can see that the incremental contribution to Fréchet loss due to imperfect shrinkage of the largest eigenvalue is equal to $(u_i' \Sigma^{1/2} u_i - \sqrt{\tilde{d}_i})^2$.

B.11 Quadratic

$$\mathcal{L}^Q(\tilde{S}, \Sigma) := \frac{1}{p} \left\| \Sigma^{-1} \tilde{S} - \mathbb{I} \right\|_F^2 = \frac{1}{p} \text{Tr} \left[\left(\Sigma^{-1} \tilde{S} - \mathbb{I} \right) \left(\Sigma^{-1} \tilde{S} - \mathbb{I} \right)' \right] \quad (\text{B.58})$$

$$= \frac{1}{p} \text{Tr} \left[\left(\Sigma^{-1} \tilde{S} - \mathbb{I} \right) \left(\tilde{S} \Sigma^{-1} - \mathbb{I} \right) \right] = \frac{1}{p} \text{Tr} \left[\left(\tilde{S} \Sigma^{-1} - \mathbb{I} \right) \left(\Sigma^{-1} \tilde{S} - \mathbb{I} \right) \right] \quad (\text{B.59})$$

$$= \frac{1}{p} \text{Tr} \left[\tilde{S} \Sigma^{-2} \tilde{S} - 2 \Sigma^{-1} \tilde{S} + \mathbb{I} \right] \quad (\text{B.60})$$

$$= \frac{1}{p} \text{Tr} \left[U \tilde{D} U' \Sigma^{-2} U \tilde{D} U' - 2 \Sigma^{-1} U \tilde{D} U' + \mathbb{I} \right] \quad (\text{B.61})$$

$$= \frac{1}{p} \text{Tr} \left[\tilde{D} U' \Sigma^{-2} U \tilde{D} - 2 U' \Sigma^{-1} U \tilde{D} + \mathbb{I} \right] \quad (\text{B.62})$$

$$= \frac{1}{p} \sum_{i=1}^p \tilde{d}_i^2 \cdot u_i' \Sigma^{-2} u_i - \frac{2}{p} \sum_{i=1}^p \tilde{d}_i \cdot u_i' \Sigma^{-1} u_i + 1 \quad (\text{B.63})$$

The first-order condition (FOC) is obtained as follows:

$$\frac{\partial \mathcal{L}^Q(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = \frac{2}{p} \tilde{d}_i \cdot u_i' \Sigma^{-2} u_i - \frac{2}{p} u_i' \Sigma^{-1} u_i \quad (\text{B.64})$$

$$\text{FOC: } \tilde{d}_i \cdot u_i' \Sigma^{-2} u_i = u_i' \Sigma^{-1} u_i \Leftrightarrow \tilde{d}_i = \frac{u_i' \Sigma^{-1} u_i}{u_i' \Sigma^{-2} u_i}. \quad (\text{B.65})$$

From Equation (B.63), we can see that the incremental contribution to Quadratic loss due to imperfect shrinkage of the largest eigenvalue is equal to

$$u_p' \Sigma^{-2} u_p \cdot \tilde{d}_p^2 - 2 u_p' \Sigma^{-1} u_p \cdot \tilde{d}_p + \frac{(u_p' \Sigma^{-1} u_p)^2}{u_p' \Sigma^{-2} u_p}. \quad (\text{B.66})$$

As with the Symmetrized Stein loss, we added a term so that (B.66) is zero for the FSOPT.

B.12 Inverse Quadratic

The algebraic manipulations are the same as above, except that Σ^{-1} becomes Σ and \tilde{S} becomes \tilde{S}^{-1} , so there is no point repeating the intermediary steps.

$$\mathcal{L}^{\text{QINV}}(\tilde{S}, \Sigma) := \frac{1}{p} \left\| \tilde{S}^{-1} \Sigma - \mathbb{I} \right\|_F^2 \quad (\text{B.67})$$

$$= \frac{1}{p} \tilde{d}_i^{-2} u_i' \Sigma^2 u_i - \frac{2}{p} \sum_{i=1}^p \tilde{d}_i^{-1} u_i' \Sigma u_i + 1 \quad (\text{B.68})$$

$$\frac{\partial \mathcal{L}^{\text{QINV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = -\frac{2}{p} \tilde{d}_i^{-3} \cdot u_i' \Sigma^2 u_i + \frac{2}{p} \tilde{d}_i^{-2} \cdot u_i' \Sigma u_i \quad (\text{B.69})$$

$$\text{FOC: } \frac{\partial \mathcal{L}^{\text{QINV}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = \frac{u_i' \Sigma^2 u_i}{u_i' \Sigma u_i}. \quad (\text{B.70})$$

From Equation (B.68), we can see that the incremental contribution to Inverse Quadratic loss due to imperfect shrinkage of the largest eigenvalue is equal to

$$\frac{u'_p \Sigma^2 u_p}{\tilde{d}_p^2} - 2 \frac{u'_p \Sigma u_p}{\tilde{d}_p} + \frac{(u'_p \Sigma u_p)^2}{u'_p \Sigma^2 u_p} . \quad (\text{B.71})$$

B.13 Generalized Frobenius

To defuse a well-known source of confusion, we use the notation $\gamma^{-1}(x)$ to signify the inverse function of the invertible function γ , and $\gamma(x)^{-1}$ to signify one divided by $\gamma(x)$. For example, if $\gamma(x) = x^3$ then $\gamma^{-1}(x) = \sqrt[3]{x}$ and $\gamma(x)^{-1} = 1/x^3$.

$$\mathcal{L}^{\gamma, \text{F}}(\tilde{S}, \Sigma) := \frac{1}{p} \text{Tr} \left[\{ \gamma(\tilde{S}) - \gamma(\Sigma) \}^2 \right] \quad (\text{B.72})$$

$$= \frac{1}{p} \text{Tr} \left[\{ \gamma(\Sigma) - \gamma(\tilde{S}) \}^2 \right] = \frac{1}{p} \text{Tr} \left[\{ \gamma(\Sigma) - \gamma(U \tilde{D} U') \}^2 \right] \quad (\text{B.73})$$

$$= \frac{1}{p} \text{Tr} \left[\{ \gamma(\Sigma) - U \gamma(\tilde{D}) U' \}^2 \right] = \frac{1}{p} \text{Tr} \left[\{ U' \gamma(\Sigma) U - \gamma(\tilde{D}) \}^2 \right] \quad (\text{B.74})$$

$$= \frac{1}{p} \text{Tr} \left[\{ U' \gamma(\Sigma) U \}^2 - 2 U' \gamma(\Sigma) U \gamma(\tilde{D}) + \{ \gamma(\tilde{D}) \}^2 \right] \quad (\text{B.75})$$

$$= \frac{1}{p} \text{Tr} \left[\gamma(\Sigma)^2 - 2 U' \gamma(\Sigma) U \gamma(\tilde{D}) + \gamma(\tilde{D})^2 \right] \quad (\text{B.76})$$

$$= \frac{1}{p} \sum_{i=1}^p \{ \gamma(\tau_i)^2 - 2 u'_i \gamma(\Sigma) u_i \gamma(\tilde{d}_i) + \gamma(\tilde{d}_i)^2 \} \quad (\text{B.77})$$

$$\frac{\partial \mathcal{L}^{\gamma, \text{F}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = -2 u'_i \gamma(\Sigma) u_i \gamma'(\tilde{d}_i) + 2 \gamma'(\tilde{d}_i) \gamma(\tilde{d}_i) \quad \forall i = 1, \dots, p \quad (\text{B.78})$$

$$\text{FOC: } \frac{\partial \mathcal{L}^{\gamma, \text{F}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \tilde{d}_i = \gamma^{-1}(u'_i \gamma(\Sigma) u_i) . \quad (\text{B.79})$$

B.14 Generalized Kullback-Leibler Divergence

$$\mathcal{L}^{\gamma, \text{KL}}(\tilde{S}, \Sigma) := \frac{1}{2p} \left\{ \text{Tr} \left[\gamma(\tilde{S})^{-1} \gamma(\Sigma) \right] - \log \det \left[\gamma(\tilde{S})^{-1} \gamma(\Sigma) \right] - p \right\} \quad (\text{B.80})$$

$$= \frac{1}{2p} \left\{ \text{Tr} \left[\gamma(U \tilde{D} U')^{-1} \gamma(\Sigma) \right] - \log \det \left[\gamma(\tilde{S})^{-1} \gamma(\Sigma) \right] - p \right\} \quad (\text{B.81})$$

$$= \frac{1}{2p} \left\{ \text{Tr} \left[U \gamma(\tilde{D})^{-1} U' \gamma(\Sigma) \right] - \log \det \left[\gamma(\tilde{S})^{-1} \gamma(\Sigma) \right] - p \right\} \quad (\text{B.82})$$

$$= \frac{1}{2p} \left\{ \text{Tr} \left[\gamma(\tilde{D})^{-1} U' \gamma(\Sigma) U \right] + \log \left[\det(\gamma(\tilde{S})) \right] - \log \left[\det(\gamma(\Sigma)) \right] - p \right\} \\ = \frac{1}{2p} \sum_{i=1}^p \left\{ \frac{u'_i \gamma(\Sigma) u_i}{\gamma(\tilde{d}_i)} + \log \left[\gamma(\tilde{d}_i) \right] - \log \left[\gamma(\tau_i) \right] - 1 \right\} \quad (\text{B.83})$$

$$\frac{\partial \mathcal{L}^{\gamma, \text{KL}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = -\frac{u'_i \gamma(\Sigma) u_i}{\gamma(\tilde{d}_i)^2} \gamma'(\tilde{d}_i) + \frac{\gamma'(\tilde{d}_i)}{\gamma(\tilde{d}_i)} \quad \forall i = 1, \dots, p \quad (\text{B.84})$$

$$\frac{\partial \mathcal{L}^{\gamma, \text{KL}}(\tilde{S}, \Sigma)}{\partial \tilde{d}_i} = 0 \Leftrightarrow \gamma(\tilde{d}_i) = u'_i \gamma(\Sigma) u_i \Leftrightarrow \tilde{d}_i = \gamma^{-1} (u'_i \gamma(\Sigma) u_i) \quad (\text{B.85})$$

B.15 Summary of Spike Contributions to Loss

To wrap up the finite-sample analysis, Table B.1 isolates the contribution of the largest eigenvalue to the overall loss, for the eight loss functions studied in Section 6.5. These have been standardized so that the FSOPT does not contribute to loss, by construction.

Loss	Contribution of Spike
Frobenius	$(u'_p \Sigma u_p - \tilde{d}_p)^2$
Inverse Stein	$\frac{u'_p \Sigma u_p}{\tilde{d}_p} - \log \left(\frac{u'_p \Sigma u_p}{\tilde{d}_p} \right) - 1$
Stein	$u'_p \Sigma^{-1} u_p \cdot \tilde{d}_p - \log(u'_p \Sigma^{-1} u_p \cdot \tilde{d}_p) - 1$
Inverse Frobenius	$\left(u'_p \Sigma^{-1} u_p - \frac{1}{\tilde{d}_p} \right)^2$
Symmetrized Stein	$u'_p \Sigma^{-1} u_p \cdot \tilde{d}_p + \frac{u'_p \Sigma u_p}{\tilde{d}_p} - 2 \sqrt{u'_p \Sigma u_p \cdot u'_p \Sigma^{-1} u_p}$
Fréchet	$\left(u'_i \Sigma^{1/2} u_i - \sqrt{\tilde{d}_i} \right)^2$
Quadratic	$u'_p \Sigma^{-2} u_p \cdot \tilde{d}_p^2 - 2 u'_p \Sigma^{-1} u_p \cdot \tilde{d}_p + \frac{(u'_p \Sigma^{-1} u_p)^2}{u'_p \Sigma^{-2} u_p}$
Inverse Quadratic	$\frac{u'_p \Sigma^2 u_p}{\tilde{d}_p^2} - 2 \frac{u'_p \Sigma u_p}{\tilde{d}_p} + \frac{(u'_p \Sigma u_p)^2}{u'_p \Sigma^2 u_p}$

Table B.1: Loss contribution of the p th shrunk eigenvalue in excess of FSOPT.

C Proofs of Theorems in Sections 4 and 5

Theorems 4.1 and 4.2 are special cases of Theorem 4.5 with $\gamma(x) = \log(x)$, resp. $\gamma(x) = \sqrt{x}$.

C.1 Quadratic

Proposition C.1. *Under Assumptions 2–5,*

$$\mathcal{L}_n^Q(\tilde{S}_n, \Sigma_n) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \left[\frac{\tilde{\varphi}(x)^2}{t^2} - 2 \frac{\tilde{\varphi}(x)}{t} + 1 \right] \cdot \theta(x, t) dH(t) dF(x) \quad (\text{C.1})$$

Proof. For simplicity, let us assume that the support of F is a single compact interval $[a, b] \subset (0, +\infty)$; the generalization to the case $\kappa > 1$ is trivial. From Appendix B.11 we have:

$$\mathcal{L}_n^Q(\tilde{S}_n, \Sigma_n) = \frac{1}{p} \sum_{i=1}^p \tilde{d}_{n,i}^2 \cdot u'_{n,i} \Sigma_n^{-2} u_{n,i} - \frac{2}{p} \sum_{i=1}^p \tilde{d}_{n,i} \cdot u'_{n,i} \Sigma_n^{-1} u_{n,i} + 1 \quad (\text{C.2})$$

$$= \frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p \left[\frac{\tilde{d}_{n,i}^2}{\tau_{n,j}^2} - 2 \frac{\tilde{d}_{n,i}}{\tau_{n,j}} + 1 \right] \cdot (u'_i v_j)^2 \quad (\text{C.3})$$

$$= \int_a^b \int_{-\infty}^{+\infty} \left[\frac{\tilde{\varphi}_n(\lambda_{n,i})^2}{\tau_{n,j}^2} - 2 \frac{\tilde{\varphi}_n(\lambda_{n,i})}{\tau_{n,j}} + 1 \right] d^2 \Theta_n(x, t), \quad (\text{C.4})$$

where Θ_n is the random bivariate function from Equation (3.4). By applying the technique from the proof of Theorem 3.1 of Ledoit and Wolf (2018), and by using Theorem 3.2 to handle the function Θ_n , it follows that

$$\mathcal{L}_n^Q(\tilde{S}_n, \Sigma_n) \xrightarrow{\text{a.s.}} \int_a^b \int_{-\infty}^{+\infty} \left[\frac{\tilde{\varphi}(x)^2}{t^2} - 2 \frac{\tilde{\varphi}(x)}{t} + 1 \right] \theta(x, t) dx dt, \quad (\text{C.5})$$

where, as per Equation (3.6),

$$\forall x \in [a, b] \quad \forall t \in \mathbb{R} \quad \theta(x, t) := \frac{cxt}{|t[1 - c - cx\check{m}_F(x)] - x|^2} \cdot \blacksquare \quad (\text{C.6})$$

Proposition C.1 lets us characterize the asymptotically optimal nonlinear shrinkage function under Quadratic loss.

Corollary C.1. *Suppose Assumptions 2–5 hold. A covariance matrix estimator \tilde{S}_n minimizes in the class of rotation-equivariant estimators the a.s. limit (C.1) of the Quadratic loss if and only if its limiting shrinkage function $\tilde{\varphi}$ verifies $\forall x \in \text{Supp}(F)$, $\tilde{\varphi}(x) = \hat{\varphi}^Q(x)$, where*

$$\forall x \in \text{Supp}(F) \quad \hat{\varphi}^Q(x) := \frac{\int_{-\infty}^{+\infty} \frac{1}{t} \cdot \theta(x, t) dH(t)}{\int_{-\infty}^{+\infty} \frac{1}{t^2} \cdot \theta(x, t) dH(t)}. \quad (\text{C.7})$$

Proof. If we fix $x \in \text{Supp}(F)$, then the marginal contribution of $\tilde{\varphi}(x)$ to the almost sure (nonrandom) limit of the loss function $\mathcal{L}_n^Q(\Sigma_n, \tilde{S}_n)$ is

$$\int_{-\infty}^{+\infty} \left[\frac{\tilde{\varphi}(x)^2}{t^2} - 2 \frac{\tilde{\varphi}(x)}{t} + 1 \right] \theta(x, t) dH(t). \quad (\text{C.8})$$

The partial derivative of (C.8) with respect to $\tilde{\varphi}(x)$ is

$$\int_{-\infty}^{+\infty} \left[\frac{2\tilde{\varphi}(x)}{t^2} - \frac{2}{t} \right] \theta(x, t) dH(t) . \quad (\text{C.9})$$

The first-order condition is

$$\varphi(x) \int_{-\infty}^{+\infty} \frac{1}{t^2} \theta(x, t) dH(t) = \int_{-\infty}^{+\infty} \frac{1}{t} \theta(x, t) dH(t) . \quad (\text{C.10})$$

The solution is

$$\varphi(x) = \frac{\int_{-\infty}^{+\infty} \frac{1}{t} \theta(x, t) dH(t)}{\int_{-\infty}^{+\infty} \frac{1}{t^2} \theta(x, t) dH(t)} . \blacksquare \quad (\text{C.11})$$

The proof of Theorem 4.3 is concluded as follows: To the unobservable quantity c corresponds the plug-in estimator p/n ; to the unobservable quantity $H(t)$ corresponds the plug-in estimator $\hat{H}_n(t) := \sum_{i=j}^p \mathbb{1}_{[\hat{\tau}_{n,j}, +\infty)}(t)/p$; and to the unobservable quantity $\theta(x)$ corresponds the plug-in estimator $\hat{\theta}_n(x, t)$ from Equation (3.7). The fact that these three unobservable quantities can be replaced with their respective plug-in counterparts at no loss asymptotically is established in the same way as in the proof of Ledoit and Wolf's (2018) Theorem 5.2. ■

C.2 Inverse Quadratic

Proposition C.2. *Under Assumptions 2–5,*

$$\mathcal{L}_n^{QINV}(\tilde{S}_n, \Sigma_n) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \left[\frac{t^2}{\tilde{\varphi}(x)^2} - 2 \frac{t}{\tilde{\varphi}(x)} + 1 \right] \cdot \theta(x, t) dH(t) dF(x) \quad (\text{C.12})$$

Proof. As before, we assume that the support of F is a single compact interval $[a, b] \subset (0, +\infty)$. From Appendix B.12 we have:

$$\mathcal{L}_n^{QINV}(\tilde{S}_n, \Sigma_n) = \frac{1}{p} \sum_{i=1}^p \tilde{d}_{n,i}^{-2} \cdot u'_{n,i} \Sigma_n^2 u_{n,i} - \frac{2}{p} \sum_{i=1}^p \tilde{d}_{n,i}^{-1} \cdot u'_{n,i} \Sigma_n u_{n,i} + 1 \quad (\text{C.13})$$

$$= \frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p \left[\frac{\tau_{n,j}^2}{\tilde{d}_{n,i}^2} - 2 \frac{\tau_{n,j}}{\tilde{d}_{n,i}} + 1 \right] \cdot (u'_i v_j)^2 \quad (\text{C.14})$$

$$= \int_a^b \int_{-\infty}^{+\infty} \left[\frac{\tau_{n,j}^2}{\tilde{\varphi}_n(\lambda_{n,i})^2} - 2 \frac{\tau_{n,j}}{\tilde{\varphi}_n(\lambda_{n,i})} + 1 \right] d^2 \Theta_n(x, t) . \quad (\text{C.15})$$

By applying the technique from the proof of Theorem 3.1 of Ledoit and Wolf (2018), and by using Theorem 3.2:

$$\mathcal{L}_n^{QINV}(\tilde{S}_n, \Sigma_n) \xrightarrow{\text{a.s.}} \int_a^b \int_{-\infty}^{+\infty} \left[\frac{t^2}{\tilde{\varphi}(x)^2} - 2 \frac{t}{\tilde{\varphi}(x)} + 1 \right] \theta(x, t) dx dt . \blacksquare \quad (\text{C.16})$$

Corollary C.2. *Under Assumptions 2–5, a covariance matrix estimator \tilde{S}_n minimizes in the class of rotation-equivariant estimators the a.s. limit (C.12) of the Inverse Quadratic loss if and*

only if its limiting shrinkage function $\tilde{\varphi}$ verifies $\forall x \in \text{Supp}(F)$, $\tilde{\varphi}(x) = \hat{\varphi}^{QINV}(x)$, where

$$\forall x \in \text{Supp}(F) \quad \hat{\varphi}^{QINV}(x) := \frac{\int_{-\infty}^{+\infty} t^2 \cdot \theta(x, t) dH(t)}{\int_{-\infty}^{+\infty} t \cdot \theta(x, t) dH(t)} . \quad (\text{C.17})$$

Proof. If we fix $x \in \text{Supp}(F)$, then the marginal contribution of $\tilde{\varphi}(x)$ to the almost sure (nonrandom) limit of the loss function $\mathcal{L}_n^{QINV}(\Sigma_n, \tilde{S}_n)$ is

$$\int_{-\infty}^{+\infty} \left[\frac{t^2}{\tilde{\varphi}(x)^2} - 2 \frac{t}{\tilde{\varphi}(x)} + 1 \right] \theta(x, t) dH(t) . \quad (\text{C.18})$$

The partial derivative of (C.18) with respect to $\tilde{\varphi}(x)$ is

$$\int_{-\infty}^{+\infty} \left[-2 \frac{t^2}{\tilde{\varphi}(x)^3} + 2 \frac{t}{\tilde{\varphi}(x)^2} \right] \theta(x, t) dH(t) . \quad (\text{C.19})$$

The first-order condition is

$$\varphi(x) \int_{-\infty}^{+\infty} t \theta(x, t) dH(t) = \int_{-\infty}^{+\infty} t^2 \theta(x, t) dH(t) . \quad (\text{C.20})$$

The solution is

$$\varphi(x) = \frac{\int_{-\infty}^{+\infty} t^2 \theta(x, t) dH(t)}{\int_{-\infty}^{+\infty} t \theta(x, t) dH(t)} . \blacksquare \quad (\text{C.21})$$

The proof of Theorem 4.4 is concluded as before: To the unobservable quantity c corresponds the plug-in estimator p/n ; to the unobservable quantity $H(t)$ corresponds the plug-in estimator $\hat{H}_n(t) := \sum_{i=j}^p \mathbb{1}_{[\hat{\tau}_{n,j}, +\infty)}(t)/p$; and to the unobservable quantity $\theta(x)$ corresponds the plug-in estimator $\hat{\theta}_n(x, t)$ from Equation (3.7). The fact that these three unobservable quantities can be replaced with their respective plug-in counterparts at no loss asymptotically is established in the same way as in the proof of Ledoit and Wolf's (2018) Theorem 5.2. ■

C.3 Generalized Frobenius

Proposition C.3. Under Assumptions 2–5,

$$\begin{aligned} \mathcal{L}_n^{\gamma, F}(\tilde{S}_n, \Sigma_n) &\xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} \gamma(t)^2 dH(t) - 2 \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) dF(x) \\ &\quad + \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \gamma(\tilde{\varphi}(x))^2 dF(x) . \end{aligned} \quad (\text{C.22})$$

Proof. For simplicity: $\text{Supp}(F) = [a, b] \subset (0, +\infty)$. From Appendix B.13:

$$\begin{aligned}\mathcal{L}_n^{\gamma, F}(\tilde{S}_n, \Sigma_n) &= \frac{1}{p} \sum_{i=1}^p \{ \gamma(\tau_{n,i})^2 - 2u'_{n,i} \gamma(\Sigma_n) u_{n,i} \gamma(\tilde{d}_{n,i}) + \gamma(\tilde{d}_{n,i})^2 \} \\ &= \frac{1}{p} \sum_{j=1}^p \gamma(\tau_{n,j})^2 - \frac{2}{p} \sum_{i=1}^p \sum_{j=1}^p \gamma(\tau_{n,j}) \gamma(\tilde{\varphi}(\lambda_{n,i})) \cdot (u'_{n,i} v_{n,j})^2 + \frac{1}{p} \sum_{i=1}^p \gamma(\tilde{\varphi}(\lambda_{n,i}))^2 \\ &= \int_{-\infty}^{+\infty} \gamma(t)^2 dH_n(t) - 2 \int_a^b \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}_n(x)) d^2 \Theta_n(x, t) + \int_a^b \gamma(\tilde{\varphi}_n(x))^2 dF_n(x).\end{aligned}\tag{C.23}$$

By applying the technique from the proof of Theorem 3.1 of Ledoit and Wolf (2018), and by using Theorem 3.2:

$$\begin{aligned}\mathcal{L}_n^{\gamma, F}(\tilde{S}_n, \Sigma_n) &\xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} \gamma(t)^2 dH(t) - 2 \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) dF(x) \\ &\quad + \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \gamma(\tilde{\varphi}(x))^2 dF(x) . \blacksquare\end{aligned}\tag{C.24}$$

Corollary C.3. Suppose Assumptions 2–5 hold. A covariance matrix estimator \tilde{S}_n minimizes in the class of rotation-equivariant estimators the a.s. limit (C.22) of the Generalized Frobenius loss if and only if its limiting shrinkage function $\tilde{\varphi}$ verifies $\forall x \in \text{Supp}(F)$, $\tilde{\varphi}(x) = \varphi^\gamma(x)$, where

$$\forall x \in \text{Supp}(F) \quad \varphi^\gamma(x) := \gamma^{-1} \left[\int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t) \right] . \tag{C.25}$$

This yields an oracle covariance matrix estimator $S_n^\gamma := U_n \text{Diag}(\varphi^\gamma(\lambda_{n,1}), \dots, \varphi^\gamma(\lambda_{n,p})) U_n'$.

Proof. If we fix $x \in \text{Supp}(F)$, then the marginal contribution of $\tilde{\varphi}(x)$ to the almost sure (nonrandom) limit of the loss function $\mathcal{L}_n^{\gamma, F}(\Sigma_n, \tilde{S}_n)$ is

$$- 2 \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) + \gamma(\tilde{\varphi}(x))^2 . \tag{C.26}$$

The partial derivative of (C.26) with respect to $\tilde{\varphi}(x)$ is

$$- 2 \int_{-\infty}^{+\infty} \gamma(t) \gamma'(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) + 2 \gamma'(\tilde{\varphi}(x)) \gamma(\tilde{\varphi}(x)) \tag{C.27}$$

The first-order condition is $\gamma(\tilde{\varphi}(x)) = \int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t)$, hence the solution is

$$\tilde{\varphi}(x) = \gamma^{-1} \left(\int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t) \right) . \blacksquare \tag{C.28}$$

The proof of Theorem 4.5 is concluded as before: To the unobservable quantity c corresponds the plug-in estimator p/n ; to the unobservable quantity $H(t)$ corresponds the plug-in estimator $\hat{H}_n(t) := \sum_{i=j}^p \mathbb{1}_{[\hat{\tau}_{n,j}, +\infty)}(t)/p$; and to the unobservable quantity $\theta(x)$ corresponds the plug-in estimator $\hat{\theta}_n(x, t)$ from Equation (3.7). The fact that these three unobservable quantities can be replaced with their respective plug-in counterparts at no loss asymptotically is established in the same way as in the proof of Ledoit and Wolf's (2018) Theorem 5.2. \blacksquare

C.4 Generalized Kullback-Leibler Divergence

Proposition C.4. *Under Assumptions 2–5,*

$$\begin{aligned} \mathcal{L}_n^{\gamma, KL}(\tilde{S}_n, \Sigma_n) &\xrightarrow{\text{a.s.}} \frac{1}{2} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \frac{\gamma(t)}{\gamma(\tilde{\varphi}(x))} \cdot \theta(x, t) dH(t) dF(x) \\ &\quad + \frac{1}{2} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \log [\gamma(\tilde{\varphi}(x))] dF(x) \\ &\quad - \frac{1}{2} \int_{-\infty}^{+\infty} \log [\gamma(t)] dH(t) - \frac{1}{2}. \end{aligned} \quad (\text{C.29})$$

Proof. For simplicity, $\text{Supp}(F) = [a, b] \subset (0, +\infty)$. From Appendix B.14:

$$\mathcal{L}_n^{\gamma, KL}(\tilde{S}_n, \Sigma_n) = \frac{1}{2p} \sum_{i=1}^p \left\{ \frac{u_i' \gamma(\Sigma) u_i}{\gamma(\tilde{\delta}_i)} + \log [\gamma(\tilde{\delta}_i)] - \log [\gamma(\tau_i)] - 1 \right\} \quad (\text{C.30})$$

$$\begin{aligned} &= \frac{1}{2} \int_a^b \int_{-\infty}^{+\infty} \frac{\gamma(t)}{\gamma(\tilde{\varphi}_n(x))} d^2 \Theta_n(x, t) + \frac{1}{2} \int_a^b \log [\gamma(\tilde{\varphi}_n(x))] dF_n(x) \\ &\quad - \frac{1}{2} \int_{-\infty}^{+\infty} \log [\gamma(t)] dH_n(t) - \frac{1}{2}. \end{aligned} \quad (\text{C.31})$$

By applying the technique from the proof of Theorem 3.1 of Ledoit and Wolf (2018), and by using Theorem 3.2:

$$\begin{aligned} \mathcal{L}_n^{\gamma, KL}(\tilde{S}_n, \Sigma_n) &\xrightarrow{\text{a.s.}} \frac{1}{2} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \frac{\gamma(t)}{\gamma(\tilde{\varphi}(x))} \cdot \theta(x, t) dH(t) dF(x) \\ &\quad + \frac{1}{2} \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \log [\gamma(\tilde{\varphi}(x))] dF(x) \\ &\quad - \frac{1}{2} \int_{-\infty}^{+\infty} \log [\gamma(t)] dH(t) - \frac{1}{2}. \end{aligned} \quad (\text{C.32})$$

Corollary C.4. *Suppose Assumptions 2–5 hold. A covariance matrix estimator \tilde{S}_n minimizes in the class of rotation-equivariant estimators the a.s. limit (C.29) of the Generalized Kullback-Leibler loss if and only if its limiting shrinkage function $\tilde{\varphi}$ verifies $\forall x \in \text{Supp}(F)$, $\tilde{\varphi}(x) = \varphi^\gamma(x)$, where $\varphi^\gamma(x)$ is defined by Equation (C.25). This results in the same oracle covariance matrix estimator $S_n^\gamma := U_n \text{Diag}(\varphi^\gamma(\lambda_{n,1}), \dots, \varphi^\gamma(\lambda_{n,p})) U_n'$ as in Corollary C.3.*

Proof. If we fix $x \in \text{Supp}(F)$, then the marginal contribution of $\tilde{\varphi}(x)$ to the almost sure (nonrandom) limit of the loss function $\mathcal{L}_n^{\gamma, KL}(\Sigma_n, \tilde{S}_n)$ is

$$\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\gamma(t)}{\gamma(\tilde{\varphi}(x))} \cdot \theta(x, t) dH(t) + \log [\gamma(\tilde{\varphi}(x))] . \quad (\text{C.33})$$

The partial derivative of (C.33) with respect to $\tilde{\varphi}(x)$ is

$$-\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\gamma(t)}{\gamma(\tilde{\varphi}(x))^2} \gamma'(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) + \frac{\gamma'(\tilde{\varphi}(x))}{2\gamma(\tilde{\varphi}(x))} \quad (\text{C.34})$$

The first-order condition is $\gamma(\tilde{\varphi}(x)) = \int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t)$, hence the solution is

$$\tilde{\varphi}(x) = \gamma^{-1} \left(\int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t) \right) . \blacksquare \quad (\text{C.35})$$

The proof of Theorem 4.6 is concluded as before, by showing that replacing the key oracle objects with their plug-in counterparts comes at no cost under large-dimensional asymptotics. ■

C.5 Singular Case

There is one small difference with the non-singular case: the support of the limiting sample spectral distribution F is now $\text{Supp}(F) = \{0\} \cup (\bigcup_{k=1}^{\kappa} [a_k, b_k])$, where (as before) $0 < a_1 < b_1 < \dots < a_{\kappa} < b_{\kappa} < \infty$.

Proposition C.5. *Under Assumptions 3-6,*

$$\begin{aligned} \mathcal{L}_n^{\gamma, F}(\tilde{S}_n, \Sigma_n) &\xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} \gamma(t)^2 dH(t) - 2 \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) dF(x) \\ &\quad - 2 \frac{c-1}{c} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(0)) \cdot \theta(0, t) dH(t) \\ &\quad + \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \gamma(\tilde{\varphi}(x))^2 dF(x) + \frac{c-1}{c} \gamma(\tilde{\varphi}(0))^2 , \end{aligned} \quad (\text{C.36})$$

where $\theta(0, t)$ is given by Equation (5.4).

Proof. For simplicity, we assume that $\text{Supp}(F) = \{0\} \cup [a, b]$, as the extension to the case $\kappa > 1$ is straightforward. Starting from the proof of Proposition C.3:

$$\begin{aligned} \mathcal{L}_n^{\gamma, F}(\Sigma_n, \tilde{S}_n) &= \frac{1}{p} \sum_{j=1}^p \gamma(\tau_{n,j})^2 - \frac{2}{p} \sum_{i=1}^p \sum_{j=1}^p \gamma(\tau_{n,j}) \gamma(\tilde{\varphi}_n(\lambda_{n,i})) \cdot (u'_{n,i} v_{n,j})^2 + \frac{1}{p} \sum_{i=1}^p \gamma(\tilde{\varphi}_n(\lambda_{n,i}))^2 \\ &= \frac{1}{p} \sum_{j=1}^p \gamma(\tau_{n,j})^2 - \frac{2}{p} \sum_{i=p-n+1}^p \sum_{j=1}^p \gamma(\tau_{n,j}) \gamma(\tilde{\varphi}_n(\lambda_{n,i})) \cdot (u'_{n,i} v_{n,j})^2 \\ &\quad - \frac{2}{p} \sum_{i=1}^{p-n} \sum_{j=1}^p \gamma(\tau_{n,j}) \gamma(\tilde{\varphi}_n(0)) \cdot (u'_{n,i} v_{n,j})^2 \\ &\quad + \frac{1}{p} \sum_{i=p-n+1}^p \gamma(\tilde{\varphi}_n(\lambda_{n,i}))^2 + \frac{1}{p} \sum_{i=1}^{p-n} \gamma(\tilde{\varphi}_n(0))^2 \end{aligned} \quad (\text{C.37})$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} \gamma(t)^2 dH_n(t) - 2 \int_a^b \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}_n(x)) d^2 \Theta_n(x, t) \\ &\quad - 2 \frac{p-n}{p} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}_n(0)) d\Theta_n(0, t) \\ &\quad + \int_a^b \gamma(\tilde{\varphi}_n(x))^2 dF_n(x) + \frac{p}{p-n} \gamma(\tilde{\varphi}_n(0))^2 . \end{aligned} \quad (\text{C.38})$$

By applying the technique from the proof of Theorem 6.1 of [Ledoit and Wolf \(2018\)](#), and by using Theorem 3 of [Ledoit and P  ch   \(2011\)](#) to handle the limit of Θ_n , it follows that:

$$\begin{aligned} \mathcal{L}_n^{\gamma, F}(\Sigma_n, \tilde{S}_n) &\xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} \gamma(t)^2 dH(t) - 2 \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(x)) \cdot \theta(x, t) dH(t) dF(x) \\ &\quad - 2 \frac{c-1}{c} \int_{-\infty}^{+\infty} \gamma(t) \gamma(\tilde{\varphi}(0)) \cdot \frac{1}{\left(1 - \frac{1}{c}\right) [1 + \check{m}_{\underline{F}}(0) t]} dH(t) \\ &\quad + \sum_{k=1}^{\kappa} \int_{a_k}^{b_k} \gamma(\tilde{\varphi}(x))^2 dF(x) + \frac{c-1}{c} \gamma(\tilde{\varphi}(0))^2. \blacksquare \end{aligned} \quad (\text{C.39})$$

Corollary C.5. *Under Assumptions 3–6, a covariance matrix estimator \tilde{S}_n minimizes in the class of rotation-equivariant estimators the a.s. limit (C.36) of the Generalized Frobenius loss if and only if its limiting shrinkage function $\tilde{\varphi}$ verifies $\forall x \in \text{Supp}(F)$, $\tilde{\varphi}(x) = \varphi^\gamma(x)$, where*

$$\forall x \in \text{Supp}(F) \quad \varphi^\gamma(x) := \gamma^{-1} \left[\int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(x, t) dH(t) \right] . \quad (\text{C.40})$$

This yields the oracle covariance matrix estimator $S_n^\gamma := U_n \text{Diag}(\varphi^\gamma(\lambda_{n,1}), \dots, \varphi^\gamma(\lambda_{n,p})) U_n'$.

For $x \in \bigcup_{k=1}^{\kappa} [a_k, b_k]$, the proof is the same as for Corollary C.3. The only pending matter is what happens when $x = 0$. The marginal contribution of $\tilde{\varphi}(0)$ to the almost sure (nonrandom) limit of the loss function $\mathcal{L}_n^{\gamma, \text{F}}(\Sigma_n, \tilde{S}_n)$ is

$$-2\frac{c-1}{c}\int_{-\infty}^{+\infty}\gamma(t)\gamma(\tilde{\varphi}(0))\cdot\theta(0,t)dH(t)+\frac{c-1}{c}\gamma(\tilde{\varphi}(0))^2. \quad (\text{C.41})$$

The partial derivative of (C.41) with respect to $\tilde{\varphi}(0)$ is

$$-2\frac{c-1}{c}\int_{-\infty}^{+\infty}\gamma(t)\gamma'(\tilde{\varphi}(0))\cdot\theta(0,t)dH(t)+2\frac{c-1}{c}\gamma'(\tilde{\varphi}(0))\gamma(\tilde{\varphi}(0)) \quad (\text{C.42})$$

The first-order condition is $\gamma(\tilde{\varphi}(0)) = \int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(0, t) dH(t)$, hence the solution is

$$\tilde{\varphi}(0) = \gamma^{-1} \left(\int_{-\infty}^{+\infty} \gamma(t) \cdot \theta(0, t) dH(t) \right) . \blacksquare \quad (\text{C.43})$$

Proof of Theorem 5.1. The proof is concluded as follows: To the unobservable quantity c corresponds the plug-in estimator p/n ; to the unobservable function $H(t)$ corresponds the plug-in estimator $\hat{H}_n(t) := \sum_{i=j}^p \mathbb{1}_{[\hat{\tau}_{n,j}, +\infty)}(t)/p$; to the unobservable function $\theta(x, t)$ corresponds the plug-in estimator $\hat{\theta}_n(x, t)$ from Equation (3.7) for $x > 0$; and to the unobservable quantity $\theta(0, t)$ corresponds the plug-in estimator $\hat{\theta}_n(0, t)$ from Equation (5.5). The fact that these four unobservables can be replaced with their respective plug-in counterparts at no loss asymptotically is established in the same way as in the proof of [Ledoit and Wolf's \(2018\)](#) Theorem 6.2. ■