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Stellar Structure and Evolution Homework 5

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1. TIDES

1.1. *a*

We define these paremeters:
mass of Earth: $M_e = 5.972 \times 10^{27}$ g, mass of Moon: $M_m = 7.349 \times 10^{25}$ g
distance between Earth and mass center: d_e
distance between Moon and mass center: d_m
radius of Earth: $R_e = 6.378 \times 10^8$ cm, radius of Moon: $R_m = 1.738 \times 10^8$ cm
orbital angular velocity: $\omega_e = \omega_m = \omega$
spin period of Earth: $P_e = 1$ day, spin period of Moon: $P_m = 27.32$ days
spin angular velocity of Earth: $\Omega_e = \frac{2\pi}{P_e}$, spin angular velocity of Moon: $\Omega_m = \frac{2\pi}{P_m}$
$q = M_m/M_e \approx 0.012$, $M = M_e + M_m = (1+q)M_e$, $d = d_e + d_m = 3.844 \times 10^{10}$ cm.

According to Newton's second law, we have:

$$\begin{cases} \frac{GM_eM_m}{(d_e+d_m)^2} = M_e\omega^2 d_e \\ \frac{GM_eM_m}{(d_e+d_m)^2} = M_m\omega^2 d_m \end{cases} \tag{1}$$

$$\therefore \begin{cases} M_e = \frac{1}{1+q}M, \ M_m = \frac{q}{1+q}M \\ d_e = \frac{q}{1+q}d, \ d_m = \frac{1}{1+q}d \\ \omega = \sqrt{\frac{GM}{d^3}} \end{cases} \tag{2}$$

Then we can calculate the angular momentum of orbit and spin by assuming $\Omega = \omega$:

$$\therefore \begin{cases} J_{e,o} = M_e d_e^2 \omega = \frac{q^2}{(1+q)^3}\sqrt{GM^3 d} \\ J_{m,o} = M_m d_m^2 \omega = \frac{q}{(1+q)^3}\sqrt{GM^3 d} \\ J_{e,s} = \frac{2}{5}M_e R_e^2 \Omega_e \\ J_{m,s} = \frac{2}{5}M_m R_m^2 \Omega_m \end{cases} \tag{3}$$

We can find that: $\frac{J_{e,o}}{J_{m,o}} = q = 0.012$, $\frac{J_{m,s}}{J_{e,s}} = q(\frac{R_m}{R_e})^2 \frac{P_e}{P_m} \approx 3.34 \times 10^{-5}$, which means that the orbital angular momentum of the Earth and the spin angular of the Moon are small compared to the other terms.

So the toal agular momentum is: $J_{total} \approx J_{m,o} + J_{e,s} \approx (2.82 + 0.71) \times 10^{41} = 3.53 \times 10^{41}$ erg·s

1.2. *b*

$$J_{total} = J_{m,o} + J_{e,s} = \frac{q}{(1+q)^{5/2}}\sqrt{GM^3 d_m} + \frac{2}{5}M_e R_e^2 \Omega_e \tag{4}$$

From conservation of angular momentum: $\frac{dJ_{total}}{dt} = 0$, then we get:

$$\frac{q}{2(1+q)^{5/2}}\sqrt{\frac{GM^3}{d_m}}\frac{d_m}{dt} + \frac{2}{5}M_e R_e^2 \frac{d\Omega_e}{dt} = 0 \tag{5}$$

$$\therefore \frac{d\Omega_e}{dt} = A\frac{d_m}{dt} < 0 \tag{6}$$

1.3. *c*

$$\begin{aligned}\frac{dP_e}{dt} = \frac{-2\pi}{\Omega_e^2}\frac{d\Omega_e}{dt} =& \frac{5\pi q}{2(1+q)^3 M_e R_e^2 \Omega_e^2}\sqrt{\frac{GM^3}{d}}\frac{d_m}{dt} \\ \approx& 1.7 \times 10^{-5}\ \text{s} = 17\mu\text{s}\end{aligned} \tag{7}$$

2. THE PRE-MAIN SEQUENCE

2.1. *a*

From conservation of energy we have:

$$L = L_{nuc} - \frac{d\Omega}{dt} - \frac{dE_{ini}}{dt} \tag{8}$$

From virial theory we have:

$$\frac{dE_{ini}}{dt} = -0.5\frac{d\Omega}{dt} \tag{9}$$

Therefore:

$$L = L_{nuc} - 0.5\frac{d\Omega}{dt} = L_{nuc} + L_{grav} \tag{10}$$

2.2. *b*

We assume the function of radius:

$$R(t) = \frac{R_0}{\lambda(t)} \tag{11}$$

The gravitational potential energy is:

$$\Omega(t) = \Omega(R_0)\frac{R_0}{R(t)} = \lambda(t)\Omega(R_0) \tag{12}$$

By assuming no nuclear sources, the luminosity is:

$$L = 0.5\frac{d\Omega}{dt} = 0.5\Omega(R_0)\frac{d\lambda(t)}{dt} \tag{13}$$

Then we have:

$$\lambda(t) = \frac{L}{0.5\Omega(R_0)}t + 1 \tag{14}$$

Therefore, the radius function with time is:

$$R(t) = \frac{R_0}{1+\frac{t}{\tau}},\ \text{where } \tau = \frac{0.5\Omega(R_0)}{L} \tag{15}$$

2.3. *c*

From previoius homeworks we know that for a polytrope of index n=1.5, the equation of state is:

$$\therefore \begin{cases} P_c = K\rho_c^{5/3} = 0.42422 \cdot GM^{1/3}R\rho_c^{5/3} \\ \rho_c = 5.99071 \cdot \bar{\rho} = 5.99071 \cdot \frac{3M}{4\pi R^3} \\ P_c = \frac{\rho_c}{m_\mu}kT_c \end{cases} \tag{16}$$

Therefore, the central temperature is:

$$T_c = 0.538 \cdot \frac{m_\mu G}{k}\frac{M}{R} \tag{17}$$

And from Eq.15 we have:

$$T_c = 0.538 \cdot \frac{m_\mu G}{k}\frac{M}{R_0}(1+\frac{t}{\tau}),\ \text{where } m_\mu = \frac{m_H}{2X + 3Y/4 + Z/2} \tag{18}$$

2.4. *d*

By assuming a uniform sphere, we can calculate the gravitational enregy:

$$\Omega(R_0) = \int G\frac{Mr^3}{R^3}\frac{4\pi r^2\rho dr}{r} \tag{19}$$

$$= \frac{3GM_0^2}{5R_0} \tag{20}$$

And we take $M_0 = M_\odot,\ L = 4L_\odot,\ R_0 = 4R_\odot,\ X = 0.7,\ Y = 0.28,\ Z = 0.02$ and plot Figure 1 and Figure 2.

2.5. *e*

From Eq.18 we can calculate: $t_{MS} = 2.48 \times 10^6$ yr. From Eq.15 we can calcultae: $R_{MS} = 0.19R_0 = 0.76R_\odot$. The thermal timescale is: $\tau_{KH} = \frac{GM^2}{2RL} \sim 1 \times 10^6$ yr.

We can find that: $t_{MS} \gtrsim \tau_{KH}$

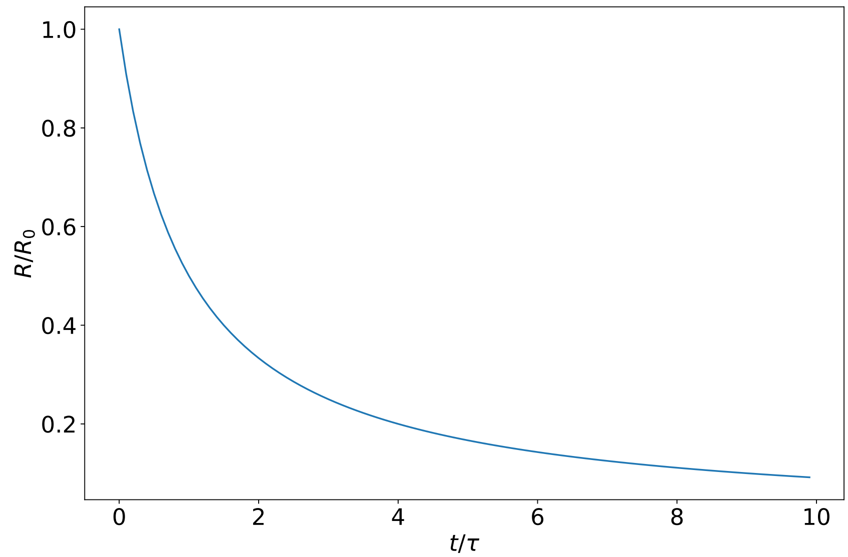

Figure 1. R(t)

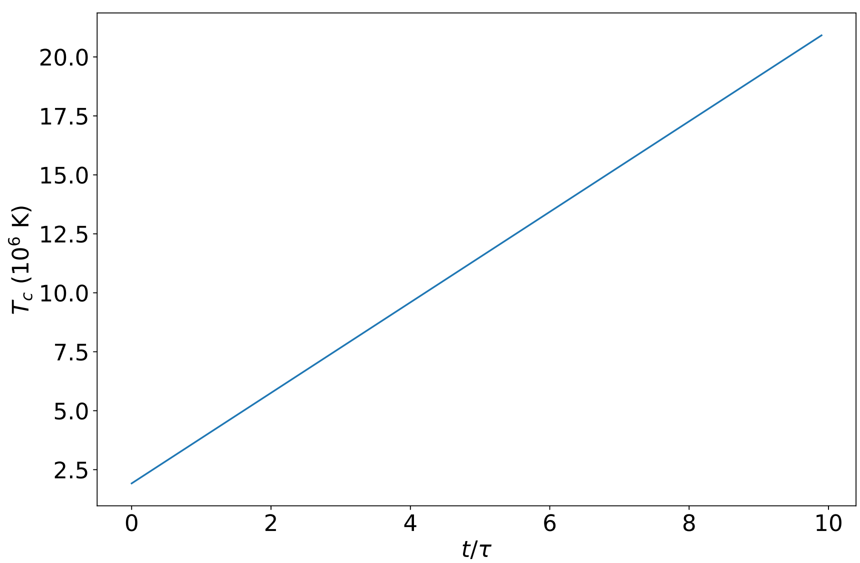

Figure 2. T_c(t)

3. SUPERNOVAE AND NEUTRON STARS

3.1. *a*

From conservation of momentum:

$$M_{neuron}V_{kick} = \frac{1\% \times E_{supernova}}{c} \tag{21}$$

We can estimate that $V_{kick} = \frac{1\% \times E_{supernova}}{1.4 M_{\odot} \cdot c} \approx 1.2 \times 10^{7}$ cm s^{-1} = 120 km s^{-1}

3.2. *b*

$$E = \frac{1}{2} I\omega^2 = \frac{2\pi^2 I}{P^2} \tag{22}$$

$$|\frac{dE}{dt}| = \frac{4\pi^2 I}{P^3}\dot{P} \tag{23}$$

$$t \sim E/|\frac{dE}{dt}| = \frac{P}{2\dot{P}} \approx 1.7 \times 10^{14} \text{ yr} \tag{24}$$

3.3. *c*

I use these value:
Lethal dose: 5 Sv, which means $E_{lethal} = 250$ J for a man with 50 kg.
The neutrino-nucleon cross section: $\sigma \approx 10^{-47}$ m^2
Atom numbers of human body: $N = 7 \times 10^{27}$

Then we can calculate the effective area: $S = \sigma N = 7 \times 10^{-16}$ cm^2, and the energy flux: $F_{lethal} = \frac{E_{lethal}}{S} \approx 4 \times 10^{24}$ erg cm^{-2}

Therefore, the distance is:

$$d_{lethal} = \sqrt{\frac{L}{4\pi F_{lethal}}} \sim 10^{14} \text{ cm} \sim 10 \text{ AU} \tag{25}$$

4. HALF-LIFE OF NI

4.1. *a*

$$\begin{cases} \frac{dN_{Ni}}{dt} = -\lambda_{Ni} n_{Ni}, & \text{where } \lambda_{Ni} = \frac{ln2}{\tau_{1/2,Ni}} \\ \frac{dN_{Co}}{dt} = -\lambda_{Co} n_{Co} + \lambda_{Ni} n_{Ni}, & \text{where } \lambda_{Co} = \frac{ln2}{\tau_{1/2,Co}} \end{cases} \tag{26}$$

4.2. *b*

Assume that the initial number density of pure Ni is n_0, then the solution of Ni is:

$$n_{Ni} = n_0 e^{-\lambda_{Ni} t} \tag{27}$$

Then we take this into equation of Co and multiples $e^{\lambda_{Co} t}$ to each side:

$$e^{\lambda_{Co} t} \frac{dN_{Co}}{dt} = - e^{\lambda_{Co} t} \lambda_{Co} n_{Co} + e^{\lambda_{Co} t - \lambda_{Ni} t} \lambda_{Ni} n_0 \tag{28}$$

$$\frac{d(e^{\lambda_{co} t} n_{Co})}{dt} = e^{\lambda_{Co} t - \lambda_{Ni} t} \lambda_{Ni} n_0 \tag{29}$$

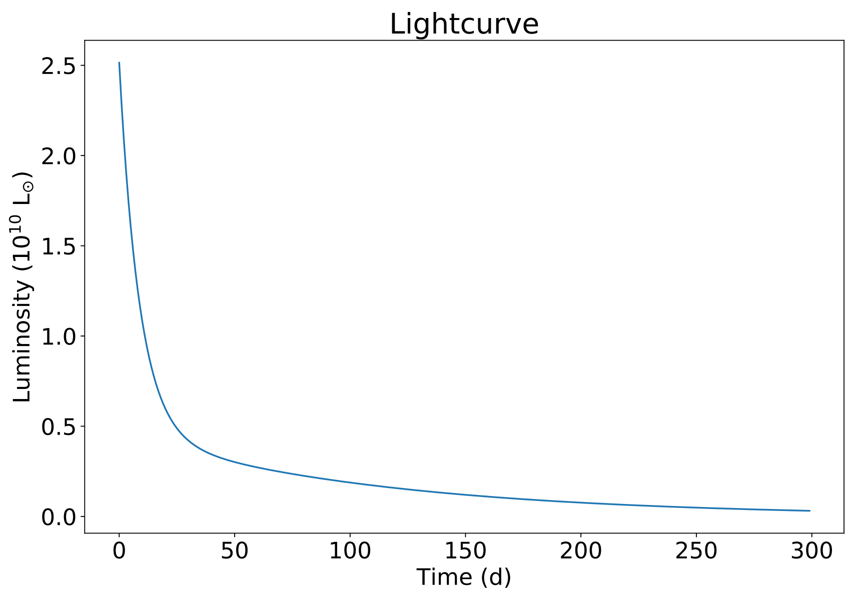

Figure 3. This is the lightcurve for an explosion that produces 1 $M_\odot$ of ^{56}Ni

Therefore, the jnumber density of Co is:

$$n_{Co} = \frac{\lambda_{Ni} n_0}{\lambda_{Ni} - \lambda_{Co}} (e^{-\lambda_{Co} t} - e^{-\lambda_{Ni} t}) \tag{30}$$

4.3. *c*

$$L(t) = Q_{Ni} \lambda_{Ni} N_{Ni} + Q_{Co} \lambda_{Co} N_{Co} \tag{31}$$

$$= Q_{Ni} \lambda_{Ni} N_0 e^{-\lambda_{Ni} t} + \frac{Q_{Co} \lambda_{Co} \lambda_{Ni} N_0}{\lambda_{Ni} - \lambda_{Co}} (e^{-\lambda_{Co} t} - e^{-\lambda_{Ni} t}) \tag{32}$$

4.4. *d*

see Figure 3.

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5. REFERENCES

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