Optimization vs Epidemics

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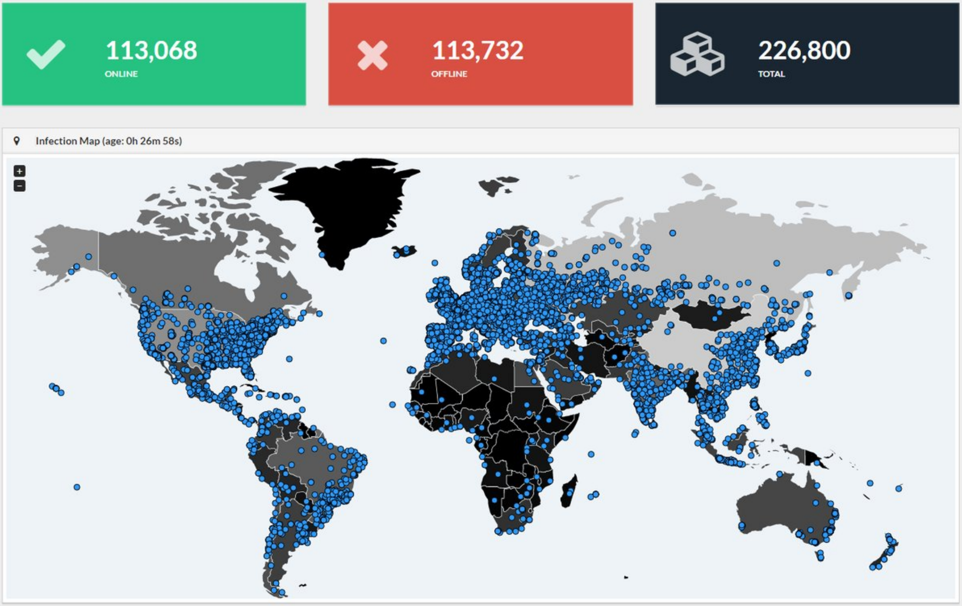
WannaCry (May 2017)



Petya (from early 2016 to June 2017)

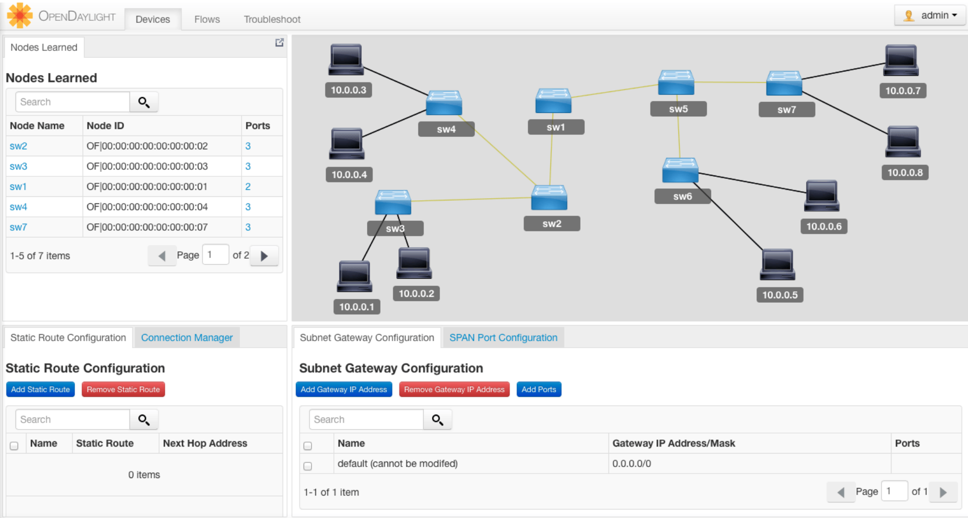
PIXELS | CHRONIQUES DES (R)ÉVOLUTIONS NUMÉRIQUES

VIE EN LIGNE

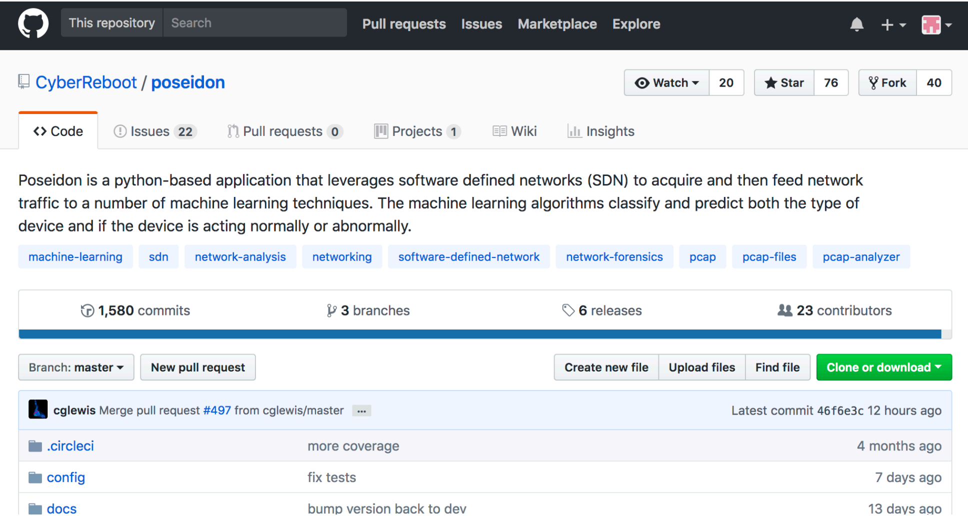
Le virus Petya a coûté plus d’un milliard d’euros aux entreprises

Des ports de marchandise à l’arrêt, des usines immobilisées et des entreprises ralenties… la facture de ce faux « rançongiciel » est salée, selon un décompte du « Monde ».

SDN: software-defined networking



Poseidon: machine learning for node health



SDN against malware

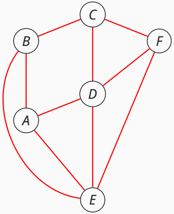
- transparent and unified data analysis
- network management by algorithms

▷ opportunity to design autonomous security systems for networks

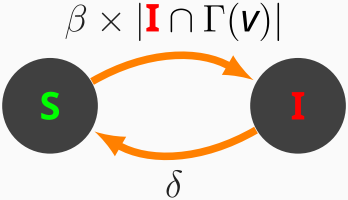
- our focus: stopping malware epidemics

Networked epidemic models

- undirected graph $G = (V, E)$ represents neighborhood of $n := |V|$ nodes
- a node is either **S**usceptible or **I**nfected



node v:



- Markov chain with 2^n states
- unique absorbing state: every node is **S**

sufficient condition for
$O(\log n)$ time to extinction:

$$\lambda_1(G) < \frac{\delta}{\beta}$$

[Prakash et al. '12]

From a theorem to a control system

How to achieve $\lambda_1(G) < \dfrac{\delta}{\beta}$?

note: β and δ can be inferred in real time with e.g. maximum likelihood estimators [Ruhi et al. '17]

A) deploy software patches to increase δ/β

$\rightarrow$ hand-crafted response against specific malware

$\rightarrow$ hard to predict the effect of a patch on β and/or δ

B) modify topology to decrease $\lambda_1(G)$

$\rightarrow$ generic response against any epidemic

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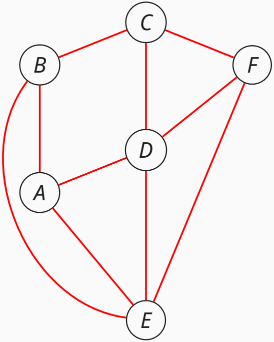
What is $\lambda_1(\boldsymbol{G})$?



$$\begin{array}{c c} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

spectrum of the above adjacency matrix:

{ 3.39, 0.19, 0.81, -0.55, -1.57, -2.27 }

The spectral radius of $G = (V, E)$

- structural param. related to connectivity

  $$\max\left(\sqrt{\Delta(G)}, \frac{2m}{n}\right) \leq \lambda_1(G) \leq \Delta(G)$$

- a graph with no edges has $\lambda_1(\mathbf{0}) = 0$

Theorem: λ_1 is monotone
if H is a subgraph of G then $\lambda_1(H) \leq \lambda_1(G)$

Theorem: λ_1 is positive homogeneous
if $\alpha > 0$ then $\lambda_1(\alpha A) = \alpha\, \lambda_1(A)$

A natural optimization problem

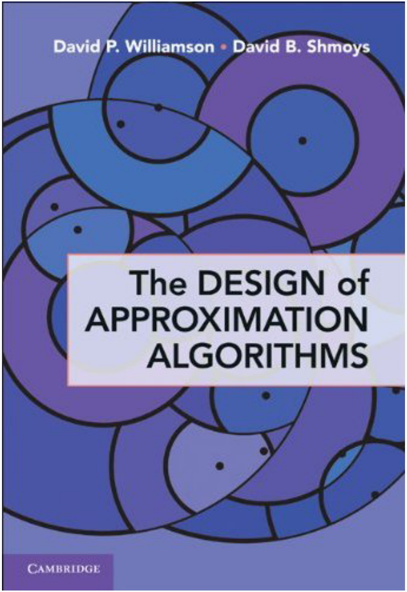
Problem: MAXIMUM λ-SPECTRAL SUBGRAPH

Instance: undirected graph $G = (V, E)$ and spectral threshold $1 \leq \lambda < \lambda_1(G)$

Task: find a subgraph $H = (V, E')$ of G with maximum number of edges and $\lambda_1(H) \leq \lambda$

- Set $\lambda = \dfrac{\delta}{\beta}$ to end a given epidemic
- NP-hard even in graphs with $\Delta \leq 3$

I always have trouble with



MλSSP as a mathematical program

$$\begin{aligned} \max \quad & \sum_{ij \in E} y_{ij} \\ \text{s.t.} \quad & \sum_{ij \in E} y_{ij} A_{ij} \preceq \lambda I \qquad (\mathcal{P}) \\ & y_{ij} \in \{0,1\}, \forall ij \in E \end{aligned}$$

- recall that $M \succeq N \iff \forall i, \lambda_i(M - N) \geq 0$
- hence $A \preceq \lambda I \iff \lambda_1(A) \leq \lambda$
- binary SDP amenable to MISDP solvers

A polytime solvable relaxation

$$
\begin{aligned}
\max \quad & \sum_{ij \in E} y_{ij} \\
\text{s.t.} \quad & \sum_{ij \in E} y_{ij} A_{ij} \preceq \lambda I \qquad\qquad (\mathcal{S}) \\
& \sum_{j \in \Gamma(i)} y_{ij} \leq \lambda^2, \ \forall i \in V \\
& y_{ij} \in [0,1], \ \forall ij \in E
\end{aligned}
$$

- valid inequality because: $\lambda_1 \leq \lambda \Rightarrow \Delta \leq \lambda^2$
- solvable by fast SDP solvers e.g. SuperSCS

Relaxation & randomized rounding

- solve relaxation and let $y^* = \arg \mathcal{S}$
- define r.v. $\mathbf{x}_{ij} \sim \text{Bernoulli}(y^*_{ij}/r)$
- let $H = \{ij \in E \,|\, x_{ij} = 1\}$
- determine $r > 1$ s.t. H is feasible w.h.p.
$\Pr(\lambda_1(\sum \mathbf{x}_{ij} A_{ij}) \leq \lambda) = 1 - 1/n$

▷ randomized r-approximation algorithm

- oldie but goodie [Raghavan & Tompson '87]

Approximate maximization

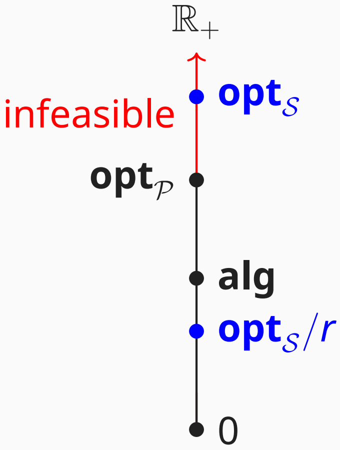


Figure: **alg** is the expected number of edges in H

Determining the approx. ratio r

- instead of designing algorithms...

▷ prove concentration for a random object!

- here: random adjacency matrices
- use concentration inequalities for sum of random matrices $\mathbf{X} = \sum_e \mathbf{X}_e$ [Tropp '15]

$$\Pr(\lambda_1(\mathbf{X}) \geq a) \leq \min_{t>0} e^{-ta} \operatorname{tr} \exp\left(\sum_e \log\left(\mathbb{E} e^{t\mathbf{X}_e}\right)\right)$$

Matrix Bernstein gives

- for $\mathbf{X} = \sum_e \mathbf{X}_e$ with $\mathbb{E}\mathbf{X} = \mathbf{0}$ and $\lambda_1(\mathbf{X}_e) \leq L$
- matrix variance: $v(\mathbf{X}) \stackrel{\text{def}}{=} \lambda_1(\sum_e \mathbb{E}\mathbf{X}_e^2)$

$$\Pr(\lambda_1(\mathbf{X}) \geq a) \leq n \exp\left(-\frac{a^2}{2v(\mathbf{X}) + 2La/3}\right)$$

- apply to $\bar{\mathbf{A}} = \sum \mathbf{x}_{ij}A_{ij} - \mathbb{E}\mathbf{x}_{ij}A_{ij}$, $a = (1 - 1/r)\lambda$

Goal: $\Pr(H \text{ is infeasible}) \leq 1/3$

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$$\Pr(\lambda_1(\mathbf{X}) \geq a) \leq n \exp\left(-\frac{a^2}{2v(\mathbf{X}) + 2La/3}\right)$$

- apply to $\bar{\mathbf{A}} = \sum \mathbf{x}_{ij} A_{ij} - \frac{y^*_{ij}}{\mathbf{r}} A_{ij}$, $a = (1 - 1/\mathbf{r})\lambda$

Goal: $\Pr(H \text{ is infeasible}) \leq 1/3$ for $\mathbf{r} = ?$

$r = O(\log n)$ when $\lambda \geq \log n$

- key property: $v(\bar{\mathbf{A}}) \leq \lambda^2/r$ (cf. valid ineq.)
- crank out the math...

Result: $\Pr(H \text{ is infeasible}) \leq 1/3$ for $r = 1 + 3\log n$ if $\lambda \geq \log n$

- from $1/3$ to $1/n$ with amplification

What about $\lambda \leq \log n$?

- spectral ver. of classical result on Δ vs ν

$$\nu(G) \geq \frac{m}{\lambda_1^2(G) - 1}$$

- implies that a maximum matching is a $(\lambda^2 - 1)$-approximation of MλSSP

Result: $O(\log^2 n)$-approx. if $\lambda \leq \log n$

Conclusion

- randomized $O(\log^2 n)$-approximation algorithm
- parallel approximate SDP solver + parallel independent rounding + power method
- super fast in practice, can exploit GPUs

Perspectives

- hardness of approx. (Max k-Cut?)
- Erdös-Rényi graphs limit our approach:

$$\lambda_1(G(n,p)) = \Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right)$$

even when $np = O(1)$ [Krivelevich et al. '01]

- maximum degree-constrained subgraph for $\lambda \leq \log n$ (from λ^2 to λ-approx?)
- better concentration bounds? [Le et al., '15]


Any questions?

Problem variants

- spectral subgraph setting with two parameters: number of edges vs spectral radius

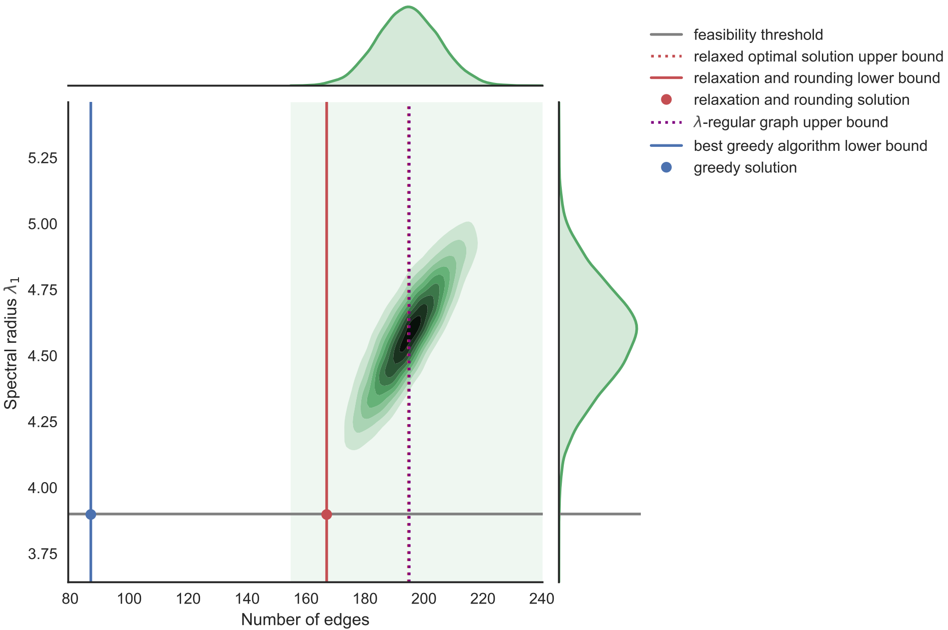
	constraint	objective
v. Mieghem et al. ’12	$\lvert E' \rvert = k$	min λ_1
Saha et al. ’15	$\lambda_1 \leq \lambda$	min $\lvert E \rvert - \lvert E' \rvert$
Zhang et al. ’15	$\lvert E' \rvert \leq k$	min λ_1
this work	$\lambda_1 \leq \lambda$	max $\lvert E' \rvert$

The Laplacian spectrum

- $\mu \leq \mu_2(H)$ instead of $\lambda_1(H) \leq \lambda$
- related to expander construction:

$$\begin{aligned} \min \ & \sum_{ij \in \bar{E}} y_{ij} \\ \text{s.t. } & L(G) + \sum_{ij \in \bar{E}} y_{ij} L_{ij} \succeq \mu P_{\mathbf{1}^\perp} \qquad (\mathcal{Q}) \\ & y_{ij} \in \{0,1\}, \ \forall ij \in \bar{E} \end{aligned}$$

- some results [Ghosh & Boyd '06, Kolla et al. '09]


feasibility threshold
relaxed optimal solution upper bound
relaxation and rounding lower bound
relaxation and rounding solution
λ-regular graph upper bound
best greedy algorithm lower bound
greedy solution
Spectral radius λ_1
Number of edges
3.75
4.00
4.25
4.50
4.75
5.00
5.25
80
100
120
140
160
180
200
220
240