Active Risk Attribution Model Specifications

Let's assume we have n securities, in m asset classes, which returns,

$$r_t = \mu \quad + \epsilon_t$$

Where $\epsilon_t \sim N(0,\ \Sigma_t)$

The tracking error can be calculated as:

$$TE_t^{\ 2} = gap_t^T \ \Sigma_t \ gap_t$$

With:
gap_t: vector of active positions (nx1) at time t
Σ_t: covariance matrix at time t

Let's consider a particular time t and ignore the t subindex to simplify the notation.

The total contribution per position can be calculated from:

$$TE \quad = \frac{gap^T \ \Sigma_{nxn} \ gap_{nx1}}{TE}$$

Where:

$$TE \quad = gap \quad \cdot \frac{\Sigma_{nxn} \ gap_{nx1}}{TE}$$

And then the total contribution from asset i to the TE will be:

$$TE_i = gap_i \cdot \left[\frac{\Sigma_{nxn} \ gap_{nx1}}{TE}\right]_i$$

For the allocation and selection effects, we consider a similar decomposition.

Risk Attribution Components

Let's suppose at a time t:

$$gap = gap_{alloc} + gap_{select}$$

Then, we can define each component $gap_{alloc,i}$ as the active position on the security i in the asset class j if the investment in the asset class were exactly in the benchmark proportions. Only a multiplicative effect may appear given that the position in the asset class may be higher or smaller than in the benchmark. Therefore:

$$gap_{alloc,i} = w_{b,i,j} * GAP_j$$

Where,
$gap_{alloc,i}$: gap allocation for security i
$w_{b,i}$: relative weight of the security i in the benchmark inside the asset class j
GAP_j : active position of the whole asset class j

The $gap_{selection} = gap - gap_{allocation}$

For example, if we have:

		Portfolio position	Benchmark position	Port Position w/r to the asset class	Bench Position w/r to the asset class	Gap Allocation	Gap selection
Equity	SPY	33%	30%	50%	50%	3%	0%
	EZU	33%	30%	50%	50%	3%	0%
FI	AGG	34%	40%				

gap_allocation_spy = 50% * 6% = 3%
gap_selection_spy = 3% - 3% = 0

		Portfolio position	Benchmark position	Port Position w/r to the asset class	Bench Position w/r to the asset class	Gap Allocation	Gap selection
Equity	SPY	27%	30%	45%	50%	0%	-3%
	EZU	33%	30%	55%	50%	0%	3%
FI	AGG	40%	40%				

gap_allocation_spy = 50% * 0% = 0%
gap_selection_spy = (27%-30%) - 0% = -3%

For the allocation and selection effects we separate the effects as follows:

$$TE^2 = gap^T \ \Sigma_{nxn} \ gap_{nx1}$$

$$TE^2 = (gap^T_{alloc} + gap^T_{selec}) \ \Sigma_{nxn} \ (gap_{alloc} + gap_{selec})$$

$$TE^2 = gap^T_{alloc} \ \Sigma_{nxn} \ gap_{alloc} + gap^T_{selec} \ \Sigma_{nxn} \ gap_{selec} + gap^T_{alloc} \ \Sigma_{nxn} \ gap_{selec} + gap^T_{selec} \ \Sigma_{nxn} \ gap_{alloc}$$

$$TE^2 \quad = \quad TE^2_{alloc} \quad + \quad TE^2_{select} \quad + \quad TE^2_{interaction}$$

We want to allocate the interaction in a way that makes sense.
When gap_alllocation is zero in a position, the addition should be zero.
When gap_selection is zero in a position, the addition should be zero.

So, separate the interaction and assign to the TE allocation and selection effects

$$gap^T_{alloc} \ \Sigma_{nxn} \ gap_{selec} + \quad gap^T_{selec} \ \Sigma_{nxn} \ gap_{alloc}$$

$$TE_{interaction, \ allocation\ lead} + TE_{interaction, \ selection \ lead}$$

And we apply the first term to TE allocation effect. By dividing by TE we obtain the final definition,

$$TE \quad = \quad \frac{TE^2_{alloc} + TE^2_{interact, \ allocation\ lead}}{TE} + \frac{TE^2_{select} \ + TE^2_{interact, \ selection \ lead}}{TE}$$

Where the first term is the TE allocation component, and the second is the TE selection component t.

$$TE_{Allocation\ effect} = \frac{TE^2_{alloc} + TE^2_{interact, \ allocation\ lead}}{TE}$$

$$TE_{Selection\ effect} = \frac{TE^2_{select} \ + TE^2_{interact, \ selection\ lead}}{TE}$$

If we come back to a previous equation, we can see the decomposition in a simpler and more natural way, passing on the interaction effect:

$$TE^2 = (gap^T_{alloc} + gap^T_{selec}) \ \Sigma_{nxn} \ (gap_{alloc} + gap_{selec})$$

$$TE^2 = gap^T_{alloc} \ \Sigma_{nxn} \ (gap_{alloc} + gap_{selec}) + \ gap^T_{selec} \ \Sigma_{nxn} \ (gap_{alloc} + gap_{selec})$$

And then, dividing by the total tracking error TE:

$$TE = \frac{gap_{alloc}^{T}\ \Sigma_{nxn}\,(gap_{alloc} + gap_{selec})}{TE} + \frac{gap_{selec}^{T}\,\Sigma_{nxn}\,(gap_{alloc} + gap_{selec})}{TE}$$

We get to the first term as the allocation effect and the second term as the selection effect. The same we found before re-allocating the interaction effect. Given that the interaction effect is widely known in performance attribution, I made my first analysis by recognizing it and deciding a way to assign it in the most appropriate way. But clearly, we get to the same result in the "simpler" approach bypassing it.

Covariance Matrix Parametrization

Individual covariances are calculated using correlation and volatilities

$$Cov(A, B) = \rho_{A,B}\sigma_A\sigma_B$$

Correlations:
Calculated using weekly returns, to minimize time differences among assets

Volatilities:
Calculated using daily returns

Decay factor:
Applied with a half-life of 40 days (about 2-months). Apply to both, volatilities and correlations

Historical window:
1 year for both, volatilities and correlations