Algorithmic Game Theory

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Research Objective

Define a system that can choose a number of users to take a tour. The number of users and the maximum number of kilometers that can be traveled is fixed. Each user has a budget that they cannot exceed. The costs of the tour are divided into proportional costs related to the distance of the tour and fixed costs.

Outline

1. Bayesian Game Settings
2. Quasilinear mechanism

2.1. Direct quasilinear mechanism

3. Vickrey-Clarke-Groves Mechanisms
4. Problem Resolution

1 Bayesian Game Settings

A Bayesian game is a type of game in which players have **incomplete information** about the other players' characteristics or strategies. In this game, each player has a prior belief about the distribution of the other players' types.

The main challenge in a Bayesian game is to find a strategy that maximizes a player's expected **utility** given their prior beliefs about the other players. They are called Bayesian because of the probabilistic analysis inherent in the game.

A Bayesian game is a tuple (N,O,Θ,p,u), where:

- N = {1,...,n} is a finite set of players;

- O = a set of outcomes;

- $\Theta = \theta_1 \times \cdots \times \theta_n$, with θ_i the set of possible types of player i;

- p : Θ→[0,1] is the probability distribution over Θ;

- $u = (u_1,...,u_n)$ is a profile of utility functions $u_i : A \times \Theta \rightarrow R$.

2 Quasilinear Mechanism

A mechanism in the quasilinear setting is a triple (A, χ ,$\mathcal{P}$), where:

- $A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to agent $i \in N$;
- $x : A \rightarrow \Pi(X)$ maps each action profile to a distribution over choices;
- $\mathcal{P}$: $A \rightarrow R^n$ maps each action profile to a payment for each agent.

The use of a quasilinear mechanism came about because its strengths:

- the mechanism can choose to charge or reward agents with an arbitrary monetary amount, so our users can choose their own budget;

- an agent's degree of preference for selecting **any choice is independent** of the amount. So our users will choose the destination of the trip just for pleasure without thinking about the amount needed. This feature is really important as it leads users to focus only on their own interests without seeking strategies based on others' choices. They have no need to lie and always proclaim the truth.

2.1 Direct quasilinear mechanism

A direct quasilinear mechanism is a pair (χ,$\wp$). It defines a mechanism in the quasilinear setting, where for each i, $A_i = \Theta_i$. The set of actions available to each player is just the set of possible preferences of the player.

3 Vickrey-Clarke-Groves Mechanisms

For the problem resolution, the choice fell on a mechanism belonging to the **Groves mechanisms**. Groves mechanisms belong to the direct quasilinear mechanisms (χ,$\wp$),for which:

- $\chi(\hat{\mathrm{v}}) = \mathrm{argmax}_{\mathrm{x}} \sum_{\mathrm{i}}^{\mathrm{n}} \mathrm{v}_{\mathrm{i}}(\mathrm{x})$;
- $\wp_{\mathrm{i}}(\hat{\mathrm{v}}) = h_i(\hat{v}_{\text{-i}}) - \sum_{i \neq j}^{n} \hat{v}_j(\chi(\hat{v}))$;

The dominant strategy of the mechanism is **truthtelling**, this mechanism allows us to make efficient choices.

What does it mean efficiency?

A quasilinear mechanism is efficient, if in equilibrium it selects a choice x such that:

$$\forall v \forall x', \quad \sum_{i=0 i}^{n} v_i(x) \quad \geq \quad \sum_{i=0 i}^{n} \quad v_i(x')$$

In VCG mechanism, h$_i$ is replace with:

- $h_i(\hat{v}_i) = \sum_{i \neq j}^{n} \hat{v}_j \ (\chi(\hat{v}_{-i}))$

Using this information, it is possible now to establish how the VCG mechanism is structured:

- $\chi(\hat{v}) = \arg\max_x \sum_i^n v_i(x)$;
- $\wp_i(\hat{v}) = \sum_{i \neq j}^{n} \ \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{i \neq j}^{n} \ \hat{v}_j \ (\chi(\hat{v}))$;

4 Problem Resolution

Goal: Organize a travel composed by k players following some rules:

- each player has a budget;
- each player is an agent represented by an arc;
- each player has fixed and variables costs proportional to travel length;
- each player declares truthfully its interests;
- the bus has a fixed number of places;
- the tour start and finish in the same place;
- the bus has maximum number of km that can be covered;

4.1 Definition of variables

- P :- set of total players;

$$\mathrm{P} = \{\text{Francesca, Paola}\}$$

- S :- subset of players joining the travel;

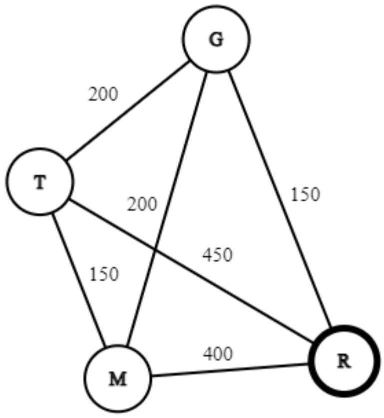
$$\mathrm{S} = \{\mathrm{S} \subseteq \mathrm{P}\text{: } |S| = \mathrm{k}\}\text{ , k is the number of players;}$$

- L :- set of locations available;
- $\mathrm{L_S}$ = start location;
- $\mathrm{T_P}$:- set of tour's path;
- $\mathrm{P_U}$:- set of players' utilities;
- MAX_L :- limit of kilometers;
- $\mathrm{D_L}$:- set of all distances for each place.
- $\mathrm{B_i}$:- Budget maximum for each player.

4.2 Overview Resolution

1. We generate all possible path
2. We apply the "traveling salesman problem", we search the shortest path of tour and eliminate all path that exceed the limit (MAX_L).
3. Calculate the sum of utilities for each path.
4. We apply the mechanism VCG to maximize first of all the utilities of tour and than apply costs to all players.
5. We repeat the process if costs exceed the budget
6. If the costs collected by the mechanism exceed the cost of tour, we divide it among the participants according to the kilometers traveled.

4.3 Example Application



- P = S = {Paola, Francesca}
- k = 2
- B_{Paola} = 500
- $B_{Francesca}$ = 500
- L = {Rome,Genoa,Milan,Turin}
- D_L = {
  Rome-Turin: 670km,
  Rome-Milan: 346km,
  Rome-Genoa: 500km,
  Milan-Turin: 143km,
  Milan-Genoa: 142km,
  Turin-Genoa: 172km,
  }
- L_S = Rome
- P_U = {
  Paola = [M=0,G=1,T=2],
  Francesca = [M=0,G=2,T=1],
  }
- MAX_L = 1600 km

1. Generation of all possible combination without repetition (who started from Rome).
   T_P = [RGMT, RG, RM, RT, RMG, RGT, RMT]

2. Result of Salesman problem:
   T_P = [RGMT, RG, RM, RT, RGM, RGT, RMT]

3. Calculate path's distance and eliminate that do not respect the constraint (we have to multiply the distance by two because we consider round trip).

   - **RT** = 670 km*2 = 1340 km ,
   - **RM** = 346 km*2 = 692 km,
   - **RG** = 500 km*2 = 1000 km,
   - **RGT** = [(RG) 500 km + (GT) 172 km]*2 = 672 km * 2 = 1344 km
   - **RGM** = [(RG) 500 km + (GM) 142 km]*2 = 642 km * 2 = 1284 km
   - **RMT** = [(RG) 500 km + (MT) 143 km]*2 = 643 km * 2 = 1286 km
   - **RGMT** = [(RG) 500 km + (GM) 142 km + (MT) 143 km]*2 = 785km * 2 = 1570km

4. Calculate utilities in descending order

   - **RGT** = 6
   - **RT** = 3
   - **RG** = 3
   - **RGM** = 3
   - **RMT** = 3
   - **RM** = 0

5. We select the first with maximum utilities: **RGT**.

6. $\chi(\hat{v}) = \arg\max_x \sum_i^n v_i(x)$;
   Travel considered = **[R G T]**,
   Paola competes for **GT** and Francesca for **RG**.
   Each agent represents the arc of the major preference in the trip.
   G T cities must be crossed, and I am obliged in doing so.
   If an agent (arc) is missed, an alternative route (if possible) is sought that crosses the two cities and minimizes costs [Costs are the weight on the arches].

7. $\wp_i(\hat{v}) = \sum_{i \neq j}^n \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{i \neq j}^n \hat{v}_j\ (\chi(\hat{v}))$;
   Brief Formula explication = Cost of the alternative tour without player's participation (and consequently without his arch) - Cost of the tour with player's participation but without your financial contribution.
   C_P = (RGMT) - (RGT - GT) =
   = 500 € - (350 € - 200 €) = 500 € - 150€ = 350€ < B_{Paola}.
   C_F = RMGT - (RGT - RG) =
   600 €- (350€- 150€) = 600€ - 200€= 400 € < $B_{Francesca}$.
   $C_T = C_P + C_F$ = 350€ + 400 € = 750€.

8. The budgets were respected however, there is a surplus that we are going to divide according to the kilometers of the destinations that were competed for.
   Surplus = C_T 750€ - (RGT) 350€ = 400€
   C per Km = Surplus / D_{L_i} = 400 € / 672km (RG + GT) = 0.59 € /km

   S_F = C per Km * RG km = 0.59€ * 500km = 296€
   S_P = C per Km * GT = 0.59€ * 172km = 104 €
   C_P - Surplus = 350 €- 296 €= 54€
   C_F - Surplus = 400 €- 104 €= 296€