1. Resuelva el problema de valores en la frontera $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, sujeta a las condiciones de frontera $u(0,t) = (10,t) = 0$; $u(x,0) = f(x)$, donde

$$f(x) = \begin{cases} x-1, & 1 \le x \le 2 \\ 11-5x, & 2 \le x \le 3 \\ 5x-19, & 3 \le x \le 4 \\ 5-x, & 4 \le x \le 5 \\ 0, & otro\ caso \end{cases}$$

¿Qué sucede con la solución del problema cuando $t$ crece?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,t) = XT \Rightarrow XT' = X''T \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

(I) $X'' + \lambda X = 0$

$\lambda = w^2$

$X'' + w^2 X = 0$

$m^2 + w^2 = 0 \qquad w = \sqrt{\lambda}$

$m = \pm wi \qquad \sqrt{\lambda} = n^2$

$X(x) = c_1 \cos nx + c_2 \operatorname{sen} nx$

$X(0) = C_1 = 0; \quad X(10) = C_2 \operatorname{sen} 10\sqrt{\lambda}$

$X_n(x) = \operatorname{sen} xn \quad , n = 1,2,3,\dots$

(II) $T' + \lambda T = 0$

$\Rightarrow \partial T / \partial t = -n^2 T$

$\Rightarrow \int \frac{\partial T}{T} = -\int n^2 \partial t \Rightarrow \ln T = -n^2 t + c$

$\Rightarrow T = e^{-n^2 t + c}$

$T_n(t) = c e^{-n^2 t} \quad , n = 1,2,3,\dots$

$u_n(x,t) = X_n(x) T_n(t) = \operatorname{sen} nx\, e^{-n^2 t} \quad , n = 1,2,3\dots$

Principio de superposición

$$u(x,t) = \sum_{n=1}^{\infty} B_n \operatorname{sen} 10x\, e^{-n^2 t} = f(x)$$

$$B_n = \frac{2}{\pi}\int_0^{10} f(x) \operatorname{sen} nx\, dx$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{10} f(x) \operatorname{sen} nx\, dx\right] \operatorname{sen} nx\, e^{-n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_1^{2} (x-1) \operatorname{sen} nx\, dx\right] \operatorname{sen} nx\, e^{-n^2 t} \Rightarrow \sum_{n=1}^{\infty}\left(\frac{2(-n\cos(2n) + \sin(2n) - \sin(n))}{\pi n^2}\right) \operatorname{sen} xn\, e^{-n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_2^{3} (11-5x) \operatorname{sen} nx\, dx\right] \operatorname{sen} nx\, e^{-n^2 t} \Rightarrow \sum_{n=1}^{\infty}\left(\frac{2(n\cos(2n) - 5(\sin(3n)) - \sin(2n) + 4n\cos(3n))}{\pi n^2}\right) \operatorname{sen} xn\, e^{-n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_3^{4} (5x-19) \operatorname{sen} nx\, dx\right] \operatorname{sen} nx\, e^{-n^2 t} \Rightarrow \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\left(\frac{-\cos(4n) - 4\cos(3n)}{n} + \frac{5\sin(4n) - 5\sin(3n)}{n^2}\right)\right) \operatorname{sen} xn\, e^{-n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_4^{5} (5-x) \operatorname{sen} nx\, dx\right] \operatorname{sen} nx\, e^{-n^2 t} \Rightarrow \sum_{n=1}^{\infty}\left(\frac{2(n\cos(4n) - \sin(5n) + \sin(4n))}{\pi n^2}\right) \operatorname{sen} xn\, e^{-n^2 t}$$

2. Resuelva los problemas de valores en la frontera a continuación.

2.1. $u_{tt} = u_{xx}, u(0,t) = u_x(1,t) = 0, u(x,0) = 1, u_t(x,0) = 0$

2.2. $u_{tt} = 2u_{xx}, u_x(0,t) = u_x(1,t) = 0, u(x,0) = x(1-x), u_t(x,0) = 0$

2.1 $\quad u_{tt} = u_{xx} \Rightarrow \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} \Rightarrow u(x,t) = XT \Rightarrow X''T = XT'' \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{T} = -\lambda$

$X'' + \lambda X = 0 \Rightarrow w^2 = \lambda$

$m^2 + w^2 = 0 \Rightarrow m = \pm wi$

$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \operatorname{sen}\sqrt{\lambda}x$

$X'(x) = -c_1\sqrt{\lambda}\operatorname{sen}\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\sqrt{\lambda} = n^2 \Rightarrow n = \lambda$

$X(0) = c_1 = 0$

$X'(1) = c_2 n \cos n = 0$

$X_n(x) = c_2 n \cos nx, \quad n = 1,2,3\ldots$

$T'' + \lambda T = 0 \Rightarrow w^2 = \lambda$

$m^2 + w^2 = 0 \Rightarrow m = \pm wi$

$T(t) = \partial_1 \cos\sqrt{\lambda}t + \partial_2 \operatorname{sen}\sqrt{\lambda}t$

$T(0) = \partial_1 = 1$

$T'(t) = -\partial_1\sqrt{\lambda}\operatorname{sen}\sqrt{\lambda}t + \partial_2\sqrt{\lambda}\cos\sqrt{\lambda}t$

$T'(0) = \partial_2 n = 0$

$T_n(t) = \partial_1 \cos nt, \quad n = 1,2,3,\ldots$

$u_n(x,t) = X_n(x)\cdot T_n(x) \Rightarrow (c_2 n \cos nx)(\partial_1 \cos nt), \quad n = 1,2,3,\ldots$

$u_n(x,t) = g_n \cos nx \cos nt, \quad n = 1,2,3,\ldots$

$u(x,t) = \sum_{n=1}^{\infty} B_n \cos nx \cos nt$

$B_n = \dfrac{2}{\pi}\int_0^1 \operatorname{sen} nx \, dx$

$u(x,t) = \sum_{n=1}^{\infty} \left[\dfrac{2}{\pi}\int_0^1 \operatorname{sen} nx \, dx\right] \cos nx \cos nt$ //

2.2 $\quad \dfrac{\partial^2 u}{\partial t^2} = 2\dfrac{\partial^2 u}{\partial x^2} \Rightarrow u(x,t) = X(x)T(t) \Rightarrow XT'' = 2X''T \Rightarrow T''/2T = X''/X = -\lambda$

$X'' + \lambda X = 0 \Rightarrow \lambda = w^2$

$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \operatorname{sen}\sqrt{\lambda}x$

$X'(x) = -c_1\sqrt{\lambda}\operatorname{sen}\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$X'(0) = c_2\sqrt{\lambda} = 0; \quad X'(1) = -c_1\sqrt{\lambda}\operatorname{sen}\sqrt{\lambda} = 0$

$\sqrt{\lambda} = n^2 \Rightarrow n = \lambda$

$X_n(x) = n \operatorname{sen} nx, \quad n = 1,2,3,\ldots$

$T'' + 2\lambda T = 0$

$T(t) = \partial_1 \cos\sqrt{\lambda}t + \partial_2 \operatorname{sen}\sqrt{\lambda}t$

$T(0) = \partial_1 = x(1-x)$

$T'(t) = -\partial_1\sqrt{\lambda}\operatorname{sen}\sqrt{\lambda}t + \partial_2\sqrt{\lambda}\cos\sqrt{\lambda}t$

$T'(0) = \partial_2\sqrt{\lambda} = 0$

$T_n(t) = -\partial_1 n \operatorname{sen} nt, \quad n = 1,2,3,\ldots$

$u_n(x,t) = W_n \operatorname{sen} nx \operatorname{sen} nt$

$u(x,t) = \sum_{n=1}^{\infty} B_n \operatorname{sen} nx \operatorname{sen} nt$

$B_n = \dfrac{2}{\pi}\int_0^1 x(1-x)\operatorname{sen} nx \, dx$

$u(x,t) = \sum_{n=1}^{\infty} \left[\dfrac{2}{\pi}\int_0^1 x(1-x)\operatorname{sen} nx \, dx\right] \operatorname{sen} xn \operatorname{sen} nt$ //

3. Sean $a, c > 0$ constantes positivas. Resuelva la ecuación $u_{tt} + au_t = c^2 u_{xx}$ , la cual representa una versión amortiguada de la ecuación de onda, sujeta a las condiciones $u(0,t) = u(1,t) = 0, u(x,0) = f(x),\ u_t(x,0) = 0.$

Condiciones de frontera $0 \leq x \leq 1$

$$u_{tt} + 2u_t = c^2 u_{xx} \Rightarrow \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow u(x,t) = XT \Rightarrow XT'' + 2XT = c^2 X''T$$

$$\Rightarrow X(T'' + 2T) = c^2 X''T \Rightarrow \frac{T'' + 2T}{T} = \frac{c^2 X''}{X} = -\lambda$$

$$T'' + 2T + \lambda T = 0$$

$$T'' + T(2+\lambda) = 0$$

$$(2+\lambda) = w^2$$

$$m^2 + w^2 = 0 \Rightarrow m = \pm wi$$

$$T(t) = c_1 \cos\sqrt{2+\lambda}\, t + c_2 \operatorname{sen}\sqrt{2+\lambda}\, t$$

$$\sqrt{2+\lambda} = n^2 \Rightarrow n = 2+\lambda$$

$$T'(t) = -c_1 n \operatorname{sen} nt + c_2 n \cos nt$$

$$T'(0) = c_2 n = 0$$

$$T_n(t) = c_1 n \operatorname{sen} nt \quad , \quad n = 1, 2, 3, \ldots$$

$$c^2 X'' + \lambda X = 0$$

$$c^2 m^2 + w^2 = 0 \Rightarrow m^2 = -w^2/c^2 \Rightarrow m = \pm\sqrt{-w^2}/c$$

$$X(x) = d_1 e^{nx} + d_2 e^{-nx}$$

$$X(0) = d_1 + d_2 = 0$$

$$X(1) = d_1 e^{n} + d_2 e^{-n} \qquad \sqrt{(n^2)} = \sqrt{\left(\frac{\sqrt{-w^2}}{c}\right)} \Rightarrow n = \frac{-w^2}{\sqrt{c}}$$

$$X_n(x) = e^{nx} + e^{-nx} \quad , \quad n = 1, 2, 3, \ldots$$

$$u_n(x,t) = \sum_{n=1}^{\infty} B_n (e^{nx} + e^{-nx}) \operatorname{sen} nt$$

$$B_n = \frac{2}{\pi} \int_0^1 x(1-x) \operatorname{sen} nx \, dx$$

$$u_n(x,t) = \sum_{n=1}^{\infty} \left[ \frac{2}{\pi} \int_0^1 x(1-x) \operatorname{sen} nx \, dx \right] (e^{nx} + e^{-nx}) \operatorname{sen} nt \quad //$$

4. Resuelva los problemas siguientes para la ecuación de Laplace sobre el cuadrado $\Omega = \{0 \leq x \leq \pi, 0 \leq y \leq \pi\}$.

4.1. $u(x,\pi) = u(0,y) = 0, u(x,0) = sen^3 x, u(\pi,y) = 0$

4.2. $u(x,\pi) = 0,\ u(0,y) = seny, u(x,0) = 0, u(\pi,y) = 0$

4.1

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u(x,y) = X(x)Y(y)$$

$$\Rightarrow X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$X(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \operatorname{sen}\sqrt{\lambda}\,x$

$X(0) = C_1 = 0$

$X(\pi) = C_2 \operatorname{sen}\sqrt{\lambda}\,\pi = 0$

$\Rightarrow \operatorname{sen}\sqrt{\lambda}\,\pi = 0 \Rightarrow \sqrt{\lambda}\,\pi = n\pi \Rightarrow \sqrt{\lambda} = n$

$\Rightarrow X_n(x) = \operatorname{sen} nx, \quad n = 1,2,3,\ldots$

$Y - \lambda Y = 0 \Rightarrow Y'' - n^2 Y = 0$

$m^2 - n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \pm n$

$\Rightarrow Y_n(y) = d_1 \cosh ny + d_2 \operatorname{senh} ny$

$\Rightarrow Y(\pi) = d_1 = 0$

$\Rightarrow Y_n(y) = \operatorname{senh} n\pi y, \quad n = 1,2,3,\ldots$

$u_n(x,y) = \operatorname{sen} nx \operatorname{senh} n\pi y$

$u_n(x,y) = \sum_{n=1}^{\infty} B_n \operatorname{sen} nx \operatorname{senh} n\pi y$

$$B_n = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen}^3 x \operatorname{sen} nx\, dx$$

$$u_n(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{\pi}\int_0^{\pi} \operatorname{sen}^3 x \operatorname{sen} nx\, dx\right] \operatorname{sen} nx \operatorname{senh} n\pi y$$

4.2

Se realiza el mismo procedimiento

$u_n(x,y) = \operatorname{sen} nx \operatorname{senh} n\pi y$

$u_n(x,y) = \sum_{n=1}^{\infty} B_n \operatorname{sen} nx \operatorname{senh} n\pi y$

$$B_n = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} y \operatorname{sen} nx\, dx$$

$$u_n(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{\pi}\int_0^{\pi} \operatorname{sen} y \operatorname{sen} nx\, dx\right] \operatorname{sen} nx \operatorname{senh} n\pi y$$