# Optimality conditions for non-convex, non-coercive autonomous variational problems with constraints

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$f : [\alpha, \beta] \times \mathbb{R}_0^+ \to \mathbb{R}$ is a lower semicontinuous, non-negative integrand (not necessarily smooth, convex or coercive).

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- necessary and sufficient conditions for the existence of the minimum in terms of the prescribed slope $\xi_0 := \dfrac{\beta - \alpha}{b - a}$

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- Clarke (TAMS 1993)

- Mariconda (Nonlin. Anal. 1994)

- Reymond (JOTA 1994)

- Crasta (Nonlin. Anal. 1996)

- Cellina - Treu - Zagatti (JDE 1996)

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$$\lim_{|z|\to+\infty} \sup_{|s|\leq R} \sup(f^{**}(s,z) - z\partial f^{**}(s,z)) = -\infty \quad \forall R \geq 0$$

(weaker than superlinearity, example: $f(z) = |z| - \sqrt{|z|}$)

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- $(\mathrm{DBR})_1$ (when $|B_{v_0}| = 0$): there exists $c \leq \mu$ such that

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- $(\mathrm{DBR})_2$ (when $|B_{v_0}| > 0$): $f(v_0(t), 0) = \mu$ for a.e. $t \in B_{v_0}$ and

$$f(v_0(t), v_0'(t)) - \mu \in v_0'(t) \; \partial f(v_0(t), v_0'(t)) \quad \text{a.e. in } A_{v_0}$$

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$$\tilde{F}(u) := \int_{\alpha}^{\beta} f\left(\tau, \frac{1}{u'(\tau)}\right) u'(\tau)\, d\tau \;+\; \mu(b - u(\beta)), \quad u \in \tilde{\Omega}$$

$$\tilde{\Omega} := \{u \in W^{1,1}(\alpha,\beta) : \; u(\alpha) = a, \; u(\beta) \leq b, \; u'(\tau) > 0 \text{ a.e. in } (\alpha,\beta)\}.$$

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Define two maps $\chi : \Omega \to \tilde{\Omega}$ and $\Psi : \tilde{\Omega} \to \Omega$ by

$$\chi_v(\tau) := a + \int_{\alpha}^{\tau} w_v'(s)\, ds \quad \text{where} \quad w_v(\tau) := \min\{t \in [a,b] : v(t) = \tau\}$$

$$\Psi_u(t) := \begin{cases} u^{-1}(t) & \text{if } \; a \leq t \leq u(s^*) \\ s^* & \text{if } \; u(s^*) \leq t \leq u(s^*) + b - u(\beta) \\ u^{-1}(t - b + u(\beta)) & \text{if } \; u(s^*) + b - u(\beta) \leq t \leq b \end{cases}$$

where $s^* := \min\{\tau \in [\alpha,\beta] : f(\tau,0) = \min_{s \in [\alpha,\beta]} f(s,0)\}.$

**Theorem 2.** (Equivalence)

If $v \in \Omega$ is a minimizer for problem $(P)$ then $\chi_v \in \tilde{\Omega}$ is a minimizer for problem $(\tilde{P})$. Vice versa, if $u \in \tilde{\Omega}$ is a minimizer for problem $(\tilde{P})$, then $\Psi_u \in \Omega$ is a minimizer for problem $(P)$.

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$$(\mathrm{EL})_1 \quad (\text{when } \; u_0(\beta) = b) \;\; \exists\, c \leq \mu \;:\; c \in \partial \tilde{f}(\tau, u_0'(\tau)) \quad \text{a.e. in } (\alpha, \beta)$$

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$$(\mathrm{EL})_2 \quad (\text{when } u_0(\beta) < b) \quad \mu \in \partial \tilde{f}(\tau, u_0'(\tau)) \quad \text{a.e. in } (\alpha, \beta)$$

where $\partial \tilde{f}(\tau, u_0'(\tau))$ denotes the subdifferential of $\tilde{f}(\tau, \cdot)$ at $u_0'(\tau)$.

**Theorem 4.** Let $f(s,\cdot)$ be continuous at 0 for every $s \in [\alpha,\beta]$.

Then, problem $(P)$ admits minimum if and only if problem $(P^{**})$ admits a minimizer $v_0 \in \Omega$ such that

$$v_0'(t) \in co(C_{v_0(t)}) \qquad \text{for a.e. } t \in A_{v_0}$$

where:

- $C_s := \{z > 0 : f(s,z) = f^{**}(s,z)\}$ denotes the contact set
- $co(C_{v_0(t)})$ denotes the convex envelope of the set $C_{v_0(t)}$ .

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**Corollary 5.** Under the same assumptions above, if $co(C_s) = (0,+\infty)$ for every $x \in [\alpha,\beta]$, then problem $(P)$ admits minimum if and only if problem $(P^{**})$ admits minimum.

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**Notations:** for every $(s,z) \in [\alpha,\beta] \times (0,+\infty)$ put

$$g^-(s,z) := f^{**}(s,z) - z f^{**}_-(s,z) \qquad g^+(s,z) := f^{**}(s,z) - z f^{**}_+(s,z)$$

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$$\lambda(s) := \inf_{z>0} g^+(s,z) \geq -\infty \qquad \Lambda(s) := \sup_{z>0} g^-(s,z) \leq +\infty$$

and for $\lambda(s) < y < \Lambda(s)$ put $\gamma(s,y) := \max\{z > 0 : \; g^-(s,z) \geq y\}$

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$$c_0 := \operatorname*{ess\,sup}_{s \in [\alpha,\beta]} \lambda(s)$$

**Theorem 6.** Let $f^{**}$ be continuous on $[\alpha, \beta] \times [0, +\infty)$ and assume that if $c_0 < \mu$ we have $1/\gamma(s, c) \in L^1(\alpha, \beta)$ for every $c \in (c_0, \mu)$.

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- *(necessity)* If problem $(P^{**})$ admits minimum then

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- *(sufficiency)* Vice versa, if (1) holds for some constant $\hat{c} \in (c_0, \mu]$ then problem $(P^{**})$ admits minimum.

- *(Lip-regularity)* Moreover, if (1) holds for some constant $\hat{c} > \max_{s \in [\alpha, \beta]} \lambda(s)$, then the minimizers are Lipschitz continuous.

- *(<u>strict monotonicity</u>)* Finally, if $v_0$ is a minimizer, we have

$$v_0'(t) > 0 \text{ a.e. in } (\alpha, \beta) \quad \Leftrightarrow \quad \int_\alpha^\beta \frac{1}{\gamma(s,\mu)}\, ds \geq b - a.$$

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Condition (1) is an upper limitation for the assigned mean slope $\xi_0$:

$$\xi_0 \leq 1 \Big/ \fint_\alpha^\beta \frac{1}{\gamma(s,\hat{c})}\, ds$$

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**Corollary 8.** If $c_0 < \mu$ and $\displaystyle\lim_{c \to c_0^+} \int_\alpha^\beta \frac{1}{\gamma(s, c)}\, ds < b - a$, then problem $(P^{**})$ admits minimum.

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Instead, if $c_0 = -\infty$ and $\displaystyle\lim_{c \to -\infty} \int_\alpha^\beta \frac{1}{\gamma(s, c)}\, ds > b - a$ then $(P^{**})$ does not admit minimum.

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More in particular:

- If $c_0 > \lambda(s)$ a.e. in $[\alpha, \beta]$ then $\displaystyle\int_\alpha^\beta \frac{1}{\gamma(s,c_0)}\, ds > 0$ and the minimum exists if and only if $\xi_0 \le 1 / \displaystyle\int_\alpha^\beta \frac{1}{\gamma(s,c_0)}\, ds$.

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- If $c_0 = \lambda(s)$ a.e. in $[\alpha, \beta]$, then the minimum exists for every $\xi_0 > 0$.

## Application to the case $f(s,z) = a(s) + h(z)$

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$$\ell := \inf_{z>0} (h(z) - zh'(z))$$

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$$\gamma(y) = \max\{z > 0 : \; h(z) - zh'(z) \geq y\}.$$

- if $\ell = -\infty$, then the minimum exists for every slope $\xi_0 > 0$ with Lipschitz continuous minimizers;

- if $\ell > -\infty$ and $a(x) < M$ a.e. in $(\alpha, \beta)$, then the minimum exists if and only if $M + \ell \leq m + h(0)$ and

$$\int_\alpha^\beta \frac{1}{\gamma(M + \ell - a(s))}\, ds \leq b - a.$$

- if $\ell > -\infty$ and $a(x) < M$ a.e. in $(\alpha, \beta)$, then the minimum exists if and only if $M + \ell \leq m + h(0)$ and

$$\int_{\alpha}^{\beta} \frac{1}{\gamma(M + \ell - a(s))}\, ds \leq b - a.$$

Moreover, if $M + \ell < m + h(0)$ and

$$\lim_{c \to (M+\ell)^+} \int_{\alpha}^{\beta} \frac{1}{\gamma(c - a(s))}\, ds < b - a$$

then the minimizers are Lipschitz continuous.

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- If $\ell = 0$, $m > 0$ and $h$ is not affine in any half-line, then the minimum exists for every slope $\xi_0 > 0$ and the minimizers are Lipschitz continuous. Instead, if $\ell = 0$, $m = 0$ and $h$ is strictly convex, then the minimum does not exists for any slope $\xi_0 > 0$.

## Application to integrands with the structure $f(s,z) = a(s)h(z)$

- If $\ell = -\infty$ the minimum exists for every slope $\xi_0 > 0$; moreover, if $m > 0$ the minimizers are Lipschitz continuous.

- If $\ell = 0$, $m > 0$ and $h$ is not affine in any half-line, then the minimum exists for every slope $\xi_0 > 0$ and the minimizers are Lipschitz continuous. Instead, if $\ell = 0$, $m = 0$ and $h$ is strictly convex, then the minimum does not exists for any slope $\xi_0 > 0$.

- If $\ell > 0$ and $a(s) < M$ a.e. in $(\alpha, \beta)$, then the minimum exists if and only if $M\ell \leq mh(0)$ and

$$\int_\alpha^\beta \frac{1}{\gamma(M\ell/a(s))}\, ds \leq b - a.$$

Moreover, if $M\ell < mh(0)$ and

$$\lim_{c \to (M\ell)^+} \int_\alpha^\beta \frac{1}{\gamma(c/a(s))}\, ds < b - a$$

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- If $-\infty < \ell < 0$ and $a(s) > m$ a.e. in $(\alpha, \beta)$, then the minimum exists if and only if

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- case $k > 0$: the minimum exists if and only if $Mk \leq m(k+1)$ and

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For instance, take $[\alpha, \beta] = [0, 1]$ and $a(s) = s + 1$. Then, $k \leq 1$ is a necessary condition for the existence of the minimum. Moreover, for $k = 1$ the minimum exists if and only if $b - a \geq 2/3$.

**Example 2.** (Newton's problem)

$$F(v) := \int_0^1 v(t) \frac{(v'(t))^3}{1 + (v'(t))^2} \, dt$$

in $\Omega := \{v \in W^{1,1}(0,1) : v(0) = 0, v(1) = \beta, v'(t) \geq 0 \text{ a.e.}\}$.

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Put $h(z) = \frac{z^3}{1+z^2}$, we have $h^{**}(z) = h(z)$ if $0 \leq z \leq 1$, $h^{**}(z) = z - \frac{1}{2}$ if $z > 1$. Hence, $\ell = -\frac{1}{2}$, $m = 0$, so $c_0 = \mu = 0$ and $\gamma(0) = 0$.

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Then, the problem does not admit minimum for any $\beta > 0$.

Indeed, the functional

$$\int_0^1 v(t)\frac{(v'(t))^3}{1+(v'(t))^2}\,dt + \frac{1}{2}(v(0))^2$$

in the class $\bar{\Omega} := \{v \in W^{1,1}(0,1) : v(0) \geq 0, v(1) = \beta, v'(t) \geq 0 \text{ a.e.}\}$ admits minimum attained at a function satisfying $v(0) > 0$.

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**Example 4.** Take $f(s,z) = (s^p + k)z^q$, $(s,z) \in [0,\beta] \times [0,+\infty)$, with $k \ge 0$, $p > 0$ and $q > 1$.

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Indeed, when $k = 0$, the minimizer is $v_0(t) = \beta t^{q/(p+q)}$ which is not Lipschitz continuous.

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Nevertheless, when $m = 0$ and $a(s) > 0$ a.e. in $(\alpha, \beta)$ we have $c_0 = -\infty$ and the existence results can be applied.