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METAMATHEMATICAL APPROACH TO PROVING THEOREMS OF DISCRETE MATHEMATICS*

Yu. V. Matiyasevich

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A new approach is described to the problems of discrete mathematics, based on the application of the theory of logical deductions. To be precise, in the study of some property $\mathfrak{R}$, with each object X possessing this property there is connected a new object, viz., a formal proof $\mathcal{P}_{\mathfrak{R}}(X)$ that X does indeed possess property $\mathfrak{R}$, carried out within the framework of some deductive system $\mathcal{R}(X)$. In what follows information on the properties of X is extracted from an analysis of the structure of $\mathcal{P}_{\mathfrak{R}}(X)$. The paper presents various deductive systems arising in the formalization of certain properties of graphs, describes possible schemes of using $\mathcal{P}_{\mathfrak{R}}(X)$, and cites examples of (previously well-known) theorems admitting of proof by such schemes.

1. In this paper we describe a certain new approach to the problems of discrete mathematics. The origin of this approach, named metamathematical, was presented by the author in [1, 2].

1.1. The main feature of the metamathematical approach is the following. Suppose that we are studying a certain property $\mathfrak{R}$ of some type of finite discrete objects. The first

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stage of the metamathematical approach consists in the formalization of property $\mathcal{R}$: we must try to find some deductive system $\mathcal{R}$ and to connect with each object X possessing property $\mathcal{R}$ a new finite discrete object, viz., a formal proof $\mathcal{P}_{\mathcal{R}}(X)$ that X possesses property $\mathcal{R}$, carried out within the framework of system $\mathcal{R}$. Subsequent study of property $\mathcal{R}$ includes the consideration, together with X, also of $\mathcal{P}_{\mathcal{R}}(X)$ and the analysis of the structures of both the objects. It is precisely the putting into consideration of the formal proofs that is the main feature of the metamathematical approach.

The new approach was advanced in the hope that the consideration of formal proofs would extend our capabilities of proving the theorems of discrete mathematics. We assume here that every proof within the scope of the metamathematical approach must admit of a restatement in traditional terms, and, therefore, the adoption of this approach does not widen the set of theorems. Thus, the metamathematical approach is not some new principle of mathematical reasoning which can be adopted by some mathematicians but proves unusable by others. In this respect it differs at the root from, say, the method of mathematical induction in arithmetic or the axiom of choice in set theory. The metamathematical approach yields only a scheme for a possible proof and points us to seeking the proof in some form or other. In this respect the metamathematical approach is similar, for instance, to the method of generating functions.

2. Let us describe what we mean by a deductive system and present examples.

2.1. An arbitrary deductive system is a quadruple of the form $\langle \mathcal{F}, \mathcal{A}, \mathcal{D}, \mathcal{H} \rangle$; here $\mathcal{F}$ is some set whose elements are called formulas; $\mathcal{A}$ is some subset of set $\mathcal{F}$, the formulas from which are called postulates; $\mathcal{D}$ is some set of predicates (called deduction rules), the domain of each of which is the set of all n formulas for some n not less than 2; $\mathcal{H}$ is a subset of set $\mathcal{F}$, the formulas from which are called final formulas. By a deduction we mean every list of formulas $F_1, \ldots, F_k$ in which for each formula F_i that is not a postulate we can find formulas $F_{j_1}, \ldots, F_{j_m}$ $(j_1, \ldots, j_m < i)$ such that the formula $\langle F_{j_1}, \ldots, F_{j_m}, F_i \rangle$ satisfies one of the deduction rules. A formula is said to be deducible if a deduction exists in which it occurs.

It is easy to see that we could simplify the terminology by eliminating the concept of a postulate and as rules we could admit one-place predicates; however, such a unification is inappropriate from the heuristic viewpoint.

2.2. We say that a given property $\mathcal{R}$ has been formalized if with every object X (from the domain in $\mathcal{R}$) we can associate some deductive system $\mathcal{R}(X)$ such that X possesses property $\mathcal{R}$ if and only if at least one of the final formulas is deducible in $\mathcal{R}(X)$.

A more traditional approach to formalization would consist in the consideration of only one fixed (i.e., not dependent on the object) deductive system (from which the concept of a final formula would have been excluded). In the case of the traditional formalization object X possesses property $\mathcal{R}$ if and only if some formula $H(X)$ (playing the role of a final formula in our formalization) is deducible in the deductive system being examined. However, experience in the development of methods for the machine proof of theorems (see [3, 4], for instance) and the small experience in the application of the metamathematical approach (see [1, 2]) show that it makes sense to construct for each object its own specific deductive system closely connected with this object. Here it usually turns out well to take formulas of very simple structure as the final formulas, which is impossible under the more traditional approach to formalization.

3. We cite examples of deductive systems arising in the formalization of certain properties of graphs. In the present paper by a graph we mean a finite unoriented graph without loops and multiple edges. Such a graph can be given by means of the set V of its vertices and of a symmetric two-place predicate $\mathfrak{J}$ which gives the relation "to be vertices connected by edge"; we shall denote this graph by the pair $\langle V, \mathfrak{J} \rangle$.

The graph $\langle V, \mathfrak{J} \rangle$ mapping the set V into the set $\{1, \ldots, n\}$, interpreted as the "set of colors," will be called an n-coloring. A coloring is said to be regular if the endpoints of each edge turn out to be colored in different colors. A graph is said to be n-colorable (n-noncolorable) if it has (does not have) a regular n-coloring. We begin by indicating several formalizations of the property "to be an n-noncolorable graph."

3.1. Each coloring is uniquely (to within a renumbering of the colors) restorable from the symmetric two-place relation "vertices x and y are colored in one color," which we denote $\mathfrak{E}xy$. We shall take it that the predicate $\mathfrak{E}$ is defined on unordered pairs of unequal vertices and we do not distinguish between the designations $\mathfrak{E}xy$ and $\mathfrak{E}yx$.

It is obvious that if predicate $\mathfrak{E}$ corresponds to some coloring, then for any three vertices x, y, and z from $\mathfrak{E}xy$ and $\mathfrak{E}yz$ follows $\mathfrak{E}xz$, or, what is the same, the formula

$$\neg \mathfrak{E}xy \vee \neg \mathfrak{E}yz \vee \mathfrak{E}xz \tag{1}$$

is valid. Conversely, if an arbitrarily given predicate satisfies all formulas of form (1), then it corresponds to some coloring.

If a coloring is regular, then predicate $\mathfrak{E}$ must be false on any pair of two vertices x and y joined by an edge. We shall write this symbolically as

$$(\mathfrak{J}xy \Rightarrow) \qquad\qquad \neg \mathfrak{E}xy. \tag{2}$$

(In speaking subsequently about "all formulas of form (2)" we shall have in mind all possible formulas of the form $\neg\mathfrak{E}xy$, where x and y are arbitrary vertices satisfying the condition written on the left. An analogous notation will be applied farther on.) On the other hand, if a predicate $\mathfrak{E}$ satisfies all formulas of form (1) and (2), then the coloring to which it corresponds is regular.

Finally, if we have to deal with an n-coloring, then in any set of $(n+1)$ vertices $x_0, \ldots, x_n$ we necessarily find two that are colored alike, and, consequently, all formulas of the form

$$\bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^{n} \mathfrak{E} x_i x_j \tag{3}$$

must be true. On the other hand, if a predicate $\mathfrak{E}$ satisfies all formulas of form (1) and (3), then it corresponds to some n-coloring.

Thus:

graph $\langle V, \mathfrak{I} \rangle$ is n-noncolorable if and only if the set of all formulas of form (1), (2), and (3) is inconsistent.

Many methods are known for testing a set of formulas for consistency. Here we shall use the so-called method of hyperresolutions introduced in [5]. Here we require, essentially, only a propositional variant of it and no acquaintance with [5] is necessary for the subsequent reading.

The method of hyperresolutions has two mutually dual forms, i.e., we can construct deductions both from purely positive disjunctions as well as from purely negative ones. The first form yields the following deductive system. The formulas are all possible disjunctions of elementary formulas of the form $\mathfrak{E}xy$; here and subsequently we do not distinguish between disjunctions differing only in the order or the repetition of terms. Postulates are formulas of form (3). Deduction rules correspond to formulas (1) and (2) and, written in traditional form, appear thus:

$$\frac{\mathfrak{A} \vee \mathfrak{E}xy \qquad \mathfrak{B} \vee \mathfrak{E}yz}{\mathfrak{A} \vee \mathfrak{B} \vee \mathfrak{E}xz}, \tag{4'}$$

$$(\mathfrak{I}xy \Rightarrow) \qquad \frac{\mathfrak{A} \vee \mathfrak{E}xy}{\mathfrak{A}}. \tag{4''}$$

The only final formula is the identically false empty disjunction $\square$. The theorem on the completeness of the method of hyperresolutions (see [5]) guarantees that the empty disjunction $\square$ is deducible if (and, obviously, only if) the set of all formulas of form (1), (2), and (3) is inconsistent. Thus we indeed are dealing with a formalization of the property "to be an n-noncolorable graph."

The second form of the method of hyperresolutions yields the following deductive system. The formulas are all possible disjunctions of elementary formulas of the form $\neg\mathfrak{E}xy$, the postulates are formulas of form (2), and the deduction rules are thus:

$$\frac{\mathfrak{A} \vee \neg\mathfrak{E}xz}{\mathfrak{A} \vee \neg\mathfrak{E}xy \vee \neg\mathfrak{E}yz}, \tag{5'}$$

$$\frac{\mathfrak{A}_1 \vee \neg \mathcal{E} x_0 x_1 \ldots \mathfrak{A}_{n(n+1)/2} \vee \neg \mathcal{E} x_{n-1} x_n}{\mathfrak{A}_1 \vee \ldots \vee \mathfrak{A}_{n(n+1)/2}}. \tag{5''}$$

The only final formula also is $\square$.

3.2. Let us now indicate two formalizations of the same property "to be an n-noncolorable graph" in terms of the relation "vertex x is colored in a color with a number smaller than that for vertex y." We shall write this predicate as $x<y$ and instead of $\neg(x<y)$ we shall write $x \geqslant y$; it is assumed that this predicate is defined for all ordered pairs of vertices not necessarily distinct.

It is easy to verify that the role of formulas (1), (2), and (3) will now be played by the formulas

$$x \geqslant x, \tag{6}$$

$$x<y \vee y<z \vee x \geqslant z, \tag{7}$$

$$(\mathcal{I} xy \Rightarrow) \qquad x<y \vee y<x, \tag{8}$$

$$x_0 \geqslant x_1 \vee x_1 \geqslant x_2 \vee \ldots \vee x_{n-1} \geqslant x_n. \tag{9}$$

Formulas of form (6) are needed only so that the set of all formulas of form (1) has its own complete analog, viz., the set of all formulas of form (6) and (7) (the predicate $<$ corresponds to some coloring if and only if it satisfies all formulas of form (6) and (7)). However, if we are interested exclusively in n-colorings, then formulas of form (6) can be eliminated since they are a special case of formulas of form (9).

The first form of the method of hyperresolutions yields the following deductive system. The formulas are all possible disjunctions of elementary formulas of the form $x<y$, the postulates are formulas of form (8), and the deduction rules are thus:

$$\frac{\mathfrak{A} \vee x<z}{\mathfrak{A} \vee x<y \vee y<z}, \tag{10}$$

$$\frac{\mathfrak{A}_1 \vee x_1<x_0 \ldots \mathfrak{A}_n \vee x_{n+1}<x_n}{\mathfrak{A}_1 \vee \ldots \vee \mathfrak{A}_n}. \tag{11}$$

The only final formula is $\square$. The theorem on the completeness of the method of hyperresolutions again guarantees us that we are indeed dealing with a formalization of the property "to be an n-noncolorable graph" we are examining.

The second form of the method of hyperresolution yields the following deductive system. The formulas are all possible disjunctions of elementary formulas of the form $x \geqslant y$, the postulates are formulas of form (9), and the deduction rules are thus:

$$\frac{\mathfrak{A} \vee x \geqslant y \quad \mathfrak{B} \vee y \geqslant z}{\mathfrak{A} \vee \mathfrak{B} \vee x \geqslant z}, \tag{12}$$

$$(\mathfrak{I}xy \Rightarrow) \qquad \frac{\mathfrak{A} \vee x \geqslant y \quad \mathfrak{B} \vee y \geqslant x}{\mathfrak{A} \vee \mathfrak{B}}. \tag{13}$$

The only final formula also is $\square$.

3.3. As a second example we consider one property of graphs, given by Vitaver [6], viz., the property "to be a graph in which under any orientation of the edges there arises an oriented path of length not less than n." We shall specify orientation by means of the antisymmetric two-place predicate $\prec$, which is defined on pairs of vertices joined by an edge and which indicates the order in which the vertices run. The antisymmetry of predicate $\prec$ corresponds to the truth of all formulas of the form

$$(\mathfrak{I}xy \Rightarrow) \qquad x \prec y \ \vee \ y \prec x, \tag{14}$$

$$(\mathfrak{I}xy \Rightarrow) \qquad x \succcurlyeq y \ \vee \ y \succcurlyeq x, \tag{15}$$

where $x \succcurlyeq$ denotes $\neg(x \prec y)$. The absence of oriented paths of length not less than n is the truth of all formulas of form

$$(\mathop{\&}_{i=1}^{n} \mathfrak{I}x_i x_{i+1} \Rightarrow) \ x_1 \succcurlyeq x_2 \vee x_2 \succcurlyeq x_3 \vee \ldots \vee x_n \succcurlyeq x_{n+1}. \tag{16}$$

For technical reasons it is convenient for us to complete the definition of predicate $\prec$ up to all pairs of vertices, reckoning it to be false on pairs of vertices not joined by an edge. Then all formulas of form

$$x_1 \succcurlyeq x_2 \vee x_2 \succcurlyeq x_3 \vee \ldots \vee x_n \succcurlyeq x_{n+1} \tag{17}$$

will be true (and, all the more, so will all formulas of form (15), being already a special case of formulas (17)).

The second form of the method of hyperresolutions yields the following deductive system: the formulas are all possible disjunctions of elementary formulas of the form $x \succcurlyeq y$, the postulates are formulas of form (16), the only deduction rule is

$$(\mathfrak{I}xy \Rightarrow) \qquad \frac{\mathfrak{A} \vee x \succcurlyeq y \quad \mathfrak{B} \vee y \succcurlyeq x}{\mathfrak{A} \vee \mathfrak{B}},$$

and the only final formula is $\square$.

3.4. As a third example we consider one property of graphs, arising in the study of the four-color conjecture. We recall that a graph is called Hamiltonian if it contains a closed non-self-intersecting path passing through all the vertices. For the sake of uniformity of formulation we shall consider complete graphs with one and two vertices as Hamiltonian as well.

From Whitney's theorem [7] (also see [8]) it follows that to prove the four-color conjecture it is enough to show that we can color all vertices of an arbitrary planar Hamiltonia graph in four colors.

We shall call a planar Hamiltonian graph polygonal (the name was suggested by the natura manner of representing such graphs on a plane). We shall say that a graph is almost polygona if to it we can append, if necessary, new edges so that the resultant graph becomes polygonal Finally, we call the graph antipolygonal if neither itself nor any of the graphs (without loops!) resulting from it by pasting together vertices (and by the corresponding identification of the edges) is almost polygonal.

The four-color conjecture can be formulated in these terms as follows: each 4-noncolorable graph is antipolygonal. Indeed, if this were not so, then a 4-noncolorable graph would be found, which after an identification of certain vertices (unnecessary) and the appending of certain edges (also unnecessary) would become planar and, obviously, as before, 4-noncolorable. In addition, it is easy to see that no 4-colorable graph can be antipolygonal, because in it we can paste together the vertices colored in one color and the resulting graph, having no more than four vertices, will obviously be almost polygonal.

Thus, the four-color conjecture can be formulated in this way:

> the property "to be a 4-noncolorable graph" is equivalent to the property "to be an antipolygonal graph."

Below we take advantage of the fact that we have succeeded in formulating the four-color conjecture as an equivalence and not as an implication, as is usual. The first one of the properties was formalized above; we proceed to the formalization of the second.

At first we consider an arbitrary planar Hamiltonian graph $\langle V, \mathfrak{J} \rangle$. We fix a certain Hamiltonian cycle. We choose some vertex as the initial one and we number all the vertices in the order in which they are encountered on one of the sides as we go round the cycle. This vertex order is described by some two-place irreflexive antisymmetric transitive predicate $\prec$:

$$x \succcurlyeq x, \tag{18}$$

$$(x \neq y \Rightarrow) \quad x \prec y \vee y \prec x, \tag{19}$$

$$x \succcurlyeq y \vee y \succcurlyeq x, \tag{20}$$

$$x \prec y \vee y \prec z \vee x \succcurlyeq z, \tag{21}$$

$x \succcurlyeq y$ denotes $\neg(x \prec y)$.

Hamiltonian cycles separate all the edges not lying on them into those lying "inside" and "outside." Edges lying on the cycle itself are separated arbitrarily into "interior" and "exterior" ones. In order to formalize this separation we shall not introduce variables for the edges but will make use of the fact that we are examining only graphs without multiple edges; thus, for us each edge is uniquely determined by a pair of vertices. By $\mathcal{S}uvxy$ we shall denote the predicate "the edge joining vertices u and v lies on that same side of the cycle as does the edge joining vertices x and y." The predicate $\mathcal{S}$ is defined on the vertex quadruple $\langle u, v, x, y \rangle$ such that $\mathfrak{I}uv \,\&\, \mathfrak{I}xy$.

$$\begin{aligned} \mathcal{G}uvxy &\Leftrightarrow \mathcal{G}vuxy \\ &\Leftrightarrow \mathcal{G}uvyx \\ &\Leftrightarrow \mathcal{G}xyuv. \end{aligned} \tag{22}$$

Analogously to what we did with the predicate $\mathcal{E}$, we shall not rewrite this property of predicate $\mathcal{G}$ as a system of disjunctions but we shall simply not distinguish the elementary formulas occurring in (22).

The predicate $\mathcal{G}$ is transitive in the following sense:

$$\neg \mathcal{G}pquv \vee \neg \mathcal{G}uvxy \vee \mathcal{G}pqxy. \tag{23}$$

In addition, of any three edges some two lie on one side of the cycle:

$$\varphi pquv \vee \varphi uvxy \vee \varphi pqxy. \tag{24}$$

The predicate φ "agrees" with the predicate $\prec$ in the condition that the edges do not intersect:

$$(\mathfrak{I}uv \,\&\, \mathfrak{I}xy \Rightarrow) \quad u \succcurlyeq x \vee x \succcurlyeq v \vee v \succcurlyeq y \vee \neg \varphi uvxy. \tag{25}$$

It is not difficult to verify that if we have succeeded in defining the predicates $\succcurlyeq$ and φ on an arbitrary graph $\langle V, \mathfrak{I} \rangle$ such that conditions (18)-(21) and (23)-(25) are fulfilled, then this graph is almost polygonal. Thus, the graph $\langle V, \mathfrak{I} \rangle$ is not almost polygonal if and only if the system (18)-(21), (23)-(25) is inconsistent. It remains to note that the graph $\langle V, \mathfrak{I} \rangle$ is antipolygonal if and only if now the system resulting from system (18)-(21), (23)-(25) by the replacement of condition (19) by a weaker condition

$$(\mathfrak{I}xy \Rightarrow) \qquad x < y \vee y < x \tag{26}$$

proves to be inconsistent.

The further construction of the deductive system is carried out as was done earlier. The formulas are all possible disjunctions of formulas of form $x<y$ and $\varphi pquv$ (under conditions $\mathfrak{I}pq \,\&\, \mathfrak{I}uv$), the postulates are formulas of form (24) and (26), the deduction rules have the form

$$\frac{\mathfrak{A} \vee x \prec x}{\mathfrak{A}}, \tag{27}$$

$$\frac{\mathfrak{A} \vee x \prec z}{\mathfrak{A} \vee x \prec y \vee y \prec z}, \tag{28}$$

$$(\mathcal{J}pq \,\&\, \mathcal{J}uv \,\&\, \mathcal{J}xy \Rightarrow) \frac{\mathfrak{A} \vee \varphi pquv \quad \mathfrak{B} \vee \varphi uvxy}{\mathfrak{A} \vee \mathfrak{B} \vee \varphi pqxy} \tag{29}$$

$$(\mathcal{J}pq \,\&\, \mathcal{J}xy \Rightarrow) \frac{\mathfrak{A} \vee u<x \quad \mathfrak{B} \vee x<v \quad \mathfrak{L} \vee v<y \quad \mathfrak{V} \vee \varphi uvxy}{\mathfrak{A} \vee \mathfrak{B} \vee \mathfrak{L} \vee \mathfrak{V}}, \tag{30}$$

and the only final formula is the formula $\square$.

4. We pass on to the description of possible proof schemes by the metamathematical approach. We restrict ourselves to a consideration of one typical form of theorems, viz., we shall examine theorems of the form $\forall X\,(\mathcal{R}(X) \Rightarrow \mathcal{Q}(X))$.

4.1. One of the proof schemes most used in discrete mathematics is proof by induction. The method of mathematical induction can, roughly speaking, be characterized thus: we lead the proof of the property of interest to us of some object to the proof of the same property for one or several simpler objects (some measure of the object's complexity is used as the induction parameter), while for the simplest objects we give a direct proof (the basis of the induction).

If often happens that the complexity measure used is in no way connected either with property $\mathcal{R}$ or with property $\mathcal{Q}$, and the possibility of choosing "natural" induction parameters is usually limited. The metamathematical approach yields many possibilities for choosing the induction parameters closely connected with property $\mathcal{R}$. Indeed, if we have succeeded in formalizaing property $\mathcal{R}$ in some deductive system $\mathfrak{R}$, then as the measure of complexity of object X we can take any measure of complexity of proof $\mathcal{P}_{\mathfrak{R}}(X)$. Let us list some potentially possible induction parameters:

a) the number of branches in $\mathcal{P}_{\mathfrak{R}}(X)$;

b) the length of the longest branch;

c) the total number of applications of the deduction rules;

d) the number of applications of some fixed rule or rules from a fixed group without regard to the number of applications of the other rules; more generally,

e) the weighted number of applications (different rules are given different weights);

f) the number of occurrences (usages) of some postulate;

g) the number of occurrences of some postulate, standing beyond the applications of a specific rule.

Essentially it is precisely by this scheme that a part of König's theorem is proved (in somewhat other terms) in [1]. Therein property $\mathfrak{R}$ is the property "to be a 2-noncolorable graph" and property $\mathcal{Q}$ is the property "to contain cycles of odd length." The first of the formalizations considered by us was taken as the formalization of property $\mathcal{R}$, but the only deductions examined were those in which no application of rule (4") stands beyond with respect to the branch of some application of rule (4') (it is easy to show that every deduction can be rearranged into a deduction with such a property). The number of applications of rule (4') is used as the induction parameter.

The "descent" by the scheme described above from more complex objects to simpler ones takes place, roughly, in the following manner:

we find the proof $\mathcal{P}_{\mathcal{R}}(X)$ with respect to object X possessing property $\mathcal{R}$,

from proof $\mathcal{P}_{\mathcal{R}}(X)$ we find a simpler formal proof $\mathcal{P}^*_{\mathcal{R}}$;

proof $\mathcal{P}^*_{\mathcal{R}}$ must be the proof of property $\mathcal{R}$ for some object Y, i.e., must be a proof of the form $\mathcal{P}_{\mathcal{R}}(Y)$;

applying the induction hypothesis, we get that object Y possesses property Q ;

we prove the implication $Q(Y) \Rightarrow Q(X)$. Here the proof $\mathcal{P}^*_{\mathcal{R}}$ is closely connected with the original proof $\mathcal{P}_{\mathcal{R}}(X)$; it is usually obtained from $\mathcal{P}_{\mathcal{R}}(X)$ by means of some simplifying rearrangements. As a result, the most difficult step in the proof can turn out to be the finding of the object Y for which the simplified proof $\mathcal{P}^*_{\mathcal{R}}$ would be a proof of property $\mathcal{R}$. (If we wish to act conversely, i.e., to first choose the object Y possessing property $\mathcal{R}$ and, next, to construct some proof $\mathcal{P}_{\mathcal{R}}(Y)$ for it, then we are faced with greater difficulties when comparing the complexities of proofs $\mathcal{P}_{\mathcal{R}}(X)$ and $\mathcal{P}_{\mathcal{R}}(Y)$.) The scheme described above apparently has a rather limited area of applicability.

4.2. The following scheme, also using induction with respect to the deduction $\mathcal{P}_{\mathcal{R}}(X)$, probably has a wider area of application. Let us assume that we have succeeded in generalizing property $\mathcal{R}$ to property $\mathcal{R}_F$ (here F is an arbitrary formula from the formalization of property $\mathcal{R}$) by such a method that:

(a) if F is a postulate, then $Q_F(X)$;

(b) if $F_1, \ldots, F_{n-1}, F_n$ are formulas satisfying one of the deduction rules, then from $Q_{F_1}(X) \mathbin{\&} \ldots \mathbin{\&} Q_{F_{n-1}}(X)$ follows $Q_{F_n}(X)$;

(c) if F is a final formula, then from $Q_F(X)$ follows $Q(X)$. It is clear that in this case we have generalized the implication $\mathcal{R}(X) \Rightarrow Q(X)$. Formally, the induction is carried out with respect to the complexity of the deduction of formula F in $\mathcal{R}$ (which we can define, for example, as the number of formulas in the longest branch or as the total number of applications of the deduction rules); moreover, by induction over this parameter we can prove the following auxiliary statement: "if formula F is deducible in $\mathcal{R}$, then X possesses property Q_F." Here item (a) corresponds to the basis of the induction, item (b) is the induction transition, and item (c) ensures the transition from the auxiliary statement to the implication $\mathcal{R}(X) \Rightarrow Q(X)$.

This proof scheme was realized in [2] by example of a part of Vitaver's theorem. This theorem states that a graph does not have vertex colorings in n colors if and only if for each orientation of its edges there appears an oriented path of length not less than n. In [2] as property $\mathfrak{R}$ was taken the property "to be an n-noncolorable graph" and the formalization was also based not on the system of formulas of form (1)-(3), but a simple resolution rule (see [4]) was taken as a deduction rule. Property Q_F differs from property Q, roughly speaking, in that formula F "permits" certain breaks in the required oriented path and some of these breaks are taken into account when computing the length of such a broken path. Here property $Q_\Box$ coincides with property Q.

4.3. Up to this point we have examined what the formalization of property $\mathfrak{R}$ yields. Let us now assume that property Q too has been formalized in some deductive system $\mathcal{G}$. In such a situation we can detail the preceding scheme in the following way. With each formula F from the formalization of property $\mathfrak{R}$ let there be associated some set $\mathfrak{N}_F$ of formulas from the formalization of property Q; moreover, for each final formula F the set $\mathfrak{N}_F$ also contains a final formula. We can try to realize the above-described scheme by taking as Q_F the property "all formulas of $\mathfrak{N}_F$ are deducible in $\mathcal{G}(X)$." Property (c) will be fulfilled thanks to the choice of sets $\mathfrak{N}_F$, while properties (a) and (b) will be proved if we succeed in finding the corresponding "insertion." (An "insertion" is directly the deduction in $\mathcal{G}$ of formulas of $\mathfrak{N}_F$, if F is a postulate, and the deduction of formulas occurring in $\mathfrak{N}_{F_n}$ from formulas occurring in $\mathfrak{N}_{F_1} \cup \ldots \cup \mathfrak{N}_{F_{n-1}}$, under the condition that F_n is deducible from $F_1, \ldots, F_{n-1}$ in $\mathfrak{R}$.) Thus, the proof of property Q will be obtained as a result of a "stepwise modelling" of the proof of $\mathfrak{R}$.

A formalization of property Q can prove useful also in view of the so-called "inventor's paradox." It consists in the following. Suppose that we wish to prove a certain property $\mathfrak{A}(n)$ (for simplicity n is a positive integer). It can happen (see [9], for example) that it is impossible to do this directly since the induction transition $\mathfrak{A}(n) \Longrightarrow \mathfrak{A}(n+1)$ is unprovable without induction. At the same time, we can find a property $\mathfrak{B}(n)$, stronger than $\mathfrak{A}(n)$ and, moreover, the implication $\mathfrak{B}(n) \Longrightarrow \mathfrak{A}(n)$, the induction's basis $\mathfrak{B}(0)$, and the induction transition $\mathfrak{B}(n) \Longrightarrow \mathfrak{B}(n+1)$ are provable without induction. Thus, in order to prove property $\mathfrak{A}$ we need at first to "invent" a stronger property $\mathfrak{B}$.

The formula

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} < \frac{1}{\sqrt{n}} \tag{31}$$

serve as an example of the "inventor's paradox." A direct attempt to prove this inequality by induction fails since the inequality

$$\frac{1}{\sqrt{n}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{n+1}} \tag{32}$$

is false. At the same time, the stronger inequality

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} < \frac{1}{\sqrt{n+1}} \tag{33}$$

is easily proved by induction because

$$\frac{1}{\sqrt{n+1}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{n+2}}. \tag{34}$$

Two mutually contradictory requirements are imposed on the choice of property $\mathcal{B}$. On the one hand, it must be stronger than property $\mathcal{A}$, but, on the other hand, it must not be too strong, otherwise not all objects would possess it.

In such a situation it can prove useful to seek a strengthening of the property in terms of restrictions on the deductions (we have in mind strong restrictions violating, in the general case, the completeness of the system). The strengthened property $\mathcal{Q}^{\sim}$ can be that a deduction of one of the final formulas exists satisfying additional requirements, for instance, of the form:

a) such-and-such a postulate or such-and-such a deduction rule is not encountered anywhere in a deduction beyond the application of a given rule;

b) one of the premises of a given rule is always a postulate (of such-and-such a type);

c) behind each application of a given rule there necessarily is the application of such-and-such a rule.

Strengthenings of such type can turn out to be exactly the "golden mean," viz., not so strong that the implication $\mathcal{R}(X) \Longrightarrow \mathcal{Q}^{\sim}(X)$ will be true and yet sufficiently strong that the induction transition would be provable directly. It is clear that it is most natural to use such strengthenings when the induction runs over some metamathematical parameter. For example, we can combine these strengthening with the preceding scheme by taking as $\mathcal{Q}^{\sim}_F$ the property: "Each formula from $\mathcal{N}_F$ has a deduction satisfying such-and-such requirements and such-and-such additional requirements." The inventor's paradox can show up here in the following manner: the rearrangement of arbitrary deductions of formulas from $\mathcal{N}_{F_1}, \ldots, \mathcal{N}_{F_{n-1}}$ into some deductions of formulas from $\mathcal{N}_{F_n}$ will be impossible, while the rearrangement of deductions (of the same formulas) satisfying certain additional requirements in the deductions of formulas from $\mathcal{N}_{F_n}$, satisfying the same requirements, will be realizable.

4.4. Up to this point we have examined the formalizations of properties $\mathcal{R}$ and $\mathcal{Q}$ separately. We now consider that we can study "joint formalizations."

We say that properties $\mathcal{R}$ and $\mathcal{Q}$ have been formalized jointly if with each object X from the (common) domain of $\mathcal{R}$ and $\mathcal{Q}$ we have associated a certain deductive system $\langle \mathcal{F}, \mathcal{A}, \mathcal{D}, \mathcal{H} \rangle$ equipped with the following additional information. In the sets $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{H}$ there are indicated subsets $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{Q}$, $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{Q}$, and $\mathcal{H}_{\mathcal{R}}$ and $\mathcal{H}_{Q}$, respectively, such that X possesses property $\mathcal{R}$ (property Q) if and only if one of the formulas from $\mathcal{H}_{\mathcal{R}}$ (respectively, from $\mathcal{H}_{Q}$) is deducible from $\mathcal{A}_{\mathcal{R}}$ (respectively, from $\mathcal{A}_{Q}$) by means of only rules from $\mathcal{D}_{\mathcal{R}}$ (respectively, $\mathcal{D}_{Q}$).

Having a joint formalization of properties $\mathcal{R}$ and Q, we can try to prove the implication $\mathcal{R}(X) \Longrightarrow Q(X)$ by means of a multistep rearrangement of the deduction corresponding to property $\mathcal{R}$ into the deduction corresponding to property Q.

We pay special attention to the special case when $\mathcal{A}_{\mathcal{R}} \supseteq \mathcal{A}_{Q}$, $\mathcal{D}_{\mathcal{R}} \supseteq \mathcal{D}_{Q}$, and $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_{Q}$. It is clear that in this case the implication $\mathcal{R}(X) \Longleftarrow Q(X)$ is valid and, thus, we are dealing with the proof of the equivalence $\mathcal{R}(X) \Longleftrightarrow Q(X)$. Such a joint formalization of properties $\mathcal{R}$ and Q indeed exists often and is natural under the condition that the implication $\mathcal{R}(X) \Longleftarrow Q(X)$ is trivially provable.

Let us assume at first that there holds not simply the inclusion $\mathcal{A}_{\mathcal{R}} \supseteq \mathcal{A}_{Q}$ but even the equality $\mathcal{A}_{\mathcal{R}} = \mathcal{A}_{Q}$. In this case the proof of the implication $\mathcal{R}(X) \Rightarrow Q(X)$ reduces to the proof of the admissibility of the rules from $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{D}_{Q}$ in $\langle \mathcal{F}, \mathcal{A}_{Q}, \mathcal{D}_{Q}; \mathcal{H}_{Q} \rangle$. Here we need to note the following.

Under the concept of a deductive system adopted by us we are interested only in the deductions of certain formulas, viz., the final ones. For such systems it is natural to say that those rules are admissible whose applications can be eliminated from any deduction terminating in a final formula (usually, those rules are said to be admissible in some system, whose applications can be eliminated from any deduction). It is also clear that we are interested in the joint admissibility of the rules, since from the admissiblity in some system of each rule from some group it does not necessarily follow that all applications of the rules of this group can be eliminated from a deduction in which they are encountered simultaneously.

Let us now consider the case when there simply holds the inclusion $\mathcal{A}_{\mathcal{R}} \supseteq \mathcal{A}_{Q}$, but, in return, there is fulfilled the equality $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{Q}$. In this case, to prove the implication $\mathcal{R}(X) \Rightarrow Q(X)$ it is sufficient to prove that any formula from $\mathcal{A}_{\mathcal{R}} \setminus \mathcal{A}_{Q}$ is deducible from $\mathcal{A}_{Q}$ by means of rules from $\mathcal{D}_{Q}$. However, this can turn out to be too severe a requirement. Since we are interested only in the deductions of final formulas, it is sufficient to prove that the postulates from $\mathcal{A}_{\mathcal{R}} \setminus \mathcal{A}_{Q}$ can be eliminated from any deduction (with postulates from $\mathcal{A}_{\mathcal{R}}$) of any final formula by rules from $\mathcal{D}_{Q}$. In such a situation we shall say also that these postulates are jointly admissible.

Finally, let us consider the general case when there hold simply the inclusions $\mathcal{A}_{\mathcal{R}} \supseteq \mathcal{A}_{Q}$ and $\mathcal{D}_{\mathcal{R}} \supseteq \mathcal{D}_{Q}$. It is clear that in this case, to prove the implication $\mathcal{R}(X) \Rightarrow Q(X)$ it is sufficient to establish the joint admissibility of the formulas from $\mathcal{A}_{\mathcal{R}} \setminus \mathcal{A}_{Q}$ and the rules from $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{D}_{Q}$ in system $\langle \mathcal{F}, \mathcal{A}_{Q}, \mathcal{D}_{Q}, \mathcal{H}_{Q} \rangle$ (this concept is defined in a natural manner).

Let us cite an example of a theorem which can be proved by the scheme described above. A comparison of the second formalization of the property "does not have vertex colorings by no more than n colors" and the formalization above of the graph property arising in Vitaver's paper shows that if we identify the predicates $<$ and $\prec$ (and we can do this now because after formalization both predicates become only formal symbols), then the formalization of the first property differs from the formalization of the second by only one additional rule

$$\frac{\mathfrak{A} \vee x \succcurlyeq y \quad \mathfrak{B} \vee y \succcurlyeq z}{\mathfrak{A} \vee \mathfrak{B} \vee x \succcurlyeq z}\,. \tag{35}$$

Thus, Vitaver's theorem is the theorem on the fact that rule (35) is admissible in a deductive system having formulas of form (16) as postulates, rule (17), and formula $\Box$ as the only final formula.

The admissibility of rule (35) can be proved as follows. Let $\mathcal{P}$ be some deduction from which we need to eliminate the application of rule (35). Since rule (17) does not contain in the conclusion any elementary formulas which would not occur in the premise, then deduction $\mathcal{P}$ can be easily rearranged into a deduction $\mathcal{P}'$ in which any application of rule (17) is found behind any application of rule (35). The fundamental step consists in proving that any application of rule (35) to the postulates yields a formula containing some postulate as a subformula. Let

$$p_1 \succcurlyeq p_2 \vee \ldots \vee p_n \succcurlyeq p_{n+1}$$

and

$$q_1 \succcurlyeq p_2 \vee \ldots \vee q_n \succcurlyeq q_{n+1}$$

be two postulates to which rule (35) is applied, where $x = p_i$,

$$p_{i+1} = y = q_j,\ q_{j+1} = z \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n).$$

We consider the case when the vertices $p_1, \ldots, p_{n+1}$ (and, analogously, the vertices $q_1, \ldots, q_{n+1}$) are pairwise distinct. In such a case the conclusion of applying rule (35) contains the two subformulas

$$p_1 \succcurlyeq p_2 \vee \ldots \vee p_{i-1} \succcurlyeq x \vee x \succcurlyeq z \vee z \succcurlyeq q_{j+1} \vee \ldots \vee q_n \succcurlyeq q_{n+1} \tag{36}$$

and

$$q_1 \succcurlyeq q_2 \vee \cdots \vee q_{j-1} \succcurlyeq y \vee y \succcurlyeq p_{i+1} \vee \cdots \vee p_n \succcurlyeq p_{n+1}. \tag{37}$$

These formulas together contain $2n-1$ elementary subformulas and, therefore, one of them contains a postulate of form (16).

The cases when among the vertices $p_1, \ldots, p_{n+1}$ or among the vertices $q_1, \ldots, q_{n+1}$ there are like ones can be treated as being only slightly more complex (compare with the proof in [2]) and we omit them here.

Since the application of rule (35) to postulates yields formulas containing the postulates, in deduction $\mathcal{P}'$ we can remove some uppermost application of rule (35) (if there are none at all, then $\mathcal{P}'$ already is the desired deduction), replace its conclusion by the postulate contained in it, and, then, "weed out" the deduction. The resulting deduction contains fewer applications of rule (35) and all of them are located beyond any application of rule (17). Repeating the process described a sufficient number of times, we finally obtain the desired deduction entirely free of applications of rule (35).

Let us now show that the well-known four-color conjecture can be stated as an assertion on the admissibility of one rule in some deductive system.

We showed above that the four-color conjecture is equivalent to an implication $\mathcal{R}(X) \Rightarrow \mathcal{Q}(X)$, where $\mathcal{R}$ is the property "to be noncolorable in four colors" and $\mathcal{Q}$ is the property "to be antipolygonal."

As the formalization of property $\mathcal{R}$ we take the first formalization in Paragraph 3,2. In our case $n = 4$ and rule (11) appears thus:

$$\frac{\mathfrak{A}_1 \vee x_1 < x_0 \qquad \mathfrak{A}_2 \vee x_2 < x_1 \quad \mathfrak{A}_3 \vee x_3 < x_2 \quad \mathfrak{A}_4 \vee x_4 < x_3}{\mathfrak{A}_1 \vee \mathfrak{A}_2 \vee \mathfrak{A}_3 \vee \mathfrak{A}_4}. \tag{38}$$

Property $\mathcal{Q}$ was formalized in Paragraph 3.4.

Comparing these formalizations of properties $\mathcal{R}$ and $\mathcal{Q}$, we see that all the postulates and deduction rules in the formalization of $\mathcal{R}$, excepting rule (II), have their own analogs in the formalization of $\mathcal{Q}$ (we now identify $<$ and $\prec$). At the same time, postulates (24) and rules (29) and (30) from the formalization of $\mathcal{Q}$ do not have analogs in the formalization of $\mathcal{R}$ since there is no analog of predicate φ.

In order to obtain a joint formalization with $\mathcal{A}_{\mathcal{R}} = \mathcal{A}_{\mathcal{Q}}$, $\mathcal{D}_{\mathcal{R}} \supseteq \mathcal{D}_{\mathcal{Q}}$, and $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_{\mathcal{Q}}$, we formally supplement the formalization of property $\mathcal{R}$ being examined with the predicate $\varphi u v x y$, with postulates of form (24), and with rules (29) and (30). We need only to verify that the extended system formalizes property $\mathcal{R}$ as before, and to do this it is enough to convince ourselves that if a graph L has a regular 4-coloring, we can define the predicate $<(\prec)$ and the predicate φ in such a way that all the formulas (7), (8), (9), (23), (24), and (26) will be true. Let us show that this indeed holds.

Let L be a graph having a 4-coloring. We paste together the vertices having a like color. The resulting graph will obviously be a part of a complete graph with four vertices. We shall now interpret the predicates $<$ and φ in the same way as in the formalization of property $\mathcal{Q}$; then all the formulas (7), (8), (23), (24), and (26) turn out to be true since a complete graph with four vertices is polygonal, while formulas of form (9) will be true since there are four vertices in all in this graph.

Thus we have shown that:

the four-color conjecture is equivalent to the following: rule (38) is admissible in a deductive system having formulas of form (24) and (26) as postulates, rules (27)-(30) as deduction rules, and formula $\Box$ as the only final formula.

The difficulty is proving the conjecture is apparently explained by the fact that rule (38) is very powerful, requiring an exponential growth in the length of the deduction for its elimination.

We now turn to the general case, i.e., we waive the assumptions $\mathcal{A}_{\mathcal{R}} \supseteq \mathcal{A}_{\mathcal{Q}}$, $\mathcal{D}_{\mathcal{R}} \supseteq \mathcal{D}_{\mathcal{Q}}$, $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_{\mathcal{Q}}$. As before we can try to prove the implication $\mathcal{R}(X) \Rightarrow \mathcal{Q}(X)$ by means of a multistep rearrangement of the deduction corresponding to property $\mathcal{R}$ into the deduction corresponding to property $\mathcal{Q}$. However, in the general case being considered the intermediate deductions will, in general, be little meaningful: they will give neither a formalization of property $\mathcal{R}$ nor a formalization of property $\mathcal{Q}$.

A multistep rearrangement of deductions can be carried out by various schemes. For instance, it is sufficient to prove the following three lemmas:

a) If $H \in \mathcal{H}_{\mathcal{R}}$ and H is deducible from $\mathcal{A}_{\mathcal{R}}$ by rules from $\mathcal{D}_{\mathcal{R}}$, then H is deducible from $\mathcal{A}_{\mathcal{Q}}$ by rules from $\mathcal{D}$.

b) If $H \in \mathcal{H}_{\mathcal{R}}$, then any deduction H from $\mathcal{A}_{\mathcal{Q}}$ by rules from $\mathcal{D}$, containing applications of rules from $\mathcal{D} \setminus \mathcal{D}_{\mathcal{Q}}$, can be rearranged into a new deduction H from $\mathcal{A}_{\mathcal{Q}}$ by rules from $\mathcal{D}$, containing a fewer number of applications of rules from $\mathcal{D} \setminus \mathcal{D}_{\mathcal{Q}}$.

c) If $H \in \mathcal{H}_{\mathcal{R}}$ and H is deducible from $\mathcal{A}_{\mathcal{Q}}$ by rules from $\mathcal{D}_{\mathcal{Q}}$, then some formula from $\mathcal{H}_{\mathcal{Q}}$ is deducible from $\mathcal{A}_{\mathcal{Q}}$ by rules from $\mathcal{D}_{\mathcal{Q}}$.

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TRANSFINITE EXPANSIONS OF ARITHMETIC FORMULAS*

G. E. Mints

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A transfinite iteration of carrying the classical existence $\neg\forall x\neg$ (denoted by $\dot{\exists} x$) forward in the form of constructive realizability $(\exists x)$ by the scheme

$$\dot{\exists} x A x \leftrightarrow \exists y (\dot{\exists} x A x \dot{\vee} A y)$$

is examined. This idea was expressed by Shanin who used it to define the majorant of arithmetic formulas. For any negative formula F the expansion F_α of (transfinite) rank α is defined as the α-fold carrying forward of the quantifiers $\exists$ and $\forall$ by the scheme described above and of the retaining at the end only quantifier-free disjunctive terms. It is proved that the constructive truth of F_α is equivalent to the existence of a recursive infinite derivation of formula F (by the classical rules) of height α. This permits the proof of F_α being a disjunctive iteration of classical arithmetic. A modification of expansions F_α, equivalent to majorants in the sense of Shanin, is indicated.

1. Preliminary Explanations

1. Our aim is to describe the immersion of classical arithmetic into the constructive one, based on the idea of replacing the ineffective existence quantifier $\neg\forall x\neg$ by a transfinite sequence of effective quantifiers $\exists x$. The inclination to "reinterpret" negative arithmetic inferences so that their proof would yield something more than just the contradiction revealed by them, is observed in various papers; we mention [1-3].

The last of these served as the source of the idea of the approach being proposed. Another important leading consideration was the Kuznetsov—Shoenfield theorem on the completeness of an ω-rule for the classical ω-truth, although this theorem is not mentioned in what follows. Here we bear in mind a modernized variant in a form acceptable to constructivists: a purely negative arithmetic formula A is true if and only if the canonic tree T_A of search for the derivation of formula A without cuts is well founded (we recall that this tree is primitive recursive).

Besides the paper by Shanin, conversations with whom had a great influence on the contents of the present paper, the remarks by the participants of the Leningrad seminar, espe-

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