THE APPLICATION OF THE METHODS OF THE THEORY OF LOGICAL DERIVATION TO GRAPH THEORY

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We give an inductive definition of the property "a graph which cannot be colored with n colors."

1. The definitions used in constructing mathematical theories are of various kinds. For example, the property $\mathscr{P}$ can be defined using properties $\mathscr{Q}$ and relations $\mathscr{R}$, previously defined, as follows:

The object X has the property $\mathscr{P}$ if and only if for every object Y with the property $\mathscr{Q}$, the objects X and Y are in the relation $\mathscr{R}$. (1)

An example of such a definition is the definition of the linear independence of vectors: "The vectors $\overline{V}_1, \ldots, \overline{V}_n$ are linearly independent if and only if for any numbers $c_1, \ldots, c_n$ such that $c_1^2 + \ldots c_n^2 \neq 0$, we have $c_1\overline{V}_1 + \ldots + c_n\overline{V}_n \neq \overline{0}$."

Other definitions have an inductive nature. For example, the property $\mathscr{P}$ can be defined using the previously defined functions $\varphi_1(x_1, \ldots, x_{m_1}), \ldots, \varphi_n(x_1, \ldots, x_{m_n})$ and the concrete objects $A_1, \ldots, A_k$ as follows:

(a) The objects $A_1, \ldots, A_k$ have the property $\mathscr{P}$;

(b) if the objects $X_1, \ldots, X_m$ have the property $\mathscr{P}$, the object $\varphi_i(X_1, \ldots, X_{m_i})$ also has the property $\mathscr{P}$ $(i = 1, \ldots, n)$; (2)

(c) no other objects have the property $\mathscr{P}$, apart from those in (a) and (b).

An example of a definition of this type is the following: "(a) The pair of numbers (1, 1) is a Fibonacci pair; (b) if (u, v) is a Fibonacci pair, (v, u + v) is also a Fibonacci pair; (c) no other pair of numbers is a Fibonacci pair apart from those in (a) and (b)."

Definitions of the second type are sometimes preferable since they permit induction proofs by constructing the objects. However, the determination of the equivalent definition of type (2) from a definition of type (1) is usually a difficult problem (if it is in general possible). In this paper we show, by the example of a definition in graph theory, that in some cases the transition is possible using the theory of logical derivation.

2. We retain the terminology introduced in [1]. By a graph we shall always mean an ordinary graph, i.e., a finite nonoriented graph without loops and multiple arcs. We shall assume that the nodes of the graph are numbered using the natural numbers (pairwise distinct, but not necessarily in sequence), and write the graph as the pair (X, U), where X is the set of nodes and U the set of pairs of nodes joined by an arc. Following [1], we write an unordered pair consisting of the nodes i and j as $\widetilde{ij}$.

We shall say that an arbitrary partition of the set X into pairwise nonintersecting subsets is a <u>quasicoloration</u> of the graph L = (X, U). If none of the subsets contains a pair of adjacent nodes, obviously the quasicoloration is a complete coloration (since in this paper we consider only complete colorations, we

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shall omit the adjective "complete" in future). We shall say that the graph L is colorable in n colors if it has a coloration in n or fewer colors.

In graph theory various necessary and sufficient conditions for the possibility of coloring graphs in not more than a given number of colors are studied. We give an inductive definition of the property "a graph which cannot be colored with n colors." Using this definition we can prove theorems on the existence of colorations, for example, we can give a new proof of the familiar Koenig theorem on bichromatic graphs. Using this definition we can also formulate Hadwiger's hypothesis (see [2] or [1], §40) and the four color hypothesis in new terms. These new formulations may be useful in investigations involving the above hypotheses.

3. Let d be the number of nodes of the graph $L = (X, U)$. To define the coloration of the graph L we establish a one-to-one correspondence between the pairs of nodes of X and the set $\mathfrak{P}_L$ consisting of $d(d-1)/2$ Boolean variables. A_{ij} and A_{ji} ($i, j \in X$, $i \neq j$) denote the same Boolean variable below corresponding to the pair consisting of the i-th and j-th nodes of L.

Let $\mathfrak{A}_L$ denote the set of all formulas of the form

$$\neg A_{ij} \vee \neg A_{jk} \vee A_{ik} \quad (i,\ j,\ k \in X,\ \ i \neq j,\ \ j \neq k,\ \ k \neq i),$$

let $\mathfrak{B}_L$ denote the set of all formulas of the form

$$\neg A_{ij} \qquad (\widetilde{ij} \in U),$$

and let $\mathfrak{C}_L^n$ ($n = 1, 2, \ldots$) denote the set of all formulas of the form

$$\bigvee_{i=1}^{n+1} \bigvee_{j=i+1}^{n+1} A_{l_i l_j} \qquad (l_1, \ldots, l_{n+1} \in X,\ l_i \neq l_j \text{ for } i \neq j).$$

Each coloration of L generates an association with the truth values of the variables of $\mathfrak{P}_L$: the variable A_{ij} ($i \neq j$) takes the value **T** (true) if the nodes i and j are colored in the same color, and the value **F** otherwise. Since the relation "the nodes i and j are colored in the same color" is transitive, by the above association of truth values all formulas of $\mathfrak{A}_L$ take the value **T**. If the given quasicoloration is a coloration, all formulas of $\mathfrak{B}_L$ also take the value **T**. If in addition the coloration uses not more than n colors, in any set of $n+1$ vertices $l_1, \ldots, l_{n+1}$ there can be found two colored in the same color and, consequently, all formulas of $\mathfrak{C}_L^n$ also take the value **T**.

It is easy to see that the converse also holds. In particular, if with all the variables of $\mathfrak{P}_L$ we associate truth values and all the formulas of $\mathfrak{A}_L$, take the value **T**, we have thus specified a quasicoloration of L. If in addition the value **T** is taken by all formulas of $\mathfrak{B}_L$, the quasicoloration is a coloration. Further, if all the formulas of $\mathfrak{C}_L^n$ take the value **T**, the coloration uses not more than n colors.

Thus, the graph L is a coloration in n colors if and only if for some set of truth values of the variables of $\mathfrak{P}_L$, all formulas of the set $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$ take the value **T**.

4. The set of formulas in the propositional calculus having the property that for a given choice of the truth values of the variables all the formulas of the set take the value T is said to be feasible. At the present time we know many different methods of recognizing the feasibility of a given finite set of formulas. We use the method of hyperresolution proposed in [3] (this method is a modification of the method of resolution [4]).

We now give some necessary definitions. Elementary formulas are Boolean variables and their negations. Following [4], we shall say that every formula of the form $W_1 \vee \ldots \vee W_l$, where $W_1, \ldots, W_l$ are elementary formulas, is a disjunction. We shall not distinguish between disjunctions which differ from each other only by permutations and repetitions of the elementary formulas. We allow the case $l = 0$ and say that the resulting disjunction is empty. By definition, the empty disjunction always takes the value **F**. We shall say that a disjunction is positive if it does not contain the sign $\neg$.

Let $s \geq 1$, P_{ij} ($1 \leq i \leq s$, $1 \leq j \leq t_j$) and R_k ($1 \leq k \leq u$) be arbitrary Boolean variables (not necessarily distinct), Q_i ($1 \leq i \leq s$) pairwise distinct Boolean variables distinct from the variables P_{ij} and R_k. Assume that these variables are associated with values such that the positive disjunctions

$$P_{i1} \vee \ldots \vee P_{it_i} \vee Q_i \qquad (i = 1, \ldots, s) \tag{3}$$

and the disjunction

$$\neg Q_1 \vee \ldots \vee \neg Q_s \vee R_1 \vee \ldots \vee R_u \tag{4}$$

take the value T. It is easy to verify that in this case the disjunction

$$P_{11} \vee \ldots \vee P_{1t_1} \vee \ldots \vee P_{s1} \vee \ldots \vee P_{st_s} \vee R_1 \vee \ldots \vee R_u \tag{5}$$

also takes the value T. We say that the disjunction (5) is the hyperresolvent of the disjunctions (3) and (4). We shall also say that the disjunction (5) is obtained from the disjunctions (3) and (4) by the hyperresolution rule and call the disjunction (4) the kernel of the given application of the hyperresolution rule.

Clearly, if to any feasible set of disjunctions we add the hyperresolvent of any of its members, the resulting set is also feasible. This process of complementing a set by the hyperresolvents of its members can be repeated and all the resulting sets are feasible if the original one is. Hence, if at any stage the empty disjunction is the hyperresolvent, the initial set is infeasible.

The process of deriving new disjunctions from an original set can be described conveniently as follows. We shall say that the list of positive disjunctions

$$Z_1, \ldots, Z_q \tag{6}$$

is a derivation from the set of disjunctions $\mathfrak{M}$ if each term in this list is either an element of $\mathfrak{M}$ or is obtained by the hyperresolution rule from preceding terms in the list and a disjunction of $\mathfrak{M}$ as kernel. The derivation (6) is said to be a derivation of the disjunction Z if $Z_q = Z$. We say that the disjunction Z is derivable from the set $\mathfrak{M}$ if there is a derivation of Z.

As shown above, if the empty disjunction is derivable from some set of disjunctions, the set is infeasible. It was proved in [3] that the converse is also true:

The empty disjunction can be derived from any infeasible set of disjunctions.

Thus, the graph L cannot be colored in n colors if and only if we can derive the empty disjunction from the set $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$.

5. In general the empty disjunction can be derived from an infeasible set in many ways. We need derivations which have special properties. We shall say that a derivation is correct if no application in it of the hyperresolution rule with a kernel consisting of a single term precedes the application of the rule with a kernel consisting of more than one term.

LEMMA. If from the set $\mathfrak{M}$ we can derive the disjunction Z_1, there is a correct derivation of Z_1 from $\mathfrak{M}$.

Proof. Let (6) be a derivation of the disjunction Z from $\mathfrak{M}$. We shall prove that each disjunction of (6) has a correct derivation from $\mathfrak{M}$. The proof is by induction on the number of disjunctions in (6).

If $Z_k (1 \leq k \leq q)$ is a disjunction of $\mathfrak{M}$, its correct derivation consists of one term – the disjunction Z_k itself. Consider now the case when the disjunction Z_k $(1 \leq k \leq q)$ is obtained from the disjunctions Z_{l_1}, ..., $Z_{l_s} (l_1, \ldots, l_s < k)$ and some disjunction of $\mathfrak{M}$ by the hyperresolution rule. We construct, by the induction hypothesis, the correct derivation

$$Y_1^i, \ldots, Y_{t_i}^i$$

of the disjunction Z_{l_i} of $\mathfrak{M}$ $(i = 1, \ldots, s)$. Let m_i be the largest number such that in the derivation of Y_1^i, ..., $Y_{m_i}^i$ there is no application of the hyperresolution rule with a kernel consisting of one term $(i = 1, \ldots, s)$. Clearly, the disjunction $Y_{m_i}^i$ has the form

$$Z_{l_i} \vee \bigvee_{j=1}^{p_i} Q_{ij},$$

where the Q_{ij} are Boolean variables such that all disjunction of the form $\neg Q_{ij}$ occur in $\mathfrak{M}$. It is easy to verify that the list

$$Y_1^1, \ldots, Y_{m_1}^1, \ldots, Y_1^s, \ldots, Y_{m_s}^s, Z_k \vee \bigvee_{i=1}^{s} \bigvee_{j=1}^{p_i} Q_{ij}$$

is a correct derivation which can be continued to a correct derivation of Z_k. The lemma is proved.

6. We give an inductive definition of a subset of the set of all graphs which cannot be colored in n colors.

Definition. (a) Every graph of the form

$$(\{l_1, \ldots, l_{n+1}\}, \{\widetilde{l_i l_j} \mid 1 \leqslant i < j \leqslant n+1\}),$$

where the $l_1, \ldots, l_{n+1}$ are pairwise distinct natural numbers (i.e., a complete graph with n + 1 nodes) is a μ_n-graph;

(b) if $L' = (X', U')$ and $L'' = (X'', U'')$ are two μ_n-graphs, and i, j, k are pairwise distinct natural numbers such that

$$\widetilde{ij} \in U', \quad \widetilde{ij} \notin U'' \quad \widetilde{jk} \notin U', \quad \widetilde{jk} \in U'',$$

then the graph

$$(X' \cup X'', \; U' \cup U'' \cup \{\widetilde{ik}\} \setminus \{\widetilde{ij}, \widetilde{jk}\})$$

is also a μ_n-graph;

(c) no other graphs are μ_n-graphs apart from those in (a) and (b).

THEOREM 1. Each μ_n-graph is not colorable in n colors; each graph which cannot be colored in n colors contains a μ_n-graph as part of it.

Proof. We prove the second part of the theorem first. Let $L = (X, U)$ be a graph which cannot be colored in n colors. Let $L^* = (X, U^*)$ be an arbitrary graph with the same set of nodes. We establish a correspondence between it and the disjunction

$$\bigvee_{\widetilde{ij} \in U^*} A_{ij}.$$

Clearly, for each positive disjunction constructed from the variables of $\mathfrak{P}_L$, we can find the graph to which it corresponds.

Let $Z_1, \ldots, Z_q$ be a correct derivation of the empty disjunction from $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$. Without loss of generality we can assume that this derivation is minimal in the sense that any list obtained by striking out any term in the derivation is not a derivation of the empty disjunction. Let m be the greatest number such that the derivation $Z_1, \ldots, Z_m$ does not contain an application of the hyperresolution rule with kernels consisting of one term. Let $L_1, \ldots, L_m$ be the graphs corresponding to the disjunctions $Z_1, \ldots, Z_m$. We shall prove by induction on k that $L_k (1 \leq k \leq m)$ is a μ_n-graph.

If the disjunction $Z_k (1 \leq k \leq m)$ occurs in $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$, since Z_k is positive, this disjunction must occur in $\mathfrak{C}_L^n$. Clearly in this case L_k is a complete graph with n + 1 vertices and is a μ_n-graph in accordance with the first point of the definition.

If the disjunction $Z_k (1 \leq k \leq m)$ is not in $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$, by the choice of m this disjunction is obtained from some disjunctions Z_s, Z_t $(s, t < k)$ by the hyperresolution rule with kernel in $\mathfrak{A}_L$. By the induction hypothesis, the graphs L_s and L_t are μ_n-graphs. It is easy to verify that in this case L_k is a μ_n-graph in accordance with the second point of the definition (we have to take L_s and L_t as L' and L'').

Each of the disjunctions $Z_{m+1}, \ldots, Z_q$ is obtained from the preceding one by the hyperresolution rule with a kernel consisting of one term from $\mathfrak{A}_L \cup \mathfrak{B}_L \cup \mathfrak{C}_L^n$. Clearly, the kernel is taken from $\mathfrak{B}_L$. It is easy to see that since Z_q is the empty disjunction, the graph L_m is part of the graph L. The second part of the theorem is proved.

Now let $L = (X, U)$ be an arbitrary μ_n-graph. We construct the list $L_1, \ldots, L_m$ of μ_n-graphs such that $L_m = L$ and each graph in the list is a μ_n-graph according to the first point of the definition or is obtained from some two previous graphs by the rule described in the second point of the definition. Let

$$L_i = (X_i, U_i) \qquad (i = 1, \ldots, m), \; X^* = \bigcup_{i=1}^{m} X_i,$$

$L^* = (X^*, U)$. Consider the list $Z_1, \ldots, Z_m$ of disjunctions which correspond to the graphs $L_1, \ldots, L_m$. It is easy to verify that this list is a derivation from $\mathfrak{A}_{L^*} \cup \mathfrak{C}_{L^*}^n$ and that it can be continued to a derivation of the empty disjunction from $\mathfrak{A}_{L^*} \cup \mathfrak{B}_{L^*} \cup \mathfrak{C}_{L^*}^n$.

Thus, the graph L^* is not colorable in n colors. The graph L^* differs from L only in that it may have isolated nodes, which, obviously, do not affect the colorability. Consequently, L also cannot be colored in n colors. The theorem is proved.

7. We give an example of the use of Theorem 1 to prove an assertion about the existence of colorations.

We give a new proof of the part "if" in the familiar theorem on bichromatic graphs (see, e.g., [1], §12).

KOENIG'S THEOREM. A graph can be colored in two colors if and only if it does not contain cyclic paths of odd length.

Clearly, by Theorem 1, it is sufficient if we show that each μ_2-graph contains a cyclic path of odd length. Obviously, any graph which is a μ_2-graph according to the first point of the definition (i.e., a complete graph with three nodes) contains a cyclic path of odd length.

Consider now the case when the graph L is obtained from the μ_2-graphs $L' = (X', U')$ and $L'' = (X'', U'')$ by the rule described in point (b) of the definition. By the induction hypothesis, the graphs L' and L" contain cyclic paths of odd length. It is easy to see that each cyclic path of odd length contains a simple cycle of odd length (see [1], §11). Suppose the graphs L' and L" contain simple cycles of odd length

$$a_1v_1 \ldots a_{s-1}v_{s-1}a_s \tag{7}$$

and

$$b_1w_1 \ldots b_{t-1}w_{t-1}b_t, \tag{8}$$

respectively, where

$$a_1, \ldots, a_s \in X', \ a_1 = a_s, \ v_1, \ldots, v_{s-1} \in U',$$
$$b_1, \ldots, b_t \in X'', b_1 = b_t, \ w_1, \ldots, w_{t-1} \in U''.$$

If the arc $\widetilde{ij}$ does not occur in the cycle (7), the cycle is in the graph L; similarly, the cycle (8) is in L if the arc $\widetilde{jk}$ is not in (8). It remains to consider the case when the arc $\widetilde{ij}$ is in the cycle (7) and the arc $\widetilde{jk}$ is in (8).

Without loss of generality we can assume that $\widetilde{ij} = v_{s-1}$, $\widetilde{jk} = w_1$, i.e., $i = a_{s-1}$, $j = a_s = a_1 = b_1 = b_t$, $k = w_2$. It is easy to see that in this case L contains the following cyclic path of odd length:

$$a_1v_1 \ldots a_{s-2}v_{s-2}i \ \widetilde{ik} \ kw_2b_3 \ldots w_{t-1}b_t.$$

We note that, using the concept of a μ_n-graph, we can also give a new proof of Vitaber's theorem (see [5], Theorem 1, or [1], § 42).

8. In conclusion we formulate two familiar hypotheses in terms of μ_n-graphs.

Hadwiger's hypothesis [2]. Each μ_n-graph contains a part homeomorphic with a complete graph having n + 1 nodes.

In § 7, we have essentially proved the validity of Hadwiger's hypothesis for n = 2.

The four color hypothesis. No μ_4-graph is plane.

The equivalence of the above formulations of the hypotheses to their traditional variants follows directly from Theorem 1.

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