Hyper-resolution principle and Nullstellensatz

Paolo Comensoli

28/12/2017

Here I will expand some details of the Lifschitz's paper *Semantical completeness theorems in logic and algebra.*[1] First of all I will briefly give the hyper-resolution principle and Robinson's theorem as stated by Lifschitz, then I will show the proof of Nullstellensatz with the aim to clarify every step.

Notation:

- *Literal:* an atomic formula or a negation of an atomic formula.
- *Clause:* a finite set of literals.
- if Γ is a clause, $\overline{\Gamma}$ is the disjunction of the elements of Γ in a fixed order.

1 Hyper-resolution principle

Let T be a set of propositional formulae, each of the form:

$$\bigwedge_{i=1}^{m} A_i \rightarrow \bigvee_{j=1}^{n} B_j$$

Where the A_i's and B_j's are literals and $m + n > 0$.
We have the calculus H_T: the objects derivable are clauses and the rules of inference are:

$$\frac{\Gamma_1 \cup \{A_1\}, \quad \cdots, \quad \Gamma_m \cup \{A_m\}}{\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{B_1, \cdots, B_n\}} \tag{1}$$

for every element of T.
Whenever $A_i \in \Gamma_i$ for some i, we call such application *trivial.* Lemma 2 at page 92 ensures that such kind of application can always be removed.

[1]American Mathematical Society, Volume 79, Number 1, May 1980, Pages: 89-96

Example Let $T = \{A \to B \vee C,\ A,\ \neg B\}$, then H_T has three rules:

$$\frac{\Gamma \cup \{A\}}{\Gamma \cup \{B, C\}} \qquad \frac{\Gamma}{\Gamma \cup \{A\}} \qquad \frac{\Gamma \cup \{B\}}{\Gamma}$$

Using such rules we get the tree:

$$\cfrac{\cfrac{\overline{\{A\}}}{\{B, C\}}}{\{C\}}$$

and we write $\varnothing \vdash_{H_T} \{C\}$

Robinson's theorem, version 4

For any clauses $\Gamma_1, \cdots, \Gamma_l, \Delta$, if in every model of T, the following is valid

$$\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l \to \overline{\Delta}$$

then there exists a derivation of some $\Delta' \subseteq \Delta$ from $\Gamma_1, \cdots, \Gamma_l$ in H_T, containing no trivial application of rules of inference.

2 Nullstellensatz

Using Hyper-resolution principle and Robinson's theorem it is possible to prove the Nullstellensatz as stated here:

Let K be a field and $f, f_1, \cdots, f_l \in K[\underline{x}]$, if in any extension of K, f vanishes at all common zeros of $f_1, \cdots, f_l$, then there exists $p \in \mathbb{N}$ and $h_1, \cdots, h_l \in K[\underline{x}]$ such that

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

To this aim, consider the first order language $\mathscr{L} = \{+, \cdot, = 0\} \cup K$ where $+, \cdot$ are binary function symbols, $= 0$ is a unary predicate symbol and the elements of K are constants; so the terms are polynomials over K and the atomic formulae are algebraic equations.

Consider the theory given by the following axioms:

$$0 = 0 \tag{2}$$

$$\neg(1 = 0) \tag{3}$$

$$r_1 = 0 \to r_2 = 0 \qquad (r_1, r_2 \texttt{ are equal polynomials}) \tag{4}$$

$$(r = 0 \wedge s = 0) \to r + s = 0 \tag{5}$$

$$r = 0 \to r \cdot s = 0 \tag{6}$$

$$r \cdot s = 0 \to (r = 0 \vee s = 0) \tag{7}$$

One can prove the equality axioms, the axioms of integral domain and the diagram of K. It follows that the models of the theory are the integral domains that contains K. Lifschitz, instead of the axiom $\neg(1 = 0)$, uses the axioms $\neg(\alpha = 0)$ for each $\alpha \in K \setminus \{0\}$, however the one taken here is equivalent and simplifies the next arguments.

The calculus H_T consists of the axiom $\{0 = 0\}$ and of the rules of inference:

$$\frac{\Gamma \cup \{1 = 0\}}{\Gamma} \tag{8}$$

$$\frac{\Gamma \cup \{r_1 = 0\}}{\Gamma \cup \{r_2 = 0\}} \qquad r_1, r_2 \texttt{ are equal polynomials} \tag{9}$$

$$\frac{\Gamma \cup \{r = 0\}, \quad \Delta \cup \{s = 0\}}{\Gamma \cup \Delta \cup \{r + s = 0\}} \tag{10}$$

$$\frac{\Gamma \cup \{r = 0\}}{\Gamma \cup \{r \cdot s = 0\}} \tag{11}$$

$$\frac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}} \tag{12}$$

Note that the rules (8-11) do not increase the number of elements of the clauses, precisely for an application

$$\frac{\Gamma \quad \Delta}{\Sigma}\,(8\text{-}11)$$

we have $|\Sigma| \leq |\Gamma \cup \Delta|$; while for rule (12) we have $\Delta = \varnothing$ and $|\Sigma| = |\Gamma| + 1$.

For any derivation in H_T there is a tree with all the applications of the rule (12) at the bottom; this is ensured by the following.

Lemma *Every derivation in H_T which contains non-trivial applications can be rearranged in such a way that any application of (12) is either the last one or it is followed by an application of (12).*

Proof: Suppose to have a non-trivial application of a rule (12) followed by a rule (x) different from (12):

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}\,,}\,12 \qquad \Delta}{\Sigma \cup \{r = 0, s = 0\}}\,x$$

Here Δ is the second premise in case (x) is the rule (10), otherwise it should be dropped. In the case that both $r = 0$ and $s = 0$ in the conclusion are different from A_1 shown in the scheme (1) of the rule (x), then $A_1 \in \Gamma$ and we can change the order as follow:

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}, \qquad \Delta}{\Sigma \cup \{r \cdot s = 0\}}\,x}{\Sigma \cup \{r = 0, s = 0\}}\,12$$

Otherwise let $\{r = 0\}$ be A_1 of (x), depending on whether (x) is (8),(9) ,(10) or (11), change the derivation according to one of the schemes below

$$\dfrac{\dfrac{\Gamma \cup \{1 \cdot s = 0\}}{\Gamma \cup \{1 = 0, s = 0\}}\,12}{\Gamma \cup \{s = 0\}}\,8 \quad \rightsquigarrow \quad \dfrac{\Gamma \cup \{1 \cdot s = 0\}}{\Gamma \cup \{s = 0\}}\,9$$

$$\dfrac{\dfrac{\Gamma \cup \{r_1 \cdot s = 0\}}{\Gamma \cup \{r_1 = 0, s = 0\}}\,12}{\Gamma \cup \{r_2 = 0, s = 0\}}\,9 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \cup \{r_1 \cdot s = 0\}}{\Gamma \cup \{r_2 \cdot s = 0\}}\,9}{\Gamma \cup \{r_2 = 0, s = 0\}}\,12$$

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}}\,12 \qquad \Delta \cup \{t = 0\}}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}}\,10 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}, \quad \dfrac{\Delta \cup \{t = 0\}}{\Delta \cup \{t \cdot s = 0\}}\,11}{\Gamma \cup \Delta \cup \{r \cdot s + t \cdot s = 0\}}\,10}{\Gamma \cup \Delta \cup \{(r + t) \cdot s = 0\}}\,9}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}}\,12$$

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}}\,12}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,11 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{(r \cdot s) \cdot t\}}\,11}{\Gamma \cup \{(r \cdot t) \cdot s = 0\}}\,9}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,12$$

By a series of application of this procedure, we obtained a derivation as required.

Now we have all the tools to give the proof of nullstellentsatz:

Proof: Let K be a field and $f, f_1 \cdots f_l \in K[\underline{x}]$ such that in any extension of K, f vanishes at all common zeros of $f_1 \cdots f_l$. This amounts to require that in every model of (2-7) the following is valid:

$$\Gamma_1 \wedge \cdots \wedge \Gamma_l \to \Delta$$

where $\Gamma_i = \{f_i = 0\}$ and $\Delta = \{f = 0\}$.
Then by Robinson's theorem:$\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$ where either $\Delta' = \varnothing$ or $\Delta' = \{f = 0\}$. In particular take a derivation without trivial applications and where all the applications of (12) are at the bottom, the existence of such tree is ensured by the lemma shown above. The tree is of the form:

$$\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\Sigma}\,8\text{ - }11}{\Delta'}\,12$$

Since all the Γ_i's are singletons, and since the rules (8-11) do not increase the size of the clauses, Σ is either a singleton or the empty set. Let's treat separately the cases of Δ'.

Case $\Delta' = \varnothing$

The only way to get $\varnothing$ out of the rules of inference is by applying (8), in this case rule (12) does not apply at all and the tree is of the form:

$$\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{1 = 0\}}\,9\text{ - }11}{\varnothing}\,8$$

1 is obtained by a series of applications of rules (9 - 11), therefore it is a linear combination of $f_1, \cdots, f_l$:

$$f^0 = 1 = \sum_{i=1}^{l} h_i \cdot f_i$$

Case $\Delta' = \{f = 0\}$

Suppose that rule (12) applies $p - 1$ times, (8) does not apply at all and the tree is of the form:

$$\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\Sigma}\ 9\text{ - }11}{\{f = 0\}}\ (p-1) \text{ applications of } 12$$

Since (12) applies and Σ is a singleton, Σ must be of the form $\{t_1 \cdot r_1 = 0\}$. Suppose that the A_1 of the i^{th} application is $r_i = t_{i+1} \cdot r_{i+1}$, then the tree is:

$$\begin{array}{c}
\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{t_1 \cdot r_1 = 0\}}\ 9\text{ - }11}{\{t_1 = 0, t_2 \cdot r_2 = 0\}}\ 12 \\
\vdots \\
\cfrac{\{t_1 = 0, \cdots t_{p-2} = 0, t_{p-1} \cdot r_{p-1} = 0\}}{\{t_1 = 0, \cdots, t_{p-1} = 0, r_{p-1} = 0\}}\ 12
\end{array}$$

Since the root of the tree must be the singleton $\Delta' = \{f = 0\}$, it is the case that for any i: $t_i = r_{p-1} = f$. Therefore $\Sigma = \{f^p = 0\}$ and we conclude

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$