Hyper-resolution principle, applications to discrete mathematics

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Contents

Introduction

In the first two sections I will expand some details of the Lifschitz's work [1]. First of all I will briefly give the hyper-resolution principle and Robinson's theorem as stated by Lifschitz, then I will show the proof of Nullstellensatz with the aim to clarify every step.

Notation:

- *Literal:* an atomic formula or a negation of an atomic formula.
- *Clause:* a finite set of literals.
- if Γ is a clause, $\overline{\Gamma}$ is the disjunction of the elements of Γ in a fixed order.

Chapter 1

Hyper-resolution principle

Let T be a set of propositional formulae, each of the form:

$$\bigwedge_{i=1}^{m} A_i \to \bigvee_{j=1}^{n} B_j$$

Where the A_i's and B_j's are literals and $m + n > 0$.

We have the calculus H_T: the objects derivable are clauses and the rules of inference are:

$$\frac{\Gamma_1 \cup \{A_1\}, \quad \cdots, \quad \Gamma_m \cup \{A_m\}}{\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{B_1, \cdots, B_n\}} \tag{1.1}$$

for every element of T. The formulae of the set T will be called the *non-logical axioms* of H_T

Example Let $T = \{A \to B \vee C,\ A,\ \neg B\}$, then H_T has three rules:

$$\frac{\Gamma \cup \{A\}}{\Gamma \cup \{B, C\}} \qquad \frac{\Gamma}{\Gamma \cup \{A\}} \qquad \frac{\Gamma \cup \{B\}}{\Gamma}$$

Using such rules we get the tree:

$$\cfrac{\cfrac{\cfrac{\varnothing}{\{A\}}}{\{B, C\}}}{\{C\}}$$

and we write $\varnothing \vdash_{H_T} \{C\}$

Given a rule in the form (1.1) and an application of it in a derivation, whenever $A_i \in \Gamma_i$ for some i, we call such application *trivial.* The following two lemmas ensure that such kind of application can always be removed.

Lemma 1.1. *Let Π be obtained from $\Sigma_1, ..., \Sigma_m$ by one application of a rule H_T. For any $\Sigma'_1 \subset \Sigma_1, ..., \Sigma'_m \subset \Sigma_m$, one of the two following holds:*
(a) $\Sigma'_i \subset \Pi$ for some i
(b) some $\Pi' \subset \Pi$ can be obtained from $\Sigma'_1, ..., \Sigma'_m$ by an application of the same rule

Proof: Consider an application of (1.1) leading from $\Sigma_1, ..., \Sigma_m$ to Π, for this application:

$$\Sigma_1 = \Gamma_1 \cup \{A_1\}, ..., \Gamma_m \cup \{A_m\}$$

$$\Pi = \Gamma_1 \cup ... \cup \Gamma_m \cup \{B_1, ..., B_n\}$$

Consider the following two cases:
(a) $A_i \notin \Sigma'_i$ form some i. Then $\Sigma'_i \subset \Gamma_i \subset \Pi$
(b) $A_i \in \Sigma'_i$ for every i. Then $\Sigma'_i = \Gamma'_i \cup \{A_i\}$ where $\Gamma'_i = \Sigma'_i \backslash \{A_i\}$, one application of (1.1) to $\Sigma'_1, ..., \Sigma'_m$ gives $\Pi' = \Gamma'_1 \cup ... \cup \Gamma'_m \cup \{B_1, ..., B_n\} \subset \Pi$

Lemma 1.2. *For every derivation of a clause Π in H_T there exists a derivation of some $\Pi' \subset \Pi$ in H_T without trivial application of rules of inference.*

Theorem 1.3 (Robinson's, version 4). *For any clauses $\Gamma_1, \cdots, \Gamma_l, \Delta$, if in every model of T, the following is valid*

$$\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l \rightarrow \overline{\Delta}$$

then there exists a derivation of some $\Delta' \subseteq \Delta$ from $\Gamma_1, \cdots, \Gamma_l$ in H_T, containing no trivial application of rules of inference.

Chapter 2

Nullstellensatz

Using Hyper-resolution principle and Robinson's theorem it is possible to prove the Nullstellensatz as stated here:

Let K be a field and $f, f_1, \cdots, f_l \in K[\underline{x}]$, if in any extension of K, f vanishes at all common zeros of $f_1, \cdots, f_l$, then there exists $p \in \mathbb{N}$ and $h_1, \cdots, h_l \in K[\underline{x}]$ such that

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

To this aim, consider the first order language $\mathscr{L} = \{+, \cdot, = 0\} \cup K$ where $+, \cdot$ are binary function symbols, $= 0$ is a unary predicate symbol and the elements of K are constants; so the terms are polynomials over K and the atomic formulae are algebraic equations.

Consider the theory given by the following axioms:

$$0 = 0 \tag{2.1}$$

$$\neg(1 = 0) \tag{2.2}$$

$$r_1 = 0 \rightarrow r_2 = 0 \qquad (r_1, r_2 \texttt{ are equal polynomials}) \tag{2.3}$$

$$(r = 0 \wedge s = 0) \rightarrow r + s = 0 \tag{2.4}$$

$$r = 0 \rightarrow r \cdot s = 0 \tag{2.5}$$

$$r \cdot s = 0 \rightarrow (r = 0 \vee s = 0) \tag{2.6}$$

One can prove the equality axioms, the axioms of integral domain and the diagram of K. It follows that the models of the theory are the integral domains that contains K. Lifschitz, instead of the axiom $\neg(1 = 0)$, uses the axioms $\neg(\alpha = 0)$ for each $\alpha \in K \setminus \{0\}$, however the one taken here is equivalent and simplifies the next arguments.

The calculus H_T consists of the axiom $\{0 = 0\}$ and of the rules of inference:

$$\frac{\Gamma \cup \{1 = 0\}}{\Gamma} \tag{2.7}$$

$$\frac{\Gamma \cup \{r_1 = 0\}}{\Gamma \cup \{r_2 = 0\}} \qquad r_1, r_2 \texttt{ are equal polynomials} \tag{2.8}$$

$$\frac{\Gamma \cup \{r = 0\}, \quad \Delta \cup \{s = 0\}}{\Gamma \cup \Delta \cup \{r + s = 0\}} \tag{2.9}$$

$$\frac{\Gamma \cup \{r = 0\}}{\Gamma \cup \{r \cdot s = 0\}} \tag{2.10}$$

$$\frac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}} \tag{2.11}$$

Note that the rules (2.7-2.10) do not increase the number of elements of the clauses, precisely for an application

$$\frac{\Gamma \quad \Delta}{\Sigma}\,(2.7\text{-}2.10)$$

we have $|\Sigma| \leq |\Gamma \cup \Delta|$; while for rule (2.11) we have $\Delta = \varnothing$ and $|\Sigma| = |\Gamma| + 1$.

For any derivation in H_T there is a tree with all the applications of the rule (2.11) at the bottom; this is ensured by the following.

Lemma 2.1. *Every derivation in H_T which contains non-trivial applications can be rearranged in such a way that any application of (2.11) is either the last one or it is followed by an application of (2.11).*

Proof: Suppose to have a non-trivial application of a rule (2.11) followed by a rule (x) different from (2.11):

$$\frac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}}\,2.11 \qquad \Delta}{\Sigma \cup \{r = 0, s = 0\}}\,x$$

Here Δ is the second premise in case (x) is the rule (2.9), otherwise it should be dropped. In the case that both $r = 0$ and $s = 0$ in the conclusion are different from A_1 shown in the scheme (1.1) of the rule (x), then $A_1 \in \Gamma$ and we can change the order as follow:

$$\frac{\dfrac{\Gamma \cup \{r \cdot s = 0\}, \qquad \Delta}{\Sigma \cup \{r \cdot s = 0\}}\,x}{\Sigma \cup \{r = 0, s = 0\}}\,2.11$$

Otherwise let $\{r = 0\}$ be A_1 of (x), depending on whether (x) is (2.7),(2.8) ,(2.9) or (2.10), change the derivation according to one of the schemes below

$$\dfrac{\dfrac{\Gamma \cup \{1 \cdot s = 0\}}{\Gamma \cup \{1 = 0, s = 0\}}\,2.11}{\Gamma \cup \{s = 0\}}\,2.7 \quad \rightsquigarrow \quad \dfrac{\Gamma \cup \{1 \cdot s = 0\}}{\Gamma \cup \{s = 0\}}\,2.8$$

$$\dfrac{\dfrac{\Gamma \cup \{r_1 \cdot s = 0\}}{\Gamma \cup \{r_1 = 0, s = 0\}}\,2.11}{\Gamma \cup \{r_2 = 0, s = 0\}}\,2.8 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \cup \{r_1 \cdot s = 0\}}{\Gamma \cup \{r_2 \cdot s = 0\}}\,2.8}{\Gamma \cup \{r_2 = 0, s = 0\}}\,2.11$$

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}}\,2.11 \qquad \Delta \cup \{t = 0\}}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}}\,2.9 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}, \quad \dfrac{\Delta \cup \{t = 0\}}{\Delta \cup \{t \cdot s = 0\}}\,2.10}{\Gamma \cup \Delta \cup \{r \cdot s + t \cdot s = 0\}}\,2.9}{\Gamma \cup \Delta \cup \{(r + t) \cdot s = 0\}}\,2.8}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}}\,2.11$$

$$\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{r = 0, s = 0\}}\,2.11}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,2.10 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup \{r \cdot s = 0\}}{\Gamma \cup \{(r \cdot s) \cdot t\}}\,2.10}{\Gamma \cup \{(r \cdot t) \cdot s = 0\}}\,2.8}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,2.11$$

By a series of application of this procedure, we obtained a derivation as required.

$\square$

Now we have all the tools to achieve the proof of nullstellentsatz:

Proof: Let K be a field and $f, f_1 \cdots f_l \in K[\underline{x}]$ such that in any extension of K, f vanishes at all common zeros of $f_1 \cdots f_l$. This amounts to require that in every model of (2.1-2.6) the following is valid:

$$\Gamma_1 \wedge \cdots \wedge \Gamma_l \to \Delta$$

where $\Gamma_i = \{f_i = 0\}$ and $\Delta = \{f = 0\}$.

Then by Robinson's theorem: $\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$ where either $\Delta' = \varnothing$ or $\Delta' =$

$\{f = 0\}$. In particular take a derivation without trivial applications and where all the applications of (2.11) are at the bottom, the existence of such tree is ensured by the lemma shown above. The tree is of the form:

$$\begin{array}{c} \dfrac{\Gamma_1, \quad ..., \quad \Gamma_l}{\Sigma}\,2.7\text{-}2.10 \\ \vdots \text{ only rules 2.11} \\ \Delta' \end{array} \tag{2.12}$$

Since all the Γ_i's are singletons, and since the rules (2.7-2.10) do not increase the size of the clauses, Σ is either a singleton or the empty set. Let's treat separately the cases of Δ'.

Case $\Delta' = \varnothing$

The only way to get $\varnothing$ out of the rules of inference is by applying (2.7), in this case rule (2.11) does not apply at all and the tree is of the form:

$$\dfrac{\dfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{1 = 0\}}\,2.8 \text{ - } 2.10}{\varnothing}\,2.7$$

1 is obtained by a series of applications of rules (2.8 - 2.10), therefore it is a linear combination of $f_1, \cdots, f_l$:

$$f^0 = 1 = \sum_{i=1}^{l} h_i \cdot f_i$$

Case $\Delta' = \{f = 0\}$

Suppose that rule (2.11) applies $p - 1$ times, (2.7) does not apply at all and the tree is of the form:

$$\dfrac{\dfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\Sigma}\,2.8 \text{ - } 2.10}{\{f = 0\}}\,(p-1) \text{ applications of } 2.11$$

Since (2.11) applies and Σ is a singleton, Σ must be of the form $\{t_1 \cdot r_1 = 0\}$.

Suppose that the A_1 of the i^{th} application is $r_i = t_{i+1} \cdot r_{i+1}$, then the tree is:

$$\cfrac{\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{t_1 \cdot r_1 = 0\}}\ 2.8 - 2.10}{\{t_1 = 0, t_2 \cdot r_2 = 0\}}\ 2.11}{\vdots}$$

$$\frac{\{t_1 = 0, \cdots t_{p-2} = 0, t_{p-1} \cdot r_{p-1} = 0\}}{\{t_1 = 0, \cdots, t_{p-1} = 0, r_{p-1} = 0\}}\ 2.11$$

Since the root of the tree must be the singleton $\Delta' = \{f = 0\}$, it is the case that for any i: $t_i = r_{p-1} = f$. Therefore $\Sigma = \{f^p = 0\}$ and we conclude

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

$\square$

Chapter 3

Applications to graph theory

In this chapter I will discuss applications of the resolution principle to graph theory, in particular we will face two different and equivalent ways of describing n-colourable graphs via resolution. The first approach relies on propositions of the kind "the vertex x has the same colour of the vertex y", this method will be used to give a constructive proof that a graph is 2-colourable if and only if it does not contain odd cycles. The second approach assigns to each colour a number and relies on proposition of the kind "the vertex x has a colour number smaller than the one of y", in other words this method introduces a partial order on the vertices and therefore it will introduce a stronger transitivity rule; this method will be used to prove Gallai-Hasse-Roy-Vitaver theorem, which gives a characterization of the chromatic number of a simple graph, by describing a partial order among vertices using directed paths and retrieving the n-colurable property by showing that such order can be used also for colours. Finally I will show another characterization of the chromatic number which generalizes the characterization of bipartite graphs and that may be used to state the four colour theorem.

3.1 Recalls of graph theory

In this section I will briefly recall some notions of graph theory, for the part on simple graphs I will adopt the notation and terminology of Diestel [8], while for directed graphs I will mainly refer to Chartrand-Zhang [9]. Some of the result shown here will be also shown in the next section using resolution methods, it might be interesting to compare the proofs and how different they are.

A simple graph G is a couple of sets (V, E) such that the elements of E are unordered couples of elements of V. The elements of V are called *vertices* and the elements of E are called *edges*. A vertex v is *incident* with an edge e if v is one of the two elements of e, the two vertices incidents an edge are called its *ends* and are said to be *adjacent*; an edge whose endpoints are x and y is denoted as xy. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

A *path* is a graph $P = (V, E)$ of the form:

$$V = \{x_0, x_1, ..., x_n\} \qquad E = \{x_0x_1, x_1x_2, ..x_{n-1}x_n\}$$

where the vertices are pairwise different, the length of the path is defined as the cardinality of E which is n; we often refer to a path by the natural sequence of the vertices, writing $P = x_0x_1...x_n$. A graph is *connected* if for any two distinct vertices, there is a path between them. A *cycle* is a path where x_n coincides with x_0; a cycle of length n will be briefly called n-cycle; if a n-cycle is a subgraph of G, we say that G has a n-cycle.

A connected graph which has no cycle is called *tree*, trees are characterized by the following theorem

Theorem 3.1 (characterization of trees)**.** *The following are equivalent:*

(1) T is a tree

(2) for any edge e of T, $T - e$ is not connected

(3) any two vertices of T are linked by a unique path

(4) T is maximally acyclic, i.e. for any two vertices u, v such that uv is not an edge of T, $T + uv$ has a cycle

Proof:

(1)⇒(2):

If uv is an edge of T such that $T - uv$ is still connected, there is a path from u to v that does not pass trough the edge uv, such path concatenated with uv makes a cycle in T.

(2)⇒(3):

If there are two distinct vertices u, v of T which are linked by two different path, there is an edge e which does not belong to both paths, then $T - e$ is still connected.

(3)⇒(1):

If it is connected and there is a cycle, for any two distinct vertices of the cycle there at least two path linking them.

(4)⇒(1):

If it's maximally acyclic it is also connected, indeed if the vertices u, v are not linked by a path, $T + uv$ would be an acyclic graph and T not maximally acyclic.

(3)⇒(4):

Let u, v be two distinct vertices of T, such that uv is not an edge of T, if u, v are connected by a path, there is a cycle in $T + uv$, hence T is maximally acyclic.

□

For any connected graph G, it is always possible to obtain a tree by only removing edges, this tree is called *spanning tree* and its existence is ensured by the following constructive proof.

Proposition 3.2 (Spanning Tree)**.** *Let $G = (V, E)$ be a connected graph, then there is a tree $T = (V, E')$ such that $E' \subseteq E$*

Proof: Consider the following procedure: If G has no cycle than it is a tree and we have obtained what we were seeking; otherwise let e be an edge of G that belongs to a cycle, then $G - e$ is still connected and we restart the procedure with $G - e$. The algorithm always ends since E is a finite set and it returns the required tree. □

A *k-colouring* of a graph $G = (V, E)$ is a map $c : V \to \{1, .., k\}$ such that $c(v) \neq c(w)$ whenever v and w are adjacent; a graph is *k-colourable* if it admits a k-colouring. The minimum positive integer k such that G is $k-$colurable is the *chromatic number* and is denoted by $\chi(G)$.

A graph $G = (V, E)$ is bipartite if V admits a partition in two subsets such that every edge has its ends in different subsets; clearly a graph is bipartite if and only if it is 2-colourable since we can assign a colour to each partition to obtain a 2-colouring and conversely whenever we have a 2-colouring we immediately obtain the partitions dividing vertices according to their colour.

Another important characterization of bipartite graph is the property of not having cycles of odd length.

Theorem 3.3. *A graph $G = (V, E)$ is bipartite if and only if it does not contain odd cycles.*

Proof: Suppose first that G is bipartite; then V can be partitioned into sets U and W, so every edge has an end in U and the other in W. Let $C = v_1 v_2 ... v_n v_1$

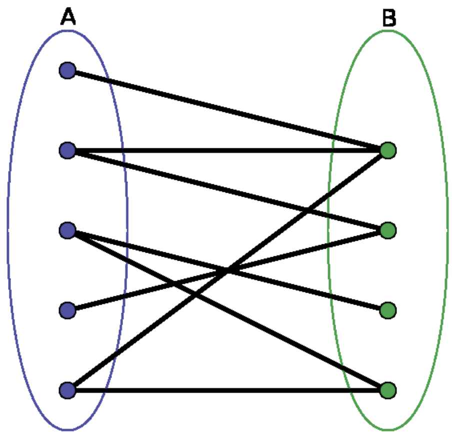


Figure 3.1: A bipartite graph coloured with two colours

be a $n-$cycle of G, if we assume $v_1 \in U$, then $v_2 \in W$, $v_3 \in U$ and so on. In particular $v_i \in U$ when i is odd and $v_i \in W$ when i is even, since $v_1 \in U$ it follows $v_n \in W$ and n is even.

Conversely, let G be a graph without odd cycles. A graph is bipartite if all its components are bipartite, so assume the G is connected. Let $T = (V, E')$ be a spanning tree of G and fix a vertex $r \in V$, for any $v \in V$ let $d(v)$ be the length of the unique path from r to v; now build the partitions U, W of V according to the following rule:

- if $d(v)$ is odd, $v \in U$
- if $d(v)$ is even, $v \in W$

Clearly $U \cap W = \varnothing$ and $U \cup W = V$, it remains to show that any $e \in E$ has its ends in different partitions. Let $xy \in E$, if it is the case that $xy \in E'$ then $d(x) = d(y) \pm 1$ and therefore they belong to different partitions. If $xy \notin E'$, then adding it to the spanning tree creates a cycle C which is even since it is also e cycle of G; thus $C - xy$ is the unique path in T from x to y and is of odd length, whence $d(x)$ and $d(y)$ have different parity. $\square$

Directed graphs

A *digraph* (or directed graph) D is a couple of sets (V, A) where the elements of A are ordered couples of V. Elements of V are still called vertices while elements of E are called *arcs* or *directed edges.* For instance a representation of the digraph $D = (\{x, y, z\}, \{(x, y), (y, x), (x, z), (z, y)\})$ is shown in fig. 3.2

If for each pair of distinct vertices u, v of a digraph D, at most one of (u, v) and (v, u) is an arc, D is an *oriented graph.* In fig. 3.2 D is not an oriented graph while D_1 and D_2 are good examples. Thus an oriented graph can be obtained from a simple graph G by assigning a direction to each one of its edges, the digraph obtained in this way is said to be an *orientation* of G; on the other hand starting from an oriented graph D we obtain the *underlying graph* by replacing all arcs (u, v) with an edge uv.

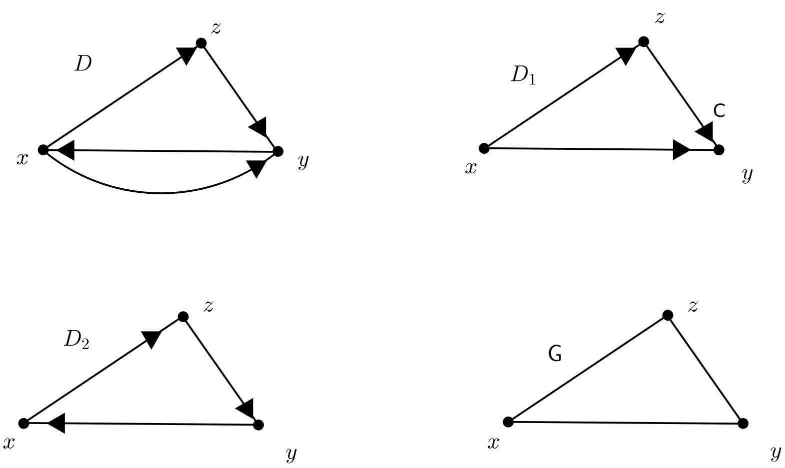


Figure 3.2: D is a generic digraph; D_1 is an acyclic oriented graph; D_2 is a directed cycle; G is the underlying graph of D, D_1, D_2

A sequence $x_0, ..., x_n$ of distinct vertices of a digraph D is a *(directed) path* if for any i (x_i, x_{i+1}) is an arc of D; a path in which x_n coincides with x_0 and is called *direct cycle*. Note that Fig. 3.2 shows that an acyclic oriented graph may have an underlying graph with a cycle. As simple graphs have spanning trees and they are easy to obtain, digraphs have spanning acyclic digraph, they also have a maximality property, i.e. whichever arc is added, it creates a directed cycle.

Lemma 3.4 (**Spanning acyclic digraph**). *Let $D = (V, A)$ be a connected digraph, then there is an acyclic oriented cycle T which is connected and obtained by D only by removing arcs.*

The orientations of a simple graph can be useful to find the chromatic number, a characterization of $\chi(G)$ was found independently by L.M. Vitaver [10](1962), M. Hasse [11](1963), B. Roy [12](1967), T. Gallai [13](1969); here I will show the proof given by Chartrand and Zhang in their book [9], and later I will show the method adopted by Matijasevič [5] which uses hyper-resolution.

Given a simple graph G and an orientation D of G, we define $\ell(D)$ as the length of the longest directed path in D.

Proposition 3.5. *There is an orientation D of $G = (V, E)$ such that*

$$\chi(G) \geq 1 + \ell(D)$$

We take a colouring map $c : V \to \{1, ..., \chi(G)\}$, we build an orientation D choosing a direction for each edge uv of G, in particular we pick the arc (u, v) if $c(u) < c(v)$. In such orientation, any path is long at most $\chi(G) - 1$ whence $\ell(D) \leq \chi(G) - 1$, that is $\chi(G) \geq 1 + \ell(D)$

□

Theorem 3.6 (The Gallai-Hasse-Roy-Vitaver theorem)**.** *For every orientation D of a graph $G = (V, E)$*

$$\chi(G) \leq 1 + \ell(D)$$

Let D be an orientation of G and D' a spanning acyclic subdigraph of D, we define a colouring map c on G by assigning to each vertex v the colour 1 plus the length of the longest directed path of D' that ends in v. Clearly the number of colours used is $1 + \ell(D)$, indeed the longest path of D' is also in D, otherwise D' would not be maximal.

Thus it remains to show that c is a proper colouring, i.e. that if $uv \in E$, $c(u) \neq c(v)$, to this aim consider an arc (u, v) of D, if it is also an arc of D' then $c(u) < c(v)$; otherwise if it is not in D', adding it creates a directed cycle which is the case only if there is a path from v to u, thus $c(v) < c(u)$

□

The following result characterizes the chromatic number of a simple graphs and is a direct consequence of the last propositions.

Corollary 3.7. *Let G be a graph and ℓ the minimum possible value of $\ell(D)$, D orientation of G, then*

$$\chi(G) = 1 + \ell$$

Proof:

Let D be an orientation of G such that

$$\ell(D) = \ell = \min\{\ell(D')\ , D' \text{ orientation of } G\}$$

then by Theorem 3.6 $\chi(G) \leq 1 + \ell$. On the other hand by Proposition 3.5 for some orientatin D' of G, $\chi(G) \geq 1 + \ell(D')$; therefore, since $\ell(D') \geq \ell$

$$\chi(G) = 1 + \ell$$

□

The figure 3.3 displays hot the Gallai-Hasse-Roy-Vitaver theorem applies to a cycle of length five, there are shown four different orientations of the cycle and to each vertex is assigned a colour based on how long is the longest directed path that reaches it. The rightmost orientation is not acyclic, every vertex is reached by a directed path of length five and they would all receive the same colour. The leftmost orientation is one that achieves the minimum $\ell(D)$ which in the case of odd cycles is 2, indeed the chromatic number for odd cycles is 3

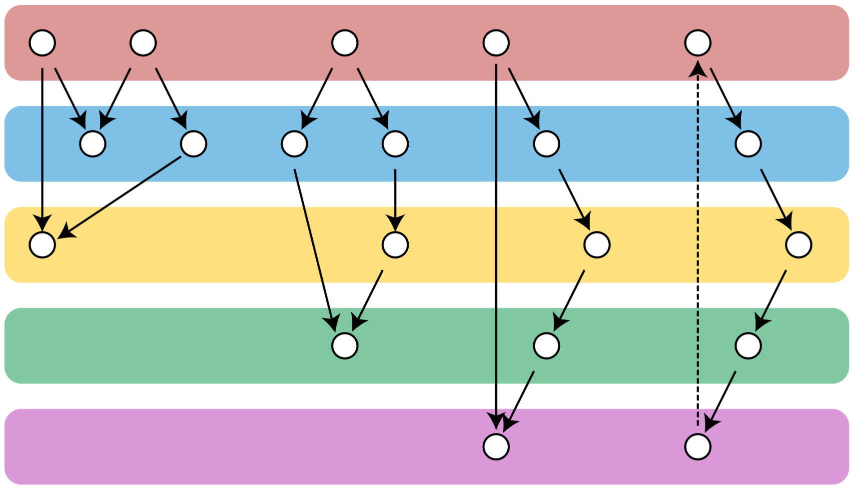


Figure 3.3

3.2 Resolution approach to colouring

Every colouring can be uniquely described (up to renaming the colours) by the symmetric binary relation ”vertices x and y have the same colour”, which we denote by $\mathscr{E}xy$. Such binary relation, in order to represent a colouring map, should respect a transitivity axiom, and be false whenever the two vertices are adjacent

$$\mathscr{E}xy \wedge \mathscr{E}yz \to \mathscr{E}xz \tag{3.1}$$

$$(xy \in E \Rightarrow) \qquad \neg\mathscr{E}xy \tag{3.2}$$

Also for an n-colouring we require that whenever we pick $n+1$ vertices at least 2 of them must share the colour, which amount to require for any $\{x_0, x_1, ..., x_n\} \subseteq V$

$$\bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^{n} \mathscr{E}x_i x_j \tag{3.3}$$

Conversely if a binary symbol satisfies the above axioms, then it corresponds to some colouring map. More formally, for every graph $G = (V, E)$, we consider a language $\mathscr{L} = \{\mathscr{E}\} \cup V \cup E$, where $\mathscr{E}$ is a binary relation symbols and $V \cup E$ are constants; the models of the formulae (3.1-3.3) are the n-colourable graphs that contain G, if G is not n-colurable the axioms are inconsistent and there are no models. The formulae (3.1-3.3) give rise to the calculus H_G which consists of the following inference rules:

$$\frac{\Gamma \cup \{\mathscr{E}xy\} \quad \Delta \cup \{\mathscr{E}yz\}}{\Gamma \cup \Delta \cup \{\mathscr{E}xz\}} \tag{3.4}$$

$$\frac{\Gamma \cup \{\mathscr{E}xy\}}{\Gamma} \quad xy \in E \tag{3.5}$$

$$\frac{\Gamma}{\Gamma \cup \{\mathscr{E}x_0x_1, ..., \mathscr{E}x_{n-1}x_n\}} \tag{3.6}$$

Note that any tree made with these rules have (3.6) as leaves; on the other hand, if there are no trivial applications, rule (3.6) appears only at leaves, indeed $\varnothing \in \Gamma$

for every non-empty Γ.

Besides axioms (3.1), (3.2), (3.3) it should be necessary to add the axiom $\mathscr{E}xx$, I preferred to keep it implicit in the definition of the proposition symbol $\mathscr{E}$ in order to keep the formalism slightly lighter.

Bipartite graphs

Proposition 3.8. *A graph $G = (V, E)$ that contains an odd cycle is not 2-colourable*

I will show that if there is an odd cycle the empty clause is derivable in H_G, i.e. $\vdash_{H_G} \varnothing$, and that therefore the propositions (3.1-3.3) are inconsistent.

Let $\{x_1, ..., x_{2n+1}\}$ be an odd cycle, by definition for $i = 1, 2, ..., 2n$ $x_ix_{i+1} \in E$ and $x_{2n+1}x_1 \in E$. Then consider the following deduction in H_G:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathscr{E}x_1x_2,\mathscr{E}x_2x_3,\mathscr{E}x_1x_3}{\mathscr{E}x_2x_3,\mathscr{E}x_1x_3}}{\mathscr{E}x_1x_3}\,(3.5) \qquad \cfrac{\cfrac{\mathscr{E}x_3x_5,\mathscr{E}x_4x_5,\mathscr{E}x_3x_5}{\mathscr{E}x_4x_5,\mathscr{E}x_3x_5}}{\mathscr{E}x_3x_5}\,(3.5)}{\mathscr{E}x_1x_5}\,(3.4)}{\begin{matrix}\vdots\\ \mathscr{E}x_1x_{2n-1}\end{matrix}} \qquad \cfrac{\cfrac{\mathscr{E}x_{2n-1}x_{2n},\mathscr{E}x_{2n}x_{2n+1},\mathscr{E}x_{2n-1}x_{2n+1}}{\mathscr{E}x_{2n}x_{2n+1},\mathscr{E}x_{2n-1}x_{2n+1}}}{\mathscr{E}x_{2n-1}x_{2n+1}}\,(3.5)}{\cfrac{\mathscr{E}x_1x_{2n+1}}{\varnothing}\,(3.5)}\,(3.4)$$

The idea shown by this tree is that, if the graph is 2-colourable with a $(2n + 1)$-cycle and c is a 2-colouring map, whenever we pick three consecutive vertices x_i, x_{i+1}, x_{i+2} of the cycle, two of them must be of the same colour and since adjacent vertices have different colours, we have $c(x_i) = c(x_{i+2})$; whence $c(x_1) = c(x_3) = c(x_5) = ... = c(x_{2n+1})$ which is impossible since $x_1x_{2n+1} \in E$.

Before proving the opposite direction, we need the following result.

Lemma 3.9. *Every derivation in H_G can be rearranged in such a way that any application of (3.5) is either the last one or it is followed by an application of (3.5).*

Proof:

$$\dfrac{xy \in E\ \dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\}}{\Gamma \cup \{\mathscr{E}ab\}}\,3.5 \qquad \Delta \cup \{\mathscr{E}bc\}}{\Gamma \cup \Delta \cup \{\mathscr{E}ac\}}\,3.4 \quad \rightsquigarrow \quad xy \in E\ \dfrac{\dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\} \quad \Delta \cup \{\mathscr{E}bc\}}{\Gamma \cup \Delta \cup \{\mathscr{E}xy, \mathscr{E}ac\}}\,3.4}{\Gamma \cup \Delta \cup \{\mathscr{E}ac\}}\,3.5$$

Proposition 3.10. *If a graph $G = (V, E)$ is not 2-colourable, then it has an odd cycle*

If G is not 2-colourbale, also every graph that contains G is not 2-colourable; by Robinson's theorem and lemma 3.9, we know that $\vdash_{H_G} \varnothing$ and in particular there is a derivation without trivial application where all the applications of (3.5) are close to the root. Now we proceed by induction on the number k of application of (3.4). If $k = 0$ the tree is of the form:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\mathscr{E}xy, \mathscr{E}yz, \mathscr{E}xz\}}\,3.6}{\{\mathscr{E}xy, \mathscr{E}yz\}}\,xz \in E}{\{\mathscr{E}xy\}}\,yz \in E}{\varnothing}\,xy \in E$$

Then $xy, xz, yz \in E$ and the 3-cycle $\{x, y, z, x\}$ is a subgraph of G.

For $k > 0$ the tree is of the form:

$$\begin{array}{c} \varnothing \qquad\qquad \varnothing \\ \vdots \qquad\qquad\ \vdots \\ \dfrac{\Gamma \cup \{\mathscr{E}xy\} \quad \Delta \cup \{\mathscr{E}yz\}}{\Sigma \cup \{\mathscr{E}xz\}}\,3.4 \\ \vdots \\ \varnothing \end{array}$$

Whence $xz \in E$ and if $\mathscr{E}ab \in \Gamma \cup \Delta$ then $ab \in E$.

Consider $E' = E \cup \{xy, yz\}$, the graph $G' = (V, E')$ has G as subgraph, thus we have the following trees in H_Gz

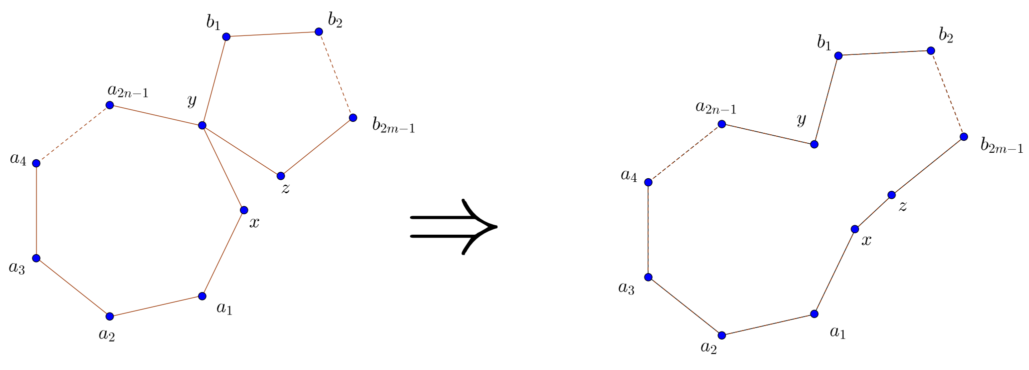
$$\begin{array}{c} \varnothing \\ \vdots \\ \dfrac{\Gamma \cup \{\mathscr{E}xy\}}{\Gamma}\, xy \in E' \\ \vdots\ \Gamma \subseteq \Sigma \\ \varnothing \end{array} \qquad \begin{array}{c} \varnothing \\ \vdots \\ \dfrac{\Delta \cup \{\mathscr{E}yz\}}{\Delta}\, yz \in E' \\ \vdots\ \Delta \subseteq \Sigma \\ \varnothing \end{array}$$

by induction hypothesis G' has two odd cycles C_1 and C_2.

If $y \notin C_1$ (or C_2), then C_2 (or C_1) is an odd cycle of G.

Thus let $y \in C_1 \cap C_2$, if $x \notin C_1$ (or $z \notin C_2$), then C_1 (or C_2) is an odd cycle of G; so consider $x \in C_1$ and $z \in C_2$, if $x \in C_2$ (or $z \in C_1$) then $\mathscr{E}xy \in \Delta$ (or $\mathscr{E}yz \in \Gamma$) and $xy \in E$ (or $yz \in E$) and C_1 (or C_2) is an odd cycle of G.

The last case to be considered is $y \in C_1 \cap C_2$, $x \in C_1 \backslash C_2$, $z \in C_2 \backslash C_1$; let $2n+1$ and $2m+1$ respectively the length of C_1 and C_2, if y is the only vertex in the intersection, by removing the edges xy, yz and adding xz we get a cycle in G of length $2(n+m)+1$.



It remains to consider when C_1 and C_2 intersect multiple times, let P and Q be the longest path respectively of C_1 and C_2 with $y \in P \cap Q$ and where the only vertices that belong to $C_1 \cap C_2$ are the endpoints of the paths. Then $P \cup Q$ is a cycle that pass trough the edges xy and yz, if the length of such cycle is even, by removing xy, yz and adding xz we get an odd cycle in G.

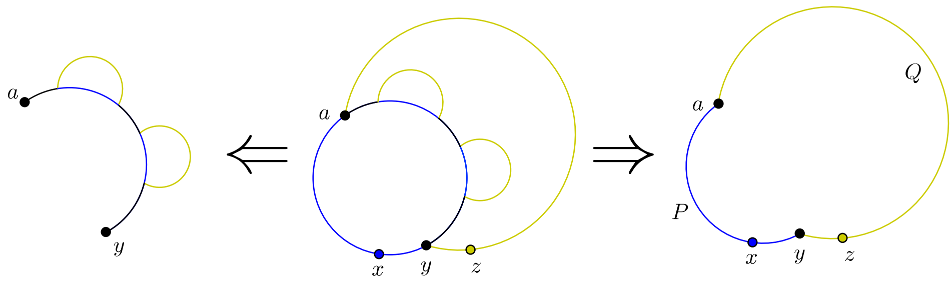


Figure 3.4: In blue $C_1 \backslash C_2$, in green $C_2 \backslash C_1$ and in black the intersection

If $P \cup Q$ is an odd cycle, consider the remaining part which namely consists of $C_1 \cap C_2$ and of $\tilde{C}_1, .., \tilde{C}_l$ cycles, then at at least one of them must be odd since $|P \cup Q|$ is odd and the following holds

$$2n + 1 + 2m + 1 - |P \cup Q| = \sum_{i=1}^{l} length(\tilde{C}_i) + 2 \cdot |C_1 \cap C_2|$$

Finally, since $xy \in P$ and $yz \in Q$, such odd cycle is also in G.

$\square$

Gallai-Hasse-Roy-Vitaver theorem

Another way to describe a proper colouring of a graph, is by introducing an order on the colours, indeed we may use the binary symbol $<$ and write $x < y$ to say "the vertex x is coloured in a colour smaller than the one of the vertex y". We shall write $x \geq y$ instead of $\neg(x < y)$. The role of the formulas (3.1), (3.2) and (3.3) is now played by

$$x \geq x \tag{3.7}$$

$$x < y \wedge y < z \rightarrow x < z \tag{3.8}$$

$$(xy \in E \Rightarrow) \qquad x < y \vee y < x \tag{3.9}$$

$$\bigvee_{i=0}^{n-1} x_i \geq x_{i+1} \tag{3.10}$$

In the formalism of hyper resolution, for a given graph $G = (V, E)$ we consider the language $\mathscr{L} = \{\geq\} \cup V \cup E$, where $V \cup E$ are constants; the models are the n-colourable graphs that contains G while the inference rules are

$$\frac{\Gamma}{\Gamma \cup \{x \geq x\}} \qquad\qquad \frac{\Gamma \cup \{x \geq y\} \quad \Sigma \cup \{y \geq z\}}{\Gamma \cup \Sigma \cup \{x \geq z\}}$$

$$xy \in E\,\frac{\Gamma \cup \{x \geq y\} \quad \Sigma \cup \{y \geq x\}}{\Gamma \cup \Sigma} \qquad\qquad \frac{\Gamma}{\Gamma \cup \{x_0 \geq x_1, ..., x_{n-1} \geq x_n\}}$$

The advantage of these rules respect the previous approach can be seen in the last rule, indeed now to describe a n-colouring it's enough to have an axiom that introduces $n-1$ literals while with (3.3) we must involve $n(n+1)/2$ vertices for each instance.

Now, fix a simple graph G and an orientation D of G, we want to describe the property of Vitaver theorem "The directed paths in D are not longer than

$\ell(D)$" where $\ell(D)$ is defined to be the length of the longest directed path in D, maintaining the same notation as in Theorem 3.6. To formalize such property we introduce the binary relation symbol $\mathrel{-\!\!\!<}$ which is defined on pairs of vertices and which indicates the existence of a directed path from one vertex to the other

$$x \mathrel{-\!\!\!<} y \Leftrightarrow \text{There is path from } x \text{ to } y$$

Since every arc can be seen as a path of length one, if there is an edge xy in the underlying graph $G = (V, E)$ of D, then it's necessary to have

$$xy \in E \Rightarrow x \mathrel{-\!\!\!<} y \vee y \mathrel{-\!\!\!<} x \tag{3.11}$$

If the end of the path is the start of another path, there exists a path which is the concatenation of the two:

$$x \mathrel{-\!\!\!<} y \wedge y \mathrel{-\!\!\!<} z \rightarrow x \mathrel{-\!\!\!<} z \tag{3.12}$$

Also, requiring that $\ell(D)$ is the maximum length of a path is equivalent to ask that there are no path of length equal to $n = \ell(D) + 1$, which amounts to ask for any set of $n + 1$ vertices

$$\neg(x_0 \mathrel{-\!\!\!<} x_1 \wedge x_1 \mathrel{-\!\!\!<} x_2 \wedge ... \wedge x_{n-1} \mathrel{-\!\!\!<} x_n)$$

and if we denote $\neg(x \mathrel{-\!\!\!<} y)$ as $x \mathrel{>\!\!\!=} y$

$$x_0 \mathrel{>\!\!\!=} x_1 \vee x_1 \mathrel{>\!\!\!=} x_2 \vee ... \vee x_{n-1} \mathrel{>\!\!\!=} x_n \tag{3.13}$$

Finally, we require D to be an acyclic orientation of G, which amounts to say that for any vertex x there is no path from x to x, that is:

$$x \mathrel{>\!\!\!=} x \tag{3.14}$$

Thus if we consider the language $\mathscr{L} = \{ \mathrel{>\!\!\!=} \} \cup V \cup E$, the models are the acyclic orientation of a subgraph of G and the rules of inference are

$$\frac{\Gamma}{\Gamma \cup \{x \succcurlyeq x\}} \qquad\qquad \frac{\Gamma \cup \{x \succcurlyeq y\} \quad \Sigma \cup \{y \succcurlyeq z\}}{\Gamma \cup \Sigma \cup \{x \succcurlyeq z\}}$$

$$xy \in E\,\frac{\Gamma \cup \{x \succcurlyeq y\} \quad \Sigma \cup \{y \succcurlyeq x\}}{\Gamma \cup \Sigma} \qquad\qquad \frac{\Gamma}{\Gamma \cup \{x_0 \succcurlyeq x_1, ..., x_{n-1} \succcurlyeq x_n\}}$$

Clearly the rules have the same shape of the ones given for $\geq$, we can therefore conclude that $\succcurlyeq$ describes an n-coloring for G where

$$n = 1 + \min\{\ell(D'), D' \text{ spanning acyclic digraph of an orientation of } G\}$$

If we consider an orientation D of G and a spanning acyclic digraph D' of D, we have $\ell(D') \leq \ell(D)$ whence

$$n \leq 1 + \ell$$

where ℓ is $\min\{\ell(D), D \text{ orientation of } G\}$ as in corollary 3.7.
Finally recalling that P ropositions 3.5 says that for some orientation D

$$\chi(G) \geq 1 + \ell(D)$$

we reach the same conclusion of Corollary 3.7, i.e.

$$\chi(G) = 1 + \ell$$

μ_n graphs

Now I will show a way to characterize n-colourable graphs proposed by Matiyasevich [5] and how it can be used to rephrase theorems such the four colour theorem. The μ_n-graphs are a family of graphs which is defined inductively, the Matiyasevich theorem says that a graph which is not n-colourable contains a μ_n-graph.

Definition 3.11. *$\mu_n - graph$*

- Every complete graph with $n + 1$ vertices is a μ_n-graph
- If $G = (V, E)$ and $G' = (V', E')$ are μ_n-graphs and a, b, c are distinct vertices such that

$$a, b \in V \quad b, c \in V'$$

and

$$ab \in E, ab \notin E' \qquad bc \in E', bc \notin E$$

Then the graph $G'' = (V \cup V', E \cup E' \cup \{ac\} \setminus \{ab, bc\})$ is a μ_n-graph.

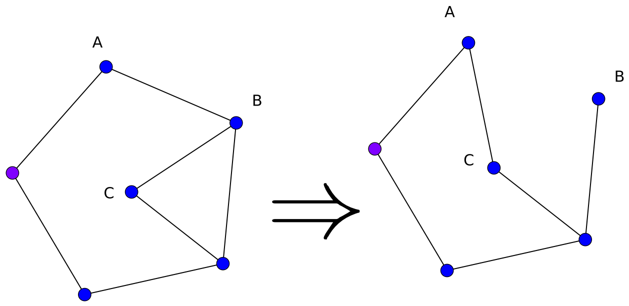
To understand better how such graphs look like, it is convenient to explore the μ_2 graphs

Proposition 3.12. *The odd cycles are μ_2 graphs*

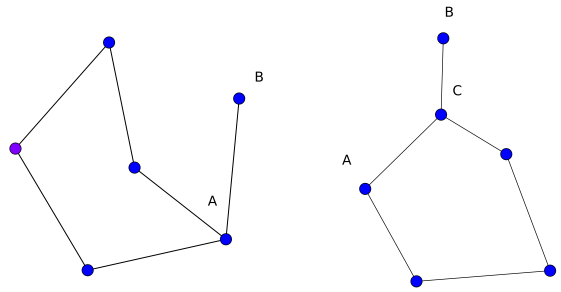
Proof:

Since the μ_2 graphs are defined by induction, the proof will proceed by induction on the number of vertices. First, the 3-cycle is the complete graph with 3 vertices then it is μ_2 by definition. Thus let $x_1 ... x_n$ be a cycle with n odd, the cycles $x_1x_2x_3$ and $x_3x_4...x_n$ are of odd length and therefore by induction they are μ_2, then the second point of the definition applies by taking $a = x_1$, $b = x_3$ and $c = x_n$, hence the original cycle is μ_2 $\square$

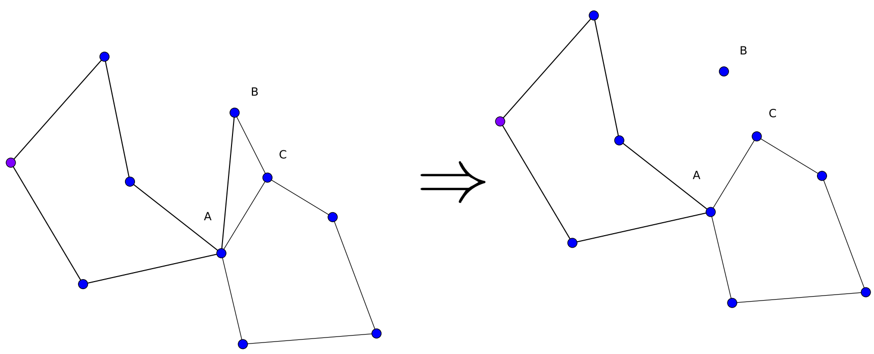
On the other hand, not all μ_2 graphs are odd cycles, the following example shows a μ_2 graph which is not a cycle and which can be built using the definition and starting by a 3-cycle and a 5-cycle



They do not even need to be connected, indeed starting from these two μ_2 graphs



we can apply the definition to make the following construction that produces a non-connected μ_2 graph



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