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Applications of hyper-resolution

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Introduction

In 1972 Matiyasevich published an article entitled *Application of the methods of the theory of logical derivation to graph theory* [9], in which the author successfully applies proof theoretic tools to graph theory, a field of mathematics that can appear quite far from logic. In that paper Matiyasevich gives an inductive definition of "a graph which cannot be colored with n colors" and uses it to formulate Hadwiger's conjecture and the four color theorem. Such formulations are said by Matiyasevich to be "useful in investigation of the above hypotheses" [9]; this may be one of the reasons that made him publish, in 1974 and 1975, two further papers [10, 11] on the argument. In those articles he generalizes his approach, and finds new applications such as Vitaver's theorem on coloring of directed graphs.

The main proof theoretic tool used by Matiyasevich is the hyper-resolution calculus, which was developed by Robinson in 1965 [1, 2, 3]. In the paper *A Machine-oriented logic based on the resolution principle* [1] Robinson defines the *resolution principle*, a single inference principle which provides, by itself, a complete system of propositional calculus. In the following papers [2, 3], he generalizes the resolution principle to hyper-resolution principle. The main advantage of the latter is that, while deduction trees in resolution calculus are binary, the trees in hyper-resolution calculus can branch more and deductions are more compact.

In 1980 Lifschitz, starting from Matiyasevich's work, applied hyper-resolution to algebra [5]; in particular he proved Hilbert's Nullstellensatz using Robinson's completeness theorem of hyper-resolution calculus. To achieve that proof, Lifschitz elaborated four equivalent versions of the completeness theorem; the last one is formulated in such a way that can be readily applied to many kinds of problems.

In this work we will see to several applications of the hyper-resolution calculus to fields of mathematics different from proof theory. In the first chapter, we will give a definition of the hyper-resolution calculus; since we are interested in applications, we will take in particular account Lifschitz's versions of the completeness theorem. The second chapter is dedicated to algebraic applications; the main goal is to see how the tools developed in chapter 1 can be applied to commutative algebra. To that aim, first we will see how it is possible to represent vector spaces and fields using hyper-resolutions, then we will use such knowledge to prove the Nullstellensatz.

In the last chapter we will see several applications to graph theory. We will focus on coloring problems, and we will see two different ways to describe a coloring map using hyper-resolution. We will move from Matiyasevich's work [9, 10, 11], and we will see his results using Lifschitz's versions of the completeness theorem. We prefer to use Lifschitz's approach since it is more direct and clearer than Matiyasevich's original one.

Chapter 1

Hyper-resolution

In 1965, Robinson presented a propositional logic which is specifically designed for use as theoretical instrument of a computer theorem-proving program [1]. A prominent feature of this logic system is that it is based on one single inference principle, the resolution principle. The formalism is based upon the notions of unsatisfiability and refutation rather than upon validity and proof. The resolution technique uses proof by contradiction, and is based on the fact that any formula in propositional logic can be transformed into an equivalent formula in conjunctive normal form. The main perk of resolution principle is its ability to avoid the major combinatorial obstacles in the theorem proving procedures [1]. The hyper-resolution principle is a natural generalization which was formulated by Robinson [2, 3] few months after the resolution principle. Starting from this principle and from Matiyasevich's work [11], Lifschitz [5] developed a way to apply the hyper-resolution principle to parts of mathematics other than proof theory.

1.1 Preliminaries

The aim of this section is to introduce the resolution calculus. First of all we introduce some elements which are necessary to define the trees of this calculus; secondly we state the resolution principle. We take for granted some basic notions such as terms, relation symbols and atomic formulae.

- A *literal:* is either an atomic formula, or has the form $\neg A$ where $\neg$ is the negation symbol and A is an atomic formula. The literals A, $\neg A$ are *complementary*, and each is the *complement* of the other. Literals will be denoted using Latin capital letters $A, B, C, ...$

- A *clause:* is a finite set of literals. Clauses are denoted by Greek capital letters $\Gamma, \Sigma, \Pi, ...$ The singleton $\{A\}$ is denoted by its element A, so that $\Gamma \cup A$ stands for $\Gamma \cup \{A\}$.

- if Γ is a clause, $\overline{\Gamma}$ is the disjunction of the elements of Γ in a fixed order.

- A *tree of clauses* is a tree to each node of which is attached a clause. The leafs are called *assumptions of the tree*, and the root *conclusion of the tree.*

- A set S of clause is *satisfiable* if there is a model of S; otherwise S is *unsatisfiable* [1].

Definition 1.1 (resolvent)**.** *Let Γ_1 and Γ_2 be two clauses, if there is a literal P such that $P \in \Gamma_1$ and $\neg P \in \Gamma_2$; then $\Pi = \Gamma_1 \cup \Gamma_2 \setminus \{P, \neg P\}$ is called the resolvent of Γ_1 and Γ_2.*

The ***resolution principle*** can be stated as follow: *If Π is unsatisfiable and is the resolvent of Γ_1 and Γ_2; then Γ_1 and Γ_2 are unsatisfiable.* A tree of clauses is called *resolution tree*, if it is binary, and each node is either a leaf or the resolvent of its two immediate predecessors.

Example 1.2.

$$\frac{A \quad \neg A}{\varnothing} \qquad \frac{\{A, P\} \quad \{B, \neg P\}}{\{A, B\}}$$

These two trees are resolution trees, both with only one application of the resolution principle.

Semantics

Here we briefly show some hints of semantics of hyper-resolution; Robinson treats semantics in [3], and we follow closely his approach.

Literals and terms become meaningful only when an *interpretation* is provided for the symbols in the language over which they are written; thus each literal and term acquires a *denotation*, that is to say *something which it denotes.* Literals always denote one or other of the two *truth values* $\textbf{true}, \textbf{false}$. Terms always denote some specific object in the *domain* of the interpretation.

Formally an interpretation is a map I of a *language* $\mathscr{L}$ onto a collection D of entities. Specifically I maps every function symbol of degree n in $\mathscr{L}$ onto a function from D^n to D, and each relation symbol of degree n onto a function from D^n to $\{\textbf{true}, \textbf{false}\}$.

The negation symbol $\neg$ denotes a function from truth values to truth values, which when applied to **true** returns **false** and conversely. The disjunction symbol $\bigvee$ denotes a function from a set of truth values Γ to truth values, defined as:

- $\bigvee \Gamma$ is **true** if there is $A \in \Gamma$ such that A is **true**
- $\bigvee \Gamma$ is **false** otherwise

Therefore $\bigvee \varnothing$ is **false**.

The clauses are intended to be read as disjunction of their elements. With this in mind, we see that the second tree of the Example 1.2 is a deduction of $A \vee B$ from $A \vee P$ and $B \vee \neg P$, which is a correct derivation also according to the rules of the natural deduction as shown by the following tree.

$$\cfrac{A \vee P \qquad \cfrac{[A]}{A \vee B} \qquad \cfrac{B \vee \neg P \qquad \cfrac{[B]}{A \vee B} \qquad \cfrac{\cfrac{[P] \quad [\neg P]}{\bot} \qquad \bot \to A \vee B}{A \vee B}}{A \vee B}}{A \vee B}$$

Also, we see that the first tree of 1.2 is a deduction of $\bigvee \varnothing$ from a literal A and is complement $\neg A$; in other words we deduce **false** if the literals are complementary.

We say that, if the empty set is the conclusion of some resolution tree, such tree is a *refutation* of its leaves; the soundness theorem ensures that if there is a refutation for a set of clauses S, such set is also unsatisfiable. Robinson's theorem [1] give us also the completeness.

Theorem 1.3 (Robinson). *If a set S of clauses is unsatisfiable, then there is a refutation of S.*

The resolution principle is generalized by the following hyper-resolution principle:

From $\Gamma_1 \cup A_1, \cdots \Gamma_m \cup A_m$ one may infer the resolvent $\Gamma_1 \cup \cdots \cup \Gamma_m \cup \Sigma$, whenever $A_1 \wedge \cdots \wedge A_m \to \overline{\Sigma}$

This inference principle gives rise to the hyper-resolution calculus, which is sound and complete [2, 3]. While the proof trees made with the resolution principle are binary, any node in a hyper-resolution tree can have any number of predecessors.

1.2 Lifschitz's versions

Matiyasevich was presumably the first to see that the hyper-resolution calculus and Robinson's theorem on its completeness can be successfully applied to discrete mathematics [11, 9]. Lifschitz applied Robinson's theorem to algebra [5]; in order to do so he defined the hyper-resolution calculus in a more convenient way, and he stated several versions of the completeness theorem which can be applied to many kinds of problems.

Let T be a set of propositional formulae, each of the form:

$$\bigwedge_{i=1}^{m} A_i \to \bigvee_{j=1}^{n} B_j \tag{1.1}$$

Where the A_i's and B_j's are literals and $m + n > 0$. For $m = 0$, the formula is an axiom and will be denoted only by the disjunction $B_1 \vee \cdots \vee B_n$. While for $n = 0$, we interpret the empty disjunction as falsum, and we denote the formula by $\neg(A_1 \wedge \cdots \wedge A_m)$. Consider the following calculus H_T: the objects derivable are clauses and the rules of inference are:

$$\frac{\Gamma_1 \cup A_1, \quad \cdots, \quad \Gamma_m \cup A_m}{\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{B_1, \cdots, B_n\}} \tag{1.2}$$

for every element of T. We write a derivation in H_T in tree form; the leaves of the tree are called *assumptions*, and the root *conclusion*.

Example 1.4.

Let $T = \{A \wedge B \to C \vee D,\ A,\ B,\ \neg C\}$, then H_T has four rules:

$$\frac{\Gamma_1 \cup A \quad \Gamma_2 \cup B}{\Gamma_1 \cup \Gamma_2 \cup \{C, D\}} \qquad \frac{\Gamma}{\Gamma \cup A} \qquad \frac{\Gamma}{\Gamma \cup B} \qquad \frac{\Gamma \cup C}{\Gamma}$$

Using such rules we get the tree:

$$\cfrac{\cfrac{\cfrac{\varnothing}{A} \quad \cfrac{\varnothing}{B}}{\{C, D\}}}{D}$$

and we write $\varnothing \vdash_{H_T} D$. When the only assumption is the empty set, as in the example, it can be dropped to write $\vdash_{H_T} D$.

Now Robinson's theorem 1.3 can be state as follows

Theorem 1.5 (Robinson's, version 1)**.** *If T is contradictory, then* $\vdash_{H_T} \varnothing$

Starting from this version, Lifschitz develops three more equivalent versions, the first one is the following

Theorem 1.6 (Robinson's, version 2)**.** *For any clauses* $\Gamma_1, \cdots, \Gamma_l, \Delta$
if $\Gamma_1, \cdots, \Gamma_l \vdash \Delta$, *then* $\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$ *for some* $\Delta' \subseteq \Delta$

Proof: Let $\Delta = \{C_1, \cdots, C_m\}$, and $T' = T \cup \{\overline{\Gamma}_1, \cdots, \overline{\Gamma}_l, \neg C_1, \cdots, \neg C_m\}$; then T' is contradictory. The calculus $H_{T'}$ consists of the rules of H_T, the axioms $\Gamma_1, \cdots \Gamma_l$ and the rules

$$\frac{\Gamma \cup C_i}{\Gamma} \tag{1.3}$$

By Robinson's theorem, there exists a derivation of $\varnothing$ in $H_{T'}$. If none of the rules (1.3) appear in such derivation, then $\Gamma_1, \cdots \Gamma_l \vdash_{H_T} \Delta'$ with $\Delta' = \varnothing$. Otherwise, consider an application of (1.3), replace every clause Σ that appears below it with

$\Sigma \cup C_i$; the result is still a derivation in $H_{T'}$, and now the conclusion also has the literal C_i. Repeat this procedures until all applications of (1.3) are eliminated, the conclusion is some subset Δ' of $\{C_1, \cdots, C_m\}$; the final tree uses only rules of H_T or some of the axioms Γ_i, then $\Gamma_1, \cdots \Gamma_l \vdash_{H_T} \Delta'$ $\square$

Example 1.7. *Let $T = \{A, \neg A\}$. Then for any B, $T \vdash A \to B$. By the second version of Robinson's theorem, either $A \vdash_{H_T} B$ or $A \vdash_{H_T} \varnothing$.*
Now the calculus H_T has two rules:

$$\frac{\Gamma}{\Gamma \cup A} \qquad\qquad \frac{\Gamma \cup A}{\Gamma}$$

Using this two rules, the clause B is not derivable from A; however using the second rule we get $A \vdash_{H_T} \varnothing$.

In the next chapters we deal with applications of hyper-resolution principle which are in first order logic, while formulae of the kind 1.1 are quantifier free. For this reason, we consider a third version of Robinson's theorem.

Theorem 1.8 (Robinson's , version 3)**.** *For any clauses $\Gamma_1, \cdots, \Gamma_l, \Delta$, if in every model of T, the following is valid*

$$\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l \to \overline{\Delta} \tag{1.4}$$

then $\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$, for some $\Delta' \subseteq \Delta$.

Proof: If 1.4 is valid in every model of T, since all the formulae in T and 1.4 are quantifier free; by completeness of the propositional calculus $T \vdash (\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l) \to \overline{\Delta}$. Then by the second version of Robinson's theorem, there is $\Delta' \subseteq \Delta$ such that $\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$ $\square$

Trivial applications

Given a rule in the form (1.2) and an application of it in a derivation, whenever $A_i \in \Gamma_i$ for some i, we call such application *trivial* according to Lifschitz's terminology. The following two lemmas ensure that such kind of applications can always be removed.

Lemma 1.9. *Let Π be obtained from $\Sigma_1, ..., \Sigma_m$ by one application of a rule H_T. For any $\Sigma'_1 \subseteq \Sigma_1, ..., \Sigma'_m \subseteq \Sigma_m$, one of the two following holds:*
(a) $\Sigma'_i \subseteq \Pi$ for some i
(b) some $\Pi' \subseteq \Pi$ can be obtained from $\Sigma'_1, ..., \Sigma'_m$ by an application of the same rule

Proof: Consider an application of (1.2) leading from $\Sigma_1, ..., \Sigma_m$ to Π, for this application:

$$\Sigma_1 = \Gamma_1 \cup A_1, ..., \Gamma_m \cup A_m$$

$$\Pi = \Gamma_1 \cup ... \cup \Gamma_m \cup \{B_1, ..., B_n\}$$

Consider the following two cases:
(a) $A_i \notin \Sigma'_i$ form some i. Then $\Sigma'_i \subseteq \Gamma_i \subseteq \Pi$
(b) $A_i \in \Sigma'_i$ for every i. Then $\Sigma'_i = \Gamma'_i \cup A_i$ where $\Gamma'_i = \Sigma'_i \backslash A_i$, one application of (1.2) to $\Sigma'_1, ..., \Sigma'_m$ gives $\Pi' = \Gamma'_1 \cup ... \cup \Gamma'_m \cup \{B_1, ..., B_n\} \subseteq \Pi$ $\square$

Lemma 1.10. *For every derivation of a clause Π in H_T there exists a derivation of some $\Pi' \subset \Pi$ in H_T without trivial application of rules of inference.*

Proof: Consider a derivation as follow

$$\cfrac{\Sigma_1 \quad \cdots \quad \cfrac{\begin{matrix}\mathscr{D}_1 & & \mathscr{D}_m \\ \vdots & & \vdots \\ \Gamma_1 \cup A_1 & \cdots & \Gamma_m \cup A_m\end{matrix}}{\Sigma_k = \Gamma_1 \cup \cdots \cup \Gamma_m \cup \{B_1, \cdots, B_m\}}\,(1) \quad \cdots \quad \Sigma_l}{\Pi}\,(2)$$

Suppose that (1) is trivial, say $A_1 \in \Gamma_1$. Then $\mathscr{D}_1$ is a derivation of Γ_1 and we can consider the following tree

$$\cfrac{\Sigma_1 \quad \cdots \quad \begin{array}{c}\mathscr{D}_1\\ \vdots \\ \Gamma_1\end{array} \quad \cdots \quad \Sigma_l}{\Pi}\ (2')$$

Now $(2')$ is not necessarily an application of a rule of inference. However, we can apply 1.9 with $\Sigma'_1 = \Sigma_1, \cdots, \Sigma'_k = \Gamma_1 \subseteq \Sigma_k \cdots \Sigma'_l = \Sigma_l$ to get either:

(a) $\Sigma'_i \subseteq \Pi$ for some i

(b) some $\Pi' \subseteq \Pi$ can be obtained from $\Sigma'_1, ..., \Sigma'_m$ by an application of (2).

If it is the case of (a), the application $(2')$ is still trivial and we repeat the procedure. We keep repeating as required, the procedure terminates since at every step at least one literal is removed. $\square$

This last lemma and Theorem 1.8 let us give a final version of Robinson's theorem.

Theorem 1.11 (Robinson's, version 4)**.** *For any clauses $\Gamma_1, \cdots, \Gamma_l, \Delta$, if in every model of T, the following is valid*

$$\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l \rightarrow \overline{\Delta}$$

then there exists a derivation of some $\Delta' \subseteq \Delta$ from $\Gamma_1, \cdots, \Gamma_l$ in H_T, containing no trivial application of rules of inference.

Chapter 2

Applications to commutative algebra

In this chapter we show how it is possible to apply the hyper-resolution principle to commutative algebra; in particular we see how two algebraic theorems follow directly from Robinson's Theorem. Before starting with such applications, the first section briefly recalls few notions of algebra in order to fix notations and terminology, the main reference for this part is Lang's book *Algebra* [7]. The second section presents actual applications of the hyper-resolution principle; first a well known fact of linear algebra is proved, here we speak about vector spaces over a field, so the only operations required are addition and multiplication by a scalar. Second a proof of Hilbert's Nullstellensatz is presented; to do so we also need to represent the products between polynomials, so more axioms are required and the resolution calculus becomes richer.

The Nullstellensatz was first proved by Hilbert [6] in 1892 , and is widely known to be a fundamental theorem for algebraic geometry. Lang, in his book, shows Rabinowitsch's proof (1929) which is so terse that the original article [8] consists only of one page. In the second section we see the proof proposed by Lifschitz [5] in 1980, this proof relies on Robinson's theorem.

2.1 Basic notions of commutative algebra

The aim of this section is to summarize few elements of algebra in order to show Rabinowitsch's proof of Hilbert's Nullstellensatz. All the notations and the proofs adopted here are from Lang's book [7]. Many basic definitions, such as fields, polynomials and ideals, are not given here; indeed the main purpose is to show a classical proof of the Nullstellensatz that can be compared with Lifschitz's one. For the same reason, many results and theorems are taken for granted, and some proofs are not given.

For a field K, the ring of the polynomials with coefficients in K and n variables is denoted as $K[x_1, ..., x_n]$. We often use vectorial notation, letting x denote the n-tuple $(x_1, ..., x_n)$ and $K[\mathbf{x}] = K[x_1, ..., x_n]$.

If K is a field and E an extension of it, we say that an element x of E is *algebraic* over K if there are $a_0, ..., a_n \in K$ not all zero, such that

$$a_0 + a_1 \cdot \alpha + ... + a_n \cdot \alpha^n = 0$$

which amounts to require that α is a root of some non-zero polynomial with coefficients in K. The extension E is called *algebraic* if every element of E is algebraic over K. Also, for $\alpha_1, ..., \alpha_n$, we denote as $K(\alpha_1, ..., \alpha_n)$ the smallest subfield of E that contains $\alpha_1, ..., \alpha_n$. The field E is said to be *finitely generated* if for some $\alpha_1, ..., \alpha_n$, we have that $E = K(\alpha_1, ..., \alpha_n)$. The following lemma gives a property about rings of polynomials that is used later.

Lemma 2.1. *Let K be a field and $K[\mathbf{x}]$ a finitely generated extension of it. If $K[\mathbf{x}]$ is a field then it is also algebraic over K*

A field K is called *algebraically closed* if every non constant polynomial in $K[\mathbf{x}]$ has a root in K. It is known that for any field K, there is an extension which is algebraically closed, algebraic over K and unique up to isomorphism; such extension will be called *algebraic closure of* K and denoted as K^a. The algebraic closure is the maximal extension, indeed a field is algebraically closed if and only if it does not have proper algebraic extensions.

Theorem 2.2. *Let K be a field and I an ideal in $K[x_1, ..., x_n]$; then either $I = K[\mathbf{x}]$ or I has a zero in K^a*

Proof: Suppose $I \neq K[\mathbf{x}]$. Then there is some maximal ideal M that contains I, and $K[\mathbf{x}]/M$ is a field. Such field is a finitely generated extension of K, because it is generated by $x_1, ...x_n$ mod M. By Lemma 2.1 the field $K[\mathbf{x}]/M$ is algebraic over K and therefore can be embedded into K^a by some map $e : K[\mathbf{x}]/M \to K^a$. Now consider the canonical map mod M $\pi : K[\mathbf{x}] \to K[\mathbf{x}]/M$, then the homomorphism $h = e \circ \pi : K[\mathbf{x}] \to K^a$ gives the desired zero of I. □

Theorem 2.2 implies that if a family of polynomials have a common zero in some field, then they have a common zero in the algebraic closure of K.

Theorem 2.3 (Hilbert's Nullstellensatz)**.** *Let K be a field and I an ideal in $K[\mathbf{x}]$, let f be a polynomial of $K[\mathbf{x}]$ such that $f(c) = 0$ for every zero of I in K^a. Then there exists a natural number p such that $f^p \in I$*

Proof: If $f = 0$, clearly $f \in I$ since any ideal is a subgroup with respect to the addition; therefore Let $f \neq 0$. We introduce a new variable y and consider the ideal I' generated by I and $1 - y \cdot f$. By Proposition 2.2 and the current assumptions, the ideal I' must be the whole polynomial ring $K[\mathbf{x}, \mathbf{y}]$. In particular, 1 is in I' and so for some $g_0, \cdots, g_l \in K[\mathbf{x}, \mathbf{y}]$ and $f_1, .., f_l \in I$

$$1 = g_0 \cdot (1 - \mathbf{y} \cdot f) + g_1 \cdot f_1 + ... + g_l \cdot f_l$$

Now let p be the max degree of the polynomials $g_1, ...g_l$ for the variable y. Then substitute y for $1/f$, and multiply both sides by f^p to clear all the denominators on the right-hand side. So by setting $h_i = f^p \cdot g_i$, we get

$$f^p = h_1 \cdot f_1 + ... + h_l \cdot f_l$$

that means $f^p \in I$ □

Later we will see a proof of this theorem that uses resolutions methods, the formulation will be slightly different as follows:

Let K be a field and $f, f_1, \cdots, f_l \in K[\mathbf{x}]$, if in any algebraic extension of K, f vanishes at all common zeros of $f_1, \cdots, f_l$, then there exists $p \in \mathbb{N}$ and $h_1, \cdots, h_l \in K[\mathbf{x}]$ such that

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

The premise of this version is equivalent of the one stated in 2.3, that says

Let f be a polynomial of $K[\mathbf{x}]$ such that $f(c) = 0$ for every zero of I in K^a.
Indeed if f vanishes at all common zeros of $f_1, \cdots, f_l$, then f belongs to the ideal I generated by the polynomials f_i. Conversely if f vanishes for every zero of I, this is true in particular for some generators $f_1, ..., f_l$ of I. Here we have used the Hilbert's basis theorem to get a finite set of generators of I. Also if this happens in all extensions, this is true in particular for K^a; and if it is true for all the zeros of K^a, then it also holds for every zeros of every extensions.

2.2 Applications

Now two concrete cases are presented; both are described by Lifschitz [5] with the purpose of establishing connections between proof theory and algebra. Besides that, the following examples show how it is possible to encode algebraic items, such as vector spaces and polynomials, into a hyper-resolution calculus. Once the calculus is properly defined, the problem becomes mainly proof theoretic, and we can use Robinson's theorem.

Linear equations

We want to give a hyper-resolution interpretation to the vector spaces over a given field K; consider the language with the following symbols
- for any $\alpha \in K$, a unary function symbol $\alpha\cdot$ for multiplication by the scalar α
- the binary function symbol $+$
- the unary relation symbol $=0$
Terms of this language are the linear forms of K, and atomic formulae are homogeneous linear equations. For any terms r, s, $r = s$ stands for $r - s = 0$. Now consider the following axioms

$$0 \cdot r = 0 \tag{2.1}$$

$$r_1 = 0 \to r_2 = 0 \qquad (r_1, r_2 \text{ represent the same linear form}) \tag{2.2}$$

$$(r = 0 \wedge s = 0) \to r + s = 0 \tag{2.3}$$

$$r = 0 \to \alpha \cdot r = 0 \tag{2.4}$$

These axioms imply all equality axioms and axioms of vector spaces over K. The models of this theory, factorized with respect to $x = y$, are precisely the vector spaces over K. We will refer to the set of all possible instances of these axioms as the set L, which gives rise to the calculus H_L.

The inference rules of H_L are

$$\frac{\Gamma}{\Gamma \cup 0 \cdot r = 0} \tag{2.5}$$

$$\frac{\Gamma \cup r_1 = 0}{\Gamma \cup r_2 = 0} \tag{2.6}$$

$$\frac{\Gamma \cup r = 0 \quad \Delta \cup s = 0}{\Gamma \cup \Delta \cup r + s = 0} \tag{2.7}$$

$$\frac{\Gamma \cup r = 0}{\Gamma \cup \alpha \cdot r = 0} \tag{2.8}$$

The following two lemmas give us information about the behavior of trees in H_L

Lemma 2.4. *The clause $\varnothing$ cannot be derived in H_L*

Proof: Rules (2.5), (2.6), (2.8) do not reduce the number of literals involved and therefore cannot be used to deduce $\varnothing$. Rule (2.7) reduces the number of literals by one, but it works with two non-empty premises and therefore $\varnothing$ cannot be inferred using it. □

Lemma 2.5. *Consider a derivation $\Gamma_1, ..., \Gamma_l \vdash_{H_L} \Delta$ without trivial applications. If Γ_i is a singleton for $i = 1, .., l$, then all the nodes of the derivation are singletons*

Proof: If the derivation has no trivial applications, the rule (2.5) can be introduced only from $\varnothing$ which is not one of the premises $\Gamma_1, ..., \Gamma_l$; therefore (2.5) does not appear. All the other rules transform singletons into a singleton, since all the premises are singletons all the other nodes of the derivations shall be singletons. □

Proposition 2.6. *If every solution of a system of linear equations* $r_1 = 0, ..., r_l = 0$ *is a solution of a linear equation* $s = 0$*; then* s *is linear combination of* $r_1, ..., r_l$

Proof: Let Δ be the singleton clause $\{s = 0\}$, for each equation $r_i = 0$ let $\Gamma_i = \{r_i = 0\}$; then the hypothesis can be written as

$$\bigwedge_{i=1}^{l} \Gamma_i \to \Delta$$

Therefore, by Robinson's Theorem, there is a derivation in H_T of some $\Delta' \subseteq \Delta$ without trivial application. By Lemma 2.4 Δ' is not the empty set, and since Δ is a singleton , $\Delta' = \Delta$ and $\Gamma_1, ..., \Gamma_l \vdash_{H_L} \Delta$. Since all the premises are singletons, by Lemma 2.5 all the nodes are singletons. The literals in H_L are linear equations and when used as premises for rules (2.6), (2.7), (2.8), the conclusion is a linear combination of the premises. Hence s is linear combination of $r_1, ..., r_l$. $\square$

Nullstellensatz

We want to prove the Nullstellensatz as stated here:

Let K *be a field and* $f, f_1, \cdots, f_l \in K[\mathbf{x}]$*, if in any extension of* K*,* f *vanishes at all common zeros of* $f_1, \cdots, f_l$*, then there exists* $p \in \mathbb{N}$ *and* $h_1, \cdots, h_l \in K[\mathbf{x}]$ *such that*

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

We need to describe the extensions of a field and the polynomials over it. To this aim, consider the first order language $\mathscr{L} = \{+, \cdot, = 0\} \cup K$ where $+, \cdot$ are binary function symbols, $= 0$ is a unary relation symbol and the elements of K are constants; so the terms are polynomials over K and the atomic formulae are algebraic equations.

Now consider the theory given by the following axioms:

$$0 = 0 \tag{2.9}$$

$$\neg(1 = 0) \tag{2.10}$$

$$r_1 = 0 \to r_2 = 0 \qquad (r_1, r_2 \text{ are equal polynomials}) \tag{2.11}$$

$$(r = 0 \wedge s = 0) \to r + s = 0 \tag{2.12}$$

$$r = 0 \to r \cdot s = 0 \tag{2.13}$$

$$r \cdot s = 0 \to (r = 0 \vee s = 0) \tag{2.14}$$

One can prove the equality axioms, the axioms of integral domain and the diagram of K. It follows that the models of the theory are the integral domains that contain K. Lifschitz, instead of the axiom $\neg(1 = 0)$, uses the axioms $\neg(\alpha = 0)$ for each $\alpha \in K \setminus \{0\}$, however the one taken here is equivalent and simplifies the next arguments.

Let N be the set of all possible instances of the axioms; the calculus H_N consists of the axiom $\{0 = 0\}$ and of the rules of inference:

$$\frac{\Gamma \cup 1 = 0}{\Gamma} \tag{2.15}$$

$$\frac{\Gamma \cup r_1 = 0}{\Gamma \cup r_2 = 0} \qquad r_1, r_2 \text{ are equal polynomials} \tag{2.16}$$

$$\frac{\Gamma \cup r = 0\,, \quad \Delta \cup s = 0}{\Gamma \cup \Delta \cup r + s = 0} \tag{2.17}$$

$$\frac{\Gamma \cup r = 0}{\Gamma \cup r \cdot s = 0} \tag{2.18}$$

$$\frac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}} \tag{2.19}$$

Note that the rules (2.15-2.18) do not increase the number of elements of the clauses, precisely for an application

$$\frac{\Gamma \quad \Delta}{\Sigma}\,(2.15\text{-}2.18)$$

we have $|\Sigma| \leq |\Gamma \cup \Delta|$; while for rule (2.19) we have $\Delta = \varnothing$ and $|\Sigma| = |\Gamma| + 1$.

For any derivation in H_N there is a tree with all the applications of the rule (2.19) at the bottom; this is ensured by the following.

Lemma 2.7. *Every derivation in H_N which contains non-trivial applications can be rearranged in such a way that any application of (2.19) is either the last one or it is followed by an application of (2.19).*

Proof: Suppose to have a non-trivial application of a rule (2.19) followed by a rule (x) different from (2.19):

$$\frac{\dfrac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}\,,}\,(2.19) \qquad \Delta}{\Sigma \cup \{r = 0, s = 0\}}\,x$$

Here Δ is the second premise in case (x) is the rule (2.17), otherwise it should be dropped. In the case that both $r = 0$ and $s = 0$ in the conclusion are different from A_1 shown in the scheme (1.2) of the rule (x), then $A_1 \in \Gamma$ and we can change the order as follows:

$$\frac{\dfrac{\Gamma \cup r \cdot s = 0\,, \qquad \Delta}{\Sigma \cup r \cdot s = 0}\,x}{\Sigma \cup \{r = 0, s = 0\}}\,(2.19)$$

Otherwise let $\{r = 0\}$ be A_1 of (x), depending on whether (x) is (2.15),(2.16), (2.17) or (2.18), change the derivation according to one of the schemes below

$$\dfrac{\dfrac{\Gamma \cup 1 \cdot s = 0}{\Gamma \cup \{1 = 0, s = 0\}}\,(2.19)}{\Gamma \cup s = 0}\,(2.15) \quad \rightsquigarrow \quad \dfrac{\Gamma \cup 1 \cdot s = 0}{\Gamma \cup s = 0}\,(2.16)$$

$$\dfrac{\dfrac{\Gamma \cup r_1 \cdot s = 0}{\Gamma \cup \{r_1 = 0, s = 0\}}\,(2.19)}{\Gamma \cup \{r_2 = 0, s = 0\}}\,(2.16) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \cup r_1 \cdot s = 0}{\Gamma \cup r_2 \cdot s = 0}\,(2.16)}{\Gamma \cup \{r_2 = 0, s = 0\}}\,(2.19)$$

$$(2.17)\,\dfrac{(2.19)\,\dfrac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}} \quad \Delta \cup t = 0}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup r \cdot s = 0, \quad \dfrac{\Delta \cup t = 0}{\Delta \cup t \cdot s = 0}\,(2.18)}{\Gamma \cup \Delta \cup r \cdot s + t \cdot s = 0}\,(2.17)}{\Gamma \cup \Delta \cup (r + t) \cdot s = 0}\,(2.16)}{\Gamma \cup \Delta \cup \{r + t = 0, s = 0\}}\,(2.19)$$

$$\dfrac{\dfrac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}}\,(2.19)}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,(2.18) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \cup r \cdot s = 0}{\Gamma \cup (r \cdot s) \cdot t = 0}\,(2.18)}{\Gamma \cup (r \cdot t) \cdot s = 0}\,(2.16)}{\Gamma \cup \{r \cdot t = 0, s = 0\}}\,(2.19)$$

By a series of application of this procedure, we obtain a derivation as required.

□

Theorem 2.8 (Hilbert's Nullstellensatz). *Let K be a field and $f, f_1, \cdots, f_l \in K[\mathbf{x}]$, if in any extension of K, f vanishes at all common zeros of $f_1, \cdots, f_l$, then there exists $p \in \mathbb{N}$ and $h_1, \cdots, h_l \in K[\mathbf{x}]$ such that*

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

Proof: If f vanishes at all common zeros of $f_1 \cdots f_l$, in every model of the axioms 2.9-2.14 the following is valid:

$$\Gamma_1 \wedge \cdots \wedge \Gamma_l \rightarrow \Delta$$

where $\Gamma_i = \{f_i = 0\}$ and $\Delta = \{f = 0\}$.
By Robinson's theorem: $\Gamma_1, \cdots, \Gamma_l \vdash_{H_N} \Delta'$ where either $\Delta' = \varnothing$ or $\Delta' = \{f = 0\}$. In particular take a derivation without trivial applications and where all the applications of (2.19) are at the bottom, the possibility of building such tree is ensured by Lemma 2.7. The tree is of the form:

$$\dfrac{\Gamma_1, \quad ..., \quad \Gamma_l}{\Sigma}\,(2.15)\text{-}(2.18)$$
$$\vdots \;\text{only rules (2.19)}$$
$$\Delta'$$

Since all the Γ_i's are singletons, and since the rules (2.15)-(2.18) do not increase the size of the clauses, Σ is either a singleton or the empty set. Let's treat separately the cases of Δ'.

Case $\Delta' = \varnothing$

The only way to get $\varnothing$ out of the rules of inference is by applying (2.15), in this case rule (2.19) does not apply at all, and the tree is of the form:

$$\dfrac{\dfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{1 = 0\}}\,(2.16)\text{ - }(\text{ }2.18)}{\varnothing}\,(2.15)$$

1 is obtained by a series of applications of rules (2.16) - (2.18), and therefore is a linear combination of $f_1, \cdots, f_l$:

$$f^0 = 1 = \sum_{i=1}^{l} h_i \cdot f_i$$

Case $\Delta' = \{f = 0\}$

Suppose that rule (2.19) applies $p-1$ times, (2.15) does not apply at all and the tree is of the form:

$$\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\Sigma}\,(2.16) - (2.18)}{\{f = 0\}}\,(p-1) \text{ applications of } (2.19)$$

Since (2.19) applies and Σ is a singleton, Σ must be of the form $\{t_1 \cdot r_1 = 0\}$. Suppose that the A_1 of the i^{th} application is $r_i = t_{i+1} \cdot r_{i+1}$, then the tree is:

$$\begin{array}{c}
\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{t_1 \cdot r_1 = 0\}}\,(2.16) - (2.18)}{\{t_1 = 0, t_2 \cdot r_2 = 0\}}\,(2.19) \\
\vdots \\
\cfrac{\{t_1 = 0, \cdots t_{p-2} = 0, t_{p-1} \cdot r_{p-1} = 0\}}{\{t_1 = 0, \cdots, t_{p-1} = 0, r_{p-1} = 0\}}\,(2.19)
\end{array}$$

Since the root of the tree must be the singleton $\Delta' = \{f = 0\}$, it is the case that for any i: $t_i = r_{p-1} = f$. Therefore $\Sigma = \{f^p = 0\}$ and we conclude

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

because f^p is obtained by a series of applications of rules (2.16) - (2.18).

$\square$

Chapter 3

Applications to graph theory

In this chapter we discuss applications of the hyper-resolution principle to graph theory, in particular we face two different and equivalent ways of describing n-colorable graphs via resolution. The first approach relies on propositions of the kind "the vertex x has the same color of the vertex y", this method is used to give a proof that a graph is 2-colorable if and only if it does not contain odd cycles. The second approach assigns to each color a number, and relies on proposition of the kind "the vertex x has a color number smaller than the one of y", in other words this method introduces a partial order on the vertices and therefore it introduces a stronger transitivity rule. This method is used to prove the Gallai-Hasse-Roy-Vitaver theorem, which gives a characterization of the chromatic number of a simple graph, by describing a partial order among vertices using directed paths, and retrieving the n-colorable property by showing that such order can be used also for colors. Both methods are described by Matiyasevich in 1975 [11]; in this article Matiyasevich shows how it is possible to apply hyper-resolutions to discrete mathematics, and applies his method to coloring problems. Last, we describe the μ_n graphs, a family of graphs introduced in 1972 by Matiyasevich [9], in order to give an inductive definition of the property "a graph which cannot be colored with n colors". This definition lets us generalize the characterization of bipartite (2-colorable) graphs, and lets state the four color theorem and the Hadwiger conjecture in terms of μ_n graphs.

3.1 Basic notions of graph theory

In this section we briefly recall some notions of graph theory; for the part on simple graphs we adopt the notation and terminology of Diestel [12], while for directed graphs the main reference is Chartrand-Zhang [13]. Some of the results shown here are also shown in the next section using resolution methods, it might be interesting to compare the proofs and how different they are.

A simple graph G is a pair of sets (V, E) such that the elements of E are unordered pairs of elements of V. The elements of V are called *vertices* and the elements of E are called *edges*. A vertex v is *incident* with an edge e if v is one of the two elements of e, the two vertices incident to an edge are called its *ends* and are said to be *adjacent*; an edge whose endpoints are x and y is denoted as xy. A graph is called *complete* if all of its vertices are pairwise adjacent; a complete graph with n vertices is denoted with K^n. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

A *path* is a graph $P = (V, E)$ of the form:

$$V = \{x_0, x_1, ..., x_n\} \qquad E = \{x_0x_1, x_1x_2, ..x_{n-1}x_n\}$$

where the vertices are pairwise different, the length of the path is defined as the cardinality of E which is n; we often refer to a path by the natural sequence of the vertices, writing $P = x_0x_1...x_n$. A graph is *connected* if for any two distinct vertices, there is a path between them which is a subgraph of the given graph. A *cycle* is a path where x_n coincides with x_0; a cycle of length n will be briefly called n-cycle; if a n-cycle is a subgraph of G, we say that G has a n-cycle. A connected graph which has no cycle is called *tree*, trees are characterized by the following theorem

Theorem 3.1 (characterization of trees)**.** *The following are equivalent:*

(1) T is a tree

(2) for any edge e of T, $T - e$ is not connected

(3) any two vertices of T are linked by a unique path

(4) T is maximally acyclic, i.e. for any two vertices u, v such that uv is not an edge of T, $T + uv$ has a cycle

Proof:

(1)⇒(2):

If uv is an edge of T such that $T - uv$ is still connected, there is a path from u to v that does not pass trough the edge uv, such path concatenated with uv makes a cycle in T.

(2)⇒(3):

If there are two distinct vertices u, v of T which are linked by two different paths, there is an edge e which does not belong to both paths, then $T - e$ is still connected.

(3)⇒(1):

If it is connected and there is a cycle, for any two distinct vertices of the cycle there are at least two paths linking them.

(4)⇒(1):

If it's maximally acyclic it is also connected, indeed if the vertices u, v are not linked by a path, $T + uv$ would be an acyclic graph and T not maximally acyclic.

(3)$\Rightarrow$(4):

Let u, v be two distinct vertices of T, such that uv is not an edge of T, if u, v are connected by a path, there is a cycle in $T + uv$, hence T is maximally acyclic. $\square$

For any connected graph G, it is always possible to obtain a tree by only removing edges; this tree is called *spanning tree* and its existence is ensured by the following proof.

Proposition 3.2 (Spanning Tree). *Let $G = (V, E)$ be a connected graph; then there is a tree $T = (V, E')$ such that $E' \subseteq E$*

Proof: Consider the following procedure: If G has no cycle then it is a tree and we have obtained what we were seeking; otherwise let e be an edge of G that belongs to a cycle, then $G - e$ is still connected and we restart the procedure with $G - e$. The algorithm always ends since E is a finite set and it returns the required tree. $\square$

A *k-coloring* of a graph $G = (V, E)$ is a map $c : V \to \{1, .., k\}$ such that $c(v) \neq c(w)$ whenever v and w are adjacent; a graph is *k-colorable* if it admits a k-coloring. The minimum positive integer k such that G is $k-$colorable is the *chromatic number* and is denoted by $\chi(G)$.

A graph $G = (V, E)$ is *bipartite* if V admits a partition into two subsets such that every edge has its ends in different subsets; clearly a graph is bipartite if and only if it is 2-colorable since we can assign a color to each partition to obtain a 2-coloring and conversely whenever we have a 2-coloring we immediately obtain the partitions dividing vertices according to their color.

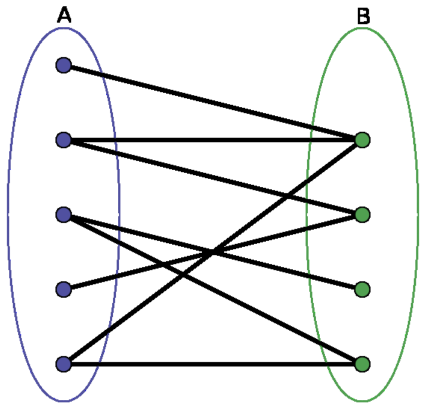


Figure 3.1: A bipartite graph colored with two colors

Another important characterization of bipartite graphs is the property of not having cycles of odd length.

Theorem 3.3. *A graph $G = (V, E)$ is bipartite if and only if it does not contain odd cycles.*

Proof: Suppose first that G is bipartite; then V can be partitioned into sets U and W, so every edge has an end in U and the other in W. Let $C = v_1v_2...v_nv_1$ be a $n-$cycle of G, if we assume $v_1 \in U$, then $v_2 \in W$, $v_3 \in U$ and so on. In particular $v_i \in U$ when i is odd and $v_i \in W$ when i is even, since $v_1 \in U$ it follows $v_n \in W$ and n is even.

Conversely, let G be a graph without odd cycles. A graph is bipartite if all its components are bipartite, so assume that G is connected. Let $T = (V, E')$ be a spanning tree of G and fix a vertex $r \in V$, for any $v \in V$ let $d(v)$ be the length of the unique path from r to v; now build the partitions U, W of V according to the following rule:

- if $d(v)$ is odd, $v \in U$
- if $d(v)$ is even, $v \in W$

Clearly $U \cap W = \varnothing$ and $U \cup W = V$, it remains to show that any $e \in E$ has its ends in different partitions. Let $xy \in E$, if it is the case that $xy \in E'$ then $d(x) = d(y) \pm 1$ and therefore they belong to different partitions. If $xy \notin E'$, then adding it to the spanning tree creates a cycle C which is even since it is also a cycle of G; thus $C - xy$ is the unique path in T from x to y and is of odd length, whence $d(x)$ and $d(y)$ have different parity. $\square$

Directed graphs

A *digraph* (or directed graph) D is a pair of sets (V, A) where the elements of A are ordered couples of V. Elements of V are still called vertices while elements of E are called *arcs* or *directed edges*. For instance a representation of the digraph $D = (\{x, y, z\}, \{(x, y), (y, x), (x, z), (z, y)\})$ is shown in Fig. 3.2

If for each pair of distinct vertices u, v of a digraph D, at most one of (u, v) and (v, u) is an arc, D is an *oriented graph*. In fig. 3.2 D is not an oriented graph while D_1 and D_2 are good examples. Thus an oriented graph can be obtained from a simple graph G by assigning a direction to each one of its edges, and the digraph obtained in this way is said to be an *orientation* of G. On the other hand starting from an oriented graph D we obtain the *underlying graph* by replacing all arcs (u, v) with an edge uv.

A sequence $x_0, ..., x_n$ of distinct vertices of a digraph D is a *(directed) path* if for any i (x_i, x_{i+1}) is an arc of D; a path in which x_n coincides with x_0 is called *directed cycle*. Note that Fig. 3.2 shows that an acyclic oriented graph may have an underlying graph with a cycle. As simple graphs have spanning trees and they are easy to obtain, digraphs have spanning acyclic digraphs; they also have a maximality property, i.e. whichever arc is added, it creates a directed cycle.

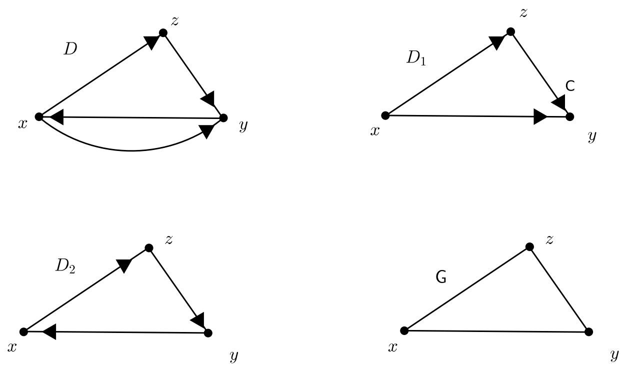


Figure 3.2: D is a generic digraph; D_1 is an acyclic oriented graph; D_2 is a directed cycle; G is the underlying graph of D, D_1, D_2

Lemma 3.4 (**Spanning acyclic digraph**). *Let $D = (V, A)$ be a connected digraph; then there is an acyclic oriented cycle T which is connected and obtained by D only by removing arcs.*

The orientations of a simple graph can be useful to find the chromatic number. A characterization of $\chi(G)$ was found independently by L.M. Vitaver [14](1962), M. Hasse [15](1963), B. Roy [16](1967), T. Gallai [17](1969); here I will show the proof given by Chartrand and Zhang in their book [13], and later I will show the method adopted by Matijasevič [11] which uses hyper-resolution.

Definition 3.5. *Let G be a simple graph and D an orientation of G; we define $\ell(D)$ as the length of a longest directed path of D*

Proposition 3.6. *There is an orientation D of $G = (V, E)$ such that*

$$\chi(G) \geq 1 + \ell(D)$$

Proof: We take a coloring map $c : V \to \{1, ..., \chi(G)\}$, we build an orientation D choosing a direction for each edge uv of G, in particular we pick the arc (u, v) if $c(u) < c(v)$. In such orientation, any path is long at most $\chi(G) - 1$ whence $\ell(D) \leq \chi(G) - 1$, that is $\chi(G) \geq 1 + \ell(D)$ $\square$

Theorem 3.7 (The Gallai-Hasse-Roy-Vitaver theorem)**.** *For every orientation D of a graph $G = (V, E)$*

$$\chi(G) \leq 1 + \ell(D)$$

Proof: Let D be an orientation of G and D' a spanning acyclic subdigraph of D, we define a coloring map c on G by assigning to each vertex v the color 1 plus the length of the longest directed path of D' that ends in v. Clearly the number of colors used is $1 + \ell(D)$, indeed the longest path of D' is also in D, otherwise D' would not be maximal.

Thus it remains to show that c is a proper coloring, i.e. that if $uv \in E$, $c(u) \neq c(v)$, to this aim consider an arc (u, v) of D, if it is also an arc of D' then $c(u) < c(v)$; otherwise if it is not in D', adding it creates a directed cycle which is the case only if there is a path from v to u, thus $c(v) < c(u)$ □

The following result characterizes the chromatic number of simple graphs and is a direct consequence of the last propositions.

Corollary 3.8. *Let G be a graph and ℓ the minimum possible value of $\ell(D)$, D orientation of G; then*

$$\chi(G) = 1 + \ell$$

Proof: Let D be an orientation of G such that

$$\ell(D) = \ell = \min\{\ell(D')\ , D' \text{ orientation of } G\}$$

then by Theorem 3.7 $\chi(G) \leq 1 + \ell$. On the other hand by Proposition 3.6 for some orientation D' of G, $\chi(G) \geq 1 + \ell(D')$; therefore, since $\ell(D') \geq \ell$

$$\chi(G) = 1 + \ell$$

□

The figure 3.3 displays how the Gallai-Hasse-Roy-Vitaver theorem applies to a cycle of length five, there are shown four different orientations of the cycle and to each vertex is assigned a color based on how long is the longest directed path that reaches it. The rightmost orientation is not acyclic, every vertex is reached by a directed path of length five and they would all receive the same color. The leftmost orientation is one that achieves the minimum $\ell(D)$, which in the case of odd cycles is 2, and the chromatic number of odd cycles is 3

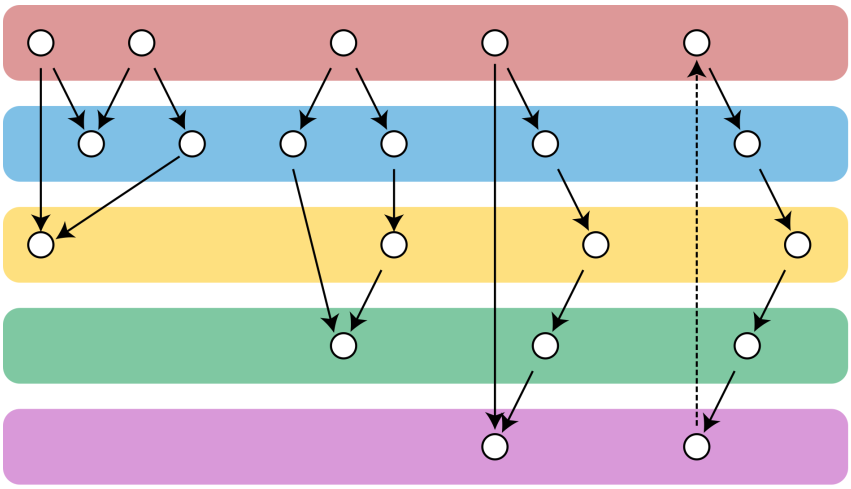


Figure 3.3

3.2 Hyper-resolution approach to coloring

Every coloring can be uniquely described (up to renaming the colors) by the symmetric binary relation "vertices x and y have the same color", which we denote by $\mathscr{E}xy$. Such binary relation, in order to represent a coloring map, should respect a symmetry axiom, a transitivity axiom, and be false whenever the two vertices are adjacent

$$\mathscr{E}xy \to \mathscr{E}yx \tag{3.1}$$

$$\mathscr{E}xy \wedge \mathscr{E}yz \to \mathscr{E}xz \tag{3.2}$$

$$(xy \in E \Rightarrow) \qquad \neg\mathscr{E}xy \tag{3.3}$$

Also for an n-coloring we require that whenever we pick $n+1$ vertices, at least 2 of them must share the color, which amounts to require for any $\{x_0, x_1, ..., x_n\} \subseteq V$

$$\bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^{n} \mathscr{E}x_i x_j \tag{3.4}$$

Conversely if a binary symbol satisfies the above axioms, then it corresponds to some coloring map. More formally, for every graph $G = (V, E)$, we consider the language $\mathscr{L} = \{\mathscr{E}\} \cup V \cup E$, where $\mathscr{E}$ is a binary relation symbol, and $V \cup E$ contains the constants. The models of the formulae (3.1)-(3.4) are the n-colorable graphs that contain G; and if G is not n-colorable, then the axioms are inconsistent and there are no models.

Definition 3.9. *Let G be a simple graph and n a positive natural number; then we call $C_{G,n}$ the set of all possible formulae of the form (3.2), (3.1), (3.3), and (3.4). The resolution calculus that comes from $C_{G,n}$ is denoted as $H_{G,n}$.*

The hyper-resolution calculus $H_{G,n}$ has the following rules

$$\frac{\Gamma \cup \mathscr{E}xy}{\Gamma \cup \mathscr{E}yx} \tag{3.5}$$

$$\frac{\Gamma \cup \mathscr{E}xy \quad \Delta \cup \mathscr{E}yz}{\Gamma \cup \Delta \cup \mathscr{E}xz} \tag{3.6}$$

$$\frac{\Gamma \cup \mathscr{E}xy}{\Gamma} \quad xy \in E \tag{3.7}$$

$$\frac{\Gamma}{\Gamma \cup \{\mathscr{E}x_0x_1, ..., \mathscr{E}x_{n-1}x_n\}} \tag{3.8}$$

Note that, since (3.8) is the only rule with possible empty premise, any tree made in $H_{G,n}$ has (3.8) as leaves. The following lemma ensures that every tree can be rearranged in such a way that the rules appear in a precise order: First the literals are introduced from $\varnothing$ by the rule (3.8); second all the symmetry rules (3.5) apply, followed by the transitivity rule (3.6); last literals are eventually removed by the elimination rule (3.7).

Lemma 3.10. *For any derivation in $H_{G,n}$, there is a tree that respects the following*

1) *(3.8) is preceded only by other applications of (3.8); also if the derivation has no trivial application, then (3.8) appears only at leaves.*

2) *(3.5) is preceded only by applications of (3.5) or (3.8)*

3) *(3.6) is not preceded by applications of (3.7)*

4) *(3.7) is followed only by applications of (3.7)*

Proof:

1) Suppose to have a rule x that precedes (3.8), switch them in the following way.

$$\cfrac{\cfrac{\Gamma \qquad \Delta}{\Sigma}\, x}{\Sigma \cup \{\mathscr{E}x_0x_1, ..., \mathscr{E}x_{n-1}x_n\}}\,(3.8) \quad \rightsquigarrow \quad \cfrac{\cfrac{\Gamma}{\Gamma \cup \{\mathscr{E}x_0x_1, ..., \mathscr{E}x_{n-1}x_n\}}\,(3.8) \qquad \Delta}{\Sigma \cup \{\mathscr{E}x_0x_1, ..., \mathscr{E}x_{n-1}x_n\}}\, x$$

Here Δ is the second premise in the case that x is (3.6), otherwise it should be dropped. Also, if the tree has no trivial application, (3.8) cannot be preceded by anything and it must be a leaf. Indeed, if we apply (3.8) to a non empty clause Γ, such application is trivial since $\varnothing \in \Gamma$.

2) Because of point 1, we can always achieve a tree where (3.8) is never applied below (3.5), it remains to check the case when (3.6) or (3.7) precedes (3.5).

$$\dfrac{\dfrac{\Gamma \cup \mathscr{E}ab \quad \Delta \cup \mathscr{E}bc}{\Gamma \cup \Delta \cup \mathscr{E}ac}\,(3.6)}{\Gamma \cup \Delta \cup \mathscr{E}ca}\,(3.5) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \cup \mathscr{E}bc}{\Delta \cup \mathscr{E}cb}\,(3.5) \quad \dfrac{\Gamma \cup \mathscr{E}ab}{\Gamma \cup \mathscr{E}ba}\,(3.5)}{\Delta \cup \Gamma \cup \mathscr{E}ca}\,(3.6)$$

$$\dfrac{xy \in E\,\dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\}}{\Gamma \cup \mathscr{E}ab}\,(3.7)}{\Gamma \cup \mathscr{E}ba}\,(3.5) \quad \rightsquigarrow \quad xy \in E\,\dfrac{\dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\}}{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ba\}}\,(3.5)}{\Gamma \cup \mathscr{E}ba}\,(3.7)$$

3) From point 1 and 2, we know that there is a tree in which the transitivity rule (3.6) cannot be followed by (3.8) nor (3.5). It remains to show how to rearrange it when (3.7) precedes (3.6)

$$\dfrac{xy \in E\,\dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\}}{\Gamma \cup \mathscr{E}ab}\,(3.7) \quad \Delta \cup \mathscr{E}bc}{\Gamma \cup \Delta \cup \mathscr{E}ac}\,(3.6) \quad \rightsquigarrow \quad xy \in E\,\dfrac{\dfrac{\Gamma \cup \{\mathscr{E}xy, \mathscr{E}ab\} \quad \Delta \cup \mathscr{E}bc}{\Gamma \cup \Delta \cup \{\mathscr{E}xy, \mathscr{E}ac\}}\,(3.6)}{\Gamma \cup \Delta \cup \mathscr{E}ac}\,(3.7)$$

4) If a derivation respects all the previous points, then the elimination rule (3.7) is either the last one or is followed only by other applications of itself. Indeed if another rule follows the elimination one, either the point 1, 2 or 3 does not hold.

$\square$

Bipartite graphs

Now we see, using resolution methods, that a graph is bipartite if and only if it does not contain odd cycles. First we show that any graph G with an odd cycle is not 2-colorable and therefore not bipartite; this can be done by directly showing that the empty set can be derived in $H_{G,2}$. Second, that whenever a graph G is not 2-colorable, there is an odd cycle in G. This second proof relies on Robinson's Theorem; given a non bipartite graph, it shows how to build an odd cycle which is a subgraph of it.

Proposition 3.11. *Let $G = (V, E)$ be a graph that contains an odd cycle; then G is not 2-colorable*

Proof: We show that if there is an odd cycle in G, then the empty clause is derivable in $H_{G,2}$, i.e. $\vdash_{H_{G,2}} \varnothing$, and that therefore the axioms 3.1 ,3.2, 3.3 and 3.4 are inconsistent.

Let $x_1, ..., x_{2n+1}$ be an odd cycle, by definition for $i = 1, 2, ..., 2n$ $x_i x_{i+1} \in E$ and $x_{2n+1}x_1 \in E$. For $i = 1, 2, ..., 2n-1$ consider the tree $\mathscr{D}_i$ in $H_{G,2}$:

$$\cfrac{x_{i+1}x_{i+2} \in E \qquad \cfrac{x_i x_{i+1} \in E \qquad \cfrac{\{\mathscr{E}x_i x_{i+1}, \mathscr{E}x_i x_{i+2}, \mathscr{E}x_{i+1}x_{i+2}\}}{\varnothing}\,(3.8)}{\{\mathscr{E}x_i x_{i+2}, \mathscr{E}x_{i+1}x_{i+2}\}}\,(3.7)}{\mathscr{E}x_i x_{i+2}}\,(3.7)$$

Using the trees $\mathscr{D}_1, ..., \mathscr{D}_{2n-1}$, we obtain the following derivation of the empty set in $H_{G,2}$:

$$
\cfrac{x_{2n+1}x_1 \in E \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\mathscr{D}_1\\ \vdots\\ \mathscr{E}x_1x_3\end{matrix} \quad \begin{matrix}\mathscr{D}_3\\ \vdots\\ \mathscr{E}x_3x_5\end{matrix}}{\mathscr{E}x_1x_5}\,(3.6) \quad \begin{matrix}\mathscr{D}_5\\ \vdots\end{matrix} \quad \cdots \quad \begin{matrix}\mathscr{D}_{2n-3}\\ \vdots\end{matrix}}{\mathscr{E}x_1x_{2n-1}} \quad \begin{matrix}\mathscr{D}_{2n-1}\\ \vdots\\ \mathscr{E}x_{2n-1}x_{2n+1}\end{matrix}}{\mathscr{E}x_1x_{2n+1}}\,(3.6)}{\mathscr{E}x_{2n+1}x_1}\,(3.5)}{\varnothing}\,(3.7)
$$

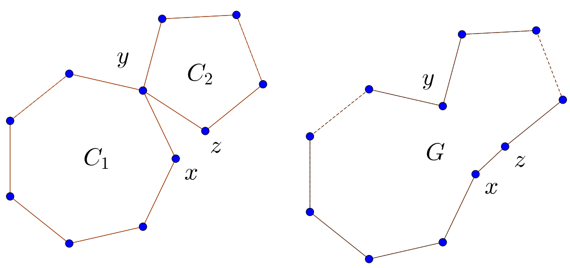
□

The idea shown by the last tree is that, if the graph is 2-colorable with a $(2n+1)$-cycle and c is a 2-coloring map, whenever we pick three consecutive vertices x_i, x_{i+1}, x_{i+2} of the cycle, two of them must be of the same color and since adjacent vertices have different colors, we have $c(x_i) = c(x_{i+2})$; whence $c(x_1) = c(x_3) = c(x_5) = ... = c(x_{2n+1})$ which is impossible since $x_{2n+1}x_1 \in E$.

The opposite direction is less trivial, and the proof requires Robinson's theorem; before showing it, we need the following lemma about odd cycles.

Lemma 3.12. *Let $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$ be two odd cycle such that $xy \in E_1 \setminus E_2$ and $yz \in E_2 \setminus E_1$; then the graph $G = C_1 \cup C_2 \setminus \{xy, yz\} \cup \{xz\}$ contains an odd cycle.*

Proof: Let $2n+1$ and $2m+1$ respectively the length of C_1 and C_2. If y is the only vertex in the intersection, then G is a cycle of length $2(n+m)+1$.



If the two cycles intersect multiple times, let $V' = V_1 \cup V_2$, $E' = E_1 \cup E_2 \backslash (E_1 \cap E_2)$, and consider the graph $G' = (V', E')$.

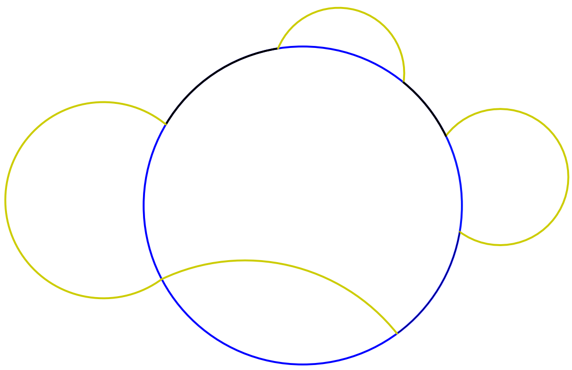


Figure 3.4: The two cycles C_1 and C_2. In blue $C_1 \backslash C_2$, in green $C_2 \backslash C_1$, and in black the $C_1 \cap C_2$. G' is obtained by removing the black part.

We notice three facts about G'. First that it is composed only by primitive cycles, say $\tilde{C}_1, \cdots, \tilde{C}_l$, that are pairwise edge-disjoint; that it has an even number of edges, since $|E'| = 2n + 1 + 2m + 1 - 2|E_1 \cap E_2|$; and that $xy, yz \in E'$, since $xy, yz \notin E_1 \cap E_2$.

If there is a cycle $\tilde{C}_i$ which is odd and does not contain either xy nor yz, then such cycle is also in G. Thus suppose that every $\tilde{C}_i$ such that $xy, yz \notin \tilde{C}_i$ is of even length. Now we distinguish two cases, first that $xy, yz \in \tilde{C}_1$, second that $xy \in \tilde{C}_1$ and $yz \in \tilde{C}_2$. In the first case $\tilde{C}_1$ is even since $|\tilde{C}_1| = |E'| - \Sigma_{i=1}^{l} |\tilde{C}_i|$; thus we take $C = \tilde{C}_1 \setminus \{xy, yz\} \cup \{xz\}$ which is an odd cycle. In the second case $|\tilde{C}_1 \cup \tilde{C}_2|$ is even, and we take $C = \tilde{C}_1 \cup \tilde{C}_2 \setminus \{xy, yz\} \cup \{xz\}$. In both cases C is an odd cycle of G. $\square$

Proposition 3.13. *If a graph $G = (V, E)$ is not 2-colorable, then it has an odd cycle*

Proof: If G is not 2-colorable, also every graph that contains G is not 2-colorable; therefore by Robinson's Theorem, we know that $\vdash_{H_{G,2}} \varnothing$. Consider a derivation of $\varnothing$ that respects Lemma 3.10, in particular that it does not have trivial application, and all the applications of the elimination rule (3.7) are close to the root. Now we proceed by induction on the number k of applications of the transitivity rule (3.6). If $k = 0$ the tree is of the form:

$$\dfrac{xy \in E \quad \dfrac{yz \in E \quad \dfrac{xz \in E \quad \dfrac{}{\{\mathscr{E}xy, \mathscr{E}yz, \mathscr{E}xz\}}\,(3.8)}{\{\mathscr{E}xy, \mathscr{E}yz\}}\,(3.7)}{\mathscr{E}xy}\,(3.7)}{\varnothing}\,(3.7)$$

Then $xy, xz, yz \in E$ and the 3-cycle xyz is a subgraph of G.

For $k > 0$ the tree is of the following form,

$$\begin{array}{c}
\begin{array}{cc}
\varnothing & \varnothing \\
\vdots\ \mathscr{D}_1 & \vdots\ \mathscr{D}_2 \\
\Gamma_1 \cup \mathscr{E}xy & \Gamma_2 \cup \mathscr{E}yz
\end{array} \\ \hline
\Sigma \cup \mathscr{E}xz \\
\vdots \\
\varnothing
\end{array}\,(3.6)$$

Every literal in $\Sigma \cup \mathscr{E}xz$ is eliminated by rule (3.7), which means that the corresponding edge is in E. Since there are no trivial applications, $\mathscr{E}xy \notin \Gamma_1$ and $\mathscr{E}yz \notin \Gamma_2$, and for any $\mathscr{E}ab \in \Gamma_i$, $ab \in E$, whence we have the derivations $\mathscr{D}'_i : \Gamma_i \vdash_{H_{G,2}} \varnothing$.

Let $E' = E \cup \{xy, yz\}$ and $G' = (V, E')$; then the derivations $\mathscr{D}_i$ and $\mathscr{D}'_i$ are valid also in $H_{G',2}$. Thus we have the two following trees in $H_{G',2}$:

$$
\begin{array}{cc}
\dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_1 \\ \Gamma_1 \cup \mathscr{E}xy\end{array}}{\begin{array}{c}\Gamma_1 \\ \vdots\ \mathscr{D}'_1 \\ \varnothing\end{array}}\, xy \in E'
&
\dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_2 \\ \Gamma_2 \cup \mathscr{E}yz\end{array}}{\begin{array}{c}\Gamma_2 \\ \vdots\ \mathscr{D}'_2 \\ \varnothing\end{array}}\, yz \in E'
\end{array}
$$

In both trees the number of applications of the rule (3.6) is certainly less than k; then by induction hypothesis G' has two odd cycles C_1 and C_2. If $C_1 = C_2$, $\mathscr{E}xy \in \Gamma_2$, and $\mathscr{E}yz \in \Gamma_1$, whence $xy, yz \in E$ and C_1 is also in G.

Let E_i be the set that contains the edges of C_i, such sets cannot coincides otherwise $C_1 = C_2$. If $xy, yz \notin E_1$, then $E_1 \subseteq E$ and C_1 is a cycle of G. Thus suppose it is not the case that $xy, yz \notin E_1$, i.e. $xy \in E_1 \vee yz \in E_1$.

Suppose that $yz \in E_1$, then $\mathscr{E}yz \in \Gamma_1$ and $yz \in E$; if $xy \notin E_2$, then $E_2 \subseteq E$, otherwise $\mathscr{E}xy \in \Gamma_2$ and $xy \in E$, in both cases $E_2 \subseteq E$ and G contains C_2. Therefore it remain the case $yz \notin E_1 \wedge xy \in E_1$. Similarly we conclude $xy \notin E_2 \wedge yz \in E_2$, whence $xy \in E_1 \setminus E_2$ and $yz \in E_2 \setminus E_1$.

Finally we can apply Lemma 3.12, the resulting graph is a subgraph of G since $E_1 \cup E_2 \setminus \{xy, yz\} \cup \{xz\} \subseteq E$, and G has the required odd cycle. $\square$

Gallai-Hasse-Roy-Vitaver theorem

Another way to describe a proper coloring of a graph, is by introducing a partial order on the colors, indeed we may use the binary symbol $<$ and write $x < y$ to say "the vertex x is colored in a color smaller than the one of the vertex y". We shall write $x \geq y$ instead of $\neg(x < y)$. The role of the formulas (3.2), (3.3) and (3.4) is now played by

$$x \geq x \tag{3.9}$$

$$x < y \wedge y < z \rightarrow x < z \tag{3.10}$$

$$(xy \in E \Rightarrow) \qquad x < y \vee y < x \tag{3.11}$$

$$\bigvee_{i=0}^{n-1} x_i \geq x_{i+1} \tag{3.12}$$

In the formalism of hyper-resolution, for a given graph $G = (V, E)$ we consider the language $\mathscr{L} = \{\geq\} \cup V \cup E$, where $V \cup E$ are constants; the models are the n-colorable graphs that contain G while the inference rules are

$$\frac{\Gamma}{\Gamma \cup x \geq x} \qquad\qquad \frac{\Gamma \cup x \geq y \quad \Sigma \cup y \geq z}{\Gamma \cup \Sigma \cup x \geq z}$$

$$xy \in E\,\frac{\Gamma \cup x \geq y \quad \Sigma \cup y \geq x}{\Gamma \cup \Sigma} \qquad\qquad \frac{\Gamma}{\Gamma \cup \{x_0 \geq x_1, ..., x_{n-1} \geq x_n\}}$$

The advantage of these rules with respect to the previous approach can be seen in the last rule, indeed now to describe a n-coloring it's enough to have an axiom that introduces $n-1$ literals while with (3.4) we must involve $n(n+1)/2$ vertices for each instance.

Now, fix a simple graph G and an orientation D of G, we want to describe the property of Vitaver's theorem "The directed paths in D are not longer than

$\ell(D)$" where $\ell(D)$ is defined to be the length of the longest directed path in D, maintaining the same notation as in Theorem 3.7. To formalize such property we introduce the binary relation symbol $\mathrel{-\!\!\prec}$ which is defined on pairs of vertices and which indicates the existence of a directed path from one vertex to the other

$$x \mathrel{-\!\!\prec} y \Leftrightarrow \text{There is a path from } x \text{ to } y$$

Since every arc can be seen as a path of length one, if there is an edge xy in the underlying graph $G = (V, E)$ of D, then it's necessary to have

$$xy \in E \Rightarrow x \mathrel{-\!\!\prec} y \vee y \mathrel{-\!\!\prec} x \tag{3.13}$$

If the end of the path is the start of another path, there exists a path which is the concatenation of the two:

$$x \mathrel{-\!\!\prec} y \wedge y \mathrel{-\!\!\prec} z \rightarrow x \mathrel{-\!\!\prec} z \tag{3.14}$$

Also, requiring that $\ell(D)$ is the maximum length of a path is equivalent to ask that there are no paths of length equal to $n = \ell(D) + 1$, which amounts to ask for any set of $n + 1$ vertices

$$\neg(x_0 \mathrel{-\!\!\prec} x_1 \wedge x_1 \mathrel{-\!\!\prec} x_2 \wedge ... \wedge x_{n-1} \mathrel{-\!\!\prec} x_n)$$

and if we denote $\neg(x \mathrel{-\!\!\prec} y)$ as $x \mathrel{\succ\!\!=} y$

$$x_0 \mathrel{\succ\!\!=} x_1 \vee x_1 \mathrel{\succ\!\!=} x_2 \vee ... \vee x_{n-1} \mathrel{\succ\!\!=} x_n \tag{3.15}$$

Finally, we require D to be an acyclic orientations of G, which amounts to say that for any vertex x there is no path from x to x, that is:

$$x \mathrel{\succ\!\!=} x \tag{3.16}$$

Thus if we consider the language $\mathscr{L} = \{ \mathrel{\succ\!\!=} \} \cup V \cup E$, the models are the acyclic orientation of a subgraph of G and the rules of inference are

$$\frac{\Gamma}{\Gamma \cup x \succcurlyeq x} \qquad \frac{\Gamma \cup x \succcurlyeq y \quad \Sigma \cup y \succcurlyeq z}{\Gamma \cup \Sigma \cup x \succcurlyeq z}$$

$$xy \in E\,\frac{\Gamma \cup x \succcurlyeq y \quad \Sigma \cup y \succcurlyeq x}{\Gamma \cup \Sigma} \qquad \frac{\Gamma}{\Gamma \cup \{x_0 \succcurlyeq x_1, ..., x_{n-1} \succcurlyeq x_n\}}$$

Clearly the rules have the same shape as the ones given for $\geq$. We can therefore conclude that $\succcurlyeq$ describes an n-coloring for G where

$$n = 1 + \min\{\ell(D'), D' \text{ spanning acyclic digraph of an orientation of } G\}$$

If we consider an orientation D of G and a spanning acyclic digraph D' of D, we have $\ell(D') \leq \ell(D)$ whence $n \leq 1 + \ell$ where ℓ is $\min\{\ell(D), D \text{ orientation of } G\}$ as in corollary 3.8. Finally recalling that Proposition 3.6 says that for some orientation D

$$\chi(G) \geq 1 + \ell(D)$$

we reach the same conclusion of Corollary 3.8, i.e.

$$\chi(G) = 1 + \ell$$

μ_n graphs

The μ_n graphs are a family of graphs defined by Matiyasevich [9] with the purpose to characterize the graphs that cannot be colored in n colors. Such graphs are built starting from the complete graphs with $n+1$ vertices, which is not n colorable; and every step of the construction ensures that the result is still not n colorable. Matiyasevich's theorem tells us that every graph that cannot be colored in n colors, contains a μ_n graph. They are defined by induction as follows

Definition 3.14. $\mu_n - graph$

- Every complete graph with $n+1$ vertices is a μ_n-graph
- If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are μ_n-graphs and a, b, c are distinct vertices such that

$$a, b \in V_1 \quad b, c \in V_2$$

$$ab \in E_1, ab \notin E_2 \qquad bc \in E_2, bc \notin E_1$$

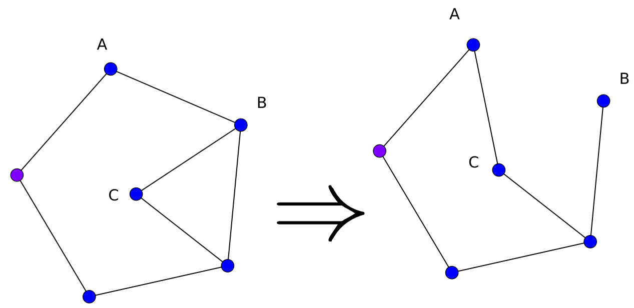
Then the graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{ac\} \setminus \{ab, bc\})$ is a μ_n-graph.

To the best of our knowledge, in the literature there are no other graphs defined in such a way; although we have found the μ_n graphs can be constructed using splitting methods which were introduced by Fleischner (1990) [18].

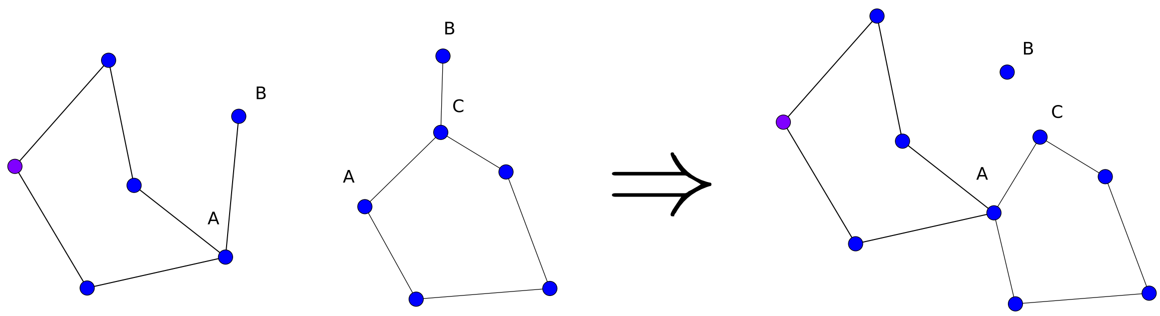
The μ_1 graphs are of little interest. The complete graph with two vertices, K^2, is a single edge; when two edges are composed according to the definition, the result is a single edge. Thus all μ_1 graphs are single edges; and every graph that contains an edge, contains a μ_1 graph. Clearly μ_1 graphs are not 1-colorable; and graphs that are not 1-colorable contain at least one edge.

μ_2 graphs are more interesting; they characterize bipartite graphs, and can give a better insight of generic μ_n graphs. Clearly all the odd cycles are μ_2 graphs;

on the other hand, not all μ_2 graphs are odd cycles, the following example shows a μ_2 graph which is not a cycle and which can be built using the definition and starting by a 3-cycle and a 5-cycle



They do not even need to be connected, indeed starting from two μ_2 graphs of the kind shown above, we obtain a μ_2 graph with an isolated vertex.



The examples suggest that μ_2 graphs always contain an odd cycle, this can be readily proved thanks to Lemma 3.12, which was used to prove that non bipartite graphs have at least an odd cycle.

Proposition 3.15. *Let G be μ_2 graph; then it contains an odd cycle.*

Proof: We proceed by induction on the number of vertices n. If $n = 3$, the only μ_2 graph is the complete graph which is a 3-cycle. If $n > 3$, let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{ac\} \setminus \{ab, bc\})$ as in Definition 3.14. By induction hypothesis, both G_1 and G_2 contain an odd cycle; then by Lemma 3.12, G has an odd cycle. $\square$

Now we move to a generic family μ_n, first we show that they cannot be colored in n colors. Second, we see that there is always a μ_n subgraph in a graph which is not n-colorable.

Proposition 3.16. *Let G be a μ_n graph; then G is not n-colorable.*

Proof: Let $G = (V, E)$; we proceed by induction on Definition 3.14. If G is a complete graph with $n+1$ vertices, clearly it is not n-colorable. Then let G_1 and G_2 as in the Definition 3.14; by inductive hypothesis they are not n-colorable, and therefore the empty set is derivable in both calculi $H_{G_1,n}$ and $H_{G_2,n}$, namely we have two deductions:

$$\mathscr{D}'_1 : \vdash_{G_1,n} \varnothing \quad \mathscr{D}'_2 : \vdash_{G_2,n} \varnothing$$

Every edge of G_1 or G_2, but ab and bc, is also in G, therefore if the literals $\mathscr{E}ab, \mathscr{E}bc$ do not appear in one of the two deductions, then such deduction is also valid in $H_{G,n}$ and G is not n-colorable. Otherwise, since G does not have the edges ab and bc, by dropping the rules that eliminates $\mathscr{E}ab$ and $\mathscr{E}bc$, we get two derivations in $H_{G,n}$

$$\mathscr{D}_1 : \vdash_{G,n} \mathscr{E}ab \quad \mathscr{D}_2 : \vdash_{G,n} \mathscr{E}bc$$

Recalling that ac is en edge of G by definition, we get the following deduction of the empty set in $H_{G,n}$, which implies that G is not n-colorable

$$\dfrac{ac \in E \qquad \dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_1 \\ \mathscr{E}ab\end{array} \qquad \begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_2 \\ \mathscr{E}bc\end{array}}{\mathscr{E}ac}\,(3.6)}{\varnothing}\,(3.7)$$

□

Theorem 3.17 (Matiyasevich). *Let G be a graph that is not n-colorable; then it has a μ_n graph as subgraph.*

The proof of this theorem is a generalization of the one seen in Proposition 3.13. It starts by applying Robinson's theorem to get $\vdash_{G,n} \varnothing$, and then it proceeds by induction on the number of the application of the transitivity rule (3.6). In this case the role of Lemma 3.12 is played by the Definition 3.14. Now the characterization of bipartite graphs comes as corollary of Matiyasevich's theorem

Corollary 3.18. *A graph G is 2-colorable if and only if it does not contain odd cycles.*

Proof: We show that G is not 2-colorable if and only if it contains an odd cycle. If G is not 2-colorable, by Theorem 3.17 it contains a μ_2 graph, which contains an odd cycle by Proposition 3.15. Conversely; since odd cycles are μ_2, and since μ_2 graphs are not 2-colorable by Proposition 3.16, if G contains an odd cycles, then it is not 2-colorable. □

Matiyasevich's theorem and μ_n graphs can be used to rewrite several problems of graph theory, for instance we can rephrase the four color Theorem as follows

Corollary 3.19. *Every planar graph is 4-colorable if and only if no μ_4 is planar.*

Proof: Suppose first that every planar graph is 4-colorable, then every graph which is not 4-colorable is not planar; by Proposition 3.16 μ_4 graphs are not 4-colorable, and therefore are not planar. Conversely suppose that no μ_4 is planar; if a graph is planar, then it does not contain μ_4 subgraphs and by Matiyasevich's theorem is 4-colorable. □

Another important problem that can be written using μ graphs is Hadwiger's conjecture (1946)[19]. In 1980 this conjecture was defined by Bollobs, Catlin and Erdös "One of the deepest unsolved problems in graph theory" [20]. Here we see first the original conjecture, and then the equivalent version stated by Matiyasevich using μ graphs [9].

Conjecture 3.20 (Hadwiger)**.** *Every graph G with chromatic number $\chi(G) = n$ has the complete graph K^n as minor*

Conjecture 3.21. *Every graph that contains a μ_n graph has K^{n+1} as minor*

Corollary 3.22. *Hadwiger's conjecture is equivalent to 3.21*

Proof: First we show that 3.21 implies 3.20. Let G be a graph such that $\chi(G) = n$, then it is not $n-1$ colorable and by Matiyasevich's theorem 3.17 it contains a μ_{n-1} graph; therefore by 3.21 it contains K^n. Conversely, suppose that G contains μ_n, by 3.16 G is not n colorable i.e. for some $m > n$, $\chi(G) = m$. By 3.20 G has K^m as minor; since K^m is complete, it has K^l as minor for any $l \leq m$ and in particular for $l = n+1$. □

Matiyasevich's version 3.21 clearly shows that Hadwiger's conjecture implies the four color theorem; indeed if μ_4 graphs has K^5 as minor, by Kuratowski's theorem they are not planar, and the four color theorem would follow from 3.19.

Conclusions

In these pages we saw that the hyper-resolution calculus is a powerful tool not only in proof theory, but also in fields apparently very different such as algebra and graph theory. A remarkable point is that proofs based on hyper-resolution and Robinson's theorem can be implemented in a proof assistant such as COQ or AGDA. For example, it may be interesting to implement the proof of the Proposition 3.13 on bipartite graphs, the expected result is a program that takes a non-bipartite graph as input, and returns an odd cycle of such graph.

The chapter about algebra showed how it is possible to represent fields and vector spaces using hyper-resolution calculus. We used this representation and Robinson's theorem to prove the Hilbert's Nullstellensatz, it may be interesting to use the same approach for other theorems about fields and polynomials. Possible further applications are about partially ordered groups; in particular, a challenging application may be Levi's theorem, which says that a partially ordered group is totally orderable if and only if it is torsion free [21].

In the last chapter we saw some problems about coloring graphs. We saw two different ways to represent the property of a graph "to be colorable with n colors"; one based on the relation among vertices $\mathscr{E}xy$ which is interpreted to be true when x and y have the same color, and another one based on the relation $x \geq y$ which is true when "x has a color-number greater or equal than the color-number of y". These two different approaches are equally good, however we saw that the first one was useful to deal with bipartite and μ_n graphs, while the second one with problems related to orientations of a graph.

In all these different applications, we can distinguish at least two different steps which are equally important; first of all one should find a good set of axioms that represent the environment of the problem. Then, one tries to get the required result out of the derivation provided by Robinson's theorem. The chapter about graphs shows that, when the second step seems not feasible, it may be useful to go back to the first one and look for another set of axioms.

Besides this last observation, we can notice that any formula of the kind $\bigwedge_{i=1}^m A_i \to \bigvee_{j=1}^n B_j$ is classically equivalent to $\bigwedge_{j=1}^n \neg B_j \to \bigvee_{i=1}^m \neg A_i$. For instance the set of axioms 3.1, 3.2, 3.3, 3.4 is equivalent to the following one

$$\begin{array}{cc} \neg\mathscr{E}yx \to \neg\mathscr{E}xy & \neg\mathscr{E}xz \to \neg\mathscr{E}xy \vee \neg\mathscr{E}yz \\ (xy \in E \Rightarrow)\ \neg\mathscr{E}xy & \neg(\neg\mathscr{E}x_0x_1 \wedge \cdots \wedge \neg\mathscr{E}x_{n-1}x_n) \end{array}$$

which gives rise to the following rules

$$\begin{array}{cc} \dfrac{\Gamma \cup \neg\mathscr{E}yx}{\Gamma \cup \neg\mathscr{E}xy} & \dfrac{\Gamma \cup \neg\mathscr{E}xz}{\Gamma \cup \{\neg\mathscr{E}xy, \neg\mathscr{E}yz\}} \\ xy \in E\,\dfrac{\Gamma}{\Gamma \cup \neg\mathscr{E}xy} & \dfrac{\Gamma_1 \cup \neg\mathscr{E}x_0x_1 \quad \cdots \quad \Gamma_n \cup \neg\mathscr{E}x_{n-1}x_n}{\Gamma_1 \cup \cdots \cup \Gamma_n} \end{array}$$

This means that, any set of axioms corresponds to at least two different hyper-resolution calculi.

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