Hyper-resolution applications

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Introduction

- In 1965, J.A. Robinson published "**A Machine-oriented logic** based on the resolution principle."
- A single inference principle that provides a **complete system** of propositional calculus.
- In 1972, Matijasevič published "Application of the methods of the theory of logical derivation to **graph theory**."
- In 1980 Vladimir Lifschitz published "Semantical completeness theorems in logic and **algebra**."

Preliminaries:

- **Atomic formulae:** a formula that contains no logical connectors.
- **Literal:** is either an atomic formula, or has the form $\neg A$ where $\neg$ is the negation symbol and A is an atomic formula. Literals will be denoted using Latin capital letters $A, B, C, \ldots$.
- **Clause:** is a finite set of literals. Clauses are denoted by Greek capital letters $\Gamma, \Sigma, \Pi, \Delta \ldots$. The singleton $\{A\}$ is denoted by its element A.
- A set S of clause is **satisfiable** if there is a model of S; otherwise S is **unsatisfiable**

Definition

- Γ_1 and Γ_2: clauses
- $P \in \Gamma_1$ and $\neg P \in \Gamma_2$
- $\Pi = \Gamma_1 \cup \Gamma_2 \setminus \{P, \neg P\}$ is called the **resolvent** of Γ_1 and Γ_2

The **resolution principle**:
If Π is unsatisfiable and is the resolvent of Γ_1 and Γ_2; then Γ_1 and Γ_2 are unsatisfiable.

Example

$$\frac{A \quad \neg A}{\varnothing} \qquad \frac{\{A, P\} \quad \{B, \neg P\}}{\{A, B\}}$$

$$A, \neg A \vdash \perp \qquad A \vee P, B \vee \neg P \vdash A \vee B$$

Lifschitz's approach

Let T be a set of propositional formulae, each of the form:

$$\bigwedge_{i=1}^{m} A_i \to \bigvee_{j=1}^{n} B_j \qquad m+n>0$$

Every element of T gives rise to a **rule of inference**:

$$\frac{\Gamma_1 \cup A_1, \quad \cdots , \quad \Gamma_m \cup A_m}{\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{B_1, \cdots , B_n\}}$$

Example

Let $T = \{A \wedge B \to C \vee D,\ A,\ B,\ \neg C\}$, then H_T has four rules:

$$\frac{\Gamma_1 \cup A \quad \Gamma_2 \cup B}{\Gamma_1 \cup \Gamma_2 \cup \{C, D\}} \qquad \frac{\Gamma}{\Gamma \cup A} \qquad \frac{\Gamma}{\Gamma \cup B} \qquad \frac{\Gamma \cup C}{\Gamma}$$

Using such rules we get the tree:

$$\cfrac{\cfrac{\cfrac{\varnothing}{A} \quad \cfrac{\varnothing}{B}}{\{C, D\}}}{D}$$

and we write $\varnothing \vdash_{H_T} D$.

Completeness theorems

Theorem (Robinson's, version 1)

If T is contradictory, i.e. $T \vdash \perp$, then $\vdash_{H_T} \varnothing$

Theorem (Robinson's , version 3)

For any clauses $\Gamma_1, \cdots, \Gamma_l, \Delta$, if in every model of T, the following is valid

$$\overline{\Gamma}_1 \wedge \cdots \wedge \overline{\Gamma}_l \rightarrow \overline{\Delta}$$

then $\Gamma_1, \cdots, \Gamma_l \vdash_{H_T} \Delta'$, for some $\Delta' \subseteq \Delta$.

Two steps:

- Find **set of axioms** that represent the environment of the problem.
- Try to get the required result **out of the derivation** provided by Robinson's theorem.

Applications to algebra: Nullstellensatz

Let $\mathbb{K}$ be a field and $f, f_1, \cdots, f_l \in \mathbb{K}[\mathbf{x}]$, if in any extension of $\mathbb{K}$, f vanishes at all common zeros of $f_1, \cdots, f_l$, then there exists $p \in \mathbb{N}$ and $h_1, \cdots, h_l \in \mathbb{K}[\mathbf{x}]$ such that

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

Consider the language $\mathscr{L} = \{+, \cdot, = 0\} \cup \mathbb{K}$.

- $+, \cdot$ are binary function
- $= 0$ is a unary relation symbol

The **terms are polynomials** over $\mathbb{K}$ and the **atomic formulae are algebraic equations**.

Axioms and rules of inference

$$0 = 0 \qquad \frac{\Gamma}{\Gamma \cup 0 = 0}$$

$$\neg(1 = 0) \qquad \frac{\Gamma \cup 1 = 0}{\Gamma}$$

$$(r = 0 \wedge s = 0) \to r + s = 0 \qquad \frac{\Gamma \cup r = 0\,, \quad \Delta \cup s = 0}{\Gamma \cup \Delta \cup r + s = 0}$$

$$r = 0 \to r \cdot s = 0 \qquad \frac{\Gamma \cup r = 0}{\Gamma \cup r \cdot s = 0}$$

$$r \cdot s = 0 \to (r = 0 \vee s = 0) \qquad \frac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}}$$

The models of these axioms are the **integral domains** that contain $\mathbb{K}$

Second step

The hypothesis: "f is a polynomial that vanishes at all common zeros of $f_1, \cdots, f_l$" is equivalent to say that in every model of the axioms:

$$\Gamma_1 \wedge \cdots \wedge \Gamma_l \rightarrow \Delta$$

where $\Gamma_i = \{f_i = 0\}$ and $\Delta = \{f = 0\}$

By **Robinson's theorem**, there is a derivation of Δ' from $\Gamma_1, \cdots, \Gamma_l$:

$$\Gamma_1, \cdots, \Gamma_l \vdash_{H_N} \Delta'$$

where either $\Delta' = \varnothing$ or $\Delta' = \{f = 0\}$.

Case $\Delta' = \varnothing$

The tree must be of the form:

$$\dfrac{\begin{array}{c} \Gamma_1, \quad \cdots, \quad \Gamma_l \\ \vdots \\ \{1 = 0\} \end{array}}{\varnothing}$$

- The other rules can only be the ones related to the axioms:

$$(r = 0 \wedge s = 0) \to r + s = 0 \qquad\qquad r = 0 \to r \cdot s = 0$$

- 1 is a linear combination of $f_1, \cdots, f_l$:

$$f^0 = 1 = \sum_{i=1}^{l} h_i \cdot f_i$$

Case $\Delta' = \{f = 0\}$

The derivations is made only of the following rules

$$\frac{\Gamma \cup r = 0, \quad \Delta \cup s = 0}{\Gamma \cup \Delta \cup r + s = 0}(1) \quad \frac{\Gamma \cup r = 0}{\Gamma \cup r \cdot s = 0}(2) \quad \frac{\Gamma \cup r \cdot s = 0}{\Gamma \cup \{r = 0, s = 0\}}(3)$$

and has the following shape

$$\cfrac{\cfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\Sigma}\text{rules (1) and (2)}}{\{f = 0\}}(p-1)\text{ applications of (3)}$$

Therefore Σ is a singleton, since the rule (3) applies

$$\Sigma = \{t_1 \cdot r_1 = 0\}$$

Case $\Delta' = \{f = 0\}$

The tree is

$$\dfrac{\dfrac{\dfrac{\Gamma_1, \quad \cdots, \quad \Gamma_l}{\{t_1 \cdot r_1 = 0\}}\text{ rules (1) and (2)}}{\{t_1 = 0, t_2 \cdot r_2 = 0\}}(3)}{\vdots}$$

$$\dfrac{\{t_1 = 0, \cdots t_{p-2} = 0, t_{p-1} \cdot r_{p-1} = 0\}}{\{t_1 = 0, \cdots, t_{p-1} = 0, r_{p-1} = 0\}}(3)$$

Since $\Delta' = \{f = 0\}$ we have $t_i = r_{p-1} = f$.
Therefore $\Sigma = \{f^p = 0\}$ and

$$f^p = \sum_{i=1}^{l} h_i \cdot f_i$$

Hyper-resolution approach to coloring

Every n-coloring of a given graph $G = (V, E)$ can be described by the **symmetric binary relation** "vertices x and y **have the same color**", which we denote by $\mathscr{E}xy$. The following axioms gives the calculus $H_{G,n}$

$$\mathscr{E}xy \to \mathscr{E}yx \qquad \frac{\Gamma \cup \mathscr{E}xy}{\Gamma \cup \mathscr{E}yx}$$

$$\mathscr{E}xy \wedge \mathscr{E}yz \to \mathscr{E}xz \qquad \frac{\Gamma \cup \mathscr{E}xy \quad \Delta \cup \mathscr{E}yz}{\Gamma \cup \Delta \cup \mathscr{E}xz}$$

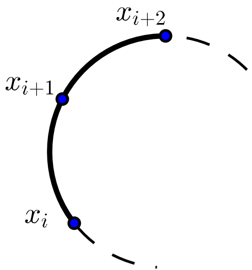
$$xy \in E \Rightarrow \qquad \neg\mathscr{E}xy \qquad \frac{\Gamma \cup \mathscr{E}xy}{\Gamma}$$

$$\bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^{n} \mathscr{E}x_i x_j \qquad \frac{\Gamma}{\Gamma \cup \{\mathscr{E}x_0x_1, \ldots, \mathscr{E}x_{n-1}x_n\}}$$

Odd cycles are not 2-colorable

Let $C = x_1 x_2 \ldots x_{2n+1}$, for any 3 consecutive vertices x_i, x_{i+1}, x_{i+2} we have the following derivation $\mathscr{D}_i$ in $H_{C,2}$:

$$\cfrac{\cfrac{\cfrac{\varnothing}{\{\mathscr{E}x_i x_{i+1}, \mathscr{E}x_i x_{i+2}, \mathscr{E}x_{i+1} x_{i+2}\}}}{\{\mathscr{E}x_i x_{i+2}, \mathscr{E}x_{i+1} x_{i+2}\}}}{\mathscr{E}x_i x_{i+2}}$$



Odd cycles are not 2-colorable

Using the derivations $\mathscr{D}_1, \ldots, \mathscr{D}_{2n-1}$, we obtain the following derivation of the empty set in $H_{G,2}$:

$$
x_{2n+1}x_1 \in E \;\; \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ \vdots\\ \mathscr{E}x_1x_3\end{matrix}\quad\begin{matrix}\mathscr{D}_3\\ \vdots\\ \mathscr{E}x_3x_5\end{matrix}}{\mathscr{E}x_1x_5}\quad \begin{matrix}\mathscr{D}_5\\ \vdots\end{matrix}\quad\begin{matrix}\cdots\\ \vdots\end{matrix}\quad\begin{matrix}\mathscr{D}_{2n-3}\\ \vdots\end{matrix}}{\mathscr{E}x_1x_{2n-1}}\qquad\begin{matrix}\mathscr{D}_{2n-1}\\ \vdots\\ \mathscr{E}x_{2n-1}x_{2n+1}\end{matrix}}{\mathscr{E}x_1x_{2n+1}}}{\mathscr{E}x_{2n+1}x_1}}{\varnothing}
$$

If G has no odd cycles, is 2-colorable

We prove the equivalent proposition:
"If a graph $G = (V, E)$ is not 2-colorable, then it has an odd cycle"
Every graph that contains G is not 2-colorable; therefore by Robinson's Theorem, we know that

$$\vdash_{H_{G,2}} \varnothing$$

Every tree in $H_{G,2}$ can be rearranged in such a way that:

- $\mathscr{E}x_1x_2 \vee \mathscr{E}x_2x_3 \vee \mathscr{E}x_1x_3$ are at the **top**
- $\mathscr{E}xy \wedge \mathscr{E}yz \to \mathscr{E}xz$ are in the **middle**
- $xy \in E \Rightarrow \quad \neg\mathscr{E}xy$ are at the **bottom**

Proof:

We proceed by **induction** on the number k of applications of the transitivity rule $\mathscr{E}xy \wedge \mathscr{E}yz \rightarrow \mathscr{E}xz$. If $k = 0$ the tree is:

$$
\cfrac{xz \in E \; \cfrac{\overline{\{\mathscr{E}xy, \mathscr{E}yz, \mathscr{E}xz\}}}{yz \in E \; \cfrac{\{\mathscr{E}xy, \mathscr{E}yz\}}{xy \in E \; \cfrac{\mathscr{E}xy}{\varnothing}}}}{}
$$

Then $xy, xz, yz \in E$ and the **3-cycle** xyz is a subgraph of G.

$k > 0$

The tree is of the following form:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_1 \\ \Gamma_1 \cup \mathscr{E}xy\end{array} \quad \begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_2 \\ \Gamma_2 \cup \mathscr{E}yz\end{array}}{\Sigma \cup \mathscr{E}xz} \\
\vdots \\
\varnothing
\end{array}$$

For each $\mathscr{E}ab \in \Sigma \cup \mathscr{E}xz$, we have $ab \in E$, then we have two derivations:

$$\mathscr{D}_1' : \Gamma_1 \vdash_{H_{G,2}} \varnothing \qquad \mathscr{D}_2' : \Gamma_1 \vdash_{H_{G,2}} \varnothing$$

made only of the rule $ab \in E \Rightarrow \neg\mathscr{E}ab$

$k > 0$

Let $E' = E \cup \{xy, yz\}$ and $G' = (V, E')$, and consider $H_{G',2}$

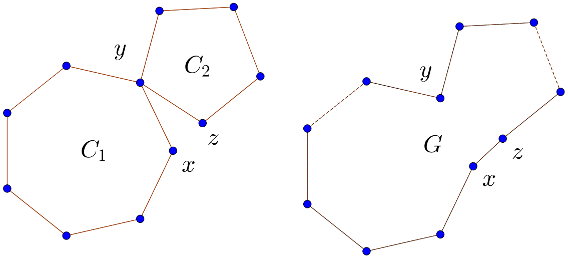
$$
\dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_1 \\ \Gamma_1 \cup \mathscr{E}xy\end{array}}{\begin{array}{c}\Gamma_1 \\ \vdots\ \mathscr{D}_1' \\ \varnothing\end{array}}\, xy \in E'
\qquad
\dfrac{\begin{array}{c}\varnothing \\ \vdots\ \mathscr{D}_2 \\ \Gamma_2 \cup \mathscr{E}yz\end{array}}{\begin{array}{c}\Gamma_2 \\ \vdots\ \mathscr{D}_2' \\ \varnothing\end{array}}\, yz \in E'
$$

By inductive hypothesis, we have **two odd cycles** in G', say C_1, C_2

$k > 0$

- Say $length(C_1) = 2n + 1$, $length(C_2) = 2m + 1$

If C_1 and C_2 are edge-disjoint



G has a cycle of length $2(n + m) + 1$ which is the **required odd cycle.**

μ_n graphs

Inductive definition by by Matijasevič:

- Every complete graph with $n+1$ vertices is a μ_n-graph
- If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are μ_n-graphs and a, b, c are distinct vertices such that

$$a, b \in V_1 \quad b, c \in V_2$$

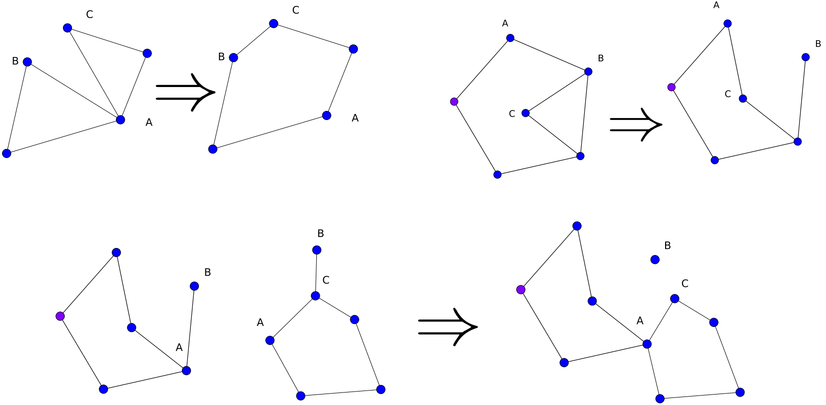
$$ab \in E_1, ab \notin E_2 \qquad bc \in E_2, bc \notin E_1$$

  Then the graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{ac\} \setminus \{ab, bc\})$ is a μ_n-graph.

Theorem (Matijasevič)

- *μ_n graphs are not n-colorable*
- *Every graphs which is not n-colorable has a μ_n graph as subgraph.*

μ_2 graphs



Conclusion

Using Matijasevič's theorem, it is possible to rephrase some problems of graph theory, for instance:

- **Four color's theorem:** No μ_4 is planar.
- **Hadwiger's conjecture:** Any graph that contains a μ_n graph has K^{n+1} as minor.

Thank you!