Optimal investment strategies and hedging of derivatives in the presence of transaction costs

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References

- E. Aurell and P. Muratore-Ginanneschi, "Financial Friction and Multiplicative Markov Market Game", *International J. of Theoretical and Applied Finance* (IJTAF) **3**, No. 3 (2000) 501-510

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- E. Aurell and P. Muratore-Ginanneschi, “ Growth Optimal Strategies with Quadratic Friction Over Finite-Time Investment Horizons”, *International J. of Theoretical and Applied Finance* (IJTAF) **7**, No. 5 (2004) 645-657

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- E. Aurell and P. Muratore-Ginanneschi," Optimal hedging of derivatives with transaction costs", *in prep.*

Market model with transaction costs

Stock and bond model

- W_t: overall investor capital

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- Time $t + dt$: $\quad W_{t+1}^{\text{Stocks}} = [u_t \rho_t + \Delta \chi_t] \, W_t$

$$W_{t+1} = [1 + \rho_t (u_t - 1) - \Delta F_\gamma(\chi_t)] \, W_t$$

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  - χ_t: Control strategy
  - ΔF_γ: Transaction costs

Optimal control problem

- Multiplicative Markov model

$$\Delta\rho_t = \frac{\rho_t + (u_t - 1)\rho_t + \Delta\chi_t}{1 + \rho_t(u_t - 1) - \Delta F_\gamma(\chi_t)} - \rho_t$$

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- For any assigned form of the transaction costs $F(\chi_t)$ determine the control strategy χ_t yielding the optimum of a chosen utility function.

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- For any assigned form of the transaction costs $F(\chi_t)$ determine the control strategy χ_t yielding the optimum of a chosen utility function.
- A natural choice of the utility is

$$\lambda(x, t|T) = \mathbb{E}_{\rho_t = x}\left\{\ln\frac{W_T}{W_t}\right\}$$

Continuum time description

- one event per time-unit

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- The dynamics is modelled by the SDE's

$$dW_t = [\mu\rho_t - \gamma\mathcal{F}(f)]\, W_t dt + \sigma\,\rho_t W_t\, dB_t$$
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$$\begin{aligned} a &= \rho_t(1-\rho_t)(\mu - \sigma^2\rho_t) \\ b &= \sigma\rho_t(1-\rho_t) \end{aligned}$$

Oseledec's theorem

The stochastic control is determined as a function of ρ_t and t, by maximizing the expectation value of the wealth growth:

$$\lambda(x, t; T) = \mathbb{E}_{\rho_t = x} \left\{ \int_t^T ds \left[V(\rho_s) - \gamma \mathcal{F}(\rho_s) \right] \right\}$$

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$$V(\rho_t) = \mu\rho_t - \frac{\sigma^2\,\rho_t^2}{2}$$

Oseledec's theorem predicts

$$\lim_{T-t\uparrow\infty} \lambda(x,t|T) = \lim_{T-t\uparrow\infty} (T-t)\ell$$

Hamilton-Bellman-Jacobi formalism

Dynamic programming equation

$$\partial_t \lambda + [f + a + \gamma x \mathcal{F}] \partial_x \lambda + \frac{b^2}{2} \partial_x^2 \lambda + V - \gamma \mathcal{F} = 0$$

$$\lambda(x, T; T) = 0$$

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If borrowing and short-selling is not allowed, the probability measure of ρ is conserved in the interval $[0, 1]$

$$\partial_x \lambda(x, t; T)|_{x=0} = \partial_x \lambda(x, t; T)|_{x=1} = 0$$

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Dimensional analysis

$$[\lambda] = [x] = 0\,, \quad [\sigma^2] = [\mu] = [f] = [1/t]\,, \quad [\gamma \mathcal{F}] = [1/t]$$

Optimal control

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$$\partial_t \lambda + a \partial_\rho \lambda + \frac{b^2}{2} \partial_\rho^2 \lambda \; + \; V = 0$$

$$\partial_\rho \lambda|_{\rho=\rho_{Max}} = -\frac{\gamma}{1 - \gamma \rho_{Max}}, \qquad \partial_\rho \lambda|_{\rho=\rho_{min}} = \frac{\gamma}{1 + \gamma \rho_{min}}$$

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- "Quadratic" transaction costs $\mathcal{F} = |f|^2$

$$\partial_t \lambda + a\, \partial_\rho \lambda + \frac{(\partial_\rho \lambda)^2}{4\, \gamma\, (1 - \rho\, \partial_\rho \lambda)} + \frac{b^2}{2} \partial_\rho^2 \lambda + \mu \rho - \frac{\sigma^2 \rho^2}{2} = 0$$

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In the presence of transaction costs

- Dimensionless order parameter

$$\varepsilon = \begin{cases} \gamma & \text{linear transaction costs} \\ \sigma^2\,\gamma & \text{quadratic transaction costs} \end{cases}$$

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- $\lambda = (T - t)\frac{\mu^2}{2\sigma^2} + \varphi(T - t, x, \varepsilon)\,, \qquad x = \rho - \rho^{\star}$

Scaling analysis

- Every quantity is rescaled by a power of ε, e.g.

$$x \Rightarrow \varepsilon^{\omega_x} x_{\varepsilon}$$

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$$x \Rightarrow \varepsilon^{\omega_x} x_\varepsilon$$

- Controls are used to match diffusion and utility.
- Example: quadratic friction

$$\begin{aligned}
O(\partial_t \lambda) = O\left(\frac{(\partial_x \lambda)^2}{4\,\gamma\,(1 - x\,\partial_x \lambda)}\right) &\quad \Rightarrow \quad \omega_\varphi - \omega_t = 2(\omega_\varphi - \omega_x) - 1 \\
O(\partial_x^2 \lambda) = O\left(\frac{(\partial_x \lambda)^2}{4\,\gamma\,(1 - x\,\partial_x \lambda)}\right) &\quad \Rightarrow \quad \omega_\varphi - 2\omega_x = 2(\omega_\varphi - \omega_x) - 1 \\
O\left(\frac{\sigma^2 x^2}{2}\right) = O\left(\frac{(\partial_x \lambda)^2}{4\,\gamma\,(1 - x\,\partial_x \lambda)}\right) &\quad \Rightarrow \quad 2\omega_x = 2(\omega_\varphi - \omega_x) - 1
\end{aligned}$$

Normal forms

- Linear friction

$$\lambda(x, T-t, \varepsilon) = (T-t)\frac{\mu^2}{2\sigma^2} + \varepsilon^{4/3}\,\varphi\Big(\frac{T-t}{\varepsilon^{2/3}}, \frac{x}{\varepsilon^{1/3}}; 1\Big) + \mathrm{h.o.t}$$

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$$\partial_t \lambda + \frac{D^2}{2}\partial_x^2 \lambda + \frac{\mu^2}{2\,\sigma^2} - \frac{\sigma^2 x^2}{2} = 0$$

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- Quadratic friction

$$\lambda(x, T-t, \varepsilon) = (T-t)\frac{\mu^2}{2\sigma^2} + \varepsilon\,\varphi(\frac{T-t}{\varepsilon^{1/2}}, \frac{x}{\varepsilon^{1/4}}; 1) + \text{h.o.t}$$

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$$\partial_t\lambda+\frac{(\partial_x\lambda)^2}{4\,\gamma}+\frac{D^2}{2}\partial_x^2\lambda+\frac{\mu^2}{2\,\sigma^2}-\frac{\sigma^2x^2}{2}=0$$

Properties of the solutions

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"Linear t.c.":
$$\ell = \frac{\mu^2}{2\sigma^2} - \frac{\left(9\, D^2\, \sigma^2\, \epsilon^2\right)^{1/3}}{2}$$
$$L^\star \propto \varepsilon^{1/3}$$

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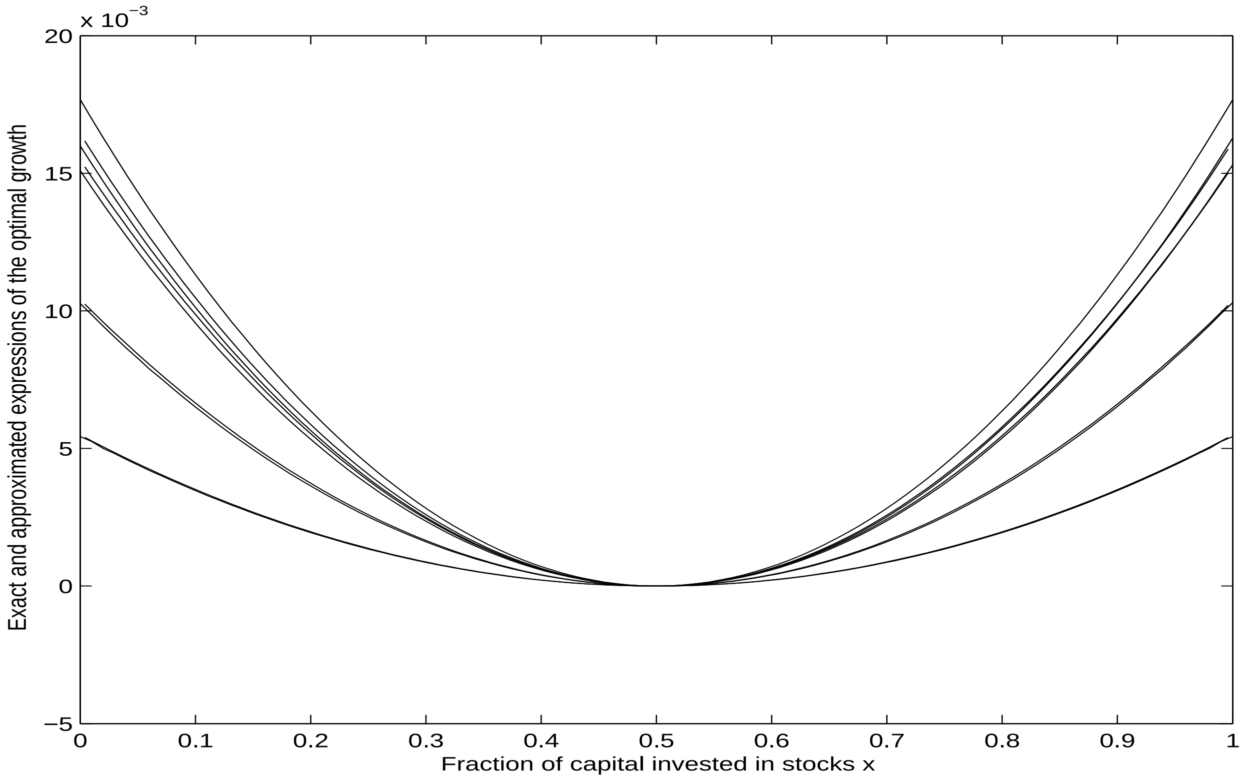
Quadratic friction: $$\ell = \frac{\mu^2}{2\sigma^2} - D^2\, \sqrt{\frac{\epsilon}{2}}$$ $$L^\star \propto \varepsilon^{1/4}$$

Control potential for quadratic friction

$$f(x, T-t) = -\partial_x \lambda(x, T-t)$$

for $\mu = \sigma^2/2$, $\sigma = 10^{-2}$ and $\gamma = 10^2$.



Inclusion of a stock derivative

Dynamical equations

$$
\begin{aligned}
dW &= W\left[(\mu\rho + \mu_d\eta)\,dt + (\sigma\rho + \sigma_d\eta)\;dB_t - (\gamma\,\mathcal{F} + \gamma_d\mathcal{F}_d)\;dt\right] \\
d\rho &= \left[f + a + \rho\,(\gamma\mathcal{F} + \gamma_d\mathcal{F}_d)\right]dt + b\,dB_t \\
d\eta &= \left[f_d + a_d + \eta\,(\gamma\mathcal{F} + \gamma_d\mathcal{F}_d)\right] + b_d\,dB_t
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\end{aligned}
$$

Derivative price

$$
\begin{aligned}
\frac{dC}{C} &= \frac{1}{C}\left[\partial_t C + \mu p \partial_p C + \frac{\sigma^2\,p^2}{2}\partial_p^2 C\right]dt + \frac{\sigma p \partial_p C}{C}\,dB_t \\
&:= \mu_d dt + \sigma_d\,dB_t
\end{aligned}
$$

Non convex utility

- In the absence of transaction costs

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- The Hessian has a zero eigenvalue
- Introducing coordinates along the marginal and stable subspaces of the Hessian gives

$$V = \mu\,\zeta + (\sigma\mu_d - \mu\sigma_d)\,\frac{\sigma_d\zeta - \sigma\vartheta}{\sigma^2 + \sigma_d^2} - \frac{\sigma^2\,\zeta^2}{2}$$

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$$V = \mu\,\zeta + (\sigma\mu_d - \mu\sigma_d)\,\frac{\sigma_d\zeta - \sigma\vartheta}{\sigma^2 + \sigma_d^2} - \frac{\sigma^2\,\zeta^2}{2}$$

- Solvability condition: Black and Scholes

$$\mu_d = \frac{\mu}{\sigma}\sigma_d \;\Rightarrow\; \partial_t C + \frac{\sigma^2 p^2}{2}\partial_p^2 C$$

Summary

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- Black and Scholes provide a solvability condition
- The hedging is defined by the solution of an Hamilton-Jacobi-equations