Supplemental material for "How nanomechanical systems can minimize dissipation"

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I. INTRODUCTION

In these supplementary notes we report the explicit formulas substantiating the interpretation of the $\mathsf{V}^{(1)} = 0$ as "slow manifold" condition of the optimal control equations in the limit of vanishing coupling. These formulas are unfortunately rather long but their derivation straightforward.

II. TRANSITION BETWEEN GAUSSIAN STATES IN A TWO DIMENSIONAL PHASE SPACE: REDUCTION TO A FINITE DIMENSIONAL PROBLEM

We list the boundary conditions defining a transition between Gaussian Boltzmann distributions

$$\rho_{\iota}(\boldsymbol{x}) = \left(\frac{\beta}{2\,\pi\,\sigma_{\iota}\,\tau}\right)^{d} \exp\left\{-\frac{\beta\,p^2}{2\,m} - \frac{\beta\,m\,q^2}{2\,\sigma_{\iota}^2\,\tau^2}\right\} \tag{1a}$$

$$\rho_{\mathrm{f}}(\boldsymbol{x}) = \left(\frac{\beta}{2\,\pi\,\sigma_{\mathrm{f}}\,\tau}\right)^{d} \exp\left\{-\frac{\beta\,p^2}{2\,m} - \frac{\beta\,m\,(q-\mathsf{q}_{\mathrm{f}})^2}{2\,\sigma_{\mathrm{f}}^2\,\tau^2}\right\} \tag{1b}$$

and the Ansätze we used to solve the optimal control equations:

- cumulant generating function of the probability density for any $t \in [0, T]$

$$\begin{aligned}\bar{\phi}(\bar{q},\bar{p},t) &\equiv \ln \int_{\mathbb{R}^{2\,d}} \mathrm{d}^d p \mathrm{d}^d q\, e^{\imath\,(q\,\bar{q}+p\,\bar{p})}\rho(q,p,t) \\ &= -\frac{\tau^2\,\mathsf{c}_{qq,t}\,\bar{q}^2}{2\,m\,\beta} - \frac{m\,\mathsf{c}_{pp,t}\,\bar{p}^2}{\beta} - \frac{\tau\,\mathsf{c}_{qp,t}\,\bar{p}\,\bar{q}}{\beta} + \frac{\imath\,\tau\,\mathsf{c}_{q,t}\,\bar{q}}{\sqrt{\beta\,m}} + \frac{\imath\,\sqrt{m}\,\mathsf{c}_{p,t}\,\bar{p}}{\sqrt{\beta}}\end{aligned} \tag{2}$$

- value function

$$V(\boldsymbol{x},t) = \frac{m\,\mathsf{v}_{qq,t}\,q^2}{2\,\tau^2} + \frac{\mathsf{v}_{qp,t}\,q\,p}{\tau} + \frac{\mathsf{v}_{pp,t}\,p^2}{2\,m} + \sqrt{\frac{m}{\beta}}\frac{\mathsf{v}_{q,t}\,q}{\tau} + \frac{\mathsf{v}_{p,t}\,p}{\sqrt{m\,\beta}} + \frac{v}{\beta} \tag{3}$$

- quadratic control potential

$$U(q,t) = \sqrt{\frac{m}{\beta}}\frac{\mathsf{u}_{q,t}\,q}{\tau} + \frac{m\,\mathsf{u}_{qq,t}\,q^2}{2\,\tau^2} \tag{4}$$

The Ansatz (II) turns the Fokker–Planck equation into three differential equations for the second order cumulants

$$\tau\,\dot{\mathsf{c}}_{qq,t} = 2\,\mathsf{c}_{qp,t} \tag{5a}$$

$$\tau\,\dot{\mathsf{c}}_{qp,t} = \mathsf{c}_{pp,t} - \mathsf{c}_{qp,t} - \mathsf{u}_{qq,t}\,\mathsf{c}_{qq,t} \tag{5b}$$

$$\tau\,\dot{\mathsf{c}}_{pp,t} = 2\,\left(1 - \mathsf{u}_{qq,t}\,\mathsf{c}_{qp,t} - \mathsf{c}_{pp,t}\,\right) \tag{5c}$$

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and two differential equations for the first order ones

$$\tau\, \dot{\mathsf{c}}_{q,t} = \mathsf{c}_{p,t} \tag{6a}$$

$$\tau\, \dot{\mathsf{c}}_{p,t} = -\mathsf{u}_{q,t} - \mathsf{u}_{qq,t}\, \mathsf{c}_{q,t} - \mathsf{c}_{p,t} \tag{6b}$$

Similarly we get from (3) three equations for the second order dual moments

$$\tau\, \dot{\mathsf{v}}_{qq,t} = 2\, \mathsf{u}_{qq,t}\, \mathsf{v}_{qp,t} - 2\, g\, \frac{(k - \mathsf{u}_{qq,t}\, \mathsf{c}_{qq,t})^2}{\mathsf{c}_{qq,t}^2} \tag{7a}$$

$$\tau\, \dot{\mathsf{v}}_{qp,t} = -\mathsf{v}_{qq,t} + \mathsf{v}_{qp,t} + \mathsf{u}_{qq,t}\, \mathsf{v}_{pp,t} \tag{7b}$$

$$\tau\, \dot{\mathsf{v}}_{pp,t} = -2\, (1 + \mathsf{v}_{qp,t} - \mathsf{v}_{pp,t}) \tag{7c}$$

and two equations for the first order ones

$$\tau\, \dot{\mathsf{v}}_{q,t} = \mathsf{u}_{qq,t}\, \mathsf{v}_{p,t} + \mathsf{u}_{q,t}\, (\mathsf{v}_{qp,t} - 2\, g\, \mathsf{u}_{qq,t}) + 2\, g\, k \frac{(\mathsf{u}_{q,t} - \mathsf{u}_{qq,t}\, \mathsf{c}_{q,t})\, \mathsf{c}_{qq,t} + k\, \mathsf{c}_{q,t}}{\mathsf{c}_{qq,t}^2} \tag{8a}$$

$$\tau\, \dot{\mathsf{v}}_{p,t} = -\mathsf{v}_{q,t} + \mathsf{v}_{p,t} + \mathsf{u}_{q,t}\, \mathsf{v}_{pp,t} \tag{8b}$$

The two finite hierarchies formed by (5), (6) on the one hand and by (7), (8) on the other are coupled by the stationarity condition for the potential. Such condition yields for the coefficients in (4):

$$\mathsf{u}_{q,t} = \frac{\mathsf{c}_{qq,t}\, (\mathsf{v}_{p,t} + \mathsf{c}_{p,t}\, \mathsf{v}_{pp,t}) - \mathsf{c}_{q,t}\, (\mathsf{c}_{qp,t}\, \mathsf{v}_{pp,t} + 2\, g\, k)}{2\, g\, \mathsf{c}_{qq,t}} \tag{9a}$$

$$\mathsf{u}_{qq,t} = \frac{\mathsf{c}_{qq,t}\, \mathsf{v}_{qp,t} + \mathsf{c}_{qp,t}\, \mathsf{v}_{pp,t} + 2\, g\, k}{2\, g\, \mathsf{c}_{qq,t}} \tag{9b}$$

The optimal control equations are the fully specified by the boundary conditions imposed by (1)

$$\mathsf{c}_{q,t_\iota} = 0 \quad \& \quad \mathsf{c}_{q,t_\mathrm{f}} = \frac{\sqrt{m\, \beta}\, \mathsf{q}_\mathrm{f}}{\tau} \tag{10a}$$

$$\mathsf{c}_{qq,t_\iota} = \sigma_\iota^2 \;\; \& \;\; \mathsf{c}_{qq,t_\mathrm{f}} = \sigma_\mathrm{f}^2 \quad \& \quad \mathsf{c}_{qp,t_\iota} = \mathsf{c}_{qp,t_\mathrm{f}} = 0 \quad \& \quad \mathsf{c}_{pp,t_\iota} = \mathsf{c}_{pp,t_\mathrm{f}} = 1 \tag{10b}$$

We notice that (10) specify a total of ten boundary conditions. The optimal control problem is hence well posed because the coupled system (5), (6), (7), (8) consists also of ten first order differential equations.

III. THE $\mathsf{V}^{(1)} = 0$ CONDITION

The condition translates into the requirement that

$$a_{2,t} = \mathsf{c}_{qq,t}\, \mathsf{v}_{qp,t} + \mathsf{c}_{qp,t}\, \mathsf{v}_{pp,t} \tag{11a}$$

$$b_{2,t} = 2\, \mathsf{c}_{qp,t} + \mathsf{c}_{qq,t}\, \mathsf{v}_{qq,t} - \mathsf{c}_{pp,t}\, \mathsf{v}_{pp,t} \tag{11b}$$

are *vanishing integrals of motion* of the of second order cumulants dynamics and that for any $\mathsf{c}_{qq,t} > 0$

$$a_{1,t} = \mathsf{v}_{p,t} + \mathsf{c}_{q,t}\, \mathsf{v}_{qp,t} + \mathsf{c}_{p,t}\, \mathsf{v}_{pp,t} \tag{12a}$$

$$b_{1,t} = 2\, \mathsf{c}_{p,t} + \mathsf{v}_{q,t} + \mathsf{c}_{q,t}\, \mathsf{v}_{qq,t} + \mathsf{c}_{p,t}\, \mathsf{v}_{qp,t} \tag{12b}$$

are *vanishing integrals of motion* of first order cumulants dynamics. The condition $\mathsf{V}^{(1)} = 0$ is satisfied if we choose the coefficients of the quadratic potential (4) to be

$$\mathsf{u}_{q,t}^{\star} = \mathsf{c}_{q,t}\, \frac{\mathsf{c}_{qq,t}\, (\mathsf{c}_{p,t} - 2\, \mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}) + 2\, \mathsf{c}_{qp,t}^2}{\mathsf{c}_{qq,t}^2} - \mathsf{c}_{p,t} \tag{13a}$$

$$\mathsf{u}^{\star}_{qq,t} = -\frac{\mathsf{c}_{qq,t}\,(\mathsf{c}_{qp,t} - 2\,\mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}) + 2\,\mathsf{c}^{2}_{qp,t}}{\mathsf{c}^{2}_{qq,t}} \tag{13b}$$

The resulting "slow manifold equations" are

$$\tau\,\dot{\mathsf{c}}_{qq,t} = 2\,\mathsf{c}_{qp,t} \tag{14a}$$

$$\tau\,\dot{\mathsf{c}}_{qp,t} = \frac{2\,\mathsf{c}^{2}_{qp,t}}{\mathsf{c}_{qq,t}} - \mathsf{c}_{pp,t} + \mathsf{v}_{pp,t} \tag{14b}$$

$$\tau\,\dot{\mathsf{c}}_{pp,t} = 2\,(1 - \mathsf{c}_{pp,t}) + 2\,\mathsf{c}_{qp,t}\frac{\mathsf{c}_{qq,t}\,(\mathsf{c}_{qp,t} - 2\,\mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}) + 2\,\mathsf{c}^{2}_{qp,t}}{\mathsf{c}^{2}_{qq,t}} \tag{14c}$$

$$\tau\,\dot{\mathsf{v}}_{pp,t} = 2\,\left(1 + \frac{\mathsf{c}_{qp,t}}{\mathsf{c}_{qq,t}}\right)\mathsf{v}_{pp,t} - 2 \tag{14d}$$

for second order cumulants and

$$\tau\,\dot{\mathsf{c}}_{q,t} = \mathsf{c}_{p,t} \tag{15a}$$

$$\tau\,\dot{\mathsf{c}}_{p,t} = 0 \tag{15b}$$

for first order ones. In what follows, we use homogenization (multiscale) perturbation theory [1] to exhibit the relation between (9) and (13) in the limit of vanishing g.

IV. "FENICHEL VARIABLES" AND ASYMPTOTIC FORM OF THE DYNAMICS FOR VANISHING g

We introduce the "Fenichel variables" [2–5]:

$$w_{2-,t} = \frac{a_{2,t} + \sqrt{g}\,b_{2,t}}{2\,g} \tag{16a}$$

$$w_{2+,t} = \frac{a_{2,t} - \sqrt{g}\,b_{2,t}}{2\,g} \tag{16b}$$

in the second order equations and

$$w_{1-,t} = \frac{a_{1,t} + \sqrt{g}\,b_{1,t}}{2\,g} \tag{17a}$$

$$w_{1+,t} = \frac{a_{1,t} - \sqrt{g}\,b_{1,t}}{2\,g} \tag{17b}$$

in the first order equations. The optimal control equations for second order cumulants become

$$\sqrt{g}\,\tau\,\dot{w}_{2-,t} = -w_{2-,t} - \frac{\mathsf{c}_{qq,t}(k + \mathsf{c}_{qp,t} - 2\,\mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}) + 2\,\mathsf{c}^{2}_{qp,t}}{\mathsf{c}_{qq,t}} + \sqrt{g}\,G_{2w-,t} \tag{18a}$$

$$\sqrt{g}\,\tau\,\dot{w}_{2+,t} = w_{2+,t} + \frac{\mathsf{c}_{qq,t}(k + \mathsf{c}_{qp,t} - 2\,\mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}) + 2\,\mathsf{c}^{2}_{qp,t}}{\mathsf{c}_{qq,t}} + \sqrt{g}\,G_{2w+,t} \tag{18b}$$

$$\tau\,\dot{\mathsf{c}}_{qq,t} = 2\,\mathsf{c}_{qp,t} \tag{18c}$$

$$\tau\,\dot{\mathsf{c}}_{qp,t} = -k - \mathsf{c}_{qp,t} + \mathsf{c}_{pp,t} - \frac{w_{2-,t} + w_{2+,t}}{2} \tag{18d}$$

$$\tau\,\dot{\mathsf{c}}_{pp,t}=2\,(1-\mathsf{c}_{pp,t})-\frac{\mathsf{c}_{qp,t}\,\left(2\,k+w_{2-,t}+w_{2+,t}\right)}{\mathsf{c}_{qq,t}}\tag{18e}$$

$$\tau\,\dot{\mathsf{v}}_{pp,t}=2\left(1+\frac{\mathsf{c}_{qp,t}}{\mathsf{c}_{qq,t}}\right)\mathsf{v}_{pp,t}-2-2\,g\,\frac{w_{2-,t}+w_{2+,t}}{\mathsf{c}_{qq,t}}\tag{18f}$$

with

$$G_{2w-,t}=w_{2-,t}\frac{\mathsf{c}_{qq,t}+2\,\mathsf{c}_{qp,t}}{2\,\mathsf{c}_{qq,t}}+w_{2+,t}\frac{\mathsf{c}_{qq,t}-2\,\mathsf{c}_{qp,t}}{2\,\mathsf{c}_{qq,t}}+\sqrt{g}\,\frac{w_{2-,t}+w_{2+,t}}{\mathsf{c}_{qq,t}}\left(k+\mathsf{c}_{pp,t}+\frac{w_{2-,t}+w_{2+,t}}{4}\right)\tag{19a}$$

$$G_{2w+,t}=w_{2-,t}\frac{\mathsf{c}_{qq,t}-2\mathsf{c}_{qp,t}}{2\,\mathsf{c}_{qq,t}}+w_{2+,t}\frac{\mathsf{c}_{qq,t}+2\,\mathsf{c}_{qp,t}}{2\,\mathsf{c}_{qq,t}}-\sqrt{g}\,\frac{w_{2-,t}+w_{2+,t}}{\mathsf{c}_{qq,t}}\left(k+\mathsf{c}_{pp,t}+\frac{w_{2-,t}+w_{2+,t}}{4}\right)\tag{19b}$$

The equations for first order cumulants in terms of Fenichel variables become

$$\sqrt{g}\,\tau\,\dot{w}_{1-,t}=-w_{1-,t}-\mathsf{c}_{p,t}+\sqrt{g}\,G_{1w-,t}\tag{20a}$$

$$\sqrt{g}\,\tau\,\dot{w}_{1,t}=w_{1+,t}-\mathsf{c}_{p,t}+\sqrt{g}\,G_{1w+,t}\tag{20b}$$

$$\tau\,\dot{\mathsf{c}}_{q,t}=\mathsf{c}_{p,t}\tag{20c}$$

$$\tau\,\dot{\mathsf{c}}_{p,t}=-\frac{2\,\mathsf{c}_{p,t}+w_{1-,t}-w_{1+,t}}{2\,\mathsf{c}_{qq,t}}\tag{20d}$$

with

$$G_{1w-,t}=\frac{w_{1-,t}}{2}-\frac{w_{1+,t}}{2}+\sqrt{g}\,k\,\frac{w_{1-,t}+w_{1+,t}}{2\,\mathsf{c}_{qq,t}}\tag{21a}$$

$$G_{1w+,t}=-\frac{w_{1-,t}}{2}+\frac{w_{1+,t}}{2}+\sqrt{g}\,k\,\frac{w_{1-,t}-w_{1+,t}}{2\,\mathsf{c}_{qq,t}}\tag{21b}$$

The new sets of equations (18) and (20) exhibit the "slow-fast" [2] nature of the optimal control dynamics in the limit of vanishing g. Namely, if we set g to zero, (18a), (18b) and (20a), (20b) become algebraic equations for $w_{i\pm,t}$ $i=1,2$. Once we insert the solutions in the remaining differential equations we obtain the dynamics driven by (13). On the other hand, if we first rescale the time variable as

$$t\mapsto\frac{t}{\sqrt{g}}\equiv\tilde{t}\tag{22}$$

and then set g to zero, we get into

$$\dot{\mathsf{c}}_{qq,\tilde{t}}=\dot{\mathsf{c}}_{qp,\tilde{t}}=\dot{\mathsf{c}}_{pp,\tilde{t}}=\dot{\mathsf{v}}_{pp,\tilde{t}}=0\tag{23}$$

and

$$\dot{\mathsf{c}}_{q,\tilde{t}}=\dot{\mathsf{c}}_{p,\tilde{t}}=0\tag{24}$$

The limit dynamics with respect to fast time (23) of the Fenichel coordinates $z_{i,\tilde{t}}$, $w_{i,\tilde{t}}$ $i=1,2$ describe then the linear stable and unstable manifolds around the "fixed point" specifying the slow manifold of the dynamics. We can encapsulate the foregoing observations into a systematic asymptotic analysis of the control equations using a two-time perturbative expansion. We implement such an expansion by inserting into (18), (20) expressions like

$$w_{2-,t}=w^{(0)}_{2-,\frac{t}{\sqrt{g}},t}+\sqrt{g}\,w^{(1)}_{2-,\frac{t}{\sqrt{g}},t}+O(g)\equiv w^{(0)}_{2-,\tilde{t},t}+\sqrt{g}\,w^{(1)}_{2-,\tilde{t},t}+O(g)\tag{25}$$

A. Two-time expansion of second order cumulants

At leading order the only variables depending upon the fast time are

$$
\begin{aligned}
w^{(0)}_{2-,\tilde{t}\,t} =& -\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}} \\
&+e^{-\frac{\tilde{t}}{\tau}}\left[w^{(0)}_{2-,0\,t}+\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}}\right]
\end{aligned}
\tag{26a}
$$

$$
\begin{aligned}
w^{(0)}_{2+,\tilde{t}\,t} =& -\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}} \\
&+e^{\frac{\tilde{t}}{\tau}}\left[w^{(0)}_{2+,0,t}+\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}}\right]
\end{aligned}
\tag{26b}
$$

We readily see that the argument of the square brackets is a function of the "slow" time t quantifying the deviation from the slow manifold. The slow manifold naturally dynamics emerges as a solvability condition for the first order correction. Namely, upon denoting by "primes" derivatives with respect to the fast variables we obtain as first order correction to (26) a six dimensional *linear* system in the *fast* time of the form

$$
\tau\,\boldsymbol{X}'_{2,\tilde{t},t}+\mathsf{M}_{2,t}\cdot\boldsymbol{X}_{2,\tilde{t},t}+\boldsymbol{F}_{2,t}=0
\tag{27}
$$

We used the notation $\boldsymbol{X}=\left[w^{(1)}_{2-,\tilde{t},t},w^{(1)}_{2+,\tilde{t},t},\mathsf{c}^{(1)}_{qq,\tilde{t},t},\mathsf{c}^{(1)}_{qp,\tilde{t},t},\mathsf{c}^{(1)}_{pp,\tilde{t},t},\mathsf{v}^{(1)}_{pp,\tilde{t},t}\right]$ and

$$
\mathsf{M}_{2,t}=\begin{bmatrix}
1 & 0 & -2\left(\dfrac{\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\right)^{2} & 1+\dfrac{4\,\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}} & -2 & 1\\
0 & -1 & 2\left(\dfrac{\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\right)^{2} & -1-\dfrac{4\,\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}} & 2 & -1\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{28}
$$

Finally the "forcing" $\boldsymbol{F}_{2,t}$ depends only upon the "slow" time t. Inspection of (33) readily shows that the matrix is constant with respect to the fast time and that the kernel of its adjoint is four-dimensional. Homogenization theory [1] guarantee us in such a case that we can use the slow time dependence of zero order terms to cancel the projection of $\boldsymbol{F}_{2,t}$ onto the kernel of $\mathsf{M}^{\dagger}_{2,t}$ in order to satisfy the conditions imposed by Fredholm's alternative. We obtain as solvability conditions for Fredholm's alternative equations (14a), (14d) together with

$$
\begin{aligned}
\tau\,\dot{\mathsf{c}}^{(0)}_{qp,t} =& \frac{2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}}-\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t} \\
&-\frac{e^{-\frac{\tilde{t}}{\tau}}}{2}\left[w^{(0)}_{2-,0\,t}+\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}}\right] \\
&-\frac{e^{\frac{\tilde{t}}{\tau}}}{2}\left[w^{(0)}_{2-,0\,t}+\frac{\mathsf{c}^{(0)}_{qq,t}\,\left(k+\mathsf{c}^{(0)}_{qp,t}-2\,\mathsf{c}^{(0)}_{pp,t}+\mathsf{v}^{(0)}_{pp,t}\right)+2\,\mathsf{c}^{(0)}_{qp,t}{}^{2}}{\mathsf{c}^{(0)}_{qq,t}}\right]
\end{aligned}
\tag{29a}
$$

$$\tau\,\dot{\mathsf{c}}^{(0)}_{pp,t} = 2\left(1-\mathsf{c}^{(0)}_{pp,t}\right) + 2\,\mathsf{c}^{(0)}_{qp,t}\,\frac{\mathsf{c}^{(0)}_{qq,t}\left(\mathsf{c}^{(0)}_{qp,t} - 2\,\mathsf{c}^{(0)}_{pp,t} + \mathsf{v}^{(0)}_{pp,t}\right) + 2\,\mathsf{c}^{(0)\,2}_{qp,t}}{\mathsf{c}^{(0)\,2}_{qq,t}}$$
$$-e^{-\frac{\tilde{t}}{\tau}}\,\frac{\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\left[w^{(0)}_{2-,0\,t} + \frac{\mathsf{c}^{(0)}_{qq,t}\left(k+\mathsf{c}^{(0)}_{qp,t} - 2\,\mathsf{c}^{(0)}_{pp,t} + \mathsf{v}^{(0)}_{pp,t}\right) + 2\,\mathsf{c}^{(0)\,2}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\right]$$
$$-e^{\frac{\tilde{t}}{\tau}}\,\frac{\mathsf{c}^{(0)}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\left[w^{(0)}_{2-,0\,t} + \frac{\mathsf{c}^{(0)}_{qq,t}\left(k+\mathsf{c}^{(0)}_{qp,t} - 2\,\mathsf{c}^{(0)}_{pp,t} + \mathsf{v}^{(0)}_{pp,t}\right) + 2\,\mathsf{c}^{(0)\,2}_{qp,t}}{\mathsf{c}^{(0)}_{qq,t}}\right] \tag{29b}$$

The solvability conditions (29) allows us to draw two conclusions. The first is that we can achieve a complete scale separation by choosing $w_{2\pm,0,t}$ so to cancel the $\tilde{t}$ dependence from the right hand side of the solvability conditions (29). Geometrically, such a choice pins the dynamics on the slow manifold equations (14). We emphasize that since $\boldsymbol{F}_{2,t}$ does not depend on the fast time, we can always choose $w^{(1)}_{2\mp,0,t}$ such that it is on the fixed point. The second is that *independently* of the parameter k the vanishing g limit of the second order cumulants control equations is described by the six dimensional slow-fast system

$$\tau\,\dot{\bar{w}}_{2-,t} = -\frac{\bar{w}_{2-,t}}{\sqrt{g}} \tag{30a}$$

$$\tau\,\dot{\bar{w}}_{2-,t} = \frac{\bar{w}_{2+,t}}{\sqrt{g}} \tag{30b}$$

$$\tau\,\dot{\mathsf{c}}_{qq,t} = 2\,\mathsf{c}_{qp,t} \tag{30c}$$

$$\tau\,\dot{\mathsf{c}}_{qp,t} = \frac{2\,\mathsf{c}^2_{qp,t}}{\mathsf{c}_{qq,t}} - \mathsf{c}_{pp,t} + \mathsf{v}_{pp,t} - \frac{\bar{w}_{2-,t}+\bar{w}_{2+,t}}{2} \tag{30d}$$

$$\tau\,\dot{\mathsf{c}}_{pp,t} = 2\left(1-\mathsf{c}_{pp,t}\right) + 2\,\mathsf{c}_{qp,t}\frac{\mathsf{c}_{qq,t}\left(\mathsf{c}_{qp,t} - 2\,\mathsf{c}_{pp,t} + \mathsf{v}_{pp,t}\right) + 2\,\mathsf{c}^2_{qp,t}}{\mathsf{c}^2_{qq,t}} - \frac{\mathsf{c}_{qp,t}}{\mathsf{c}_{qq,t}}(\bar{w}_{2-,t}+\bar{w}_{2+,t}) \tag{30e}$$

$$\tau\,\dot{\mathsf{v}}_{pp,t} = 2\left(1+\frac{\mathsf{c}_{qp,t}}{\mathsf{c}_{qq,t}}\right)\mathsf{v}_{pp,t} - 2 \tag{30f}$$

1. *Two-time expansion of first order cumulants*

Proceeding in full analogy with the previous section we find at leading order in the two-times expansion

$$\tau\,\dot{w}_{1-,\tilde{t},t} = -w_{1-,\tilde{t},t} - \mathsf{c}^{(0)}_{p,\tilde{t},t} \tag{31a}$$

$$\tau\,\dot{w}_{1+,\tilde{t},t} = w_{1-,\tilde{t},t} - \mathsf{c}^{(0)}_{p,\tilde{t},t} \tag{31b}$$

$$\mathsf{c}'_{q,\tilde{t},t} = 0 \tag{31c}$$

$$\mathsf{c}'_{p,\tilde{t},t} = 0 \tag{31d}$$

Hence first order cumulants in the large time scale approximation do not depend upon the fast time. Upon writing the first order correction in the form of the linear differential equation in the fast time

$$\tau\,\boldsymbol{X}'_{1,\tilde{t},t} + \mathsf{M}_{1,t}\cdot\boldsymbol{X}_{2,\tilde{t},t} + \boldsymbol{F}_{1,t} = 0 \tag{32}$$

We used the notation $\boldsymbol{X} = \left[w^{(1)}_{1-,\tilde{t},t}, w^{(1)}_{1+,\tilde{t},t}, \mathsf{c}^{(1)}_{q,\tilde{t},t}, \mathsf{c}^{(1)}_{p,\tilde{t},t}\right]$ and

$$\mathsf{M}_{1,t} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

The solvability conditions for the linear system are then specified by (15). We arrive therefore to the slow-fast system

$$\tau\, \dot{\bar{w}}_{1-,t} = -\frac{\bar{w}_{1-,t}}{\sqrt{g}} \tag{34a}$$

$$\tau\, \dot{\bar{w}}_{1+,t} = \frac{\bar{w}_{1+,t}}{\sqrt{g}} \tag{34b}$$

$$\tau\, \dot{\mathsf{c}}_{q,t} = 2\, \mathsf{c}_{p,t} \tag{34c}$$

$$\tau\, \dot{\mathsf{c}}_{p,t} = \frac{\bar{w}_{1-,t} - \bar{w}_{1+,t}}{2\, \mathsf{c}_{qq,t}} \tag{34d}$$

governing the limit of vanishing g of the optimal control equations for the first order cumulants.

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