PROBLEM 4

PART 1 + PART 3

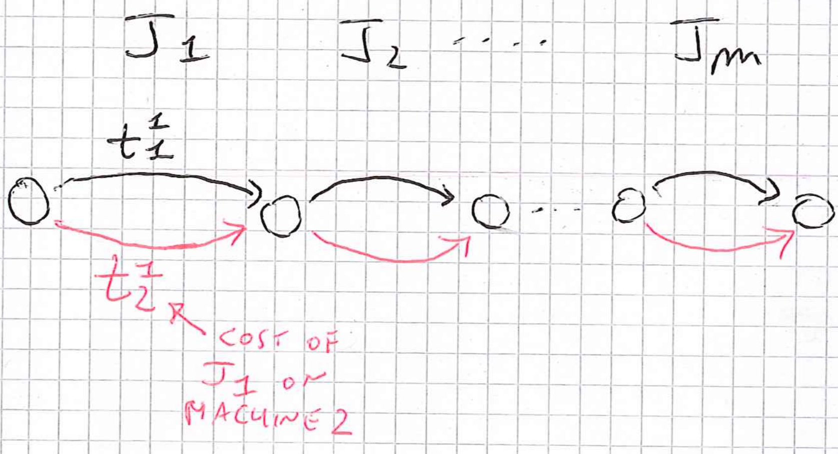
MAIN INTUITION:

THIS PROBLEM CAN ENCODE (REDUCTION) THE PROBLEM OF SCHEDULING UNRELATED MACHINES:

- NO α-APX TRUTHFUL FOR $\alpha < 2$
- TRIVIAL 2-APX USING VCG

} 2 UNRELATED MACHINES

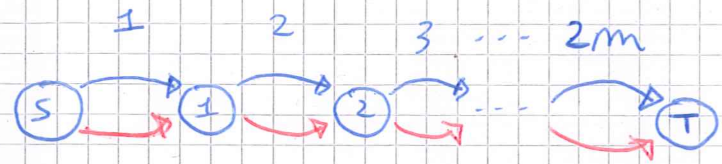
VISUALLY:



PROBLEM ~~5~~ 4

SUPPOSE A TRUTHFUL MECHANISM FOR MIN MAXCOST EXISTS

CONSIDER THIS INSTANCE:



IN WHICH ALL EDGE COSTS ARE $\underline{1}$

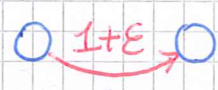
WE HAVE $2m$ NODES, WITHOUT COUNTING (S), SO

$$OPT = m$$

THE MECHANISM (A, P) SELECTS m BLUE EDGES, AND m RED.

NOW CHANGE RED COSTS AS FOLLOWS

EDGES SELECTED IN PREVIOUS CASE (ALL COSTS = 1)



EDGES NOT SELECTED IN PREVIOUS CASE (ALL COSTS = 1)

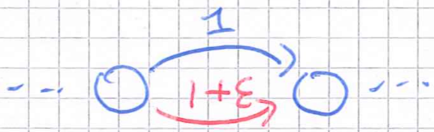
WE HAVE TWO POSSIBLE COSTS FOR PLAYER i = RED

$$c_i = (\dots, \underset{m}{1}, \dots)$$

$$c_i^e \qquad e \in E_i$$

$\hat{c}_i$ WHERE $\hat{c}_i^e = \begin{cases} \varepsilon & \text{IF SELECTED IN } c_i \\ 1+\varepsilon & \text{OTHERWISE} \end{cases}$

FOR $\hat{c}_i$, AND BLUE PLAYER UNCHANGED, THE OPTIMUM MUST SELECT ALL ε EDGES, AND ABOUT HALF OF THE OTHERS



WHICH ARE m – SO $OPT \leq (1+\varepsilon)\frac{m}{2}$ AND IT MUST SELECT K EDGES OF COST $1+\varepsilon$ FOR $K \geq 1$

TRUTHFULNES REQUIRES THAT, IF c_i IS THE TRUE COST, THEN REPORTING $\hat{c}_i$ IS NOT BENEFICIAL, AND ALSO THE OPPOSITE CASE – THAT IS

TRUE $\qquad\qquad\qquad\qquad$ FALSE

$$c_i \qquad\qquad\qquad\qquad \hat{c}_i$$

$$\boxed{P(c_i, c_{-i}) - m \geq P(\hat{c}_i, c_{-i}) - (m+k)}$$

$$\hat{c}_i \qquad\qquad\qquad\qquad c_i$$

$$\boxed{P(\hat{c}_i, c_{-i}) - \varepsilon^{m} - k(1+\varepsilon) \geq P(c_i, c_{-i}) - \varepsilon m}$$

SUMMING UP THE TWO INEQUALITIES WE GET

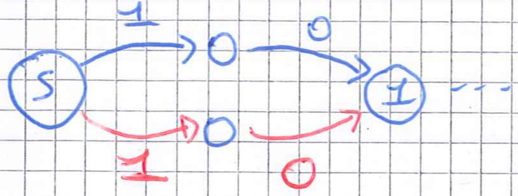
$$-m - \varepsilon m - k - k\varepsilon \geq -m - k - \varepsilon m$$

$$\Updownarrow$$

$$-k\varepsilon \geq 0$$

WHICH IS A CONTRADICTION WITH $k \geq 1$ _

IF WE DO NOT ALLOW PARALLEL EDGES, THE FOLLOWING IS AN EQUIVALENT INSTANCE



INAPX: SUPPOSE (A,P) IS α-APX FOR $\alpha < 2$.

IN THE PREVIOUS ISTANCES:

ALL 1'S $\Rightarrow$ ONE PLAYER MUST GET AT LEAST 1 EDGE, AND AT MOST m

(SAY IT IS THE RED)

$\Uparrow$

CHANGE COSTS OF THIS PLAYER AS BEFORE

$$\hat{C}_i^e \to \begin{cases} \varepsilon \text{ IF } e \text{ WAS TAKEN} \\ 1+\varepsilon \text{ IF } e \text{ WAS NOT TAKEN} \end{cases}$$

IN $\hat{C}_i$ PLAYER i GETS SOME OF ε-EDGES AND AT LEAST ℓ OF $(1+\varepsilon)$-EDGES

LET k = # OF ε-EDGES

$\hat{k}$ = # OF ε-EDGES SELECTED IN $\hat{C}_i$

IF $\ell = 0$ IN $\hat{C}_i$: THE MECHANISM IS AT LEAST

$$\frac{m}{OPT'} - APX$$

WHERE $OPT' \leq \left\lceil \frac{m}{2} \right\rceil (1+\varepsilon)$

FOR m LARGE ENOUGH

$$\frac{m}{OPT'} > \alpha \quad \text{FOR ANY } \alpha < 2$$

IF $\ell \geq 1$ WE VIOLATE TRUTHFULNESS:

$$\boxed{\overset{C_i}{\overbrace{P_i - k}} \geq \overset{\hat{C}_i}{\overbrace{\hat{P}_i - \hat{k} - \ell}}}$$

$$\boxed{\overset{\hat{C}_i}{\overbrace{\hat{P}_i - \hat{k}\varepsilon - \ell(1+\varepsilon)}} \geq \overset{C_i}{\overbrace{P_i - k\varepsilon}}}$$

$$-k-\hat{k}\varepsilon-\ell-\ell\varepsilon \geq -\hat{k}-\ell-k\varepsilon$$

$$\Updownarrow$$

$$-k-\hat{k}\varepsilon-\ell\varepsilon \geq -\hat{k}-k\varepsilon$$

$$\Updownarrow$$

$$\underbrace{(\hat{k}-k)+\varepsilon(k-\hat{k})}_{(k-\hat{k})(\varepsilon-1)} \geq \varepsilon\ell$$

$$\underset{\substack{\vee\!\shortmid \\ 0}}{(k-\hat{k})}\underset{\substack{\wedge \\ 0}}{(\varepsilon-1)} \geq \varepsilon\ell$$

CONTRADICTI[ON] WITH $\ell \geq 1$

BECAUSE $\hat{k}$ IS THE NUMBER OF ε- EDGES CHOSEN IN $\hat{C}_i$, AND k IS THE NUMBER OF ε-EDGES

PROBLEM 4 (PART 2)

THE VCG MECHANISM COMPUTING THE SHORTEST PATH (MINIMIZE THE SUM OF EDGE COSTS) IS TRUTHFUL (VCG MECHANISMS ARE TRUTHFUL)

WE SHOW IT IS AN m-APX FOR MINMAX COST:

Q = SOL OF VCG (SHORTEST PATH)

Q^* = OPTIMUM FOR MINMAX COST

THEN

$\max_i c_i(Q) \leq$ (MAX COST AT MOST SUM ALL COSTS)

$\sum_i c_i(Q) \leq$ (Q MINIMIZES SUM OF COSTS)

$\sum_i c_i(Q^*) \leq$ (IN ANY SOL SUM ALL COSTS AT MOST $m \cdot$ MAX COST)

$m \cdot \max_i c_i(Q^*) =$ (Q^* OPTIMAL FOR MIN MAX COST)

$m \cdot \mathrm{OPT}$