

Reversibility and Mixing Time for Logit Dynamics with Concurrent Updates

Vincenzo Auletta, Diodato Ferraioli, Francesco Pasquale,
Paolo Penna, and Giuseppe Persiano

Dipartimento di Informatica “Renato M. Capocelli”
Università di Salerno
auletta,ferraioli,pasquale,penna,giuper@dia.unisa.it

July 13, 2012

Abstract

Logit dynamics [Blume, Games and Economic Behavior, 1993] is a randomized best response dynamics where at every time step a player is selected uniformly at random and she chooses a new strategy according to the “logit choice function”, i.e. a probability distribution biased towards strategies promising higher payoffs, where the bias level corresponds to the degree of rationality of the agents. While the logit choice function is a very natural behavioral model for *approximately rational* agents, the specific revision process that selects one single player per time step seems less justified. In this paper we thus focus on the dynamics where at every time step *every* player *simultaneously* updates her strategy according to the logit choice function. We call such a dynamics the “all-logit”, as opposed to the classical “one-logit” dynamics.

The all-logit dynamics for a game induces an ergodic Markov chain over the set of strategy profiles which is significantly different from the Markov chain induced in the one-logit case. In this paper we first highlight similarities and differences between the two dynamics with some simple examples of two-player games; we then give a characterization of the class of games such that the Markov chains induced by the all-logit dynamics are reversible and we show it is a subclass of potential games; finally, we analyze the mixing time of the all-logit dynamics for a well-known coordination game.

1 Introduction

This paper considers the classical game-theoretic scenario of n selfish players, each one with a set of possible actions or strategies trying to maximize her own payoff or utility. The utility obtained by a player depends not only on her strategy but also on the strategies adopted by the other players. Of this classical scenario, we are interested in the dynamics by which players update their own strategies eventually bringing the system to reach a stable state. Any such dynamics can be seen as composed of two ingredients:

- **Selection rule:** by which the set of players that update their strategy is determined;
- **Update rule:** by which the selected players update their strategy.

Roughly speaking, the classical notion of a Nash equilibrium can be seen as the stable state of the dynamics that composes the *best response* with a selection rule that selects one player at the time. In the best response update rule, the selected player picks the strategy that, given the current strategies of the other players, guarantees the highest utility.

In this paper, we study a specific class of randomized update rules called the *logit choice function* [12, 7, 16] which is a type of noisy best response that models in a clean and tractable way the limited knowledge (or bounded rationality) of the players in terms of a parameter β (in similar models studied in Physics, β is the inverse of the temperature). Intuitively, a low value of β (that is, high temperature) models the situation where players choose their strategies “nearly at random”; a high value of β (that is, low temperature and entropy) models players that “almost surely” play their best response; that is, they pick the strategies yielding higher payoffs with higher probability. The logit choice function can be coupled with different selection rules so to give different dynamics. For example, in the *logit* dynamics [7] at every time step a single player is selected uniformly at random and the selected player updates her strategy according to the logit choice function. The remaining players are not allowed to revise their strategies in this time step. One of the appealing features of the logit dynamics is that it naturally describes an ergodic Markov chain. This means that the underlying Markov chain admits a *unique stationary distribution* which we take as solution concept. This distribution describes the long-run behavior of the system (which states appear more frequently over a long run). The interplay between the noise and the underlying game naturally determines the system behavior: (i) As the noise becomes “very large” the equilibrium point is “approximately” the uniform distribution; (ii) As the noise vanishes the stationary distribution concentrates on so called stochastically stable states which, for certain classes of games, correspond to pure Nash equilibria.

A distinctive feature of our work is the focus on the *mixing time* of the underlying Markov chain; that is, the time necessary to reach the stationary distribution. Our general approach consists in studying the long-term behavior of a system of n selfish players by looking at the stationary distribution of Markov chain induced by the specific dynamics that we take as descriptive of the behavior of selfish players with bounded rationality. This conceptual framework though is meaningful only if stationarity is reached quickly (in our approach “quickly” means in time polynomial in the number of players) so that observables measured at stationarity are descriptive of the system. On the other hand, if too long (that is, time exponential in the number of players) is taken to reach equilibrium then one can say that the system is never in the stationarity and measures taken at stationarity do not say anything about the system.

While the logit choice function is a very natural behavioral model for approximately rational agents, the specific selection rule that selects one single player per time step seems less justified.

Ideally, the selected equilibrium point should be the “natural” result of the game and of the players rationality level, and *not* of the selection rule. Therefore a natural question arises

What happens if *concurrent* updates are allowed?

For example, it is easy to construct games for which the best response converges to a Nash equilibrium when only one player is selected at each step and does not converge to any state when more players are chosen to concurrently update their strategies. Motivated by this, we study a dynamics in which *all* players update their strategies at every time step and the update rule is the logit choice function. We call such a dynamics *all-logit*, as opposed to the classical (*one*-)logit dynamics in which only one player at a time is allowed to move. Admittedly, this dynamics might be considered even less natural than the logit dynamics. However, it can be regarded as the “most different” version of noisy best response dynamics and thus as a sort of “worst-case” scenario in terms of robustness of logit dynamics. Furthermore, this dynamics has been also considered in [1] under the name of “instantaneous learning”. The reason for this term is to consider the scenario in which players revise their strategies whenever they learn that another player has changed strategy. Note also that concurrent updates can be the result of players updating their strategies based on outdated information.

Our contribution. Our goal is to understand the effect of the concurrent selection rule on the logit choice function and to study this dynamics (concurrent move + logit choice function) for every possible inverse noise β and see which properties of the original (one-)logit dynamics are preserved, and which are not. We compare the stationary distribution and the mixing time of the two dynamics (one-logit vs all-logit) for the same game.

As a warm-up, we discuss a few classical two-player games for which (1) the stationary distributions of the one-logit and of the all-logit are the same but the mixing times are significantly different, (2) the stationary distributions are different but the mixing times are the same, or (3) the stationary distributions and the mixing times are essentially the same. In particular one cannot infer that the two dynamics converge to the equilibrium in approximately the same time if the equilibrium is the same. Conversely, it can happen that the two dynamics converge in (asymptotically) the same time though to very different equilibria.

We then study *reversibility* of the dynamics, an important property of stochastic processes which is also useful to obtain explicit formulas for the stationary distribution. We *characterize* the class of games for which the all-logit dynamics (that is, the Markov chain resulting from the all-logit dynamics) is reversible as a proper (though natural) subclass of potential games that we name *social potential games*. Social potential games generalize graphical games (games that are played by the nodes of a network and each node plays the same game with all of its neighbors). This class of games includes the games used to model the diffusion of technology in a social network [17, 18]. As a by-product, one obtains that the all-logit dynamics of two-player potential games are reversible, while not all potential games result in a reversible all-logit dynamics. This is to be compared to the well-known result saying that one-logit dynamics of every potential game is reversible with stationary distribution being the Gibbs measure [7]. The Gibbs measure gives an intuitive characterization of the most “likely” states as those with minimal potential. One of the tools we develop for our characterization yields a closed formula for the stationary distribution of reversible all-logit dynamics.

Finally, we give the first bounds on the mixing time of the all-logit. We start by giving a *general* upper bound on the mixing time of the all-logit in terms of the *cumulative utility* of the game. We then look at the well-known n -player Ising model on the complete graph (also called the Curie-Weiss model in Statistical Physics) and derive an upper bound on the mixing time that is tighter than the one obtained from our general upper bound. We complement the upper

bound for the Ising model on the complete graph with a lower bound. The two bounds show that the mixing time is constant for $\beta = O(1/n^2)$, polynomial in n for $\beta = O(\log n/n^2)$, and exponential for $\beta = \Omega(1/n)$. The mixing time for β between $\log n/n^2$ and $1/n$ is still open.

Related works. The all-logit dynamics for strategic games has been studied by Alós-Ferrer and Netzer in [1] which can be seen as complementary to our work. Specifically, in [1] the authors study *general* selection rules (including the selection rule of the all-logit) and investigate conditions for which a state is *stochastically stable* under the dynamics resulting from combining the selection rule with logit-choice function. A stochastically stable state is a state that has non-zero probability as the noise vanishes. On the contrary, we focus on a very particular selection rule but consider the whole range of values of the noise. Our bounds are quantitative in the sense that we express the stationary distribution and the mixing time as a function of the inverse noise β . One of the results in [1] says that, for a class of selection rules that does not include the one of the all-logit, the set of stochastically stable states of the corresponding dynamics is a subset of Nash equilibria. They also derive an explicit formula for the stationary distribution of logit dynamics with general selection rules and a consequent characterization of stochastically stable states.

In contrast, the (one-)logit dynamics has been actively studied starting from the work of Blume [7] that showed that for 2×2 coordination games, the long-term behavior of the system is concentrated in the risk dominant equilibrium (see [9]). The mixing time and the metastability of the (one-)logit dynamics for strategic games has been studied by [4, 3, 5]. Much work has been devoted to the study of the (one-)logit for graphical coordination games as they are used to model the spread of a new technology in a social network [8, 18]. A general upper bound on the mixing time of the (one-)logit graphical coordination game is given by Berger et al. [6]. Montanari and Saberi [15] instead studied the hitting time of the highest potential configuration in graphical coordination games and relate this quantity to a connectivity property of the underlying network.

2 Definitions

In this section we briefly recall some standard game-theoretic notation and we formally define the Markov chain induced by the all-logit dynamics.

Strategic games. Let $\mathcal{G} = ([n], S_1, \dots, S_n, u_1, \dots, u_n)$ be a finite normal-form strategic game. The set $[n] = \{1, \dots, n\}$ is the player set, S_i is the set of *strategies* for player $i \in [n]$, $S = S_1 \times S_2 \times \dots \times S_n$ is the set of *strategy profiles* and $u_i: S \rightarrow \mathbb{R}$ is the *utility* function of player $i \in [n]$.

We adopt the standard game-theoretic notation and denote by S_{-i} the set $S_{-i} = S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$ and, for $\mathbf{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in S_{-i}$ and $y \in S_i$, we denote by (\mathbf{x}, y) the strategy profile $(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \in S$.

Potential games [14] constitute an important class of games. We say that function $\Phi: S \rightarrow \mathbb{R}$ is an *exact potential* (or simply a *potential*) for game \mathcal{G} if for every $i \in [n]$ and every $\mathbf{x} \in S_{-i}$

$$u_i(\mathbf{x}, y) - u_i(\mathbf{x}, z) = \Phi(\mathbf{x}, z) - \Phi(\mathbf{x}, y)$$

for all $y, z \in S_i$. A game \mathcal{G} that admits a potential is called a *potential game*.

Logit choice function. We study the interaction of n players of a strategic game \mathcal{G} that update their strategy according to the *logit choice function* [12, 7, 16] described as follows:

from profile $\mathbf{x} \in S$ player $i \in [n]$ updates her strategy to $y \in S_i$ with probability

$$\sigma_i(y | \mathbf{x}) = \frac{e^{\beta u_i(\mathbf{x}_{-i}, y)}}{\sum_{z \in S_i} e^{\beta u_i(\mathbf{x}_{-i}, z)}}. \quad (1)$$

In other words, the logit choice function leans towards strategies promising higher utility. The parameter $\beta \geq 0$ is a measure of how much the utility influences the choice of the player.

All-logit. In this paper we consider the *all-logit* dynamics, by which *all* players *concurrently* update their strategy. Most of the previous works have focused on dynamics where at each step *one* player is chosen uniformly at random and she updates her strategy by following the logit choice function. We call that dynamics *one-logit*, to distinguish it from the *all-logit*.

The all-logit dynamics induces a Markov chain over the set of strategy profiles whose transition probability $P(\mathbf{x}, \mathbf{y})$ from profile $\mathbf{x} = (x_1, \dots, x_n)$ to profile $\mathbf{y} = (y_1, \dots, y_n)$ is

$$P(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \sigma_i(y_i | \mathbf{x}) = \frac{e^{\beta \sum_{i=1}^n u_i(\mathbf{x}_{-i}, y_i)}}{\prod_{i=1}^n \sum_{z \in S_i} e^{\beta u_i(\mathbf{x}_{-i}, z)}}. \quad (2)$$

Sometimes it is useful to write the transition probability from \mathbf{x} to \mathbf{y} in terms of the *cumulative utility* of \mathbf{x} with respect to \mathbf{y} defined as $U(\mathbf{x}, \mathbf{y}) = \sum_i u_i(\mathbf{x}_{-i}, y_i)$. Indeed, by observing that

$$\prod_{i=1}^n \sum_{z \in S_i} e^{\beta u_i(\mathbf{x}_{-i}, z)} = \sum_{\mathbf{z} \in S} \prod_{i=1}^n e^{\beta u_i(\mathbf{x}_{-i}, z_i)},$$

we can rewrite (2) as

$$P(\mathbf{x}, \mathbf{y}) = \frac{e^{\beta U(\mathbf{x}, \mathbf{y})}}{D(\mathbf{x})}, \quad (3)$$

where $D(\mathbf{x}) = \sum_{\mathbf{z} \in S} e^{\beta U(\mathbf{x}, \mathbf{z})}$. For a potential game \mathcal{G} with potential Φ , we can define the *cumulative potential* of \mathbf{x} with respect to \mathbf{y} as $\Psi(\mathbf{x}, \mathbf{y}) = \sum_i \Phi(\mathbf{x}_{-i}, y_i)$. Simple algebraic manipulations show that, for a potential game, we can rewrite the transition probabilities in (3) as

$$P(\mathbf{x}, \mathbf{y}) = \frac{e^{-\beta \Psi(\mathbf{x}, \mathbf{y})}}{T(\mathbf{x})},$$

where $T(\mathbf{x}) = \sum_{\mathbf{z} \in S} e^{-\beta \Psi(\mathbf{x}, \mathbf{z})}$.

It is easy to see that a Markov chain with transition matrix (2) is ergodic. Indeed, for example, ergodicity follows from the fact that all entries of the transition matrix are strictly positive.

Mixing time. An ergodic Markov chain has a unique stationary distribution π and for every starting profile \mathbf{x} the distribution $P^t(\mathbf{x}, \cdot)$ of the chain at time t converges to π as t goes to infinity. The *mixing time* is a measure of how long it takes to get close to the stationary distribution from the *worst-case* starting profile

$$t_{\text{mix}}(\varepsilon) = \inf \{t \in \mathbb{N} : \|P^t(\mathbf{x}, \cdot) - \pi\|_{\text{TV}} \leq \varepsilon \text{ for all } \mathbf{x} \in S\},$$

where $\|P^t(\mathbf{x}, \cdot) - \pi\|_{\text{TV}} = \frac{1}{2} \sum_{\mathbf{y} \in S} |P^t(\mathbf{x}, \mathbf{y}) - \pi(\mathbf{y})|$ is the *total variation distance*. We will usually write t_{mix} for $t_{\text{mix}}(1/4)$. We refer the reader to [11] for a more detailed description of notational conventions about Markov chains and mixing times.

3 Warm-up: two-player games

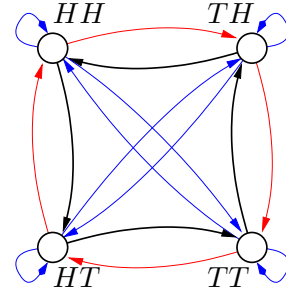
A first natural question is whether there is any relation between the one-logit and the all-logit in terms of stationary distribution and mixing time. In this section we study stationary distribution and mixing time of the all-logit dynamics for three simple two-player games and we compare them with the corresponding results for the one-logit case. The analysis of the three games highlights that, in some sense, “everything can happen”. Indeed, in the *Matching Pennies* example the stationary distribution of the all-logit dynamics is the same as the stationary distribution of the one-logit one, but mixing times are completely different; in the *two-site Ising* example the stationary distribution is different from the one-logit case, but the mixing time is asymptotically the same; in the *Prisoner’s Dilemma* example both stationary distribution and mixing time have the same *qualitative* behavior as in the one-logit case. In addition we observe that, for 2×2 games, an important observable of the one-logit dynamics (the *magnetization*) is preserved when we look at the all-logit dynamics.

Matching Pennies. We start by considering the classical matching pennies game. Here the first player wants to coordinate while the second player prefers not to. The payoff matrix is

| | H | T |
|-----|----------|----------|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

According to (1) and (2), the transition matrix of the Markov chain induced by the all-logit dynamics is

$$P = \begin{pmatrix} & \begin{array}{c|cc} & HH & HT & TH & TT \end{array} \\ \begin{array}{c} HH \\ HT \\ TH \\ TT \end{array} & \begin{array}{cccc} p(1-p) & (1-p)^2 & p^2 & p(1-p) \\ p^2 & p(1-p) & p(1-p) & (1-p)^2 \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ p(1-p) & p^2 & (1-p)^2 & p(1-p) \end{array} \end{pmatrix}$$



where $p = 1/(1 + e^{2\beta})$.¹

The transition matrix is doubly-stochastic so the uniform distribution is stationary (as for the one-logit case) and it is easy to prove that the mixing time is $\Theta(1/p) = \Theta(e^{2\beta})$ (while in the one-logit case it was upper bounded by a constant independent of β).

Notice that the black arrows in the picture draw a cycle over the four states, so the chain becomes more and more *periodic* as β goes to infinity.

Two-site Ising game. This game models the interaction of two particles each one having a possible magnetization (thus two states are possible, $+$ and $-$) and the game assigns a higher utility to profiles in which the two players have the same magnetization. For this example, we look at the case in which there is no external magnetic field and thus the utilities of profiles $++$ and $--$ are equal. This game is a special case of a two-player coordination game (the class

¹In the pictures of this section, black arrows indicate probabilities going to 1 while blue and red arrows indicate probabilities going to 0, as β goes to infinity. Red arrows go to zero faster than blue ones.

of games studied by [7]) with no risk-dominant strategy [9]. The utilities of the two-site Ising game are the following

| | | |
|---|--------|--------|
| | + | - |
| + | +1, +1 | -1, -1 |
| - | -1, -1 | +1, +1 |

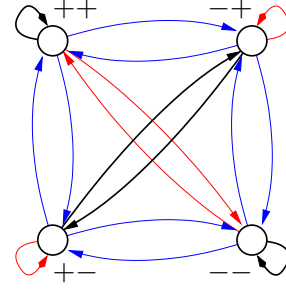
It can be easily seen that the game is a potential game with potential Φ such that

$$\Phi(++) = \Phi(--)= -1 \quad \text{and} \quad \Phi(+-) = \Phi(-+) = 1.$$

It is well known that the stationary distribution of the one-logit of a potential game is the Gibbs distribution. For the two-site Ising game, the Gibbs distribution assigns to $\mathbf{x} \in \{+, -\}^2$ probability $e^{-\beta\phi(\mathbf{x})}/Z$, where $Z = \sum_{\mathbf{x} \in \{+, -\}^2} e^{-\beta\phi(\mathbf{x})}$ is the *partition function*.

The transition matrix of the Markov chain induced by the all-logit dynamics is

$$P = \begin{pmatrix} & ++ & +- & -+ & -- \\ ++ & (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ +- & p(1-p) & p^2 & (1-p)^2 & p(1-p) \\ -+ & p(1-p) & (1-p)^2 & p^2 & p(1-p) \\ -- & p^2 & p(1-p) & p(1-p) & (1-p)^2 \end{pmatrix}$$



where $p = 1/(1 + e^{2\beta})$ again.

The fact that the transition matrix is doubly-stochastic implies that the stationary distribution of the all logit is uniform. The chain is reversible and the mixing time is $\Theta(1/p) = \Theta(e^{2\beta})$ (as in the one-logit case).

Prisoner's Dilemma. Finally, we consider the following payoff matrix of a general two-player symmetric game

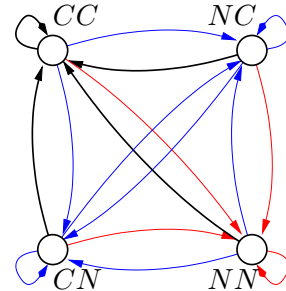
| | | |
|---|--------|--------|
| | C | N |
| C | a, a | c, d |
| N | d, c | b, b |

In order for this payoff matrix to model the Prisoner's Dilemma (C ="Confess", N ="Not confess") we want:

1. $a > d$ so that CC is a Nash equilibrium;
2. $b < c$ so that NN is not Nash equilibrium;
3. $2a < c + d < 2b$ so that NN is the social optimum and CC is the worst social profile.

The transition matrix of the Markov chain induced by the all-logit dynamics is

$$P = \begin{pmatrix} & CC & CN & NC & NN \\ CC & (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ CN & (1-p)(1-q) & p(1-q) & q(1-p) & pq \\ NC & (1-p)(1-q) & q(1-p) & p(1-q) & pq \\ NN & (1-q)^2 & q(1-q) & q(1-q) & q^2 \end{pmatrix}$$



where $p = 1/(1 + e^{(a-d)\beta})$ is the probability a player does not confess (N) at next step given the other player is currently confessing (C) and $q = 1/(1 + e^{(c-b)\beta})$ is the probability a player does not confess (N) at the next step given the other player is currently not confessing (N). From conditions 1 and 2, it follows that both p and q go to 0 as β goes to infinity. Notice that all the black arrows in the picture point at state CC .

Observe that if $p = q$ (i.e. if $a - d = c - b$) then the transition matrix becomes

$$P = \begin{pmatrix} & CC & CN & NC & NN \\ \begin{matrix} CC \\ CN \\ NC \\ NN \end{matrix} & \begin{matrix} (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \end{matrix} \end{pmatrix}$$

In this case the stationary distribution is $\pi(CC) = (1-p)^2$, $\pi(NN) = p^2$ and $\pi(CN) = \pi(NC) = p(1-p)$. Moreover, the transition probability to a state does not depend on the previous state, thus after just one step the chain is stationary. In the general case the stationary distribution is

$$\pi(CC) = \frac{(1-q)^2}{(1+p-q)^2} \quad \pi(NC) = \pi(CN) = \frac{p(1-q)}{(1+p-q)^2} \quad \pi(NN) = \frac{p^2}{(1+p-q)^2}.$$

The stationary probability of state CC goes to 1 as β goes to infinity, and it is easy to prove that the mixing time is upper bounded by a constant independent of β .

Two-player potential games. As we have seen from the examples above the stationary of the one-logit and of the all-logit are markedly different. Notice though that for both the examples above that are potential games (Ising game and Prisoner's dilemma) the all-logit dynamics is reversible. This is no coincidence and we shall see that the all-logit is reversible for all two-player potential games. Actually, in Section 4, we will characterize the class of games for which the all-logit dynamics is reversible. It turns out that it is a subclass of potential games, which we will call *social* potential games, and that includes all two-player potential games. In this section, we discuss some further relations between the one-logit and the all-logit of two-player games, starting from the magnetization.

Let \mathcal{G} be a game in which each player has two strategies, denoted by 0 and 1. We define the *magnetization* $M(\mathbf{x})$ of profile \mathbf{x} of \mathcal{G} as the difference between the number of players that play 0 and the number of players that play 1 in \mathbf{x} . The *expected magnetization* $\mathbb{E}_1(M)$ of the one-logit of \mathcal{G} is the expected value of $M(\mathbf{x})$ for \mathbf{x} distributed according to the stationary distribution of the one-logit. Similarly, the *expected magnetization* $\mathbb{E}_A(M)$ of the all-logit of \mathcal{G} is the expected value of $M(\mathbf{x})$ for \mathbf{x} distributed according to the stationary distribution of the all-logit.

In several games the magnetization has an interesting interpretation. The magnetization in the Ising game is naturally interpreted as the magnetization that one expects to observe once stationarity is reached. For the simple case of no external magnetic field that we have discussed above, it is easy to see that the expected magnetization is 0 both for the one-logit and for the all-logit. For another example, coordination games on a network have been used to study the diffusion of a new technology (or of a new social norm, in general) in a social network (see [18]). Here each player can choose whether to stay with the old technology (corresponding to strategy 0) or switch to the new technology (corresponding to strategy 1). In this context the magnetization is simply the difference between the number of users that stayed with old technology and the number of adopters of the new technology.

We next show that for all 2×2 potential games \mathcal{G} , it holds that $\mathbb{E}_1(M) = \mathbb{E}_A(M)$. A 2×2 game \mathcal{G} is a two-player game in which each player has two strategies, called 0 and 1. Let $\Phi : \{0, 1\}^2 \rightarrow \mathbb{R}$ be a potential for the 2×2 game \mathcal{G} . For $\mathbf{x} \in \{0, 1\}^2$ set $\pi_1(\mathbf{x}) = e^{\beta \cdot \Phi(\mathbf{x})}$ and $Z_1 = \sum_{\mathbf{x} \in \{0, 1\}^2} \pi_1(\mathbf{x})$. As we have already recalled, the Gibbs measure $\pi_1(\mathbf{x})/Z_1$ is the stationary distribution of the one-logit of a potential game with potential Φ (see [7]). The expected magnetization $\mathbb{E}_1(M)$ of the one-logit of \mathcal{G} is thus

$$\mathbb{E}_1(M) = 2 \cdot \frac{\pi_1(00) - \pi_1(11)}{Z_1}.$$

To study the magnetization of the all-logit, we set $\pi_A(\mathbf{x})$ as

$$\begin{aligned} \pi_A(00) &= (\pi_1(00) + \pi_1(01)) \cdot (\pi_1(00) + \pi_1(10)) \\ \pi_A(01) &= (\pi_1(00) + \pi_1(01)) \cdot (\pi_1(01) + \pi_1(11)) \\ \pi_A(10) &= (\pi_1(00) + \pi_1(10)) \cdot (\pi_1(10) + \pi_1(11)) \\ \pi_A(11) &= (\pi_1(01) + \pi_1(11)) \cdot (\pi_1(10) + \pi_1(11)) \end{aligned}$$

and $Z_A = \sum_{\mathbf{x} \in \{0, 1\}^2} \pi_A(\mathbf{x})$ is the partition function of the all-logit. From Theorem 4.7, it is not difficult to see that, for a 2×2 potential game, $\pi_A(\mathbf{x})/Z_A$ is the stationary distribution of the all-logit. The expected magnetization $\mathbb{E}_A(M)$ of the all-logit of \mathcal{G} is thus

$$\mathbb{E}_A(M) = 2 \cdot \frac{\pi_A(00) - \pi_A(11)}{Z_A}.$$

We observe that

$$\begin{aligned} Z_A &= (\pi_A(00) + \pi_A(01)) + (\pi_A(10) + \pi_A(11)) \\ &= (\pi_1(00) + \pi_1(01)) \cdot Z_1 + (\pi_1(10) + \pi_1(11)) \cdot Z_1 \\ &= Z_1^2 \end{aligned}$$

That is, the partition function of the all-logit is the square of the partition function of the one-logit. On the other hand,

$$\begin{aligned} \pi_A(00) - \pi_A(11) &= (\pi_1(00) + \pi_1(01)) \cdot (\pi_1(00) + \pi_1(10)) \\ &\quad - (\pi_1(01) + \pi_1(11)) \cdot (\pi_1(10) + \pi_1(11)) \\ &= (\pi_1(00) - \pi_1(11)) \cdot Z_1 \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{E}_A(M) &= 2 \cdot \frac{\pi_A(00) - \pi_A(11)}{Z_A} \\ &= 2 \cdot \frac{(\pi_1(00) - \pi_1(11)) \cdot Z_1}{Z_A} \\ &= 2 \cdot \frac{\pi_1(00) - \pi_1(11)}{Z_1} \\ &= \mathbb{E}_1(M). \end{aligned}$$

4 Reversibility and stationary distribution

Reversibility is an important property of Markov chains and, in general, of stochastic processes. Roughly speaking, for a reversible Markov chain the stationary frequency of transitions from a

state x to a state y is equal to the stationary frequency of transitions from y to x . It is easy to see that the one-logit for a game \mathcal{G} is reversible if and only if \mathcal{G} is a potential game. As we shall see, this does not hold for the all-logit. Indeed, we will prove that the class of games for which the all-logit is reversible is a subclass of potential games: those games that are the “sum” of two-player potential games.

4.1 Reversibility criteria

Let \mathcal{M} be a Markov chain with transition matrix P and state set S . \mathcal{M} is *reversible* with respect to a distribution π if, for every pair of states $x, y \in S$, the following *detailed balance condition* holds

$$\pi(x)P(x, y) = \pi(y)P(y, x).$$

It is easy to see that if \mathcal{M} is reversible with respect to π then π is also stationary. The Kolmogorov reversibility criterion allows us to establish the reversibility of a process directly from the transition probabilities. Before stating the criterion, we introduce the following notation. A *directed path* Γ from state $x \in S$ to state $y \in S$ is a sequence of states $\langle x_0, x_1, \dots, x_\ell \rangle$ such that $x_0 = x$ and $x_\ell = y$. The probability $\mathbf{P}(\Gamma)$ of path Γ is defined as $\mathbf{P}(\Gamma) = \prod_{j=1}^{\ell} P(x_{j-1}, x_j)$. The *inverse of path* $\Gamma = \langle x_0, x_1, \dots, x_\ell \rangle$ is the path $\Gamma^{-1} = \langle x_\ell, x_{\ell-1}, \dots, x_0 \rangle$. Finally, a *cycle* C is simply a path from a state x to itself. We are now ready to state Kolmogorov’s reversibility criterion (for a proof see, for example, [10]).

Theorem 4.1 (Kolmogorov’s Reversibility Criterion). *An irreducible Markov chain \mathcal{M} with transition matrix P and state space S is reversible if and only if for every cycle C it holds that*

$$\mathbf{P}(C) = \mathbf{P}(C^{-1}).$$

We have the following lemma.

Lemma 4.2. *Let \mathcal{M} be an irreducible Markov chain with transition probability P and state space S . \mathcal{M} is reversible if and only if for every pair of states $x, y \in S$, there exists a constant $c_{x,y}$ such that for all paths Γ from x to y , it holds that*

$$\frac{\mathbf{P}(\Gamma)}{\mathbf{P}(\Gamma^{-1})} = c_{x,y}.$$

Proof. Fix $x, y \in S$ and consider two paths, Γ_1 and Γ_2 , from x to y . Let C_1 and C_2 be the cycles $C_1 = \Gamma_1 \circ \Gamma_2^{-1}$ and $C_2 = \Gamma_2 \circ \Gamma_1^{-1}$, where \circ denotes the concatenation of paths. If \mathcal{M} is reversible then, by the Kolmogorov Reversibility Criterion, $\mathbf{P}(C_1) = \mathbf{P}(C_2)$. On the other hand,

$$\mathbf{P}(C_1) = \mathbf{P}(\Gamma_1) \cdot \mathbf{P}(\Gamma_2^{-1}) \quad \text{and} \quad \mathbf{P}(C_2) = \mathbf{P}(\Gamma_2) \cdot \mathbf{P}(\Gamma_1^{-1}).$$

Thus

$$\frac{\mathbf{P}(\Gamma_1)}{\mathbf{P}(\Gamma_1^{-1})} = \frac{\mathbf{P}(\Gamma_2)}{\mathbf{P}(\Gamma_2^{-1})}.$$

For the other direction, fix $z \in S$ and, for all $x \in S$, set $\tilde{\pi}(x) = c_{z,x}/Z$, where $Z = \sum_x c_{z,x}$ is the normalizing constant. Now consider any two states $x, y \in S$ of \mathcal{M} , let Γ_1 be any path from z to x and set $\Gamma_2 = \Gamma_1 \circ \langle x, y \rangle$ (that is, Γ_2 is Γ_1 concatenated with the edge (x, y)). We

have that

$$\begin{aligned}
\frac{\tilde{\pi}(x)}{\tilde{\pi}(y)} &= \frac{c_{z,x}}{c_{z,y}} \\
&= \frac{\mathbf{P}(\Gamma_1)}{\mathbf{P}(\Gamma_1^{-1})} \cdot \frac{\mathbf{P}(\Gamma_2)}{\mathbf{P}(\Gamma_2^{-1})} \\
&= \frac{\mathbf{P}(\Gamma_1)}{\mathbf{P}(\Gamma_1^{-1})} \cdot \frac{\mathbf{P}(\Gamma_1^{-1}) \cdot P(y, x)}{\mathbf{P}(\Gamma_1) \cdot P(x, y)} \\
&= \frac{P(y, x)}{P(x, y)}
\end{aligned}$$

and therefore \mathcal{M} is reversible with respect to $\tilde{\pi}$. \square

The above lemma is very useful as it also gives an expression for the stationary distribution π . Specifically, if we fix a state z , then for all x , $\pi(x)$ is proportional to $P(x, z)/P(z, x)$

4.2 Reversibility implies potential games

In this section we prove that if the all-logit for a game \mathcal{G} is reversible then \mathcal{G} is a potential game.

The following lemma shows a condition on the cumulative utility of a game \mathcal{G} that is necessary and sufficient for the reversibility of the all-logit of \mathcal{G} .

Lemma 4.3. *The all-logit for game \mathcal{G} is reversible if and only if the following property holds for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$:*

$$U(\mathbf{x}, \mathbf{y}) - U(\mathbf{y}, \mathbf{x}) = \left(U(\mathbf{x}, \mathbf{z}) + U(\mathbf{z}, \mathbf{y}) \right) - \left(U(\mathbf{y}, \mathbf{z}) + U(\mathbf{z}, \mathbf{x}) \right). \quad (4)$$

Proof. To prove the only if part, pick any three $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$ and consider paths $\Gamma_1 = \langle \mathbf{x}, \mathbf{y} \rangle$ $\Gamma_2 = \langle \mathbf{x}, \mathbf{z}, \mathbf{y} \rangle$. From Lemma 4.2 we have that reversibility implies

$$\frac{\mathbf{P}(\Gamma_1)}{\mathbf{P}(\Gamma_1^{-1})} = \frac{\mathbf{P}(\Gamma_2)}{\mathbf{P}(\Gamma_2^{-1})}$$

whence

$$\frac{e^{\beta U(\mathbf{x}, \mathbf{y})}}{D(\mathbf{x})} \frac{D(\mathbf{y})}{e^{\beta U(\mathbf{y}, \mathbf{x})}} = \frac{e^{\beta U(\mathbf{x}, \mathbf{z})}}{D(\mathbf{x})} \frac{e^{\beta U(\mathbf{z}, \mathbf{y})}}{D(\mathbf{z})} \frac{D(\mathbf{y})}{e^{\beta U(\mathbf{y}, \mathbf{z})}} \frac{D(\mathbf{z})}{e^{\beta U(\mathbf{z}, \mathbf{x})}}.$$

which in turn implies 4.

As for the if part, let us fix state $\mathbf{z} \in S$ and define $\tilde{\pi}(\mathbf{x}) = \frac{P(\mathbf{z}, \mathbf{x})}{Z \cdot P(\mathbf{x}, \mathbf{z})}$, where Z is the normalizing constant. For any $\mathbf{x}, \mathbf{y} \in S$, we have

$$\frac{\tilde{\pi}(\mathbf{x})}{\tilde{\pi}(\mathbf{y})} = \frac{P(\mathbf{z}, \mathbf{x})}{P(\mathbf{x}, \mathbf{z})} \cdot \frac{P(\mathbf{y}, \mathbf{z})}{P(\mathbf{z}, \mathbf{y})} = \frac{e^{\beta U(\mathbf{z}, \mathbf{x})}}{e^{\beta U(\mathbf{x}, \mathbf{z})}} \cdot \frac{e^{\beta U(\mathbf{y}, \mathbf{z})}}{e^{\beta U(\mathbf{z}, \mathbf{y})}} \cdot \frac{D(\mathbf{x})}{D(\mathbf{y})} = \frac{e^{\beta U(\mathbf{y}, \mathbf{x})}}{e^{\beta U(\mathbf{x}, \mathbf{y})}} \cdot \frac{D(\mathbf{x})}{D(\mathbf{y})} = \frac{P(\mathbf{y}, \mathbf{x})}{P(\mathbf{x}, \mathbf{y})},$$

where the first equality follows from the definition of $\tilde{\pi}$, the second and the fourth follow from (3) and the third follows from (4). Therefore, the detailed balance equation holds for $\tilde{\pi}$ and thus the Markov chain is reversible. \square

We are now ready to prove that the all-logit is reversible only for potential games.

Theorem 4.4. *If the all-logit for game \mathcal{G} is reversible, then \mathcal{G} is a potential game.*

Proof. We show that if the all-logit is reversible then the utility improvement over any cycle of length 4 is 0. The theorem then follows by Theorem A.1.

Consider circuit $\Gamma = \langle \mathbf{x}, \mathbf{z}, \mathbf{y}, \mathbf{w} \rangle$ and let i be the player in which \mathbf{x} and \mathbf{z} differ and let j be the player in which \mathbf{z} and \mathbf{y} differ. Then \mathbf{y} and \mathbf{w} differ in player i and \mathbf{w} and \mathbf{x} differ in player j . In other words, $\mathbf{z} = (\mathbf{x}_{-i}, y_i) = (\mathbf{y}_{-j}, x_j)$ and $\mathbf{w} = (\mathbf{x}_{-i}, y_j) = (\mathbf{y}_{-i}, x_i)$. Therefore we have that

$$\begin{aligned} U(\mathbf{x}, \mathbf{y}) &= \sum_{k \neq i, j} u_k(\mathbf{x}) + u_i(\mathbf{z}) + u_j(\mathbf{w}) & U(\mathbf{y}, \mathbf{x}) &= \sum_{k \neq i, j} u_k(\mathbf{y}) + u_i(\mathbf{w}) + u_j(\mathbf{z}) \\ U(\mathbf{x}, \mathbf{z}) &= \sum_{k \neq i, j} u_k(\mathbf{x}) + u_i(\mathbf{z}) + u_j(\mathbf{x}) & U(\mathbf{z}, \mathbf{y}) &= \sum_{k \neq i, j} u_k(\mathbf{z}) + u_i(\mathbf{z}) + u_j(\mathbf{y}) \\ U(\mathbf{y}, \mathbf{z}) &= \sum_{k \neq i, j} u_k(\mathbf{y}) + u_i(\mathbf{y}) + u_j(\mathbf{z}) & U(\mathbf{z}, \mathbf{x}) &= \sum_{k \neq i, j} u_k(\mathbf{z}) + u_i(\mathbf{x}) + u_j(\mathbf{z}) \end{aligned}$$

By plugging the above expressions into (4) and rearranging terms, we obtain

$$\left(u_i(\mathbf{z}) - u_i(\mathbf{x}) \right) + \left(u_j(\mathbf{y}) - u_j(\mathbf{z}) \right) + \left(u_i(\mathbf{w}) - u_i(\mathbf{y}) \right) + \left(u_j(\mathbf{x}) - u_j(\mathbf{w}) \right) = 0$$

which shows that $I(\Gamma) = 0$. □

4.3 A necessary and sufficient condition for reversibility

In the previous section we have established that the all-logit is reversible only for potential games and therefore, from now on, we only consider potential games \mathcal{G} with potential function Φ . In this section we present in Theorem 4.6 a necessary and sufficient condition for reversibility that involves the potential and the cumulative potential. The condition will then be used in the next section to prove that the n -player games that are the sum of two player potential games are exactly the games whose all-logit is reversible.

We start by re-writing Lemma 4.3 in terms of cumulative potential.

Lemma 4.5. *The all-logit is reversible if and only if for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$:*

$$\Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}) = \left(\Psi(\mathbf{x}, \mathbf{z}) + \Psi(\mathbf{z}, \mathbf{y}) \right) - \left(\Psi(\mathbf{y}, \mathbf{z}) + \Psi(\mathbf{z}, \mathbf{x}) \right). \quad (5)$$

We are now ready to prove a necessary and sufficient condition for reversibility that involves potential and cumulative potential.

Theorem 4.6. *The all-logit for \mathcal{G} is reversible if and only if \mathcal{G} has potential function Φ such that*

$$\Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}) = (n - 2) (\Phi(\mathbf{x}) - \Phi(\mathbf{y})) \quad \text{for every } \mathbf{x}, \mathbf{y} \in S. \quad (6)$$

Proof. Clearly (6) implies (5). As for the other direction, we proceed by induction on the Hamming distance between \mathbf{x} and \mathbf{y} . Let \mathbf{x} and \mathbf{y} be two profiles at Hamming distance 1; that is, \mathbf{x} and \mathbf{y} differ in only one player, say j . This implies that $(y_j, \mathbf{x}_{-j}) = \mathbf{y}$ and $(x_j, \mathbf{y}_{-j}) = \mathbf{x}$. Moreover, for $i \neq j$, $(y_i, \mathbf{x}_{-i}) = \mathbf{x}$ and $(x_i, \mathbf{y}_{-i}) = \mathbf{y}$. Thus,

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}) &= \sum_i \left(\Phi(y_i, \mathbf{x}_{-i}) - \Phi(x_i, \mathbf{y}_{-i}) \right) \\ &= \left(\Phi(y_j, \mathbf{x}_{-j}) - \Phi(x_j, \mathbf{y}_{-j}) \right) + \sum_{i \neq j} \left(\Phi(y_i, \mathbf{x}_{-i}) - \Phi(x_i, \mathbf{y}_{-i}) \right) \\ &= \left(\Phi(\mathbf{y}) - \Phi(\mathbf{x}) \right) + (n - 1) \left(\Phi(\mathbf{x}) - \Phi(\mathbf{y}) \right) = (n - 2) \left(\Phi(\mathbf{x}) - \Phi(\mathbf{y}) \right). \end{aligned}$$

Now assume that the claim holds for any pair of profiles at Hamming distance $k < n$ and let \mathbf{x} and \mathbf{y} be two profiles at distance $k + 1$. Let j be any player such that $x_j \neq y_j$ and let

$\mathbf{z} = (y_j, \mathbf{x}_{-j})$: \mathbf{z} is at distance at most k from \mathbf{x} and from \mathbf{y} . Then, by (5) and by the inductive hypothesis, we have

$$\begin{aligned}\Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}) &= \left(\Psi(\mathbf{x}, \mathbf{z}) + \Psi(\mathbf{z}, \mathbf{y}) \right) - \left(\Psi(\mathbf{y}, \mathbf{z}) + \Psi(\mathbf{z}, \mathbf{x}) \right) \\ &= (n-2) \left(\Phi(\mathbf{x}) + \Phi(\mathbf{z}) - \Phi(\mathbf{y}) - \Phi(\mathbf{z}) \right) = (n-2) \left(\Phi(\mathbf{x}) - \Phi(\mathbf{y}) \right).\end{aligned}$$

□

The above lemma and Lemma 4.2 allow us to express the stationary distribution for reversible all-logit dynamics in the following way. We remind the reader that $T(\mathbf{x})$ has been defined as $T(\mathbf{x}) = \sum_{\mathbf{z} \in S} e^{-\beta \Psi(\mathbf{x}, \mathbf{z})}$.

Lemma 4.7. *Let \mathcal{G} be a potential game with potential function Φ and reversible all-logit. Then the stationary distribution of the all-logit for \mathcal{G} is*

$$\pi(\mathbf{x}) \propto e^{(n-2)\beta \Phi(\mathbf{x})} \cdot T(\mathbf{x}). \quad (7)$$

Proof. Fix any profile \mathbf{y} . The detailed balance equation gives for every $\mathbf{x} \in S$

$$\frac{\pi(\mathbf{x})}{\pi(\mathbf{y})} = \frac{P(\mathbf{y}, \mathbf{x})}{P(\mathbf{x}, \mathbf{y})} = e^{\beta(\Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}))} \frac{T(\mathbf{x})}{T(\mathbf{y})}.$$

By Lemma 4.6 we have

$$\pi(\mathbf{x}) = e^{(n-2)\beta \Phi(\mathbf{x})} \cdot T(\mathbf{x}) \left(\frac{\pi(\mathbf{y})}{e^{(n-2)\beta \Phi(\mathbf{y})} \cdot T(\mathbf{y})} \right).$$

Since the term in parenthesis does not depend on \mathbf{x} the lemma follows. □

We end the section by making an observation that will be used for proving the main result in the following.

Observation 4.8. *Let Φ be a potential function satisfying (6). Then for all $\mathbf{x}, \mathbf{y} \in S$,*

$$\sum_{i: x_i \neq y_i} \left(\Phi(y_i, \mathbf{x}_{-i}) - \Phi(x_i, \mathbf{y}_{-i}) \right) = (h-2) (\Phi(\mathbf{x}) - \Phi(\mathbf{y})),$$

where h is the Hamming distance between \mathbf{x} and \mathbf{y} .

4.4 Social potential games

In this section we prove that the games whose all-logit is reversible are exactly those potential games whose potential can be written as a sum of two-player potentials. We call these games *social potential games*.

A potential $\Phi : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$ is a *two-player potential* if there exist $u, v \in [n]$ such that, for any $\mathbf{x}, \mathbf{y} \in S$ with $x_u = y_u$ and $x_v = y_v$ we have $\Phi(\mathbf{x}) = \Phi(\mathbf{y})$. In other words, Φ is a function of only its u -th and v -th argument. We say that potential game \mathcal{G} with potential Φ is a *social potential game* if there exist N two-player potentials Φ_1, \dots, Φ_N such that $\Phi = \Phi_1 + \dots + \Phi_N$. It is easy to see that generality is not lost by further requiring that $1 \leq l \neq l' \leq N$ implies $(u_l, v_l) \neq (u_{l'}, v_{l'})$, where u_l and v_l are the two players of potential Φ_l .

At every social potential game \mathcal{G} with potential $\Phi = \Phi_1 + \dots + \Phi_N$, we can associate a *social graph* G that has a vertex for each player of \mathcal{G} and has edge (u, v) iff there exists l such that potential Φ_l depends on players u and v . In other words, we can see the players of a social potential game as sitting at the vertices of the social graph G and each player playing a (possibly different) two-player potential game with each one of her neighbors.

The following theorem holds.

Theorem 4.9. *The all-logit of a social potential game is reversible.*

Proof. We prove that any two-player potential satisfies (6) and then observe that the sum of two potentials satisfying (6) also satisfies (6).

Let Φ be a two-player potential and let u and v be its two players. Then we have that for $w \neq u, v$, $\Phi(y_w, \mathbf{x}_{-w}) = \Phi(\mathbf{x})$ and that $\Phi(y_u, x_{-u}) = \Phi(x_v, y_{-v})$ and $\Phi(y_v, x_{-v}) = \Phi(x_u, y_{-u})$. Thus

$$\begin{aligned}\Psi(\mathbf{x}, \mathbf{y}) &= \Phi(y_u, \mathbf{x}_{-u}) + \Phi(y_v, \mathbf{x}_{-v}) + (n-2)\Phi(\mathbf{x}) \\ \text{and} \\ \Psi(\mathbf{y}, \mathbf{x}) &= \Phi(x_v, \mathbf{y}_{-v}) + \Phi(x_u, \mathbf{y}_{-u}) + (n-2)\Phi(\mathbf{y}) \\ &= \Phi(y_u, \mathbf{x}_{-u}) + \Phi(y_v, \mathbf{x}_{-v}) + (n-2)\Phi(\mathbf{y})\end{aligned}$$

□

Next we prove that if an n -player potential Φ satisfies (6) and thus the all-logit is reversible then Φ can be written as the sum of at most $N = \binom{n}{2}$ two-player potentials, Φ_1, \dots, Φ_N . We do so by describing an effective procedure that constructs the N two-player potentials.

Without loss of generality, we assume that each strategy set S_i includes strategy 0 and denote by $\mathbf{0}$ the strategy profile consisting of n 0's. Moreover, we fix an arbitrary ordering $(u_1, v_1), \dots, (u_N, v_N)$ of the N unordered pairs of players. For a potential Φ we define the sequence $\vartheta_0, \dots, \vartheta_N$ of potentials as follows: $\vartheta_0 = \Phi$ and, for $i = 1, \dots, N$, set

$$\vartheta_i = \vartheta_{i-1} - \Phi_i \tag{8}$$

where, for $\mathbf{x} \in S$, $\Phi_i(\mathbf{x})$ is defined as

$$\Phi_i(\mathbf{x}) = \vartheta_{i-1}(x_{u_i}, x_{v_i}, \mathbf{0}_{-u_i v_i}).$$

Observe that, for $i = 1, \dots, N$, Φ_i is a two-player potential and its players are u_i and v_i . By summing for $i = 1, \dots, N$ in (8) we obtain

$$\sum_{i=1}^N \vartheta_i = \sum_{i=0}^{N-1} \vartheta_i - \sum_{i=1}^N \Phi_i.$$

Thus

$$\Phi - \vartheta_N = \sum_{i=1}^N \Phi_i.$$

The next two lemmas prove that, if Φ satisfies (6), then ϑ_N is identically zero. This implies that Φ is the sum of at most N non-zero two-player potentials and thus a social potential game.

A ball $B(r, \mathbf{x})$ of radius $r \leq n$ centered in $\mathbf{x} \in S$ is the subset of S containing all profiles \mathbf{y} that differ from \mathbf{x} in at most r coordinates.

Lemma 4.10. *For any n -player potential function Φ and for any ordering of the pairs of players, $\vartheta_N(\mathbf{x}) = 0$ for every $\mathbf{x} \in B(2, \mathbf{0})$.*

Proof. We distinguish three cases based on the distance of \mathbf{x} from $\mathbf{0}$.

$\mathbf{x} = \mathbf{0}$: for every $i \geq 1$, we have

$$\vartheta_i(\mathbf{0}) = \vartheta_{i-1}(\mathbf{0}) - \Phi_i(\mathbf{0}) = \vartheta_{i-1}(\mathbf{0}) - \vartheta_{i-1}(\mathbf{0}) = 0.$$

\mathbf{x} is at distance 1 from $\mathbf{0}$: That is, there exists $u \in [n]$ such that $\mathbf{x} = (x_u, \mathbf{0}_{-u})$, with $x_u \neq 0$.

Let us denote by $t(u)$ the smallest t such that the t -th pair contains u . We next show that for $i \geq t(u)$, $\vartheta_i(\mathbf{x}) = 0$. Indeed, we have that if u is a component of the i -th pair then

$$\vartheta_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \Phi_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \vartheta_{i-1}(\mathbf{x}) = 0;$$

On the other hand, if u is not a component of the i -th pair then

$$\vartheta_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \Phi_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \vartheta_{i-1}(\mathbf{0}) = \vartheta_{i-1}(\mathbf{x});$$

\mathbf{x} is at distance 2 from 0: That is, there exist u and v such that $\mathbf{x} = (x_u, x_v, \mathbf{0}_{-uv})$, with $x_u, x_v \neq 0$.

Let t be the index of the pair (u, v) . Notice that $t \geq t(u), t(v)$. We show that $\vartheta_t(\mathbf{x}) = 0$ and that this value does not change for all $i > t$. Indeed, we have

$$\vartheta_t(\mathbf{x}) = \vartheta_{t-1}(\mathbf{x}) - \Phi_t(\mathbf{x}) = \vartheta_{t-1}(\mathbf{x}) - \vartheta_{t-1}(\mathbf{x}) = 0;$$

If instead neither of u and v belongs to the i -th pair, with $i > t$, then we have

$$\vartheta_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \Phi_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \vartheta_{i-1}(\mathbf{0}) = \vartheta_{i-1}(\mathbf{x});$$

Finally, suppose that the i -th pair, for $i > t$, contains exactly one of u and v , say u . Then we have

$$\vartheta_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \Phi_i(\mathbf{x}) = \vartheta_{i-1}(\mathbf{x}) - \vartheta_{i-1}(x_u, \mathbf{0}_{-u}).$$

We conclude the proof by observing that $i - 1 \geq t \geq t(u)$ and thus, by the previous case, $\vartheta_{i-1}(x_u, \mathbf{0}_{-u}) = 0$. \square

The next lemma shows that if a potential Φ satisfies (6) and is constant in a ball of radius 2, then it is constant everywhere.

Lemma 4.11. *Let Φ be a function that satisfies (6). If there exist $\mathbf{x} \in S$ and $c \in \mathbb{R}$ such that $\Phi(\mathbf{y}) = c$ for every $\mathbf{y} \in B(2, \mathbf{x})$, then $\Phi(\mathbf{y}) = c$ for every $\mathbf{y} \in S$.*

Proof. Fix $h > 2$ and suppose that $\Phi(\mathbf{z}) = c$ for every $\mathbf{z} \in B(h-1, \mathbf{x})$. Consider $\mathbf{y} \in B(h, \mathbf{x}) \setminus B(h-1, \mathbf{x})$ and observe that $(y_i, \mathbf{x}_{-i}) \in B(h-1, \mathbf{x})$ and $(x_i, \mathbf{y}_{-i}) \in B(h-1, \mathbf{x})$ for every i such that $x_i \neq y_i$. From Observation 4.8, we have $(h-2)(\Phi(\mathbf{y}) - \Phi(\mathbf{x})) = 0$ which implies $\Phi(\mathbf{y}) = \Phi(\mathbf{x}) = c$. \square

We can thus conclude that if the all-logit of a potential game \mathcal{G} is reversible then \mathcal{G} is a social potential game. By combining this result with Theorem 4.4 and Theorem 4.9, we obtain

Theorem 4.12. *The all-logit of game \mathcal{G} is reversible if and only if \mathcal{G} is a social potential game.*

5 Mixing time

The all-logit dynamics for a strategic game has the property that, for every pair of profiles \mathbf{x}, \mathbf{y} and for every value of β , the transition probability from \mathbf{x} to \mathbf{y} is strictly positive. In order to give upper bounds on the mixing time, we will use the following simple well-known lemma (see e.g. Theorem 11.5 in [13]).

Lemma 5.1. *Let P be the transition matrix of an ergodic Markov chain with state space Ω . For every $y \in \Omega$ let us name $\alpha_y = \min\{P(x, y) : x \in \Omega\}$ and $\alpha = \sum_{y \in \Omega} \alpha_y$. Then the mixing time of P is $t_{\text{mix}} = \mathcal{O}(1/\alpha)$.*

In this section we first give an upper bound holding for every strategic game. We will then focus on a specific game, the *Ising game* on the clique (also known as the Curie-Weiss model in statistical physics) and we will give a refined version of the upper bound and a lower bound.

For a strategic game \mathcal{G} , in Section 2 we defined the cumulative utility function for the ordered pair of profiles (\mathbf{x}, \mathbf{y}) as $U(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n u_i(\mathbf{x}_{-i}, y_i)$. Let us name ΔU the size of the range of U ,

$$\Delta U = \max\{U(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in S\} - \min\{U(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in S\}.$$

By using Lemma 5.1 we can give a simple upper bound on the mixing time of the all-logit dynamics for \mathcal{G} as a function of β and ΔU .

Theorem 5.2 (General upper bound). *For every strategic game \mathcal{G} the mixing time of the all-logit dynamics for \mathcal{G} is $\mathcal{O}(e^{\beta \Delta U})$.*

Proof. Let P be the transition matrix of the all-logit dynamics for \mathcal{G} and let $\mathbf{x}, \mathbf{y} \in S$ be two profiles. From (3) we have that

$$P(\mathbf{x}, \mathbf{y}) = \frac{e^{\beta U(\mathbf{x}, \mathbf{y})}}{\sum_{\mathbf{z} \in S} e^{\beta U(\mathbf{x}, \mathbf{z})}} = \frac{1}{\sum_{\mathbf{z} \in S} e^{\beta(U(\mathbf{x}, \mathbf{z}) - U(\mathbf{x}, \mathbf{y}))}} \geq \frac{1}{|S| e^{\beta \Delta U}}.$$

Hence for every $\mathbf{y} \in S$ it holds that

$$\alpha_{\mathbf{y}} \geq \frac{e^{-\beta \Delta U}}{|S|}$$

and $\alpha = \sum_{\mathbf{y} \in S} \alpha_{\mathbf{y}} \geq e^{-\beta \Delta U}$. The thesis then follows from Lemma 5.1. \square

5.1 Ising game on the clique

In this section we prove upper and lower bounds on the mixing time of the all-logit dynamics for the Ising game on the clique. In such a game, every player has two strategies, $+1$ and -1 , and the utility of player $i \in [n]$ is the sum of the number of players playing the same strategy as i , minus the number of players playing the opposite strategy, i.e. the utility of player $i \in [n]$ at profile $\mathbf{x} = (x_1, \dots, x_n) \in \{-1, +1\}^n$ is

$$u_i(\mathbf{x}) = x_i \sum_{j \neq i} x_j.$$

It is easy to see that such a game is a potential game with potential function

$$\Phi(\mathbf{x}) = - \sum_{\{i, j\} \in \binom{[n]}{2}} x_i x_j.$$

Due to the high level of symmetry of the game, the potential of a profile \mathbf{x} depends only on the *number* of players playing ± 1 . If we name $k_{\mathbf{x}} := \sum_{i=1}^n x_i$ the *magnetization* of \mathbf{x} we can write the potential of \mathbf{x} as

$$\Phi(\mathbf{x}) = - \frac{k_{\mathbf{x}}^2 - n}{2}.$$

The upper bound. Observe that, for the Ising model on the clique we have $\Delta U = 2n(n-1)$, hence by using Theorem 5.2 we get directly that

$$t_{\text{mix}} = \mathcal{O}\left(e^{2\beta n(n-1)}\right). \quad (9)$$

Hence it follows that mixing time is $\mathcal{O}(1)$ for $\beta = \mathcal{O}(1/n^2)$ and it is $\mathcal{O}(\text{poly}(n))$ for $\beta = \mathcal{O}(\log n/n^2)$.

In what follows we show that factor “2” at the exponent in (9) can be removed and that a slightly better upper bound can be given for $\beta > \log n/n$.

Lemma 5.3. *For every $\mathbf{x}, \mathbf{y} \in \Omega$ it holds that*

$$P(\mathbf{x}, \mathbf{y}) \geq q^{(n+|k_{\mathbf{y}}|)/2} (1-q)^{(n-|k_{\mathbf{y}}|)/2}$$

where

$$q = \frac{1}{1 + e^{2\beta(n-1)}}.$$

Proof. Consider a profile $\mathbf{y} \in \{-1, +1\}^n$ and let $k_{\mathbf{y}}$ be its magnetization. Remember that the number of players playing $+1$ and -1 in \mathbf{y} can be written as $\frac{n+k_{\mathbf{y}}}{2}$ and $\frac{n-k_{\mathbf{y}}}{2}$, respectively. If \mathbf{y} has positive magnetization $k_{\mathbf{y}} > 0$, i.e. if the number of players playing $+1$ is larger than the number of players playing -1 , then the profile that minimizes $P(\mathbf{x}, \mathbf{y})$ is profile $\mathbf{x}_- = (-1, \dots, -1)$ where every player plays -1 . If we name

$$q = \frac{e^{-\beta(n-1)}}{e^{-\beta(n-1)} + e^{\beta(n-1)}} = \frac{1}{1 + e^{2\beta(n-1)}}$$

the probability that a player in \mathbf{x}_- chooses strategy $+1$ for the next round, we have that

$$P(\mathbf{x}_-, \mathbf{y}) = q^{\frac{n+k_{\mathbf{y}}}{2}} (1-q)^{\frac{n-k_{\mathbf{y}}}{2}}.$$

On the other hand, if \mathbf{y} has negative magnetization $k_{\mathbf{y}} < 0$, $P(\mathbf{x}, \mathbf{y})$ is minimized when $\mathbf{x} = \mathbf{x}_+ = (+1, \dots, +1)$ and, since q is also the probability that a player in \mathbf{x}_+ chooses strategy -1 for the next round, we have that

$$P(\mathbf{x}_+, \mathbf{y}) = q^{\frac{n-k_{\mathbf{y}}}{2}} (1-q)^{\frac{n+k_{\mathbf{y}}}{2}}$$

and the thesis follows. \square

Now we can give an upper bound on the mixing time by using lemmata 5.1 and 5.3

Theorem 5.4 (Upper bound). *The mixing time of the all-logit dynamics for the Ising model on the clique is*

$$t_{\text{mix}} = \mathcal{O}\left(ne^{\beta n^2}\right).$$

If $\beta \geq \log n/n$ the mixing time is

$$t_{\text{mix}} = \mathcal{O}\left(\frac{ne^{\beta n^2}}{2^n}\right).$$

Proof. From Lemma 5.3 it follows that for every $\mathbf{y} \in \{-1, +1\}^n$ we have

$$\alpha_{\mathbf{y}} = \min\{P(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \{-1, +1\}^n\} \geq q^{(n+|k_{\mathbf{y}}|)/2} (1-q)^{(n-|k_{\mathbf{y}}|)/2}.$$

Hence

$$\alpha = \sum_{\mathbf{y} \in \{-1, +1\}^n} \alpha_{\mathbf{y}} \geq \sum_{\mathbf{y} \in \{-1, +1\}^n} q^{(n+|k_{\mathbf{y}}|)/2} (1-q)^{(n-|k_{\mathbf{y}}|)/2}. \quad (10)$$

Now observe that there are $\binom{n}{\frac{n-k}{2}}$ profiles with magnetization k , and since $q \leq 1/2$, the largest terms in (10) are the ones with magnetization as close to zero as possible. In order to give a lower bound to α we will thus consider only terms with magnetization $k = 0$, when n is even, and terms with magnetization $k = \pm 1$, when n is odd.

Case n even: If we consider only profiles with magnetization $k = 0$ in (10) we have that

$$\alpha \geq \binom{n}{n/2} [q(1-q)]^{n/2}.$$

By using a standard lower bound for the binomial coefficient (see e.g. Lemma 9.2 in [13]) we have that

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}.$$

As for $[q(1-q)]^{n/2}$ we have that

$$\begin{aligned} q(1-q) &= \frac{1}{1+e^{2\beta(n-1)}} \cdot \frac{1}{1+e^{-2\beta(n-1)}} \\ &= \frac{1}{e^{2\beta(n-1)} + 2 + e^{-2\beta(n-1)}} \\ &= \frac{1}{e^{2\beta(n-1)} (1 + 2e^{-2\beta(n-1)} + e^{-4\beta(n-1)})} \end{aligned} \quad (11)$$

Now observe that for every $\beta \geq 0$ we can bound $1 + 2e^{-2\beta(n-1)} + e^{-4\beta(n-1)} \leq 4$. Thus we have that

$$[q(1-q)]^{n/2} \geq \frac{1}{2^n e^{\beta n(n-1)}}. \quad (12)$$

Hence

$$\alpha \geq \binom{n}{n/2} [q(1-q)]^{n/2} \geq \frac{1}{(n+1)e^{\beta n(n-1)}}.$$

And by using Lemma 5.1 we have

$$t_{\text{mix}} = \mathcal{O}\left(ne^{\beta n(n-1)}\right).$$

If β is large enough, say $\beta \geq \log n/n$, in (11) we can bound

$$1 + 2e^{-2\beta(n-1)} + e^{-4\beta(n-1)} \leq 1 + \frac{1}{n}.$$

Thus, in this case we have that

$$[q(1-q)]^{n/2} \geq \frac{1}{e^{\beta n(n-1)} (1 + 1/n)^{(n/2)}} \geq \frac{1}{e^{\beta n(n-1)} \cdot \sqrt{e}}. \quad (13)$$

Hence $\alpha \geq \frac{2^n}{(n+1)e^{1/2+\beta n(n-1)}}$ and

$$t_{\text{mix}} = \mathcal{O}\left(\frac{ne^{\beta n(n-1)}}{2^n}\right).$$

Case n odd: If we consider only profiles with magnetization ± 1 in (10) we get

$$\alpha \geq 2 \binom{n}{\frac{n+1}{2}} q^{\frac{n+1}{2}} (1-q)^{\frac{n-1}{2}} = 2 \binom{n}{\frac{n+1}{2}} \sqrt{\frac{q}{1-q}} [q(1-q)]^{n/2}.$$

Now observe that

$$\sqrt{\frac{q}{1-q}} = e^{-\beta(n-1)} \quad \text{and} \quad \binom{n}{\frac{n+1}{2}} \geq \frac{1}{2} \cdot \frac{2^n}{n+1}.$$

By using bounds (12) and (13) for $[q(1-q)]^{n/2}$ we get $t_{\text{mix}} = \mathcal{O}\left(ne^{\beta(n^2-1)}\right)$ for every $\beta \geq 0$ and $t_{\text{mix}} = \mathcal{O}\left(\frac{ne^{\beta(n^2-1)}}{2^n}\right)$ for $\beta \geq \log n/n$. \square

The lower bound. A key function for the all-logit dynamics for a potential game with potential function Φ is

$$\Upsilon(\mathbf{x}, \mathbf{y}) = (n-2)\Phi(\mathbf{x}) + \Psi(\mathbf{x}, \mathbf{y})$$

where $\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \Phi(\mathbf{x}_{-i}, y_i)$. Indeed, from (7) it follows that for a “social potential game”

$$\pi(\mathbf{x})P(\mathbf{x}, \mathbf{y}) = \frac{1}{Z} e^{\beta((n-2)\Phi(\mathbf{x}) + \Psi(\mathbf{x}, \mathbf{y}))}.$$

In order to give a lower bound on the mixing time, we first show that, for the Ising model on the clique, $\Upsilon(\mathbf{x}, \mathbf{y})$ is symmetric and can be written as a function of the magnetization of the two profiles and the Hamming distance between them.

Lemma 5.5. *Let $\mathbf{x}, \mathbf{y} \in \{-1, +1\}^n$ be two profiles with magnetization $k_{\mathbf{x}}$ and $k_{\mathbf{y}}$ respectively and let $d_{\mathbf{x}, \mathbf{y}}$ be their Hamming distance, i.e. the number of players where they differ. Then*

$$(n-2)\Phi(\mathbf{x}) - \Psi(\mathbf{x}, \mathbf{y}) = k_{\mathbf{x}}k_{\mathbf{y}} + 2d_{\mathbf{x}, \mathbf{y}} - n.$$

Proof. We already know that $\Phi(\mathbf{x}) = \frac{n-k_{\mathbf{x}}^2}{2}$. In order to evaluate $\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \Phi(\mathbf{x}_{-i}, y_i)$ let us name a, b and c as follows

$$\begin{aligned} a &= \#\{i \in [n] : x_i = y_i\}; \\ b &= \#\{i \in [n] : x_i = +1, y_i = -1\}; \\ c &= \#\{i \in [n] : x_i = -1, y_i = +1\}. \end{aligned}$$

In other words, a is the number of players playing the same strategy in profiles \mathbf{x} and \mathbf{y} , b is the number of players playing $+1$ in \mathbf{x} and -1 in \mathbf{y} , and c the number of players playing -1 in \mathbf{x} and $+1$ in \mathbf{y} . It holds that

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n \Phi(\mathbf{x}_{-i}, y_i) \\ &= a \frac{n - k_{\mathbf{x}}^2}{2} + b \frac{n - (k_{\mathbf{x}} - 2)^2}{2} + c \frac{n - (k_{\mathbf{x}} + 2)^2}{2} \\ &= \frac{1}{2} ((a + b + c)(n - k_{\mathbf{x}}^2) + 4(b - c)k_{\mathbf{x}} - 4(b + c)). \end{aligned} \quad (14)$$

Now observe that $a + b + c = n$, $2(b - c) = k_{\mathbf{x}} - k_{\mathbf{y}}$, and $(b + c) = d_{\mathbf{x}, \mathbf{y}}$. Hence from (14) we get

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{y}) &= \frac{1}{2} (n(n + k_{\mathbf{x}}^2) + 2(k_{\mathbf{x}} - k_{\mathbf{y}})k_{\mathbf{x}} - 4d_{\mathbf{x}, \mathbf{y}}) \\ &= \frac{n^2}{2} - \frac{n-2}{2}k_{\mathbf{x}}^2 - k_{\mathbf{x}}k_{\mathbf{y}} - 2d_{\mathbf{x}, \mathbf{y}}. \end{aligned} \quad (15)$$

Thus

$$(n-2)\Phi(\mathbf{x}) - \Psi(\mathbf{x}, \mathbf{y}) = k_{\mathbf{x}}k_{\mathbf{y}} + 2d_{\mathbf{x}, \mathbf{y}} - n.$$

□

Since the Hamming distance between two profiles is at most n , from the above lemma we get the following observation.

Observation 5.6. *Let \mathbf{x}, \mathbf{y} be two profiles with $k_{\mathbf{x}}k_{\mathbf{y}} \leq 0$, then $(n-2)\Phi(\mathbf{x}) - \Psi(\mathbf{x}, \mathbf{y}) \leq n$.*

Now we can give a lower bound on the mixing time by using the bottleneck-ratio technique.

Theorem 5.7 (Lower bound). *The mixing time of the all-logit dynamics for the Ising model on the clique is*

$$t_{\text{mix}} = \Omega \left(\frac{e^{\beta n(n-2)}}{4^n} \right).$$

Proof. Let $S \subseteq \{-1, +1\}^n$ be the set of profiles with negative magnetization

$$S = \{\mathbf{x} \in \{-1, +1\}^n : k_{\mathbf{x}} < 0\}$$

and observe that $\pi(S) \leq 1/2$. From Observation 5.6 we have that for every $\mathbf{x} \in S$ and $\mathbf{y} \in \bar{S}$ it holds that

$$\pi(\mathbf{x})P(\mathbf{x}, \mathbf{y}) = \frac{1}{Z} e^{\beta[(n-2)\Phi(\mathbf{x}) - \Psi(\mathbf{x}, \mathbf{y})]} \leq e^{\beta n}/Z. \quad (16)$$

Moreover, if we name \mathbf{x}_- the profile where everyone is playing -1 we have that

$$\pi(S) \geq \pi(\mathbf{x}_-) \geq \frac{1}{Z} e^{-2\beta\Phi(\mathbf{x}_-)} = \frac{1}{Z} e^{\beta n(n-1)}. \quad (17)$$

Hence, by using bounds (16) and (17), and the fact that the size of S is at most 2^{n-1} , we can bound the bottleneck at S with

$$B(S) = \frac{Q(S, \bar{S})}{\pi(S)} = \frac{\sum_{\mathbf{x} \in S} \sum_{\mathbf{y} \in \bar{S}} \pi(\mathbf{x})P(\mathbf{x}, \mathbf{y})}{\pi(S)} \leq \frac{2^{2n-2} e^{\beta n}}{e^{\beta n(n-1)}} = \frac{2^{2n-2}}{e^{\beta n(n-2)}}.$$

By using the bottleneck-ratio theorem (see e.g. Theorem 7.3 in [11]) it follows that

$$t_{\text{mix}} = \Omega \left(\frac{e^{\beta n(n-2)}}{2^{2n}} \right).$$

□

Remarks. In this section we proved upper and lower bounds on the mixing time of the all-logit dynamics for the Ising model on the clique. In particular, the upper bound shows that for $\beta = \mathcal{O}(1/n^2)$ the mixing time is constant and for $\beta = \mathcal{O}(\log n/n^2)$ it is at most polynomial. The lower bound shows that, for every constant $\varepsilon > 0$, if $\beta > (1 + \varepsilon)(\log 4)/n$ the mixing time is exponential. When β is between $\Theta(\log n/n^2)$ and $\Theta(1/n)$ we still cannot say if mixing is polynomial or exponential.

6 Conclusions and open problems

In this paper we studied some properties of the dynamics induced by the logit choice function when all players play concurrently and we compared the results with the case of the one-logit dynamics. It is well-known that the class of games whose one-logit is reversible is the class of potential games. We showed that the class of games such that the all-logit is reversible is a natural subclass of potential games that we called social potential games. We also derived the first general upper bound on the mixing time of the all-logit and specific upper and lower bound for the well-studied Curie-Weiss model.

In the one-logit dynamics one player is selected uniformly at random at each step while in the all-logit one every player updates her strategy at every step. A natural generalization is to consider a probability distribution μ over the family of subsets of players, such that at each step a subset of players is chosen for the update according to μ . It would be interesting to see how stationary distributions and mixing times are affected by μ , and to find out whether there is any observable of the dynamics that, when measured at stationarity, does not depend on μ . Some interesting results along that direction have been obtained in [1, 2].

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A A characterization of potential games

In this section we review a characterization of potential games in terms of the utilities. Let \mathcal{G} be a game. A *circuit* $\Gamma = \langle s_0, \dots, s_\ell \rangle$ is a sequence of strategy profiles such that $s_0 = s_\ell$, $s_h \neq s_k$ for $1 \leq h \neq k \leq \ell$ and, for $k = 1, \dots, \ell$, there exists player i_k such that s_{k-1} and s_k differ only for player i_k . For such a circuit Γ we define the *utility improvement* $I(\Gamma)$ as

$$I(\Gamma) = \sum_{k=1}^{\ell} [u_{i_k}(s_k) - u_{i_k}(s_{k-1})] .$$

The following theorem holds.

Theorem A.1 (Monderer and Shapley [14]). *A game \mathcal{G} is a potential game if and only if $I(\Gamma) = 0$ for all circuits of length 4.*