Improved bead sort using the Fourier Transform

Paolo Bettelini

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1 Introduction

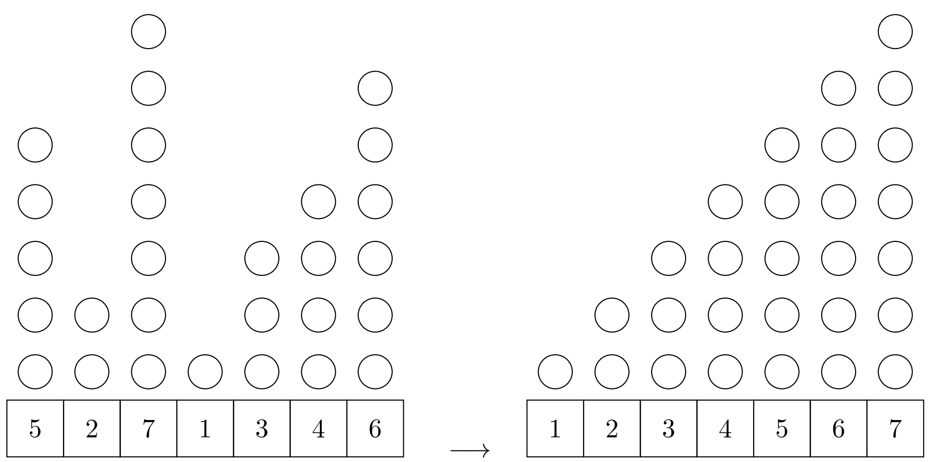
1.1 Abstract

Bead sort has always been an interesting, but useless, sorting algorithm for integers. It is able to sort numbers without ever directly comparing them. This paper presents my attempt to make it possibly more efficient for software when sorting large arrays of numbers.

1.2 Bead sort

Bead sort[1] or *gravity sort* is a natural sorting algorithm. This algorithm is primarily used to sort integers, but can be extended to sort the rationals.

The basic idea is to represent the unsorted integers in a matrix (1a), where each columns has n beads. We then shift every bead to the right side of the matrix until they collide with another bead, as if they were affected by gravity. The final step is to count the elements in each columns, forming the final ordered sequence (1b).



(a) Unsorted beads (b) Sorted beads

Figure 1: Beads matrix for $S = 5, 2, 7, 1, 3, 4, 6$

The time complexity of this algorithm ranges from $O(1)$ to $O(P)$ where $P = \sum S_k$.

2 Algorithm

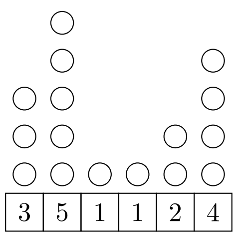
2.1 Core idea

A classical implementation of the bead sort would execute the following steps:

1. Construct the initial beads matrix
2. Count the amount of elements in each row
3. Construct the final beads matrix using the values of Step 2.
4. Count the amount of elements in each row

My idea is to represents the beads matrix as a sum of sine functions. Each frequency represents a row, and its amplitude the amount of elements in it.

For example



$$6\sin(t) + 4\sin(2t) + 3\sin(3t) + 2\sin(4t) + \sin(5t)$$

By using the Fourier Transform on the function representing the matrix we can retrieve the amplitudes of the different frequencies and construct the final beads matrix.

2.2 Implementation

Given a sequence of unordered natural numbers

$$S = \{n_k\}, \quad 0 < k \leq |S|$$

we want to find a sequence

$$\hat{S} = \{m_l\}, \quad 0 < l \leq |\hat{S}|$$

which is the ordered version of S.

We define a time-dependent function $X(t)$

$$X(t) = \sum_{k=1}^{|S|} \sum_{f=1}^{n_k} \sin(ft)$$

By applying the Fourier Transform we get a frequency-dependent function

$$\hat{X}(\xi) = \mathcal{F}\{X(t)\}$$

We can now give an explicit formula for m_l

$$m_l + 1 = \min(\xi \in \mathbb{N}) \quad \text{such that} \quad |\hat{X}(\xi)| < l$$

Note that

$$\hat{X}(|S| + 1) = 0$$

2.3 Computation

2.3.1 Time Complexity of $X(t)$

The function $X(t)$ has time complexity $O(N \cdot P)$ where $P = \sum S_k$ for its construction. We can reduce this complexity by noting that $X(t)$ contains a geometric series.

$$
\begin{aligned}
\sum_{f=1}^{n_k} \sin(ft) &= \Im \sum_{f=1}^{n_k} e^{ift} \\
&= \Im \left(e^{it} \frac{e^{itn_k} - 1}{e^{it} - 1} \right) \\
&= \Im \left(e^{it} \frac{e^{in_k t/2}(e^{in_k t/2} - e^{-in_k t/2})}{e^{it/2}(e^{it/2} - e^{-it/2})} \right) \\
&= \Im \left(e^{it} \frac{e^{in_k t/2}(2i \sin(n_k t/2))}{e^{it}(2i \sin(t/2))} \right) \\
&= \Im \left(e^{i(n_k+1)t/2} \frac{\sin(n_k t/2)}{\sin(t/2)} \right) \\
&= \Im \left((\cos((n_k+1)t/2) + i \sin((n_k+1)t/2)) \frac{\sin(n_k t/2)}{\sin(t/2)} \right) \\
&= \frac{\sin(n_k t/2)}{\sin(t/2)} \sin((n_k+1)t/2)
\end{aligned}
$$

Thus,

$$
X(t) = \sum_{k=1}^{|S|} \frac{\sin(n_k t/2)}{\sin(t/2)} \sin((n_k+1)t/2)
$$

The time complexity of $X(t)$ is now $O(N)$.

2.3.2 Time Complexity of $\hat{X}(\xi)$

The Fourier Transform of $X(t)$ is given by

$$
\begin{aligned}
\hat{X}(\xi) &= \frac{1}{\max - \min} \int_{\min}^{\max} e^{-2\pi it\xi} X(t)\, dt \\
&= \frac{1}{\max - \min} \int_{\min}^{\max} e^{-2\pi it\xi} \sum_{k=1}^{|S|} \frac{\sin(n_k t/2)}{\sin(t/2)} \sin((n_k+1)t/2)\, dt
\end{aligned}
$$

Since $|S|$ is finite we can apply an interchange of summation and integration

$$
\hat{X}(\xi) = \frac{1}{\max - \min} \sum_{k=1}^{|S|} \int_{\min}^{\max} \frac{\sin(n_k t/2)}{\sin(t/2)} \sin((n_k+1)t/2) e^{-2\pi it\xi}\, dt
$$

If $\hat{X}(\xi)$ has a closed-form, then its time complexity is the one of $X(t)$, which is $O(N)$. Alternatively, one may use the FFT algorithm, which has time complexity $O(N \log(N))$.

2.3.3 Time Complexity of m_l

The computation of m_l requires a scan of the frequencies given by $|\hat{X}(\xi)|$. The amount of frequencies to consider is equal to the maximum integer in the input. Since the amplitude of the frequencies are ordered it may be optimized using a binary search, giving a time complexity of $O(Q)$ where Q is the maximum integer.

2.4 Total space complexity

If a closed-form for $\hat{X}(\xi)$ is found, the total space complexity of the algorithm is $O(1)$.