Fibrational topology

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1 Fibrational topology

Note: The category **Top** of topological spaces is the Grothendieck construction of the indexed category

$$\mathcal{T}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$$

that assigns to each set the poset of topologies on it, with reindexing given by inverse image.

$$\mathbf{Set} \longrightarrow \mathbf{Poset}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & (P_X, \subseteq) \arrow[d, bend right, "\exists f^*"'] \arrow[d, bend left, "(f^*)^{-1}"] \\
Y & (P_Y, \subseteq)
\end{tikzcd}$$

where $(P_X, \subseteq)$ is the collection of all subframes of $\mathcal{P}(X)$.
We have

$$\begin{tikzcd}
P_X \arrow[d, hook] & P_Y \arrow[l, dashed] \arrow[d, hook] \\
\mathcal{P}(\mathcal{P}(X)) & \mathcal{P}(\mathcal{P}(Y)) \arrow[l, "\exists f^*"]
\end{tikzcd}$$

and also the subframe inclusion

$$\begin{tikzcd}
\mathcal{P}(Y) \arrow[r, "f^*=f^{-1}"] & \mathcal{P}(X) \\
O_Y \arrow[u, hook] \arrow[r] & f^*(O_Y) \arrow[u, hook]
\end{tikzcd}$$

where $f^*(O_Y) = \{f^{-1}(U) \mid U \in O_Y\}$.
For any $O_Y \subseteq \mathcal{P}(Y)$ subframe, $\exists f^*(O_Y)$ is the "subspace topology" on X induced by P_Y via f (even when f is not an inclusion). $\exists f^*(O_Y)$ is a subframe of $\mathcal{P}(X)$ since O_Y is a subframe of $\mathcal{P}(Y)$ and $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ is a frame homomorphism.

Dually, we can consider the right adjoint $(f^*)^{-1}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(Y))$ which sends

$$O_X \to (f^*)^{-1}(O_X) = \{V \in \mathcal{P}(Y) \mid \underbrace{f^*(V)}_{f^{-1}} \in O_X\}$$

This gives the "quotient topology" on Y since f^{-1} is a frame induced by O_X via f.
Note that $(f^*)^{-1}$ also restricts to a map $P_X \to P_Y$.

1.1 General categorical setting

Let $\mathcal{C}$ be a category with pullbacks such that $\forall C$, $\mathrm{Sub}_{\mathcal{C}}(C)$ is a frame and for any $f\colon d \to c$,

$$f^*\colon \mathrm{Sub}_{\mathcal{C}}(d) \to \mathrm{Sub}_{\mathcal{C}}(c)$$

is a frame homomorphism

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Poset}$$

$$\begin{tikzcd}
c \arrow[r] \arrow[d, "f"'] & P_c \subseteq \mathcal{P}(\mathrm{Sub}_{\mathcal{C}}(c)) \arrow[d, bend right, "\exists f^*"'] \arrow[d, bend left, "(f^*)^{-1}"] \\
d & P_d \subseteq \mathcal{P}(\mathrm{Sub}_{\mathcal{C}}(d))
\end{tikzcd}$$

with the pullback $f*\colon \mathrm{Sub}_{\mathcal{C}}(d) \to \mathrm{Sub}_{\mathcal{C}}(c)$ along f.
One can develop much abstract topology in this setting (e.g. by taking P_c to be the collection of all subframes of $\mathrm{Sub}_{\mathcal{C}}(c)$). Duality exchanges the role of left and right adjoints.
This is strictly related to the theory of internal locales.