!id curves-basic-results-definitions

!gen-page true

!section Basic results

!snippet curve-equivalence-same-support

Proposition
Let $\varphi_1 \colon I_1 \to \mathbb{R}^n$ and $\varphi_2 \colon I_2 \to \mathbb{R}^n$ be two curves. If $\varphi_1 \sim \varphi_2$ then φ_1 and φ_2 have the same support.

!endsnippet

!snippet lipschitz-curve-is-rectifiable

Proposition
Let $\varphi \colon I \to \mathbb{R}^n$ be a curve that is lipschitz. Then, φ is rectifiable.

!endsnippet

!snippet lipschitz-curve-is-rectifiable-proof

Proof
We have that

$$\forall t, s \in I, ||\varphi(s) - \varphi(t)|| \le L|t - s|$$

thus

$$\begin{aligned} L(\varphi, P) &= \sum_{i=1}^{n} ||\varphi(t_i) - \varphi(t_{i-2})|| \\ &\le \sum_{i=1}^{n} L|t_i - t_{i-1}| = L(b-a) \end{aligned}$$

meaning that $L(\varphi) \le L(b-a)$. ∎

!endsnippet

!snippet curve-length-example1

Example
Let $\varphi(t) = (\cos t, \sin t)$ defined on $I = [0, \pi]$. We have

$$\begin{aligned} ||\varphi(t) - \varphi(s)|| &= \left\{ (\cos t - \cos s)^2 + (\sin t - \sin s)^2 \right\}^{1/2} \\ &= \left\{ (-\sin \alpha(t-s))^2 + (\cos \beta(t-s))^2 \right\}^{1/2} \\ &\le \sqrt{2}|t-s| \end{aligned}$$

We thus have that $L(\varphi) \le \sqrt{2}\pi$.

!endsnippet

!snippet curve-length-upper-bound

Proposition

$$\left|\left| \int_a^b \varphi(t)\, dt \right|\right| \le \int_a^b ||\varphi(t)||\, dt$$

where the integral of a vector is the vector of the integrals.

!endsnippet

!snippet curve-length-upper-bound-proof

Proof
Let

$$v = \left(\int_a^b \varphi_1(t)\, dt, \cdots, \int_a^b \varphi_n(t)\, dt \right)$$

then

$$\begin{aligned} \left|\left| \int_a^b \varphi(t)\, dt \right|\right| &= ||v||^2 = \langle v, v \rangle = \sum_{i=1}^{n} v_i \int_a^b \varphi_i(t)\, dt \\ &= \int_a^b \sum_{i=1}^{n} v_i \varphi_i(t)\, dt \\ &= \int_a^b \langle v, \varphi(t) \rangle\, dt \\ &\le \int_a^b ||v|| \cdot ||\varphi(t)||\, dt \\ &= ||v|| \int_a^b ||\varphi(t)||\, dt \end{aligned}$$

∎

!endsnippet

!section Curve length formula

!snippet curve-length-formula-theorem

Theorem Curve length formula
Let $\varphi \colon I \to \mathbb{R}^n$ with $\varphi \in \mathcal{C}^1(I)$. Then, φ is rectifiable and

$$L(\varphi) = \int_a^b ||\varphi'(t)||\, dt$$

!endsnippet

!snippet curve-length-formula-proof

Proof Curve length formula
Clearly the function is lipschitz.

1.
$$\begin{aligned} L(\varphi, P) &= \sum_{i=1}^{n} ||\varphi(t_i) - \varphi(t_{i-1})|| \\ &= \sum_{i=1}^{n} \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t)\, dt \right|\right| \\ &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ||\varphi'(t)||\, dt \\ &= \int_a^b ||\varphi'(t)||\, dt \end{aligned}$$

   We thus have

$$L(\varphi) \le \int_a^b ||\varphi'(t)||\, dt$$

2. we will now prove the other direction. Since $[a, b]$ is compact and φ is continuous, then it is uniformly continuous. Hence,

$$\forall \varepsilon > 0, \exists \delta > 0 \,|\, \delta > |t - s| \implies ||\varphi(t) - \varphi(s)|| < \varepsilon$$

   We now take a partition P such that $|t_i - t_{i-1} < \delta$. We then have

$$\begin{aligned} \int_{t_{i-1}}^{t_i} ||\varphi'(t)||\, dt &= \int_{t_{i-1}}^{t_i} ||\varphi'(t) - \varphi'(t_{i-1}) + \varphi'(t_{i-1})||\, dt \\ &\le \int_{t_{i-1}}^{t_i} ||\varphi'(t) - \varphi'(t_{i-1})||\, dt + \int_{t_{i-1}}^{t_i} ||\varphi'(t_{i-1})||\, dt \\ &\le \varepsilon(t_i - t_{i-1}) + \varphi'(t_{i-1})(t_i - t_{i-1}) = \varepsilon(t_i - t_{i-1}) + ||(t_i - t_{i-1})\varphi'(t_{i-1})|| \\ &= \varepsilon(t_i - t_{i-1}) + \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t_{i-1})\, dt \right|\right| = \varepsilon(t_i - t_{i-1}) + \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t_{i-1}) - \varphi'(t) - \varphi'(t)\, dt \right|\right| \\ &= \varepsilon(t_i - t_{i-1}) + \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t_{i-1}) - \varphi'(t)\, dt \int_{t_{i-1}}^{t_i} \varphi'(t)\, dt \right|\right| \\ &\le \varepsilon(t_i - t_{i-1}) + \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t_{i-1}) - \varphi'(t)\, dt \right|\right| + \left|\left| \int_{t_{i-1}}^{t_i} \varphi'(t)\, dt \right|\right| \\ &\le \varepsilon(t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} \varphi'(t_{i-1}) - \varphi'(t)\, dt + ||\varphi(t_i) - \varphi(t_{i-1})|| \\ &\le 2\varepsilon(t_i - t_{i-1}) + ||\varphi(t_i) - \varphi(t_{i-1})|| \end{aligned}$$

   by adding we obtain

$$\begin{aligned} \int_a^b ||\varphi'(t)||\, dt &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ||\varphi'(t)||\, dt \\ &\le 2\varepsilon(b-a) + \sum_{i=1}^{n} ||\varphi(t_i) - \varphi(t_{i-1})|| \\ &\le 2\varepsilon + L(\varphi) \\ \int_a^b ||\varphi'(t)||\, dt &\le L(\varphi) \end{aligned}$$

meaning that the equivalence is true. ∎

!endsnippet

!snippet curve-length-formula-bidimensional-case

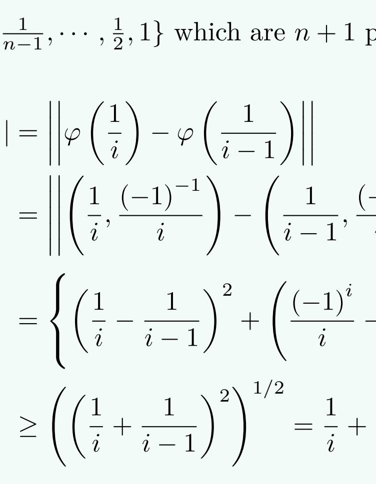
For the bidimensional case $\varphi(t) = (t, f(t))$ we have

$$\int_a^b ||\varphi'(t)||\, dt = \int_a^b ||(1, f'(t))||\, dt = \int_a^b \sqrt{1 + \frac{df}{dt}^2}\, dt$$

!endsnippet

!snippet non-rectifiable-curve-example1

Example
The curve $\varphi \colon [0, 1] \to \mathbb{R}^2$ defined as a segment which bounces between the two bisector at the point $1/i$, and coming closer to the origin, is not rectifiable.



Consider the partition $P = \{0, \frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{2}, 1\}$ which are $n + 1$ point. Consider

$$\begin{aligned} ||\varphi(t_i) - \varphi(t_{i-1})|| &= \left|\left| \varphi\left(\frac{1}{i}\right) - \varphi\left(\frac{1}{i-1}\right) \right|\right| \\ &= \left|\left| \left(\frac{1}{i}, \frac{(-1)^{-1}}{i}\right) - \left(\frac{1}{i-1}, \frac{(-1)^{i-1}}{i-1}\right) \right|\right| \\ &= \left\{ \left(\frac{1}{i} - \frac{1}{i-1}\right)^2 + \left(\frac{(-1)^i}{i} - \frac{(-1)^{i-1}}{i-1}\right)^2 \right\}^{1/2} \\ &\ge \left(\left(\frac{1}{i} + \frac{1}{i-1}\right)^2 \right)^{1/2} = \frac{1}{i} + \frac{1}{i-1} \ge \frac{2}{i} \end{aligned}$$

then, the sum

$$\sum_{i=1}^{n} ||\varphi(t_i) - \varphi(t_{i-1})|| \ge \sum_{i=1}^{n} \frac{2}{i}$$

diverges for $n \to \infty$.

!endsnippet