Algebra II

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Contents

1 Teoremi di isomorfismo su quozienti di spazi vettoriali

Let V be a vector space over $\mathbb{K}$ and W be a linear subspace of V.

We have a map

$$\pi\colon V \to V/W$$

defined as

$$\pi(v) \triangleq v + W \in V/W$$

which is a linear map.

Indeed,

1.
$$\pi(0_V) = 0_V + W = w + W$$

2.
$$\begin{aligned} \pi(v_1 + v_2) &= \pi(v_1) + \pi(v_2) \\ (v_1 + v_2) + W &= (v_1 + W) + (v_2 + W) \end{aligned}$$

3.
$$\pi(\lambda v) = (\lambda v) + W = \lambda\,(v + W)$$

We now consider a morphism $\varphi\colon V_1 \to V_2$ between vector spaces. We know that its kernel is a subspace of V_1. We now construct a new morphism

$$\overline{\varphi}\colon V_1/\ker_\varphi \to V_2$$

such that

$$\overline{\varphi}(v + \ker_\varphi) \triangleq \varphi(v)$$

We need to ensure that such mapping is well-defined. Let $v' \in v + \ker_\varphi$, meaning that $v' = v + w$ with $w \in \ker_\varphi$.

$$\begin{aligned} \overline{\varphi}(v' + \ker_\varphi) &= \varphi(v') = \varphi(v + w) = \varphi(v) + \varphi(w) \\ &= \varphi(v) = \overline{\varphi}(v + \ker_\varphi) \end{aligned}$$

We now show that it is also linear:

1.
$$\overline{\varphi}(0_{V_1} + \ker_\varphi) = \varphi(0_{V_1}) = 0_{V_2}$$

2.

$$
\begin{aligned}
\overline{\varphi}((v_1 + \ker_\varphi) + (v_2 + \ker_\varphi)) &= \overline{\varphi}((v_1 + v_2) + \ker_\varphi) \\
&= \varphi(v_1 + v_2) = \varphi(v_1 + v_2) \\
&= \overline{\varphi}(v_1 + \ker_\varphi) + \overline{\varphi}(v_2 + \ker_\varphi)
\end{aligned}
$$

3.

$$
\overline{\varphi}(\lambda(v + \ker_\varphi)) = \lambda\left(\overline{\varphi}(v + \ker_\varphi)\right)
$$

Il seguente diagramma commuta e π è suriettiva in quanto $v + \ker_\varphi = \pi(v)$.

$$
\begin{tikzcd}
V_1 \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"'] & V_2 \\
V_1/\ker_\varphi \arrow[ur, bend right, "\overline{\varphi}"'] &
\end{tikzcd}
$$

Quindi $\varphi = \overline{\varphi} \circ \pi$.

Teorema First isomorphism theorem

Let $\varphi \colon V_1 \to V_2$ be a morphism between vector spaces.

$$
\overline{\varphi} \colon V_1/\ker_\varphi \to \mathrm{im}_\varphi
$$

is an isomorphism of vector spaces, meaning

$$
V_1/\ker \cong \mathrm{im}_\varphi
$$

Proof First isomorphism theorem

We need to show that the morphism is both surjective and injective:

1. let $v_2 \in \mathrm{im}_\varphi$. We want to find a $v_1 \in V_1$ such that $v_2 = \varphi(v_1)$. This is precisely

$$
\overline{\varphi}(v_1 + \ker_\varphi)
$$

2. we want to show that the kernel is trivial.

$$
\begin{aligned}
\ker_{\overline{\varphi}} &= \{v + \ker_\varphi \,|\, \overline{\varphi}(v + \ker_\varphi) = 0_{V_2}\} \\
&= \{v + \ker_\varphi \,|\, v \in \ker_\varphi\} \\
&= 0_{V_1} + \ker_\varphi
\end{aligned}
$$

since $v + \ker_\varphi = \ker_\varphi$ and we can just choose 0_{V_1}.

Esempio

Consider a vector space $V = W_1 \oplus W_2$ with $W_1, W_2 \leq V$ and consider the mappings

$$
p_1 \colon V \to W_1, \quad p_2 \colon V \to W_2
$$

Using the diagrams with $\overline{p_1}, \pi_1$ and $\overline{p_2}, \pi_2$, we have

$$
W_1 \cong V/W_2, \quad W_2 \cong V/W_1
$$

since $W_2 = \ker_{p_1}$ and $W_1 = \ker_{p_2}$.

Teorema Second isomorphism theorem

Let V be a vector space over $\mathbb{K}$ and $U, W \leq V$. Then,

$$\frac{W}{W \cap U} \cong \frac{W+U}{U}$$

Proof Second isomorphism theorem

We apply the first isomoprhism theorem. Construct a surjective mapping

$$\varphi \colon \frac{W}{W \cap U} \to W + U$$

such that $\ker_\varphi = U$. We first note that

$$\frac{W}{W \cap U} \leq V/U$$

and so we define

$$\varphi(w) \triangleq w + U \in V/U$$

We need to show that it is linear (todo). It is surjective as

$$\mathrm{Im}_\varphi = \frac{W+U}{U}$$

since $w + u + U = w + U = \varphi(w)$. We now need to study that it is injective

$$\begin{aligned} \ker_\varphi &= \{w \in W \mid w + U = 0_{V/U} = 0_V + U\} \\ &= \{w \in W \mid w \in U\} = W \cap U \end{aligned}$$

since $w + U = 0_V + U$ means that $w \in U$.

Notiamo che U potrebbe non essere sottospazio di W quindi non possiamo rimpiazzare $W+U$ con W/U.