Differentiation

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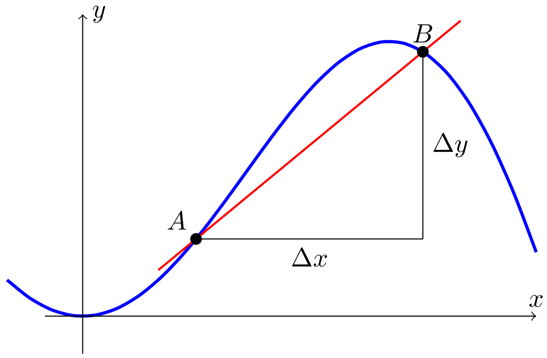
Contents

1 Definition

1.1 Tangent



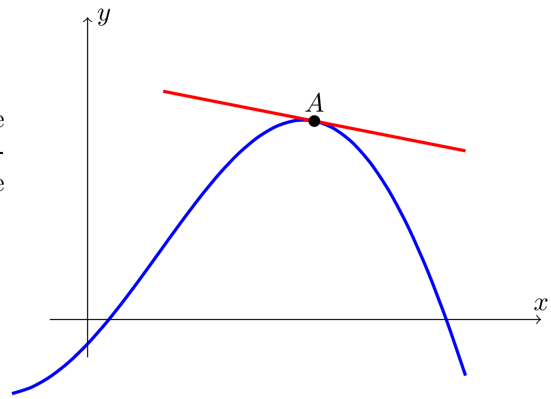
The mean slope of a function f between a point A and B is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}$$

As we make A and B closer to eachother, Δx decreases. As Δx decreases the mean slope is more representative of the rate of change of f in the interval $[A; B]$.

When Δx is infinitely small, we have the precise slope of a given point on the function. This slope is represented by the tangent line, which is parallel to the given point.

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$



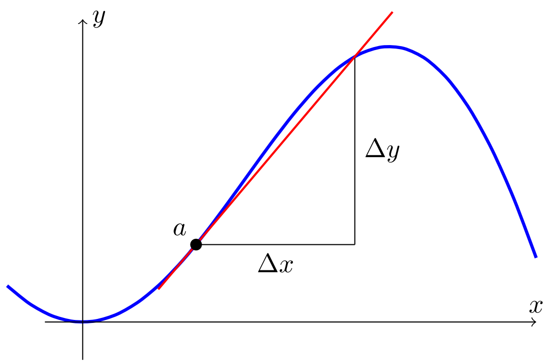
1.1.1 Subtangent

This is the subtangent

1.2 Derivative

The derivative of a function $f(x)$ is another function $f'(x)$ which represents the rate of change of $f(x)$. In other words, $f'(x)$ represents the slope at each x of $f(x)$.

We define $f'(x)$ by taking the limit of the slope for every x.



We define the derivative as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$f'(x) = \lim_{h \to x} \frac{f(h) - f(x)}{x - h}$$

Using the derivative, the tangent line at $x = a$ is given by

$$y = f'(a)(x - a) + f(a)$$

2 Interpretation

2.1 Rate of Growth

Since the derivative $f'(x)$ represents the rate of change of $f(x)$, assuming that $f(a)$ is defined.

- If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$
- If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$
- If $f'(a) = 0$, then $f(x)$ is critical at $x = a$
- If $f'(a)$ is not defined, then $f(x)$ is critical at $x = a$ (sharp corner)

A critical point is when the function is stationary.

A function increases on an interval I iff

$$\forall x_1, x_2 \in I f(x_1) < f(x_2)$$

and decreases iff

$$\forall x_1, x_2 \in I f(x_1) > f(x_2)$$

2.2 First Derivative Test

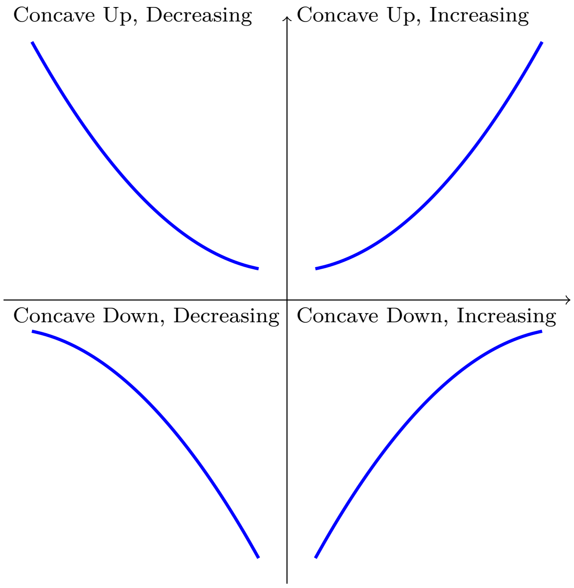
A critical point $f'(c) = 0$ does not generally imply that $x = c$ is a minimum or a maximum.

Let $f(x)$ be critical at $x = c$

- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right $x = c$ is a maximum
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right $x = c$ is a minimum
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither.

A function may also change sign when it is undefined.

2.3 Concavity



Functions may present **concavity**

- $f(x)$ is **concave up** on an interval I iff all of the tangents on I are below the graph.
- $f(x)$ is **concave down** on an interval I iff all of the tangents on I are above the graph.

- $f''(x) > 0$ for all x in some interval I then $f(x)$ is concave up on I
- $f''(x) < 0$ for all x in some interval I then $f(x)$ is concave down on I

This works because when the function is concave up, it increases or decreases more and more. So $f'(x)$ tells us that $f(x)$ is increasing or decreasing, and $f''(x)$ tells us the rate at which the increment is increasing or the decrease is decrementing. The same goes for when the function is concave down.

An **inflection point** is a point where the function is continuous and the concavity at that point changes. Hence, when $f''(x)$ changes sign we have an inflection point.

2.4 Second Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ such that $f'(x) = 0$ and that $f''(x)$ is continuous around $x = c$.

- If $f''(x) < 0$ then $x = c$ is a maximum.
- If $f''(x) > 0$ then $x = c$ is a minumum.
- If $f''(x) = 0$ then $x = c$ could be a maximum, minimum or neither.

3 Absolute Extrema

When looking for an absolute extrema in a function $f(x)$, asking when $f'(x) = 0$ is not enough since the function may not be continuous and have a maxima at a discontinuity point.

4 Rules for differentiation

$$\frac{d}{dx}(n) = 0$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}^*$$

$$\frac{d}{dx}\left(n \cdot f(x)\right) = n\frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}(f + g) = f' + g'$$

Product Rule

$$\frac{d}{dx}(f \cdot g) = g'f + gf'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(f^g) = f^g\left(\frac{f'g}{f} + g'\ln f\right)$$

5 L'Hôpital Rule

Given the function $f(x)$ and $g(x)$ which are differentiable in an open interval I except possibly at $x = c$, if

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty, \quad g(x \in I) \neq 0$$

then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

if such limit exists.

6 Intermediate value Theorem

A function f continuous on an interval $[a;b]$ will take every value in the interval $[f(a);f(b)]$.

7 Bolzano's Theorem

If $f(x)$ is continuous on $[a;b]$ and $f(a) \cdot f(b) < 0$ then there is a root.

$$f(a) \cdot f(b) < 0 \implies \exists c \in [a;b] \mid f(c) = 0$$

8 Weierstrass Theorem

If $f(x)$ is continuous in $[a;b]$ then the function will have a maxima and a minima.

9 Rolle's Theorem

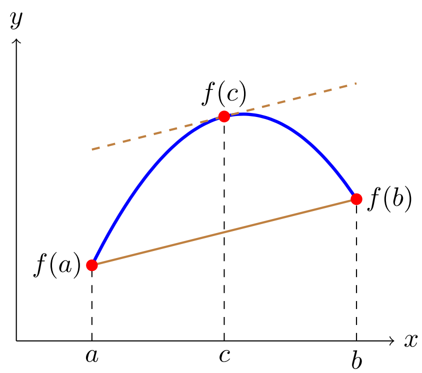
Suppose that $f(x)$ is continuous on $[a;b]$ and differentiable on $(a;b)$.

$$f(a) = f(b) \implies \exists\, c \,|\, f'(c) = 0, \quad a < c < b$$

10 Mean Value Theorem

Suppose $f(x)$ is a function continuous on $[a;b]$ and differentiable on $(a;b)$, there there exist a number c such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}, \quad a < c < b$$



The mean value on the interval can be represented by the secant line. What this means is that the interval contains a point whose tangent is equal to the secant.

Note that if $f(a) = f(b)$ this is Rolle's theorem.

11 Chain Rule

11.1 Definition

If z depends on y, and y depends on x, then z also depends on x.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

which is equivalent to

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

11.2 Proof

Assuming that z and y are differentiable in x

$$\begin{aligned}
\frac{dz}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x} \\
&= \left(\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right) \\
&= \left(\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx}
\end{aligned}$$

As $\Delta x \to 0$ also $\Delta y \to 0$, so we can replace Δx with Δy

$$\begin{aligned}
\frac{dz}{dx} &= \left(\lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx} \\
&= \frac{dz}{dy} \cdot \frac{dy}{dx}
\end{aligned}$$

12 Differentials

Given a function $y = f(x)$ we call dy and dx differentials and their relationship is

$$dy = f'(x)dx$$

If we are given just $f(x)$ then the differentials would be df and dx

$$df = f'(x)dx$$