Let's look at a simple example. We are going to derive the Fourier series of a function $f(x)$ defined as such:

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ +1 & \text{if } 0 < x < \pi \end{cases}$$

The period of this function is $T = 2\pi$. We can already simplify the $\frac{2\pi}{T}$ term, leaving us with

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

First, we need to find a_n. Simplifying $\frac{2\pi}{T}$ and $\frac{T}{2}$ we get

$$a_n = \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

Looking at the graph we notice that we can split the integral into two parts at $x = 0$. On the left part, the function is $-\cos(nx)$, while on the right part the function is $\cos(nx)$.

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{0} -\cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx \\
&= -\frac{1}{\pi} \int_{-\pi}^{0} \cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx \\
&= -\frac{1}{\pi} \left[\frac{\sin(xn)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{\sin(xn)}{n}\right]_{0}^{-\pi} \\
&= -\frac{1}{\pi} \left[\frac{\sin(\pi n)}{n}\right] + \frac{1}{\pi} \left[\frac{\sin(\pi n)}{n}\right] \\
&= \left(\frac{1}{\pi} - \frac{1}{\pi}\right) \left[\frac{\sin(\pi n)}{n}\right] \\
&= 0
\end{aligned}$$

a_n is always going to be 0. (note). We can remove the $a_n \cos(nx)$ and $\frac{a_0}{2}$ terms from the series.
Now for b_n

$$b_n = \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Again, we split the integral into two parts

$$\begin{aligned}
b_n &= -\frac{1}{\pi} \int_{-\pi}^{0} \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin(nx)\, dx \\
&= -\frac{1}{\pi} \left[\frac{-\cos(xn)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{-\cos(xn)}{n}\right]_{0}^{-\pi} \\
&= -\frac{1}{\pi} \left[-\frac{1}{n} + \frac{\cos(\pi n)}{n}\right] + \frac{1}{\pi} \left[\frac{-\cos(\pi n)}{n} + \frac{1}{n}\right] \\
&= -\frac{1}{\pi} \left[\frac{\cos(\pi n) - 1}{n}\right] + \frac{1}{\pi} \left[\frac{1 - \cos(\pi n)}{n}\right] \\
&= \frac{2}{\pi} \cdot \frac{1 - \cos(\pi n)}{n} \\
&= \frac{2 - 2\cos(\pi n)}{\pi n}
\end{aligned}$$