A general theory of self-similarity

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Motivation

We have seen how coinduction is useful in defining *behaviour*. Now we will concentrate on the fact that coinductive constructions capture topological objects: notable examples are

> The real line $\mathbb{R}$, or at least the positive real line $\mathbb{R}^+$, may be characterized as the terminal coalgebra for an endofunctor
>
> Let Pos be the category of posets. Consider the endofunctor
>
> $$F_1 : \mathrm{Pos} \to \mathrm{Pos}$$
>
> that acts by ordinal product? with ω,
>
> $$F_1 : X \mapsto \omega \cdot X,$$
>
> where $\omega \cdot X$ is $\omega \times X$ with the lexicographic order.
>
> ***Proposition***. *The terminal coalgebra of F_1 is order isomorphic to the non-negative real line $\mathbb{R}^+$, with its standard order.*
>
> ***Proof***. This is theorem 5.1 in (Pavlovic–Pratt 1999).

but also the p-adic integers:

The p-adic Integers as Final Coalgebra

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Abstract. We express the classical p-adic integers $\hat{\mathbb{Z}}_p$, as a metric space, as the final coalgebra to a certain endofunctor. We realize the addition and the multiplication on $\hat{\mathbb{Z}}_p$ as the coalgebra maps from $\hat{\mathbb{Z}}_p \times \hat{\mathbb{Z}}_p$.

The ring of p-adic integers is characterised as the terminal coalgebra for the endofunctor sending an ultrametric space (X, d) into $\frac{1}{p}X$, the disjoint union of p copies of X, each piece of which has the ultrametric rescaled by a factor p.

There's more! One can neatly characterise the profinite topology on $\hat{\mathbb{Z}}_p$ with the universal property of the terminal coalgebra, also defining sum and product (and their continuity is automatic).

Cantor takes a walk

Another interesting problem is the following: study "systems of linear equations" in a category with sums and products (more generally, in a 2-rig or a rig category)

A discrete equational system can be thought of as a system of linear equations such as

$$\begin{aligned} x_1 &= 2x_1 + 5x_2 + x_3 && (3) \\ x_2 &= x_2 && (4) \\ x_3 &= 4x_1 + x_2. && (5) \end{aligned}$$

Better, it can be thought of as a *categorification* of such a system: the variables x_i represent spaces, addition is coproduct, and the equalities are really isomorphisms. General equational systems can also be thought of as a categorification of such systems of equations—but a more subtle one.

The Cantor set is an example of solution of the single equation $A \cong A + A$:

The **Cantor set** is the topological space $2^{\mathbb{N}^+}$, that is, the product $2 \times 2 \times \cdots$ of countably infinitely many copies of the discrete two-point space $2 = \{0, 1\}$. (Here $\mathbb{N}^+$ is the set $\{1, 2, \ldots\}$ of positive integers.) The Cantor set is often regarded as a subset of the real interval $[0, 1]$ via the embedding

$$(m_n)_{n \geq 1} \longmapsto \sum_{n \geq 1} 2m_n \cdot 3^{-n}$$

($m_n \in \{0, 1\}$), but here we will only consider it as an abstract topological space.

The Cantor set satisfies an 'equation': $2^{\mathbb{N}^+} = 2^{\mathbb{N}^+} + 2^{\mathbb{N}^+}$. More precisely, there is a canonical isomorphism

$$\iota : 2^{\mathbb{N}^+} \xrightarrow{\sim} 2^{\mathbb{N}^+} + 2^{\mathbb{N}^+},$$

where $\iota(0, m_2, m_3, \ldots)$ is the element $(m_2, m_3, \ldots)$ of the first copy of $2^{\mathbb{N}^+}$, and $\iota(1, m_2, m_3, \ldots)$ is the element $(m_2, m_3, \ldots)$ of the second copy of $2^{\mathbb{N}^+}$. The pair $(2^{\mathbb{N}^+}, \iota)$ has, moreover, a universal property: it is terminal among all pairs (X, ξ) where X is a topological space and $\xi : X \longrightarrow X + X$ is any (continuous) map.

(I was motivated by the following question: in a differential 2-rig $\partial 1 \cong \partial 1 + \partial 1$: so

$$\begin{array}{ccc} \partial 1 & \cong & \partial 1 + \partial 1 \\ \downarrow & & \downarrow \\ C & \cong & C + C \end{array}$$

the horizontal maps are invertible: what does this mean?)

We can also describe a space of walks:

Consider walks on the natural numbers, of the following type:

- start at some position n
- with each tick of the clock, take one step left or one step right—unless at position 0, in which case stay there
- continue forever.

(One might consider imposing a different rule at 0; see Example 10.4.)

Let W_n be the set of all walks starting at position n. Formally, W_n is the set of elements $(a_0, a_1, \ldots) \in \mathbb{N}^{\mathbb{N}}$ such that $a_0 = n$ and for all $r \in \mathbb{N}$, *either* $a_r > 0$ and $a_{r+1} \in \{a_r - 1, a_r + 1\}$, *or* $a_r = a_{r+1} = 0$. There is a (profinite) topology on W_n generated by taking, for each $n, a_0, \ldots, a_n \in \mathbb{N}$, the set of all walks beginning $(a_0, \ldots, a_n)$ to be closed. So we have a family $(W_n)_{n \in \mathbb{N}}$ of spaces, and this is the 'topological object' that we will characterize by a universal property.

$$\overset{\curvearrowright}{0} \leftarrow 1 \leftrightarrows 2 \leftrightarrows 3 \leftrightarrows 4 \leftrightarrows \cdots$$

(In fact, W_0 is the one-point space, so ι_0 is the identity.)

These isomorphisms can be expressed as follows. The family $W = (W_n)_{n\in\mathbb{N}}$ is an object of the category $\mathcal{C} = \mathbf{Top}^{\mathbb{N}}$ of sequences of spaces. There is an endofunctor G of $\mathcal{C}$ defined by

$$(G(X))_n = \begin{cases} X_{n-1} + X_{n+1} & \text{if } n > 0 \\ X_0 & \text{if } n = 0 \end{cases} \tag{6}$$

($X \in \mathcal{C}$, $n \in \mathbb{N}$). We have just observed that there is a canonical isomorphism $\iota : W \xrightarrow{\sim} G(W)$; that is, (W, ι) is a fixed point of G. The universal property is that (W, ι) is the terminal G-coalgebra. Again, this can be proved directly and follows from later theory.

(Of the many types of walk that could be considered, this one is of special interest: in a certain sense, the sequence $(W_n)_{n\geq 1}$ has period 6. See [Lei4] and compare [Bla] and [FL].)

We look for a general theory encompassing all these examples:

In the Cantor set example, $\mathcal{C} = \mathbf{Top}$, and in the walks example, $\mathcal{C} = \mathbf{Top}^{\mathbb{N}}$. In both, then, $\mathcal{C} = \mathbf{Top}^A$ for some set A. We write objects of $\mathbf{Top}^A$ as indexed families $(X_a)_{a\in A}$.

In the Cantor set example, the functor $G : \mathcal{C} \longrightarrow \mathcal{C}$ is defined by $G(X) = X + X$, and in the walks example, G is defined by (6). In both, G has the following property: for each $a \in A$, the space $(G(X))_a$ is a finite sum of spaces X_b ($b \in A$). More precisely, there is a family $(M_{b,a})_{b,a\in A}$ of natural numbers such that for all $X \in \mathbf{Top}^A$ and $a \in A$,

$$(G(X))_a = \sum_{b\in A} M_{b,a} \times X_b.$$

These are *finite* sums, that is, $\sum_{b\in A} M_{b,a} < \infty$ for all $a \in A$. It makes no difference for now if we take $M_{b,a}$ to be a finite set rather than a natural number, and for reasons of functoriality that emerge later, it will be better if we do so.

Thus, in both examples the category $\mathcal{C}$ and the endofunctor G are determined by a set A and a matrix of sets $M = (M_{b,a})_{b,a\in A}$. This suggests the following definition.

Definition 1.4 **A discrete equational system** is a pair (A, M) where A is a set and M is a family $(M_{b,a})_{b,a\in A}$ of sets such that for each $a \in A$, the disjoint union $\sum_{b\in A} M_{b,a}$ is finite.

All in all, a discrete equational system (**DES**) is an endoprofunctor $M : A \times A \to \mathsf{Set}$ satisfying the additional request that the coproduct $\sum_b M(b,a)$ is a finite set (note: this is stronger than asking each $M(b,a)$ to be finite, because A must also be finite).

In the space of walks case over $\mathbb{N}$, from the endoprofunctor we obtain precisely the graph

$$\overset{\curvearrowright}{0} \leftarrow 1 \leftrightarrows 2 \leftrightarrows 3 \leftrightarrows 4 \leftrightarrows \cdots$$

Example 1.5 (One-variable systems) A discrete equational system (A, M) in which A is a one-element set amounts to just a finite set M. If M has n elements then the induced endofunctor $M \otimes -$ of **Top** is $X \longmapsto n \times X$. In the Cantor set example, $n = 2$.

Example 1.6 (Walks) The walks example corresponds to the discrete equational system (A, M) in which $A = \mathbb{N}$ and

$$|M_{b,a}| = \begin{cases} 1 & \text{if } a > 0 \text{ and } b = a \pm 1 \\ 1 & \text{if } a = b = 0 \\ 0 & \text{otherwise} \end{cases}$$

$(b, a \in \mathbb{N})$. The induced endofunctor $M \otimes -$ is exactly the functor G defined earlier.

Now for the interesting part: it is possible to produce a graph from a DES in a canonical way:

A discrete equational system (A, M) can also be viewed as a graph. Call an element $m \in M_{b,a}$ a **sector** of type b in a, and write $m : b \longrightarrow\!\!\!\!\!+\; a$. Then there is one sector of type b in a for each copy of X_b appearing in the expression (7) for $(M \otimes X)_a$. The (directed) graph corresponding to (A, M) has the elements of A as its vertices and the sectors as its edges (Figure 1.1). The finiteness condition on M is that each $a \in A$ contains only finitely many sectors, or equivalently that each vertex is at the head of only finitely many edges.

All in all, we can summarise the story as follows: the correspondence

$$\{\text{promonads } M : A \times A \to \mathsf{Set}\} \cong \text{ cats with } A \text{ as set of objects}$$

can be made more general because a generic endoprofunctor (so not a monoid in $\mathsf{Prof}(A, A)$, just an object) corresponds to a mere graph, which is compositive precisely when M is a monoid, i.e. a promonad.

This is just the good old principle "categories are monads in **Span**": graphs are endo-1-cells in **Span**.

Explicitly, every DES (A, M) defines a graph, having vertices the elements of A, and having an edge $m : b \rightsquigarrow a$ for every element $m \in M(b, a)$. This procedure is reversible, because every graph determines an endoprofunctor M on the set of its vertices: $M(e, f)$ is just the set of edges $e \to f$.

From here, it's all downhill:

Discrete systems and coalgebras

Definition 1.7 Let (A, M) be a discrete equational system and $\mathcal{E}$ a category with finite sums. An **M-coalgebra** (in $\mathcal{E}$) is a coalgebra for the endofunctor $M \otimes -$ of $\mathcal{E}^A$. A **universal solution** of (A, M) (in $\mathcal{E}$) is a terminal M-coalgebra.

When $\mathcal{E}$ is **Set** or **Top**, or more generally if $\mathcal{E}$ has enough limits, every discrete equational system has a universal solution. This can be constructed as follows.

For every object a define

$$
\begin{aligned}
I_a &= \{\dots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a\} \\
&= \{(\vec{a}, \vec{m}) \mid m_k \in M(a_k, a_{k-1})\}
\end{aligned}
$$

(Compare this with the nerve construction: every category $\mathcal{C}$ has an associated simplicial set, the *nerve* of $\mathcal{C}$, whose n-dimensional simplices are the sets of composabe n-tuples of morphisms in $\mathcal{C}$; here, we are taking -possibly infinite-simplices in the nerve of the category of elements $\int M$ of $M : A \times A \to$ Set.)

Now the correspondence $a \mapsto I_a$ s a functor $A \to \mathbf{Set}$, and

$$
(M \otimes I)_a = \sum_{b \in A} M(b, a) \times I_b
$$

or rather $\{\dots \to b_2 \to b_1 \to b_0 = b \to a\}$ which is just another way to say I_a, so that, by construction, $(M \otimes I)_\bullet \cong I_\bullet$.

Remark. Each I_a so defined has a natural choice for a topology, where a closed set $C_{\bar{v}}$ is the set of all subsets of I_a whose head is $\bar{v} = (v_1, \dots, v_n)$, as

$n, v_1, \ldots, v_n$ run over *finite* list of composable "fake arrows" $m \in M$.

Problem is that, precisely because of their universal property, all sets I_a turn out to be **totally disconnected**!

In fact, we have a "splitting" isomorphism

$$I_a \cong (M \otimes I)_a \cong \sum_{b \in A} M(b,a) \times I_b$$

and by virtue of the way in which **Set**-copowers are defined in a generic category, this is in turn equal to

$$\sum_{b, m: b \to a} I_b$$

i.e. to a **disjoint sum of copies of** I_b.

How to circumvent this evidently annoying limitation (we would like to describe even objects whose topology is more complicated)?

Ideally, the "problem" was the total disconnection of the domain A: taking into account a nondiscrete category (say $\mathbb{A}$) instead of a set, nstead of the sum $\sum_b M(b,a) \times I_b$ we consider not the coproduct

$$\sum_b M_{ba} \times I_b$$

any more, but instead the coend

$$(M \otimes I)_a = \int^b M(b,a) \times I_b$$

which is no more, no less than the composition of the profunctor $M : \mathbb{A} \rightsquigarrow \mathbb{A}$ with the profunctor $I : 1 \rightsquigarrow \mathbb{A}$.

Nondiscrete systems

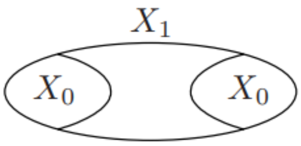
The real interval

In 1999, Peter Freyd [Fre2] found a new characterization of the real interval $[0,1]$. The interval is isomorphic to two copies of itself joined end to end, and Freyd's theorem says that it is universal as such.

The result of joining two copies of $[0,1]$ end to end is naturally described as the interval $[0,2]$, and then multiplication by 2 gives a bijection $[0,1] \longrightarrow [0,2]$, which may be written as

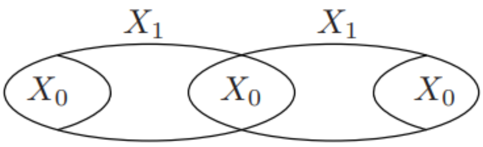
$$\iota_1 : \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \xrightarrow{\sim} \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet . \tag{10}$$

Now let $\mathcal{C}$ be the category whose objects are diagrams $X_0 \overset{u}{\underset{v}{\rightrightarrows}} X_1$ where X_0 and X_1 are sets and u and v are injections with disjoint images. (For now we consider only sets; we consider spaces later.) An object $X = (X_0, X_1, u, v)$ of $\mathcal{C}$ can be drawn as



where the copies of X_0 on the left and the right are the images of u and v respectively. A map $X \longrightarrow X'$ in $\mathcal{C}$ consists of functions $X_0 \longrightarrow X'_0$ and $X_1 \longrightarrow X'_1$ making the evident two squares commute.

Given $X \in \mathcal{C}$, we can form a new object $G(X)$ of $\mathcal{C}$ by gluing two copies of X end to end:



. (12)

Theorem 2.1 (Freyd) *(I, ι) is the terminal G-coalgebra.*

There is also a topological version of Freyd's theorem. Let $\mathcal{C}'$ be the category whose objects are diagrams $X_0 \overset{u}{\underset{v}{\rightrightarrows}} X_1$ of topological spaces and continuous closed injections with disjoint images, and whose maps are pairs of continuous maps making the evident squares commute. (A map of topological spaces is **closed** if the direct image of every closed subset is closed.) Define an endofunctor G' of $\mathcal{C}'$ by the same pushout diagram (13) as before. Define a G'-coalgebra (I, ι) as before, with the Euclidean topology on $[0,1]$.

Theorem 2.2 (Topological Freyd) *(I, ι) is terminal in the category of G'-coalgebras.*

(We have to choose u, v to be closed maps, because otherwise the topology on $[0,1]$ turns out to be trivial.)

Julia sets

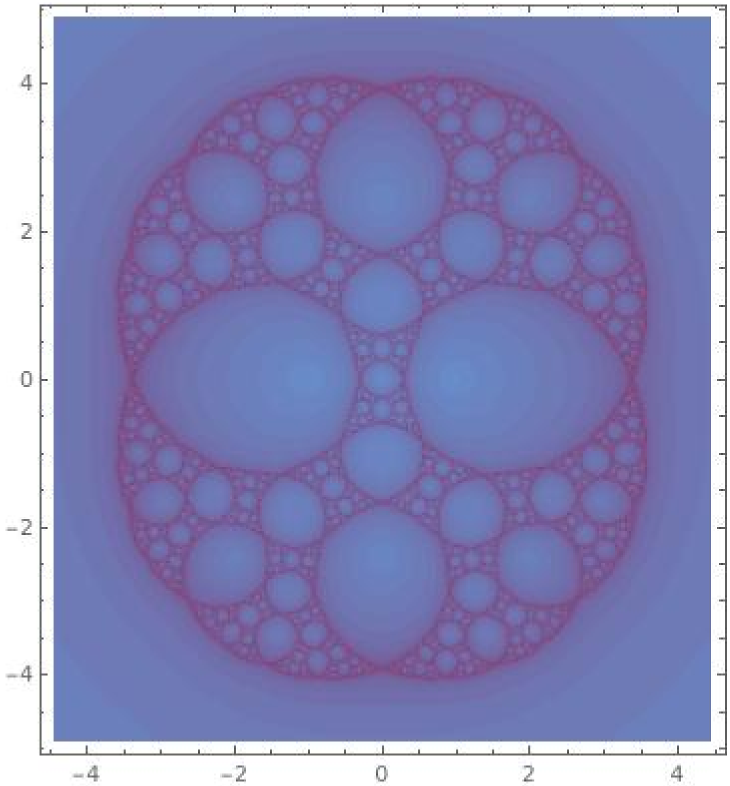
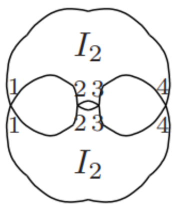


Figure 2.2(a) shows the Julia set of the function $z \longmapsto (2z/(1+z^2))^2$. Write I_1 for this Julia set, regarded as an abstract topological space. Evidently I_1 has reflectional symmetry in a horizontal axis, so may be written

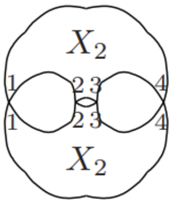
$I_1 \cong$  (14)

Let $\mathcal{C}$ be the category whose objects are diagrams

$$\begin{tikzcd}[row sep=small]
 & X_1 \\
X_0 \arrow[r, shift left=3, "u_1"] \arrow[r, shift left=1, "u_2"] \arrow[r, shift right=1, "u_3"] \arrow[r, shift right=3, "u_4"'] \arrow[dr, shift left=3, "v_1"] \arrow[dr, shift left=1, "v_2"] \arrow[dr, shift right=1, "v_3"] \arrow[dr, shift right=3, "v_4"'] & X_2 \\
 & X_3
\end{tikzcd}$$

of topological spaces and continuous closed injections such that u_1, u_2, u_3 and u_4 have disjoint images, and similarly v_1, v_2, v_3 and v_4. Let G be the endofunctor of $\mathcal{C}$ corresponding to the right-hand sides of (14)–(17); for instance,

$$(G(X))_1 =$$



$$= (X_2 + X_2)/\sim$$

for a certain equivalence relation $\sim$. (The picture of $(G(X))_1$ is drawn as if X_0 were a single point.) Then, conjecturally, (14)–(17) give an isomorphism $\iota : I \xrightarrow{\sim} G(I)$ and (I, ι) is the terminal G-coalgebra. If true, this means that

It was important to *restrict* the domain category for G, to a full subcategory of $\mathbf{Top}^A$, defined by **functors satisfying a certain exactness property.** This will be a main theme of the discussion:

"Solvable" NESs are determined by profunctors that are componentwise flat

For both the real interval and the Julia set, the category $\mathcal{C}$ is not $\mathbf{Set}^A$ or $\mathbf{Top}^A$ for any set A (as it was for discrete systems); rather, it is a full subcategory of $[\mathbb{A}, \mathbf{Set}]$ or $[\mathbb{A}, \mathbf{Top}]$ for some small category $\mathbb{A}$. In the case of the interval,

$$\mathbb{A} = \left(0 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} 1 \right), \tag{18}$$

and in the case of the Julia set,

$$\mathbb{A} = \left(\begin{tikzcd}[row sep=small]
 & 1 \\
0 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[dr, shift left=3] \arrow[dr, shift left=1] \arrow[dr, shift right=1] \arrow[dr, shift right=3] & 2 \\
 & 3
\end{tikzcd} \right). \tag{19}$$

This can be axiomatised with a clean, elegant request on what pro/functors we want to study

Definition 2.3 Let $\mathbb{A}$ be a small category. A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is **nondegenerate** (or **componentwise flat**) if the functor

$$- \otimes X : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

preserves finite connected limits. The full subcategory of $[\mathbb{A}, \mathbf{Set}]$ formed by the nondegenerate functors is written $\langle \mathbb{A}, \mathbf{Set} \rangle$.

Example 2.5 (Julia set) Here $\mathbb{A}$ is given by (19). In the gluing formula (15) for I_2, the one-point space I_0 appears 8 times (Figure 2.3), I_1 does not appear at all, I_2 appears twice, and I_3 appears once, so

$$|M(0,2)| = 8, \qquad |M(1,2)| = 0, \qquad |M(2,2)| = 2, \qquad |M(3,2)| = 1.$$

So, for instance, if $X \in [\mathbb{A}, \mathbf{Top}]$ then

$$(M \otimes X)_2 = (8 \times X_0 + 2 \times X_2 + X_3)/\sim$$

where $\sim$ identifies the 8 copies of X_0 with their images in X_2 and X_3. Again it can be shown that $M \otimes -$ restricts to an endofunctor of $\mathcal{C} = \langle \mathbb{A}, \mathbf{Top} \rangle$ and that this is the endofunctor G described earlier.

($\sim$ is the equivalence relation on the coend $\int^k M(k,2) \times X_k$ that defines $(M \otimes X)_2$).

Remark. Due to the way we compute left Kan extensions as coends (the integral above is exactly the Yoneda extension of M, computed on X, and evaluated on 2), the finiteness condition turns into a condition on the category of elements $\int M$ of M.

Definition 2.6 A presheaf $Y : \mathbb{B}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is **finite** if its category of elements is finite. A module $M : \mathbb{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}$ is **finite** if for each $a \in \mathbb{A}$, the presheaf $M(-, a)$ is finite.

Explicitly, M is finite if for each $a \in \mathbb{A}$ there are only finitely many diagrams of the form

$$b' \xrightarrow{f} b \overset{m}{\longrightarrow\!\!\!\!\!\!\!+\;\;} a.$$

Certainly this holds if, as in the interval example, the category $\mathbb{A}$ and the sets $M(b, a)$ are finite.

Finally, we have the definition of a **NES** (or just an **ES**):

Definition 2.7 Let $\mathbb{A}$ and $\mathbb{B}$ be small categories. A module $M : \mathbb{B} \looparrowright \mathbb{A}$ is **nondegenerate** if $M(b, -) : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate for each $b \in \mathbb{B}$.

Definition 2.8 An **equational system** is a small category $\mathbb{A}$ together with a finite nondegenerate module $M : \mathbb{A} \looparrowright \mathbb{A}$.

We might more precisely say 'finite-colimit equational system'. The discrete equational systems are precisely the equational systems $(\mathbb{A}, M)$ in which the category $\mathbb{A}$ is discrete (Example 4.5).

Definition 2.10 A topological space S is **realizable** if there exist an equational system $(\mathbb{A}, M)$ with universal solution (I, ι), and an object $a \in \mathbb{A}$, such that $S \cong I(a)$. It is **discretely realizable** (respectively, **finitely realizable**) if $\mathbb{A}$ can be taken to be discrete (respectively, finite).

(Instead of 'realizable', we might more precisely say 'corecursively realizable by finite colimits'.)

Conjecture 2.11 *The Julia set $J(f)$ of any complex rational function f is finitely realizable.*

There is a well-developed general theory of coalgebras for endofunctors, but for endofunctors $M \otimes -$ arising from equational systems, the theory has a special

flavour (§5). In a loose way it resembles homological algebra; we use terms such as *complex*, *double complex* and *resolution*. We develop this theory and prove that the endofunctor of $[\mathbb{A}, \mathbf{Set}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$, and similarly for **Top**, as was assumed in the introductory sections.

For the impatient listener: nondegeneracy is a flatness property for a diagram, and it can be stated in terms of said flatness conditions: a functor is *componentwise flat*, i.e. precomposition $- \diamond X$ (regarding X as a profunctor from/to the point and $\diamond$ as composition of profunctors) preserves connected limits if and only if any of b,c,d below are satisfied.

Theorem 4.11 (Nondegenerate functors) *Let $\mathbb{A}$ be a small category. The following conditions on a functor $X : \mathbb{A} \longrightarrow$ **Set** are equivalent:*

a. X is nondegenerate

b. every finite connected diagram in $\mathbb{E}(X)$ admits a cone

c. X satisfies **ND1** *and* **ND2**

d. X is a sum of flat functors.

where, $\mathbb{E}(X) = \int X$ is the category of elements of X, and conditions **ND1, ND2** are "amalgamation properties":

ND1 given

$$\begin{tikzcd} a \arrow[dr, "f"'] & & a' \arrow[dl, "f'"] \\ & b & \end{tikzcd}$$

in $\mathbb{A}$ and $x \in X(a)$, $x' \in X(a')$ such that $fx = f'x'$, there exist a commutative square

$$\begin{tikzcd} & c \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ a \arrow[dr, "f"'] & & a' \arrow[dl, "f'"] \\ & b & \end{tikzcd}$$

and $z \in X(c)$ such that $x = gz$ and $x' = g'z$

ND2 given $a \overset{f}{\underset{f'}{\rightrightarrows}} b$ in $\mathbb{A}$ and $x \in X(a)$ such that $fx = f'x$, there exist a fork

$$c \xrightarrow{g} a \overset{f}{\underset{f'}{\rightrightarrows}} b \tag{21}$$

and $z \in X(c)$ such that $x = gz$. (A diagram (21) is a **fork** if $fg = f'g$.)

These conditions work as recognition principles for flatness: see Ecclesiastes 1:9

None of this theory is new: it goes back to Grothendieck and Verdier [GV] and Gabriel and Ulmer [GU], and was later developed by Weberpals [Web], Lair [Lair], Ageron [Age], and Adámek, Borceux, Lack, and Rosický [ABLR]. More general statements of much of what follows can be found in [ABLR].

In the particular case where the domain of the presheaf X is a twisted product category $\mathcal{C}^o \times \mathcal{C}$ (so X is a profunctor) one can state **ND1** and **ND2** as follows:

ND1 any commutative square of solid arrows

$$
\begin{tikzcd}
& b \arrow[d, dotted, "p", "|" marking] \arrow[ddl, "m"', "|" marking] \arrow[ddr, "m'", "|" marking] & \\
& d \arrow[dl, dotted, "g"] \arrow[dr, dotted, "g'"'] & \\
a \arrow[dr, "f"'] & & a' \arrow[dl, "f'"] \\
& c, &
\end{tikzcd}
$$

can be filled in by dotted arrows to a commutative diagram as shown, and

ND2 any diagram $b \xrightarrow{m} a \underset{f'}{\overset{f}{\rightrightarrows}} c$ with $fm = f'm$ can be extended to a diagram

$$
\begin{tikzcd}
& d \arrow[d, dotted, "e"] \\
b \arrow[ur, dotted, "p", "|" marking] \arrow[r, "m"', "|" marking] & a \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] \\
& c
\end{tikzcd}
$$

in which the triangle commutes and the right-hand column is a fork.

Yes. But how do we *build* a solution to a NES?

Simple: with homological algebra.

Coalgebras & resolutions

A coalgebra can be thought of as a kind of iterative system [Adá]. To see this in our context, let $\mathbb{A}$ be any small category, $M : \mathbb{A} \nrightarrow \mathbb{A}$ any module, and (X, ξ) a coalgebra for the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$. Let $a_0 \in \mathbb{A}$ and $x_0 \in X(a_0)$. The map

$$\xi_{a_0} : X(a_0) \longrightarrow (M \otimes X)(a_0) = \Big(\sum_{a_1} M(a_1, a_0) \times X(a_1)\Big)/\sim$$

sends x_0 to

$$\xi_{a_0}(x_0) = (a_1 \overset{m_1}{\nrightarrow} a_0) \otimes x_1$$

for some $a_1 \in \mathbb{A}$, $m_1 \in M(a_1, a_0)$ and $x_1 \in X(a_1)$. (To represent $\xi_{a_0}(x_0)$ as $m_1 \otimes x_1$ requires a choice; there are in general many such representations.)

ξ_{a_0} is a map defined from Xa_0 to the quotient (defining the composition of profunctors); the choice one has to make is of a representative in an equivalence class. This choice can be repeated in order to *resolve* a give x:

Similarly, we may write

$$\xi_{a_1}(x_1) = (a_2 \overset{m_2}{\nrightarrow} a_1) \otimes x_2.$$

Continuing in this way, we obtain a diagram

$$\cdots \overset{m_{n+1}}{\nrightarrow} a_n \overset{m_n}{\nrightarrow} \cdots \overset{m_2}{\nrightarrow} a_1 \overset{m_1}{\nrightarrow} a_0 \tag{22}$$

and a sequence $x_\bullet = (x_n)_{n \in \mathbb{N}}$ with $x_n \in X(a_n)$ and

$$\xi_{a_n}(x_n) = m_{n+1} \otimes x_{n+1}$$

for all $n \in \mathbb{N}$. The diagram (22) together with the sequence $x_\bullet$ will be called a **resolution** $(a_\bullet, m_\bullet, x_\bullet)$ of x_0. I will also call $x_\bullet$ a resolution of x_0 **along** the diagram (22).

We say that a resolution is a choice of a transversal set of representatives in various equivalence classes built out of the NES: first, choose $[m_1 \otimes x_1] = \xi_{a_0}(x_0)$ where $m_1 \in M(a_1, a_0)$ for some a_1, which determines $\xi_{a_1} = [m_2 \otimes x_1]$ for some $m_2 \in M(a_2, a_1)$, which...

An obvious problem at this point is the uniqueness of the resolution constructed in this way: this is equivalent to ask when, exactly, two elements $m \otimes x, n \otimes y$ in $\sum M(b, a) \otimes X_b$ are identified with respect to the equivalence relation that defines the coend. It turns out that

Lemma 5.1 (Equality in $M \otimes X$) *Let $\mathbb{A}$ be a small category, let $M : \mathbb{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{A}$, and let $X \in \langle \mathbb{A}, \mathbf{Set} \rangle$. Take module elements*

$$
\begin{tikzcd}
b \arrow[dr, "m"', "|" marking] & & b' \arrow[dl, "m'", "|" marking] \\
& a &
\end{tikzcd}
$$

and $x \in X(b)$, $x' \in X(b')$. Then $m \otimes x = m' \otimes x' \in (M \otimes X)(a)$ if and only if there exist a commutative square

$$
\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
b \arrow[dr, "m"', "|" marking] & & b' \arrow[dl, "m'", "|" marking] \\
& a &
\end{tikzcd}
$$

and an element $z \in X(c)$ such that $fz = x$ and $f'z = x'$.

A **complex** in $(\mathbb{A}, M)$ is a diagram (22), abbreviated as $(a_\bullet, m_\bullet)$. A **map** $(a_\bullet, m_\bullet) \longrightarrow (a'_\bullet, m'_\bullet)$ of complexes is a sequence $f_\bullet = (f_n)_{n \in \mathbb{N}}$ of maps in $\mathbb{A}$ such that the diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "m_{n+1}", "|" marking] & a_n \arrow[r, "m_n", "|" marking] \arrow[d, "f_n"] & \cdots \arrow[r, "m_2", "|" marking] & a_1 \arrow[r, "m_1", "|" marking] \arrow[d, "f_1"] & a_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, "m'_{n+1}"', "|" marking] & a'_n \arrow[r, "m'_n"', "|" marking] & \cdots \arrow[r, "m'_2"', "|" marking] & a'_1 \arrow[r, "m'_1"', "|" marking] & a'_0
\end{tikzcd}
$$

commutes. For each $a \in \mathbb{A}$ there is a category $\mathcal{I}(a)$ whose objects are the complexes $(a_\bullet, m_\bullet)$ satisfying $a_0 = a$, and whose maps $f_\bullet$ are those satisfying $f_0 = 1_a$.

Now for the essential (*weak*) uniqueness of a resolution:

Proposition 5.2 (Essential uniqueness of resolutions) *Let $\mathbb{A}$ be a small category, $M : \mathbb{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{A}$ a module, and (X, ξ) a coalgebra for the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$, with X nondegenerate. Let $a \in \mathbb{A}$ and $x \in X(a)$. Then the category $\mathbf{Reso}(x)$ is connected.*

Corollary 5.3 (Resolving complex) *Take* $(\mathbb{A}, M)$, (X, ξ), $a \in \mathbb{A}$ *and* $x \in X(a)$ *as in Proposition 5.2. Then any two complexes along which* x *can be resolved lie in the same connected-component of* $\mathcal{I}(a)$.

Proof The complexes along which x can be resolved are the objects of $\mathcal{I}(a)$ in the image of the forgetful functor $\mathbf{Reso}(x) \longrightarrow \mathcal{I}(a)$. □

Coalgebras for nondegenerate modules

We still have to prove that for any equational system $(\mathbb{A}, M)$, the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$, and similarly with **Top** in place of **Set**. The set-theoretic case is straightforward.

Proposition 5.4 (Set-theoretic endofunctor) *Let* $\mathbb{A}$ *be a small category and* $M : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}$ *a nondegenerate module. Then the endofunctor* $M \otimes -$ *of* $[\mathbb{A}, \mathbf{Set}]$ *restricts to an endofunctor of* $\langle \mathbb{A}, \mathbf{Set} \rangle$.

Nondegeneracy of M is also a *necessary* condition, since for each $b \in \mathbb{A}$ the representable $\mathbb{A}(b, -)$ is nondegenerate, and $M \otimes \mathbb{A}(b, -) = M(b, -)$.

The proof is extremely simple: Y is nondegenerate iff $- \otimes Y$ preserves connected limits, so one has to show that if $- \otimes M$, $- \otimes X$ preserve connected limits, $- \otimes (M \otimes X)$ does the same; this is evident, because composition is associative.

$$\lim_{\leftarrow i} (Y_i \otimes M \otimes X) \cong \left(\lim_{\leftarrow i} (Y_i \otimes M) \right) \otimes X \cong \left(\lim_{\leftarrow i} Y_i \right) \otimes M \otimes X,$$

With a little bit of care and some point-set topology malarkey we now get that the underlying set of a topological universal solution to an ES is a set-theoretic universal solution:

Proposition 5.8 (Topological endofunctor) *Let* $(\mathbb{A}, M)$ *be an equational system. Then the endofunctor* $M \otimes -$ *of* $[\mathbb{A}, \mathbf{Top}]$ *restricts to an endofunctor of* $\langle \mathbb{A}, \mathbf{Top} \rangle$.

Proposition 5.9 (Top vs Set) *Let* $(\mathbb{A}, M)$ *be an equational system. The forgetful functor*

$$U_* : \mathbf{Coalg}(M, \mathbf{Top}) \longrightarrow \mathbf{Coalg}(M, \mathbf{Set})$$

has a left adjoint, and if (I, ι) *is a universal solution in* **Top** *then* $U_*(I, \iota)$ *is a universal solution in* **Set**.

Conversely, we will see later that any universal solution in **Set** carries a natural topology, and is then the universal solution in **Top**.

This perspective elegantly turns the existence-uniqueness of solutions to a DES into a triviality:

Example 5.10 (Discrete systems) When $\mathbb{A}$ is discrete, most of the results of this section become trivial. Every **Set**- or **Top**-valued functor on a discrete category is nondegenerate, so $\langle \mathbb{A}, \mathbf{Set} \rangle = [\mathbb{A}, \mathbf{Set}]$ and $\langle \mathbb{A}, \mathbf{Top} \rangle = [\mathbb{A}, \mathbf{Top}]$.

Let $M : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}$ be a module and (X, ξ) an M-coalgebra in **Set**. Then every element $x \in X(a)$ $(a \in \mathbb{A})$ has a unique resolution, and $\mathbf{Reso}(x)$ is the terminal category $\mathbf{1}$. As we saw in §1, every discrete equational system $(\mathbb{A}, M)$ has a universal solution in both **Top** and **Set**; and in accordance with Proposition 5.9, the universal solution in **Top** is the universal solution in **Set**, suitably topologized.

Construction of the universal solution

Interestingly enough, the assumptions under which a solution to an ES exists are as sharp as possible: an ES has a set-theoretic solution if and only if it has a topological solution, if and only if the system satisfies a certain solvability condition **S** divided in two parts;

S1 for every commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "m_3", "\shortmid"{marking}] & a_2 \arrow[r, "m_2", "\shortmid"{marking}] \arrow[d, "f_2"] & a_1 \arrow[r, "m_1", "\shortmid"{marking}] \arrow[d, "f_1"] & a_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, "p_3", "\shortmid"{marking}] & b_2 \arrow[r, "p_2", "\shortmid"{marking}] & b_1 \arrow[r, "p_1", "\shortmid"{marking}] & b_0 \\
\cdots \arrow[r, "m_3'"', "\shortmid"{marking}] & a_2' \arrow[r, "m_2'"', "\shortmid"{marking}] \arrow[u, "f_2'"'] & a_1' \arrow[r, "m_1'"', "\shortmid"{marking}] \arrow[u, "f_1'"'] & a_0', \arrow[u, "f_0'"']
\end{tikzcd}$$

there exists a commutative square

$$\begin{tikzcd}
 & a_0 \arrow[dr, "f_0"] & \\
\cdot \arrow[ur] \arrow[dr] & & b_0 \\
 & a_0' \arrow[ur, "f_0'"'] &
\end{tikzcd}$$

in $\mathbb{A}$, and

S2 for every serially commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "m_3", "\shortmid"{marking}] & a_2 \arrow[r, "m_2", "\shortmid"{marking}] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "f_2'"] & a_1 \arrow[r, "m_1", "\shortmid"{marking}] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_1'"] & a_0 \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_0'"] \\
\cdots \arrow[r, "p_3"', "\shortmid"{marking}] & b_2 \arrow[r, "p_2"', "\shortmid"{marking}] & b_1 \arrow[r, "p_1"', "\shortmid"{marking}] & b_0,
\end{tikzcd}$$

there exists a fork $\cdot \longrightarrow a_0 \underset{f_0'}{\overset{f_0}{\rightrightarrows}} b_0$ in $\mathbb{A}$.

In **S2**, 'serially commutative' means that $f_{n-1}m_n = p_n f_n$ and $f'_{n-1}m_n = p_n f'_n$ for all $n \geq 1$.

Example: if M is the hom-profunctor of a category $\mathbb{A}$, condition **S** says that $\mathbb{A}$ is componentwise filtered, i.e. every connected component of $\mathbb{A}$ is cofiltered:

Example 6.1 For any small category $\mathbb{A}$ there is a module $M : \mathbb{A} \looparrowright \mathbb{A}$ defined by $M(b, a) = \mathbb{A}(b, a)$, and $(\mathbb{A}, M)$ is an equational system as long as $\sum_b \mathbb{A}(b, a)$ is finite for each $a \in \mathbb{A}$. Condition **S** says that $\mathbb{A}$ is componentwise cofiltered; so, for instance, the equational system obtained by taking $\mathbb{A} = (0 \rightrightarrows 1)$ has no universal solution. If $\mathbb{A}$ *is* componentwise cofiltered then the universal solution is the functor $\mathbb{A} \longrightarrow$ **Top** constant at the one-point space, with its unique coalgebra structure.