

Reference Cards

Monoidal and enriched derivators

Friday 9th June, 2017 11:47

A nice feature of rings is that they behave like monoidal categories with one object (or vice versa).

- Any monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ (lax is enough) induces a **base change** 2-functor

$$F_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{W}\text{-Cat}$$

that sends a \mathcal{V} -category \mathcal{C} into the \mathcal{W} -category having the same objects of \mathcal{C} and where $(F_*\mathcal{C})(X, Y) = F(\mathcal{C}(X, Y))$.

- The structural 2-cells of F induce the monoidal structure on $F_*\mathcal{C}$.
- Monoidal transformations are induced accordingly (the definition is straightforward): a natural transformation $\beta : F \rightarrow G$ induces a 2-natural transformation between the 2-functors F_* and G_* with 'restricted' components.

It seems that this construction could be applied to $\mathcal{V} \rightarrow \mathbf{Set}$ to generate the **underlying** functor $U : \mathcal{V}\text{-Cat} \rightarrow \mathbf{Cat}$, but the fact is that $\text{hom}(J, -)$ is seldom monoidal.

The assignment described above that sends \mathcal{V} into $\mathcal{V}\text{-Cat}$, F to F_* and β to β_* is a 2-functor

$$(-)\text{-Cat} : \text{Cat}_{\otimes} \longrightarrow 2\text{-Cat}$$

A suitable 2-categorical Grothendieck construction gives rise then to a **universal fibration**

$$\begin{array}{c} \text{EnCat} \\ \downarrow p \\ \text{Cat}_{\otimes} \end{array}$$

whose fiber over \mathcal{V} is the 2-category of \mathcal{V} -categories.

- This is no different from the construction of the fibration **Mod** \rightarrow **Ring** whose fiber over the ring R is the category $R\text{-Mod}$ of modules over R . This is the canonical fibration for $F : \mathbf{Ring} \rightarrow \mathbf{Cat}$, and **Mod** $= \int_1 F$.
- General definitions pertaining the Grothendieck construction apply here and we have a definition on functors and natural transformations.

- A morphism $(\mathcal{V}, \mathcal{C}) \rightarrow (\mathcal{W}, \mathcal{D})$ in EnCat is given by a pair $u : \mathcal{V} \rightarrow \mathcal{W}$ and a functor $f : u_*\mathcal{C} \rightarrow \mathcal{D}$. Composition is given by

$$(vu)_*\mathcal{C} = v_*u_*\mathcal{C} \xrightarrow{v_*f} v_*\mathcal{D} \xrightarrow{g} \mathcal{E}$$

- A 2-cell $\alpha : (u, f) \rightarrow (v, g)$ is defined for two parallel 1-cells $(\mathcal{V}, \mathcal{C}) \rightarrow (\mathcal{W}, \mathcal{D})$ as a pair $\beta : u \rightarrow v$ (which is monoidal) and α is a 2-cell

$$\begin{array}{ccc}
 u_*\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
 \searrow \beta_* & \Downarrow \alpha & \nearrow g \\
 & v_*\mathcal{C} &
 \end{array}$$

All the forgetful functors $U_{\mathcal{V}} : \mathcal{V}\text{-Cat} \rightarrow \text{Cat}$ glue together to form a functor

$$U : \text{EnCat} \rightarrow \text{Cat}$$

defined by $U(\mathcal{V}, \mathcal{C}) = U_{\mathcal{V}}(\mathcal{C})$ = the underlying unenriched category of \mathcal{C} .
All the compatibility checks are straightforward.

Recall that a monoidal prederivator is a strict 2-functor $\mathbb{E} : \text{Cat}^{\text{op}} \rightarrow \text{Cat}_{\otimes}$.
 A prederivator enriched over \mathbb{E} is a 2-functor \mathbb{D} such that $p \circ \mathbb{D} = \mathbb{E}$.

The essential of this definition is: an enriched derivator specifies an $\mathbb{E}(J)$ -enriched category $\mathbb{D}(J)$ for each $J \in \text{Cat}$, and this specification is 2-functorial in J . Graphically,

$$\begin{array}{ccccc} \text{Cat}^{\text{op}} & \xrightarrow{\mathbb{D}} & \text{EnCat} & \xrightarrow{U} & \text{CAT} \\ & \searrow \mathbb{E} & \downarrow p & & \\ & & \text{Cat}_{\otimes} & & \end{array}$$

The composition $U \circ \mathbb{D}$ is the prederivator **underlying** the enriched prederivator \mathbb{D} .

Defining a morphism of enriched prederivators is notationally quite painful, but the definition is clear: it's a pseudonatural transformation between 2-functors $\text{Cat}^{\text{op}} \rightarrow \text{EnCat}$.

From the definition of morphism in EnCat it follows that we have to specify a pseudonatural transformation $F : \mathbb{D} \rightarrow \mathbb{D}'$ whose components $F_I : \mathbb{D}(I) \rightarrow \mathbb{D}'(I)$ satisfy the commutativity

$$\begin{array}{ccc} \mathbb{E}(u)_* \mathbb{D}(K) & \longrightarrow & \mathbb{E}(u)_* \mathbb{D}'(K) \\ \downarrow & \swarrow & \downarrow \\ \mathbb{D}(J) & \xrightarrow{F_J} & \mathbb{D}'(J) \end{array}$$

for each $u : J \rightarrow K$, where we exceptionally denoted $\mathbb{E}(u)$ the action of \mathbb{E} on u .

(the yoga is: as a monoidal functor, $\mathbb{E}(u)$ turns $\mathbb{D}(K)$ into a $\mathbb{E}(J)$ -enriched category, and then the square above is the only way to compare them according to the def. of morphisms in EnCat).

A general result in enriched stuff is:

Theorem

Given a 2v adjunction $\mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ where \mathcal{E} is monoidal and \mathcal{D} is \mathcal{E} -tensorial. Then \mathcal{D} is also \mathcal{E} -cotensorial and canonically \mathcal{E} -enriched.

We want to show that this is the base case of a theorem on derivators:

Theorem for derivators

Let \mathbb{E} be a monoidal derivator, and $b\mathcal{D}$ tensorial over \mathbb{E} . If there is a 2v adjunction inducing the tensoring,

$$(\otimes, \text{hom}_l, \text{hom}_r) : \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D}$$

then \mathbb{D} is canonically \mathbb{E} -enriched and cotensorial.

From the definition of an 2v adjunction for derivators we get that each $\mathbb{D}(K)$ is tensored over $\mathbb{E}(K)$ and part of a 2v adjunction

$$(\otimes, \text{HOM}_{l, \mathbb{D}(K)}, \text{HOM}_{r, \mathbb{D}(K)}) : \mathbb{E}(K) \times \mathbb{D}(K) \rightarrow \mathbb{D}(K)$$

Using the result for plain categories we get that each $\mathbb{D}(K)$ is enriched over $\mathbb{E}(K)$, and we prove that it is coherently so: hom_r will give all the needed coherence.

As a general tenet, if you can do something in model categories you can do it in derivators:

If \mathcal{M}, \mathcal{N} are combinatorial model categories, \mathcal{M} is also monoidal, and \mathcal{N} is \mathcal{M} -tensored, then the derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored, cotensored and enriched over the monoidal derivator $\mathbb{D}_{\mathcal{M}}$.

This applies to **sSet**-model categories, **Sp**-model categories, **dg_k**-model categories. . .

The Grothendieck construction

The previous construction of p makes heavy use of the Grothendieck construction for 2-categories. We recall it starting from its 0-dimensional counterpart.

For a functor $F : I \rightarrow \mathbf{Set}$ all you need to know is in any of these equivalent universal properties:

$$\begin{array}{ccc}
 \int_0 F \longrightarrow \mathbf{Set}_* & \int_0 F \longrightarrow I^{\mathrm{op}} & \int_0 F \longrightarrow * \\
 \downarrow & \swarrow & \swarrow \\
 I \xrightarrow{F} \mathbf{Set} & * \xrightarrow{[F]} [I, \mathbf{Set}] & I \longrightarrow \mathbf{Set} \\
 & \downarrow y & \downarrow [*] \\
 & &
 \end{array}$$

There is a fibration $p : \int_0 F \rightarrow I$ such that $p^{-1}i$ is the set $F(i)$.

For a functor $F : I \rightarrow \mathbf{Cat}$ we define $\int_1 F$ as the category of pairs $(i, X \in F(i))$, and a morphism $(i, X) \rightarrow (j, Y)$ to be a pair (f, u) such that $f : i \rightarrow j$ and $u : F(f)X \rightarrow Y$ in $F(j)$. Composition is defined as

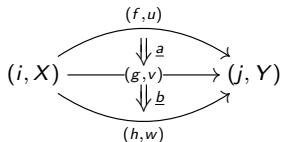
$$\begin{aligned} (i, X) &\xrightarrow{(f, u)} (j, Y) \xrightarrow{(g, v)} (k, Z) \\ (i, X) &\xrightarrow{(g \circ f, v \circ F(g)u)} (k, Z) \end{aligned}$$

Again there is a fibration $p : \int_1 F \rightarrow I$ such that $p^{-1}i$ is a category isomorphic to $F(i)$.

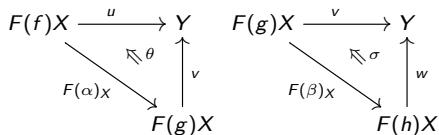
For a 2-functor $F : I \rightarrow 2\text{-}\mathbf{Cat}$, things go as expected but the definition is quite daunting: $\int F$ has $\int_1 F$ as underlying 1-category (in a similar manner, $\int_1 F$ has $\int_0 F$ as set of objects); 2-cells and their two compositions (horizontal and vertical) are defined as follows

- A 2-cell $(i, X) \begin{array}{c} \xrightarrow{(f,u)} \\ \Downarrow \underline{a} \\ \xrightarrow{(g,v)} \end{array} (j, Y)$ is a pair (α, θ) such that $\alpha : f \rightarrow g$ is a 2-cell in I and $\theta : v.F(\alpha)_X \rightarrow u$ is a 2-cell in $F(j)$.

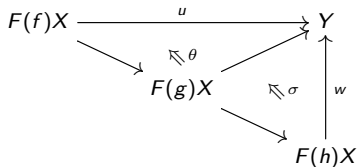
- Horizontal composition is defined for cells



i.e. for diagrams of 2-cells like



as the pasting



giving that $(\beta, \sigma) \circ_v (\alpha, \theta) = (\beta \circ_v \alpha, (\sigma * F(\alpha)_X) \circ \theta)$.

- Vertical composition is defined for cells

$$\begin{array}{ccccc}
 (i, X) & \xrightarrow{(f_1, u_1)} & (j, Y) & \xrightarrow{(g_1, v_1)} & (k, Z) \\
 & \Downarrow \underline{a} & & \Downarrow \underline{b} & \\
 (i, X) & \xrightarrow{(f_2, u_2)} & (j, Y) & \xrightarrow{(g_2, v_2)} & (k, Z)
 \end{array}$$

i.e. for diagrams of 2-cells like

$$\begin{array}{ccc}
 F(f_1)X & \xrightarrow{\quad} & Y \\
 & \searrow \theta & \uparrow \\
 & & F(f_2)X
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(g_1)Y & \xrightarrow{\quad} & Z \\
 & \searrow \sigma & \uparrow \\
 & & F(g_2)Y
 \end{array}$$

as the pasting

$$\begin{array}{ccccc}
 F(g_1 f_1)X & \xrightarrow{F(g_1)u_1} & F(g_1)Y & \xrightarrow{v_1} & Z \\
 & \searrow F(g_1)\theta & \downarrow & \searrow \sigma & \uparrow v_2 \\
 & & F(g_1 f_2)X & \xrightarrow{\text{nat}} & F(g_2)Y \\
 & & \searrow F(\beta)_{f_2}X & & \uparrow F(g_2)u_2 \\
 & & & & F(g_2 f_2)X.
 \end{array}$$