Bicategories of automata, automata in bicategories

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Prolegomena

Fix a monoidal category $(\mathcal{K}, \otimes)$.

Definition

A Mealy machine (of input I and output O) in $\mathcal{K}$ is a span

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

Definition

A Moore machine (of input I and output O) in $\mathcal{K}$ is a span

$$E \xleftarrow{d} E \otimes I \,,\; E \xrightarrow{s} O$$

Prolegomena

```
record MealyObj I O : Set (o ⊔ l ⊔ e) where
  field
    E : Obj
    d : E ⊗ I ⇒ E
    s : E ⊗ I ⇒ O
```

Mealy and Moore

Definition

The category of Mealy machines[1] has objects the Mealy machines as above, (E, d, s), and morphisms $(E, d, s) \to (F, d', s')$ the $f : E \to F$ such that

$$\begin{tikzcd}
E \arrow[d, "f"'] & E \otimes I \arrow[l, "d"'] \arrow[r, "s"] \arrow[d, "f \otimes I"] & O \arrow[d, equal] \\
F & F \otimes I \arrow[l, "d'"'] \arrow[r, "s'"'] & O
\end{tikzcd}$$

[1] All definitions from now on can be Moore-ified without effort.

Mealy and Moore

Let P be an monoidal monad on $\mathcal{K}$; the Kleisli category of P becomes monoidal; dually, if P is opmonoidal, Elienberg-Moore (not the same Moore!) becomes monoidal.

Definition

A P_λ-machine is a Mealy machine in $Kl(P)$; a P-machine is a Mealy machine in $EM(P)$.

Why care?

- nondeterminism (if $P =$powerset, $Kl(P) = \textbf{Rel}$);
- additional structure on objects (they are P-algebras).

Mealy and Moore

Theorem
The category of Mealy machines fits into a (strict, 2-)pullback in **Cat**

$$\begin{tikzcd} \mathbf{Mly}(I, O) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}(- \otimes I) \arrow[d] \\ ((- \otimes I)/O) \arrow[r] & \mathcal{K} \end{tikzcd}$$

(A similar result holds for Moore: replace the comma $((- \otimes I)/O)$ with the slice $\mathcal{K}/O$.)

X-automata

(cf. Adámek-Trnková)

Instead of considering a span

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

consider the action of a generic endofunctor $X : \mathcal{K} \to \mathcal{K}$:

$$E \longleftarrow XE \longrightarrow O$$

X-algebra obj. of comma

which yields at once a pullback characterization of X-automata, ...

X-automata

$$\begin{tikzcd} X\text{-}\mathbf{Mly} \arrow[r, "U'"] \arrow[d, "V'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X/O) \arrow[d, "V"] \\ \mathbf{Alg}(X) \arrow[r, "U"'] & \mathcal{K} \end{tikzcd}$$

...and in particular when $X \dashv R$

Theorem
The category of X-automata is cocomplete when $\mathcal{K}$ is, with colimits created by a canonical functor $X\text{-}\mathbf{Mly} \to \mathcal{K}$; it is complete when $\mathcal{K}$ is.

X-automata

From [Mac Lane, V.6, Ex. 3]: in every strict pullback of categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "U'"] \arrow[d, "V'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "V"] \\ \mathcal{C} \arrow[r, "U"'] & \mathcal{K} \end{tikzcd}$$

if U creates, and V preserves, limits of a given shape $\mathcal{J}$, then U' creates limits of shape $\mathcal{J}$.

But $\mathbf{Alg}(X) \to \mathcal{K}$ creates all limits, and $X/O \to \mathcal{K}$ all connected limits; thus the problem boils down to find a terminal object (products follow).

X-automata

Claim: the terminal object of X-**Mly** is the terminal coalgebra for the functor

$$A \mapsto RA \times O$$

which (Adámek theorem) is $O_\infty = \prod_{n \geqslant 1} R^n O$, with structural morphisms given by deletion of first factor, and projection over first factor:

$$d_\infty : X\Big(\prod_{n \geqslant 1} R^n O\Big) \to \prod_{n \geqslant 1} R^n O$$
$$s_\infty : X\Big(\prod_{n \geqslant 1} R^n O\Big) \to O$$

A similar line of reasoning leads to the terminal object in X-**Mre** being $O_\infty = \prod_{n \geqslant 0} R^n O$.

X-automata

How to induce a terminal morphism

$$
\begin{tikzcd}
E \arrow[d, "!_E"'] & XE \arrow[l, "d"'] \arrow[r, "s"] \arrow[d, "X!_E"'] & O \arrow[d, equal] \\
O_\infty & XO_\infty \arrow[l, "d_\infty"] \arrow[r, "s_\infty"'] & O
\end{tikzcd}
$$

$!_E : E \to O_\infty$ is defined as

$$d_\infty : \text{mate}\big(\cdots \to XXXE \xrightarrow{XXd} XXE \xrightarrow{Xd} XE \xrightarrow{s} O\big)$$

$$s_\infty : \text{mate}\big(\cdots \to XXXE \xrightarrow{XXd} XXE \xrightarrow{Xd} XE \xrightarrow{s} O\big)$$

X-automata

Products are computed as pullbacks along terminal maps, but the latter are computed as in $\mathcal{K}$:

$$\begin{tikzcd} P_\infty \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "\bar{s}_{F,\infty}"] \\ E \arrow[r, "\bar{s}_{E,\infty}"'] & O_\infty \end{tikzcd}$$

The pullback P_∞ can be thought as the bisimulation object for X-machines. As a corollary,

- **Mly**(I, O) complete with terminal object $[I^+, O]$ ($\leftarrow$ free semigroup).
- **Mre**(I, O) complete with terminal object $[I^*, O]$ ($\leftarrow$ free monoid).

Behaviour as an adjunction

Assume the forgetful $U : \mathbf{Alg}(X) \to \mathcal{K}$ has a lef adjoint F.

There is a composite of adjoint functors

$$\begin{tikzcd} \mathcal{K}_{/O_\infty} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(X)_{/(O_\infty, d_T)} \arrow[l, shift left, "\tilde{U}"] \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & X\text{-}\mathbf{Mre} \arrow[l, shift left, "B"] \end{tikzcd}$$

where B is a ‘behaviour’ functor defined as

$$(E, d, s) \mapsto (!_E : E \to O_\infty)$$

and its left adjoint L is determined through the ‘free’ Moore machine on a X-algebra over the terminal O_∞.

Bicategories

So far so good.

It's all fun and games until someone ~~loses an eye~~uses a bicategory.

- under which assumptions is $\mathbf{Mly}(I, O)$ the hom-category of a bicategory?
- similar question for Mealy.
- A monoidal category is just$^{\mathrm{TM}}$ a bicategory with a single object; but then what is a Mealy automaton in a bicategory $\mathbb{B}$?

Bicategories of Automata

The bicategory $\underline{\textbf{Mly}}$

Let $\mathcal{K}$ be a Cartesian category. Define a bicategory $\underline{\textbf{Mly}}_{\mathcal{K}}$ as follows

1. its **objects** are the same objects of $\mathcal{K}$;
2. its **1-cells** $I \to O$ are the Mealy machines (E, d, s), i.e. the objects of the category $\textbf{Mly}_{\mathcal{K}}(I, O)$;
3. its **2-cells** are Mealy machine morphisms defined *ibid.*;
4. the composition of 1-cells $- \circ -$ is defined as [postponed];
5. the **vertical** composition of 2-cells is the composition of Mealy machine morphisms $f : E \to F$;
6. the **horizontal** composition of 2-cells is the operation defined thanks to bifunctoriality of $- \circ -$;
7. the associator and the unitors are inherited from $\mathcal{K}$.

Composition of 1-cells

Given two Mealy machines $(E, d, s) : I \to J$ and $(F, d', s') : J \to K$,

$$E \xleftarrow{d} E \times I \xrightarrow{s} J$$

$$F \xleftarrow{d'} F \times J \xrightarrow{s'} K$$

define their composition $(E \times F, d' \diamondsuit d, s' \diamondsuit s) : I \to K$ as

$$s' \diamondsuit s = s' \cdot (E \times s) : F \times E \times I \to F \times J \to K$$

$$d' \diamondsuit d = \langle d \cdot \pi_F, d' \cdot (E \times s) \rangle : F \times E \times I \to F \times E$$

where

$$E \xleftarrow{d} E \times I \xleftarrow{\pi_F} E \times F \times I \xrightarrow{E \times s} F \times J \xrightarrow{d'} F$$

Composition of 1-cells

Proof of associativity is bureaucracy:

$$
\begin{aligned}
(d_1 \diamondsuit d_2) \diamondsuit d_3 &= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1, d_1 \cdot (E_1 \times s_2)\rangle \cdot (E_1 \times E_2 \times s_3)\rangle \\
&= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3), d_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3)\rangle\rangle \\
&= \langle d_3 \cdot \pi_{12}, \underline{\langle d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3)}, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle\rangle \\
d_1 \diamondsuit (d_2 \diamondsuit d_3) &= \langle\langle d_3 \cdot \pi_2, d_2 \cdot (E_2 \times s_3)\rangle \cdot \pi_1, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle \\
&= \langle\langle d_3 \cdot \pi_2 \cdot \pi_1, d_2 \cdot (E_2 \times s_3) \cdot \pi_1\rangle, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle \\
&= \langle d_3 \cdot \pi_{12}, \langle \underline{d_2 \cdot (E_2 \times s_3) \cdot \pi_1}, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle\rangle \\
s_1 \diamondsuit (s_2 \diamondsuit s_3) &= s_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \\
&= s_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3) \\
(s_1 \diamondsuit s_2) \diamondsuit s_3 &= s_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3)
\end{aligned}
$$

Unitality follows a similar (simpler) strategy.

Corollar(ies)

- there are categories $\underline{\mathbf{Mly}}_{\mathbf{Set}}, \underline{\mathbf{Mly}}_{\mathbf{Cat}}, \underline{\mathbf{Mly}}_{\mathbf{Top}}, \underline{\mathbf{Mly}}_{\mathbf{Pos}}, \underline{\mathbf{Mly}}_{\mathbf{Mon}}, \dots$
- if $\mathcal{K}$ is Cartesian closed, all right/left Kan extensions/lifts exist;
- as a corollary, the terminal objects $[I^+, O], [I^*, O]$ (and even before, I^+, I^*) can be characterised as Kan extensions!

? the assignment $\mathcal{K} \mapsto \underline{\mathbf{Mly}}_{\mathcal{K}}$ is (2-)functorial $\mathbf{CCat} \to 2\text{-}\mathbf{Cat}$ (careful with the 2-cells, Eugene).

? Guitart defines a 'bicategory of Mealy machines' as $\mathbf{Spn}_F(\mathbf{Mon})$, spans in **Cat** between monoids whose left leg is a fibration. Interesting adjunctions with our $\underline{\mathbf{Mly}}$'s?

Automata in bicategories

A monoidal category is just a bicategory with a single object. What is a machine inside a bicategory $\mathbb{B}$ with objects $A, B, C \dots$?

Definition

A bicategorical Moore machine consists of a span of 2-cells in $\mathbb{B}$

$$e \xLeftarrow{\delta} e \circ i \xRightarrow{\sigma} o$$

or rather a diagram of 2-cells

$$\begin{tikzcd}
 & A \arrow[d, "i"'] \arrow[dd, bend right=60, "e"'] \arrow[dd, bend left=60, "o"] & \\
{} \arrow[r, phantom, "\Leftarrow_{\delta}"] & A \arrow[d, "e"'] \arrow[r, phantom, "\Rightarrow^{\sigma}"] & {} \\
 & B &
\end{tikzcd}$$

for objects $A, B \in \mathbb{B}$.

Examples

- the mere fact that the 2-cells δ, σ exist implies that i is an endomorphism;
- so, iterated compositions $i \circ \cdots \circ i$ make sense as much as iterated tensor powers $I \otimes \cdots \otimes I$ made sense in $\mathcal{K}$;
- one can find examples in
  - categories, functors and natural transformations;
  - categories, functors and lax transformations;
  - categories, profunctors and 2-cells (a fortiori, in **Rel**);
  - sets and metric relations;
  - topological, approach, closure spaces,...

Behaviour as a Kan extension

A bicategorical Moore machine in $\mathbb{B}$ of fixed input and output i, o is a diagram

$$\begin{tikzcd}
& A \arrow[dl, "i"'] \arrow[dr, dotted, "e"] \arrow[dr, bend left=50, "o"] \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] \arrow[dr, phantom, shift left=3, "\sigma\Nearrow"] & \\
A \arrow[rr, dotted, "e"'] & {} & B
\end{tikzcd}$$

The 'terminal way' of filling such a span is the right extension of the output cell along the free monad $i^\natural$ on the input:

- from the unit $\boldsymbol{\eta} : 1_A \Rightarrow i^\natural$, get $\mathrm{Ran}_i \Rightarrow \mathrm{Ran}_1 = 1_A$, and thus $\sigma : \mathrm{Ran}_i o \Rightarrow o$;
- from the multiplication $\boldsymbol{\mu} : i^\natural \circ i^\natural \Rightarrow i^\natural$ get $\mathrm{Ran}_{i^\natural} \Rightarrow \mathrm{Ran}_{i^\natural} \circ \mathrm{Ran}_{i^\natural}$, and thus

$$\delta : \mathrm{Ran}_{i^\natural} o \circ i \xRightarrow{\mathrm{Ran}_{i^\natural} o * \boldsymbol{\eta}} \mathrm{Ran}_{i^\natural} o \circ i^\natural \xRightarrow{\quad\quad} \mathrm{Ran}_{i^\natural} o$$

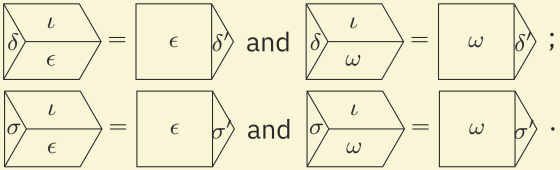
Intertwiners

Definition

An intertwiner $(u, v) : (e, \delta, \sigma)_{A,B} \nrightarrow (e', \delta', \sigma')_{A',B'}$ consists of a pair of 1-cells $u : A \to A'$, $v : B \to B'$ and a triple of 2-cells ι, ϵ, ω disposed as follows:

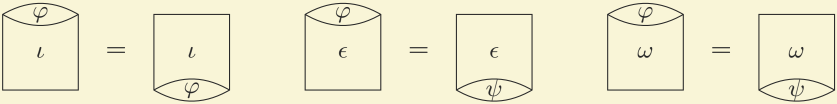
$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow_{\iota}"] & A' \arrow[d, "i'"] \\
A \arrow[r, "u"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "e"'] \arrow[dr, phantom, "\swarrow_{\epsilon}"] & A' \arrow[d, "e'"] \\
B \arrow[r, "v"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "o"'] \arrow[dr, phantom, "\swarrow_{\omega}"] & A' \arrow[d, "o'"] \\
B \arrow[r, "v"'] & B'
\end{tikzcd}
$$

such that:



Definition

Let $(u, v), (u', v') : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ be two parallel intertwiners between bicategorical Mealy machines; a 2-cell $(\varphi, \psi) : (u, v) \Rightarrow (u', v')$ consists of a pair of 2-cells $\varphi : u \Rightarrow u'$, $\psi : v \Rightarrow v'$ such that the following identities hold true:



Intertwiners

Specialized to the monoidal case, the previous two definitions become

- **morphisms** of type $\iota : I' \otimes U \to V \otimes I, \epsilon : E' \otimes U \to V \otimes E, \omega : O' \otimes U \to V \otimes O;$
- such that

$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$
$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

- **pairs** $f : U \to U'$ and $g : V \to V'$ such that

$$\begin{tikzcd}
E' \otimes I' \otimes U \arrow[r, "d' \otimes U"] \arrow[d, "E' \otimes I' \otimes f"'] & E' \otimes U \arrow[d, "E' \otimes f"] \\
E' \otimes I' \otimes U' \arrow[r, "d' \otimes U'"'] & E' \otimes U'
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes E \otimes I \arrow[r, "V \otimes d"] \arrow[d, "g \otimes E \otimes I"'] & V \otimes E \arrow[d, "g \otimes E"] \\
V' \otimes E \otimes I \arrow[r, "V' \otimes d"'] & V' \otimes E
\end{tikzcd}$$

Open problems

Of other bicategories

In 1974 Guitart defined a bicategory of Mealy machines:

- the objects are categories $M, N, \dots$ (actually, monoids);
- the 1-cells are spans

$$M \xleftarrow{D} \mathcal{E} \xrightarrow{S} N$$

  where D is a fibration and S is a functor.
- composition of 1-cells is as in **Span**.
- G then proves that **MAC** is the Kleisli bicategory of the diagram monad $C \mapsto \mathbf{Cat}/\!\!/C$;

We conjecture the existence of a left pseudo-adjoint L in

$$L : \underline{\mathbf{Mly}_{\mathbf{Cat}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{MAC} : G$$

Nondeterminism in equipments

- In **Rel**, $R = \mathrm{Ran}_{I^\natural} O$ is the relation defined as

$$(a, b) \in R \iff \forall a' \in A.((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

  This relation expresses *reachability* of b from a:

$$a\,R\,b \iff \Big((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a'\,O\,b\Big)$$

- Passing from automata in **Cat** to automata in Prof accounts for a form of nondeterminism; one can conjecture to be able to address *nondeterministic* BA in $\mathbb{B}$ as *deterministic* BA in a proarrow equipment.