HEARTS AND TOWERS IN STABLE ∞-CATEGORIES

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Abstract. We exploit the equivalence between t-structures and normal torsion theories on stable ∞-categories to unify two apparently separated constructions in the theory of triangulated categories: the characterization of *bounded t-structures* in terms of their hearts and *semiorthogonal decompositions* on triangulated categories. In the stable ∞-categorical context both notions stem from a single construction, the *Postnikov tower* of a morphism induced by a $\mathbb{Z}$-equivariant *multiple* (bireflective and normal) *factorization system* $\{\mathbb{F}_i\}_{i\in J}$. For $J = \mathbb{Z}$ with its obvious self-action, we recover the notion of Postnikov towers in a triangulated category endowed with a t-structure, and give a proof of the the abelianity of the *heart* in the ∞-stable setting. For J is a finite totally ordered set, we recover the theory of *semiorthogonal decompositions*.

Contents

Memories of a forebear paper. The purpose of this brief subsection is to give an account of results and notation from our previous [FL14], of which the present work is a natural continuation. We borrow from [FL14] for what concerns categories and higher categories, functors, simplicial sets, and in particular for what concerns the theory of quasicategorical factorization systems, t-structures on stable ∞-categories and related topics. Since the aim of this section is only to provide a reference, statements will be recalled in a somehow sketchy form; we refer the reader to [FL14] for the fully rigorous version of these results.

The starting point is the following rephrasing of [CHK85, Prop. **2.2**].

Proposition. Let $\mathbf{C}$ be a category with a terminal object. There exists an antitone Galois connection $\Phi \dashv \Psi$ between the poset $\mathrm{Rex}(\mathbf{C})$ of reflective subcategories of $\mathbf{C}$ and the poset $\mathrm{PF}(\mathbf{C})$ of prefactorization systems on $\mathbf{C}$. The morphism Ψ sends $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ to the reflexive subcategory $\mathcal{M}/1 = \{B \in \mathbf{C} \mid (B \to 1) \in \mathcal{M}\}$, and Φ is

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defined by sending a refelexive subcategory **B** of **C** to the prefactorization system right generated by $\hom(\mathbf{B}) \subseteq \hom(\mathbf{C})$.

Starting from this, we deduce in [FL14, Thm. **3.13**] that

Theorem. Let **C** be a stable ∞-category. There is a bijective correspondence between the class of normal torsion theories $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ on **C** and the class of t-structures on **C**.

Remark. The partial order on $\text{PF}(\mathbf{C})$is given by $\mathbb{F}_{=(\mathcal{E},\mathcal{M})} \preceq \mathbb{F}'_{=(\mathcal{E}',\mathcal{M}')}$ iff $\mathcal{M} \subset \mathcal{M}'$ (or equivalently, $\mathcal{E}' \subset \mathcal{E}$). We endow the collection of normal torsion theories (and then the collection of t-structures on **C**) with the induced order.

Finally, we recall a simple and yet pervasive result in the context of normal torsion theories.

Lemma (Sator Lemma)**.** In a pointed quasicategory **C**, an initial arrow $0 \to A$ lies in a class $\mathcal{E}$ or $\mathcal{M}$ of a bireflective factorization system $\mathbb{F}$ if and only if the terminal arrow $A \to 0$ lies in the same class.

1. Posets with $\mathbb{Z}$-actions and $\mathbb{Z}$-equivariant factorization systems.

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Laozi LXIII

We begin by recalling a number of definitions about $\mathbb{Z}$-actions on partially ordered sets. Here the group of integers $\mathbb{Z}$ is seen as an ordered group, with the usual ordering, and the $\mathbb{Z}$-actions we are going to consider are therefore a particular case of the notion of partially ordered group actions on partially ordered sets. We refer the interested reader to [Bly05, Gla99, Fuc63] for more on the general case.

Definition 1.1. A $\mathbb{Z}$-poset is a partially ordered set $(P, \leq)$ together with a group action

$$\begin{aligned} +\colon P \times \mathbb{Z} &\to P \\ (x, n) &\mapsto x + n \end{aligned}$$

which is a morphism of partially ordered sets.

In the above definition, the partial order on $\mathbb{Z} \times P$ is the lexicographic one. This is actually the partial order realising $(\mathbb{Z} \times P, \leq)$ as the product of $(\mathbb{Z}, \leq)$ and $(P, \leq)$ in the category **Pos** of posets. One immediately sees that a $\mathbb{Z}$-poset is equivalently the datum of a poset $(P, \leq)$ together with a monotone bijection $\rho\colon P \to P$ such that $x \leq \rho(x)$ for any x in P. Namely, ρ and the action are related by the identity $\rho(x) = x + 1$

Example 1.2. The poset $(\mathbb{Z}, \leq)$ of integer numbers with their usual order is a $\mathbb{Z}$-poset with the action given by the usual sum of integer numbers. The poset $(\mathbb{R}, \leq)$ of real numbers with their usual order is a $\mathbb{Z}$-poset for the action given by the sum of real numbers with integer numbers (seen as a subring of real numbers).

Remark 1.3. If $(P, \leq)$ is a finite poset, then the only $\mathbb{Z}$-action it carries is the trivial one. Indeed, if $\rho \colon P \to P$ is the monotone bijection associated with the $\mathbb{Z}$-action, one sees that ρ is of finite order by the finiteness of P. Therefore there exists an $n \geq 1$ such that $\rho^n = \mathrm{id}_P$. It follows that, for any x in P,

$$x \leq x + 1 \leq \cdots \leq x + n = x$$

and so $x = x + 1$.

Lemma 1.4. If $k \in P$ is a $\leq$-maximal element in the $\mathbb{Z}$-poset $(P, \leq)$, then it is a $\mathbb{Z}$-fixed point. The same statement holds for a minimal element of P.

Proof. Since the $\mathbb{Z}$-action on P is monotone, $k \leq k + 1$ and so $k + 1 = k$ by maximality. The same kind of argument applies in the dual case, and in the particular case where P has a maximum $\max(P)$ or a minimum $\min(P)$. □

Remark 1.5. Given a poset P we can always define a partial order on the set $P \cup \{-\infty, +\infty\}$ which extends the partial order on P by the rule $-\infty \leq x \leq +\infty$ for any $x \in P$.

Lemma 1.6. If $(P, \leq)$ is a $\mathbb{Z}$-poset, then $(P \cup \{\pm\infty\}, \leq)$ carries a natural $\mathbb{Z}$-action extending the $\mathbb{Z}$-action on P, by declaring both $-\infty$ and $+\infty$ to be $\mathbb{Z}$-fixed points.

Proof. Adding a fixed point always gives an extension of an action, so we only need to check that the extended action is compatible with the partial order. This is equivalent to checking that also on $P \cup \{\pm\infty\}$ the map $x \to x + 1$ is a monotone bijection such that $x \leq x + 1$, which is immediate. □

Posets with $\mathbb{Z}$-actions naturally form a category, whose morphisms are $\mathbb{Z}$-equivariant morphisms of posets. More explicitly, if $(P, \leq)$ and $(Q, \leq)$ are $\mathbb{Z}$-posets, then a morphism of $\mathbb{Z}$-posets between them is a morphism of posets $\varphi \colon (P, \leq) \to (Q, \leq)$ such that

$$\varphi(x + n) = \varphi(x) + n,$$

for any $x \in P$ and any $n \in \mathbb{Z}$.

Lemma 1.7. The choice of an element x in a $\mathbb{Z}$-poset P is equivalent to the datum of a $\mathbb{Z}$-equivariant morphism $\varphi \colon (\mathbb{Z}, \leq) \to (P, \leq)$. Moreover x is a $\mathbb{Z}$-fixed point if and only if the corresponding morphism φ factors $\mathbb{Z}$-equivariantly through $(*, \leq)$, where $*$ denotes the terminal object of **Pos**.

Proof. To the element x one associates the $\mathbb{Z}$-equivariant morphism φ_x defined by $\varphi_x(n) = x + n$. To the $\mathbb{Z}$-equivariant morphism φ one associates the element $x_\varphi = \varphi(0)$. It is immediate to check that the two constructions are inverse each other. The proof of the second part of the statement is straightforward. □

Lemma 1.8. Let $\varphi \colon (\mathbb{Z}, \leq) \to (P, \leq)$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets. Then φ is either injective or constant.

Proof. Assume φ is not injective. then there exist two integers n and m with $n > m$ such that $\varphi(n) = \varphi(m)$. By $\mathbb{Z}$-equivariancy we therefore have

$$x_\varphi + (n - m) = x_\varphi,$$

with $n - m \geq 1$ and $x_\varphi = \varphi(0)$. The conclusion then follows by the same argument used in Remark **1.3**. □

Lemma 1.9. Let $\varphi\colon (P, \leq) \to (Q, \leq)$ be a morphism of $\mathbb{Z}$-posets. Assume Q has a minimum and a maximum. Then φ extends to a morphism of $\mathbb{Z}$-posets $(P \cup \{\pm\infty\}, \leq) \to (Q, \leq)$ by setting $\varphi(-\infty) = \min(Q)$ and $\varphi(+\infty) = \max(Q)$.

Proof. Since $\min(Q)$ and $\max(Q)$ are $\mathbb{Z}$-fixed points by Lemma **1.4**, the extended φ is a morphism of $\mathbb{Z}$-sets. Moreover, since $\min(Q)$ and $\max(Q)$ are the minimum and the maximum of Q, respectively, the extended φ is indeed a morphism of posets, and so it is a morphism of $\mathbb{Z}$-posets. □

1.1. **The natural $\mathbb{Z}$-action on t-structures.** The main reason why we are interested in the theory of $\mathbb{Z}$-poset is the following result which appears as Remark **2.13** in [FL14]:

Remark 1.10. Let $\mathbf{C}$ be a stable ∞-category. Then, the collection $\mathrm{TS}(\mathbf{C})$ of all t-structures (see [FL14, Def. **2.9**]) on $\mathbf{C}$ is a poset with respect to following relation: given two t-structures $\mathsf{t}_a = (\mathbf{C}_{\geq_a 0}, \mathbf{C}_{<_a 0})$[1] and $\mathsf{t}_b = (\mathbf{C}_{\geq_b 0}, \mathbf{C}_{<_b 0})$, one has $\mathsf{t}_a \preccurlyeq \mathsf{t}_b$ iff $\mathbf{C}_{<_a 0} \subseteq \mathbf{C}_{<_b 0}$. The ordered group $\mathbb{Z}$ acts on $\mathrm{TS}(\mathbf{C})$ with the generator $+1$ mapping a t-structure $\mathsf{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ to the t-structure $\mathsf{t}[1] = (\mathbf{C}_{\geq 0}[1], \mathbf{C}_{<0}[1])$. Since $\mathsf{t} \preccurlyeq \mathsf{t}[1]$ one sees that $\mathrm{TS}(\mathbf{C})$ is naturally a $\mathbb{Z}$-poset.

Remark 1.11. If $\mathsf{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ is a t-structure on $\mathbf{C}$, it is customary to write $\mathbf{C}_{\geq 1}$ for $\mathbf{C}_{\geq 0}[1]$ and $\mathbf{C}_{<1}$ for $\mathbf{C}_{<0}[1]$, so that $\mathsf{t}[1] = (\mathbf{C}_{\geq 1}, \mathbf{C}_{<1})$, see [FL14, Def. **2.9** (ii)].

It is therefore natural to consider families of t-structures on $\mathbf{C}$ indexed by a $\mathbb{Z}$-poset J, as in the following

Definition 1.12. Let $(J, \leq)$ be a $\mathbb{Z}$-poset. A *J-family* of t-structures on an stable ∞-category $\mathbf{C}$ is a $\mathbb{Z}$-equivariant morphism of posets $\mathsf{t}\colon J \to \mathrm{TS}(\mathbf{C})$.

More explicitly, a J-family is a family $\{\mathsf{t}_j\}_{j \in J}$ of t-structures on $\mathbf{C}$ such that

(1) $\mathsf{t}_i \preccurlyeq \mathsf{t}_j$ if $i \leq j$ in J;
(2) $\mathsf{t}_{i+1} = \mathsf{t}_i[1]$ for any $i \in J$.

Notation 1.13. For $i \in J$, will write $\mathbf{C}_{\leq i}$ and $\mathbf{C}_{>i}$ for $\mathbf{C}_{\leq_i 0}$ and $\mathbf{C}_{<_i 0}$, respectively. With this notation we have that $\mathsf{t}_i = (\mathbf{C}_{\geq i}, \mathbf{C}_{<i})$. Note that, by $\mathbb{Z}$-equivariance, this notation is consistent. Namely $\mathsf{t}_{i+1} = \mathsf{t}_i[1]$ implies $\mathbf{C}_{\geq_{i+1} 0} = \mathbf{C}_{\geq_i 0}[1]$ and so

$$\mathbf{C}_{\geq i+1} = \mathbf{C}_{\geq i}[1].$$

Similarly, one has

$$\mathbf{C}_{<i+1} = \mathbf{C}_{<i}[1].$$

Note that in this notation the condition $\mathsf{t}_i \preccurlyeq \mathsf{t}_j$ for $i \leq j$ translates to the very natural condition $\mathbf{C}_{<i} \subseteq \mathbf{C}_{<j}$ for $i \leq j$.

[1]The reader baffled by this notation is invited to look at Notation **1.13**.

Example 1.14. A $\mathbb{Z}$-family of t-structures is, by Lemma **1.7**, equivalent to the datum of a t-structure $\mathfrak{t}_0 = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$. One has $\mathfrak{t}_1 = (\mathbf{C}_{\geq 1}, \mathbf{C}_{<1})$ consistently with the notations in Remark **1.10**.

Example 1.15. A $\mathbb{R}$-family of t-structures is the datum of a t-structure $\mathfrak{t}_x = (\mathbf{C}_{\geq x}, \mathbf{C}_{<x})$ on $\mathbf{C}$ for any $x \in \mathbb{R}$ in such a way that $\mathfrak{t}_{x+1} = \mathfrak{t}_x[1]$. Such a structure is called a *slicing* of $\mathbf{C}$ in [Bri07].[2]

Example 1.16. By taking $J = \mathrm{TS}(\mathbf{C})$ and $\mathfrak{t}$ to be the identity of $\mathrm{TS}(\mathbf{C})$ one sees that the whole $\mathrm{TS}(\mathbf{C})$ can be looked at as a particular J-family of t-structures on $\mathbf{C}$.

Remark 1.17. The poset $\mathrm{TS}(\mathbf{C})$ has a minimum and a maximum given by

$$\min(\mathrm{TS}(\mathbf{C})) = (\mathbf{C}, \mathbf{0}); \qquad \max(\mathrm{TS}(\mathbf{C})) = (\mathbf{0}, \mathbf{C}).$$

which correspond under the bijection of [FL14, Thm **3.13**] to the maximal and minimal factorizations on $\mathbf{C}$ respectively, and will be called the *trivial* factorizations/t-structures.

Hence, by Lemma **1.9**, any J-family of t-structures $\mathfrak{t}\colon J \to \mathrm{TS}(\mathbf{C})$ extends to a $(J \cup \{\pm\infty\})$-family by setting $\mathfrak{t}_{-\infty} = (\mathbf{C}, \mathbf{0})$ and $\mathfrak{t}_{+\infty} = (\mathbf{0}, \mathbf{C})$.

Definition 1.18. Let $\mathfrak{t}$ be a J-family of t-structures. For i and j in J we set

$$\mathbf{C}_{[i,j)} = \mathbf{C}_{\geq i} \cap \mathbf{C}_{<j}.$$

Consistently with Remark **1.17**, we also set

$$\mathbf{C}_{[i,+\infty)} = \mathbf{C}_{\geq i}; \qquad \mathbf{C}_{[-\infty,i)} = \mathbf{C}_{<i}$$

for any i in J. We say that $\mathbf{C}$ is *J-bounded* if

$$\mathbf{C} = \bigcup_{i,j \in J} \mathbf{C}_{[i,j)}.$$

Similarly, we say that $\mathbf{C}$ is *J-left-bounded* if $\mathbf{C} = \bigcup_{i \in J} \mathbf{C}_{[i,+\infty)}$ and *J-right-bounded* if $\mathbf{C} = \bigcup_{i \in J} \mathbf{C}_{[-\infty,i)}$

Remark 1.19. Since $\mathbf{C}_{[i,j)} = \mathbf{C}_{[i,+\infty)} \cap \mathbf{C}_{[-\infty,j)}$ one immediately sees that $\mathbf{C}$ is J-bounded if and only if $\mathbf{C}$ is both J-left- and J-right-bounded.

Remark 1.20. As it is natural to expect, if $i \geq j$, then $\mathbf{C}_{[i,j)}$ is contractible. Namely, since $j \leq i$ one has $\mathbf{C}_{<j} \subseteq \mathbf{C}_{<i}$ and so

$$\mathbf{C}_{[i,j)} = \mathbf{C}_{\geq i} \cap \mathbf{C}_{<j} \subseteq \mathbf{C}_{\geq i} \cap \mathbf{C}_{<i} = \mathbf{C}_{\geq_i 0} \cap \mathbf{C}_{<_i 0}$$

which corresponds to the contractible subcategory of zero objects in $\mathbf{C}$.

Remark 1.21. Let $\mathfrak{t}$ be a $\mathbb{Z}$-family of t-structures on $\mathbf{C}$. Then $\mathbf{C}$ is $\mathbb{Z}$-bounded (resp., $\mathbb{Z}$-left-bounded, $\mathbb{Z}$-right-bounded) if and only if $\mathbf{C}$ is bounded (resp., left-bounded, right-bounded)with respect to the t-structure $\mathfrak{t}_0$, according to the classical definition of boundedness as given, e.g., in [BBD82].

[2] Actually, Bridgeland's notion of slicing is stricter than the the one we are giving here, since a suitable finiteness condition on the multiple factorizations associated with the slicing is imposed in [Bri07].

Remark 1.22. If $\mathfrak{t}$ is an $\mathbb{R}$-family of t-structures on $\mathbf{C}$, then one can define

$$\mathbf{C}_x = \bigcap_{\epsilon>0} \mathbf{C}_{[x,x+\epsilon)}.$$

These subcategories $\mathbf{C}_x$ are the *slices* of $\mathbf{C}$ in the terminology of [Bri07].

Remark 1.23. For any i, j, h, k in J with $j \le h$ one has

$$\mathbf{C}_{[i,j)} \subseteq \mathbf{C}_{[h,k)}^{\perp},$$

i.e., $\mathbf{C}(X, Y)$ is contractible whenever $X \in \mathbf{C}_{[h,k)}$ and $Y \in \mathbf{C}_{[i,j)}$ (one says that $\mathbf{C}_{[i,j)}$ is *right-orthogonal* to $\mathbf{C}_{[h,k)}$). Indeed, since $\mathbf{C}_{<j} = \mathbf{C}_{<_j 0} = \mathbf{C}_{\ge_j 0}^{\perp} = \mathbf{C}_{\ge j}^{\perp}$, and passing to the orthogonal reverses the inclusions, we have

$$\mathbf{C}_{[i,j)} \subseteq \mathbf{C}_{<j} = \mathbf{C}_{\ge j}^{\perp} \subseteq \mathbf{C}_{\ge h}^{\perp} \subseteq \mathbf{C}_{[h,k)}^{\perp}.$$

2. Multiple factorization systems and towers.

> Go to, let us go down, and there confound their language, that they may not understand one another's speech.
>
> Genesis 11:7

2.1. **Multiple factorizations associated with J-families of t-structures.** The $\mathbb{Z}$-poset $\mathrm{TS}(\mathbf{C})$ of t-structures on a stable ∞-category $\mathbf{C}$ is naturally identified with the $\mathbb{Z}$-poset $\mathrm{FS}(\mathbf{C})$ of normal factorization systems on $\mathbf{C}$ by [FL14, Thm **3.13**]: to the factorization system $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ it corresponds the t-structure defined by

$$\mathbf{C}_{\ge 0} = \{X \in \mathrm{Ob}(\mathbf{C}) \text{ such that } 0 \to X \text{ is in } \mathcal{E}\}$$
$$\mathbf{C}_{<0} = \{X \in \mathrm{Ob}(\mathbf{C}) \text{ such that } X \to 0 \text{ is in } \mathcal{M}\}.$$

In particular a J-family of t-structures is the same thing as a J-family of normal factorization systems. In the remainder of this section, $\mathfrak{t}$ will be a fixed J-family of t-structures on $\mathbf{C}$ and $\mathbb{F}$ will denote the corresponding J-family of factorization systems.

For any (finite) ascending chain $i_1 \le i_2 \le \cdots \le i_k$ in J we have a corresponding chain of factorization systems

$$\mathbb{F}_{i_1} \preceq \cdots \preceq \mathbb{F}_{i_k}$$

Spelled out explicitly, this means that if we write $\mathbb{F}_i = (\mathcal{E}_i, \mathcal{M}_i)$ then we have two chains –any of which determines the other– in $\hom(\mathbf{C})$:

$$\mathcal{M}_{i_1} \subseteq \mathcal{M}_{i_2} \subseteq \cdots \subseteq \mathcal{M}_{i_k},$$
$$\mathcal{E}_{i_1} \supseteq \mathcal{E}_{i_2} \supseteq \cdots \supseteq \mathcal{E}_{i_k}.$$

Remark 2.1. It is convenient to recall that the orthogonal classes $\mathcal{E}, \mathcal{M}$ in a normal factorization system $\mathbb{F}$ are closed under composition and satisfy the 3-for-2 property, see [FL14] for details.

Lemma 2.2. The chain $i_1 \le i_2 \le \cdots \le i_k$ determines a k-fold factorization system in $\mathbf{C}$. Namely, every arrow $f\colon X \to Y$ in $\mathbf{C}$ can be uniquely factored into a composition

$$X \xrightarrow{\mathcal{E}_{i_k}} Z_{i_k} \xrightarrow{\mathcal{E}_{i_{k-1}} \cap \mathcal{M}_{i_k}} Z_{i_{k-1}} \to \cdots \to Z_{i_2} \xrightarrow{\mathcal{E}_{i_1} \cap \mathcal{M}_{i_2}} Z_{i_1} \xrightarrow{\mathcal{M}_{i_1}} Y,$$

where the superscripts denote the classes of morphisms the arrows belong to.

Proof. For $k = 1$ this is the definition of factorization system: given $f\colon X \to Y$, we have its $\mathbb{F}_{i_1}$-factorization

$$X \xrightarrow{\mathcal{E}_{i_1}} Z_{i_1} \xrightarrow{\mathcal{M}_{i_1}} Y.$$

Then we work inductively on k. Given an arrow $f\colon X \to Y$ we first consider its $\mathbb{F}_{i_k}$-factorization

$$X \xrightarrow{\mathcal{E}_{i_k}} Z_{i_k} \xrightarrow{\mathcal{M}_{i_k}} Y,$$

and then observe that the chain $i_1 \le \cdots \le i_{k-1}$ induces a $(k-1)$-ary factorization system on $\mathbf{C}$, which we can use to decompose $Z_{i_k} \to Y$ as

$$Z_{i_k} \xrightarrow{\mathcal{E}_{i_{k-1}}} Z_{i_{k-1}} \xrightarrow{\mathcal{E}_{i_{k-2}} \cap \mathcal{M}_{i_{k-1}}} Z_{i_{k-2}} \to \cdots \to Z_{i_2} \xrightarrow{\mathcal{E}_{i_1} \cap \mathcal{M}_{i_2}} Z_{i_1} \xrightarrow{\mathcal{M}_{i_1}} Y,$$

and we are only left to prove that $Z_{i_k} \to Z_{i_{k-1}}$ is actually in $\mathcal{E}_{i_{k-1}} \cap \mathcal{M}_{i_k}$. This is an immediate consequence of the 3-for-2 property for the class $\mathcal{M}_{i_1}$. Namely, since $\mathcal{M}_{i_1} \subseteq \mathcal{M}_{i_2} \subseteq \cdots \subseteq \mathcal{M}_{i_k}$, and $\mathcal{M}_{i_k}$ is closed for composition, the morphism $Z_{i_{k-1}} \to Y$ is in $\mathcal{M}_{i_k}$. Then the 3-for-2 property applied to

$$Z_{i_k} \to Z_{i_k - 1} \to Y$$

concludes the proof. □

Lemma 2.3. Let i, j be elements in J and let X be an object in $\mathbf{C}_{\ge j}$ (see Definition **1.18**). If a morphism $f\colon X \to Y$ is in $\mathcal{E}_i \cap \mathcal{M}_j$, then $\operatorname{cofib}(f)$ is $\mathbf{C}_{[i,j)}$.

Proof. Since X is in $\mathbf{C}_{\ge j}$,

$$0 \to X \xrightarrow{f} Y$$

is the $(\mathcal{E}_j, \mathcal{M}_j)$-factorization of $0 \to Y$. Since the factorization system $\mathbb{F}_j$ is normal, we have the following diagram where the square is a pullout (see [FL14, Notation **2.4**]):

$$\begin{tikzcd}
0 \arrow[r, "\mathcal{E}_j"] & X \arrow[r, "\mathcal{M}_j"] \arrow[d, "\mathcal{E}_j"'] & Y \arrow[d, "\mathcal{E}_j"] \\
& 0 \arrow[r, "\mathcal{M}_j"] & \operatorname{cofib}(f) \arrow[d, "\mathcal{M}_j"] \\
& & 0
\end{tikzcd}$$

(see [FL14, Prop **3.10**]). Therefore, $\operatorname{cofib}(f)$ is in $\mathbf{C}_{<j}$. On the other hand, f is in $\mathcal{E}_i$, which is closed under pushouts, and so $0 \to \operatorname{cofib}(f)$ is in $\mathcal{E}_i$, i.e., $\operatorname{cofib}(f)$ is in $\mathbf{C}_{\ge i}$. □

Corollary 2.4. Let $i_1 \leq i_2 \leq \cdots \leq i_k$ an ascending chain in J. Then for any object Y in $\mathbf{C}$, the arrows $f_j \colon Y_{i_j} \to Y_{i_{j-1}}$ in the k-fold factorization of the initial morphism $0 \to Y$ are such that $\mathrm{cofib}(f_j) \in \mathbf{C}_{[i_{j-1}, i_j)}$, where we have set $i_{k+1} = +\infty$ and $Y_{+\infty} = 0$ (and, similarly, $i_0 = -\infty$ and $Y_{-\infty} = Y$) consistently with Remark **1.17**.

Proof. From the k-fold factorization

$$0 \xrightarrow{\mathcal{E}_{i_k}} Y_{i_k} \xrightarrow{\mathcal{E}_{i_{k-1}} \cap \mathcal{M}_{i_k}} Y_{i_{k-1}} \to \cdots \to Y_{i_2} \xrightarrow{\mathcal{E}_{i_1} \cap \mathcal{M}_{i_2}} Y_{i_1} \xrightarrow{\mathcal{M}_{i_1}} Y,$$

and from the fact that $\mathcal{E}_{i_1} \supseteq \mathcal{E}_{i_2} \supseteq \cdots \supseteq \mathcal{E}_{i_k}$ and each class $\mathcal{E}_{i_j}$ is closed for composition, we see that Z_{i_j} is in $\mathbf{C}_{i_j}$ and the previous lemma applies. □

Lemma 2.5. Let $i \leq j$ be elements in J and let $f \colon X \to Y$ be a morphism in $\mathbf{C}$. If X is in $\mathbf{C}_{[j,+\infty)}$ and $\mathrm{cofib}(f)$ is in $\mathbf{C}_{[i,j)}$ then $0 \to X \xrightarrow{f} Y$ is the $(\mathcal{E}_j, \mathcal{M}_j)$-factorization of the initial morphism $0 \to Y$ and Y is in $\mathbf{C}_{[i,+\infty)}$. In particular f is in $\mathcal{E}_i \cap \mathcal{M}_j$.

Proof. Since X is in $\mathbf{C}_{\geq j}$, the morphism $0 \to X$ is in $\mathcal{E}_j$, and so (reasoning up to equivalence) to show that $0 \to X \to Y$ is the $(\mathcal{E}_j, \mathcal{M}_j)$-factorization of $0 \to Y$ we are reduced to showing that $f \colon X \to Y$ is in $\mathcal{M}_j$. Since $\mathrm{cofib}(f)$ is in $\mathbf{C}_{[i,j)}$, we have in particular that $\mathrm{cofib}(f) \to 0$ is in $\mathcal{M}_j$ and so $0 \to \mathrm{cofib}(f)$ is in $\mathcal{M}_j$ by the Sator lemma. Then we have a homotopy pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "\mathcal{M}_j"] \\ Y \arrow[r] & \mathrm{cofib}(f) \end{tikzcd}$$

and so f is in $\mathcal{M}_j$ by the fact that $\mathcal{M}_j$ is closed under pullbacks. Now let $X \to T \to Y$ be the $(\mathcal{E}_i, \mathcal{M}_i)$-factorization of f. Then, by normality of the factorization systems, we can consider the diagram

$$\begin{tikzcd} 0 \arrow[d, "\mathcal{E}_j"'] & & & \\ X \arrow[r, "\mathcal{E}_j"] \arrow[d, "\mathcal{E}_i \cap \mathcal{M}_j"'] & 0 \arrow[d, "\mathcal{E}_i \cap \mathcal{M}_j"] & & \\ T \arrow[r, "\mathcal{E}_j"] \arrow[d, "\mathcal{M}_i"'] & U \arrow[r, "\mathcal{E}_i \cap \mathcal{M}_j"] \arrow[d, "\mathcal{M}_i"] & 0 \arrow[d, "\mathcal{M}_i"] & \\ Y \arrow[r, "\mathcal{E}_j"] & \mathrm{cofib}(f) \arrow[r, "\mathcal{E}_i \cap \mathcal{M}_j"] & V \arrow[r, "\mathcal{M}_i"] & 0 \end{tikzcd}$$

where all the squares are pullouts, and where we have used the Sator lemma, the fact that the classes $\mathcal{E}$ are closed for pushouts while the classes $\mathcal{M}$ are closed for pullbacks, and the 3-for-2 property for both classes. By assumption $\mathrm{cofib}(f)$ is in $\mathbf{C}_{[i,j)}$, hence we have that $0 \to \mathrm{cofib}(f)$ is in $\mathcal{E}_i$ and so by the 3-for-2 property also $U \to \mathrm{cofib}(f)$ is in $\mathcal{E}_i$. But then $U \to \mathrm{cofib}(f)$ is in $\mathcal{E}_i \cap \mathcal{M}_i$ and so is an isomorphism. Hence also $T \to Y$ is an isomorphism and therefore $0 \to Y$ is in $\mathcal{E}_i$, i.e., Y is in $\mathbf{C}_{\geq i}$. Finally, since $i \leq j$, we have $\mathcal{E}_j \subseteq \mathcal{E}_i$ and so $0 \to X$ in $\mathcal{E}_i$. By the 3-for-2 property this implies that f is in $\mathcal{E}_i$. Since we have shown that f is in $\mathcal{M}_j$ we have that f is in $\mathcal{E}_i \cap \mathcal{M}_j$. □

Lemma 2.6. Let Y an object in $\mathbf{C}$ and let $i_1 \leq i_2 \leq \cdots \leq i_k$ be an ascending chain in J. If a factorization

$$0 \xrightarrow{f_{k+1}} Y_{i_k} \xrightarrow{f_k} Y_{i_{k-1}} \to \cdots \to Y_{i_2} \xrightarrow{f_2} Y_{i_1} \xrightarrow{f_1} Y,$$

of the initial morphism $0 \to Y$ is such that $\mathrm{cofib}(f_j)$ is in $\mathbf{C}_{[i_{j-1}, i_j)}$ (with $i_{k+1} = +\infty$ and $i_0 = -\infty$) then this factorization is the k-fold factorization of $0 \to Y$ associated with the chain $i_1 \leq \cdots \leq i_k$.

Proof. By unicity of the k-fold factorization we only need to prove that $f_j \in \mathcal{E}_{i_{k-1}} \cap \mathcal{M}_{i_k}$, which is immediate by repeated application of Lemma **2.5**. □

Definition 2.7 (Postnikov tower)**.** Let $f\colon X \to Y$ be a morphism in $\mathbf{C}$ and let $i_1 \leq i_2 \leq \cdots \leq i_k$ be an ascending chain in J. We say that a factorization

$$X \xrightarrow{f_{k+1}} Z_{i_k} \xrightarrow{f_k} Z_{i_{k-1}} \to \cdots \to Z_{i_2} \xrightarrow{f_2} Z_{i_1} \xrightarrow{f_1} Y,$$

of f is a *Postnikov tower* of f relative to the chain $\{i_j\} = \{i_1 \leq i_2 \leq \cdots \leq i_k\}$ if for any $j = 1, \ldots, k+1$ one has $\mathrm{cofib}(f_j) \in \mathbf{C}_{[i_{j-1}, i_j)}$ (with $i_{k+1} = +\infty$ and $i_0 = -\infty$).

Proposition 2.8. Let $f\colon X \to Y$ be a morphism in $\mathbf{C}$ and let $i_1 \leq i_2 \leq \cdots \leq i_k$ be an ascending chain in J. Then a Postnikov tower for f relative to $\{i_j\}$, denoted $♖_{\{i_j\}}(f)$, exists and it is unique up to isomorphisms.

Proof. Consider the pullout diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
Y \arrow[r] & \mathrm{cofib}(f)
\end{tikzcd}$$

By Corollary **2.4**, the k-fold factorization

$$0 \xrightarrow{\varphi_{k+1}} A_{i_k} \xrightarrow{\varphi_k} A_{i_{k-1}} \to \cdots \to A_{i_2} \xrightarrow{\varphi_2} A_{i_1} \xrightarrow{\varphi_1} \mathrm{cofib}(f)$$

of the initial morphism $0 \to \mathrm{cofib}(f)$ is such that $\mathrm{cofib}(\varphi_{i_j}) \in \mathbf{C}_{[i_{j-1}, i_j)}$. Pulling back this factorization along $Y \to \mathrm{cofib}(f)$ we obtain a factorization

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f_{k+1}"'] & 0 \arrow[d, "\varphi_{k+1}"] \\
Z_{i_k} \arrow[r] \arrow[d, "f_k"'] & A_{i_k} \arrow[d, "\varphi_k"] \\
\vdots \arrow[d, "f_2"'] & \vdots \arrow[d, "\varphi_2"] \\
Z_{i_1} \arrow[r] \arrow[d, "f_1"'] & A_{i_1} \arrow[d, "\varphi_1"] \\
Y \arrow[r] & \mathrm{cofib}(f)
\end{tikzcd}$$

of f, and the pasting of pullout diagrams

$$\begin{tikzcd}
Z_{i_j} \arrow[r] \arrow[d, "f_j"'] & A_{i_j} \arrow[r] \arrow[d, "\varphi_j"] & 0 \arrow[d] \\
Z_{i_{j-1}} \arrow[r] & A_{i_{j-1}} \arrow[r] & \mathrm{cofib}(\varphi_j)
\end{tikzcd}$$

shows that $\mathrm{cofib}(f_j) = \mathrm{cofib}(\varphi_j)$ and so $\mathrm{cofib}(f_j) \in \mathbf{C}_{[i_{j-1}, i_j)}$. This proves the existence of the Postnikov tower. To prove uniqueness, start with a Postnikov tower $♖_{\{i_j\}}(f)$ for f and push it out along $Y \to \mathrm{cofib}(f)$ to obtain a Postnikov tower for the initial morphism $0 \to \mathrm{cofib}(f)$. By Lemma **2.6**, this is the k-fold factorization of $0 \to \mathrm{cofib}(f)$ associated with the chain $\{i_j\}$ and so $♖_{\{i_j\}}(f)$ is precisely the Postnikov tower constructed in the first part of the proof. Note how the pullout axiom of stable ∞-categories plays a crucial role. □

Remark 2.9. A Postnikov tower for f relative to an ascending chain $\{i_j\}$ can be equivalently defined as a factorization of f such that $\mathrm{fib}(f) \in \mathbf{C}_{[i_{j-1}-1, i_j-1)}$, for any $j = 0, \dots, k+1$.

Remark 2.10. It's an unavoidable temptation to think of the Postnikov tower $♖_{\{i_j\}}(f)$ relative to an ascending chain $\{i_j\}$ as the k-fold factorization of f associated with the chain $\{i_j\}$. As the following example shows, when f is not an initial morphism this is in general not true. Let $J = \mathbb{Z}$ and take an ascending chain consisting of solely the element 0. Now take a morphism $f \colon X \to Y$ between two elements in $\mathbf{C}_{[-1,0)}$. The object $\mathrm{cofib}(f)$ will lie in $\mathbf{C}_{[-1,+\infty)}$, since $\mathcal{E}_{-1}$ is closed for pushouts, but in general it will not be an element in $\mathbf{C}_{[0,+\infty)}$. In other words, we will have, in general, a nontrivial $(\mathcal{E}_0, \mathcal{M}_0)$-factorization of the initial morphism $0 \to \mathrm{cofib}(f)$. Pulling this back along $Y \to \mathrm{cofib}(f)$ we obtain the Postnikov tower $X \xrightarrow{f_2} Z \xrightarrow{f_1} Y$ of f, and this factorization will be nontrivial since its pushout is nontrivial. It follows that (f_2, f_1), cannot be the $(\mathcal{E}_0, \mathcal{M}_0)$-factorization of f. Indeed, by the 3-for-2 property of $\mathcal{M}_0$, the morphism f is in $\mathcal{M}_0$, so its $(\mathcal{E}_0, \mathcal{M}_0)$-factorization is trivial.

3. Hearts of t-structures (the transitive case).

> I watched a snail crawl along the edge of a straight razor. That's my dream. That's my nightmare. Crawling, slithering, along the edge of a straight razor... and surviving.
>
> Col. Walter E. Kurtz

We now focus in the case $J = \mathbb{Z}$. As indicated in remark **1.7** this is equivalent to considering a single distinguished t-structure $\mathfrak{t} = \mathfrak{t}_0$ on the stable ∞-category $\mathbf{C}$, since every other element can be obtained as the j-shifting $\mathfrak{t}_j = \mathfrak{t}_0[j]$. As the set of indices for our family of t-structures is the ordered set of integers, we will always consider complete ascending chains of the form

$$n < n+1 < n+2 < \cdots n+k-1$$

in what follows. In particular Proposition **2.8** becomes

Proposition 3.1. Let $f\colon X \to Y$ be a morphism in $\mathbf{C}$. Then for any integer n and any positive integer k there exists a unique Postnikov tower for f associated with the ascending chain $n < n+1 < \cdots < n+k-1$. Denoting this tower by

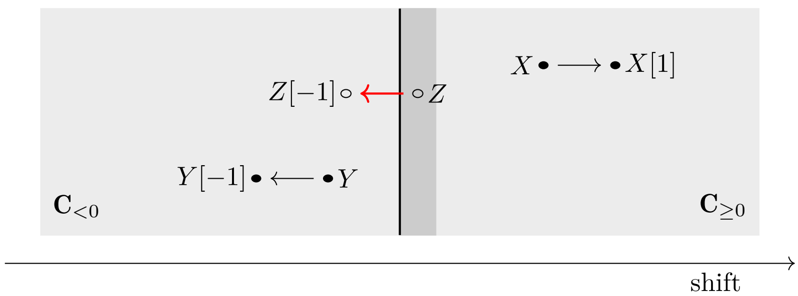
$$X \xrightarrow{f_{n+k}} Z_{n+k-1} \xrightarrow{f_{n+k-1}} Z_{n+k-2} \to \cdots \to Z_{n+1} \xrightarrow{f_n} Z_n \xrightarrow{f_{n-1}} Y,$$

one has $\mathrm{cofib}(f_j) \in \mathbf{C}_{[j,j+1)}$ for any $j = n, \ldots, n+k-1$, $\mathrm{cofib}(f_{n-1}) \in \mathbf{C}_{<n}$ and $\mathrm{cofib}(f_{n+k}) \in \mathbf{C}_{\geq n+k}$.

Since $\mathbf{C}_{[j,j+1)} = \mathbf{C}_{[0,1)}[j]$ for any $j \in \mathbb{Z}$ the above Proposition suggests to focus on the subcategory $\mathbf{C}_{[0,1)}$ of $\mathbf{C}$.

Definition 3.2. Let $\mathbf{C}$ be a stable ∞-category equipped with a t-structure $\mathfrak{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$; the *heart* $\mathbf{C}^{\heartsuit}$ of $\mathfrak{t}$ is the subcategory $\mathbf{C}_{[0,1)}$ of $\mathbf{C}$.

Remark 3.3. There is a rather evocative pictorial representation of the heart of a t-structure, manifestly inspired by [Bri07]: if we depict $\mathbf{C}_{<0}$ and $\mathbf{C}_{\geq 0}$ as contiguous half-planes, like in the following picture,



then the action of the shift functor can be represented as an horizontal shift, and the closure properties of the two classes $\mathbf{C}_{\geq 0}, \mathbf{C}_{<0}$ under positive and negative shifts are a direct consequence of the shape of these areas. With these notations, an object Z is in the heart of $\mathfrak{t}$ if it lies in a "boundary region", i.e. if it lies in $\mathbf{C}_{\geq 0}$, but $Z[-1]$ lies in $\mathbf{C}_{<0}$.

Having introduced this notation, we can rephrase the existence of the tower for f as follows: given a morphism $f\colon X \to Y$ in $\mathbf{C}$, for any integer n and any positive integer k there exists a unique factorization of f

$$X \xrightarrow{f_{n+k}} Z_{n+k-1} \xrightarrow{f_{n+k-1}} Z_{n+k-2} \to \cdots \to Z_{n+1} \xrightarrow{f_n} Z_n \xrightarrow{f_{n-1}} Y,$$

such that $\mathrm{cofib}(f_j) \in \mathbf{C}^{\heartsuit}[j]$ for any $j = n, \ldots, n+k-1$, $\mathrm{cofib}(f_{n-1}) \in \mathbf{C}_{<n}$ and $\mathrm{cofib}(f_{n+k}) \in \mathbf{C}_{\geq n+k}$.

The content of this statement becomes more interesting when $\mathbf{C}$ is *bounded* with respect to the t-structure $\mathfrak{t}$ (see Definition **1.18**). Namely, if $\mathbf{C}$ is bounded, then the $(\mathcal{E}_n, \mathcal{M}_n)$-factorizations of an initial morphism $0 \to Y$ are trivial (see Definition **1.17**) for $|n| \gg 0$. As an immediate consequence, the morphism $X \xrightarrow{f_{n+k}} Z_{n+k-1}$ and $Z_n \xrightarrow{f_{n-1}} Y$ in the Postnikov tower of f associated with the chain $n < n+1 < \cdots < n+k-1$ are isomorphisms for $n \ll 0$ and $k \gg 0$. One noticed, as it is obvious, that the class of isomorphisms in $\mathbf{C}$ is closed under transfinite composition, this leads to the following

Proposition 3.4. Let $\mathbf{C}$ be an ∞-stable category which is bounded with respect to a given t-structure $\mathfrak{t}$. Then for any morphism $f\colon X \to Y$ in $\mathbf{C}$ there exists an integer n_0 and a positive integer k_0 such that for any integer $n \leq n_0$ and any positive integer k with $k \geq n_0 - n + k_0$ there exists a unique factorization of f

$$X \xrightarrow{\sim} Z_{n+k-1} \xrightarrow{f_{n+k-1}} Z_{n+k-2} \to \cdots \to Z_{n+1} \xrightarrow{f_n} Z_n \xrightarrow{\sim} Y$$

such that $\mathrm{cofib}(f_j) \in \mathbf{C}^{\heartsuit}[j]$ for any $j = n, \ldots, n+k-1$.

Remark 3.5. By uniqueness in Proposition **3.4**, one has a well defined $\mathbb{Z}$-factorization

$$X = \lim(Z_j) \to \cdots \to Z_{j+1} \xrightarrow{f_j} Z_j \xrightarrow{f_{j-1}} Z_{j-1} \to \cdots \to \mathrm{colim}(Z_j) = Y$$

with with j ranging over the integers, $\mathrm{cofib}(f_j) \in \mathbf{C}^{\heartsuit}[j]$ for any $j \in \mathbb{Z}$ and with f_m being an isomorphism for $|j| \gg 0$. We will refer to this factorization as the *$\mathbb{Z}$-Postnikov tower* of f. Notice how the boundedness of $\mathbf{C}$ has played an essential role: when $\mathbf{C}$ is not bounded, one still has Postnikov towers for any finite ascending chain, but in general they do not stabilise.

Remark 3.6. Since we know that the Postnikov tower of an initial morphism is its k-fold $(\mathcal{E}_j, \mathcal{M}_j)$-factorization, we see that in a stable ∞-category $\mathbf{C}$ which is bounded with respect to a t-structure $\mathfrak{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ the $\mathbb{Z}$-Postnikov tower of $0 \to Y$,

$$0 = \lim(Y_j) \to \cdots \to Y_{j+1} \xrightarrow{f_j} Y_j \xrightarrow{f_{j-1}} Y_{j-1} \to \cdots \to \mathrm{colim}(Y_j) = Y$$

is such that $f_j \in \mathcal{E}_j \cap \mathcal{M}_{j+1}$ for any $j \in \mathbb{Z}$. It follows that an object Y is in $\mathbf{C}_{\geq 0}$ if and only if the $\mathbb{Z}$-Postnikov tower of $0 \to Y$ satisfies $\operatorname{cofib}(f_j) = 0$ for any $j < 0$, while Y is in $\mathbf{C}_{<0}$ if and only if $\operatorname{cofib}(f_j) = 0$ for any $j \geq 0$.

3.1. **Abelianity of the heart.** In the following section we present a complete proof of the fact that the heart of a t-structure, as defined in [Lur11, Def. **1.2.1.11**], is an abelian ∞-category. That is, $\mathbf{C}^{\heartsuit}$ is homotopy equivalent to its homotopy category $h\mathbf{C}^{\heartsuit}$, which is an abelian category; this is the higher-categorical counterpart of a classical result, first proved in [BBD82, Thm. **1.3.6**], which only relies on properties stated in terms of normal torsion theories in a stable ∞-category. We begin with the following

Definition 3.7 (Abelian ∞-category)**.** An *abelian ∞-category* is a quasicategory $\mathbf{A}$ such that

i) the hom-space $\mathbf{A}(X, Y)$ is a homotopically discrete infinite loop space for any X, Y, i.e., there exists an infinite sequence of ∞ >-groupoids $Z_0, Z_1, Z_2, \dots$, with $Z_0 \cong \mathbf{C}(X, Y)$ and homotopy equivalences $Z_i \cong \Omega Z_{i+1}$ for any $i \geq 0$, such that $\pi_n Z_0 = 0$ for any $n \geq 1$;
ii) $\mathbf{A}$ has a zero object, (homotopy) kernels, cokernels and biproducts;
iii) for any morphism f in $\mathbf{A}$, the natural morphism from the *coimage* of f to the *image* (see Definition **3.15**) of f is an equivalence.

Remark 3.8. Axiom (i) is the homotopically-correct version of $\mathbf{A}(X, Y)$ being an abelian group. For instance, if the abelian group is $\mathbb{Z}$, then the corresponding homotopy discrete space is the Eilenberg-MacLane spectrum $\mathbb{Z}, K(\mathbb{Z}, 1), K(\mathbb{Z}, 2), \dots$. The homotopy category of such a $\mathbf{A}$ is an abelian category in the classical sense (note that $\mathbf{A}(X, Y)$ being homotopically discrete is necessary in order that kernels and cokernels in $\mathbf{A}$ induce kernels and cokernels in $h\mathbf{A}$). Moreover, since the hom-spaces $\mathbf{A}(X, Y)$ are homotopically discrete, the natural morphism $\mathbf{A} \to h\mathbf{A}$ is actually an equivalence.

The rest of the section is devoted to the proof of the following result:

Theorem 3.9. The heart $\mathbf{C}^{\heartsuit}$ of a t-structure $\mathfrak{t}$ on a stable ∞-category $\mathbf{C}$ is an abelian ∞-category; its homotopy category $h\mathbf{C}^{\heartsuit}$ is the abelian category arising as the heart of the t-structure $h(\mathfrak{t})$ on the triangulated category $h\mathbf{C}$.

Lemma 3.10. For any X and Y in $\mathbf{C}^{\heartsuit}$, the hom-space $\mathbf{C}^{\heartsuit}(X, Y)$ is a homotopically discrete infinite loop space.

Proof. Since $\mathbf{C}^{\heartsuit}$ is a full subcategory of $\mathbf{C}$, we have $\mathbf{C}^{\heartsuit}(X, Y) = \mathbf{C}(X, Y)$, which is an infinite loop space since $\mathbf{C}$ is a ∞-stable category.

So we are left to prove that $\pi_n \mathbf{C}(X, Y) = 0$ for $n \geq 1$. Since $\pi_n \mathbf{C}(X, Y) = \pi_{n-1} \Omega \mathbf{C}(X, Y) = \pi_{n-1} \mathbf{C}(X, Y[-1])$, this is equivalent to showing that $\mathbf{C}(X, Y[-1])$ is contractible. Since X and Y are objects in $\mathbf{C}^{\heartsuit}$, we have $X \in \mathbf{C}_{[0,1)}$ and $Y[-1] \in \mathbf{C}_{[-1,0)}$. But $\mathbf{C}_{[-1,0)}$ is right-orthogonal to $\mathbf{C}_{[0,1)}$ (see Remark **1.23**), therefore $\mathbf{C}(X, Y[-1])$ is contractible. □

The subcategory $\mathbf{C}^\heartsuit$ inherits the 0 object and biproducts from $\mathbf{C}$, so in order to prove it is is ∞-abelian we are left to prove that it has kernels and cokernels, and that the canonical morphism from the coimage to the image is an equivalence.

Lemma 3.11. Let $f\colon X \to Y$ be a morphism in $\mathbf{C}^\heartsuit$. Then $\mathrm{fib}(f)$ is in $\mathbf{C}_{<1}$ and $\mathrm{cofib}(f)$ is in $\mathbf{C}_{\geq 0}$

Proof. Since both $X \to 0$ and $Y \to 0$ are in $\mathcal{M}[1]$, by the 3-for-2 property also f is in $\mathcal{M}[1]$. Since $\mathcal{M}[1]$ is closed for pullbacks, $\mathrm{fib}(f) \to 0$ is in $\mathcal{M}[1]$ and so $\mathrm{fib}(f)$ is in $\mathbf{C}_{<1}$. The proof for $\mathrm{cofib}(f)$ is completely dual. □

Definition 3.12. Denote by

$$0 \xrightarrow{\mathcal{E}} \ker(f) \xrightarrow{\mathcal{M}} \mathrm{fib}(f)$$

the $(\mathcal{E},\mathcal{M})$-factorization of the morphism $0 \to \mathrm{fib}(f)$ and by

$$\mathrm{cofib}(f) \xrightarrow{\mathcal{E}[1]} \mathrm{coker}(f) \xrightarrow{\mathcal{M}[1]} 0$$

the $(\mathcal{E}[1],\mathcal{M}[1])$-factorization of the morphism $\mathrm{cofib}(f) \to 0$. We call $\ker(f)$ and $\mathrm{coker}(f)$ the *kernel* and the *cokernel* of f in $\mathbf{C}^\heartsuit$.

Remark 3.13. Since $\mathrm{cofib}(f)[-1] \cong \mathrm{fib}\, f$, one can equivalently define $\mathrm{coker}(f)$ by declaring the $(\mathcal{E},\mathcal{M})$-factorization of $\mathrm{fib}(f) \to 0$ to be $\mathrm{fib}(f) \xrightarrow{\mathcal{E}} \mathrm{coker}(f)[-1] \xrightarrow{\mathcal{M}} 0$. Similarly, one can define $\ker(f)$ by declaring the $(\mathcal{E}[1],\mathcal{M}[1])$-factorization of $0 \to \mathrm{cofib}(f)$ to be $0 \xrightarrow{\mathcal{E}[1]} \ker(f)[1] \xrightarrow{\mathcal{M}[1]} \mathrm{cofib}(f)$. By normality of the factorization system we therefore have the homotopy commutative diagram

$$\begin{tikzcd}
0 \arrow[r, "\mathcal{E}"] & \ker(f) \arrow[r, "\mathcal{M}"] \arrow[d, "\mathcal{E}"'] & \mathrm{fib}(f) \arrow[d, "\mathcal{E}"'] \\
& 0 \arrow[r, "\mathcal{M}"] & \mathrm{coker}(f)[-1] \arrow[d, "\mathcal{M}"'] \\
& & 0
\end{tikzcd}$$

whose square subdiagram is a homotopy pullout.

Lemma 3.14. Both $\ker(f)$ and $\mathrm{coker}(f)$ are in $\mathbf{C}^\heartsuit$.

Proof. By construction $\ker(f)$ is in $\mathbf{C}_{\geq 0}$, so we only need to show that $\ker(f)$ is in $\mathbf{C}_{<1}$. By definition of $\ker(f)$, we have that $\ker(f) \to \mathrm{fib}(f)$ is in $\mathcal{M}$. Since $\mathcal{M}[-1] \subseteq \mathcal{M}$, we have that also $\ker(f)[-1] \to \mathrm{fib}(f)[-1]$ is in $\mathcal{M}$. By Lemma **3.11**, $\mathrm{fib}(f)[-1] \to 0$ is in $\mathcal{M}$ and so we find that also $\ker(f)[-1] \to 0$ is in $\mathcal{M}$. The proof for $\mathrm{coker}(f)$ is perfectly dual. □

By definition of $\ker(f)$ and $\operatorname{coker}(f)$, the defining diagram of $\operatorname{fib}(f)$ and $\operatorname{cofib}(f)$ can be enlarged as

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] \arrow[rr, bend left, "k_f"] & \operatorname{fib}(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] & & \\
& & 0 \arrow[r] & Y \arrow[r] \arrow[rr, bend right, "c_f"'] & \operatorname{cofib}(f) \arrow[r] & \operatorname{coker}(f) \arrow[r] & 0,
\end{tikzcd}$$

where k_f and c_f are morphisms in $\mathbf{C}^{\heartsuit}$.

Definition 3.15. Let $f\colon X \to Y$ be a morphism in $\mathbf{C}^{\heartsuit}$. The *image* $\mathsf{im}(f)$ and the *coimage* $\mathsf{coim}(f)$ of f are defined as $\mathsf{im}(f) = \ker(c_f)$ and $\mathsf{coim}(f) = \operatorname{coker}(k_f)$.

The following lemma shows that $\ker(f)$ does indeed have the defining property of a kernel:

Lemma 3.16. The homotopy commutative diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, "k_f"] \arrow[d] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd}$$

is a pullback diagram in $\mathbf{C}^{\heartsuit}$.

Proof. A homotopy commutative diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd}$$

between objects in the heart is in particular a homotopy commutative diagram in $\mathbf{C}$ so it is equivalent to the datum of a morphism $k'\colon K \to \operatorname{fib}(f)$ in $\mathbf{C}$, with K an object in $\mathbf{C}^{\heartsuit}$. By the orthogonality of $(\mathcal{E}, \mathcal{M})$, this is equivalent to a morphism $\tilde{k}\colon K \to \ker(f)$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathcal{E}"'] & \ker(f) \arrow[d, "\mathcal{M}"] \\
K \arrow[r, "k'"'] \arrow[ur, "\tilde{k}"] & \operatorname{fib}(f)
\end{tikzcd} .$$

□

There is, obviously, a dual result showing that $\operatorname{coker}(f)$ is indeed a cokernel.

Lemma 3.17. The homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r, "c_f"] & \operatorname{coker}(f) \end{tikzcd}$$

is a pushout diagram in $\mathbf{C}^{\heartsuit}$.

Lemma 3.18. For $f\colon X \to Y$ a morphism in $\mathbf{C}$, there is a homotopy commutative diagram where all squares are homotopy pullouts:

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d, "\mathcal{E}"'] \arrow[rr, bend left, "k_f"] & \operatorname{fib}(f) \arrow[r] \arrow[d, "\mathcal{E}"'] & X \arrow[r] \arrow[d, "\mathcal{E}"'] & 0 \arrow[d, "\mathcal{E}[1]"'] & \\ 0 \arrow[r] & \operatorname{coker}(f)[-1] \arrow[r] \arrow[d, "\mathcal{M}"'] & Z_f \arrow[r] \arrow[d, "\mathcal{M}[1]"'] & \ker(f)[1] \arrow[r] \arrow[d, "\mathcal{M}[1]"'] & 0 \arrow[d, "\mathcal{M}[1]"] \\ & 0 \arrow[r] & Y \arrow[r] \arrow[rr, bend right, "c_f"'] & \operatorname{cofib}(f) \arrow[r] & \operatorname{coker}(f) \end{tikzcd}$$

uniquely determining an object $Z_f \in \mathbf{C}^{\heartsuit}$.

Proof. Define Z_f as the homotopy pullout

$$\begin{tikzcd} \operatorname{fib}(f) \arrow[r] \arrow[d, "\mathcal{E}"'] & X \arrow[d, "\mathcal{E}"] \\ \operatorname{coker}(f)[-1] \arrow[r] & Z_f \end{tikzcd}$$

Here the vertical arrow on the right is in $\mathcal{E}$ since the vertical arrow on the left is in $\mathcal{E}$ by definition of $\operatorname{coker}(f)$ (see Remark **3.13**) and $\mathcal{E}$ is preserved by pushouts. Next, paste on the left of this diagram the pullout given by Remark **3.13** and build the rest of the diagram by taking pullbacks or pushouts. Use again Remark **3.13** and the fact that $\mathcal{M}[1]$ is closed under pullbacks to see that $Z_f \to Y$ is in $\mathcal{M}[1]$. Finally, we have

$$0 \xrightarrow{\mathcal{E}} X \xrightarrow{\mathcal{E}} Z_f \xrightarrow{\mathcal{M}[1]} Y \xrightarrow{\mathcal{M}[1]} 0,$$

and so Z_f is in $\mathbf{C}^{\heartsuit}$. □

Proposition 3.19. There is an isomorphism $\mathsf{im}(f) \cong \mathsf{coim}(f)$.

Proof. By definition, $\mathsf{im}(f)$ and $\mathsf{coim}(f)$ are defined by the factorizations

$$0 \xrightarrow{\mathcal{E}} \mathsf{im}(f) \xrightarrow{\mathcal{M}} \operatorname{fib}(c_f)$$

and

$$\mathrm{cofib}(k_f) \xrightarrow{\mathcal{E}[1]} \mathsf{coim}(f) \xrightarrow{\mathcal{M}[1]} 0$$

The diagram in Lemma **3.18** shows that we have $\mathrm{fib}(c_f) = Z_f = \mathrm{cofib}(k_f)$. Therefore, what we need to exhibit are the $(\mathcal{E}, \mathcal{M})$ factorizations of $0 \to Z_f$ and the $(\mathcal{E}[1], \mathcal{M}[1])$ factorization of $Z_f \to 0$. Since Z_f is an object in $\mathbf{C}^\heartsuit$, these are

$$0 \xrightarrow{\mathcal{E}} Z_f \xrightarrow{\mathrm{id}_{Z_f}} Z_f$$

and

$$Z_f \xrightarrow{\mathrm{id}_{Z_f}} Z_f \xrightarrow{\mathcal{M}[1]} 0,$$

respectively, thus giving $\mathsf{im}(f) \cong Z_f \cong \mathsf{coim}(f)$. □

3.2. **Abelian subcategories as hearts.** In the above section we have seen as the choice of a t-structure $\mathfrak{t}$ in a stable ∞-category $\mathbf{C}$ picks up an ∞-abelian subcategory $\mathbf{C}^\heartsuit$ of $\mathbf{C}$. It is natural to wonder whether a vice versa to this statement is true, i.e., whether given an ∞-abelian subcategory $\mathbf{A}$ of $\mathbf{C}$ it is possible or not to determine a t-structure $\mathfrak{t}$ on $\mathbf{C}$ whose heart is $\mathbf{C}$. Proposition **3.4** and Remark **3.5** tell us that, if we require that $\mathbf{C}$ is bounded with respect to the t-structure then $\mathbf{A}$ has to satisfy the following constraint: for any morphism $f\colon X \to Y$ in $\mathbf{C}$ there exists a unique $\mathbb{Z}$-factorization

$$X = \lim(Z_j) \to \cdots \to Z_{j+1} \xrightarrow{f_j} Z_j \xrightarrow{f_{j-1}} Z_{j-1} \to \cdots \to \mathrm{colim}(Z_j) = Y$$

with with j ranging over the integers, $\mathrm{cofib}(f_j) \in \mathbf{A}[j]$ for any $j \in \mathbb{Z}$ and with f_j being an isomorphism for $|j| \gg 0$.

We will call such a factorization an $\mathbf{A}$-*weaved* $\mathbb{Z}$-Postnikov tower for f. Moreover, by using Remark **3.6**, it is not hard to prove that the existence and uniqueness of $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov towers is also a sufficient condition, as we show next.

Proposition 3.20. Let $\mathbf{A}$ be an abelian full subcategory of a stable ∞-categrory $\mathbf{C}$, such that any morphism $f\colon X \to Y$ in $\mathbf{C}$ has a unique $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower. Let $\mathbf{C}_{\mathbf{A},\geq 0}$ be the full subcategory of $\mathbf{C}$ on those objects Y such that the $\mathbb{Z}$-factorization

$$0 = \lim(Y_j) \to \cdots \to Y_{j+1} \xrightarrow{f_j} Y_j \xrightarrow{f_{j-1}} Y_{j-1} \to \cdots \to \mathrm{colim}(Y_j) = Y$$

of the initial morphism $0 \to Y$ is such that $\mathrm{cofib}(f_j) = 0$ for any $j < 0$, and let $\mathbf{C}_{\mathbf{A},<0}$ be the full subcategory of $\mathbf{C}$ on those objects Y such that $\mathrm{cofib}(f_j) = 0$ for any $j \geq 0$. Then $\mathfrak{t}_{\mathbf{A}} = (\mathbf{C}_{\mathbf{A},\geq 0}, \mathbf{C}_{\mathbf{A},<0})$ is a t-structure on $\mathbf{C}$, the stable ∞-category $\mathbf{C}$ is bounded with respect to $\mathfrak{t}_{\mathbf{A}}$, and the heart of $\mathfrak{t}_{\mathbf{A}}$ is (equivalent to) $\mathbf{A}$.

Proof. To begin with, notice that if $\{f_j\}$ is an $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $f\colon X \to Y$, then the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $f[k]\colon X[k] \to Y[k]$ is given by the sequence $\{\varphi_j\}$ with $\varphi_j = f_{j-k}[k]$ for any $j \in \mathbb{Z}$. Namely, $\mathrm{cofib}(\varphi_j) = \mathrm{cofib}(f_{j-k})[k] \in \mathbf{A}[j-k+k] = \mathbf{A}[j]$. It follows that an object Y of $\mathbf{C}$ is in $\mathbf{C}_{\mathbf{A},\geq k}$, i.e., in $\mathbf{C}_{\mathbf{A},\geq 0}[k]$, if and only if the morphisms f_j in the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to Y$ satisfy $\mathrm{cofib}(f_j) = 0$ for any $j < k$, and Y is in $\mathbf{C}_{\mathbf{A},<k}$ if and only if

the f_j satisfy $\mathrm{cofib}(f_j) = 0$ for any $j \geq k$. In particular one immediately sees that $\mathbf{C}_{\mathbf{A},\geq 1} \subseteq \mathbf{C}_{\mathbf{A},\geq 0}$ and $\mathbf{C}_{\mathbf{A},<-1} \subseteq \mathbf{C}_{\mathbf{A},<0}$. Next, consider an object Y in $\mathbf{C}$ and let

$$0 = \lim(Y_j) \to \cdots \to Y_{j+1} \xrightarrow{f_j} Y_j \xrightarrow{f_{j-1}} Y_{j-1} \to \cdots \to \mathrm{colim}(Y_j) = Y$$

be the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower of its initial morphism. Set $Y_{\geq 0} = Y_0$. Since

$$0 = \lim(Y_j) \to \cdots \to Y_2 \xrightarrow{f_1} Y_1 \xrightarrow{f_0} Y_0 = Y_{\geq 0} \xrightarrow{\mathrm{id}_{Y_{\geq 0}}} Y_{\geq 0} \to \cdots \to Y_{\geq 0}$$

is a $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to Y_{\geq 0}$, we have $Y_{\geq 0} \in \mathbf{C}_{\mathbf{A},\geq 0}$. Let $Y_{<0}$ be the finer of the morphism $Y_{\geq 0} \to Y$ so that we have a homotopy pullout

$$\begin{tikzcd} Y_{\geq 0} \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Y_{<}0 \end{tikzcd}$$

By construction, the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $Y_{\geq 0} \to Y$ is

$$Y_{\geq 0} \to \cdots \to Y_{\geq 0} \xrightarrow{\mathrm{id}_{Y_{\geq 0}}} Y_{\geq 0} = Y_0 \xrightarrow{f_{-1}} Y_{-1} \xrightarrow{f_{-2}} Y_{-2} \to \cdots \to \mathrm{colim}(Y_j) = Y.$$

Pushing this out along $Y \to Y_{<0}$ we obtain the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to Y_{<0}$, which shows that $Y_{<0} \in \mathbf{C}_{\mathbf{A},<0}$. Finally, let X be an object in $\mathbf{C}_{\mathbf{A},\geq 0}$, let Y be an object in $\mathbf{C}_{\mathbf{A},<0}$; we want to show that $\mathbf{C}(X,Y)$ is contractible. Since, for any $n \geq 0$, $\pi_n\mathbf{C}(X,Y) = \pi_0\mathbf{C}(X, Y[-n])$ and $Y[-n] \in \mathbf{C}_{\mathbf{A},<0}$ as soon as $Y \in \mathbf{C}_{\mathbf{A},<0}$, it suffices to show that $\pi_0\mathbf{C}(X,Y) = 0$, i.e., that if $f\colon X \to Y$ is a morphism in $\mathbf{C}$, then $f \simeq 0$. Since $X \in \mathbf{C}_{\mathbf{A},\geq 0}$, if X is nonzero then the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to X$ has the form

$$0 \to \cdots \to 0 \xrightarrow{f_{k+n}} X_{k+n} \xrightarrow{f_{k+n-1}} X_{k+n-1} \to \cdots \to X_{k+1} \xrightarrow{f_k} X \xrightarrow{\mathrm{id}_X} X \to \cdots \to X$$

with f_{k+n} and f_k which are not isomorphisms, for some $k \geq 0$ and $n \geq 0$. We say that X has $\mathbf{A}$-*length* equal to $n+1$. When $X = 0$ we say that X has $\mathbf{A}$-lenght equal to 0. Similarly, since $Y \in \mathbf{C}_{\mathbf{A},<0}$, the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to Y$ has the form

$$0 \to \cdots \to 0 \xrightarrow{f_{-h-1}} Y_{-h-1} \xrightarrow{f_{-h-2}} Y_{-h-2} \to \cdots \to Y_{-h-m} \xrightarrow{f_{-h-m-1}} Y \xrightarrow{\mathrm{id}_Y} Y \to \cdots \to Y$$

with f_{-h-1} and f_{-h-m-1} which are not isomorphisms, for some $h \geq 0$ and $m \geq 0$. We say that Y has $\boldsymbol{A}$-*length* equal to $m+1$ Pulling this back along $0 \to Y$ we obtain an $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $Y[-1]$:

$$\begin{aligned} Y[-1] \to \cdots \to Y[-1] \xrightarrow{g_{-h-1}} & Z_{-h-1} \xrightarrow{g_{-h-2}} Z_{-h-2} \to \cdots \\ & \cdots \to Z_{-h-m+1} \xrightarrow{g_{-h-m}} Z_{-h-m} \xrightarrow{g_{-h-m-1}} 0 \to 0 \to \cdots \to 0 \end{aligned}$$

By the above discussion the $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $Y \to 0$ is therefore

$$\begin{aligned} Y \to \cdots \to Y \xrightarrow{\varphi_{-h}} & Y^{-h} \xrightarrow{\varphi_{-h-1}} Y^{-h-1} \to \cdots \\ & \cdots \to Y^{-h-m+2} \xrightarrow{\varphi_{-h-m+1}} Y^{-h-m+1} \xrightarrow{\varphi_{-h-m}} 0 \to 0 \to \cdots \to 0, \end{aligned}$$

where we have set $Y^j = Z_{j-1}[1]$ and $\varphi_j = g_{j-1}[1]$. Note that, consistently, $\mathrm{cofib}(\varphi_j) \in \mathbf{A}[j]$. The proof is now easily concluded by induction on

$$l_{X,Y} = \mathrm{length}_{\mathbf{A}}(X) + \mathrm{length}_{\mathbf{A}}(Y).$$

Namely, for $\mathrm{length}_{\mathbf{A}}(X) = 0$ or $\mathrm{length}_{\mathbf{A}}(Y) = 0$ any morphism from X to Y is trivial zero. For X and Y of positive length, the first case is $\mathrm{length}_{\mathbf{A}}(X) = \mathrm{length}_{\mathbf{A}}(Y) = 1$, i.e., $m = n = 0$. In this case we have $X = \mathrm{cofib}(0 \xrightarrow{f_k} X) \in \mathbf{A}[k]$ and $Y = \mathrm{fib}(Y \xrightarrow{\varphi_{-h}} 0) = \mathrm{cofib}(\varphi_{-h})[-1] \in \mathbf{A}[-h-1]$. Since $\mathbf{A}[-1]$ is right-orthogonal to $\mathbf{A}[k]$ for any $h, k \geq 0$, we see that any morphism $f\colon X \to Y$ is homotopy equivalent to 0 in this case, too. For both $\mathrm{length}_{\mathbf{A}}(X) \geq 1$ and $\mathrm{length}_{\mathbf{A}}(Y) \geq 1$, with at least one of the two strictly greater than 1, let $f\colon X \to Y$ be a morphism, and consider the two morphisms $\varphi_{-h} \circ f\colon X \to Y^{-h}$ and $f \circ f_k\colon X_{k+1} \to Y$. Since $Y^{-h} \in \mathbf{C}_{\mathbf{A},<0}$ and $\mathrm{length}_{\mathbf{A}}(Y^{-h}) \leq \mathrm{length}_{\mathbf{A}}(Y) - 1$, while $X_{k+1} \in \mathbf{C}_{\mathbf{A},\geq 0}$ and $\mathrm{length}_{\mathbf{A}}(X_{k+1}) \leq \mathrm{length}_{\mathbf{A}}(X) - 1$, we have by induction that both $\varphi_{-h} \circ f$ and $f \circ f_k$ are zero. Hence f factors both as $X \to \mathrm{fib}(\varphi^{-h}) \to Y$ and as $X \to \mathrm{cofib}(f_k) \to Y$. Since $\mathrm{fib}(\varphi^{-h}) = \mathrm{cofib}(\varphi^{-h})[-1] \in \mathbf{A}[-h-1] \subseteq \mathbf{C}_{\mathbf{A},<0}$ and $\mathrm{length}_{\mathbf{A}}(A_{-h-1}) = 1$ for any $A_{-h-1} \in \mathbf{A}[-h-1]$, while $\mathrm{cofib}(f_k) \in \mathbf{A}[k] \subseteq \mathbf{C}_{\mathbf{A},\geq 0}$ and $\mathrm{length}_{\mathbf{A}}(A_k) = 1$ for any $A_k \in \mathbf{A}[k]$, we see that either $\mathrm{length}_{\mathbf{A}}(X) + \mathrm{length}_{\mathbf{A}}(\mathrm{fib}(\varphi^{-h}))$ or $\mathrm{length}_{\mathbf{A}}(\mathrm{cofib}(f_k)) + \mathrm{length}_{\mathbf{A}}(Y)$ is strictly smaller than $\mathrm{length}_{\mathbf{A}}(X) + \mathrm{length}_{\mathbf{A}}(Y)$. So the inductive hypothesis applies and $f \simeq 0$. This concludes the proof that $\mathfrak{t}_{\mathbf{A}}$ is a t-structure on $\mathbf{C}$.

To see that $\mathbf{C}$ is bounded with respect to this t-structure, notice that, by definition of $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower, for any object Y in $\mathbf{C}$ the morphisms f_j in the factorization of $0 \to Y$ are isomorphisms for $|j| \gg 0$. Therefore, there exists two integers $k_{\min}$ and $k_{\max}$, depending on Y such that Y is an object in $\mathbf{C}_{\mathbf{A},\geq k_{\min}} \cap \mathbf{C}_{\mathbf{A},<k_{\max}}$, which proves that $\mathbf{C}$ is bounded with respect to $\mathfrak{t}_A$.

Finally, objects in $\mathbf{C}_{\mathbf{A},\geq 0} \cap \mathbf{C}_{\mathbf{A},<1}$ are precisely those Y in $\mathbf{C}$ whose $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower contains the single nontrivial stage

$$0 \cong Z_1 \xrightarrow{f_0} Z_0 \cong Y.$$

From this we see that $Y \cong \mathrm{cofib}(f_0)$ which is an object in $\mathbf{A}$ and so $\mathbf{C}_{\mathbf{A},\geq 0} \cap \mathbf{C}_{\mathbf{A},<1} \subseteq \mathbf{A}$. Vice versa, if Y is in $\mathbf{A}$, then

$$0 \to \cdots \to 0 \xrightarrow{0} 0 \to Y \xrightarrow{f_0 = \mathrm{id}_Y} Y \cdots \to Y$$

is an $\mathbf{A}$-weaved $\mathbb{Z}$-Postnikov tower for $0 \to Y$, showing that $\mathbf{A} \subseteq \mathbf{C}_{\mathbf{A},\geq 0} \cap \mathbf{C}_{\mathbf{A},<1}$.

$\square$

4. Semiorthogonal decompositions

La vie c'est ce qui se décompose à tout moment; c'est une perte monotone de lumière, une dissolution insipide dans la nuit, sans sceptres, sans auréoles, sans nimbes.

E. Cioran, *Précis de décomposition.*

At the opposite end of the transitive case $J = \mathbb{Z}$ is the *finite case*, where J is a finite totally ordered set. As we are going to show, this too is a well investigated case in the literature: J-familes of t-structures with a finite J capture (and slightly generalize) the notion of *semiorthogonal decompositions* for the stable ∞-category $\mathbf{C}$ (see [BO95,Kuz11] for the notion of semiorthogonal decomposition in the classical triangulated context).

To fix notations for this section, let $J = \Delta_{k-1}$ be the totally ordered set on k elements seen as a poset, i.e., $J = \{i_1, i_2, \dots, i_k\}$ with $i_1 \leq i_2 \leq \cdots \leq i_k$, and let $\mathsf{t}\colon \Delta_{k-1} \to \mathrm{TS}(\mathbf{C})$ be a $\mathbb{Z}$-equivariant Δ_{k-1}-family of t-structures on $\mathbf{C}$. We also set, for any $j = 1, \dots, k+1$,

$$\mathbf{A}_j = \mathbf{C}_{[i_{j-i}, i_j)}$$

where, as usual, $i_0 = -\infty$ and $i_{k+1} = +\infty$. We have that any morphism $f\colon X \to Y$ in $\mathbf{C}$ has a unique factorization

$$X \xrightarrow{f_{k+1}} Z_{i_k} \xrightarrow{f_k} Z_{i_{k-1}} \to \cdots \to Z_{i_2} \xrightarrow{f_2} Z_{i_1} \xrightarrow{f_1} Y,$$

with $\mathrm{cofib}(f_j) \in \mathbf{A}_j$, and $\mathbf{A}_j \subseteq \mathbf{A}_h^{\perp}$, for any $1 \leq j < h \leq k+1$.

What we are left to investigate are therefore the special features of the t-structures $\mathsf{t}_{i_j} = (\mathbf{C}_{\geq i_j}, \mathbf{C}_{<i_j})$ coming from the finiteness assumption on J.

4.1. The $\mathbb{Z}$-fixed points in ts($\mathbf{C}$).

As we noticed in Remark **1.3**, a $\mathbb{Z}$-action on a finite poset J is necessarily trivial. By $\mathbb{Z}$-equivariance of $\mathsf{t}\colon \Delta_{k-1} \to \mathrm{TS}(\mathbf{C})$ we have therefore that all the t-structures t_{i_j} are $\mathbb{Z}$-fixed points for the natural $\mathbb{Z}$-action on $\mathrm{TS}(\mathbf{C})$. And a rather pleasant (at a first sight unexpected) fact is that fixed points of the $\mathbb{Z}$-action on $\mathrm{TS}(\mathbf{C})$ are precisely those t-structures $\mathsf{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ for which $\mathbf{C}_{\geq 0}$ is a stable ∞-subcategory of $\mathbf{C}$. We will make use of the following

Lemma 4.1. Let $\mathbf{B}$ be a full ∞-subcategory of the stable ∞-category $\mathbf{C}$; then, $\mathbf{B}$ is a stable ∞-subcategory of $\mathbf{C}$ if and only if $\mathbf{B}$ is closed under shifts in both directions and under pushouts in $\mathbf{C}$.

Proof. The "only if" part is trivial, so let us prove the "if" part.

First of all let us see that under these assumptions $\mathbf{B}$ is closed under fibers. This is immediate: if $f\colon X \to Y$ is an arrow in $\mathbf{B}$ (i.e. an arrow of $\mathbf{C}$ between objects of $\mathbf{B}$, by fullness), then $f[-1]$ is again in $\mathbf{B}$ since $\mathbf{B}$ is closed with respect to the left shift. Since $\mathbf{B}$ is closed under pushouts in $\mathbf{C}$, also $\mathrm{fib}(f) = \mathrm{cofib}(f[-1])$ is in $\mathbf{B}$. It remains to show how this implies that $\mathbf{B}$ is actually stable, i.e. it is closed under all finite limits and satisfies the pullout axiom. Unwinding the assumptions on $\mathbf{B}$, this boils down to showing that in the square

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

the pullback B of $f, g \in \hom(\mathbf{B})$ done in $\mathbf{C}$ is actually an object of $\mathbf{B}$; indeed, once showed this, the square above will satisfy the pullout axiom in $\mathbf{C}$, so *a fortiori* it will have the universal property of a pushout in $\mathbf{B}$. To this aim, let us consider the enlarged diagram of pullout squares in $\mathbf{C}$

$$\begin{tikzcd} Z[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\star"] & \mathrm{fib}(g) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathrm{fib}(f) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

The objects $Z[-1], \mathrm{fib}(f)$ and $\mathrm{fib}(g)$ lie in $\mathbf{B}$ by the first part of the proof, so the square ($\star$) is in particular a pushout of morphism in $\mathbf{B}$; by assumption, this entails that $B \in \mathbf{B}$. □

Remark 4.2. Obviously, a completely dual statement can be proved in a completely dual fashion: a full ∞-subcategory $\mathbf{B}$ of an ∞-stable category $\mathbf{C}$ is a stable ∞-subcategory if and only if it is closed under shifts in both directions and under pullbacks in $\mathbf{C}$.

Proposition 4.3. Let $\mathsf{t} = (\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ be a t-structure on a stable ∞-category $\mathbf{C}$; then the following conditions are equivalent:

(1) t is a fixed point for the $\mathbb{Z}$-action on $\textsc{ts}(\mathbf{C})$, i.e., $\mathsf{t}[1] = \mathsf{t}$ (or, equivalently, $\mathbf{C}_{\geq 1} = \mathbf{C}_{\geq 0}$);
(2) $\mathbf{C}_{\geq 0}$ is a stable ∞-subcategory of $\mathbf{C}$.

Proof. The implication '(2) implies (1)' is obvious. Namely, if $\mathbf{C}_{\geq 0}$ is a stable ∞-subcategory of $\mathbf{C}$, then it is closed under shifts in both directions. Therefore $\mathbf{C}_{\geq 1} = \mathbf{C}_{\geq 0}[1] \subseteq \mathbf{C}_{\geq 0}$. Since, by definition of t-structure, $\mathbf{C}_{\geq 1} \subseteq \mathbf{C}_{\geq 0}$, we have $\mathbf{C}_{\geq 1} = \mathbf{C}_{\geq 0}$. To prove that '(1) implies (2)', assume $\mathbf{C}_{\geq 1} = \mathbf{C}_{\geq 0}$. This means that not only $\mathbf{C}_{\geq 0}[1] \subseteq \mathbf{C}_{\geq 0}$ as for any t-structure, but also $\mathbf{C}_{\geq 0} \subseteq \mathbf{C}_{\geq 0}[1]$, which implies that $\mathbf{C}_{\geq 0}[-1] \subseteq \mathbf{C}_{\geq 0}$. Therefore $\mathbf{C}_{\geq 0}$ is closed under shifts in both directions. By Lemma **4.1**, we then have only to show that $\mathbf{C}_{\geq 0}$ is closed under pushouts in $\mathbf{C}$ to conclude that $\mathbf{C}_{\geq 0}$ is a ∞-stable subcategory of $\mathbf{C}$. Consider a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r] & P \end{tikzcd}$$

in $\mathbf{C}$ with A, B and C in $\mathbf{C}_{\geq 0}$, and let $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ be the normal torsion theory associated to $\mathfrak{t}$. This means, in particular, that

$$\mathbf{C}_{\geq 0} = 0/\mathcal{E} = \{X \in \mathbf{C} \text{ such that } \left[\begin{smallmatrix} 0 \\ \downarrow \\ X \end{smallmatrix}\right] \in \mathcal{E}\}$$

(see [FL14, Def. **??**]). Since A and C are in $\mathbf{C}_{\geq 0}$ we therefore have that both $0 \to A$ and $0 \to C$ are in $\mathcal{E}$. But $\mathcal{E}$ has the 3-for-2 property, so also $A \to C$ is $\mathcal{E}$. Since $\mathcal{E}$ is closed for pushouts, this implies that also $B \to P$ in in $\mathcal{E}$. But $0 \to B$ in in $\mathcal{E}$ since B is in $\mathbf{C}_{\geq 0}$, and therefore also $0 \to P$ is in $\mathcal{E}$, i.e., P is in $\mathbf{C}_{\geq 0}$. □

Remark 4.4. Dually, one can equivalentlly characterise $\mathbb{Z}$-fixed points in $\mathrm{TS}(\mathbf{C})$ as those t-structures $(\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ for which $\mathbf{C}_{<0}$ is a stable ∞-subcategory of $\mathbf{C}$..

Proposition **4.3** and remark **4.4** characterize $\mathbb{Z}$-fixed points on $\mathrm{TS}(\mathbf{C})$ as the t-structures with stable classes $\mathbf{C}_{\geq 0}$ and $\mathbf{C}_{<0}$. By the correspondence between t-structures and normal factorization systems, one should expect that these should be equally characterised as the normal factorization systems $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ for which the classes $\mathcal{E}$ and $\mathcal{M}$ are "stable on both sides", i.e., are closed both for pullbacks and for pushouts. This is indeed precisely what happens, as the following theorem shows.

Theorem 4.5. Let $\mathfrak{t}$ be a t-structure on a stable ∞-category $\mathbf{C}$ and let $\mathbb{F} = (\mathcal{E}, \mathcal{M})$ be the corresponding normal factorization system; then the following conditions are equivalent:

(1) $\mathfrak{t}[1] = \mathfrak{t}$;
(2) $\mathbf{C}_{\geq 0}$ is a stable ∞-category;
(3) $\mathbf{C}_{<0}$ is a stable ∞-category;
(4) $\mathcal{E}$ is closed under pullback;
(5) $\mathcal{M}$ is closed under pushout.

Proof. The only implication we need to prove is that '(1) is equivalent to (4)'. Assume $\mathcal{E}$ is closed under pullbacks. Then for any X in $\mathbf{C}_{\geq 0}$ we have that $0 \to X$ is in $\mathcal{E}$, and so $X[-1] \to 0$ is in $\mathcal{E}$. By the Sator lemma this implies that $0 \to X[-1]$ is in $\mathcal{E}$, i.e., that $X[-1]$ is in $\mathbf{C}_{\geq 0}$. This shows that $\mathbf{C}_{\geq 0}[-1] \subseteq \mathbf{C}_{\geq 0}$ and therefore that $\mathfrak{t}[1] = \mathfrak{t}$. Conversely, assume $\mathfrak{t}[1] = \mathfrak{t}$, and consider a morphism $f : X \to Y$ in $\mathcal{E}$. For any morphism $B \to Y$ in $\mathbf{C}$ consider the diagram

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & B \arrow[r] & Y \arrow[r] & \mathrm{cofib}(f)
\end{tikzcd}$$

where all the squares are pullouts in $\mathbf{C}$. Since f is in $\mathcal{E}$ and $\mathcal{E}$ is closed for pushouts, also $0 \to \mathrm{cofib}(f)$ is in $\mathcal{E}$. This means that $\mathrm{cofib}(f)$ is in $\mathbf{C}_{\geq 0}$ and so, since we are assuming that $\mathbf{C}_{\geq 0} = \mathbf{C}_{\geq 0}[-1]$, also $\mathrm{fib}(f) = \mathrm{cofib}(f)[-1]$ is in $\mathbf{C}_{\geq 0}$, i.e., $0 \to \mathrm{fib}(f)$ is in $\mathcal{E}$. By the Sator lemma, $\mathrm{fib}(f) \to 0$ is in $\mathcal{E}$, which is closed for pushouts, and so $A \to B$ is in $\mathcal{E}$. □

Remark 4.6. In the literature, a factorization system $(\mathcal{E}, \mathcal{M})$ for which the class $\mathcal{E}$ is closed for pullbacks is sometimes called an *exact reflective* factorization, see, e.g., [CHK85]. This is equivalent to saying that the associated reflection functor is left exact (this is called a *localization* in the jargon of [CHK85]). Dually, one characterizes *colocalizations* of a category $\mathbf{C}$ with an initial object as *coexact coreflective* factorizations where the right class $\mathcal{M}$ of $\mathbb{F}$ is closed under pushouts. Therefore, in the ∞-stable case, we see that a $\mathbb{Z}$-fixed point in $\mathrm{TS}(\mathbf{C})$ is a t-structure $(\mathbf{C}_{\geq 0}, \mathbf{C}_{<0})$ such that the truncation functors $\tau_{\geq 0}\colon \mathbf{C} \to \mathbf{C}_{\geq 0}$ and $\tau_{<0}\colon \mathbf{C} \to \mathbf{C}_{<0}$ respectively form a colocalizations and a localization of $\mathbf{C}$. In the terminology of [BR07] we therefore find that in the ∞-stable case $\mathbb{Z}$-fixed point in $\mathrm{TS}(\mathbf{C})$ correspond to *hereditary torsion pairs* on $\mathbf{C}$. Since we have seen that for a $\mathbb{Z}$-fixed point in $\mathrm{TS}(\mathbf{C})$ both $\mathbf{C}_{\geq 0}$ and $\mathbf{C}_{<0}$ are ∞-stable categories, this result could be deduced also from [Lur11, Prop. **1.1.4.1**]: a left (resp., right) exact functor between stable ∞-categories is also right (resp., left) exact.

4.2. **Semiorthogonal decompositions in stable ∞-categories.** We can now precisely relate semiorthogonal decompositions in a stable ∞-category $\mathbf{C}$ to Δ_{k-1}-families of t-structures on $\mathbf{C}$. The only thing we still need is the following definition, which is an immediate adaptation to the stable ∞-context of the classical definition given for triangulated categories (see, e.g., [BO95, Kuz11]).

Definition 4.7. Let $\mathbf{C}$ be an ∞-stable category. A *semiorthogonal decomposition* with k classes on $\mathbf{C}$ is the datum of $k+1$ ∞-stable subcategories $\mathbf{A}_1$, $\mathbf{A}_2$,…, $\mathbf{A}_{k+1}$ of $\mathbf{C}$ such that

(1) one has $\mathbf{A}_i \subseteq \mathbf{A}_j^{\perp}$ for $i < j$ (semiorthogonality);
(2) for any object Y in $\mathbf{C}$ there exists a unique $\{\mathbf{A}_i\}$-weaved Postnikow tower, i.e., a factorization of the initial morphism $0 \to Y$ as

$$0 = Y_0 \to \cdots \to Y_{j+1} \xrightarrow{f_j} Y_j \xrightarrow{f_{j-1}} Y_{j-1} \to \cdots \to Y_{k+1} = Y$$

with $\mathrm{cofib}(f_j) \in \mathbf{A}_j$ for any $j = 1, \dots, k+1$.

Remark 4.8. Since $\{\mathbf{A}_i\}$-weaved Postnikow towers are preserved by pullouts, one can equivalently require that any morphism $f\colon X \to Y$ in $\mathbf{C}$ has a unique factorization of the form

$$X = Z_0 \to \cdots \to Z_{j+1} \xrightarrow{f_j} Z_j \xrightarrow{f_{j-1}} Z_{j-1} \to \cdots \to Z_{k+1} = Y$$

with $\mathrm{cofib}(f_j) \in \mathbf{A}_j$ for any $j = 1, \dots, k+1$.

Theorem 4.9. Let $\mathbf{C}$ be an ∞-stable category. Then the datum of a semiorthogonal decompositions with k classes on $\mathbf{C}$ is equivalent to the datum of a $\mathbb{Z}$-equivariant Δ_{k-1}-family of t-structures on $\mathbf{C}$

Proof. Let us start with a $\mathbb{Z}$-equivariant Δ_{k-1}-family of t-structures t, and write $i_1 < i_2 < \cdots < i_k$ for the elements of Δ_{k-1} and $\mathsf{t}_{i_j} = (\mathbf{C}_{\geq i_j}, \mathbf{C}_{<i_j})$ for the corresponding t-structures on $\mathbf{C}$. Then, setting $\mathbf{A}_j = \mathbf{C}_{[i_{j-1}, i_j)}$ we have semiorthogonality between the $\mathbf{A}_j$'s and the existence of $\{\mathbf{A}_j\}$-weaved Postnikow towers by the general argument recalled at the beginning of this section. So we are only left to prove that the subcategories $\mathbf{A}_j$ are stable. This is immediate: by Theorem **4.5**

both the ∞-subcategories $\mathbf{C}_{\geq i_{j-1}}$ and $\mathbf{C}_{<i_j}$ are stable, and so also their intersection is stable (see, [**?**]). Vice versa, if we start with a semiorthogonal decomposition, then repeating verbatim the argument in the proof of Proposition **3.20** one defines a $\mathbb{Z}$-equivariant Δ_{k-1}-family of t-structures on $\mathbf{C}$. □

Remark 4.10. By Remark **4.6**, we recover in the ∞-stable setting the well known fact (see [BR07, **IV.4**]) that semiorthogonal decompositions with a single class correspond to *hereditary torsion pairs* on the category.

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