


Bicategories of automata, automata in bicategories

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Prolegomena

Fix a monoidal category (\mathcal{K}, \otimes) .

Definition

A **Mealy machine** (of input I and output O) in \mathcal{K} is a span

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

Definition

A **Moore machine** (of input I and output O) in \mathcal{K} is a span

$$E \xleftarrow{d} E \otimes I, E \xrightarrow{s} O$$

Prolegomena



```
1 record MealyObj I 0 : Set (o ⊔ l ⊔ e) where
2   field
3     E : Obj
4     d : E ⊗ I ⇒ E
5     s : E ⊗ I ⇒ 0
```



Mealy.agda



XMealy.agda



Mealy/Bicategory.agda

Mealy and Moore

Definition

The **category** of Mealy machines¹ has objects the Mealy machines as above, (E, d, s) , and morphisms $(E, d, s) \rightarrow (F, d', s')$ the $f: E \rightarrow F$ such that

$$\begin{array}{ccccc} E & \xleftarrow{d} & E \otimes I & \xrightarrow{s} & O \\ \downarrow f & & \downarrow f \otimes I & & \parallel \\ F & \xleftarrow{d'} & F \otimes I & \xrightarrow{s'} & O \end{array}$$

¹All definitions from now on can be Moore-ified without effort.

Mealy and Moore

Let P be an monoidal monad on \mathcal{K} ; the Kleisli category of P becomes monoidal; dually, if P is opmonoidal, Eilenberg-Moore (not the same Moore!) becomes monoidal.

Definition

A P_λ -machine is a Mealy machine in $Kl(P)$; a P -machine is a Mealy machine in $EM(P)$.

Why care?

- **nondeterminism** (if $P = \text{powerset}$, $Kl(P) = \mathbf{Rel}$);
- additional **structure** on objects (they are P -algebras).

Mealy and Moore

Theorem

The category of Mealy machines fits into a (strict, 2-)pullback in

Cat

$$\begin{array}{ccc} \mathbf{Mly}(I, O) & \longrightarrow & \mathbf{Alg}(- \otimes I) \\ \downarrow & \lrcorner & \downarrow \\ ((- \otimes I)/O) & \longrightarrow & \mathcal{K} \end{array}$$

(A similar result holds for Moore: replace the comma

$((- \otimes I)/O)$ with the **slice** \mathcal{K}/O .)

X-automata

(cf. Adámek-Trnková)

Instead of considering a span

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

consider the action of a generic **endofunctor** $X : \mathcal{K} \rightarrow \mathcal{K}$:

$$\begin{array}{ccccc} E & \xleftarrow{\quad} & XE & \xrightarrow{\quad} & O \\ & \swarrow & & \searrow & \\ & \text{X-algebra} & & \text{obj. of comma} & \end{array}$$

which yields at once a pullback characterization of X-automata, ...

X-automata

X-Mly fits in a strict 2-pullback

$$\begin{array}{ccc} X\text{-}\mathbf{Mly} & \xrightarrow{U'} & (X/O) \\ \downarrow v' & \lrcorner & \downarrow v \\ \mathbf{Alg}(X) & \xrightarrow{U} & \mathcal{K} \end{array}$$

...and in particular when $X \dashv R$

Theorem

The category of X -automata is **cocomplete** when \mathcal{K} is, with colimits created by a canonical functor $X\text{-}\mathbf{Mly} \rightarrow \mathcal{K}$; it is **complete** when \mathcal{K} is.

X-automata

From [Mac Lane, V.6, Ex. 3]: in every strict pullback of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U'} & \mathcal{B} \\ v' \downarrow & \lrcorner & \downarrow v \\ \mathcal{C} & \xrightarrow{U} & \mathcal{K} \end{array}$$

if U creates, and V preserves, limits of a given shape \mathcal{J} , then U' creates limits of shape \mathcal{J} .

But $\mathbf{Alg}(X) \rightarrow \mathcal{K}$ creates all limits, and $X/O \rightarrow \mathcal{K}$ all connected limits; thus the problem boils down to find a **terminal object** (products follow).

X-automata

Claim: the terminal object of $X\text{-}\mathbf{Mly}$ is the **terminal coalgebra** for the functor

$$A \mapsto RA \times O$$

which (Adámek theorem) is $O_\infty = \prod_{n \geq 1} R^n O$, with structural morphisms given by **deletion** of first factor, and **projection** over first factor:

$$d_\infty : X\left(\prod_{n \geq 1} R^n O\right) \rightarrow \prod_{n \geq 1} R^n O$$

$$s_\infty : X\left(\prod_{n \geq 1} R^n O\right) \rightarrow O$$

A similar line of reasoning leads to the terminal object in $X\text{-}\mathbf{Mre}$ being $O_\infty = \prod_{n \geq 0} R^n O$.

X-automata

How to induce a terminal morphism

$$\begin{array}{ccccc}
 E & \xleftarrow{d} & XE & \xrightarrow{s} & O \\
 \downarrow !_E & & \downarrow X!_E & & \parallel \\
 O_\infty & \xleftarrow{d_\infty} & XO_\infty & \xrightarrow{s_\infty} & O
 \end{array}$$

$!_E : E \rightarrow O_\infty$ is defined as

$$d_\infty : \text{mate}(\cdots \rightarrow XXXE \xrightarrow{XXd} XXE \xrightarrow{Xd} XE \xrightarrow{s} O)$$

$$s_\infty : \text{mate}(\cdots \rightarrow XXXE \xrightarrow{XXd} XXE \xrightarrow{Xd} XE \xrightarrow{s} O)$$

X-automata

Products are computed as pullbacks along terminal maps:

$$\begin{array}{ccc} P_{\infty} & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow \bar{s}_{F,\infty} \\ E & \xrightarrow{\bar{s}_{E,\infty}} & O_{\infty} \end{array}$$

The pullback P_{∞} can be thought as the **bisimulation object** for X-machines. As a corollary,

- **Mly**(I, O) complete with terminal object $[I^+, O]$ (\leftarrow free **semigroup**).
- **Mre**(I, O) complete with terminal object $[I^*, O]$ (\leftarrow free **monoid**).

Behaviour as an adjunction

Assume the forgetful $U : \mathbf{Alg}(X) \rightarrow \mathcal{K}$ has a left adjoint F .

There is a composite of adjoint functors

$$\mathcal{K}/O_\infty \begin{array}{c} \xrightarrow{\tilde{F}} \\ \xleftarrow[\tilde{U}]{\perp} \end{array} \mathbf{Alg}(X)/(O_\infty, d_T) \begin{array}{c} \xrightarrow{L} \\ \xleftarrow[B]{\perp} \end{array} X\text{-}\mathbf{Mre}$$

where B is a ‘behaviour’ functor defined as

$$(E, d, s) \mapsto (!_E : E \rightarrow O_\infty)$$

and its left adjoint L is determined through the ‘free’ Moore machine on a X -algebra over the terminal O_∞ .

Bicategories

So far so good.

It's all fun and games until someone ~~loses an eye~~ uses a bicategory.

- under which assumptions is $\mathbf{Mly}(I, O)$ the hom-category of a bicategory?
- similar question for Mealy.
- A monoidal category is justTM a bicategory with a single object; but then what is a Mealy automaton in a bicategory \mathbb{B} ?

Bicategories of Automata

The bicategory Mly

Let \mathcal{K} be a Cartesian category. Define a bicategory Mly $_{\mathcal{K}}$ as follows

1. its **objects** are the same objects of \mathcal{K} ;
2. its **1-cells** $I \rightarrow O$ are the Mealy machines (E, d, s) , i.e. the objects of the category **Mly** $_{\mathcal{K}}(I, O)$;
3. its **2-cells** are Mealy machine morphisms defined *ibid.*;
4. the composition of 1-cells $- \circ -$ is defined as [postponed];
5. the **vertical** composition of 2-cells is the composition of Mealy machine morphisms $f : E \rightarrow F$;
6. the **horizontal** composition of 2-cells is the operation defined thanks to bifactoriality of $- \circ -$;
7. the associator and the unitors are inherited from \mathcal{K} .

Composition of 1-cells

Given two Mealy machines $(E, d, s) : I \rightarrow J$ and $(F, d', s') : J \rightarrow K$,

$$E \xleftarrow{d} E \times I \xrightarrow{s} J$$

$$F \xleftarrow{d'} F \times J \xrightarrow{s'} K$$

define their composition $(E \times F, d' \diamond d, s' \diamond s) : I \rightarrow K$ as

$$s' \diamond s = s' \cdot (E \times s) : F \times E \times I \rightarrow F \times J \rightarrow K$$

$$d' \diamond d = \langle d \cdot \pi_F, d' \cdot (E \times s) \rangle : F \times E \times I \rightarrow F \times E$$

where $E \xleftarrow{d} E \times I \xleftarrow{\pi_F} E \times F \times I \xrightarrow{E \times s} F \times J \xrightarrow{d'} F$

Composition of 1-cells

Proof of associativity is bureaucracy:

$$\begin{aligned}(d_1 \diamond d_2) \diamond d_3 &= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1, d_1 \cdot (E_1 \times s_2) \rangle \cdot (E_1 \times E_2 \times s_3) \rangle \\&= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3), d_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3) \rangle \rangle \\&= \langle d_3 \cdot \pi_{12}, \langle \underline{d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3)}, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \rangle \rangle \\d_1 \diamond (d_2 \diamond d_3) &= \langle \langle d_3 \cdot \pi_2, d_2 \cdot (E_2 \times s_3) \rangle \cdot \pi_1, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \rangle \\&= \langle \langle d_3 \cdot \pi_2 \cdot \pi_1, d_2 \cdot (E_2 \times s_3) \cdot \pi_1 \rangle, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \rangle \\&= \langle d_3 \cdot \pi_{12}, \langle \underline{d_2 \cdot (E_2 \times s_3) \cdot \pi_1}, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \rangle \rangle \\s_1 \diamond (s_2 \diamond s_3) &= s_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3))) \\&= s_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3) \\(s_1 \diamond s_2) \diamond s_3 &= s_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3)\end{aligned}$$

Unitality follows a similar (simpler) strategy.

Corollar(ies)

- there are categories

$\underline{\mathbf{Mly}}_{\mathbf{Set}}, \underline{\mathbf{Mly}}_{\mathbf{Cat}}, \underline{\mathbf{Mly}}_{\mathbf{Top}}, \underline{\mathbf{Mly}}_{\mathbf{Pos}}, \underline{\mathbf{Mly}}_{\mathbf{Mon}}, \dots$

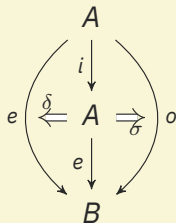
- if \mathcal{K} is Cartesian **closed**, all right/left Kan extensions/lifts exist;
- as a corollary, the terminal objects $[I^+, O], [I^*, O]$ (and even before, I^+, I^*) can be characterised as Kan extensions!
- ? the assignment $\mathcal{K} \mapsto \underline{\mathbf{Mly}}_{\mathcal{K}}$ is (2-)functorial $\mathbf{CCat} \rightarrow 2\text{-}\mathbf{Cat}$ (careful with the 2-cells, Eugene).
- ? Guitart defines a ‘bicategory of Mealy machines’ as $\mathbf{Spn}_F(\mathbf{Mon})$, spans in \mathbf{Cat} between monoids whose left leg is a fibration. Interesting adjunctions with our $\underline{\mathbf{Mly}}$ ’s?

Automata in bicategories

A monoidal category is just a bicategory with a single object.
 What is a machine **inside** a bicategory \mathbb{B} with objects $A, B, C \dots$? A **bicategorical Moore machine** consists of a span of 2-cells in \mathbb{B}

$$e \xleftarrow{\delta} e \circ i \xrightarrow{\sigma} o$$

or rather a diagram of 2-cells



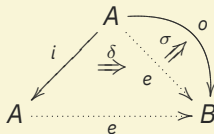
for objects $A, B \in \mathbb{B}$.

Examples

- the mere fact that the 2-cells δ, σ exist implies that i is an endomorphism;
- so, iterated compositions $i \circ \dots \circ i$ make sense as much as iterated tensor powers $I \otimes \dots \otimes I$ made sense in \mathcal{K} ;
- one can find examples in
 - **categories**, functors and natural transformations;
 - **categories**, functors and lax transformations;
 - categories, **profunctors** and 2-cells (a fortiori, in **Rel**);
 - sets and **metric relations**;
 - **topological**, approach, closure **spaces**,...

Behaviour as a Kan extension

A bicategorical Moore machine in \mathbb{B} of fixed input and output i, o is a diagram



The ‘terminal way’ of filling such a span is the **right extension** of the output cell along the **free monad** i^{\sharp} on the input:

- from the **unit** $\eta : 1_A \Rightarrow i^{\sharp}$, get $\text{Ran}_i \Rightarrow \text{Ran}_1 = 1_A$, and thus $\sigma : \text{Ran}_i o \Rightarrow o$;
- from the **multiplication** $\mu : i^{\sharp} \circ i^{\sharp} \Rightarrow i^{\sharp}$ get $\text{Ran}_{i^{\sharp}} \Rightarrow \text{Ran}_{i^{\sharp}} \circ \text{Ran}_{i^{\sharp}}$, and thus

$$\delta : \text{Ran}_{i^{\sharp}} o \circ i \xrightarrow{\text{Ran}_{i^{\sharp}} o * \eta} \text{Ran}_{i^{\sharp}} o \circ i^{\sharp} \xrightarrow{\quad} \text{Ran}_{i^{\sharp}} o$$

Intertwiners

Definition

An **intertwiner** $(u, v) : (e, \delta, \sigma)_{A,B} \multimap (e', \delta', \sigma')_{A',B'}$ consists of a pair of 1-cells $u : A \rightarrow A', v : B \rightarrow B'$ and a triple of 2-cells ι, ϵ, ω disposed as follows:

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xrightarrow{u} & A' \\ \downarrow i & \swarrow \iota & \downarrow i' \\ A & \xrightarrow{u} & A' \end{array} &
 \begin{array}{ccc} A & \xrightarrow{u} & A' \\ \downarrow e & \swarrow \epsilon & \downarrow e' \\ B & \xrightarrow{v} & B' \end{array} &
 \begin{array}{ccc} A & \xrightarrow{u} & A' \\ \downarrow o & \swarrow \omega & \downarrow o' \\ B & \xrightarrow{v} & B' \end{array}
 \end{array}$$

such that:

$$\begin{array}{lcl}
 \begin{array}{|c|} \hline \delta \\ \hline \end{array} \begin{array}{|c|} \hline \iota \\ \hline \end{array} \begin{array}{|c|} \hline \epsilon \\ \hline \end{array} = \begin{array}{|c|} \hline \epsilon \\ \hline \end{array} \begin{array}{|c|} \hline \delta \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \delta \\ \hline \end{array} \begin{array}{|c|} \hline \iota \\ \hline \end{array} \begin{array}{|c|} \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \omega \\ \hline \end{array} \begin{array}{|c|} \hline \delta \\ \hline \end{array} ; \\
 \begin{array}{|c|} \hline \sigma \\ \hline \end{array} \begin{array}{|c|} \hline \iota \\ \hline \end{array} \begin{array}{|c|} \hline \epsilon \\ \hline \end{array} = \begin{array}{|c|} \hline \epsilon \\ \hline \end{array} \begin{array}{|c|} \hline \sigma \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \sigma \\ \hline \end{array} \begin{array}{|c|} \hline \iota \\ \hline \end{array} \begin{array}{|c|} \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \omega \\ \hline \end{array} \begin{array}{|c|} \hline \sigma \\ \hline \end{array} .
 \end{array}$$

Intertwiners

Definition

Let $(u, v), (u', v') : (e, \delta, \sigma) \multimap (e', \delta', \sigma')$ be two parallel intertwiners between bicategorical Mealy machines; a **2-cell** $(\varphi, \psi) : (u, v) \Rightarrow (u', v')$ consists of a pair of 2-cells $\varphi : u \Rightarrow u'$, $\psi : v \Rightarrow v'$ such that the following identities hold true:

$$\begin{array}{ccc} \begin{array}{|c|} \hline \varphi \\ \hline \iota \\ \hline \end{array} = \begin{array}{|c|} \hline \iota \\ \hline \varphi \\ \hline \end{array} & \begin{array}{|c|} \hline \varphi \\ \hline \epsilon \\ \hline \end{array} = \begin{array}{|c|} \hline \epsilon \\ \hline \psi \\ \hline \end{array} & \begin{array}{|c|} \hline \varphi \\ \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \omega \\ \hline \psi \\ \hline \end{array} \end{array}$$

Intertwiners

Specialized to the monoidal case, the previous two definitions become

- **morphisms** of type

$$\iota : I' \otimes U \rightarrow V \otimes I, \epsilon : E' \otimes U \rightarrow V \otimes E, \omega : O' \otimes U \rightarrow V \otimes O;$$

- such that

$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

- **pairs** $f : U \rightarrow U'$ and $g : V \rightarrow V'$ such that

$$\begin{array}{ccc} E' \otimes I' \otimes U & \xrightarrow{d' \otimes U} & E' \otimes U \\ \downarrow E' \otimes I' \otimes f & & \downarrow E' \otimes f \\ E' \otimes I' \otimes U' & \xrightarrow{d' \otimes U'} & E' \otimes U' \end{array} \qquad \begin{array}{ccc} V \otimes E \otimes I & \xrightarrow{V \otimes d} & V \otimes E \\ \downarrow g \otimes E \otimes I & & \downarrow g \otimes E \\ V' \otimes E \otimes I & \xrightarrow{V' \otimes d} & V' \otimes E \end{array}$$

Open problems

Of other bicategories

In 1974 Guitart defined a **bicategory of Mealy machines**:

- the objects are categories M, N, \dots (actually, monoids);
- the 1-cells are spans

$$M \xleftarrow{D} \mathcal{E} \xrightarrow{S} N$$

where D is a fibration and S is a functor.

- composition of 1-cells is as in **Span**.
- G then proves that **MAC** is the **Kleisli bicategory** of the diagram monad $C \mapsto \mathbf{Cat} \parallel C$;

We conjecture the existence of a **left pseudo-adjoint** L in

$$L : \mathbf{Mly}_{\mathbf{Cat}} \xrightleftharpoons{\perp} \mathbf{MAC} : G$$

Nondeterminism in equipments

- In **Rel**, $R = \text{Ran}_{I^\natural} O$ is the relation defined as

$$(a, b) \in R \iff \forall a' \in A. ((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

This relation expresses *reachability* of b from a :

$$a R b \iff \left((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a' O b \right)$$

- Passing from automata in **Cat** to automata in Prof accounts for a form of nondeterminism; one can conjecture to be able to address *nondeterministic* BA in \mathbb{B} as *deterministic* BA in a **proarrow equipment**.

Thanks!

We're still in the process of finishing this up.

Get in touch if you have ideas / want to join!

