Profunctorial Semantics I

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Algebraic structures

A group is a set equipped with operations

- $m : G \times G \to G$
- $i : G \to G$
- $e : 1 \to G$

...

you know the drill

Algebraic structures

Theorem (Higman-Neumann 1953)
A group is a set equipped with a single binary operation $/ : G \times G \to G$ subject to the single equation

$$x/((((x/x)/y)/z)/(((x/x)/x)/z)) = y$$

for every $x, y, z \in X$.

Well.

This is awkward.

The theory of equationally definable classes of algebras, initiated by Birkhoff in the early thirties, is [...] hampered in its usefulness by two defects. [...T]he second is the awkwardness inherent in the presentation of an equationally definable class in terms of operations and equations.

Quite recently, Lawvere, by introducing the notion - closely akin to the clones P. Hall - of an algebraic theory, rectified the second defect.

Definition

An operator domain is a sequence $\underline{\Omega} = (\Omega_n \mid n \in \mathbb{N})$; the elements of Ω_n are called operations of arity n.

Definition

An interpretation $\underline{E}$ of an operator domain $\underline{\Omega}$ consists of a pair $(E, (f_\omega \mid \omega \in \Omega_n, n \in \mathbb{N}))$ where $f_\omega : E^n \to E$ is an n-ary operation on the set E called the *carrier* of $\underline{E}$.

An operator domain can be represented as a (rooted) graph: for example, for groups

$$\begin{tikzcd} 0 \arrow[r, "e"'] & 1 \arrow[loop above, "i"] & 2 \arrow[l, "m"] \end{tikzcd}$$

Way better to use functors.

A Lawvere theory is an identity-on-objects functor $p : \mathsf{Fin}^{\mathsf{op}} \to \mathcal{L}$ that commutes with finite products.

Unwinding the definition:

- $\mathcal{L}$ is a category with the same objects as Fin, the category of finite sets and functions;
- p is a functor that acts trivially on objects
- The only thing that can change between Fin and $\mathcal{L}$ is the number of morphisms $[n] \to [m]$.

Equivalently: p is a promonad on the opposite of Fin, regarded as an object of the bicategory of profunctors, that preserves the monoidal structure. $\mathcal{L}$ is the Kleisli object of p.

$$\left\{ \begin{array}{c} \text{identity on obj} \\ \text{left adjoints} \\ p{:}[\mathcal{L},\mathsf{Set}]\to[\mathsf{Fin}^{\mathsf{op}},\mathsf{Set}] \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{monads in Prof} \\ p{:}\mathsf{Fin}^{\mathsf{op}} \rightsquigarrow \mathsf{Fin}^{\mathsf{op}} \end{array} \right\}$$

- The trivial theory is the identity funtor $1_{\mathsf{Fin}} : \mathsf{Fin}^{\mathsf{op}} \to \mathsf{Fin}^{\mathsf{op}}$
- Since p preserves products, it is uniquely determined by its value on $[1]$. This means that if $p : \mathsf{Fin}^{\mathsf{op}} \to \mathcal{L}$ is a Lawvere theory, then every object of $\mathcal{L}$ is X^n if $p[1] = X$.
- The only difference between Fin and $\mathcal{L}$ is thus the set of morphisms $[n] \to [m]$, added on top of those in Fin.

$$\mathcal{L}_{\mathsf{Grp}} = \begin{tikzcd} {[0]} \arrow[r, "e"'] & {[1]} \arrow[loop above, "i"] & {[2]} \arrow[l, "m"] \end{tikzcd}$$

A model for a Lawvere theory p is a product-preserving functor $\ell : \mathcal{D} \to \mathrm{Set}$.

The category $\mathrm{Mod}(p)$ for a Lawvere theory is a full, reflective subcategory of the category $[\mathcal{L}, \mathrm{Set}]$ of all functors $\mathcal{D} \to \mathrm{Set}$.

Theorem
The following conditions are equivalent:

- *ℓ is a model for a Lawvere theory $\mathcal{L}$;*
- *The composition $\ell \circ p$ preserves finite products;*
- *The composition $\ell \circ p$ is representable (with respect to the inclusion J : Fin $\to$ Set), i.e.*

$$\ell(X[n]) \cong \mathit{Set}(J[n], A)$$

for some $A \in$ Set.

As a consequence, the square

$$\begin{tikzcd}
\mathsf{Mod}(p) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{D}, \mathsf{Set}]} \arrow[d, "_\circ X"] \\
\mathsf{Set} \arrow[r, "{[J,1]}"'] & {[\mathsf{Fin}^{\mathsf{op}}, \mathsf{Set}]}
\end{tikzcd}$$

is a pullback.

- $\mathsf{Mod}(p)$ is a reflective subcategory of $[\mathcal{D}, \mathsf{Set}]$. We write $r_! \dashv r$ for the resulting adjunction.
- The functor u is monadic, with left adjoint f.
- This sets up a functor

$$\mathfrak{M} : \mathsf{Th}_L(\mathsf{Fin}) \to \mathsf{Mnd}_{<\omega}(\mathsf{Set})$$

because the monad uf above is finitary.

There is an equivalence of categories between $\mathsf{Th}_L(\mathsf{Fin})$ and $\mathsf{Mnd}_{<\omega}(\mathsf{Set})$.

We have to construct a functor in the opposite direction, $\mathfrak{Z} : \mathsf{Mnd}_{<\omega}(\mathsf{Set}) \to \mathsf{Th}_L(\mathsf{Fin})$; given T, we consider the composition $\mathsf{Fin} \hookrightarrow \mathsf{Set} \xrightarrow{F^T} \mathsf{Set}^T$ and its bo-ff factorization, in a square

$$\begin{tikzcd} \mathcal{L}^{\mathsf{op}} \arrow[r, "ff"] & \mathsf{Set}^T \\ \mathsf{Fin} \arrow[u, "bo"] \arrow[r, "J"'] & \mathsf{Set} \arrow[u, "F^T"'] \end{tikzcd}$$

- the left vertical arrow is a Lawvere theory almost by definition.
- Set^T has the universal property of the category of $\mathcal{L}$-models.

Theories as promonads

There is a 2-monad $\tilde{S} : \mathsf{Prof} \to \mathsf{Prof}$ whose algebras are exactly promonoidal categories.

Given a profunctor $p : \mathcal{A} \rightsquigarrow \mathcal{B}$ between promonoidal categories $(\mathcal{A}, \mathfrak{P}, J_A), (\mathcal{B}, \mathfrak{Q}, J_B)$:

- p is a pseudo-$\tilde{S}$-algebra morphism;
- The cocontinuous left adjoint $\hat{p}$ associated to p is strong monoidal with respect to the convolution monoidal product on presheaf categories;

Assume the promonoidal structures $\mathfrak{P}, \mathfrak{Q}$ on A, B are representable; then, the conditions above are in turn equivalent to

- Both mates $p^{\triangleleft} : A \to PB$ che $p^{\triangleright} : B \to P^*A$ are strong monoidal wrt convolution on their codomains.

Theories as $[\mathsf{Fin}, \mathsf{Set}]$-categories

Theorem

$$[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set})$$

If the LHS is endowed with the monoidal structure induced by composition of endofunctors; this is called the substitution monoidal product of functors $F, G : \mathsf{Fin} \to \mathsf{Set}$:

$$F * G : m \mapsto \int^{n} Fn \times (Gm)^n$$

The substitution monoidal product is a highly non-symmetric, right closed monoidal structure (not left closed).

The category $[\mathsf{Fin}, \mathsf{Set}]$ works as base of enrichment.

Theories as $[\mathsf{Fin}, \mathsf{Set}]$-categories

From [Garner]

From now on we blur the distinction between the categories $[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set})$:

- A finitary monad is a monoid in $\mathsf{End}_{<\omega}(\mathsf{Set})$, i.e. a $\mathsf{End}_{<\omega}(\mathsf{Set})$-category with a single object, i.e. a $[\mathsf{Fin}, \mathsf{Set}]$-category with a single object.
- A Lawvere theory is a $[\mathsf{Fin}, \mathsf{Set}]$-category that is absolute (Cauchy-, Karoubi-)complete as an enriched category and generated by a single object.
- Lawvere theories form a reflective subcategory in finitary monads; reflection is the enriched Cauchy completion functor.

Theories as $[\mathbf{Fin}, \mathbf{Set}]$-categories

Equivalently,

- A Lawvere $[\mathsf{Fin}, \mathsf{Set}]$-category is an enriched category where every object A is the tensor $y[n] \odot X$ for a distinguished object $X \cong y[1] \odot X$. All such categories are enriched-Cauchy complete.
- A $[\mathsf{Fin}, \mathsf{Set}]$-category is a special kind of cartesian multicategory: one where a multimorphism $f : X_1 \dots X_n \to Y$ is such that $X_1 = X_2 = \dots = X_n$.

Generalisations/extensions:

- let $\mathbb{N}$ be the discrete category over natural numbers;
- let $\mathbf{P}$ be the groupoid of natural numbers;

The categories $[\mathbb{N}, \mathsf{Set}]$ and $[\mathbf{P}, \mathsf{Set}]$ become monoidal with respect to substitution products $*_N, *_P$:

$$F *_N G = \coprod_{k\in\mathbb{N}} G_k \times \coprod_{\vec{n}|\sum n_i = n} X_{n_1} \times \cdots \times X_{n_k}$$

$$F *_P G = \int^{k,\vec{n}} Y_k \times X_{n_1} \times \cdots \times X_{n_k} \times \mathbf{P}\left(\sum n_i, n\right)$$

PRO(P)S

$*_N$ and $*_P$-monoids are respectively non-symmetric and symmetric operads.

- A PRO is an identity-on-objects strong monoidal functor $p : \mathbb{N} \to \mathcal{P}$. $\mathcal{P}$ is possibly non-cartesian.
- A PROP is an identity-on-objects strong monoidal functor $p : \mathbb{N} \to \mathcal{P}$. $\mathcal{P}$ is symmetric monoidal.

These are, of course, other examples of promonoidal promonads.

PRO(P)s and operads

Every PRO $p : \mathsf{Fin}^{\mathsf{op}} \to \mathcal{T}$ gives rise to the operad $O(\mathcal{T}) = (\mathcal{T}(n, 1) \mid n \in \mathbb{N})$. 2. Conversely, any operad $(\mathcal{O}(n) \mid n \in \mathbb{N})$ gives rise to a pro $T(\mathcal{O})$, where

$$T(\mathcal{O})(n, m) = \coprod_{k_1 + ... + k_m = n} \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_m).$$

(It would be helpful to imagine a picture of m trees stacked vertically.)

If we begin with an operad $\mathcal{O}$, we have $\mathcal{O} = O(T(\mathcal{O}))$. (This is because $T(\mathcal{O})(n, 1) = \mathcal{O}(n)$, according to the above formula.)

On the other hand, if we start with a PRO $\mathcal{T}$, then there exists a canonical map of PROs $T(O(\mathcal{T})) \to \mathcal{T}$, given by, for each n and m, a canonical function

$$\coprod_{k_1+\cdots+k_m=n} T(k_1,1) \times \cdots \times T(k_m,1) \to T(n,m) \qquad (\star)$$

induced from the monoidal product on T.

This sets up an adjunction

$$T : \mathsf{Opd}[\mathsf{S}] \leftrightarrows \mathsf{PRO}[\mathsf{P}] : O$$

with fully faithful left adjoint, so that [symmetric] operads can be regarded as a PRO[P]s $\mathcal{T}$ such that each function (*) is bijective.

Re-enact [Garner] away from Set.

Let $\mathcal{V}$ be a locally presentable base of enrichment; let $\mathfrak{F}(\mathcal{V})$ be the subcategory of finitely presentable objects:

- $\mathfrak{F}(\mathcal{V})$ is the free finite weighted cocompletion of the point;
- There is a strong monoidal equivalence of categories

$$[\mathfrak{F}(\mathcal{V}), \mathcal{V}] \cong [\mathcal{V}, \mathcal{V}]_{<\omega}$$

  between functors $\mathfrak{F}(\mathcal{V}) \to \mathcal{V}$ and finitary endo-$\mathcal{V}$-functors;

- The $\mathcal{V}$-substitution product on LHS is

$$F * G = A \mapsto \int^B FB \otimes_{\mathcal{V}} (GA)^B$$

- There is an equivalence of categories between finitary $\mathcal{V}$-monads and enriched-Cauchy-complete categories generated by a single object under iterated finite powers.
- Models for a Lawvere theory correspond to algebras for the associated finitary monad; free models are free agebras are representables in

$$\begin{aligned} Alg(T, \mathcal{C}) &= [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-Cat}(T, \mathcal{C}) \\ &\cong [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-Cat}(\hat{T}, \mathcal{C}) \\ &= Mod(\hat{T}, \mathcal{C}) \end{aligned}$$

class of lims	finite $\times$	$\mathbb{D}$-limts	finite powers	weighted $\mathbb{D}$-limits	bicat $\times$
theory	$\mathsf{Fin}^{\mathsf{op}}$	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$
semantics	Set	Set	$\mathcal{V}$	$\mathcal{V}$	Prof
eq. with	finitary	$\mathbb{D}$-accessible	$[\mathfrak{F}(V), V]$-monoids	$[?, V]$-monoids	???

Profunctorial semantics

- Characterise the free carbicat $\mathbb{CB}(*)$ on a singleton: see link here);
- Check if the univ property of Fin remains true for $\mathbb{CB}(*)$;
- Take $\mathbb{CB}(*) = F$, and consider its free cocompletion in the bicolimit sense
- Prove that

$$\begin{aligned}[PF, PF] &\cong [\mathbb{CB}(*), PF] \\ &\cong PF\end{aligned}$$

monoidally; $\odot$-monoids := monoids in PF wrt composition in $[PF, PF]$.

Profunctorial semantics

- Prove that there is a syntax-VS-semantics adjunction here: theories are promonoidal promonads T on (a 1-skeleton of) $\mathbb{CB}(*)$, and models are carbicat homomorphisms $\mathsf{Kl}(T) \to \mathsf{Prof}$. There is an equivalence

$$\{\text{theories}\} \cong \{\text{??? monads}\}$$

- Let PROs come into play: analogue of the adjunction between PROs and operads.

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