# Triangulated Categories

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Topology 2

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Problem/Wish: Define an invariant of complexes (e.g. simplicial ones) X, Y, allowing to decide whether their *geometric realizations* |X|, |Y| are homotopy equivalent.

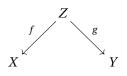
• *Homology* of complexes is a too coarse invariant:

There exist topological spaces X, Y having degree-wise isomorphic homologies but X is neither homeomorphic, nor homotopy equivalent to Y.

• **Theorem (Whitehead).** Simplicial complexes X and Y have homotopy equivalent geometric realizations |X| and |Y| iff there exists a third s.c. Z and simplicial maps  $f: Z \longrightarrow X$ ,  $g: Z \longrightarrow Y$  being *quasi-isomorphisms* (i.e.

$$f_*: H_i(Z) \longrightarrow H_i(X)$$
  $g_*: H_i(Z) \longrightarrow H_i(Y)$ 

is an isomorphism for all  $i \ge 0$ ).

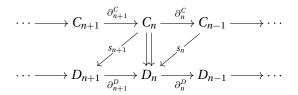


# Definition (Homotopy)

Given chain complexes  $C = \{C_n, \partial_n^C\}$ ,  $\mathcal{D} = \{D_n, \partial_n^{\mathcal{D}}\}$ , we define two chain maps  $\varphi$ ,  $\psi$  to be homotopic if there exists a chain map of degree +1,  $s_n : C_{n-1} \longrightarrow D_n$  such that

$$\psi_n - \varphi_n = \partial_{n+1}^{\mathcal{D}} \circ s_n + \Sigma_{n-1} \circ \partial_n^{\mathcal{C}}$$

*for all*  $n \in \mathbb{Z}$ :



• *Being homotopic* is an equivalence relation in every  $\hom_{\mathbf{Ch}(\mathcal{A})}(\mathcal{C}, \mathcal{D})$  which we write  $\varphi \sim \psi$ : define the homotopy category  $\mathbf{K}(\mathcal{A})$  as  $\mathrm{Ob}(\mathbf{K}(\mathcal{A})) = \mathrm{Ob}(\mathbf{Ch}(\mathcal{A}))$ ,  $\hom_{\mathbf{K}(\mathcal{A})}(\mathcal{C}, \mathcal{D}) = \hom_{\mathbf{Ch}(\mathcal{A})}(\mathcal{C}, \mathcal{D})/\sim$ .

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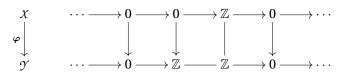
# **Problem!** In passing to the homotopy category abelian structure gets lost:

## Theorem

 $\mathbf{K}(A)$  is not an abelian category (even if A is).

## Sketch of Proof.

Build a counterexample in A = Ab; the chain map



doesn't admit a kernel in the homotopy category (both complexes  $\mathcal{K} = (\ker \varphi_n, i_n)$  and  $\mathrm{id}_X$  share the UMP of  $\ker \varphi$ , but they are *not* homotopic, contradiction  $\{ t \}$ ).

...having lost the abelian structure, we can't talk anymore about exact sequences. (Rough) Idea: Try to replace "exact sequence"

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

by "exact triangle"

$$A \longrightarrow B \longrightarrow C \longrightarrow "A^+$$
".

(whatever "plus" means).

#### Definition

Given an additive category A and an auto-equivalence  $S: A \longrightarrow A$  to be called suspension of A, a triangle (wrt S) in A is a sequence in A like

$$A \longrightarrow B \longrightarrow C \longrightarrow S(A)$$

A morphism of triangles is a triple (f, g, h) making every square below commutative.

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$$

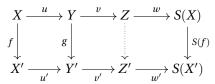
$$f \downarrow \qquad g \downarrow \qquad h \downarrow \qquad \downarrow S(f)$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} S(X')$$

# Definition (Triangulated Category)

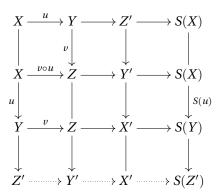
A triangulated category is an additive category T, with a suspension functor S, where we can find a family of triangles (called distinguished triangles, from now on d.t.s) subject to the following axioms:

- Every triangle isomorphic to a d.t. is itself a d.t.;
- For all  $X \in \mathrm{Ob}_{\mathfrak{T}}$ , the triangle  $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow S(X)$  is distinguished;
- To any  $f: X \longrightarrow Y$  in  $\mathfrak T$  we can associate a d.t. in which f is the first arrow,  $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow S(X)$ ;
- (rot) If  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$  is a d.t., then  $Y \xrightarrow{v} Z \xrightarrow{w} S(X) \xrightarrow{-S(u)} S(Y)$  is a d.t., and vice versa;
- (com) Given d.t.s  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$  and  $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} S(X')$ , every commutative diagram



... and finally

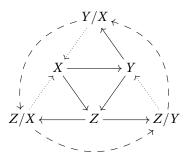
• (oct) Given d.t.s  $X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow S(X)$ ,  $Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow S(Y)$  and  $X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow S(X)$  there exists a d.t.  $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow S(Z')$ which makes the following diagram commutative:



[Notation: the image of A via S is usually denoted A[1] for short; the image  $S^k(A)$ , for  $S^k = S \circ \cdots \circ S$  is denoted A[k] for short.

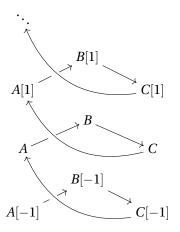
A couple of observations about the axioms:

• (oct), the most strange axiom can be alternatively stated as the completability/commutativity of the following 3D diagram:



the rough idea is that (oct) gives a weaker analogue of the "freshman algebraist's isomorphism"  $(Z/X)/(Y/X)\cong Z/Y$  for a sequence of subobjects  $X\hookrightarrow Y\hookrightarrow Z$ ; dotted arrows are the  $A/B\longrightarrow S(A)$  part of the triangle. (oct) asks that that diagram can be completed with curved dashed arrows, and the resulting external triangle  $Y/X\longrightarrow Z/X\longrightarrow Z/Y\longrightarrow S(Y/X)$  is a d.t.

(rot) is the *rotation axiom*: it encodes the fact that we can present d.t.s as "towers"



and  $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$  is a d.t. iff  $B \xrightarrow{\nu} C \longrightarrow A[1] \xrightarrow{-\nu[1]} B[1]$  is a d.t. (and so on for finitely many shifts).

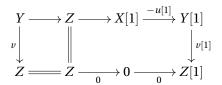
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We can easily draw triangulated analogues of classical results in an abelian category A, now without speaking of exact sequences in A:

• Let  $(\mathfrak{T},[1])$  be triangulated. Given a d.t.  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$  then  $v \circ u = 0$  and  $w \circ v = 0$ , namely composition of adiacent arrows in a d.t. is the zero arrow.

## Proof.

Thanks to (rot) it's enough to prove that  $v \circ u = 0$ . Again from (rot),  $Y \longrightarrow Z \longrightarrow X[1] \xrightarrow{-u[1]} Y[1]$  is a d.t.; in diagram



the unique arrow to 0 completes to a morphism of triangles (v, id, 0); now  $v[1] \circ (-u[1]) = -(v \circ u)[1] = 0$  implies  $v \circ u = 0$ , qed.

• (Triangulated l.e.s.) Let  $(\mathfrak{T},[1])$  be triangulated. Let  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$  be a d.t. For any  $T \in \mathrm{Ob}_{\mathfrak{T}}$  there is a l.e.s. of abelian groups

$$\begin{array}{c} \cdots \longrightarrow \hom_{\mathcal{T}}(T,X[k]) \xrightarrow{u[k]_*} \hom_{\mathcal{T}}(T,Y[k]) \xrightarrow{\nu[k]_*} \\ \xrightarrow{\nu[k]_*} \hom_{\mathcal{T}}(T,Z[k]) \xrightarrow{w[k]_*} \hom_{\mathcal{T}}(T,X[k+1]) \longrightarrow \cdots \end{array}$$

 $\bullet$  (Triangulated 5 Lemma) Let  $(\mathfrak{T},[1])$  be triangulated. Given a morphism of triangles as in

$$\begin{array}{c|c} X \stackrel{u}{\longrightarrow} Y \stackrel{v}{\longrightarrow} Z \stackrel{w}{\longrightarrow} X[1] \\ f \downarrow & g \downarrow & \downarrow h & \downarrow f[1] \\ X' \stackrel{u}{\longrightarrow} Y' \stackrel{v}{\longrightarrow} Z' \stackrel{w}{\longrightarrow} X'[1] \end{array}$$

such that f and g are isomorphisms, then also h is.

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The rest of the note is devoted to prove the following

### Theorem

 $\mathbf{K}(A)$  is a triangulated category.

First of all we have to define the *S* shift endofunctor: given  $k \in \mathbb{Z}$  define  $[k]: C \mapsto C[k]$ 

$$\begin{cases} (\mathcal{C}[k])_n = C_{n-k} \\ \partial_n^{\mathcal{C}[k]} = (-1)^k \partial_{n-k}^{\mathcal{C}} \end{cases}$$

Consider in particular [1]: it's easily seen to pass to the quotient in  $\mathbf{K}(\mathcal{A})$ ; hence it defines an autoequivalence (its inverse is [-1])  $\mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$ . Recall now the definition of the mapping cone of a chain map: Given a chain map  $\varphi \colon \mathcal{X} \longrightarrow \mathcal{Y}$ , its *mapping cone* is the chain complex  $M(\varphi)$  defined by

$$\begin{cases} M(\varphi)_n = (\chi[1] \oplus \mathcal{Y})_n = X_{n-1} \oplus Y_n \\ \partial_n^{\mathcal{M}(\varphi)} = \begin{pmatrix} -\partial_{n-1}^{\chi} & 0 \\ \varphi_{n-1} & \partial_n^{\mathcal{Y}} \end{pmatrix} \end{cases}$$

Notice that there exist a canonical exact sequence

$$0 \longrightarrow \mathcal{Y} \stackrel{lpha}{\longrightarrow} M(oldsymbol{arphi}) \stackrel{eta}{\longrightarrow} X[1] \longrightarrow 0$$

where  $\alpha$ ,  $\beta$  are the obvious injection and projection

$$egin{aligned} & lpha(oldsymbol{arphi}) \colon \mathcal{Y} \longrightarrow M(oldsymbol{arphi}) & lpha(oldsymbol{arphi})_n := \left(egin{aligned} 0 \ \mathrm{id}_{Y_n} 
ight) \ & eta(oldsymbol{arphi}) \colon M(oldsymbol{arphi}) \longrightarrow X[1] & eta(oldsymbol{arphi})_n := \left(egin{aligned} \mathrm{id}_{X_{n-1}} & 0 \end{array}
ight). \end{aligned}$$

Notice also that  $\alpha(\varphi)$  and  $\beta(\varphi)$  pass to homotopy, meaning that  $\varphi \sim \psi$  implies  $\alpha(\varphi) \sim \alpha(\psi)$ ,  $\beta(\varphi) \sim \beta(\psi)$  ( $M(\varphi) \cong M(\psi)$ , via  $\theta = \begin{pmatrix} \operatorname{id}_{X_n} & 0 \\ s_n & \operatorname{id}_{Y_{n+1}} \end{pmatrix}$  so the claim is  $\theta \circ \alpha(\varphi) \cong \alpha(\psi)$ , obvious).

## Definition

We call a d.t. in the homotopy category any triangle isomorphic to one of the form

$${\mathcal X} \xrightarrow{\quad \boldsymbol{\varphi} \quad} {\mathcal Y} \xrightarrow{\quad \boldsymbol{\alpha}(\boldsymbol{\varphi}) \quad} {M}(\boldsymbol{\varphi}) \xrightarrow{\beta(\boldsymbol{\varphi}) \quad} {\mathcal X}[1]$$

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- For any complex X consider the zero arrow  $\varphi \colon X \longrightarrow 0$ . Then the mapping cone  $M(\varphi)$  coincides with X[1]. In a similar fashion  $M(\psi) \cong \mathcal{Y}$  for  $\psi \colon 0 \longrightarrow \mathcal{Y}$ .
- Given X, Y objects of  $\mathcal{A}$  and  $f: X \longrightarrow Y$ , seen as degree-zero-concentrated complexes, the mapping cone M(f) coincides with the complex having f concentrated in degree 0,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \stackrel{f}{\longrightarrow} Y \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

- The identity chain map of  $M(\mathrm{id}_X)$ , being  $X=(X_n,\partial_n)$  a complex, is nullhomotopic via the homotopy  $s_n=\left(\begin{smallmatrix}0&\mathrm{id}_{X_{n-1}}\\0&0\end{smallmatrix}\right);$
- The mapping cone of a quasi-isomorphism  $\varphi: X \longrightarrow \mathcal{Y}$  is acyclic.

Let's now check that the axioms of triangulated category hold in  $\mathbf{K}(\mathcal{A})$ .

 Axioms 1 to 3 hold by definition; in particular for axiom 2 one can always find a d.t.

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow M(\mathrm{id}_X) \longrightarrow X[1]$$

which is directly equivalent to

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1].$$

( $M(id_x)$  is nullhomotopic).

• To prove (rot) let's consider a d.t.

$$X \xrightarrow{f} \mathcal{Y} \xrightarrow{\alpha(\varphi)} M(\varphi) \xrightarrow{\beta(\varphi)} X[1].$$

and prove that the triangle

$$\mathcal{Y} \xrightarrow{\quad \boldsymbol{\varphi} \quad} M(\boldsymbol{\varphi}) \xrightarrow{\beta(\boldsymbol{\varphi}) \quad} \boldsymbol{X}[1] \xrightarrow{\quad \boldsymbol{\varphi}[1] \quad} \boldsymbol{\mathcal{Y}}[1]$$

is distinguished by finding a d.t. which is isomorphic to that one.

Consider the d.t. associated to  $M(\varphi)$ :

define

$$\epsilon \colon X[1] \xrightarrow{\qquad \qquad} M(\alpha(\varphi))$$

$$X_{n-1} \longmapsto \begin{pmatrix} \neg\varphi_{n-1} \\ \operatorname{id}_{X_{n-1}} \end{pmatrix} Y_{n-1} \oplus X_{n-1} \oplus Y_n$$

$$\eta \colon M(\alpha(\varphi)) \xrightarrow{\qquad \qquad} X[1]$$

$$Y_{n-1} \oplus X_{n-1} \oplus Y_n \longmapsto \begin{pmatrix} \operatorname{id}_{X_{n-1}} & 0 \end{pmatrix}$$

and recall that  $M(\alpha(\varphi)) = \left( \mathcal{Y}[1] \oplus M(\varphi), \begin{pmatrix} -\partial_{n-1}^{\gamma} & 0 & 0 \\ 0 & -\partial_{n-1}^{\chi} & 0 \\ \mathrm{id}_{\gamma_{n-1}} & \varphi_{n-1} & \partial_{n}^{\gamma} \end{pmatrix} \right)$ . Now all squares are easily seen to commute (possibly up to homotopy):

$$\bullet \ \beta(\alpha(\varphi)) \circ \epsilon = (\operatorname{id}_{Y_{n-1}} \circ \circ) \left( \operatorname{id}_{X_{n-1}} \circ \circ \circ \right) = -\varphi_{n-1},$$

•  $\epsilon \circ \beta(\varphi) \sim \alpha(\alpha(\varphi))$  via the map  $\begin{pmatrix} 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : M(\varphi)_n \longrightarrow M(\alpha(\varphi))_{n+1}$  in such a way that

$$\begin{pmatrix} -\partial_n^{\mathcal{N}} & 0 & 0 \\ 0 & -\partial_n^{\mathcal{N}} & 0 \\ \mathrm{id}_{Y_{n+1}} & \varphi_n & \partial_{n+1}^{\mathcal{N}} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 & -\mathrm{id}_{Y_{n-1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\partial_{n-1}^{\mathcal{X}} & 0 \\ \varphi_{n-1} & \partial_n^{\mathcal{N}} \end{pmatrix} = \begin{pmatrix} -\varphi_{n-1} & 0 \\ 0 & 0 \\ 0 & -\mathrm{id} \end{pmatrix} = \boldsymbol{\epsilon} \circ \beta(\boldsymbol{\varphi}) - \alpha(\alpha(\boldsymbol{\varphi}))$$

•  $\beta(\varphi) = \eta \circ \alpha(\alpha(\varphi))$  holds by definition, and  $-\varphi[1] \circ \eta \sim \beta(\alpha(\varphi))$  via the homotopy  $(0 \circ \operatorname{id}_{Y_n}) : M(\alpha(\varphi))_n \longrightarrow Y_n$ :

$$\beta(\alpha(\boldsymbol{\varphi})) - (-\boldsymbol{\varphi}[1] \circ \boldsymbol{\eta}) = \left( \begin{smallmatrix} \operatorname{id}_{\gamma_{n-1}} & \boldsymbol{\varphi}[1] & 0 \end{smallmatrix} \right) = \\ - \partial_n^{\mathcal{Y}} \left( \begin{smallmatrix} 0 & 0 & \operatorname{id} \end{smallmatrix} \right) + \left( \begin{smallmatrix} 0 & 0 & \operatorname{id} \end{smallmatrix} \right) \left( \begin{smallmatrix} -\partial_{n-1}^{\mathcal{Y}} & 0 & 0 \\ 0 & -\partial_{n-1}^{\mathcal{X}} & 0 \\ \operatorname{id}_{\gamma_{n-1}} & \varphi_{n-1} & \partial_n^{\mathcal{Y}} \end{smallmatrix} \right)$$

Last but not least,  $\eta$ ,  $\epsilon$  are mutually inverses up to homotopy:  $\eta \circ \epsilon = \mathrm{id}_{\chi[1]}$  on the nose, while  $\epsilon \circ \eta \sim \mathrm{id}_{M(\alpha(\varphi))}$  via the homotopy  $s_n = \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ :

$$\epsilon_n \circ \eta_n - \mathrm{id}_{M(\alpha(\varphi))} = \begin{pmatrix} -\mathrm{id} - \varphi_{n-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} -\partial_n^{\gamma} & 0 & 0 \\ 0 & -\partial_n^{\chi} & 0 \\ 0 & -\partial_n^{\chi} & 0 \\ \mathrm{id}_{Y_{n+1}} & \varphi_n & \partial_{n+1}^{\gamma} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\partial_{n-1}^{\gamma} & 0 & 0 \\ 0 & -\partial_{n-1}^{\chi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M(\alpha(\varphi))_{n+1} \longrightarrow M(\alpha(\varphi))_n \xrightarrow{\delta_{n+1}^{M(\alpha(\varphi))}} M(\alpha(\varphi))_{n-1} \longrightarrow M(\alpha(\varphi))_{n-1}$$

$$M(\alpha(\varphi))_{n+1} \xrightarrow{\delta_{n+1}^{M(\alpha(\varphi))}} M(\alpha(\varphi))_n \longrightarrow M(\alpha(\varphi))_{n-1}$$

(com) axiom can be verified over d.t.s only; suppose

is a morphism of triangles up to the dotted arrow. Left square commutes up to a family of maps  $\{s_n \colon X_n \longrightarrow Y'_{n+1}\}$  such that  $\gamma_n \circ u_n - u'_n \circ f_n = \partial_{n+1}^{\gamma'} \circ s_n + s_{n-1} \circ \partial_n^{\chi}$ . Define

$$h_n: X_{n-1} \oplus Y_n \longrightarrow \begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix} \longrightarrow X'_{n-1} \oplus Y'_n$$

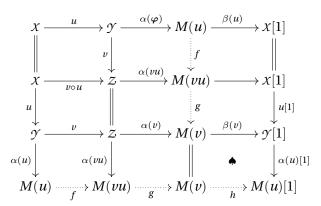
It is a morphism of complexes because of the commutativity of

$$\begin{array}{c|c} X_n \oplus Y_{n+1} & \longrightarrow \begin{pmatrix} -\partial & 0 \\ u & \partial \end{pmatrix} & \longrightarrow X_{n-1} \oplus Y_n \\ \begin{pmatrix} \varphi_n & 0 \\ s_n & \gamma_{n+1} \end{pmatrix} & & & & & & & \\ \begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix} \\ X'_n \oplus Y'_{n+1} & \longrightarrow \begin{pmatrix} -\partial & 0 \\ u' & \partial \end{pmatrix} & \longrightarrow X'_{n-1} \oplus Y'_n \end{array}$$

Let's see that  $h \circ \alpha(u) = \alpha(u') \circ \gamma$  and  $\beta(u') \circ h = \varphi[1] \circ \beta(u)$ : in fact they easily follow from

$$\begin{cases} \begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix} \\ \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} \gamma_n = \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix} \end{cases} = \begin{pmatrix} \varphi_{n-1} & 0 \\ \varphi_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} \varphi_{n-1} & 0 \\ \varphi_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} \varphi_{n-1} & 0 \\ \varphi_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} \varphi_{n-1} & 0 \\ \varphi_{n-1} & 0 \end{pmatrix}$$

Octahedral axiom is the longest and messiest to verify: consider



Let's define dotted arrows

$$f \colon M(u) \xrightarrow{\qquad} M(v \circ u)$$

$$X_{n-1} \oplus Y_n \mapsto X_{n-1} \oplus Z_n$$

$$g \colon M(v \circ u) \xrightarrow{\qquad} M(v)$$

$$X_{n-1} \oplus Z_n \mapsto X_{n-1} \oplus Z_n$$

$$h \colon M(v) \xrightarrow{\qquad} M(u)[1]$$

$$Y_{n-1} \oplus Z_n \mapsto X_{n-2} \oplus Y_{n-1}$$

(notice that with this definition of h square  $\spadesuit$  automatically commute). The resulting triangle turns out to be isomorphic to

$$M(u) \xrightarrow{\quad f \quad} M(v \circ u) \xrightarrow{\alpha(f) \quad} M(f) \xrightarrow{\quad \beta(f) \quad} M(u)[1]$$

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Consider the diagram

$$\begin{array}{c|c} M(u) & \stackrel{f}{\longrightarrow} M(v \circ u) \stackrel{g}{\longrightarrow} M(v) \stackrel{h}{\longrightarrow} M(u)[1] \\ & \parallel & \parallel & \sigma \Big| \uparrow \tau & \parallel \\ M(u) & \stackrel{f}{\longrightarrow} M(v \circ u) \xrightarrow[\alpha(f)]{} M(f) \xrightarrow[\beta(f)]{} M(u)[1] \end{array}$$

We shall find  $\sigma$ ,  $\tau$  (mutually inverses up to homotopy) such that

$$\begin{cases} \beta(f) \circ \sigma \sim h \\ h \circ \tau \sim \beta(f) \\ \sigma \circ g \sim \alpha(f) \\ \tau \circ \alpha(f) \sim g \end{cases}$$

Notice that first and the fourth identity hold on the nose as soon as we define

$$\sigma_n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y_{n-1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{Z_n} \end{pmatrix}, \quad \tau_n = \begin{pmatrix} 0 & \mathrm{id}_{Y_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_n} \end{pmatrix}$$

A homotopy between  $\alpha(f)$  and  $\sigma \circ g$  is seen to be  $\begin{pmatrix} \operatorname{id}_{x_{n-1}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ; to conclude the proof, let's find a homotopy between  $\beta(f)$  and  $h \circ \tau$ : their difference in degree n is  $\begin{pmatrix} \operatorname{id}_{x_{n-2}} & 0 & 0 & 0 \\ 0 & 0 & -u_{n-1} & 0 \end{pmatrix}$ , hence a homotopy between the two is easily seen to be

$$\left(\begin{smallmatrix}0&0&\mathrm{id}_{X_{n-1}}&0\\0&0&&0\end{smallmatrix}\right):X_{n-2}\oplus Y_{n-1}\oplus X_{n-1}\oplus Z_n\longrightarrow X_{n-1}\oplus Y_n$$

Finally,  $\sigma$  and  $\tau$  are mutually inverses up to homotopy:  $\tau \circ \sigma = id$  on the nose. Inverse composition gives

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{Y_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_n} \end{pmatrix}$$

This completes the proof of the octahedral axiom, and the proof that  $\mathbf{K}(\mathcal{A})$  is triangulated by the shift functor and mapping-cones triangles as d.t.s

# **Towards Derived Cats**

## A problem arises.

By definition, any d.t.  $X \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\alpha(\varphi)} M(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{X}[1]$  in  $\mathbf{K}(\mathcal{A})$  gives rise to an exact sequence  $0 \longrightarrow \mathcal{Y} \longrightarrow M(\varphi) \longrightarrow \mathcal{X}[1] \longrightarrow 0$  in  $\mathbf{Ch}(\mathcal{A})$ .

The converse is not true! (consider  $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$  (complexes conc. in degree zero), if there exists a d.t.

 $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[1]$ , notice that there are no non-zero arrows  $\mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[1]$ , hence w = 0; then  $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , contradiction).

Solution: pass to the derived category.