Reference Cards

Monoidal and enriched derivators

Friday 9th June, 2017 11:47

A nice feature of rings is that they behave like monoidal categories with one object (or vice versa).

- Any monoidal functor $F : \mathcal{V} \to \mathcal{W}$ (lax is enough) induces a base change 2-functor

$$F_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{W}\text{-Cat}$$

  that sends a $\mathcal{V}$-category $\mathcal{C}$ into the $\mathcal{W}$-category having the same objects of $\mathcal{C}$ and where $(F_*\mathcal{C})(X, Y) = F(\mathcal{C}(X, Y))$.
- The structural 2-cells of F induce the monoidal structure on $F_*\mathcal{C}$.
- Monoidal transformations are induced accordingly (the definition is straightforward): a natural transformation $\beta : F \to G$ induces a 2-natural transformation between the 2-functors F_* and G_* with 'restricted' components.

It seems that this construction could be applied to $\mathcal{V} \to \mathbf{Set}$ to generate the underlying functor $U : \mathcal{V}\text{-Cat} \to \text{Cat}$, but the fact is that $\hom(J, -)$ is seldom monoidal.

The assignment described above that sends $\mathcal{V}$ into $\mathcal{V}$-Cat, F to F_* and β to β_* is a 2-functor

$$(-)\text{-Cat} : \text{Cat}_{\otimes} \longrightarrow 2\text{-Cat}$$

A suitable 2-categorical Grothendieck construction gives rise then to a universal fibration

$$\begin{tikzcd} \text{EnCat} \arrow[d, "p"] \\ \text{Cat}_{\otimes} \end{tikzcd}$$

whose fiber over $\mathcal{V}$ is the 2-category of $\mathcal{V}$-categories.

- This is no different from the construction of the fibration **Mod** $\to$ **Ring** whose fiber over the ring R is the category R-Mod of modules over R. This is the canonical fibration for F : **Ring** $\to$ **Cat**, and **Mod** $= \int_1 F$.
- General definitions pertaining the Grothendieck construction apply here and we have a definition on functors and natural transformations.

- A morphism $(\mathcal{V}, \mathcal{C}) \to (\mathcal{W}, \mathcal{D})$ in EnCat is given by a pair $u : \mathcal{V} \to \mathcal{W}$ and a functor $f : u_*\mathcal{C} \to \mathcal{D}$. Composition is given by

$$(vu)_*\mathcal{C} = v_*u_*\mathcal{C} \xrightarrow{v_*f} v_*\mathcal{D} \xrightarrow{g} \mathcal{E}$$

- A 2-cell $\alpha : (u, f) \to (v, g)$ is defined for two parallel 1-cells $(\mathcal{V}, \mathcal{C}) \to (\mathcal{W}, \mathcal{D})$ as a pair $\beta : u \to v$ (which is monoidal) and α is a 2-cell

$$\begin{tikzcd} u_*\mathcal{C} \arrow[rr, "f"] \arrow[dr, "\beta_*"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathcal{D} \\ & v_*\mathcal{C} \arrow[ur, "g"'] & \end{tikzcd}$$

All the forgetful functors $U_{\mathcal{V}} : \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}$ glue together to form a functor

$$U : \mathsf{EnCat} \to \mathsf{Cat}$$

defined by $U(\mathcal{V}, \mathcal{C}) = U_{\mathcal{V}}(\mathcal{C}) =$the underlying unenriched category of $\mathcal{C}$.
All the compatibility check are straightforward.

Recall that a monoidal prederivator is a strict 2-functor $\mathbb{E} : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{Cat}_{\otimes}$. A prederivator enriched over $\mathbb{E}$ is a 2-functor $\mathbb{D}$ such that $p \circ \mathbb{D} = \mathbb{E}$.

The essential of this definition is: an enriched derivator specifies an $\mathbb{E}(J)$-enriched category $\mathbb{D}(J)$ for each $J \in \mathsf{Cat}$, and this specification is 2-functorial in J. Graphically,

$$\begin{tikzcd}
\mathsf{Cat}^{\mathsf{op}} \arrow[r, "\mathbb{D}"] \arrow[dr, bend right, "\mathbb{E}"'] & \mathsf{EnCat} \arrow[r, "U"] \arrow[d, "p"] & \mathsf{CAT} \\
& \mathsf{Cat}_{\otimes} &
\end{tikzcd}$$

The composition $U \circ \mathbb{D}$ is the prederivator underlying the enriched prederivator $\mathbb{D}$.

Defining a morphism of enriched prederivators is notationally quite painful, but the definition is clear: it's a pseudonatural transformation between 2-functors $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{EnCat}$.

From the definition of morphism in EnCat it follows that we have to specify a pseudonatural transformation $F : \mathbb{D} \to \mathbb{D}'$ whose components $F_I : \mathbb{D}(I) \to \mathbb{D}'(I)$ satisfy the commutativity

$$\begin{tikzcd} \mathbb{E}(u)_*\mathbb{D}(K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathbb{E}(u)_*\mathbb{D}'(K) \arrow[d] \\ \mathbb{D}(J) \arrow[r, "F_J"'] & \mathbb{D}'(J) \end{tikzcd}$$

for each $u : J \to K$, where we exceptionally denoted $\mathbb{E}(u)$ the action of $\mathbb{E}$ on u.
(the yoga is: as a monoidal functor, $\mathbb{E}(u)$ turns $\mathbb{D}(K)$ into a $\mathbb{E}(J)$-enriched category, and then the square above is the only way to compare them according to the def. of morphisms in EnCat).

A general result in enriched stuff is:

Theorem

Given a 2v adjunction $\mathcal{E} \times \mathcal{D} \overset{\curvearrowright}{\longrightarrow} \mathcal{D}$ where $\mathcal{E}$ is monoidal and $\mathcal{D}$ is $\mathcal{E}$-tensored. Then $\mathcal{D}$ is also $\mathcal{E}$-cotensored and canonically $\mathcal{E}$-enriched.

We want to show that this is the base case of a theorem on derivators:

Theorem for derivators

Let $\mathbb{E}$ be a monoidal derivator, and *bD* tensored over $\mathbb{E}$. If there is a 2v adjunction inducing the tensoring,

$$(\otimes, \mathsf{hom}_l, \mathsf{hom}_r) : \mathbb{E} \times \mathbb{D} \to \mathbb{D}$$

then $\mathbb{D}$ is canonically $\mathbb{E}$-enriched and cotensored.

From the definition of an 2v adjunction for derivators we get that each $\mathbb{D}(K)$ is tensored over $\mathbb{E}(K)$ and part of a 2v adjunction

$$(\otimes, \mathrm{HOM}_{l,\mathbb{D}(K)}, \mathrm{HOM}_{r,\mathbb{D}(K)}) : \mathbb{E}(K) \times \mathbb{D}(K) \to \mathbb{D}(K)$$

Using the result for plain categories we get that each $\mathbb{D}(K)$ is enriched over $\mathbb{E}(K)$, and we prove that it is coherently so: $\hom_r$ will give all the needed coherence.

As a general tenet, if you can do something in model categories you can do it in derivators:

If $\mathcal{M}, \mathcal{N}$ are combinatorial model categories, $\mathcal{M}$ is also monoidal, and $\mathcal{N}$ is $\mathcal{M}$-tensored, then the derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored, cotensored and enriched over the monoidal derivator $\mathbb{D}_{\mathcal{M}}$.

This applies to **sSet**-model categories, **Sp**-model categories, $\mathbf{dg}_k$-model categories...

The Grothendieck construction

The previous construction of p makes heavy use of the Grothendieck construction for 2-categories. We recall it starting from its 0-dimensional counterpart.

For a functor $F : I \to \mathbf{Set}$ all you need to know is in any of these equivalent universal properties:

$$\begin{tikzcd} \int_0 F \arrow[r] \arrow[d] & \mathbf{Set}_* \arrow[d, "U"] \\ I \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \int_0 F \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & I^{\mathrm{op}} \arrow[d, "y"] \\ * \arrow[r, "\ulcorner F \urcorner"'] & {[I, \mathbf{Set}]} \end{tikzcd} \qquad \begin{tikzcd} \int_0 F \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & * \arrow[d, "\ulcorner * \urcorner"] \\ I \arrow[r] & \mathbf{Set} \end{tikzcd}$$

There is a fibration $p : \int_0 F \to I$ such that $p^{-1} i$ is the set $F(i)$.

For a functor $F : I \to \mathbf{Cat}$ we define $\int_1 F$ as the category of pairs $(i, X \in F(i))$, and a morphism $(i, X) \to (j, Y)$ to be a pair (f, u) such that $f : i \to j$ and $u : F(f)X \to Y$ in $F(j)$. Composition is defined as

$$(i, X) \stackrel{(f,u)}{\to} (j, Y) \stackrel{(g,v)}{\to} (k, Z)$$
$$(i, X) \stackrel{(g.f, v.F(g)u)}{\to} (k, Z)$$

Again there is a fibration $p : \int_1 F \to I$ such that $p^{-1}i$ is a category isomorphic to $F(i)$.

For a 2-functor $F : I \to 2\text{-}\mathbf{Cat}$, things go as expected but the definition is quite daunting: $\int F$ has $\int_1 F$ as underlying 1-category (in a similar manner, $\int_1 F$ had $\int_0 F$ as set of objects); 2-cells and their two compositions (horizontal and vertical) are defined as follows

- A 2-cell $(i, X) \underset{(g,v)}{\overset{(f,u)}{\Downarrow \underline{a}}} (j, Y)$ is a pair (α, θ) such that $\alpha : f \to g$ is a 2-cell in I and $\theta : v.F(\alpha)_X \to u$ is a 2-cell in $F(j)$.

- Horizontal composition is defined for cells

$$\begin{tikzcd}
(i,X) \arrow[rr, bend left, "{(f,u)}"] \arrow[rr, "{(g,v)}" description] \arrow[rr, bend right, "{(h,w)}"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow \underline{a}"] \arrow[rr, phantom, shift right=3ex, "\Downarrow \underline{b}"] & & (j,Y)
\end{tikzcd}$$

i.e. for diagrams of 2-cells like

$$\begin{tikzcd}
F(f)X \arrow[r, "u"] \arrow[dr, "F(\alpha)_X"'] & Y \arrow[d, phantom, shift right=4ex, "\Leftarrow\theta"] \\
& F(g)X \arrow[u, "v"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(g)X \arrow[r, "v"] \arrow[dr, "F(\beta)_X"'] & Y \arrow[d, phantom, shift right=4ex, "\Leftarrow\sigma"] \\
& F(h)X \arrow[u, "w"']
\end{tikzcd}$$

as the pasting

$$\begin{tikzcd}
F(f)X \arrow[rr, "u"] \arrow[dr] & \arrow[d, phantom, "\Leftarrow\theta"] & Y \\
& F(g)X \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Leftarrow\sigma"] & \\
& & F(h)X \arrow[uu, "w"']
\end{tikzcd}$$

giving that $(\beta, \sigma) \circ_v (\alpha, \theta) = (\beta \circ_v \alpha, (\sigma * F(\alpha)_X) \circ \theta)$.

- Vertical composition is defined for cells

$$\begin{tikzcd}[column sep=huge]
(i, X) \arrow[r, bend left, "{(f_1, u_1)}"] \arrow[r, bend right, "{(f_2, u_2)}"'] \arrow[r, phantom, "\Downarrow \underline{a}"] & (j, Y) \arrow[r, bend left, "{(g_1, v_1)}"] \arrow[r, bend right, "{(g_2, v_2)}"'] \arrow[r, phantom, "\Downarrow \underline{b}"] & (k, Z)
\end{tikzcd}$$

i.e. for diagrams of 2-cells like

$$\begin{tikzcd}
F(f_1)X \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\Nwarrow \theta"] & Y & F(g_1)Y \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\Nwarrow \sigma"] & Z \\
& F(f_2)X \arrow[u] & & F(g_2)Y \arrow[u]
\end{tikzcd}$$

as the pasting

$$\begin{tikzcd}
F(g_1 f_1)X \arrow[r, "F(g_1)u_1"] \arrow[dr, "F(g_1)F(\alpha)_X"'] \arrow[r, phantom, shift right=4ex, "\Nwarrow F(g_1)\theta"] & F(g_1)Y \arrow[r, "v_1"] \arrow[d] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\Nwarrow \sigma"] & Z \\
& F(g_1 f_2)X \arrow[dr, "F(\beta)_{f_2 X}"'] \arrow[r, phantom, "\mathrm{nat}"] & F(g_2)Y \arrow[u, "v_2"'] \\
& & F(g_2 f_2)X. \arrow[u, "F(g_2)u_2"']
\end{tikzcd}$$