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Research Statement

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My research interests span across various topics; of course, this does not mean I know each such topic: just that I'm eager to learn it.

I started as an "algebraic topologist", but I soon decided I'd better leave algebraic topology to better mathematicians than I am: to me, mathematics pertains to language, not to forms. More: mathematics is a language, and following [**?**] languages are mathematical objects. Studying how this duality shapes the way we think is one of my main interests nowadays.

1 Works on stable homotopy theory

At the beginning of my PhD I was able to demonstrate that

Result 1

> In the setting of stable $(\infty, 1)$-categories the theory of t-structures is subordinated to a flexible and expressive calculus of factorization systems.

The upshot of this result is that t-structures yield the correct notion of co/reflective co/-localization in a stable $(\infty, 1)$-category, constituting a fundamental piece of the category theory expressed by a "stable homotopy theoretic universe".

The classical theory of t-structures, initiated in [**?**], has now countless applications in stable homotopy theory, representation theory of algebras, and the theory of 'homotopy coherent' algebraic structures [**?**]. Following the idea that stable phenomena become way simpler in the setting of higher categories, it has been natural to employ a neat characterization of co/reflective pairs of categories as suitable *factorization systems*: this led to the central theorem of [**?**], showing that in the setting of stable ∞-categories a t-structure on (the homotopy category of) an ∞-category $\mathbf{C}$ is completely determined by a *∞-categorical factorization system* $(\mathcal{E}, \mathcal{M})$, again in $\mathbf{C}$, such that the following two properties are satisfied:

1. (reflectivity) the 3-for-2 property holds for both classes $\mathcal{E}, \mathcal{M}$; this entails that the subcategories of "cofibrants" $\left\{X \in \mathbf{C} \mid \begin{bmatrix} 0 \\ \downarrow \\ X \end{bmatrix} \in \mathcal{E}\right\}$ and "fibrants" $\left\{Y \in \mathbf{C} \mid \begin{bmatrix} Y \\ \downarrow \\ 0 \end{bmatrix} \in \mathcal{M}\right\}$ are respectively a coreflective and a reflective subcategory of $\mathbf{C}$, forming the so-called *aisle* $\mathbf{C}_{\geq}$ and *coaisle* $\mathbf{C}_{<}$ of a *t*-structure.
2. (normality) the reflection $X \to RX$ is the cofiber of the coreflection map $SX \to X$, and the coreflection is the fiber of the reflection, in a pullback-pushout (for short, a *pullout*) diagram

$$\begin{tikzcd} SX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "\ulcorner" very near end] & X \arrow[d] \\ 0 \arrow[r] & RX. \end{tikzcd}$$

Such factorization systems are called *normal torsion theories* building on previous work of [**?**, **?**].[1]

1.1 *t*-structures on stable ∞-categories

This result led me to a torough study of how the classical theory of *t*-structures, with all its host of applications, gets reshaped by this eminently structural reformulation (informally called the 'torsiocentric' perspective henceforth).

Result 2

All the classical theory (abelianity of the *heart* $\mathbf{C}_{\geq} \cap \mathbf{C}_{<}[1]$ of a *t*-structures, the theory of *semiorthogonal decompositions* on derived categories, Postnikov towers in the stable homotopy category, and more) can be expressed in the language of factorization system, giving also elegant new insights.

These results are extensively described in a joint work [**?**] with D. Fiorenza and G. Marchetti.

In this paper we prove that under the torsiocentric perspective two apparently disconnected constructions, *Postnikov towers* (induced by the heart of a *t*-structure) and *semiorthogonal decompositions*, acquire an intrinsic description as, respectively, orbits and fixed points of the $\mathbb{Z}$-action; it all boils down to specialize to these two extremal particular cases the construction [**?**, Def. 2.7] of the "tower" $\mathrm{tow}(f)$ of a morphism $f\colon X \to Y$ with respect to a $\mathbb{Z}$-equivariant family of *t*-structures $\mathbf{t}\colon J \to \mathrm{TS}(\mathbf{C})$.

In a second moment, towards the end of my Ph.D., I concentrated my study on the theory of *recollements* of *t*-structures and this work culminated in the paper [**?**]; there, I

[1] It is important to note that such a simple statement is a direct consequence of the more natural environment offered by ∞-categories to join homotopy theory and category theory. Indeed, the above equivalence proved in [**?**], called the *Rosetta Stone* theorem, becomes way more cumbersome when stated in the language of triangulated categories: the issue was adressed in a subsequent work, [**?**]. In such a setting, we defined a notion of *triangulated factorization system* and of *triangulated normal torsion theory* on a triangulated category $\mathcal{T}$, but the first is not a true factorization system on the triangulated category, and in the second the normality property is stated in terms of *triangulated* homotopy co/cartesian squares, a different (and way worse-behaved) notion than the neat $(\infty, 1)$-categorical one.

Thus, we doubt that the triangulated notion of normal torsion theory is of any interest outside the proof of [**?**, 2.11]. What makes the point of [**?**] is instead a more general result of which this is the shadow when homotopies have been quotiented out: to show this we define notion of *derivator factorization systems* in the 2-category PDer, describing them as algebras for a suitable strict 2-monad and prove that the Rosetta stone theorem still holds true: for a stable derivator $\mathbb{D}$, a certain class of derivator factorization systems (the *normal derivator torsion theories*) correspond bijectively with *t*-structures on the base $\mathbb{D}(e)$ of the derivator –which is always triangulated and plays the rôle of $\mathcal{T}$ above.

tried to reformulate the following classical construction: given an arrangement of stable $(\infty, 1)$-categories like

$$\mathbf{D}^0 \underset{\overset{\longleftarrow}{i_L}}{\overset{\overset{i_R}{\longleftarrow}}{\xrightarrow{\ i\ }}} \mathbf{D} \underset{\overset{\longleftarrow}{q_L}}{\overset{\overset{q_R}{\longleftarrow}}{\xrightarrow{\ q\ }}} \mathbf{D}^1$$

where $i_L \dashv i \dashv i_R$ and $q_L \dashv q \dashv q_R$ are adjunctions and suitable 'exactness' properties hold (see [**?**, Def. 3.1]), it is possible to *glue* two *t*-structures $\mathbf{t}_0, \mathbf{t}_1$, respectively on the categories $\mathbf{D}^0, \mathbf{D}^1$ to a *t*-structure $\mathbf{t}_0 \wr \mathbf{t}_1$ on $\mathbf{D}$; this formalism, introduced in [**?**] is of capital importance in algebraic geometry for the theory of perverse sheaves, having also applications in intersection homology [**?**, **?**, **?**] and representation theory [**?**, **?**].

2 Plans for future research

2.1 Factorization systems on ∞-categories

My current research (especially during the last year) slightly detached from these topics, as soon as I began to better investigate the foundations of the language I used (what was in fact my primary interest in the first place); I believe that a complete axiomatization of the theory of factorization systems on $(\infty, 1)$-categories is an urgent objective, since apart from its independent interest, it would clarify certain natural constructions in stable and unstable homotopy theory (this track of research has already been explored to some extent, and quite fruitfully by a series of works by W. Chachólski [**?**, **?**] and Farjoun [**?**] in a way that resembles a theory of *unstable t-structures*; it was however only in the framework of quasicategories that Joyal and Anel proved [**?**] a factorization theoretic version of *Blakers-Massey theorem* for quasicategories).

As shown in [**?**] in the setting of $(\infty, 1)$-categories Blakers-Massey-like theorems are neatly described by the internal language of the $(\infty, 1)$-category of spaces **Spc**, referring to a $\mathbb{Z}$-indexed family of factorization systems on **Spc** (a *slicing* in the terminology of [**?**]).

A rather intriguing development of this circle of ideas involves the *$\mathcal{P}$-local version* of Blakers-Massey theorem, together with its rational and rational stable counterparts. Both perspectives permit to build an enticing connection between factorization systems and rational/chromatic homotopy theory.

More precisely, if $\mathcal{P} \sqcup \mathcal{Q}$ is a partition of the set $\mathbb{P}$ of primes in two disjoint subsets, we say that a $\mathbb{Z}$-module G is *$\mathcal{P}$-local* is each $g \in G$ has finite order, which is not divisible for any prime in $\mathcal{P}^c = \mathcal{Q}$. This allows to call $(\mathcal{P}, n)$*-connected* a simply connected space whose $\pi_{[0,n]}(X)$ is a $\mathcal{Q}$-group; it can be easily seen that if $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ then a $(\mathcal{Q}_2, n)$-connected space is also $(\mathcal{Q}_1, n)$-connected; in the same fashion, all $(\mathcal{Q}, n+1)$-connected are also $(\mathcal{Q}, n)$-connected. Similar definitions apply to the stable homotopy (or rather, stable ∞-)category $\mathbf{Sp}$ of spectra, wrt the homotopy groups of an object $E \in \mathbf{Sp}$, and to a map of spectra. This allows to reformulate the $\mathcal{P}$-local, rational, stable $\mathcal{P}$-local, and rational stable analogues of Blakers-Massey theorem as results involving a transfinite family of factorization systems twisted by the powerset of $\mathbb{P}$. These objects are called *arithmetic slicings*.

Task 1

The $\mathcal{P}$-local, rational, stable $\mathcal{P}$-local, and rational stable $\mathcal{P}$-local analogues of

Blakers-Massey theorem are theorems about arithmetic slicings.

Notably, a similar point of view has been taken in the setting of Homotopy Type Theory in [**?**, **?**].

Another promising field of application involves *non-commutative geometry* and stable $(\infty, 1)$-categories. In the last few years there have been several fertile exchanges between the two disciplines: [**?**] outlined a zoology of interesting model structures on categories of cubical C^*-algebras, whereas [**?**] sheds a light on some interesting connections between *dendroidal sets* and higher-dimensional categories of C^*-algebras, and [**?**] gives a proof of the existence on C^*-**Cat** (C^*-*categories* and $*$-functors) of a cofibrantly generated simplicial symmetric monoidal model structure that is analogous to the "folk" model structure on **Cat**, whose weak equivalences are the equivalences of categories. All these approaches probably fit in a complicated web of Quillen adjunctions giving "the" theory of $(\infty, 1)$-noncommutative spaces whose theory of $(\infty, 1)$-factorization systems and geometry are deeply intertwined.

Task 2

Is there a "convenient" stable $(\infty, 1)$-category for noncommutative geometry, enlightening some aspects of its foundations, as well as basic (as well as non basic) facts of noncommutative geometry and KK-theory?

The fact that I've mainly concentrated on pure category-theoretic topics until now does not mean I'm not interested in more concrete applications to algebraic topology/geometry. I've had the opportunity to speak with prof. Ivo dell'Ambrogio about this problem; we both have the feeling that it should be possible to characterize a stable, presentable $(\infty, 1)$-category KK^G exhibiting the universal property of a localization

$$\begin{tikzcd} C^*\text{-Alg} \arrow[r, "w"] \arrow[d, "F"'] & KK^G \arrow[dl, dotted, "\bar{F}"] \\ \mathbf{Ab} & \end{tikzcd}$$

where F is any functor sending suitable G-equivariant homotopy equivalences to isomorphisms.

2.2 Work on abstract factorization systems

Currently, there are several notions of factorization system adapted to various models for higher category theory, this will eventually yield a unified, model-independent description of t-structures arising in stable homotopy theory, homological algebra, algebraic geometry etc.: a t-structure in a DG-category is, for example, the exact counterpart of a normal torsion theory in the world of *enriched* factorization systems (see [**?**, **?**]), whereas a t-structure in a stable model category is a normal torsion theory in the setting of *homotopy* factorization systems (see [**?**, **?**]).

In a recent joint work [**?**] with S. Virili, I managed to apply the torsio-centric approach to the setting of stable derivators, thus showing the analogue of **Result 1** in this framework, and completing a nontrivial part of the model-independency statement:

Result 3

We provide the 2-category of stable derivators with a notion of *factorization system* closing the circle between 1-categorical, $(\infty, 1)$-categorical and triangulated torsion theories). In particular, we offer a formal description as algebras for a *tractable* strict 2-monad.

(in [**?**] we were able Further development of this track of research have recently been obtained in [**?**], where

Result 4

I outlined how reflective sub-prederivators can be equivalently described as objects of fractions, reflective factorization systems, and objects of algebras for idempotent monads.

In [**?**] we managed to prove that when the derivator is stable, a suitable subclass of *coherent* normal torsion theories (the derivator-normal torsion theories) correspond bijectively with t-structures on the underlying category of the derivator, and under relatively mild assumptions induce t-structure on each category $\mathbf{D}(J)$ and t-exact functors thereof. With this result in hand, it is clearly visible a deeper pattern, and an expressive theory of derivators emerges. This led me to investigate deeper the 2-category of derivators with the category theorist's, instead than with the homotopy theorist's eyes.

2.3 Formal category theory of derivators

The above circle of ideas led the to tackle a very specific and challenging problem:

Task 3

Is there a *Yoneda structure* on the 2-category **PDer**, accounting for the possibility to perform all known categorical constructions in it, while at the same time minding of the homotopy-theoretic origin of derivators?

Working towards the solution of this problem led me to study (a lot of) 2-category theory.

Result 5

In [**?**] I sketch the definition of a locally presentable and accessible object of a 2-category with a Yoneda structure. This was a necessary step to establish what a locally presentable/accessible derivator is -the question was raised at the end of [**?**] in order to understand what is a possible form of an *adjoint functor theorem* for derivators.

The ubiquity and usefulness of adjoint functor theorems in category theory is well-known. Having such a basic result in derivator theory would yield a powerful tool to tackle open problems in stable homotopy and algebraic geometry.

Result 6

In [**?**] I prove an equivalence between different frameworks of formal category theory, whenever they are well-behaved enough.

This has been a necessary step towards the determination of a Yoneda structure on derivators, because showing its existence and properties proved to be a very difficult task; the main theorem in this last paper (a result of independent interest in pure category theory) allows us to deduce the existence of a Yoneda structure from an apparently weaker piece of data –a suitably nice pseudomonad playing the rôle of an abstract presheaf construction.

More precisely, in [**?**] we prove an equivalence between cocomplete Yoneda structures and certain proarrow equipments on a 2-category $\mathbf{K}$. In order to do this, we recognize the presheaf construction of a cocomplete Yoneda structure as a relative, lax idempotent monad sending each admissible 1-cell $f : A \to B$ to an adjunction $\mathbb{P}_! f \dashv \mathbb{P}^* f$. Each cocomplete Yoneda structure on $\mathbf{K}$ arises in this way from a relative lax idempotent monad "with enough adjoint 1-cells", whose domain generates the ideal of admissibles, and the Kleisli category of such a monad equips its domain with proarrows. We call these structures "yosegi".

The main conjecture in [**?**, §6] is that the 2-category **PDer** (and its subcategory Der of derivators) possesses a sufficiently rich structure to cast our characterization of accessibility and presentability, at least in some form. The fact that there is a *variable* Yoneda structure on the 2-category of prederivators allowing for such a discussion is the central result of [**?**] (but note that the author there only considers *pseudo*functors –whereas strictness is a customary request working with derivators–, and transformations of mixed variance). This Yoneda structure is *representable* in the sense of [**?**, **?**], and the representing object is the discrete derivator associated to a sufficiently big category **SET** –sufficiently big sets so that $\mathbf{D}(J) \in \mathbf{SET}$.

A moment of reflection shows that this choice shall be discarded for all homotopy-theoretic purposes, as **SET** does not contain enough information to remind the homotopy theoretic origin of $\mathbf{D}$. We conjecture that Street's one is only a member of a complicated web of other Yoneda structures:

- The Renaudin (from [**?**]) Yoneda structure, built taking the derivator generated by the Kan-Quillen model structure on sSet; this is again a representable Yoneda structure, and the canonical map $\mathbf{sSet}_* \to \mathbf{sSet}$ (of the associated derivators, for Kan-Quillen model structures on both categories) classifies discrete opfibrations. sense of [Web07].

- The *Cisinski-Tabuada* Yoneda structure, (following [**?**]; there, the authors prove that for every stable derivator $\mathbf{D}$ there is an action $\mathrm{Sp} \times \mathbf{D} \to \mathbf{D}$ that can be promoted to a two-variable adjunction, using Brown representability). The internal-hom part of this two-variable adjunction now gives a canonical enrichment

  $$\langle \_ \, , = \rangle : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathrm{Sp}$$

  of $\mathbf{D}$ over spectra, hence there's a well defined homotopy class of a spectrum for the image of $X \in \mathbf{D}(J)$ under the Yoneda map: the map $y_{\mathbf{D}} : \mathbf{D} \to [\![\mathbf{D}^{\mathrm{op}}, \mathrm{Sp}]\!]$ has components $y_{\mathbf{D},J} : \mathbf{D}(J) \to \mathbf{PDer}(\mathbf{D}^{\mathrm{op}}, \mathrm{Sp}^J)$ and $y_{\mathbf{D},J}(X)$ sends $X \in \mathbf{D}(J)$ to the morphism of prederivators $\langle \_ , X\rangle$ that sends $Y \in \mathbf{D}^{\mathrm{op}}(I)$ into $\langle \pi_J^* Y, (\pi_{I^{\mathrm{op}}}^*)^{\mathrm{op}} X\rangle$.

2.4 Concreteness and homotopy theory

I am occasionally interested in the foundation of homotopy theory, and in particular in how homotopical algebra relates to set theory: in a joint work with I. di Liberti (Masaryk University, Brno) [**?**]:

Result 7

We propose a fairly general method to show that under certain assumptions on a model category $\mathcal{M}$, its homotopy category $\text{ho}(\mathcal{M})$ cannot be concrete with respect to the universe where it is assumed to be locally small.

This result, echoing and generalizing a famous theorem by P.J. Freyd, will hopefully be the first step towards a theory of *∞-concrete* $(\infty, 1)$-*categories*.

Task 4

In the world of ∞-categories, the notion of concreteness probably 'splits' into a countable spectrum of stronger and stronger *n*-concreteness conditions, in such a way that being *∞-concrete* is the strongest way in which a $(\infty, 1)$-category can stray from being concrete.

This is mildly related to my prior interest in the theory of factorization systems: a rather technical but conceptually deep result proved in [**?**] is that the notion of concreteness for a category **C** is tightly related with a set-theoretical condition of the class of 'generalized subobjects' of objects of **C**; under relatively mild assumptions, this is a condition on monomorphisms of **C**, and this is the starting point to build an analogy between the (classical, 1-categorical) notion of monic in a 1-category and the fact that this notion breaks into a similar countable spectrum of *n-monic arrow* in a $(\infty, 1)$-category.

2.5 Applied Category Theory

With the passing of time I have cultivated a parallel interest in the field of computer science. I am an essentially self-taught programmer, I am eagerly learning the fundamentals of Haskell from Allen-Moronuki's [**?**]. I am also eagerly learning Agda and Idris (see idris-ct to get an idea of why category theory is done better in FP).

Functional programming has notoriously a rather steep learning curve, but my experience in category theory is making the process very fun and smooth (parsers are monads, and what else they could be?). I have already had the opportunity to work with many people on the frontier between category theory and computer science, during the last *Applied Category Theory* running seminar. Pure mathematics taught me to strive for rigor, definiteness and simplicity; among mathematicians, category theory is well-known for its care for efficiency (you want to prove a single theorem once and for all). It is evident how strongly this paradigm resonates with the leading philosophy of FP.

During the ACT seminars, we wanted to reach a better understanding of [**?**], where a certain subcategory of profunctors, equivariant with respect to a monoidal structure on their domain X, arise as the category of algebras of a *promonad* on X. The insight of B. Milewski was that these structures describe what is Haskell is known as a *lens*.

2.6 Functorial semantics

Yet to come