Turtles all the way down

A gentle introduction to Higher Category Theory

Università degli studi di Bologna

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\bullet \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Phi"] & \bullet \arrow[d, equal] \\
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Composition of these "fat" 2-arrows $\Phi \colon f \Rightarrow g$ satisfies relations analogous to the 1-dimensional case:

- **Associativity** of the composition: $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ if $f \xRightarrow{\alpha} g \xRightarrow{\beta} h \xRightarrow{\gamma} k$;
- Existence of an **identity** 2-arrow $1_f \colon f \Rightarrow f$;
- (**interchange law**) 2-cells are 2-dimensional objects: they can be composed horizontally and vertically.

$$
\begin{tikzcd}
N \arrow[r, "A"] \arrow[d, equal] \arrow[dr, phantom, "\Theta"] & M \arrow[r, "C"] \arrow[d, equal] \arrow[dr, phantom, "\Xi"] & P \arrow[d, equal] \\
N \arrow[r, "B"'] \arrow[d, equal] \arrow[dr, phantom, "\Psi"] & M \arrow[r, "D"'] \arrow[d, equal] \arrow[dr, phantom, "\Omega"] & P \arrow[d, equal] \\
N \arrow[r, "E"'] & M \arrow[r, "F"'] & P
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
N \arrow[rr, "CA"] \arrow[d, equal] \arrow[drr, phantom, "\Xi\boxminus\Theta"] & & P \arrow[d, equal] \\
N \arrow[rr, "DB"'] \arrow[d, equal] \arrow[drr, phantom, "\Omega\boxminus\Psi"] & & P \arrow[d, equal] \\
N \arrow[rr, "FE"'] & & P
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
 \arrow[d, mapsto] & \arrow[d, mapsto] \\
 &
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\end{tikzcd}
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(Exercise: compare this with the Eckmann-Hilton argument. They are different!)

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- **Cat** (**the category of all small categories**) is the archetypal example of a 2-category (categories are the dots/objects, functors are lines/1-arrows, and functors can be linked on their own right via natural transformations, 2-arrows);

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- Any (strict) **monoidal category M** can be regarded as a 2-category having a single object $*$, where $\mathbf{M}(*, *)$ consists of the set of objects of **M**, composition of 1-arrows is the tensor product, and 2-arrows are arrows of **M**, composed accordingly.

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- The category of **stacks** on a topological space or a Grothendieck site, the category of all categories **enriched** in a fixed monoidal category **V**, the category of **bimodules** over commutative rings, as well as the categories of all monoids, groups, rings can be regarded as 2-categories: in the bimodule case,

$$ {}_R M_S \colon R \to S, \qquad \varphi \colon {}_R M_S \Rightarrow {}_R N_S. $$

What are we looking at?

What's the deal with a **structure** where

- a set of objects can be connected by **arrows**
- each two of which can be connected by **2-arrows**,
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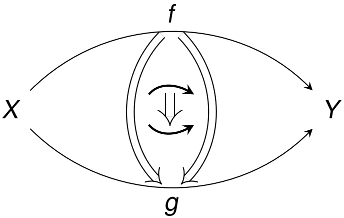

«The Eye was rimmed with fire, but was itself glazed, yellow as a cat's, watchful and intent, and the black slit of its pupil opened on a pit, a window into nothing»

I got 99 problems and n-**Cat** is one

- Which is the right approach? **Internalization** or **enrichment**?

  En. If the category V is symmetric monoidal then V-**Cat** is too, and so the enrichment process can be iterated. In particular, starting with $V_0 = (\mathbf{Sets}, \times)$, we obtain cartesian monoidal categories V_n defined by $V_{n+1} = V_n$-**Cat**. These V_n are the categories n-**Cat** of n-categories and n-functors.

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  - In a **strict 2/double-category** the composition of 1-arrows is associative on the nose: $(fg)h = f(gh)$ (e.g. composition of functors).
  - In a **weak 2/double-category** composition of 1-arrows is associative only up to a specified invertible 2-arrow, and so is the identity map (e.g. homotopy relation between paths).

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- Functors can respect structure on the nose (**strict** 2-functors), up to a specified isomorphism (**weak** 2-functors) or up to a specified 2-arrow which can be non-invertible (**lax** 2-functors).

The coherence problem

4. (Mac Lane's Theorem) Prove that every diagram commutes.

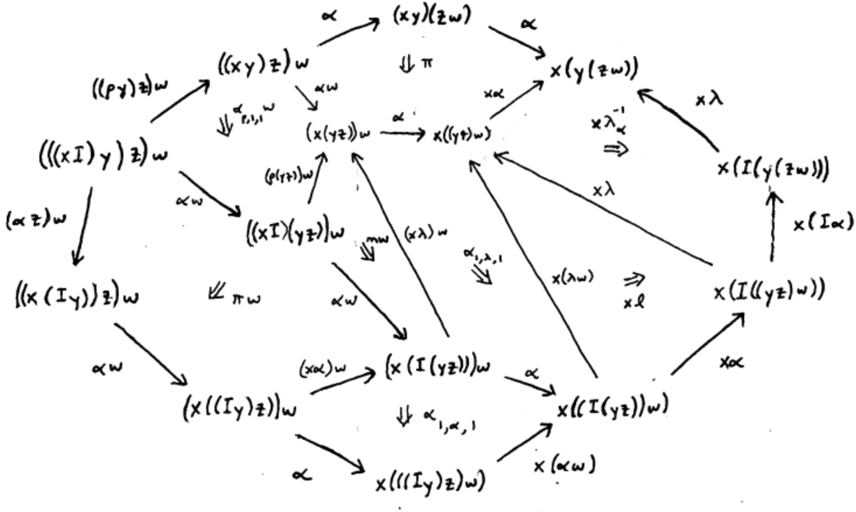
- **Coherence** is widely recognized as the most annoying problem in higher category theory: when composition is defined up to some specified operations, each of which is defined up some other operations, each of which. . . keeping a precise track of all these information is impossible and gives a (computationally) intractable theory, as n grows bigger:

> *The definition of a **tetracategory** (objects with arrows, with arrows, with arrows, with arrows, and every k-cell composes up to a specified $(k-1)$-iso) is **51 pages** long!*

51 pages containing *only* bare, (handwritten in the original version) commutative diagrams (see below) are **too much** even for the most stubborn category-theorist!

We would really like to take a different path.

The coherence problem



The comparison trap: is n=n?

n-cats arose in lots of different areas of mathematics (Algebra, Geometry, Representation Theory, Mathematical Physics, Logic...) for several (apparently) unrelated purposes.

Each of these areas adapted its own definition of higher category to its standards and its particular aim (algebras for a monad, presheaves on some category, 1-categories enriched in $(n-1)$-categories...), so there is a **comparison problem**: how can the red n-category be compared with the blue n-category?

Conceptually you have to use an induction procedure (the collection of all n-categories must be something like an $(n+1)$-category); practically this can't be done, since **induction goes in the wrong direction**:

> *Two different models for n-categories can be compared only if we already know how to compare $(n+1)$-categories.*

In the bigynnyng G. made $\boldsymbol{B}$ and Π_1

Theorem (A bit of Homotopy Hypothesis)

We can find an “equivalence of homotopy theories”

$$\Pi_1 : \mathbf{Top}_1 \leftrightarrows \mathbf{Gpd} : \boldsymbol{B}$$

between topological 1-types and 1-groupoids, realized by the adjunction between the fundamental groupoid and the classifying space.

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- A *topological 1-type* is a space such that $\pi_n(X, x_0) = (0)$ for $n > 1$.
- The fundamental groupoid of a space is the category having objects its points and arrows homotopy classes of paths with fixed endpoints.
- The classifying space $\boldsymbol{B}G$ of a groupoid G is the geometric realization of its nerve NG (compose adjunctions: $\mathbf{Gpd} \underset{N}{\leftrightarrows} \mathbf{sSet} \underset{r}{\leftrightarrows} \mathbf{Top}$)

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A topological n-type is easily defined; Grothendieck [Gro] now claims that, whatever a n-groupoid is, and whatever a $(n+1)$- category is, the $(n+1)$-category of n-groupoids must be **weakly** equivalent to the category of n-types, via an adjunction

$$\Pi_n : \mathbf{Top}_n \leftrightarrows n\text{-}\mathbf{Gpd} : \boldsymbol{B}_n$$

The full Homotopy Hypothesis

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On the other hand Homotopy Theory is a powerful (albeit extremely abstract) tool: accepting the

> *[∞**-categorical Dogma**] Whatever "∞-groupoids" are, they are (weakly) equivalent to the category of ∞-types [i.e. all topological spaces] via an adjunction*
>
> $$\Pi_\infty : \mathbf{Top} \leftrightarrows \infty\text{-}\mathbf{Gpd} : \boldsymbol{B}$$

the "problem of definition" is solved.

Accepting this dogma, as was silently done immediately after Grothendieck's suggestion, means that we accept the Homotopy Hypothesis (**HH**).

Now that we have managed to estimate the difficulty of the problem, we would like to circumvent it by cheating a little.

If an ∞-groupoid is a category with arrows on each level, and all of them are invertible, it's quite easy to find a notation taking into account that an ∞-category has all k-cells invertible for $k > n$. We say that

- An $(\infty, 0)$-category consists of an ∞-groupoid; the homotopy hypotesis amounts to say that each ∞-groupoid "is" a topological space.
- An $(\infty, 1)$-category is an ∞-category having objects, arrows between them (some of them non-invertible), and where all k-arrows, for $2 \leq k \leq \infty$ are invertible; this amounts to say that each $\hom(X, Y)$ is a **space** $\mathrm{Map}(X, Y)$, i.e. an $(\infty, 0)$-category;
- *Mutatis mutandis* to define (∞, r)-categories.

This is the key identification:

> *An $(\infty, 1)$-category is a category* ***enriched*** *over (a model for the homotopy theory of) topological spaces.*

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Localization theory for categories is the exact formal analogue in the world of "many-object-monoids": for $S \subseteq \mathrm{Mor}(\mathbf{C})$,

$$\mathbf{Cat}_{\times,S}(\mathbf{C},\mathbf{D}) \cong \mathbf{Cat}(\mathbf{C}[S^{-1}],\mathbf{D})$$

Theorem

This localization always exists in a suitably large universe ([Gabriel-Zisman], 1967), but in general it is *extremely* difficult to describe explicitly the set $\mathbf{C}[S^{-1}](X,Y)$.

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▸ The other is that GZ-localization becomes far more easier in the presence of fibrations and cofibrations we have an equivalence $\mathbf{C}\left[\mathrm{w\kappa}^{-1}\right] \cong \mathbf{C}_{\mathrm{cf}}/\simeq$, where $\mathbf{C}_{\mathrm{cf}}$ is the subcategory of fibrant-cofibrant objects.

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Two homotopical categories $(\mathbf{C}, \mathrm{w\kappa})$, $(\mathbf{D}, \mathrm{w\kappa}')$ are Quillen equivalent if they are linked by an adjunction

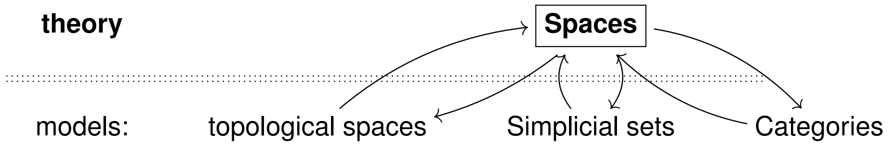
$$F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$$

which becomes a plain categorical equivalence once GZ-localized.

The simmetry hidden in the theory entails that even the class of such homotopical categories has an homotopical structure on its own: **being Quillen equivalent is kind of like being homotopic**. (This is a rather deep result proved in [Rezk]).

So we are allowed to think that two models (i.e. two categories equipped with a class of "weak equivalences") describe the same homotopy theory if they are in the same "homotopy class": this is *the homotopy theory of homotopy theories*.

▸ Among other things, Quillen shows that simplicial sets and (certain) topological spaces really are different models for the same homotopy theory: there is a Quillen equivalence (given by the nerve-realization paradigm) **sSet** $\leftrightarrows$ **Top**. One of the motivation of Grothendieck in writing *Pursuing Stacks* was to classify *all* the models for the homotopy theory of spaces.



In light of the above remarks, it seems reasonable to think that all models for higher-categories will have a homotopical flavour (this acquires a precise meaning, for example exploiting the theory of Grothendieck's derivators).

▸ But there is more! A homotopical flavour is the exact tool needed to solve the red-blue comparison problem. Indeed,

> *Given two models (red and blue) of n-category, suppose we are able to give to the collection of all red n-categories and blue n-categories a model structure; then the red theory is equivalent to the blue theory iff there is a Quillen equivalence*

$$L: n-\mathbf{Cat} \leftrightarrows n-\mathbf{Cat}: R$$

Various equivalent (see [Bergner] - [Camarena]) models have been discovered:

- (Complete) Segal spaces [Rezk], [Dwyer-Kan];
- Segal categories [Rezk];
- Simplicial categories [Bergner];
- Simplicial sets and quasicategories [Boardman-Vogt-Joyal-Lurie];
- Sheaves on framed manifolds (?) [Ayala-Rozenblyum].

At this point two things could have happened:

- We could have developed a complete theory of all these models and differences between them (this seems to have been the idea of the pioneers in the pre-Lurie era) and develop a *model-free* theory of higher categories;
- We could have elected a privileged model, and then completely describe all the technicalities in this explicit language, building higher category theory on the backbone of this theory.

Lurie's **Higher Topos Theory / Higher Algebra** stems from the choice of the second path, and reached a (fairly) complete description of the whole elementary (limits, adjunctions, presentability issues,. . .) and less elementary (monoidal categories, Grothendieck fibrations, operads and homological algebra. . .) category theory. Algebra and Algebraic Geometry.

The pattern followed by Lurie can be easily explained:

- Isolate the notion of classical category theory that you want to generalize (e.g., *monoidal categories*, *limits* or *adjoint functors*);
- Find a way to express that classical notion which uses simplicial sets (e.g.: a limit for $D\colon J \to \mathbf{C}$ is the terminal object in the category of functors $\mathrm{Fun}_D([0] \star J, \mathbf{C})$);
- Generalize the former notion to encode it in the theory of all simplicial sets: given $f\colon J \to C$, what should the terminal object in the simplicial set $n \mapsto \mathbf{sSet}_f([n] \star J, C)$ be? It is the quasicategorical limit of the simplicial map $f\colon J \to C$!
- In the same way adjunctions can be characterized as suitable bicartesian bifibrations, monoidal categories are suitable Grothendieck fibrations, ...
- All these definitions are given in such a way that the higher analogue of a notion induces the classical notion when all higher coherences are modded out (quotienting out hom-wise the ∞-category).

So what?

Is the homotopy hypothesis true? Well, it depends. The desired equivalence

$$n\text{-}\mathbf{Types} \leftrightarrows n\text{-}\mathbf{Gpd}$$

can't be obtained for $n = 3, 4, \dots$ (this is a nontrivial result of Simpson: $\Pi_3 S^2$ can't be equivalent to $\boldsymbol{B}\mathbf{G}$ for a strict 3-groupoid $\mathbf{G}$, and more generally strict n-groupoids can realize via $\boldsymbol{B}(-)$ only homotopy n-types with trivial Whitehead brackets).

Appealing weak structures gives a theory of terrifying complexity, since one has to keep track *during every operation* of the coherence cells in higher order.

In the case $n = \infty$ the HH is accepted as a kind of postulate (the moral rule is: everything behaves as if it was true, and if something is not true without accepting the HH, then simply forget about it); in Lurie's work HH is true simply because an $(\infty, 1)$-category is *defined* as a suitable (homotopy-theory-of-space)-enriched category.

So what?

*Before asking "why should I use **higher** X", you better ask yourself: "why should I use X"? Then, you should use higher X to do the same things you did with X, only **higher**.*

- Higher categories allow the (complete!) classification of n-dimensional Topological Field Theories (but don't tell physicists!);
- They shed a light on various constructions in Homotopy Theory (whose real nature is algebraic) and on various construction in Algebra (whose real nature is homotopical: derived categories); the theory of $(\infty, 0)$-cats is the theory of spaces (up to homotopy).
- They are useful in Logic: two proofs are "equivalent" when they are sort of *homotopic*.
- Again, they are useful in Physics: higher categories help to classify things like formal monoidal deformations of derived categories (but again, don't tell a physicist that D-branes are derived cats!).

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