

Profunctorial Semantics I

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Algebraic structures

A **group** is a set equipped with operations

- $m : G \times G \rightarrow G$
- $i : G \rightarrow G$
- $e : 1 \rightarrow G$

...

you know the drill

Algebraic structures

Theorem (Higman-Neumann 1953)

A **group** is a set equipped with a single binary operation $/ : G \times G \rightarrow G$ subject to the single equation

$$x / (((x/x)/y)/z) / (((x/x)/x)/z) = y$$

for every $x, y, z \in X$.

Well.

This is awkward.

The theory of equationally definable classes of algebras, initiated by Birkhoff in the early thirties, is [...] hampered in its usefulness by two defects. [...] the second is the awkwardness inherent in the presentation of an equationally definable class in terms of operations and equations.

Quite recently, Lawvere, by introducing the notion - closely akin to the clones P. Hall - of an algebraic theory, rectified the second defect.

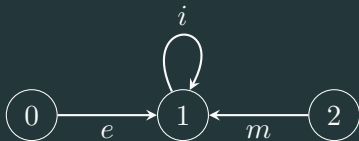
Definition

An **operator domain** is a sequence $\underline{\Omega} = (\Omega_n \mid n \in \mathbb{N})$; the elements of Ω_n are called **operations** of **arity** n .

Definition

An **interpretation** \underline{E} of an operator domain $\underline{\Omega}$ consists of a pair $(E, (f_\omega \mid \omega \in \Omega_n, n \in \mathbb{N}))$ where $f_\omega : E^n \rightarrow E$ is an n -ary operation on the set E called the *carrier* of \underline{E} .

An operator domain can be represented as a (rooted) graph: for example, for groups



Way better to use functors.

A **Lawvere theory** is an identity-on-objects functor $p : \mathbf{Fin}^{\mathbf{o}} \rightarrow \mathcal{L}$ that commutes with finite products.

Unwinding the definition:

- \mathcal{L} is a category with the same objects as \mathbf{Fin} , the category of finite sets and functions;
- p is a functor that acts trivially on objects
- The only thing that can change between \mathbf{Fin} and \mathcal{L} is the number of morphisms $[n] \rightarrow [m]$.

Equivalently: p is a **promonad** on the opposite of \mathbf{Fin} , regarded as an object of the bicategory of profunctors, that preserves the monoidal structure. \mathcal{L} is the Kleisli object of p .

$$\left\{ \begin{array}{l} \text{identity on obj} \\ \text{left adjoints} \\ p: [\mathcal{L}, \mathbf{Set}] \rightarrow [\mathbf{Fin}^{\mathbf{o}}, \mathbf{Set}] \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{l} \text{monads in Prof} \\ p: \mathbf{Fin}^{\mathbf{o}} \rightsquigarrow \mathbf{Fin}^{\mathbf{o}} \end{array} \right\}$$

- The trivial theory is the identity functor $1_{\mathbf{Fin}} : \mathbf{Fin}^{\circ} \rightarrow \mathbf{Fin}^{\circ}$
- Since p preserves products, it is **uniquely determined by its value on $[1]$** . This means that if $p : \mathbf{Fin}^{\circ} \rightarrow \mathcal{L}$ is a Lawvere theory, then every object of \mathcal{L} is X^n if $p[1] = X$.
- The only difference between \mathbf{Fin} and \mathcal{L} is thus the set of morphisms $[n] \rightarrow [m]$.

The theory of groups is generated by

$$\mathcal{L}_{\text{Grp}} =$$

and their compositions/products.

A **model** for a Lawvere theory p is a product-preserving functor $\ell : \mathcal{L} \rightarrow \mathbf{Set}$.

The category $\mathbf{Mod}(p)$ for a Lawvere theory is a full, **reflective** subcategory of the category $[\mathcal{L}, \mathbf{Set}]$ of all functors $\mathcal{L} \rightarrow \mathbf{Set}$.

Theorem

The following conditions are equivalent:

- *ℓ is a model for a Lawvere theory \mathcal{L} ;*
- *The composition $\ell \circ p$ preserves finite products;*
- *The composition $\ell \circ p$ is representable (with respect to the inclusion $J : \mathbf{Fin} \rightarrow \mathbf{Set}$), i.e.*

$$\ell(X[n]) \cong \mathbf{Set}(J[n], A)$$

for some $A \in \mathbf{Set}$.

As a consequence of the previous theorem, the square

$$\begin{array}{ccc}
 \text{Mod}(p) & \xrightarrow{r} & [\mathcal{L}, \text{Set}] \\
 u \downarrow & & \downarrow - \circ X \\
 \text{Set} & \xrightarrow{[J, 1]} & [\text{Fin}^0, \text{Set}]
 \end{array}$$

is a pullback.

- $\text{Mod}(p)$ is a **reflective** subcategory of $[\mathcal{L}, \text{Set}]$. We write $r_! \dashv r$ for the resulting adjunction.
- The functor u is **monadic**, with left adjoint f .
- This sets up a functor

$$\mathfrak{M} : \text{Th}_L(\text{Fin}) \rightarrow \text{Mnd}_{<\omega}(\text{Set})$$

because the monad uf above is finitary.

Proof of monadicity

- **Monadicity of u :** a monadic functor has a left adjoint, reflects isomorphisms, and creates u -split coequalizers (those parallel pairs that u sends to split coequalizers, have a coequalizer, that u preserves).
Apart from the existence of f , all properties are stable under pullback.
- **u commutes with filtered colimits:** it is representable by a finitely presentable object.

$$u(\ell) = \ell[1] \cong [\mathcal{L}, \mathbf{Set}](y[1], \ell)$$

Proof of reflectiveness

The category $\text{Mod}(p)$ is reflective:

A functor $F : \mathcal{L} \rightarrow \text{Set}$ preserves products if and only if it is orthogonal with respect to all σ_{AB} in

$$\begin{array}{ccc} yA \amalg yB & \xrightarrow{\forall t} & F \\ \forall \sigma_{AB} \downarrow & \nearrow & \\ y(A \times B) & & \end{array}$$

Theorem

Let \mathcal{E} be a locally presentable category and $\Sigma \subset \text{hom}(\mathcal{E})$ a set of morphism with (finitely) presentable domain; then the subcategory of Σ -orthogonal object is always reflective and (finitely) accessibly embedded.

Proof of monadicity

- Being conservative is stable under pullback: conservative functors are a right orthogonal class, precisely ι^\perp where $\iota : \{0 \rightarrow 1\} \rightarrow \{0 \cong 1\}$.
- Creating coequalizers of u -split pairs is stable under pullback:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{s} & \mathcal{B} \\ u \downarrow & & \downarrow p^* \\ \mathcal{C} & \xrightarrow[t]{} & \mathcal{L} \end{array}$$

if p^* creates them, so does u .

- Every inverse image is monadic.

$$\mathrm{Th}_L(\mathbf{Fin}) \cong \mathrm{Mnd}_{<\omega}(\mathbf{Set})$$

Construct a functor in the opposite direction,

$$\exists : \mathrm{Mnd}_{<\omega}(\mathbf{Set}) \rightarrow \mathrm{Th}_L(\mathbf{Fin});$$

given T , we consider the composition $\mathbf{Fin} \hookrightarrow \mathbf{Set} \xrightarrow{F^T} \mathbf{Set}^T$
and its bo-ff factorization,

$$\begin{array}{ccc} \mathcal{L}^0 & \xrightarrow{ff} & \mathbf{Set}^T \\ \uparrow b & & \uparrow F^T \\ \mathbf{Fin} & \xrightarrow{J} & \mathbf{Set} \end{array}$$

- the left vertical arrow is a Lawvere theory.

- $\mathbf{Set}^T \cong \mathcal{L}\text{-models}$:

$$\begin{array}{ccc} \mathbf{Set}^T & \xrightarrow{\quad} & [\mathcal{L}, \mathbf{Set}] \\ \downarrow & \lrcorner & \downarrow [b^0, \mathbf{Set}] \\ \mathbf{Set} & \xrightarrow{[J, 1]} & [\mathbf{Fin}^0, \mathbf{Set}] \end{array}$$

Theories as promonads

There is a 2-monad $\tilde{S} : \mathbf{Prof} \rightarrow \mathbf{Prof}$ whose algebras are exactly promonoidal categories.

Given a profunctor $p : \mathcal{A} \rightsquigarrow \mathcal{B}$ between promonoidal categories $(\mathcal{A}, \mathfrak{P}, J_A), (\mathcal{B}, \mathfrak{Q}, J_B)$:

- p is a **pseudo- \tilde{S} -algebra** morphism;
- The cocontinuous left adjoint \hat{p} associated to p is **strong monoidal** with respect to the convolution monoidal product on presheaf categories;

If $\mathfrak{P}, \mathfrak{Q}$ on \mathcal{A}, \mathcal{B} are representable then

- Both mates $p^{\triangleleft} : \mathcal{A} \rightarrow P\mathcal{B}$ and $p^{\triangle} : \mathcal{B} \rightarrow P^*\mathcal{A}$ are **strong monoidal** wrt convolution on their codomains.

Theories as $[\mathbf{Fin}, \mathbf{Set}]$ -categories

Theorem

$$[\mathbf{Fin}, \mathbf{Set}] \cong \mathbf{End}_{<\omega}(\mathbf{Set})$$

Proof: use Yoneda lemma.

Equivalence is monoidal; the \circ -transported structure is called the **substitution** monoidal product of functors

$F, G : \mathbf{Fin} \rightarrow \mathbf{Set}$:

$$F \ominus G : m \mapsto \int^n F n \times (G m)^n$$

Substitution is (highly!) non-symmetric, right closed monoidal structure (not left closed).

The category $[\mathbf{Fin}, \mathbf{Set}]$ works as base of enrichment.

Theories as $[\mathbf{Fin}, \mathbf{Set}]$ -categories

From [Garner]

From now on we blur the distinction between the categories $[\mathbf{Fin}, \mathbf{Set}] \cong \mathbf{End}_{<\omega}(\mathbf{Set}) = \mathcal{W}$:

- A **finitary monad** is a monoid in \mathcal{W} , i.e. a \mathcal{W} -category with a single object;
- A Lawvere theory is a \mathcal{W} -category that is **absolute** (=Cauchy-, =Karoubi-)**complete** as an enriched category and generated by a single object.

Lawvere theories form a reflective subcategory in finitary monads; reflection is the enriched **Cauchy completion** functor.

Theories as \mathcal{W} -categories

In this perspective there is no difference between a Lawvere theory and its associated monad: they are the very same thing, up to a Cauchy-completion operation.

(The Cauchy completion of a monoid in \mathbf{Cat} is rarely a monoid: take the “generic idempotent” $M = \{1, e\}$ and split $e : * \rightarrow *$ as $r : 0 \hookrightarrow * : s$).

In order to add all \mathcal{W} -absolute colimits, at least all tensors $y[n] \odot X$ must be added to the single object X .

Theories as \mathcal{W} -categories

Equivalently,

- A Lawvere \mathcal{W} -category is an enriched category where every object A is the tensor $y[n] \odot X$ for a distinguished object $X \cong y[1] \odot X$. All such categories are \mathcal{W} -absolute complete.
- A \mathcal{W} -category is a special kind of **cartesian multicategory**: one where a multimorphism $f : X_1 \dots X_n \rightarrow Y$ is such that $X_1 = X_2 = \dots = X_n$.

Generalisations/extensions:

- let \mathbb{N} be the **discrete** category over natural numbers;
- let \mathbf{P} be the **groupoid** of natural numbers;

The categories $[\mathbb{N}, \mathbf{Set}]$ and $[\mathbf{P}, \mathbf{Set}]$ become monoidal with respect to substitution products \ominus_N, \ominus_P :

$$F \ominus_N G : n \mapsto \coprod_{k \in \mathbb{N}} G_k \times \coprod_{\vec{n} : \sum n_i = n} X_{n_1} \times \cdots \times X_{n_k}$$

$$F \ominus_P G : n \mapsto \int^{k, \vec{n}} Y_k \times X_{n_1} \times \cdots \times X_{n_k} \times \mathbf{P}(\sum n_i, n)$$

PRO(P)S

\ominus_N and \ominus_P -monoids are respectively non-symmetric and symmetric operads.

- A **PRO** is an identity-on-objects strong monoidal functor $p : \mathbb{N}^0 \rightarrow \mathcal{P}$. \mathcal{P} is possibly non-cartesian.
- A **PROP** is an identity-on-objects strong monoidal functor $p : \mathbb{N}^0 \rightarrow \mathcal{P}$. \mathcal{P} is symmetric monoidal.

Still examples of promonoidal promonads and symmetric promonoidal promonads.

PRO(P)s and operads

Every PRO $p : \mathbb{N}^0 \rightarrow \mathcal{T}$ gives rise to the operad $O(\mathcal{T}) = (\mathcal{T}(n, 1) \mid n \in \mathbb{N})$.

Conversely, any operad $(\mathcal{O}(n) \mid n \in \mathbb{N})$ gives rise to a pro $T(\mathcal{O})$, where

$$T(\mathcal{O})(n, m) = \coprod_{k_1 + \dots + k_m = n} \mathcal{O}(k_1) \times \dots \times \mathcal{O}(k_m).$$

(It would be helpful to imagine a picture of m trees stacked vertically.)

If we begin with an operad \mathcal{O} , we have $\mathcal{O} = O(T(\mathcal{O}))$. (This is because $T(\mathcal{O})(n, 1) = \mathcal{O}(n)$, according to the above formula.)

On the other hand, if we start with a PRO \mathcal{T} , then there exists a canonical map of PROs $T(O(\mathcal{T})) \rightarrow \mathcal{T}$, given by, for each n and m , a canonical function

$$\coprod_{k_1 + \dots + k_m = n} \mathcal{T}(k_1, 1) \times \dots \times \mathcal{T}(k_m, 1) \rightarrow \mathcal{T}(n, m) \quad (\star)$$

induced from the monoidal product on \mathcal{T} .

This sets up an adjunction

$$T : \mathbf{Opd}[\mathbf{S}] \rightleftarrows \mathbf{PRO}[\mathbf{P}] : O$$

with fully faithful left adjoint, so that [symmetric] operads can be regarded as a PRO[P]s \mathcal{T} such that each function (\star) is bijective.

The evil plan

Re-enact [Garner] away from \mathbf{Set} .

Let \mathcal{V} be a locally presentable base of enrichment; let $\mathfrak{F}(\mathcal{V})$ be the subcategory of finitely presentable objects:

- $\mathfrak{F}(\mathcal{V})$ is the free **finite weighted** cocompletion of the point;
- There is a strong monoidal equivalence of categories

$$[\mathfrak{F}(\mathcal{V}), \mathcal{V}] \cong [\mathcal{V}, \mathcal{V}]_{<\omega}$$

between functors $\mathfrak{F}(\mathcal{V}) \rightarrow \mathcal{V}$ and finitary endo- \mathcal{V} -functors;

The evil plan

- \mathcal{V} -substitution is

$$F * G = A \mapsto \int^B FB \otimes_{\mathcal{V}} (GA)^B \leftarrow \mathcal{V}\text{-power}$$

- Equivalence between **finitary \mathcal{V} -monads** and **enriched-Cauchy-complete categories generated by a single object** under iterated finite powers.
- **Models** for a Lawvere theory correspond to **algebras** for the associated finitary monad; free models are free algebras are representables in

$$\begin{aligned}\mathrm{Alg}(T, \mathcal{C}) &= [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-Cat}(T, \mathcal{C}) \\ (\text{Cauchy compl.}) &\cong [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-Cat}(\hat{T}, \mathcal{C}) \\ &= \mathrm{Mod}(\hat{T}, \mathcal{C})\end{aligned}$$

The evil plan

class of lims	finite \times	\mathbb{D} -lims	finite powers	weighted \mathbb{D} -limits	bicat \times
basic theory	\mathbf{Fin}^0	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$
semantics in	Set	Set	\mathcal{V}	\mathcal{V}	Prof
eq. with _ monads	finitary	\mathbb{D} -accessible	$[\mathfrak{F}(V), V]$ -monoids	$[?, V]$ -monoids	???

Profunctorial semantics

- Characterise the free carbicat $\mathbb{CB}(\ast)$ on a singleton: see [link here](#));
- Check if the univ property of \mathbf{Fin} remains true for $\mathbb{CB}(\ast)$;
- Take $\mathbb{CB}(\ast) = F$, and consider its **free cocompletion** in the bicolimit sense
- Prove that

$$\begin{aligned}[PF, PF] &\cong [\mathbb{CB}(\ast), PF] \\ &\cong PF\end{aligned}$$

monoidally; \odot -**monoids** := monoids in PF wrt composition in $[PF, PF]$.

Profunctorial semantics

- Prove that there is a **syntax-VS-semantics** adjunction here: theories are promonoidal promonads T on (a 1-skeleton of) $\mathbb{CB}(\ast)$, and models are carbicat homomorphisms $\mathbf{Kl}(T) \rightarrow \mathbf{Prof}$. There is an equivalence

$$\{\text{theories}\} \cong \{\text{??? monads}\}$$

- Let PROs come into play: analogue of the adjunction between PROs and operads.

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