Profunctorial Semantics I

Fosco Loregian

March 21, 2020

Algebraic structures

A group is a set equipped with operations

- $m : G \times G \to G$
- $i : G \to G$
- $e : 1 \to G$

...

you know the drill

Algebraic structures

Theorem (Higman-Neumann 1953)
A group is a set equipped with a single binary operation $/ : G \times G \to G$ subject to the single equation

$$x/((((x/x)/y)/z)/(((x/x)/x)/z)) = y$$

for every $x, y, z \in X$.

Well.

This is awkward.

The theory of equationally definable classes of algebras, initiated by Birkhoff in the early thirties, is [...] hampered in its usefulness by two defects. [...T]he second is the awkwardness inherent in the presentation of an equationally definable class in terms of operations and equations.

Quite recently, Lawvere, by introducing the notion - closely akin to the clones P. Hall - of an algebraic theory, rectified the second defect.

Definition
An operator domain is a sequence $\underline{\Omega} = (\Omega_n \mid n \in \mathbb{N})$; the elements of Ω_n are called operations of arity n.

Definition
An interpretation $\underline{E}$ of an operator domain $\underline{\Omega}$ consists of a pair $(E, (f_\omega \mid \omega \in \Omega_n, n \in \mathbb{N}))$ where $f_\omega : E^n \to E$ is an n-ary operation on the set E called the *carrier* of $\underline{E}$.

An operator domain can be represented as a (rooted) graph: for example, for groups

$$\begin{tikzcd} 0 \arrow[r, "e"'] & 1 \arrow[loop above, "i"] & 2 \arrow[l, "m"] \end{tikzcd}$$

Way better to use functors.

A Lawvere theory is an identity-on-objects functor $p : \mathsf{Fin}^{\mathsf{o}} \to \mathcal{L}$ that commutes with finite products.

Unwinding the definition:

- $\mathcal{L}$ is a category with the same objects as Fin, the category of finite sets and functions;
- p is a functor that acts trivially on objects
- The only thing that can change between Fin and $\mathcal{L}$ is the number of morphisms $[n] \to [m]$.

Equivalently: p is a promonad on the opposite of Fin, regarded as an object of the bicategory of profunctors, that preserves the monoidal structure. $\mathcal{L}$ is the Kleisli object of p.

$$\left\{ \begin{array}{c} \text{identity on obj} \\ \text{left adjoints} \\ p{:}[\mathcal{L},\mathsf{Set}]{\to}[\mathsf{Fin}^{\mathsf{o}},\mathsf{Set}] \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{monads in Prof} \\ p{:}\mathsf{Fin}^{\mathsf{o}}{\rightsquigarrow}\mathsf{Fin}^{\mathsf{o}} \end{array} \right\}$$

- The trivial theory is the identity funtor $1_{\mathsf{Fin}} : \mathsf{Fin}^{\mathsf{o}} \to \mathsf{Fin}^{\mathsf{o}}$
- Since p preserves products, it is uniquely determined by its value on $[1]$. This means that if $p : \mathsf{Fin}^{\mathsf{o}} \to \mathcal{L}$ is a Lawvere theory, then every object of $\mathcal{L}$ is X^n if $p[1] = X$.
- The only difference between Fin and $\mathcal{L}$ is thus the set of morphisms $[n] \to [m]$.

The theory of groups is generated by

$$\mathcal{L}_{\mathsf{Grp}} = \begin{tikzcd} {[0]} \arrow[r, "e"'] & {[1]} \arrow[loop above, "i"] & {[2]} \arrow[l, "m"] \end{tikzcd}$$

and their compositions/products.

A model for a Lawvere theory p is a product-preserving functor $\ell : \mathcal{L} \to \mathsf{Set}$.

The category $\mathsf{Mod}(p)$ for a Lawvere theory is a full, reflective subcategory of the category $[\mathcal{L}, \mathsf{Set}]$ of all functors $\mathcal{L} \to \mathsf{Set}$.

Theorem
The following conditions are equivalent:

- *ℓ is a model for a Lawvere theory $\mathcal{L}$;*
- *The composition $\ell \circ p$ preserves finite products;*
- *The composition $\ell \circ p$ is representable (with respect to the inclusion J : Fin $\to$ Set), i.e.*

$$\ell(X[n]) \cong \mathit{Set}(J[n], A)$$

for some $A \in$ Set.

Proof of reflectiveness

The category Mod(p) is reflective:

A functor $F : \mathcal{L} \to$ Set preserves products if and only if it is orthogonal with respect to all σ_{AB} in

$$\begin{tikzcd}
yA \amalg yB \arrow[r, "\forall t"] \arrow[d, "\forall \sigma_{AB}"'] & F \\
y(A \times B) \arrow[ur, dotted]
\end{tikzcd}$$

Theorem
Let $\mathcal{E}$ be a locally presentable category and $\Sigma \subset \hom(\mathcal{E})$ a set of morphism with (finitely) presentable domain; then the subcategory of Σ-orthogonal object is always reflective and (finitely) accessibly embedded.

As a consequence of the previous theorem, the square

$$\begin{tikzcd} \mathsf{Mod}(p) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "\_\circ X"] \\ \mathsf{Set} \arrow[r, "{[J,1]}"'] & {[\mathsf{Fin}^{\mathsf{o}}, \mathsf{Set}]} \end{tikzcd}$$

is a pullback.

- $\mathsf{Mod}(p)$ is a reflective subcategory of $[\mathcal{L}, \mathsf{Set}]$. We write $r_! \dashv r$ for the resulting adjunction.
- The functor u is monadic, with left adjoint f.
- This sets up a functor

$$\mathfrak{M} : \mathsf{Th}_L(\mathsf{Fin}) \to \mathsf{Mnd}_{<\omega}(\mathsf{Set})$$

because the monad uf above is finitary.

Proof of monadicity

- Monadicity of u: a monadic functor has a left adjoint, reflects isomorphisms, and creates u-split coequalizers (those parallel pairs that u sends to split coequalizers, have a coequalizer, that u preserves).
  Apart from the existence of f, all properties are stable under pullback.
- u commutes with filtered colimits: it is representable by a finitely presentable object.

$$u(\ell) = \ell[1] \cong [\mathcal{L}, \mathsf{Set}](y[1], \ell)$$

Proof of monadicity

- Being conservative is stable under pullback: conservative functors are a right orthogonal class, precisely $\iota^\perp$ where $\iota : \{0 \to 1\} \to \{0 \cong 1\}$.
- Creating coequalizers of u-split pairs is stable under pullback:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "s"] \arrow[d, "u"'] & \mathcal{B} \arrow[d, "p^*"] \\ \mathcal{C} \arrow[r, "t"'] & \mathcal{L} \end{tikzcd}$$

  if p^* creates them, so does u.
- Every inverse image is monadic.

$\mathsf{Th}_L(\mathbf{Fin}) \cong \mathsf{Mnd}_{<\omega}(\mathbf{Set})$

Construct a functor in the opposite direction,

$$\mathfrak{Z} : \mathsf{Mnd}_{<\omega}(\mathsf{Set}) \to \mathsf{Th}_L(\mathsf{Fin});$$

given T, we consider the composition $\mathsf{Fin} \hookrightarrow \mathsf{Set} \xrightarrow{F^T} \mathsf{Set}^T$ and its bo-ff factorization,

$$\begin{tikzcd}
\mathcal{L}^{\mathsf{o}} \arrow[r, "ff"] & \mathsf{Set}^T \\
\mathsf{Fin} \arrow[u, "b"] \arrow[r, "J"'] & \mathsf{Set} \arrow[u, "F^T"']
\end{tikzcd}$$

- the left vertical arrow is a Lawvere theory.
- $\mathsf{Set}^T \cong \mathcal{L}$-models:

$$\begin{tikzcd}
\mathsf{Set}^T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "{[b^{\mathsf{o}}, \mathsf{Set}]}"] \\
\mathsf{Set} \arrow[r, "{[J, 1]}"'] & {[\mathsf{Fin}^{\mathsf{o}}, \mathsf{Set}]}
\end{tikzcd} .$$

Theories as promonads

There is a 2-monad $\tilde{S} : \mathsf{Prof} \to \mathsf{Prof}$ whose algebras are exactly promonoidal categories.

Given a profunctor $p : \mathcal{A} \rightsquigarrow \mathcal{B}$ between promonoidal categories $(\mathcal{A}, \mathfrak{P}, J_A), (\mathcal{B}, \mathfrak{Q}, J_B)$:

- p is a pseudo-$\tilde{S}$-algebra morphism;
- The cocontinuous left adjoint $\hat{p}$ associated to p is strong monoidal with respect to the convolution monoidal product on presheaf categories;

If $\mathfrak{P}, \mathfrak{Q}$ on $\mathcal{A}, \mathcal{B}$ are representable then

- Both mates $p^{\lhd} : \mathcal{A} \to P\mathcal{B}$ che $p^{\rhd} : \mathcal{B} \to P^*\mathcal{A}$ are strong monoidal wrt convolution on their codomains.

Theories as $[\mathsf{Fin}, \mathsf{Set}]$-categories

Theorem

$$[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set})$$

Proof: use Yoneda lemma.

Equivalence is monoidal; the $\circ$-transported structure is called the substitution monoidal product of functors $F, G : \mathsf{Fin} \to \mathsf{Set}$:

$$F \ominus G : m \mapsto \int^n Fn \times (Gm)^n$$

Substitution is (highly!) non-symmetric, right closed monoidal structure (not left closed).

The category $[\mathsf{Fin}, \mathsf{Set}]$ works as base of enrichment.

Theories as $[\mathbf{Fin}, \mathbf{Set}]$-categories

From [Garner]

From now on we blur the distinction between the categories $[\mathrm{Fin}, \mathrm{Set}] \cong \mathrm{End}_{<\omega}(\mathrm{Set}) = \mathcal{W}$:

- A finitary monad is a monoid in $\mathcal{W}$, i.e. a $\mathcal{W}$-category with a single object;
- A Lawvere theory is a $\mathcal{W}$-category that is absolute (=Cauchy-, =Karoubi-)complete as an enriched category and generated by a single object.

Lawvere theories form a reflective subcategory in finitary monads; reflection is the enriched Cauchy completion functor.

Theories as $\mathcal{W}$-categories

In this perspective there is no difference between a Lawvere theory and its associated monad: they are the very same thing, up to a Cauchy-completion operation.

(The Cauchy completion of a monoid in Cat is rarely a monoid: take the “generic idempotent” $M = \{1, e\}$ and split $e : * \to *$ as $r : 0 \leftrightarrows * : s$).

In order to add all $\mathcal{W}$-absolute colimits, at least all tensors $y[n] \odot X$ must be added to the single object X.

Theories as $\mathcal{W}$-categories

Equivalently,

- A Lawvere $\mathcal{W}$-category is an enriched category where every object A is the tensor $y[n] \odot X$ for a distinguished object $X \cong y[1] \odot X$. All such categories are $\mathcal{W}$-absolute complete.
- A $\mathcal{W}$-category is a special kind of cartesian multicategory: one where a multimorphism $f : X_1 \dots X_n \to Y$ is such that $X_1 = X_2 = \cdots = X_n$.

Generalisations/extensions:

- let $\mathbb{N}$ be the discrete category over natural numbers;
- let $\mathbf{P}$ be the groupoid of natural numbers;

The categories $[\mathbb{N}, \mathsf{Set}]$ and $[\mathbf{P}, \mathsf{Set}]$ become monoidal with respect to substitution products $\ominus_N, \ominus_P$:

$$F \ominus_N G : n \mapsto \coprod_{k \in \mathbb{N}} G_k \times \coprod_{\vec{n} : \sum n_i = n} X_{n_1} \times \cdots \times X_{n_k}$$

$$F \ominus_P G : n \mapsto \int^{k,\vec{n}} Y_k \times X_{n_1} \times \cdots \times X_{n_k} \times \mathbf{P}\big(\sum n_i, n\big)$$

PRO(P)S

$\ominus_N$ and $\ominus_P$-monoids are respectively non-symmetric and symmetric operads.

- A PRO is an identity-on-objects strong monoidal functor $p : \mathbb{N}^{\mathsf{o}} \to \mathcal{P}$. $\mathcal{P}$ is possibly non-cartesian.
- A PROP is an identity-on-objects strong monoidal functor $p : \mathbb{N}^{\mathsf{o}} \to \mathcal{P}$. $\mathcal{P}$ is symmetric monoidal.

Still examples of promonoidal promonads and symmetric promonoidal promonads.

PRO(P)s and operads

Every PRO $p : \mathbb{N}^{\mathrm{o}} \to \mathcal{T}$ gives rise to the operad $O(\mathcal{T}) = (\mathcal{T}(n,1) \mid n \in \mathbb{N})$.

Conversely, any operad $(\mathcal{O}(n) \mid n \in \mathbb{N})$ gives rise to a pro $T(\mathcal{O})$, where

$$T(\mathcal{O})(n,m) = \coprod_{k_1+\ldots+k_m=n} \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_m).$$

(It would be helpful to imagine a picture of m trees stacked vertically.)

If we begin with an operad $\mathcal{O}$, we have $\mathcal{O} = O(T(\mathcal{O}))$. (This is because $T(\mathcal{O})(n,1) = \mathcal{O}(n)$, according to the above formula.)

On the other hand, if we start with a PRO $\mathcal{T}$, then there exists a canonical map of PROs $T(O(\mathcal{T})) \to \mathcal{T}$, given by, for each n and m, a canonical function

$$\coprod_{k_1+\cdots+k_m=n} \mathcal{T}(k_1,1) \times \cdots \times \mathcal{T}(k_m,1) \to \mathcal{T}(n,m) \qquad (\star)$$

induced from the monoidal product on $\mathcal{T}$.

This sets up an adjunction

$$T : \mathsf{Opd}[\mathsf{S}] \leftrightarrows \mathsf{PRO}[\mathsf{P}] : O$$

with fully faithful left adjoint, so that [symmetric] operads can be regarded as a PRO[P]s $\mathcal{T}$ such that each function (*) is bijective.

The evil plan

Re-enact [Garner] away from Set.

Let $\mathcal{V}$ be a locally presentable base of enrichment; let $\mathfrak{F}(\mathcal{V})$ be the subcategory of finitely presentable objects:

- $\mathfrak{F}(\mathcal{V})$ is the free finite weighted cocompletion of the point;
- There is a strong monoidal equivalence of categories

$$[\mathfrak{F}(\mathcal{V}), \mathcal{V}] \cong [\mathcal{V}, \mathcal{V}]_{<\omega}$$

  between functors $\mathfrak{F}(\mathcal{V}) \to \mathcal{V}$ and finitary endo-$\mathcal{V}$-functors;

The evil plan

- $\mathcal{V}$-substitution is
$$F * G = A \mapsto \int^{B} FB \otimes_{\mathcal{V}} (GA)^{B} \leftarrow \mathcal{V}\text{-power}$$
- Equivalence between finitary $\mathcal{V}$-monads and enriched-Cauchy-complete categories generated by a single object under iterated finite powers.
- Models for a Lawvere theory correspond to algebras for the associated finitary monad; free models are free agebras are representables in

$$\begin{aligned} \mathsf{Alg}(T, \mathcal{C}) &= [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-}\mathsf{Cat}(T, \mathcal{C}) \\ (\text{Cauchy compl.}) &\cong [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-}\mathsf{Cat}(\hat{T}, \mathcal{C}) \\ &= \mathsf{Mod}(\hat{T}, \mathcal{C}) \end{aligned}$$

The evil plan

class of lims	finite $\times$	$\mathbb{D}$-limts	finite powers	weighted $\mathbb{D}$-limits	bicat $\times$
basic theory	Fin^o	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$	completion of $\{*\}$
semantics in	Set	Set	$\mathcal{V}$	$\mathcal{V}$	Prof
eq. with _ monads	finitary	$\mathbb{D}$-accessible	$[\mathfrak{F}(V), V]$-monoids	$[?, V]$-monoids	???

Profunctorial semantics

- Characterise the free carbicat $\mathbb{CB}(*)$ on a singleton: see link here);
- Check if the univ property of Fin remains true for $\mathbb{CB}(*)$;
- Take $\mathbb{CB}(*) = F$, and consider its free cocompletion in the bicolimit sense
- Prove that

$$\begin{aligned}[PF, PF] &\cong [\mathbb{CB}(*), PF] \\ &\cong PF\end{aligned}$$

monoidally; $\odot$-monoids := monoids in PF wrt composition in $[PF, PF]$.

Profunctorial semantics

- Prove that there is a syntax-VS-semantics adjunction here: theories are promonoidal promonads T on (a 1-skeleton of) $\mathbb{CB}(*)$, and models are carbicat homomorphisms $\mathsf{Kl}(T) \to \mathsf{Prof}$. There is an equivalence

$$\{\text{theories}\} \cong \{\text{??? monads}\}$$

- Let PROs come into play: analogue of the adjunction between PROs and operads.

Bibliography

- Lawvere, F. William. "Functorial semantics of algebraic theories." Proceedings of the National Academy of Sciences of the United States of America 50.5 (1963): 869.
- Linton, Fred EJ. "Some aspects of equational categories." Proceedings of the Conference on Categorical Algebra. Springer, Berlin, Heidelberg, 1966.
- Garner, Richard. "Lawvere theories, finitary monads and Cauchy-completion." Journal of Pure and Applied Algebra 218.11 (2014): 1973-1988.
- Nishizawa, Koki, and John Power. "Lawvere theories enriched over a general base." Journal of Pure and Applied Algebra 213.3 (2009): 377-386.
- Hyland, Martin, and John Power. "The category theoretic understanding of universal algebra: Lawvere theories and monads." Electronic Notes in Theoretical Computer Science 172 (2007): 437-458.