Differential 2-rigs

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Motivations

Galois theory asserts that to each polynomial equation one can attach a finite group, the Galois group of the polynomial, in such a way that the roots of a polynomial can be found via algebraic operations and root extraction if and only if its Galois group is solvable.

A group G is solvable if there exists a chain of subgroups $1 \leq G_1 \leq \cdots \leq G_n \leq G$ with the property that each G_i is normal in G_{i+1}, and each quotient G_{i+1}/G_i is abelian.

Soon after Galois' tragic death people started wondering if a similar theorem could be attained for differential equations. The work of Lie and Liouville goes precisely in the direction of establishing such conditions.

Turns out that this is possible, paying a price:

- the group attached to a differential equation is not finite any more;
- it has a non-trivial topology, and the "correct" subgroups to consider are the closed ones;
- sometimes, such groups are algebraic manifolds of infinite dimension.

What we now call Lie groups was nothing but the attempt by S. Lie to classify the objects arising as Galois groups of differential equations.

The theory of reductive algebraic groups arose as a way to understand the operation of adding a solution to $y' = y$ to the ring of polynomials. (=exponential elements; they live in rings of power series).

A differential algebra over k is a k-algebra R endowed with a endomorphism $d : R \to R$ that is k-linear, and satisfies the Leibniz rule:

$$d(a.b) = da.b + a.db$$

A differential equation in R is an equation of the form $F(y, y^{(1)}, y^{(2)}, \ldots) = 0$ where F is a polynomial with coefficients in R, and $y^{(1)} := dy$, $y^{(n)} = d(y^{(n-1)})$.

A differential extension of R is a bigger differential k-algebra $F \supset R$ obtained from R by adding solutions to differential equations;

(Liouville) Solvability of the Galois group of a differential extension $F \supset R$ allows the possibility to solve differential equations in R, finding their solutions in F, by means of 'elementary operations':

- Ring operations
- addition of integrals (=solutions to $y' = a$, $a \in R$)
- addition of exponentials (=solutions to $y' = by$, $b \in R$).

Slogan
Abstracting the theory of differential rings is a secure path to unexpectedly profound mathematics.

Guided by this, the plan is to transform rings into (rig) categories and endomorphisms $d : R \to R$ into functors $\partial : \mathcal{R} \to \mathcal{R}$, and the Leibniz condition

$$\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$$

into an isomorphism natural in A, B.

Rings

Definition

A 2-rig is a category $\mathcal{C}$ such that

- $\mathcal{C}$ admits finite coproducts, denoted $A \cup B$;
- $\mathcal{C}$ admits a monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is bilinear, i.e. the functors $A \otimes -$ and $- \otimes B$ commute with coproducts.

In simple terms, we want to capture a notion that categorifies rigs (rings without additive inverses).

A 2-rig is a particular instance of a more general notion, first introduced by Laplaza: a category $\mathcal{C}$ with two monoidal structures $\otimes, \oplus$, such that '$\otimes$ distributes over $\oplus$'.

As natural as his axiomatics may seem, the precise formalisation of a coherence theorem for a distributive category requires a lot of effort and numerous diagrams: see Laplaza for the precise definition.

Many of our examples will also satisfy additional assumptions:

- A commutative 2-rig is a category $\mathcal{C}$, where $\otimes$ is also symmetric;
- A monoidally cocomplete 2-rig is a category $\mathcal{C}$, which moreover admits all small colimits, and such that $A \otimes -$ and $- \otimes B$ distribute over all colimits;
- A closed 2-rig is a category $\mathcal{C}$ such that each $A \otimes -$ and $- \otimes B$ have right adjoints; in this case, of course, they preserve all colimits that exist in $\mathcal{C}$.

Combinations are possible: it is clear what a commutative closed 2-rig is.

The following are examples of 2-rigs:

- The category $(\mathrm{Set}, \times, 1)$ of sets and functions is a commutative closed 2-rig; more generally, all cartesian closed categories with coproducts $(\mathcal{A}, \times, 1)$ are bicartesian categories.
- The category $(\mathrm{Mod}_R, \otimes, R)$ of modules over a ring R is a commutative closed 2-rig.
- The category of (real or complex) topological vector bundles over a topological space X, equipped with the tensor product of vector bundles is a 2-rig (where $\cup$ is the direct sum of vector bundles taking the bundle associated with fiberwise Vect-coproduct).

- Given a monoidal category $(\mathcal{A}, \oplus, j)$ the category $([\mathcal{A}^{\mathrm{op}}, \mathrm{Set}], *, yj = \mathcal{A}(j, -))$ of presheaves over $\mathcal{A}$ endowed with the Day convolution monoidal structure

$$F * G := \int^{U,V\in\mathcal{A}} FU \times GV \times \mathcal{A}(U \oplus V, -)$$

  is a closed 2-rig.

Note that $[\mathcal{A}^{\mathrm{op}}, \mathrm{Set}]$ is closed no matter what $\oplus$ is, and the internal hom can be computed as

$$\{G, H\} : A \mapsto \int_X \mathrm{Set}(GX, H(A \oplus X))$$

Derivations

Definition (Derivation on a 2-rig)

A derivation on a 2-rig is a functor $\partial : \mathcal{C} \to \mathcal{C}$ having the following properties:

- $\partial(A \cup B) \cong \partial A \cup \partial B$, and naturally so; this means that ∂ is a strong monoidal functor with respect to the $\cup$ monoidal structure.
- $\partial(A \otimes B) \cong \partial A \otimes B \cup A \otimes \partial B$ and naturally so.

$\partial(A \otimes B) \cong \partial(A) \otimes B \cup A \otimes \partial B$ and naturally so

This second condition deserves to be spelled out completely: it means that ∂ is equipped with a 2-cell $\wr$ filling the diagram

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\otimes"'] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "(\partial\otimes\mathcal{C},\mathcal{C}\otimes\partial)"] \arrow[d, phantom, "\wr\Uparrow"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\cup"] \\
\mathcal{C} \arrow[rr, "\partial"'] & & \mathcal{C}
\end{tikzcd}$$

where $\Delta_{\mathcal{C}\times\mathcal{C}}$ is the diagonal functor $(A, B) \mapsto (A, B, A, B)$ and $(\partial \otimes \mathcal{C}, \mathcal{C} \otimes \partial)$ does the obvious thing.

The cell $\mathfrak{l}$ is called the leibnizator, has components $\mathfrak{l}_{AB} : \partial(A \otimes B) \Rightarrow \partial A \otimes B \cup A \otimes \partial B$, and it is subject to the following coherence conditions:

Compatibility with the right distributor:

$$\begin{tikzcd}
\partial((Y \cup Z) \otimes X) \arrow[rr, "\delta^{\mathrm{R}}"] \arrow[d, "\mathfrak{l}"'] & & \partial(Y \otimes X \cup Z \otimes X) \arrow[d, "\wr"] \\
(Y \cup Z) \otimes \partial X \cup \partial(Y \cup Z) \otimes X \arrow[d, "\delta^{\mathrm{R}} \cup \delta^{\mathrm{R}}"'] & & \partial(Y \otimes X) \cup \partial(Z \otimes X) \arrow[d, "\mathfrak{l} \cup \mathfrak{l}"] \\
Y \otimes \partial X \cup Z \otimes \partial X \cup \partial Y \otimes X \cup \partial Z \otimes X \arrow[rr, "\sim"'] & & \partial Y \otimes X \cup Y \otimes \partial X \cup \partial Z \otimes X \cup Z \otimes \partial X
\end{tikzcd}$$

where the unnamed isomorphisms are symmetries of $\cup$ or arising from the strong $\cup$-monoidality of ∂;

Compatibility with the left distributor:

$$\begin{tikzcd}
\partial(X \otimes (Y \cup Z)) \arrow[rr, "\delta^{\mathsf{L}}"] \arrow[d, "\mathfrak{l}"'] & & \partial(X \otimes Z \cup Y \otimes Z) \arrow[d] \\
\partial X \otimes (Y \cup Z) \cup X \otimes \partial(Y \cup Z) \arrow[d, "\delta^{\mathsf{L}} \cup \delta^{\mathsf{L}}"'] & & \partial(X \otimes Y) \cup \partial(X \otimes Z) \arrow[d, "\mathfrak{l} \cup \mathfrak{l}"] \\
\partial X \otimes Y \cup \partial X \otimes Z \cup X \otimes \partial Y \cup X \otimes \partial Z \arrow[rr] & & \partial X \otimes Y \cup X \otimes \partial Y \cup \partial X \otimes Z \cup X \otimes \partial Z
\end{tikzcd}$$

where the unnamed isomorphisms are symmetries of $\cup$ or arising from the strong $\cup$-monoidality of ∂;

Compatibility with the right and left annullator:

$$
\begin{tikzcd}
\partial(X \otimes 0) \arrow[rr, "\sim"] \arrow[d, "\mathfrak{l}_{X0}"'] & & \partial 0 \arrow[d, "\wr"] \\
\partial X \otimes 0 \cup X \otimes \partial 0 \arrow[r] & 0 + 0 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial(0 \otimes X) \arrow[rr, "\sim"] \arrow[d, "\mathfrak{l}_{0X}"'] & & \partial 0 \arrow[d, "\wr"] \\
\partial 0 \otimes X \cup 0 \otimes \partial X \arrow[r] & 0 + 0 \arrow[r] & 0
\end{tikzcd}
$$

where the unnamed isomorphisms come from the fact that $A \otimes -$ and $- \otimes B$ preserve the initial object for all $A, B \in \mathcal{C}$;

Compatibility with the right and left $\otimes$-unitor:

$$
\begin{tikzcd}
\partial(X \otimes I) \arrow[r, "\partial\rho_X"] \arrow[d, "\mathfrak{l}_{XI}"'] & \partial X \arrow[ddl, bend left, "\iota_{\partial X}"] \\
\partial X \otimes I \cup X \otimes \partial I \arrow[d, "\rho_{\partial X} \cup X \otimes \partial I"'] & \\
\partial X \cup X \otimes \partial I &
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial(I \otimes X) \arrow[r, "\partial\lambda_X"] \arrow[d, "\mathfrak{l}_{IX}"'] & \partial X \arrow[ddl, bend left, "\iota_{\partial X}"] \\
\partial I \otimes X \cup I \otimes \partial X \arrow[d, "\partial I \otimes X \cup \lambda_{\partial X}"'] & \\
\partial I \otimes X \cup \partial X &
\end{tikzcd}
$$

(the properties of ∂I, where I is the $\otimes$-monoidal unit, are quite a subtle business; we will come back to this later; note in particular that no axiom entails that $\partial I = 0$.).

Compatibility with the associator:

$$
\begin{tikzcd}
\partial((A \otimes B) \otimes C) \arrow[rr, "\partial\alpha"] \arrow[d, "\mathfrak{l}_{A\otimes B, C}"'] & & \partial(A \otimes (B \otimes C)) \arrow[d, "\mathfrak{l}_{A, B\otimes C}"] \\
\partial(A \otimes B) \otimes C \cup (A \otimes B) \otimes \partial C \arrow[dr, "\mathfrak{l}_{A,B} \otimes C \cup (A \otimes B) \otimes \partial C"'] & & \partial A \otimes (B \otimes C) \cup A \otimes \partial(B \otimes C) \arrow[dl] \\
& \partial A \otimes B \otimes C \cup A \otimes \partial B \otimes C \cup A \otimes B \otimes \partial C &
\end{tikzcd}
$$

First remarks

Directly from these definitions we can easily see that

- The derivative of the initial object $\varnothing$ must be the initial object (because ∂ preserves the empty coproduct, or because of the Leibniz rule);
- by induction on n,

$$\partial(A^{\otimes n}) \cong n \cdot A^{\otimes(n-1)} \otimes \partial A$$

  where $n \cdot -$ sends an object $X \in \mathcal{C}$ to the n-fold coproduct $X \cup \cdots \cup X$.
- again by induction,

$$\partial^n(X \otimes Y) = \coprod_{k=0}^{n} \binom{n}{k} \cdot \partial^{n-k} X \otimes \partial^k Y$$

  where $\binom{n}{k}$ is the set of k-elements subsets of $\{1, \ldots, n\}$.

Examples

Edge cases

Any 2-rig $\mathcal{C}$, endowed with the trivial derivation $\mathcal{C} \to \mathcal{C}$ that is the constant functor at the initial object (regarded as empty coproduct).

Let P be a distributive lattice; the identity functor $P \to P$ is, trivially, a derivation (because every element of P is $\vee$-idempotent).

Idempotents

let $\mathcal{C}$ be a 2-rig; an object $E \in \mathcal{C}$ is $\cup$-idempotent if $E \cup E \cong E$. Given an idempotent object, the endofunctor $\partial_E : A \mapsto E \otimes A$ is a derivation, since

$$\begin{aligned} \partial_E(A \otimes B) &= E \otimes A \otimes B \cong (E \cup E) \otimes A \otimes B \\ &= (E \otimes A) \otimes B \cup A \otimes (E \otimes B). \end{aligned}$$

More generally, given a derivation ∂ on a 2-rig $\mathcal{C}$, the endofunctor $A \mapsto \partial A \cup E \otimes A$ is a derivation: the former is a particular case of this construction whn $\partial = 0$ is the trivial derivation.

Polynomials

Let $\mathcal{C}$ be a 2-rig; let $Y : \mathcal{C} \to \mathcal{C}$ a strong monoidal endofunctor, we define a category of $\mathcal{C}$-valued polynomials:

- objects are 'polynomials' $\sum_{i=0}^{d} A_i \otimes Y^i$, regarded as endofunctors $\mathcal{C} \to \mathcal{C}$, with the convention that $Y^0 = 1_{\mathcal{C}}$ and the action on an object X is given by

$$X \mapsto \sum_{i=0}^{d} A_i \otimes Y^i(X) \in \mathcal{C};$$

- morphisms are natural transformations of functors.

The category $\mathcal{C}[Y]$ so obtained is a 2-rig where the sum is 'component-wise', and the $\otimes$-product is a similar 'Cauchy product' of polynomials.

Polynomials

let $\mathcal{C}$ be a 2-rig; $\mathcal{C}[Y]$ becomes a differential 2-rig if we endow $\mathcal{C}$ with the trivial derivation, and we put $\partial Y = I$, suitably extended on a generic expression $\sum_{i=0}^{d} A_i \otimes Y^i$ by linearity and Leibniz rule:

$$\partial\left(\sum_{i=0}^{d} A_i \otimes Y^i\right)) \cong \sum_{i=1}^{d} i \cdot A_i \otimes Y^{i-1}$$

let $\mathcal{C}$ be a differential 2-rig, with derivation denoted $a \mapsto \partial a$. One can define the 2-rig of differential polynomials with coefficients in $\mathcal{C}$ introducing an infinite set of 'variables' $\mathcal{Y} := \{Y, Y^{(1)}, Y^{(2)} \dots, Y^{(n)}, \dots\}$ and defining

$$\mathcal{C}[\mathcal{Y}] := \varinjlim \left(\mathcal{C} \to \mathcal{C}[Y] \to \mathcal{C}[Y, Y^{(1)}] \to \mathcal{C}[Y, Y^{(1)}, Y^{(2)}] \to \dots\right)$$

where we define inductively $\mathcal{C}[Y, Z] := \mathcal{C}[Y][Z]$, and the derivation as $\partial : Y^{(i)} \mapsto Y^{(i+1)}$.

A co-Heyting algebra is a bounded distributive lattice K such that $x \vee - : K \to K$ has a left adjoint $-\backslash x$ for all $x \in K$:

$$y \backslash x \leq z \iff y \leq x \vee z$$

Define $\partial x := x \wedge \lrcorner x$ it's easy to see that $\partial : K \to K$ is a derivation when K is regarded as a distributive 2-rig. Leibniz rule takes the form

$$\partial(a \wedge b) = (\partial a \wedge b) \vee (a \wedge \partial b).$$

Example: lattices of closed subsets of topological spaces.

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Example: lattice of subtoposes of a given topos $\mathcal{E}$; define the boundary $\partial\mathcal{A}$ of a subtopos $\mathcal{A} \subseteq \mathcal{E}$ in this lattice and then in turn the boundary ∂T of the geometric theory T that $\mathcal{A}$ classifies (see Caramello, 2009).

Results

Theorem
Let $\mathcal{C}$ be a 2-rig, and M a internal semigroup with multiplication $m : M \otimes M \to M$; then the map $\partial m : \partial M \otimes M \cup M \otimes \partial M \to \partial M$ splits as a pair of maps

$$\begin{cases} i_R : \partial M \otimes M \to \partial M \\ i_L : M \otimes \partial M \to \partial M \end{cases}$$

Then, i_R (resp., i_L) is a right (resp., left) action of M over ∂M.

- Let $\mathcal{C}$ be a 2-rig with a natural number object $\mathbf{N}$; then, $\mathbf{N}$ is a monoid in a canonical way, with respect to the morphism $\mathbf{N} \times \mathbf{N} \to \mathbf{N} : \lambda pq.s^p q$.
  If $\mathcal{C}$ is a differential bicartesian category, then $\partial\mathbf{N}$ is a Lawvere dynamical system.
- Let $\mathcal{C}$ be an elementary topos, regarded as a bicartesian category; if $\mathcal{C}$ has a differential structure, the derivative $\partial\Omega$ of the subobject classifier is a module for the monoid $(\Omega, \wedge, \mathrm{true})$.

Theorem
Let $\mathcal{C}$ be a category that satisfies the following assumptions:

- *additive category with byproducts noted $\oplus$;*
- *it has the structure of a differential 2-rig with multiplicative structure $\odot$, and a derivation ∂ that is $\oplus$-linear and $\odot$-Leibniz.*

Then there exists a canonical extension $\bar{\partial}$ of ∂ to the additive presheaf category $\hat{\mathcal{C}} = [\mathcal{C}, Ab]$ (coproduct-linear and convolution-Leibniz), that hence becomes a differential 2-rig.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "\partial"] \arrow[d, "y"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{C}^{\mathrm{op}} \arrow[d, "y"] \\
\check{\mathcal{C}} \arrow[r, dotted, "\bar{\partial}"'] & \check{\mathcal{C}}
\end{tikzcd}$$

Recall the equivalence

$$[\mathrm{Fin}, \mathrm{Set}] \cong [\mathrm{Set}, \mathrm{Set}]_\omega$$

given by left Kan extension along the embedding $J : \mathrm{Fin} \to \mathrm{Set}$, where at the right hand side we put *finitary* endofunctors of Set.

This equivalence can be promoted to a monoidal equivalence, if $[\mathrm{Set}, \mathrm{Set}]_\omega$ is considered a monoidal category with respect to composition.

Given $F : \mathrm{Fin} \to \mathrm{Set}$, $T_F = Lan_J F$ is the associated finitary functor and there exists a unique monoidal structure $\diamond$ such that

$$Lan_J(F \diamond G) \cong Lan_J F \circ Lan_J G$$

Theorem (The chain rule)

Let $F, G : \mathsf{Fin} \to \mathsf{Set}$ *be non-*Σ*-species, and let* ∂ *be a derivation with respect to the cartesian monoidal structure; then,*

$$\begin{cases} \partial T_G(T_F A) \times \partial T_F A \cong \partial(T_G \circ T_F) \\ \partial(G \diamond F) = Lan_J \partial G(Fn) \times \partial Fn. \end{cases}$$

Questions

Vices and virtues of being $\partial 1$

In a differential ring, using the Leibniz rule: $d1 = d1 + d1$, which entails $d1 = 0$.

In a rig things are way more complicated: one has to postulate that $d1 = 0$, or derivation isn't even well-defined:

$$da = d(a \cdot 1) = da + a.d1 = da + \sum_{k=1}^{\infty} a.d1$$

Something similar happens in a differential 2-rig:

$$\partial I \cong \partial I \cup \partial I$$

from which we get the idea that ∂I is 'either empty or big' (e.g., in Set, ∂I is empty or -at least- countable.)

Vices and virtues of being $\partial 1$

Theorem
There is no nontrivial finite colimit-preserving derivation on the 2-rig $(\mathrm{Fin}, \times, 1)$ *of finite sets and functions: such* $\partial : \mathrm{Fin} \to \mathrm{Fin}$ *is completely determined by its action on the point, so that*

$$\partial A \cong \partial(A \cdot 1) \cong A \cdot \partial 1.$$

Same in the category of finite dimensional vector spaces, where $d = \dim V = 2d$ has 0 (thus the zero space) as unique solution. Same in every category with a choice of dimension $\mathcal{C} \to \mathbb{N}$ for objects.

Vices and virtues of being $\partial 1$

Sometimes $\partial 1 = \partial 1 + \partial 1$ is forced to have just trivial solutions due to naturality;

Let ∂ be a derivation in a category of functors $\mathcal{A} \to \mathit{Set}$; then, $\partial 1$ is a functor $\mathcal{A} \to \mathit{Set}$ such that $F \cong F + F$, and naturally so; such functors must be constant on connected components of $\mathcal{A}$.

The unbearable largeness of $\partial 1$

Use Yoneda lemma.

Consider the hom-set $\hom(\partial 1, Z)$ for a generic object Z;

$\partial 1 \cong \partial 1 + \partial 1$ yields

$$\hom(\partial 1, Z) \cong \hom(\partial 1 + \partial 1, Z) \cong \hom(\partial 1, Z) \times \hom(\partial 1, Z).$$

So, $\hom(\partial 1, Z)$ can only be empty, a singleton or infinite.

Thus if a category is finite $\hom(\partial 1, Z)$ is either empty or a singleton, and in particular it must be a singleton when $Z = \partial 1$.

Coalgebras

If $\partial 1 \cong \partial 1 + \partial 1$, this means that $\partial 1$ is naturally a coalgebra for the ”leave it or double it” functor $S : A \mapsto A + A$, in such a way that there is a unique map

$$\begin{array}{ccc} \partial 1 & \cong & \partial 1 + \partial 1 \\ \downarrow & & \downarrow \\ C & \cong & C + C \end{array}$$

between $\partial 1$ and the terminal coalgebra of S; but wait, in the category of topological space C is the Cantor set! What just happened here?

Napier objects

As an endofunctor, ∂ might have interesting fixed points, and there is a standard procedure to build its initial algebra and terminal coalgebra.

Initial algebras are trivial, in that $\partial 0 = 0$ by using the Leibniz rule. On the other hand, the triviality of terminal coalgebras is governed by the shape of $\partial 1$:

$$1 \leftarrow \partial 1 \leftarrow \partial\partial 1 \leftarrow \partial\partial\partial 1 \leftarrow \dots$$

and the first ordinal λ for which the transition morphism $v : \partial^{\lambda} 1 \leftarrow \partial^{\lambda+1} 1$ is invertible realises the terminal coalgebra.

Derivatives of models

Let $(\mathcal{C}, J)$ be a *Grothendieck monoidal site* i.e. a Grothendieck topology such that the category of sheaves is monoidal; let $\partial : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ be a convolution-derivation; if ∂ is such that

$$\boxed{\text{a big black hole of ignorance}}$$

then, ∂ restricts to a derivation on the category of J-sheaves (that is itself monoidal).

Derivatives of models

Let $\mathcal{T}$ be an algebraic theory of some sort, with the property that *Mod*$(\mathcal{T})$ is monoidal; let $\partial : \hat{\mathcal{T}} \to \hat{\mathcal{T}}$ be a convolution-derivation; if ∂ is such that

$$\boxed{\text{a big black hole of ignorance}}$$

then ∂ sends a $\mathcal{T}$-model to a $\mathcal{T}$-model.