Research statement

July 25, 2018

Works on stable ∞-categories

At the beginning of my PhD I was able to demonstrate that

> **Result 1**
>
> In the setting of stable $(\infty, 1)$-categories the theory of t-structures is subordinated to a flexible and expressive calculus of factorization systems.

Somehow, t-structures give the correct notion of *co/reflective co/localization* in stable $(\infty, 1)$-categorical setting, hence constitute a milestone of the category theory expressed by a 'stable homotopy theoretic universe'. The classical theory, initiated in [**BBD82**], has now countless applications in stable homotopy theory, representation theory of algebras, and the theory of 'homotopy coherent' algebraic structures [**Lur11**]. In this spirit, it has been natural to employ well known techniques linking co/reflective pairs of categories of a given $\mathcal{C}$ to suitable factorization systems on $\mathcal{C}$: this led to the central theorem of [**Lor16**], showing that in the setting of stable ∞-categories a t-structure on (the homotopy category of an ∞-category) $\mathcal{C}$ is completely determined by a ∞-*categorical factorization system* $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ such that

1. the 3-for-2 property holds for both classes $\mathcal{E}, \mathcal{M}$; this entails that the subcategories of "cofibrants" $\left\{X \in \mathcal{C} \mid \left[\begin{smallmatrix} 0 \\ \downarrow \\ X \end{smallmatrix}\right] \in \mathcal{E}\right\}$ and "fibrants" $\left\{Y \in \mathcal{C} \mid \left[\begin{smallmatrix} Y \\ \downarrow \\ 0 \end{smallmatrix}\right] \in \mathcal{M}\right\}$ are respectively a coreflective and a reflective subcategory of $\mathcal{C}$, forming the *aisle* $\mathcal{C}_{\geq}$ and *coaisle* $\mathcal{C}_{<}$ of a t-structure.
2. The reflection $X \to RX$ is the cofiber of the coreflection $SX \to X$ (and the coreflection is the fiber of the reflection), in a pullback-pushout diagram

$$\begin{tikzcd} SX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "\ulcorner" very near end] & X \arrow[d] \\ 0 \arrow[r] & RX. \end{tikzcd}$$

Such factorization systems are called *normal torsion theories* building on the previous work of [**CHK85, RT07**].

This result led me to a torough study of how the classical theory of t-structures, with all its host of applications, gets reshaped by this eminently structural reformulation (informally called the 'torsio-centric' perspective henceforth).

> **Result 2**
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> All the classical theory (the abelianity of hearts, the theory of *semiorthogonal decompositions*, stable Postnikov decompositions, and more) can be expressed in the language of factorization system, giving also elegant new insights.

These results are extensively described in a joint work [**FLM15**] with D. Fiorenza and G. Marchetti. More in detail, we have been able to prove that under the "torsio-centric" perspective two apparently disconnected constructions, *Postnikov towers* (induced by the heart of a t-structure) and *semiorthogonal decompositions*, acquire an intrinsic description as, respectively, orbits and fixed points of the $\mathbb{Z}$-action; it all boils down to specialize to these two extremal particular cases the construction [**FLM15**, Def. 2.7] of the "tower" $♖(f)$ of a morphism $f\colon X \to Y$ with respect to a $\mathbb{Z}$-equivariant family of t-structures $\mathfrak{t}\colon J \to \mathrm{TS}(\mathcal{C})$.

My study then concentrated on the theory of *recollements* of t-structures and culminated in the paper [**FL15a**]; there, I tried to reformulate the following classical construction: given a "recollement" of stable $(\infty, 1)$-categories, i.e. a diagram of stable $(\infty, 1)$-categories

$$\mathbf{D}^0 \underset{\underset{i_L}{\longleftarrow}}{\overset{\overset{i_R}{\longleftarrow}}{\xrightarrow{\;i\;}}} \mathbf{D} \underset{\underset{q_L}{\longleftarrow}}{\overset{\overset{q_R}{\longleftarrow}}{\xrightarrow{\;q\;}}} \mathbf{D}^1$$

where $i_L \dashv i \dashv i_R$ and $q_L \dashv q \dashv q_R$ and suitable "exactness" properties hold (see [**FL15a**, Def. 3.1]), it is possible to *glue* two t-structures $\mathfrak{t}_0, \mathfrak{t}_1$, respectively on the categories $\mathbf{D}^0, \mathbf{D}^1$ to a t-structure $\mathfrak{t}_0 \curlyvee \mathfrak{t}_1$ on $\mathbf{D}$; this formalism, introduced in [**BBD82**] is of capital importance in algebraic geometry and in the theory of perverse sheaves, having also applications in intersection homology [**Pfl01, GM80, GM83**] and representation theory [**PS88, KW01**].

Plans for future research.

Factorization systems in $(\infty, 1)$-categories

My current research (especially during the last year) slightly detached from these topics, as soon as I began to better investigate the foundations of the language I used (what was in fact my primary interest in the first place); I believe that a complete axiomatization of the theory of factorization systems on $(\infty, 1)$-categories is an urgent objective, since apart from its independent interest, it would clarify certain "natural" constructions in stable and unstable homotopy theory (this track of research has already been explored to some extent, and quite fruitfully by a series of works by W. Chachólski [**Cha96, Cha97**] and Farjoun [**Far95**], in a way that resembles a theory of "unstable t-structures"; it was however only in the framework of quasicategories that Joyal and Anel proved [**ABFJ17**] a factorization theoretic version of *Blakers-Massey theorem* for quasicategories).

As shown in [**ABFJ17**], in the setting of $(\infty, 1)$-categories Blakers-Massey-like theorems are neatly described by the internal language of the $(\infty, 1)$-category of spaces $\mathcal{S}$, referring to a $\mathbb{Z}$-indexed family of factorization systems on $\mathcal{S}$ (a *slicing* in the terminology of [**FL15b**]).

A rather intriguing speculation involves the $\mathcal{P}$-local version of Blakers-Massey theorem, and hence also its rational and rational-stable counterpart.

If $\mathcal{P} \sqcup \mathcal{Q}$ is a partition of the set $\mathbb{P}$ of primes in $\mathbb{Z}$, we say that a $\mathbb{Z}$-module G is $\mathcal{P}$-*local* is each $g \in G$ has finite order, which is not divisible for any prime in $\mathcal{P}^c = \mathcal{Q}$.

This allows to call $(\mathcal{P}, n)$-*connected* a simply connected space whose $\pi_{[0,n]}(X)$ is a $\mathcal{Q}$-group; it can be easily seen that if $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ then a $(\mathcal{Q}_2, n)$-connected space is also $(\mathcal{Q}_1, n)$-connected; in the same fashion, all $(\mathcal{Q}, n+1)$-connected are also $(\mathcal{Q}, n)$-connected. Similar definitions apply to the stable homotopy (or rather, stable ∞-)category $\mathbf{Sp}$ of spectra, to the homotopy groups of an object $E \in \mathbf{Sp}$, and to a map of spectra. This allows to reformulate the $\mathcal{P}$-local, rational, stable $\mathcal{P}$-local, and rational stable analogues of Blakers-Massey theorem as results involving a "$\widetilde{2^{\mathbb{P}}} \times_{\text{lex}} \mathbb{Z}$-*slicing*" (see [**FL15b**]), where for each partially ordered set S, the set $\widetilde{S}$ denotes the trivial $\mathbb{Z}$-action on S. These objects are called arithmetic slicings.

Task 1

The $\mathcal{P}$-local, rational, stable $\mathcal{P}$-local, and rational stable $\mathcal{P}$-local analogues of Blakers-Massey theorem are theorems about arithmetic slicings.

Currently, there are several notions of factorization system adapted to various models for higher category theory; this yields a somewhat unified, model-independent description of t-structures arising in stable homotopy theory, homological algebra, algebraic geometry etc.: for example, a t-structure in a DG-category is the exact counterpart of a normal torsion theory in the world of *enriched* factorization systems (see [**Day74, LW14**]), whereas a t-structure in a stable model category is a normal torsion theory in the setting of *homotopy* factorization systems (see [**Bou77, Joy08**]).

For what concerns applications of the theory of factorization systems, in a recent joint work [**LV17**] with S. Virili I managed to apply the torsio-centric approach to the setting of *stable derivators*, thus showing a similar characterization theorem in this framework, and completing a nontrivial part of this model-independency statement (in [**LV17**] we were able to provide the 2-category of stable derivators with a notion of factorization system closing the circle between 1-categorical, $(\infty, 1)$-categorical and triangulated torsion theories). Further development of this track of research have recently been obtained in [**Lor18**], where I outlined how reflections of prederivators (see [**Gro13**]; a derivator is a suitably complete prestack on the 2-category of small categories) can be equivalently described as objects of fractions, reflective factorization systems, and objects of algebras for idempotent monads.

In [**LV17**], we define factorization systems in the 2-category **Der** of derivators, and we offer a formal description as algebras for a "tractable" strict 2-monad; moreover, we prove that when the derivator is stable, a suitable subclass of "coherent" normal torsion theories (the *derivator-normal torsion theories*) correspond bijectively with t-structures on the underlying category of the derivator, and under relatively mild assumptions induce t-structure on *each* category $\mathbf{D}(J)$ and t-exact functors thereof. With this result in hand, it is clearly visible a deeper pattern; in particular, we formulate the following question:

Task 2

Is it possible to find a general, model-independent theory of factorization systems in $(\infty, 1)$-categories, to be applied to the theory of t-structures (or elsewhere)?

A first step in this respect is to describe precisely the relationship between the existing notions of factorization system in different models of $(\infty, 1)$-categories.

This track of research spurred my interest for *formal* category theory and in particular for its applications to the foundations of derivator theory, that I perceive as somewhat neglected despite its (numerous) applications.

Formal category theory of derivators

I am currently tackling the following problem:

Task 3

The notion of a presentable and accessible (pre)derivator shall rely on purely formal properties of a *Yoneda structure* on the 2-category of such objects.

This seemingly formal result can provide a deeper glance to a model independent higher-categorical adjoint functor theorem (it has to be noted that derivators currently lack such an important piece of internal theory): I would like to outline the existence of this Yoneda structure and deduce various consequences that also unify the existent approaches to derivators "of small presentation" described in [**Ren09**] and [**MR16**]. Relying on this, I plan to obtain an elegant theory of *monads* on derivators (especially the part dealing with *accessible* monads), extending some results of [**LN16**].

This track of research concretized in a joint work with I. Di Liberti [**LL18**], where we engineered a definition for an accessible and a presentable object of a suitably nice 2-category $\mathcal{K}$. The main conjecture in [**LL18**, §6] is that the 2-category **PDer** (and its subcategory **Der** of derivators) possesses a sufficiently rich structure to cast our characterization of accessibility and presentability, at least in some form. There is a "variable" Yoneda structure on the 2-category $[\mathbf{Cat}^{\mathrm{op}}, \mathbf{CAT}]$ of prederivators allowing for such a discussion. This is precisely the leading idea of [**Str81**] (but note that the author there only considers *pseudo*functors –whereas strictness is a customary request working with derivators–, and transformations of mixed variance).

We make the (somehow bold) conjecture that this Yoneda structure is "representable" in the sense of [**Web07**] by the discrete derivator associated to a sufficiently big category of sets SET –sufficiently big so that $\mathbb{D}(J) \in \mathrm{SET}$. The presheaf construction here is embodied by a(n extra)natural transformation $P : \mathbb{D} \to [\![\mathbb{D}^{\mathrm{op}}, \mathrm{SET}]\!]$ (**Der** is cartesian closed, see [**Gro11**]).

Of course, being the representative object a *discrete* derivator, this choice shall be discarded for all homotopy-theoretic purposes. In fact, Street's one is presumably only one among a complicated web of Yoneda structures: we can conjecture the existence of

- The *Renaudin* (from [**Ren09**]) Yoneda structure, built taking the derivator generated by the Kan-Quillen model structure on sSet; this is again a representable Yoneda structure, and the canonical map $\mathrm{sSet}_* \to \mathrm{sSet}$ (of the associated derivators, for Kan-Quillen model structures on both categories) classifies discrete opfibrations in the sense of [**Web07**].
- The *Cisinski-Tabuada* Yoneda structure, (following [**CT11**]; there, the authors prove that for every stable derivator $\mathbb{D}$ there is an action $\mathrm{Sp} \times \mathbb{D} \to \mathbb{D}$ that can be promoted to a two-variable adjunction, using Brown representability). The internal-hom part of this two-variable adjunction now gives a canonical enrichment

$$\langle \_, = \rangle : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathrm{Sp}$$

of $\mathbb{D}$ over spectra, hence there's a well defined homotopy class of a spectrum for the image of $X \in \mathbb{D}(J)$ under the Yoneda map: the map $y_{\mathbb{D}} : \mathbb{D} \to [\![\mathbb{D}^{\mathrm{op}}, \mathrm{Sp}]\!]$ has components $y_{\mathbb{D},J} : \mathbb{D}(J) \to \mathbf{PDer}(\mathbb{D}^{\mathrm{op}}, \mathrm{Sp}^J)$ and $y_{\mathbb{D},J}(X)$ sends $X \in \mathbb{D}(J)$ to the morphism of prederivators $\langle \_, X \rangle$ that sends $Y \in \mathbb{D}^{\mathrm{op}}(I)$ into $\langle \pi_J^* Y, (\pi_{I^{\mathrm{op}}}^*)^{\mathrm{op}} X \rangle$.

In both these structure it is possible to instantiate an analogue of a derivator *having small presentation* ([**Ren09**] defines precisely *dérivateurs de pétite présentation* as localizations of the derivators $J \mapsto \mathrm{sSet}^J$ and $J \mapsto \mathrm{Sp}^J$ respectively associated to sSet (with Kan-Quillen model structure) and Sp (with, say, Bousfield-Friedlander model structure). This evidently echoes the main theorem of [**LL18**].

This problem spurred a certain interest for the abstract theory of Yoneda structure, as the definition proposed in [**SW78**] and recently refined by [**Web07**], proved to be a real fountain of information. Ongoing discussion with I. Di Liberti, subsequent to the redaction of [**LL18**], suggested that there shall be various "contexts" in which it is possible to formalize various parts of classical category theory, and in particular

- a *Yoneda context* allowed us to define accessible and presentable objects (see [**LL18**, §3]).
- a *Gabriel-Ulmer context* allows to instantiate a weak form of Gabriel-Ulmer duality, removing the asymmetry arising from two different, generally non-equivalent definitions of presentability (briefly: in a Gabriel-Ulmer context every presheaf object is accessible, see [**LL18**, §4]).
- an *Isbell context* allows to concoct an abstract form of Isbell duality, and all the consequences thereof (e.g. the "Isbell envelope" of an object); the settings where it is possible to build Isbell duality consist of "ambidextrous" Yoneda structures, where two adjoint presheaf constructions $P^\sharp \dashv P$ are given. The representability of these functors has direct consequences on how much the Yoneda structure is well-behaved.

Concreteness and homotopy theory

I am occasionally interested in the foundation of homotopy theory, and in particular in how homotopical algebra relates to set theory: in a joint work with I. di Liberti [**LL17**] we propose a fairly general method to show that under certain assumptions on a model category $\mathcal{M}$, its homotopy category $\mathrm{ho}(\mathcal{M})$ cannot be concrete. This result, echoing and generalizing a lot a famous theorem by Freyd, will hopefully be the first step towards a theory of '∞-concrete' $(\infty, 1)$-categories. The paper has been accepted on Journal of Homotopy and Related Structures.

Task 4

> In the realm of $(\infty, 1)$-categories the notion of concreteness probably 'splits' into a countable spectrum of stronger and stronger n-concreteness conditions, in such a way that being ∞-*concrete* is the strongest way in which a $(\infty, 1)$-category can stray from being concrete.

This is mildly related to my prior interest in the theory of factorization systems: a rather technical but conceptually deep result proved in [**Isb64**] is that the notion of concreteness for a category $\mathcal{C}$ is tightly related with a set-theoretical condition of the class of 'generalized subobjects' of objects of $\mathcal{C}$; under relatively mild assumptions, this is a condition on monomorphisms of $\mathcal{C}$, and this is the starting point to build an analogy between the (classical, 1-categorical) notion of monic in a 1-category and the fact that this notion breaks into a similar countable spectrum of *n-monic arrow* in a $(\infty, 1)$-category.

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