Triangulated Categories

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Topology 2

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Problem/Wish: Define an invariant of complexes (e.g. simplicial ones) X, Y, allowing to decide whether their *geometric realizations* $|X|, |Y|$ are homotopy equivalent.

- *Homology* of complexes is a too coarse invariant:

  > *There exist topological spaces X, Y having degree-wise isomorphic homologies but X is neither homeomorphic, nor homotopy equivalent to Y.*

- **Theorem (Whitehead).** Simplicial complexes X and Y have homotopy equivalent geometric realizations $|X|$ and $|Y|$ iff there exists a third s.c. Z and simplicial maps $f\colon Z \longrightarrow X$, $g\colon Z \longrightarrow Y$ being *quasi-isomorphisms* (i.e.

  $$f_*\colon H_i(Z) \longrightarrow H_i(X) \quad g_*\colon H_i(Z) \longrightarrow H_i(Y)$$

  is an isomorphism for all $i \geq 0$).

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

Definition (Homotopy)

Given chain complexes $\mathcal{C} = \{C_n, \partial_n^{\mathcal{C}}\}$, $\mathcal{D} = \{D_n, \partial_n^{\mathcal{D}}\}$, we define two chain maps φ, ψ to be homotopic *if there exists a chain map of degree $+1$, $s_n \colon C_{n-1} \longrightarrow D_n$ such that*

$$\psi_n - \varphi_n = \partial_{n+1}^{\mathcal{D}} \circ s_n + \Sigma_{n-1} \circ \partial_n^{\mathcal{C}}$$

for all $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^{C}"] & C_n \arrow[r, "\partial_n^{C}"] \arrow[dl, "s_{n+1}"'] \arrow[d, shift left] \arrow[d, shift right] & C_{n-1} \arrow[r] \arrow[dl, "s_n"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^{D}"'] & D_n \arrow[r, "\partial_n^{D}"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

- *Being homotopic* is an equivalence relation in every $\hom_{\mathbf{Ch}(\mathcal{A})}(\mathcal{C}, \mathcal{D})$ which we write $\varphi \sim \psi$: define the homotopy category $\mathbf{K}(\mathcal{A})$ as $\mathrm{Ob}(\mathbf{K}(\mathcal{A})) = \mathrm{Ob}(\mathbf{Ch}(\mathcal{A}))$, $\hom_{\mathbf{K}(\mathcal{A})}(\mathcal{C}, \mathcal{D}) = \hom_{\mathbf{Ch}(\mathcal{A})}(\mathcal{C}, \mathcal{D})/\sim$.

Problem! In passing to the homotopy category abelian structure gets lost:

Theorem

$\mathbf{K}(\mathcal{A})$ *is not an abelian category (even if $\mathcal{A}$ is).*

Sketch of Proof.

Build a counterexample in $\mathcal{A} = \mathbf{Ab}$; the chain map

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\varphi"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, dash] & 0 \arrow[r] \arrow[d] & \cdots \\
\mathcal{Y} & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

doesn't admit a kernel in the homotopy category (both complexes $\mathcal{K} = (\ker \varphi_n, i_n)$ and $\mathrm{id}_{\mathcal{X}}$ share the UMP of $\ker \varphi$, but they are *not* homotopic, contradiction $\lightning$). $\square$

…having lost the abelian structure, we can't talk anymore about exact sequences.
(Rough) Idea: Try to replace "exact sequence"

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

by "exact triangle"

$$A \longrightarrow B \longrightarrow C \longrightarrow \text{“}A^{+}\text{”}.$$

(whatever "plus" means).

Definition

Given an additive category $\mathcal{A}$ and an auto-equivalence $S\colon \mathcal{A} \longrightarrow \mathcal{A}$ to be called suspension of $\mathcal{A}$, a triangle (wrt S) in $\mathcal{A}$ is a sequence in $\mathcal{A}$ like

$$A \longrightarrow B \longrightarrow C \longrightarrow S(A)$$

A morphism of triangles is a triple (f, g, h) making every square below commutative.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & S(X) \arrow[d, "S(f)"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & S(X') \end{tikzcd}$$

Definition (Triangulated Category)

A *triangulated category* *is an additive category* $\mathcal{T}$*, with a suspension functor* S*, where we can find a family of triangles (called distinguished triangles, from now on d.t.s) subject to the following axioms:*

- *Every triangle isomorphic to a d.t. is itself a d.t.;*
- *For all* $X \in \mathrm{Ob}_{\mathcal{T}}$*, the triangle* $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow S(X)$ *is distinguished;*
- *To any* $f : X \longrightarrow Y$ *in* $\mathcal{T}$ *we can associate a d.t. in which* f *is the first arrow,* $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow S(X)$*;*
- *(rot) If* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$ *is a d.t., then* $Y \xrightarrow{v} Z \xrightarrow{w} S(X) \xrightarrow{-S(u)} S(Y)$ *is a d.t., and vice versa;*
- *(com) Given d.t.s* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$ *and* $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} S(X')$*, every commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dotted] & S(X) \arrow[d, "S(f)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & S(X')
\end{tikzcd}$$

…and finally

- (oct) Given d.t.s $X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow S(X)$, $Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow S(Y)$ and $X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow S(X)$ there exists a d.t. $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow S(Z')$ which makes the following diagram commutative:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, no head] & Y \arrow[r] \arrow[d, "v"'] & Z' \arrow[r] \arrow[d] & S(X) \arrow[d, no head] \\
X \arrow[r, "v \circ u"] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, no head] & Y' \arrow[r] \arrow[d] & S(X) \arrow[d, "S(u)"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, no head] & S(Y) \arrow[d] \\
Z' \arrow[r, dotted] & Y' \arrow[r, dotted] & X' \arrow[r, dotted] & S(Z')
\end{tikzcd}
$$

[Notation: the image of A via S is usually denoted $A[1]$ for short; the image $S^k(A)$, for $S^k = S \circ \cdots \circ S$ is denoted $A[k]$ for short.]

A couple of observations about the axioms:

- (oct), the most strange axiom can be alternatively stated as the completability/commutativity of the following 3D diagram:

$$\begin{tikzcd}
 & & Y/X \arrow[dl, dotted] \arrow[ddll, dashed, bend right] & & \\
 & X \arrow[rr] \arrow[dr] & & Y \arrow[ul] \arrow[dl] & \\
Z/X \arrow[ur, dotted] \arrow[rrrr, dashed, bend right] & & Z \arrow[ll] \arrow[rr] & & Z/Y \arrow[ul, dotted] \arrow[uull, dashed, bend right]
\end{tikzcd}$$

  the rough idea is that (oct) gives a weaker analogue of the "freshman algebraist's isomorphism" $(Z/X)/(Y/X) \cong Z/Y$ for a sequence of subobjects $X \hookrightarrow Y \hookrightarrow Z$; dotted arrows are the $A/B \longrightarrow S(A)$ part of the triangle. (oct) asks that that diagram can be completed with curved dashed arrows, and the resulting external triangle $Y/X \longrightarrow Z/X \longrightarrow Z/Y \longrightarrow S(Y/X)$ is a d.t.

(rot) is the *rotation axiom*: it encodes the fact that we can present d.t.s as "towers"

$$
\begin{tikzcd}
\ddots & & \\
& B[1] \arrow[dr] & \\
A[1] \arrow[ur] & & C[1] \arrow[uull, bend left] \\
& B \arrow[dr] & \\
A \arrow[ur] & & C \arrow[uull, bend left] \\
& B[-1] \arrow[dr] & \\
A[-1] \arrow[ur] & & C[-1] \arrow[uull, bend left]
\end{tikzcd}
$$

and $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a d.t. iff $B \xrightarrow{v} C \longrightarrow A[1] \xrightarrow{-v[1]} B[1]$ is a d.t. (and so on for finitely many shifts).

We can easily draw triangulated analogues of classical results in an abelian category $\mathcal{A}$, now without speaking of exact sequences in $\mathcal{A}$:

- Let $(\mathcal{T}, [1])$ be triangulated. Given a d.t. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ then $v \circ u = 0$ and $w \circ v = 0$, namely composition of adiacent arrows in a d.t. is the zero arrow.

Proof.

Thanks to (rot) it's enough to prove that $v \circ u = 0$. Again from (rot), $Y \longrightarrow Z \longrightarrow X[1] \xrightarrow{-u[1]} Y[1]$ is a d.t.; in diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, equal] & X[1] \arrow[r, "-u[1]"] & Y[1] \arrow[d, "v[1]"] \\ Z \arrow[r, equal] & Z \arrow[r, "0"'] & 0 \arrow[r, "0"'] & Z[1] \end{tikzcd}$$

the unique arrow to 0 completes to a morphism of triangles $(v, \mathrm{id}, 0)$; now $v[1] \circ (-u[1]) = -(v \circ u)[1] = 0$ implies $v \circ u = 0$, qed. $\square$

- (Triangulated l.e.s.) Let $(\mathcal{T}, [1])$ be triangulated. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a d.t. For any $T \in \mathrm{Ob}_{\mathcal{T}}$ there is a l.e.s. of abelian groups

$$\cdots \longrightarrow \hom_{\mathcal{T}}(T, X[k]) \xrightarrow{u[k]_*} \hom_{\mathcal{T}}(T, Y[k]) \xrightarrow{v[k]_*}$$
$$\xrightarrow{v[k]_*} \hom_{\mathcal{T}}(T, Z[k]) \xrightarrow{w[k]_*} \hom_{\mathcal{T}}(T, X[k+1]) \longrightarrow \cdots$$

- (Triangulated 5 Lemma) Let $(\mathcal{T}, [1])$ be triangulated. Given a morphism of triangles as in

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & X'[1] \end{tikzcd}$$

such that f and g are isomorphisms, then also h is.

The rest of the note is devoted to prove the following

Theorem

$\mathbf{K}(\mathcal{A})$ *is a triangulated category.*

First of all we have to define the S shift endofunctor: given $k \in \mathbb{Z}$ define $[k] \colon \mathcal{C} \mapsto \mathcal{C}[k]$

$$\begin{cases} (\mathcal{C}[k])_n = C_{n-k} \\ \partial_n^{\mathcal{C}[k]} = (-1)^k \partial_{n-k}^{\mathcal{C}} \end{cases}$$

Consider in particular $[1]$: it's easily seen to pass to the quotient in $\mathbf{K}(\mathcal{A})$; hence it defines an autoequivalence (its inverse is $[-1]$) $\mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$.
Recall now the definition of the mapping cone of a chain map:
Given a chain map $\varphi \colon \mathcal{X} \longrightarrow \mathcal{Y}$, its *mapping cone* is the chain complex $M(\varphi)$ defined by

$$\begin{cases} M(\varphi)_n = (\mathcal{X}[1] \oplus \mathcal{Y})_n = X_{n-1} \oplus Y_n \\ \partial_n^{\mathcal{M}(\varphi)} = \begin{pmatrix} -\partial_{n-1}^{\mathcal{X}} & 0 \\ \varphi_{n-1} & \partial_n^{\mathcal{Y}} \end{pmatrix} \end{cases}$$

Notice that there exist a canonical exact sequence

$$0 \longrightarrow \mathcal{Y} \xrightarrow{\alpha} M(\varphi) \xrightarrow{\beta} \mathcal{X}[1] \longrightarrow 0$$

where α, β are the obvious injection and projection

$$\begin{aligned} \alpha(\varphi)\colon \mathcal{Y} \longrightarrow M(\varphi) &\qquad \alpha(\varphi)_n := \begin{pmatrix} 0 \\ \mathrm{id}_{Y_n} \end{pmatrix} \\ \beta(\varphi)\colon M(\varphi) \longrightarrow \mathcal{X}[1] &\qquad \beta(\varphi)_n := \begin{pmatrix} \mathrm{id}_{X_{n-1}} & 0 \end{pmatrix}. \end{aligned}$$

Notice also that $\alpha(\varphi)$ and $\beta(\varphi)$ pass to homotopy, meaning that $\varphi \sim \psi$ implies $\alpha(\varphi) \sim \alpha(\psi)$, $\beta(\varphi) \sim \beta(\psi)$ ($M(\varphi) \cong M(\psi)$, via $\boldsymbol{\theta} = \begin{pmatrix} \mathrm{id}_{X_n} & 0 \\ s_n & \mathrm{id}_{Y_{n+1}} \end{pmatrix}$ so the claim is $\theta \circ \alpha(\varphi) \cong \alpha(\psi)$, obvious).

Definition

We call a d.t. in the homotopy category any triangle isomorphic to one of the form

$$\mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\alpha(\varphi)} M(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{X}[1]$$

- For any complex $\mathcal{X}$ consider the zero arrow $\varphi\colon \mathcal{X} \longrightarrow 0$. Then the mapping cone $M(\varphi)$ coincides with $\mathcal{X}[1]$. In a similar fashion $M(\psi) \cong \mathcal{Y}$ for $\psi\colon 0 \longrightarrow \mathcal{Y}$.
- Given X, Y objects of $\mathcal{A}$ and $f\colon X \longrightarrow Y$, seen as degree-zero-concentrated complexes, the mapping cone $M(f)$ coincides with the complex having f concentrated in degree 0,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \xrightarrow{\ f\ } Y \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

- The identity chain map of $M(\mathrm{id}_{\mathcal{X}})$, being $\mathcal{X} = (X_n, \partial_n)$ a complex, is nullhomotopic via the homotopy $s_n = \begin{pmatrix} 0 & \mathrm{id}_{X_{n-1}} \\ 0 & 0 \end{pmatrix}$;
- The mapping cone of a quasi-isomorphism $\varphi\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is acyclic.

Let's now check that the axioms of triangulated category hold in $\mathbf{K}(\mathcal{A})$.

- Axioms 1 to 3 hold by definition; in particular for axiom 2 one can always find a d.t.

$$\mathcal{X} \xrightarrow{\mathrm{id}_\mathcal{X}} \mathcal{X} \longrightarrow M(\mathrm{id}_\mathcal{X}) \longrightarrow \mathcal{X}[1]$$

  which is directly equivalent to

$$\mathcal{X} \xrightarrow{\mathrm{id}} \mathcal{X} \longrightarrow 0 \longrightarrow \mathcal{X}[1].$$

  ($M(\mathrm{id}_\mathcal{X})$ is nullhomotopic).

- To prove (rot) let's consider a d.t.

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{\alpha(\varphi)} M(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{X}[1].$$

  and prove that the triangle

$$\mathcal{Y} \xrightarrow{\varphi} M(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{X}[1] \xrightarrow{-\varphi[1]} \mathcal{Y}[1]$$

  is distinguished by finding a d.t. which is isomorphic to that one.

Consider the d.t. associated to $M(\varphi)$:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\alpha(\varphi)"] \arrow[d, equal] & M(\varphi) \arrow[r, "\beta(\varphi)"] \arrow[d, equal] & \mathcal{X}[1] \arrow[r, "-\varphi[1]"] \arrow[d, shift left, "\epsilon"] & \mathcal{Y}[1] \arrow[d, equal] \\
\mathcal{Y} \arrow[r, "\alpha(\varphi)"'] & M(\varphi) \arrow[r, "\alpha(\alpha(\varphi))"'] & M(\alpha(\varphi)) \arrow[r, "\beta(\alpha(\varphi))"'] \arrow[u, shift left, "\eta"] & \mathcal{Y}[1]
\end{tikzcd}$$

define

$$\epsilon\colon \mathcal{X}[1] \longrightarrow M(\alpha(\varphi))$$

$$X_{n-1} \xmapsto[\begin{pmatrix} -\varphi_{n-1} \\ \mathrm{id}_{X_{n-1}} \\ 0 \end{pmatrix}]{} Y_{n-1} \oplus X_{n-1} \oplus Y_n$$

$$\eta\colon M(\alpha(\varphi)) \longrightarrow \mathcal{X}[1]$$

$$Y_{n-1} \oplus X_{n-1} \oplus Y_n \xmapsto[\begin{pmatrix} 0 & \mathrm{id}_{X_{n-1}} & 0 \end{pmatrix}]{} X_{n-1}$$

and recall that $M(\alpha(\varphi)) = \left(\mathcal{Y}[1] \oplus M(\varphi), \begin{pmatrix} -\partial^{\mathcal{Y}}_{n-1} & 0 & 0 \\ 0 & -\partial^{X}_{n-1} & 0 \\ \mathrm{id}_{Y_{n-1}} & \varphi_{n-1} & \partial^{\mathcal{Y}}_{n} \end{pmatrix} \right)$. Now all squares are easily seen to commute (possibly up to homotopy):

- $\beta(\alpha(\varphi)) \circ \epsilon = \begin{pmatrix} \mathrm{id}_{Y_{n-1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} -\varphi_{n-1} \\ \mathrm{id}_{X_{n-1}} \\ 0 \end{pmatrix} = -\varphi_{n-1}$,
- $\epsilon \circ \beta(\varphi) \sim \alpha(\alpha(\varphi))$ via the map $\begin{pmatrix} 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : M(\varphi)_n \longrightarrow M(\alpha(\varphi))_{n+1}$ in such a way that

$$\begin{pmatrix} -\partial^{\mathcal{Y}}_{n} & 0 & 0 \\ 0 & -\partial^{X}_{n} & 0 \\ \mathrm{id}_{Y_{n+1}} & \varphi_{n} & \partial^{\mathcal{Y}}_{n+1} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 & -\mathrm{id}_{Y_{n-1}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\partial^{X}_{n-1} & 0 \\ \varphi_{n-1} & \partial^{\mathcal{Y}}_{n} \end{pmatrix} = \begin{pmatrix} -\varphi_{n-1} & 0 \\ 0 & 0 \\ 0 & -\mathrm{id} \end{pmatrix} = \epsilon \circ \beta(\varphi) - \alpha(\alpha(\varphi))$$

- $\beta(\varphi) = \eta \circ \alpha(\alpha(\varphi))$ holds by definition, and $-\varphi[1] \circ \eta \sim \beta(\alpha(\varphi))$ via the homotopy $\begin{pmatrix} 0 & 0 & \mathrm{id}_{Y_n} \end{pmatrix} : M(\alpha(\varphi))_n \longrightarrow Y_n$:

$$\beta(\alpha(\varphi)) - (-\varphi[1] \circ \eta) = \begin{pmatrix} \mathrm{id}_{Y_{n-1}} & \varphi[1] & 0 \end{pmatrix} = \\ - \partial^{\mathcal{Y}}_{n} \begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} -\partial^{\mathcal{Y}}_{n-1} & 0 & 0 \\ 0 & -\partial^{X}_{n-1} & 0 \\ \mathrm{id}_{Y_{n-1}} & \varphi_{n-1} & \partial^{\mathcal{Y}}_{n} \end{pmatrix}$$

Last but not least, $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ are mutually inverses up to homotopy: $\boldsymbol{\eta} \circ \boldsymbol{\epsilon} = \mathrm{id}_{X[1]}$ on the nose, while $\boldsymbol{\epsilon} \circ \boldsymbol{\eta} \sim \mathrm{id}_{M(\alpha(\boldsymbol{\varphi}))}$ via the homotopy $s_n = \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$$\epsilon_n \circ \eta_n - \mathrm{id}_{M(\alpha(\boldsymbol{\varphi}))} = \begin{pmatrix} -\mathrm{id} & -\varphi_{n-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id} \end{pmatrix} =$$

$$\begin{pmatrix} -\partial_n^{\mathcal{Y}} & 0 & 0 \\ 0 & -\partial_n^{\mathcal{X}} & 0 \\ \mathrm{id}_{Y_{n+1}} & \varphi_n & \partial_{n+1}^{\mathcal{Y}} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\partial_{n-1}^{\mathcal{Y}} & 0 & 0 \\ 0 & -\partial_{n-1}^{\mathcal{X}} & 0 \\ \mathrm{id}_{Y_n} & \varphi_{n-1} & \partial_n^{\mathcal{Y}} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{tikzcd}
M(\alpha(\boldsymbol{\varphi}))_{n+1} \arrow[r] & M(\alpha(\boldsymbol{\varphi}))_n \arrow[r, "\partial_{n+1}^{M(\alpha\boldsymbol{\varphi})}"] \arrow[dl, "s_n"'] \arrow[d, shift right, "\boldsymbol{\epsilon}\circ\boldsymbol{\eta}"'] \arrow[d, shift left, "\mathrm{id}"] & M(\alpha(\boldsymbol{\varphi}))_{n-1} \arrow[dl, "s_{n-1}"] \\
M(\alpha(\boldsymbol{\varphi}))_{n+1} \arrow[r, "\partial_{n+1}^{M(\alpha\boldsymbol{\varphi})}"'] & M(\alpha(\boldsymbol{\varphi}))_n \arrow[r] & M(\alpha(\boldsymbol{\varphi}))_{n-1}
\end{tikzcd}$$

- (com) axiom can be verified over d.t.s only; suppose

$$\begin{tikzcd} \mathcal{X} \arrow[r, "u"] \arrow[d, "\varphi"'] & \mathcal{Y} \arrow[r, "\alpha(u)"] \arrow[d, "\gamma"] & M(u) \arrow[r, "\beta(u)"] \arrow[d, dotted, "h"'] & \mathcal{X}[1] \arrow[d, "\varphi[1]"] \\ \mathcal{X}' \arrow[r, "u'"'] & \mathcal{Y}' \arrow[r, "\alpha(u')"'] & M(u') \arrow[r, "\beta(u')"'] & \mathcal{X}'[1] \end{tikzcd}$$

is a morphism of triangles up to the dotted arrow. Left square commutes up to a family of maps $\{s_n \colon X_n \longrightarrow Y'_{n+1}\}$ such that $\gamma_n \circ u_n - u'_n \circ f_n = \partial^{\mathcal{Y}'}_{n+1} \circ s_n + s_{n-1} \circ \partial^{\mathcal{X}}_n$. Define

$$h_n \colon \; X_{n-1} \oplus Y_n \xrightarrow{\begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix}} X'_{n-1} \oplus Y'_n$$

It is a morphism of complexes because of the commutativity of

$$\begin{tikzcd} X_n \oplus Y_{n+1} \arrow[r, "\begin{pmatrix} -\partial & 0 \\ u & \partial \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \varphi_n & 0 \\ s_n & \gamma_{n+1} \end{pmatrix}"'] & X_{n-1} \oplus Y_n \arrow[d, "\begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix}"] \\ X'_n \oplus Y'_{n+1} \arrow[r, "\begin{pmatrix} -\partial & 0 \\ u' & \partial \end{pmatrix}"] & X'_{n-1} \oplus Y'_n \end{tikzcd}$$

Let's see that $h \circ \alpha(u) = \alpha(u') \circ \gamma$ and $\beta(u') \circ h = \varphi[1] \circ \beta(u)$: in fact they easily follow from

$$\begin{cases} \begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix} \\ \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} \gamma_n = \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix} \end{cases} \qquad \begin{cases} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{n-1} & 0 \\ s_{n-1} & \gamma_n \end{pmatrix} = \begin{pmatrix} \varphi_{n-1} & 0 \end{pmatrix} \\ \varphi_{n-1} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_{n-1} & 0 \end{pmatrix} \end{cases}$$

- Octahedral axiom is the longest and messiest to verify: consider

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "u"] \arrow[d, equal] & \mathcal{Y} \arrow[r, "\alpha(\varphi)"] \arrow[d, "v"'] & M(u) \arrow[r, "\beta(u)"] \arrow[d, dotted, "f"] & \mathcal{X}[1] \arrow[d, equal] \\
\mathcal{X} \arrow[r, "v \circ u"'] \arrow[d, "u"'] & \mathcal{Z} \arrow[r, "\alpha(vu)"] \arrow[d, equal] & M(vu) \arrow[r] \arrow[d, dotted, "g"] & \mathcal{X}[1] \arrow[d, "u[1]"] \\
\mathcal{Y} \arrow[r, "v"] \arrow[d, "\alpha(u)"'] & \mathcal{Z} \arrow[r, "\alpha(v)"] \arrow[d, "\alpha(vu)"'] & M(v) \arrow[r, "\beta(v)"] \arrow[d, equal] \arrow[dr, phantom, "\spadesuit"] & \mathcal{Y}[1] \arrow[d, "\alpha(u)[1]"] \\
M(u) \arrow[r, dotted, "f"'] & M(vu) \arrow[r, dotted, "g"'] & M(v) \arrow[r, dotted, "h"'] & M(u)[1]
\end{tikzcd}$$

Let's define dotted arrows

$$f\colon M(u) \longrightarrow M(v\circ u)$$
$$X_{n-1}\oplus Y_n \xmapsto[\begin{pmatrix}\mathrm{id}_{X_{n-1}} & 0\\ 0 & v_n\end{pmatrix}]{} X_{n-1}\oplus Z_n$$

$$g\colon M(v\circ u) \longrightarrow M(v)$$
$$X_{n-1}\oplus Z_n \xmapsto[\begin{pmatrix}u_{n-1} & 0\\ 0 & \mathrm{id}_{Z_n}\end{pmatrix}]{} Y_{n-1}\oplus Z_n$$

$$h\colon M(v) \longrightarrow M(u)[1]$$
$$Y_{n-1}\oplus Z_n \xmapsto[\alpha(u)[1]\circ\beta(v)=\begin{pmatrix}0 & 0\\ \mathrm{id}_{Y_{n-1}} & 0\end{pmatrix}]{} X_{n-2}\oplus Y_{n-1}$$

(notice that with this definition of h square ♠ automatically commute). The resulting triangle turns out to be isomorphic to

$$M(u)\xrightarrow{f} M(v\circ u)\xrightarrow{\alpha(f)} M(f)\xrightarrow{\beta(f)} M(u)[1]$$

Consider the diagram

$$\begin{tikzcd} M(u) \arrow[r, "f"] \arrow[d, equal] & M(v\circ u) \arrow[r, "g"] \arrow[d, equal] & M(v) \arrow[r, "h"] \arrow[d, shift right, "\boldsymbol{\sigma}"'] & M(u)[1] \arrow[d, equal] \\ M(u) \arrow[r, "f"'] & M(v\circ u) \arrow[r, "\alpha(f)"'] & M(f) \arrow[r, "\beta(f)"'] \arrow[u, shift right, "\boldsymbol{\tau}"'] & M(u)[1] \end{tikzcd}$$

We shall find $\boldsymbol{\sigma}, \boldsymbol{\tau}$ (mutually inverses up to homotopy) such that

$$\begin{cases} \beta(f)\circ\boldsymbol{\sigma} \sim h \\ h\circ\boldsymbol{\tau} \sim \beta(f) \\ \boldsymbol{\sigma}\circ g \sim \alpha(f) \\ \boldsymbol{\tau}\circ\alpha(f) \sim g \end{cases}$$

Notice that first and the fourth identity hold on the nose as soon as we define

$$\sigma_n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y_{n-1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{Z_n} \end{pmatrix}, \quad \tau_n = \begin{pmatrix} 0 & \mathrm{id}_{Y_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_n} \end{pmatrix}$$

A homotopy between $\alpha(f)$ and $\boldsymbol{\sigma} \circ g$ is seen to be $\begin{pmatrix} \mathrm{id}_{X_{n-1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$; to conclude the proof, let's find a homotopy between $\beta(f)$ and $h \circ \boldsymbol{\tau}$: their difference in degree n is $\begin{pmatrix} \mathrm{id}_{X_{n-2}} & 0 & 0 & 0 \\ 0 & 0 & -u_{n-1} & 0 \end{pmatrix}$, hence a homotopy between the two is easily seen to be

$$\begin{pmatrix} 0 & 0 & \mathrm{id}_{X_{n-1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \longrightarrow X_{n-1} \oplus Y_n$$

Finally, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are mutually inverses up to homotopy: $\boldsymbol{\tau} \circ \boldsymbol{\sigma} = \mathrm{id}$ on the nose. Inverse composition gives

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{Y_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_n} \end{pmatrix}$$

hence we need the homotopy $\begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

This completes the proof of the octahedral axiom, and the proof that $\mathbf{K}(\mathcal{A})$ is triangulated by the shift functor and mapping-cones triangles as d.t.s $\square$

Towards Derived Cats

A problem arises.

By definition, any d.t. $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\alpha(\varphi)} M(\varphi) \xrightarrow{\beta(\varphi)} \mathcal{X}[1]$ in $\mathbf{K}(\mathcal{A})$ gives rise to an exact sequence $0 \longrightarrow \mathcal{Y} \longrightarrow M(\varphi) \longrightarrow \mathcal{X}[1] \longrightarrow 0$ in $\mathbf{Ch}(\mathcal{A})$.

The converse is not true! (consider $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ (complexes conc. in degree zero), if there exists a d.t. $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[1]$, notice that there are no non-zero arrows $\mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[1]$, hence $w = 0$; then $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, contradiction).

Solution: pass to the derived category.