Homotopy Theory in C^*-**Alg**

Fosco Loregian

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Model categories fruitfully extend Algebraic Topology to the realm of "things that resemble spaces":

A model category is a triple $(\mathbf{C}, \mathrm{W{\scriptsize K}}, \mathrm{F{\scriptsize IB}}, \mathrm{C{\scriptsize OF}})$ where $\mathrm{W{\scriptsize K}}$ is a class of arrows to be turned into isomorphisms, and $\mathrm{F{\scriptsize IB}}, \mathrm{C{\scriptsize OF}}$ are two additional classes ensuring that the localized category $\mathrm{H{\scriptsize O}}(\mathbf{C}) =: \mathbf{C}[\mathrm{W{\scriptsize K}}^{-1}]$ is not as badly-behaved as might happen.

Examples come either from Topology and Algebra:

- Topological spaces and Serre fibrations/cofibrations;
- Simplicial sets and Kan fibrations/cofibrations;
- Chain complexes of R-modules and degree-wise epimorphisms as fibrations.

$$\boxed{\mathrm{W{\scriptsize K}} \cap \mathrm{F{\scriptsize IB}} \Rightarrow \mathrm{C{\scriptsize OF}} \qquad\qquad \mathrm{W{\scriptsize K}} \cap \mathrm{C{\scriptsize OF}} \Rightarrow \mathrm{F{\scriptsize IB}}}$$

Localization theory in a nutshell

- Everybody is comfortable with the construction building $\mathbb{Q}$ out of $\mathbb{Z}$;
- Everybody is comfortable with the construction building $R[S^{-1}]$, out of a commutative domain R: for S a multiplicative system S,

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "i"'] & S \\ R[S^{-1}] \arrow[ur, "\exists! \bar{f}"'] & \end{tikzcd}$$

- Noncommutative analogue is contained in B. Stenström, *Rings Of Quotients*.

Localization theory for categories is the exact formal analogue: for $S \subseteq \mathrm{Mor}(\mathbf{C})$,

$$\begin{tikzcd} \mathbf{C} \arrow[r, "f"] \arrow[d, "i"'] & \mathbf{D} \\ \mathbf{C}[S^{-1}] \arrow[ur, "\exists! \bar{f}"'] & \end{tikzcd}$$

- It always exists ([Zisman], 1967);
- Examples: $h\mathbf{Top}$, complexes modulo homotopy, derived categories, etc.

What's nearer to a space, albeit not being one, than a commutative C^*-algebra? Recall Gel'fand-Naimark duality:

$$\mathrm{C}^*\text{-}\mathbf{Alg}_c \cong \mathbf{LCHaus}.$$

Look for Algebraic-Topological-methods in C^*-algebra theory: is there an **homotopical calculus** on C^*-**Alg** (C^*-$\mathbf{Alg}^{\mathrm{op}}$ "plays the rôle" of noncommutative spaces)?

Main Theorem in [UUye]'s paper: There is one![1] But...

- (Andersen-Grodal) This can't come from a Quillen model structure on C^*-**Alg**: so we look for a weaker form of Homotopical Calculus, still fitting our needs.
- The notion we need is that of category with fibrant objects (cfo for short) introduced by Kenneth Brown in his PhD thesis [Brown].

[1] Indeed, an entire countable family [Uuye, private communication]!

C^*-**Alg** admits a fibrant structure, obtained from a “trivial” truncation of the classical model structure (**Top**, homotopy equivalences, Serre fibrations):

- Weak equivalences are maps $X \to Y$ inducing a bijection between $\pi_0(X) \to \pi_0(Y)$;
- Fibrations are maps $E \to B$ having the RLP with respect to diagrams

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ {[0,1]} \arrow[r] & B \end{tikzcd}$$

Natural question: Does C^*-**Alg** admit a model structure, and in the end a cfo structure induced by a model structure?

- Enrichment over cubical C^*-spaces:

$$\mathrm{C}^*\text{-}\mathbf{Alg} \xrightarrow{\text{yon}} \mathbf{Sets}^{\mathrm{C}^*\text{-}\mathbf{Alg}} = \mathrm{C}^*\text{-}\mathbf{Spc} \rightarrowtail \boxdot\mathrm{C}^*\text{-}\mathbf{Spc}$$

- Categorification of the notion of C^*-algebra $\Rightarrow$ C^*-category [Dell’Ambrogio].

Definition (Category with Weak Equivalences)

A finitely (co)complete category $\mathbf{C}$ with a distinguished class of morphisms $\mathrm{W{\scriptstyle K}} \subseteq \mathrm{Mor}(\mathbf{C})$

- containing all isomorphisms of $\mathbf{C}$,
- closed under composition
- satisfies the *two-out-of-three property*:

  For f, g any two composable morphisms of $\mathbf{C}$, if any two of $\{f, g, g \circ f\}$ are in $\mathrm{W{\scriptstyle K}}$, then so is the third.

Let $(\mathbf{C}, \mathrm{W{\scriptstyle K}})$ be a category with weak equivalences. Distinguish a second class of arrows $\mathrm{F{\scriptstyle IB}}$ and call them fibrations; call acyclic fibration any $f \in \mathrm{W{\scriptstyle K}} \cap \mathrm{F{\scriptstyle IB}}$.

A *path object* for an object $B \in \mathrm{Ob}_{\mathbf{C}}$ consists of a factorization

$$B \xrightarrow{\quad s \quad} B^I \xrightarrow{\langle d_0, d_1 \rangle} B \times B$$

of the canonical arrow $\Delta \colon B \to B \times B \colon b \mapsto (b, b)$, where $\langle d_0, d_1 \rangle \in \mathrm{F{\scriptstyle IB}}$, $s \in \mathrm{W{\scriptstyle K}}$.

Definition (Category with Fibrant Objects)

A category with fibrant objects (cfo for short) is a triple $(\mathbf{C}, \mathrm{WK}, \mathrm{FIB})$ where $(\mathbf{C}, \mathrm{WK})$ is a category with weak equivalences, and where FIB is a class of maps such that

CF1 (Closure) Any isomorphism is a fibration; FIB is closed under composition.

CF2 (Stability) FIB and $\mathrm{FIB} \cap \mathrm{WK}$ are *stable under pullback*: if $f: A \twoheadrightarrow B$ is a(n acyclic) fibration, and $u: X \to B$ is any arrow, then in the diagram

$$\begin{tikzcd} A \times_B X \arrow[r] \arrow[d, two heads] & A \arrow[d, two heads, "f"] \\ X \arrow[r, "u"'] & B \end{tikzcd}$$

the arrow $A \times_B X \to X$ is again a(n acyclic) fibration.

CF3 (Paths) For all $B \in \mathrm{Ob}_{\mathbf{C}}$ there exists at least a path object $(B^I, s, \langle d_0, d_1 \rangle)$, and $B \mapsto B^I$ is functorial.

CF4 (Fibrance) The unique arrow $B \to *$ to the terminal object is a fibration.

- We will often write $\xrightarrow{\sim}$ to denote a weak equivalence, and $\twoheadrightarrow$ to denote a fibration.

- In a cfo the projection maps $A \times B \to A, B$ are fibrations –they can be obtained pulling back $A \to * \leftarrow B$;

$\Rightarrow$ For any B the composition maps $d_i = \pi_i \circ \langle d_0, d_1 \rangle$, $i = 0, 1$, are acyclic fibrations $B^I \to B$.

First fundamental theorem of Homotopical Algebra:

Lemma (Brown's Factorization Lemma)

The couple $(\mathrm{W\scriptstyle K}, \mathrm{F\scriptstyle IB})$ is a *factorization system* in **C**, i.e. any arrow $u \colon A \to B$ can be factored as the composition $p \circ i$ of a weak equivalence i and a fibration p; this fibration is acyclic if and only if u was a weak equivalence.

$\rightsquigarrow$ Resembles the fact that in **Top** any arrow canbe factored as the composition of a Hurewicz fibration and a homotopy equivalence / Hurewicz cofibration and a homotopy equivalence:

$$X \overset{j}{\rightarrowtail} \mathrm{cyl}(f) \xrightarrow[\sim]{r} Y$$

$$X \xrightarrow[\sim]{\nu} N(f) \overset{\rho}{\twoheadrightarrow} Y$$

where $N(f) := X \times_Y Y^I$, $\mathrm{cyl}(f) = \big((X \wedge I) \amalg Y\big)/(x \sim f(x))$.

Second fundamental theorem of Homotopical Algebra: The localized category $\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathrm{W\scriptstyle K}^{-1}]$ can be described via homotopies between maps in $\mathbf{C}$:

Definition (Homotopy relation)

$f, g \colon A \rightrightarrows B$ are called *(right) homotopic* if there exists a third arrow $h \colon A \to B^I$ such that $d_0 \circ h = f, d_1 \circ h = g$.
"f, g are right homotopic via h" is denoted $f \rtimes_h g$.

- $\rtimes_h$ is a binary equivalence relation on $\hom_{\mathbf{C}}(A, B)$; it is not a two-sided congruence: there exists a dual notion of *left* homotopy ${}_h\ltimes$ between maps and $\rtimes_h$ is a congruence iff the two coincide (example: **Top**).
- Homotopic maps become equal in $\mathrm{Ho}(\mathbf{C})$:

$$\gamma(f) = \gamma(d_0)\gamma(h) = \gamma(s)^{-1}\gamma(h) = \gamma(d_1)\gamma(h) = \gamma(g).$$

$\mathrm{Ho}(\mathbf{C})$ can be in a certain sense *approximated* with a category $\pi\mathbf{C}$ obtained from $\mathbf{C}$:

- Objects in $\pi\mathbf{C}$ are the same as in $\mathbf{C}$ and $\mathrm{Ho}(\mathbf{C})$;
- $\hom_{\pi\mathbf{C}}(A, B) = \hom_{\mathbf{C}}(A, B)/\bowtie$, where $f \bowtie g$ iff there exist a weak equivalence $t \colon X \to A$ and a homotopy h such that $f \circ t \rtimes_h g \circ t$.

Let A, B objects of a cfo $\mathbf{C}$. Then there exists a canonical isomorphism

$$\hom_{\mathrm{Ho}(\mathbf{C})}(A, B) \cong \varinjlim_{\pi\mathbf{C}/A} \hom_{\pi\mathbf{C}}(-, B)$$

where $\pi\mathbf{C}/A$ contains as objects $[t]\colon X \to A$ homotopy classes of weak equivalences in $\pi\mathbf{C}$, and an arrow $[t\colon X \to A] \to [s\colon Y \to A]$ consists of a homotopy class of arrows $X \to Y$ making the obvious triangle commute.

- "Manageable" characterization of $\hom_{\mathrm{Ho}(\mathbf{C})}(A, B)$ (there's something –the existence of a calculus of fraction permitting this identification– we have kept hidden in the closet). Useful because

Corollary 1: any arrow $f\colon A \to B$ in $\mathrm{Ho}(\mathbf{C})$ can be written as a *right fraction* $\gamma(f') \circ \gamma(t)^{-1}$ where $t \in \mathrm{W\scriptstyle K}$:

$$\begin{tikzcd} & A' \arrow[dl, "t"'] \arrow[dr, "f'"] & \\ A \arrow[rr, dotted, "f"'] & & B \end{tikzcd}$$

Corollary 2: if $f, g\colon A \rightrightarrows B$, then $\gamma(f) = \gamma(g)$ iff there exists a weak equivalence t which coequalizes both.

Stable Homotopy Theory

We call a cfo $\mathbf{C}$ pointed if it admits a zero object.

Definition

Let $\mathbf{C}$ be a pointed cfo. Define the *loop object* of $B \in \mathrm{Ob}_{\mathbf{C}}$, denoted $\Omega^I(B)$, as the typical fibre of $\langle d_0, d_1 \rangle \colon B^I \to B \times B$, once a particular path object B^I has been chosen for B.

Theorem

The correspondence $B \mapsto \Omega^I(B)$ defines a functor $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$. The loop object of B is an internal group in $\mathrm{Ho}(\mathbf{C})$; iterated loop objects $\Omega^k(B)$ are internal abelian groups.

Fulcrum of the proof: $B \mapsto \Omega^I B$ is a well posed definition, functorial up-to-homotopy (hence defines a mere functor $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$, *not* a functor $\mathbf{C} \to \mathbf{C}$).

- Topological analogue: $\pi_1(X, x)$ is a group only up to homotopy;
- Proof really technical, commutativity of iterated loop spaces exploits Eckmann-Hilton argument.

Stable Homotopy Theory

General problem: given a category $\mathbf{E}$ and an endofunctor Ω, find $\mathbf{E} \hookrightarrow \mathcal{SW}_\Omega(\mathbf{E})$ and an autoequivalence $\hat{\Omega}$ such that $\hat{\Omega}|_{\mathbf{E}} = \Omega$.

Our particular case: a pointed cfo $\mathbf{C}$ and $\mathbf{E} = \mathrm{Ho}(\mathbf{C})$, Ω the loop-object functor. $\mathcal{SW}_\Omega(\mathrm{Ho}(\mathbf{C})) =: \mathsf{SHo}(\mathbf{C})$: its objects are pairs $(A, n) \in Ob(\mathbf{C}) \times \mathbb{Z}$, and arrows $(A, n) \to (B, m)$ correspond to the colimit set

$$\varinjlim_{k \in \mathbb{N}} [\Omega^{n+k} A, \Omega^{m+k} B]_{\mathbf{C}}$$

- Well posed construction (composition is obtained by UMP of $\varinjlim$, associativity exploits cofinality of $\iota_K : \mathbb{N}_{\geq K} \hookrightarrow \mathbb{N}$);
- The same cofinality entails that $\{\Omega^i : (A, n) \mapsto (A, n+i)\}$ is a $\mathbb{Z}$-indexed family of endofunctors: $\Omega^i \circ \Omega^j = \Omega^{i+j}$, hence $(\Omega^i)^{-1} = \Omega^{-i}$, and $\Omega = \Omega^1$ is invertible.
- The stable homotopy category is triangulated in the sense of Verdier by $\Sigma = \Omega^{-1}$. Distinguished triangles are of the form

$$(\Omega B, n) \to (F, n) \to (E, n) \to (B, n),$$

where $E \twoheadrightarrow B$, $F \hookrightarrow E$ is the homotopy inclusion of the typical fibre, $\Omega B \to F$ can be obtained via [Brown, Prop. I.4.4].

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where $E \twoheadrightarrow B$, $F \hookrightarrow E$ is the homotopy inclusion of the typical fibre, $\Omega B \to F$ can be obtained via [Brown, Prop. I.4.4].
- A pointed cfo $\mathbf{C}$ is said to be stable if the loop functor Ω is already an autoequivalence of $\mathrm{Ho}(\mathbf{C})$. If $\mathbf{C}$ is a stable cfo, then $\Omega^0 \colon \mathrm{Ho}(\mathbf{C}) \to \mathsf{SHo}(\mathbf{C})$ is already an equivalence, and it induces a triangulated structure.
- Triangulated structure = starting point for axiomatic homology theory:

Definition (Homology theory)

A *homology theory* in a pointed cfo is a homological functor $\mathcal{H} \colon \mathsf{SHo}(\mathbf{C}) \to \mathbf{Ab}$.

$\mathcal{H}$-equivalences	$\leftrightsquigarrow$	quasi-isomorphism
$\mathcal{H}$-acyclic objects	$\leftrightsquigarrow$	homologically trivial "spaces"

Application: [Uuye, 2.3, 2.4] $\mathcal{H}$ = "Axiomatic" topological K-theory.

A cfo structure on **Top**

Top $=$*compactly generated Hausdorff spaces* or any other closed symmetric monoidal subcategory of topological spaces such that C^*-**Alg** $\leq$ **Top** (any metric space –C^*-algebras are such– is obviously Hausdorff and compactly generated).

Definition (0-Fibration)

An arrow $p\colon E \to B$ is a 0-fibration if it has the RLP with respect to the inclusion $\{0\} \hookrightarrow [0,1]$. (fibrations lift paths given the initial point). We denote the class of 0-fibrations as $\mathrm{FIB}_{\mathbf{Top},0}$.

Definition (0-Weak equivalence)

An arrow $f \in \hom_{\mathbf{Top}}(A,B)$ is a weak 0-equivalence if $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is a bijection. We denote the class of weak 0-equivalences as $\mathrm{WK}_{\mathbf{Top},0}$.

Theorem

The triple $(\mathbf{Top}, \mathbf{Wk}_{\mathbf{Top},0}, \mathbf{Fib}_{\mathbf{Top},0})$ is a cfo, denoted π_0-Top for short.

The only two non-evident properties are the existence of enough (functorial) path objects and the closure of $\mathrm{FIB}_{\mathbf{Top},0} \cap \mathrm{WK}_{\mathbf{Top},0}$ under base change:

- $B^I = \mathbf{Top}([0,1], B)$. The diagonal map can be factored as a composition

$$B \xrightarrow{i} B^I \xrightarrow{p} B \times B$$

where $i\colon B \xrightarrow{\sim} B^I$ sends b to the constant path $\gamma_b(t) \equiv b$, and p sends γ in $p(\gamma) = (\gamma(0), \gamma(1))$ in such a way that $(p \circ i)(x) = p(i(x)) = (x, x) = \Delta(x)$; p is a fibration because it is a product of fibrations ($p = \mathrm{ev}_0 \times \mathrm{ev}_1$).

- Let p be an acyclic fibration, then [Uuye, Prop. 2.10] it is surjective; hence in the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \pi_0(A \times_B E) \arrow[r] \arrow[d] & \pi_0(E) \arrow[d, "\pi_0(p)", "\sim"'] \\
* \arrow[r] & \pi_0(A) \arrow[r] & \pi_0(B).
\end{tikzcd}$$

$\pi_0(A \times_B E) \to \pi_0(A)$ is again surjective, and injective by direct check (suppose $[f(x)] = [f(y)]$, so there exists a continuous arc $\gamma\colon fx \leftrightsquigarrow fy$, and a continuous arc starting at x in $A \times_B E$, say α, such that $f(\alpha(t)) = \gamma$: this is responsible of $[x] = [y]$).

C^*-algebras in a nutshell

- A Banach space is an internal $\mathbb{C}$-vector space in **CNor**, the category of normed vector spaces, complete wrt their norm. They form a category **Ban**.
- A Banach algebra is an internal algebra in **Ban**.
- A *-algebra is a Banach algebra endowed with an involutory conjugate-linear antimorphism of algebras, $(ab)^* = b^* a^*$.

C^*-**Alg** = full subcategory of *-**Alg** made of those algebras such that $\|a^* a\| = \|a\|^2$.

It is a complete and cocomplete, **Top**-concrete and **Top**-enriched, symmetric monoidal (with respect to $- \otimes_{\max} -$) category;

▶ Gel'fand duality becomes an equivalence of enriched categories [Borceux, II.6.7].

cfo structure on C^*-**Alg**

- For any $A, B \in \mathrm{C}^*$-**Alg**, and X a compact Hausdorff space,

$$\hom_{\mathbf{Top}}(X, \hom_{\mathrm{C}^*}(A, B)) \cong \hom_{\mathrm{C}^*}(A, C(X) \otimes B)$$

($B^X = \hom_{\mathbf{Top}}(X, B)$ can be given the structure of a C^*-algebra thanks to compactness of X, adjoint nonsense does the rest).

Definition (Weak C^*-equivalence)

$t\colon A \to B$ a weak C^*-equivalence if for any $D \in \mathrm{C}^*$-**Alg** the induced map

$$t_\#\colon \hom_{\mathrm{C}^*}(D, A) \to \hom_{\mathrm{C}^*}(D, B)$$

is a weak equivalence in π_0-**Top**.

Definition (C^*-fibration)

$p\colon E \to B$ is a (Schochet) C^*-fibration if for any $D \in \mathrm{C}^*$-**Alg** the induced map

$$p_\#\colon \hom_{\mathrm{C}^*}(D, E) \to \hom_{\mathrm{C}^*}(D, B)$$

is a fibration in π_0-**Top**.

cfo structure on C^*-**Alg**

- Define a category $\pi_0\mathrm{C}^*$-**Alg** starting from C^*-**Alg** with the same objects as C^*-**Alg** and
$$\hom_{\pi_0\mathrm{C}^*\text{-}\mathbf{Alg}}(A,B) := \pi_0\big(\hom_{\mathrm{C}^*}(A,B)\big).$$
- if $X \in$ **LCTop**, homotopy classes of continuous maps $X \to Y$ in **Top** can be identified with arcwise connected components of the map space Y^X
- $t \in \mathrm{Mor}(\mathrm{C}^*\text{-}\mathbf{Alg})$ is a weak equivalence if and only if $\pi_0(t) \in \pi_0\mathrm{C}^*$-**Alg** is an invertible map.

Notation. In [Schochet]'s paper our π_0-fibrations are called π_0-*co*fibrations:

> RLP *asked to our π_0-fibrations* $\iff$ LLP *asked to* $\mathit{Spec}\,B \to \mathit{Spec}\,A$ *to be a cofibration in* **Top** *(***LCHaus** $\subset$ **Top** *in our notations), via Gel'fand duality.*

A similar incomplete notational-dualization gives the suspension functor $\Sigma = \mathbb{S}^1 \wedge -$ as the Gel'fand dual of $C_*(\mathbb{S}^1) \otimes -$.

$$B^{\mathbb{S}^1} = \hom_{\mathbf{Top}_*}(\mathbb{S}^1, B) \cong C_*(\mathbb{S}^1 \wedge \mathrm{Spec}\,B)$$

(Lemma **5.1** of exposition).

cfo structure on C^*-**Alg**

Because of that we are led to define the path object on C^*-**Alg** in such a way it corresponds to the path object $(\mathrm{Spec}\, B)^I$ in π_0-**Top**: for any $B \in \mathrm{C}^*$-**Alg** define $B^I := C_*([0,1]) \otimes B$.

Theorem (Uuye, thm. 2.11)

The triple $(\mathrm{C}^*\text{-}\mathbf{Alg}, \mathrm{W{\small K}}_{\mathrm{C}^*}, \mathrm{F{\small IB}}_{\mathrm{C}^*})$ is a pointed cfo. The homotopy category $\mathrm{H{\small O}}(\mathbf{C})$ is the category $\pi_0\mathrm{C}^*$-**Alg** defined before; we denote the triple $(\mathrm{C}^*\text{-}\mathbf{Alg}, \mathrm{W{\small K}}_{\mathrm{C}^*}, \mathrm{F{\small IB}}_{\mathrm{C}^*})$ as $\pi_0\mathrm{C}^*$-**Alg** (refer to it as the "π_0 structure").

Proof.

The only thing we can't derive directly from the existence of π_0-**Top** is that any Schochet fibration is a surjective map, but this follows easily "mimicking" the simple proof given for π_0-**Top** (see [Uuye], Prop. 2.10). □

Interlude: Model Categories

▶ model category = a cfo $(\mathbf{C}, \mathrm{W\scriptstyle K}, \mathrm{F\scriptstyle IB})$ endowed with an additional class of arrows $\mathrm{C\scriptstyle OF}$, cofibrations, having suitable stability and lifting properties with respect to fibrations.

- $(\mathbf{C}, \mathrm{W\scriptstyle K})$ is a category with weak equivalences;
- $\mathrm{W\scriptstyle K}, \mathrm{F\scriptstyle IB}, \mathrm{C\scriptstyle OF}$ are stable under taking retracts;
- For any commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ Y \arrow[r] & W, \end{tikzcd}$$

  where $i \in \mathrm{C\scriptstyle OF}$ and $p \in \mathrm{F\scriptstyle IB}$, if either i or p is acyclic, then there exists a lifting $Y \to Z$ (acyclic fibrations/cofibrations have the *right/left lifting property* —RLP, LLP for short— with respect to fibrations/cofibrations);
- $(\mathrm{W\scriptstyle K} \cap \mathrm{F\scriptstyle IB}, \mathrm{C\scriptstyle OF})$, $(\mathrm{F\scriptstyle IB}, \mathrm{W\scriptstyle K} \cap \mathrm{C\scriptstyle OF})$ are (weak) factorization systems in $\mathbf{C}$.

Proposition

The loop functor $\Omega\colon \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg}) \to \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg})$ preserves finite products.

Remark

Ω doesn't commute with *infinite* products $\iff$ it can't have a left adjoint, but the construction of arbitrary products and coproducts in $\mathrm{C}^*\text{-}\mathbf{Alg}$ if very far from explicit: can one prove it directly?

Theorems **I.1.1** and **I.2.2** in [Quillen] say that if $\mathbf{C}$ is a model category, the loop functor $\Omega|_c\colon \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}}) \to \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}})$ must be a right adjoint, but...

Theorem (Andersen-Grodal)

The loop functor $\Omega\colon \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg}) \to \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg})$ doesn't admit a left adjoint.

Proof.

Notice (i) If $\Omega\colon \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg}_c) \to \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg}_c)$ doesn't admit a left adjoint then $\Omega\colon \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg}) \to \mathrm{Ho}(\mathrm{C}^*\text{-}\mathbf{Alg})$ doesn't admit it too and (ii) by Gel'fand-Naimark duality it suffices to show that the (topological) suspension functor $\Sigma\colon \mathbf{CHaus}_* \to \mathbf{CHaus}_*$ doesn't admit a *right* adjoint. $\square$

In the end the whole proof boils down to show that $\Sigma\colon X \mapsto X \wedge \mathbb{S}^1$ doesn't admit a right adjoint.

In particular let's show that the functor $A \mapsto [\Sigma A, \mathbb{S}^1]$ is not representable. If it is, then $[\Sigma A, \mathbb{S}^1] \cong [A, Y]$ for some $Y \in \mathrm{Ho}(\mathbf{CHaus}_*)$.

$$[A, Y]_{\mathbf{Top}_*} \cong [A, Y]_{\mathbf{CHaus}_*} \cong [\Sigma A, \mathbb{S}^1]_{\mathbf{CHaus}_*} \cong [A, \Omega\mathbb{S}^1]_{\mathbf{Top}_*}$$

Yoneda lemma $\Rightarrow$ this must come from an isomorphism $f\colon Y \to \Omega\mathbb{S}^1$ in $\mathrm{Ho}(\mathbf{CHaus}_*)$, namely from a weak (homotopy) equivalence $Y \to \Omega\mathbb{S}^1$, inducing degree-wise isomorphisms between homotopy groups $\pi_i(Y)$ and $\pi_i(\Omega\mathbb{S}^1)$. The space $\mathbb{S}^1$ being a $K(\mathbb{Z}, 1)$, this boils down to say that $\pi_i(Y) \cong \pi_{i+1}(\mathbb{S}^1)$ for $i = 0, 1$.
Now $\pi_0(Y) \cong \pi_0(\Omega\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, hence Y must have an infinite number of arcwise connected components.

This is the desider contradiction (and notice that we can't conclude now, because there is plenty of compact spaces having an infinite number of *non-open* c.c.).

In the composition $Y \xrightarrow{\sim} \Omega S^1 \xrightarrow{\deg} \mathbb{Z}$ the function deg is an isomorphism in **TopGrp**, hence the preimage of open sets in $\mathbb{Z}$ (=singletons) must be open, and is exactly one of the arcwise connected components of Y.

Does C^*-**Alg** admit a model structure?

Choosing the right theory depends on a balancing act: you can

A. either get a simple theory (category of fibrant objects on C^*-algebras) and lose the power (model structure) of your language

B. or get a powerful theory (model structure) and lose the simplicity (enlarge C^*-**Alg** to a richer but more complicated category).

Uuye's path is A., what about B.?

- Take the path of categorification and enrichment: a C^*-category is a category **C** enriched over the symmetric monoidal category of (complex) C^*-algebras. Proposition **4.15** in [Dell'Ambrogio] shows that the (2-)category of C^*-categories is cofibrantly generated, and endowed with a simplicial enrichment over the monoidal model category $\mathbf{SSet}_{\mathrm{Quil}}$ of simplicial sets.
- Take the path of smoothing and enrichment: a cubical set is a presheaf on the "cube category", $\boxdot^{\mathrm{op}} \to \mathbf{Sets}$. Consider the classical Yoneda embedding $\mathrm{yon}\colon \mathrm{C}^*\text{-}\mathbf{Alg}^{\mathrm{op}} \to \mathbf{Sets}^{\mathrm{C}^*\text{-}\mathbf{Alg}}\colon A \mapsto \hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(A, -)$, and the subcategory $\boxdot\mathrm{C}^*$-Spc of *cubical set-valued presheaves* $\mathrm{Fun}(\mathrm{C}^*\text{-}\mathbf{Alg}, \boxdot\mathbf{Sets}) \cong \mathrm{Fun}(\mathrm{C}^*\text{-}\mathbf{Alg}, \mathbf{Sets})^{\boxdot^{\mathrm{op}}}$, obtaining the category of *cubical* C^**-spaces*. [Østvær] gives $\boxdot\mathrm{C}^*$-Spc four different model structures, and then studies their homotopy category in the stable and unstable version.

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