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## **Green's Functions and Field Theory (Fermions)**

In most cases of interest, the first few orders of perturbation theory cannot provide an adequate description of an interacting many-particle system. For this reason, it becomes essential to develop systematic methods for solving the Schrödinger equation to all orders in perturbation theory.

### **6 PICTURES**

As a preliminary step, we shall introduce three important pictures (Schrödinger, interaction, and Heisenberg) that are useful in analyzing the second-quantized form of the Schrödinger equation [Eqs. (1.41) and (1.60)].

#### **SCHRÖDINGER PICTURE**

The usual elementary description of quantum mechanics assumes that the state vectors are time dependent, whereas the operators are time independent and are

constructed by the familiar rules from the corresponding classical quantities. The Schrödinger equation therefore takes the form

$$i\hbar \frac{\partial}{\partial t} |\Psi_s(t)\rangle = \hat{H} |\Psi_s(t)\rangle \quad (6.1)$$

where  $\hat{H}$  is assumed to have no explicit time dependence. Since Eq. (6.1) is a first-order differential equation, the initial state at  $t_0$  determines the subsequent behavior, and a formal solution is readily obtained by writing

$$|\Psi_s(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\Psi_s(t_0)\rangle \quad (6.2)$$

Here the exponential of an operator is defined in terms of its power-series expansion. Furthermore,  $\hat{H}$  is hermitian so that the exponential represents a unitary operator. Given the solution to the Schrödinger equation at the time  $t_0$ , the unitary transformation in Eq. (6.2) generates the solution at time  $t$ .

#### INTERACTION PICTURE

Assume, as is usually the case, that the hamiltonian is time independent and can be expressed as the sum of two terms

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \quad (6.3)$$

where  $\hat{H}_0$  acting alone yields a soluble problem. How can we now include *all* the effects of  $\hat{H}_1$ ? Define the interaction state vector in the following way

$$|\Psi_I(t)\rangle \equiv e^{i\hat{H}_0 t/\hbar} |\Psi_s(t)\rangle \quad (6.4)$$

which is merely a unitary transformation carried out at the time  $t$ . The equation of motion of this state vector is easily found by carrying out the time derivative

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle &= -\hat{H}_0 e^{i\hat{H}_0 t/\hbar} |\Psi_s(t)\rangle + e^{i\hat{H}_0 t/\hbar} i\hbar \frac{\partial}{\partial t} |\Psi_s(t)\rangle \\ &= e^{i\hat{H}_0 t/\hbar} [-\hat{H}_0 + \hat{H}_0 + \hat{H}_1] e^{-i\hat{H}_0 t/\hbar} |\Psi_I(t)\rangle \end{aligned}$$

and we therefore obtain the following set of equations in the interaction picture

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = \hat{H}_1(t) |\Psi_I(t)\rangle \quad (6.5)$$

$$\hat{H}_1(t) \equiv e^{i\hat{H}_0 t/\hbar} \hat{H}_1 e^{-i\hat{H}_0 t/\hbar}$$

In general,  $\hat{H}_0$  does not commute with  $\hat{H}_1$ , so that the proper order of these operators is very important. An arbitrary matrix element in the Schrödinger picture may be written as

$$\langle \Psi'_s(t) | \hat{O}_s | \Psi_s(t) \rangle = \langle \Psi'_I(t) | e^{i\hat{H}_0 t/\hbar} \hat{O}_s e^{-i\hat{H}_0 t/\hbar} | \Psi_I(t) \rangle \quad (6.6)$$

which suggests the following definition of an operator in the interaction picture

$$\hat{O}_I(t) \equiv e^{i\hat{H}_0 t/\hbar} \hat{O}_S e^{-i\hat{H}_0 t/\hbar} \quad (6.7)$$

Equations (6.4) and (6.7) show that the operators  $\hat{O}_I(t)$  and the state vectors  $|\Psi_I(t)\rangle$  both depend on time in the interaction picture. The important point here is that the time dependence of the operators is particularly simple. Differentiate Eq. (6.7) with respect to time.

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{O}_I(t) &= e^{i\hat{H}_0 t/\hbar} (\hat{O}_S \hat{H}_0 - \hat{H}_0 \hat{O}_S) e^{-i\hat{H}_0 t/\hbar} \\ &= [\hat{O}_I(t), \hat{H}_0] \end{aligned} \quad (6.8)$$

Here the time independence of the Schrödinger operator has been used along with the observation that any function of an operator commutes with the operator itself. Consider a representation in which  $\hat{H}_0$  is diagonal.

$$\hat{H}_0 = \sum_k \hbar\omega_k c_k^\dagger c_k \quad (6.9)$$

The time dependence of the creation and destruction operators in the interaction picture can be determined from the differential equation

$$i\hbar \frac{\partial}{\partial t} c_{kI}(t) = e^{i\hat{H}_0 t/\hbar} [c_{kS}, \hat{H}_0] e^{-i\hat{H}_0 t/\hbar} = \hbar\omega_k c_{kI}(t)$$

which is easily solved to yield

$$c_{kI}(t) = c_k e^{-i\omega_k t} \quad (6.10a)$$

along with its adjoint

$$c_{kI}^\dagger(t) = c_k^\dagger e^{i\omega_k t} \quad (6.10b)$$

Thus the time occurs only in a complex phase factor, which means that the operator properties of  $c_I(t)$  and  $c_I^\dagger(t)$  are just the same as in the Schrödinger picture. In particular, the commutation relations of  $c_k$  and  $c_k^\dagger$  are simply the canonical ones from Chap. 1. Furthermore, any operator in the Schrödinger picture may be expressed in terms of the complete set  $c_k$  and  $c_k^\dagger$ , and the corresponding operator in the interaction picture is obtained with the substitution  $c_k \rightarrow c_{kI}(t)$ ,  $c_k^\dagger \rightarrow c_{kI}^\dagger(t)$ . This last result follows from the identity

$$1 = e^{-i\hat{H}_0 t/\hbar} e^{i\hat{H}_0 t/\hbar}$$

which may be inserted between each operator in the Schrödinger picture.

We shall now try to solve the equations of motion in the interaction picture. Define a unitary operator  $\hat{U}(t, t_0)$  that determines the state vector at time  $t$  in terms of the state vector at the time  $t_0$ .

$$|\Psi_I(t)\rangle = \hat{U}(t, t_0) |\Psi_I(t_0)\rangle \quad (6.11)$$

Evidently,  $\hat{U}$  must satisfy the relation

$$\hat{U}(t_0, t_0) = 1 \quad (6.12)$$

For finite times  $\hat{U}(t, t_0)$  can be constructed explicitly by using the Schrödinger picture:

$$\begin{aligned} |\Psi_I(t)\rangle &= e^{i\hat{H}_0 t/\hbar} |\Psi_S(t)\rangle = e^{i\hat{H}_0 t/\hbar} e^{-i\hat{H}(t-t_0)/\hbar} |\Psi_S(t_0)\rangle \\ &= e^{i\hat{H}_0 t/\hbar} e^{-i\hat{H}(t-t_0)/\hbar} e^{-i\hat{H}_0 t_0/\hbar} |\Psi_I(t_0)\rangle \end{aligned}$$

which therefore identifies

$$\hat{U}(t, t_0) = e^{i\hat{H}_0 t/\hbar} e^{-i\hat{H}(t-t_0)/\hbar} e^{-i\hat{H}_0 t_0/\hbar} \quad (\text{finite times}) \quad (6.13)$$

Since  $\hat{H}$  and  $\hat{H}_0$  do not commute with each other, the order of the operators must be carefully maintained. Equation (6.13) immediately yields several general properties of  $\hat{U}$

1.  $\hat{U}^\dagger(t, t_0) \hat{U}(t, t_0) = \hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) = 1$

which implies that  $\hat{U}$  is unitary:

$$\hat{U}^\dagger(t, t_0) = \hat{U}^{-1}(t, t_0) \quad (6.14)$$

2.  $\hat{U}(t_1, t_2) \hat{U}(t_2, t_3) = \hat{U}(t_1, t_3)$  (6.15)

which shows that  $\hat{U}$  has the group property, and

3.  $\hat{U}(t, t_0) \hat{U}(t_0, t) = 1$

which implies that

$$\hat{U}(t_0, t) = \hat{U}^\dagger(t, t_0) \quad (6.16)$$

Although Eq. (6.13) is the formal solution to the problem posed by Eq. (6.11), it is not very useful for computational purposes. Instead we shall construct an integral equation for  $\hat{U}$ , which can then be solved by iteration. It is clear from Eqs. (6.5) and (6.11) that  $\hat{U}$  satisfies a differential equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H}_1(t) \hat{U}(t, t_0) \quad (6.17)$$

Integrate this equation from  $t_0$  to  $t$

$$\hat{U}(t, t_0) - \hat{U}(t_0, t_0) = -\frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}_1(t') \hat{U}(t', t_0)$$

This result, combined with the boundary condition (6.12), yields an integral equation

$$\hat{U}(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}_1(t') \hat{U}(t', t_0) \quad (6.18)$$

If  $\hat{U}$  were a  $c$ -number function, Eq. (6.18) would be a Volterra integral equation, because the independent variable  $t$  appears as the upper limit of the integral. Under very broad conditions Volterra equations may be solved by iteration, and the solution is guaranteed to converge, no matter how large the kernel.<sup>1</sup> There is no assurance that the present operator equation has the same properties; nevertheless we shall attempt to solve Eq. (6.18) by iteration, always maintaining the proper ordering of the operators. The solution thus takes the form

$$\begin{aligned}\hat{U}(t, t_0) = 1 + \left(\frac{-i}{\hbar}\right) \int_{t_0}^t dt' \hat{H}_1(t') + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^t dt' \\ \int_{t_0}^{t'} dt'' \hat{H}_1(t') \hat{H}_1(t'') + \dots \quad (6.19)\end{aligned}$$

Consider the third term in this expansion. It may be rewritten as

$$\begin{aligned}\int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t') \hat{H}_1(t'') \\ = \frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t') \hat{H}_1(t'') + \frac{1}{2} \int_{t_0}^t dt'' \int_{t''}^t dt' \hat{H}_1(t') \hat{H}_1(t'') \quad (6.20)\end{aligned}$$

since the last term on the right is just obtained by reversing the order of the integrations, as illustrated in Fig. 6.1. We now change dummy variables in this

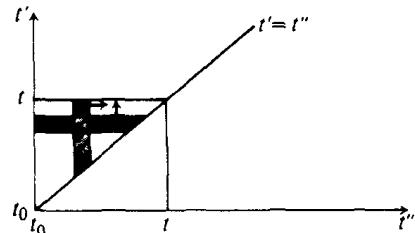


Fig. 6.1 Integration regions for second-order term in  $\hat{U}(t, t_0)$ .

second term, interchanging the labels  $t'$  and  $t''$ , and the second term of Eq. (6.20) therefore becomes

$$\frac{1}{2} \int_{t_0}^t dt'' \int_{t''}^t dt' \hat{H}_1(t') \hat{H}_1(t'') = \frac{1}{2} \int_{t_0}^t dt' \int_{t'}^t dt'' \hat{H}_1(t'') \hat{H}_1(t')$$

These two terms may now be recombined to give

$$\begin{aligned}\int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t') \hat{H}_1(t'') = \frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^t dt'' \\ \times [\hat{H}_1(t') \hat{H}_1(t'') \theta(t' - t'') + \hat{H}_1(t'') \hat{H}_1(t') \theta(t'' - t')] \quad (6.21)\end{aligned}$$

<sup>1</sup> See, for example, F. Smithies, "Integral Equations," p. 31, Cambridge University Press, Cambridge, 1962.

where the step function [Eq. (3.28)] is essential because the operators  $\hat{H}_1$  do not necessarily commute at different times. Equation (6.21) has the characteristic feature that the operator containing the latest time stands farthest to the left. We call this a *time-ordered product of operators*, denoted by the symbol  $T$ . Thus Eq. (6.21) can be rewritten as

$$\int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t') \hat{H}_1(t'') \equiv \frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' T[\hat{H}_1(t') \hat{H}_1(t'')] \quad (6.22)$$

This result is readily generalized, and the resulting expansion for  $\hat{U}$  becomes

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \left( \frac{-i\hbar}{\hbar} \right)^n \frac{1}{n!} \int_{t_0}^t dt_1 \cdots \int_{t_0}^{t_n} dt_n T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] \quad (6.23)$$

where the  $n=0$  term is just the unit operator.<sup>1</sup> The proof of Eq. (6.23) is as follows. Consider the  $n$ th term in this series. There are  $n!$  possible time orderings of the labels  $t_1 \cdots t_n$ . Pick a particular one, say,  $t_1 > t_2 > t_3 \cdots > t_n$ . Any other time ordering gives the same contribution to  $\hat{U}$ . This result is easily seen by relabeling the dummy integration variables  $t_i$  to agree with the previous ordering, and then using the symmetry of the  $T$  product under interchange of its arguments:

$$T[\cdots \hat{H}_1(t_i) \cdots \hat{H}_1(t_j) \cdots] = T[\cdots \hat{H}_1(t_j) \cdots \hat{H}_1(t_i) \cdots] \quad (6.24)$$

Equation (6.24) follows from the definition of the  $T$  product, which puts the operator at the latest time farthest to the left, the operator at the next latest time next, and so on, since the prescription holds equally well for both sides of Eq. (6.24). In this way, Eq. (6.23) reproduces the iterated series of Eq. (6.19).

#### HEISENBERG PICTURE

The state vector in the Heisenberg picture is defined as

$$|\Psi_H(t)\rangle \equiv e^{i\hat{H}t/\hbar} |\Psi_S(t)\rangle \quad (6.25)$$

Its time derivative may be combined with the Schrödinger equation (6.1) to yield

$$i\hbar \frac{\partial}{\partial t} |\Psi_H(t)\rangle = 0 \quad (6.26)$$

which shows that  $|\Psi_H\rangle$  is time independent. Since an arbitrary matrix element in the Schrödinger picture can be written as

$$\langle \Psi'_S(t) | \hat{O}_S | \Psi_S(t) \rangle = \langle \Psi'_H | e^{i\hat{H}t/\hbar} \hat{O}_S e^{-i\hat{H}t/\hbar} | \Psi_H \rangle \quad (6.27)$$

<sup>1</sup> Equation (6.23) is sometimes written as a formal time-ordered exponential

$$\hat{U}(t, t_0) = T \left[ \exp \left[ -i\hbar^{-1} \int_{t_0}^t dt' \hat{H}_1(t') \right] \right]$$

since the power-series expansion reproduces Eq. (6.23) term by term.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

a general operator in the Heisenberg picture is given by

$$\hat{O}_H(t) \equiv e^{i\hat{H}t/\hbar} \hat{O}_S e^{-i\hat{H}t/\hbar} \quad (6.28)$$

Note that  $\hat{O}_H(t)$  is a complicated object since  $\hat{H}$  and  $\hat{O}_S$  in general do not commute. We see that the Heisenberg picture ascribes all the time dependence to the operators, whereas the corresponding state vectors are time independent. In contrast, the operator  $\hat{O}_S$  in the Schrödinger picture is time independent, and the time derivative of Eq. (6.28) yields

$$i\hbar \frac{\partial}{\partial t} \hat{O}_H(t) = e^{i\hat{H}t/\hbar} [\hat{O}_S, \hat{H}] e^{-i\hat{H}t/\hbar} = [\hat{O}_H(t), \hat{H}] \quad (6.29)$$

This important result determines the equation of motion of any operator in the Heisenberg picture. In particular, if  $\hat{O}_S$  commutes with  $\hat{H}$ , the right side vanishes identically, and  $\hat{O}_H$  is a constant of the motion.

Equation (6.28) can be rewritten in terms of interaction-picture operators [Eq. (6.7)]

$$\hat{O}_H(t) = e^{i\hat{H}t/\hbar} e^{-i\hat{H}_0 t/\hbar} \hat{O}_I(t) e^{i\hat{H}_0 t/\hbar} e^{-i\hat{H}t/\hbar} \quad (6.30)$$

and the formal solution for the operator  $\hat{U}$  [Eq. (6.13)] yields

$$\hat{O}_H(t) = \hat{U}(0,t) \hat{O}_I(t) \hat{U}(t,0) \quad (6.31)$$

In addition, the various definitions show that

$$|\Psi_H\rangle = |\Psi_S(0)\rangle = |\Psi_I(0)\rangle \quad (6.32)$$

$$\hat{O}_S = \hat{O}_H(0) = \hat{O}_I(0)$$

so that all three pictures coincide at the time  $t = 0$ . The stationary solutions to the Schrödinger equation have a definite energy, and the corresponding state vectors in the Heisenberg picture satisfy the time-independent form of the Schrödinger equation

$$\hat{H} |\Psi_H\rangle = E |\Psi_H\rangle \quad (6.33)$$

These state vectors are therefore the *exact eigenstates* of the system and are naturally very complicated for an interacting system. Equation (6.32) and the definition of the operator  $\hat{U}$  together lead to the relation

$$|\Psi_H\rangle = |\Psi_I(t_0)\rangle = \hat{U}(0,t_0) |\Psi_I(t_0)\rangle \quad (6.34)$$

which allows us to construct these exact eigenstates from the interaction state vectors at the time  $t_0$  with the unitary operator  $\hat{U}$ .

#### ADIABATIC "SWITCHING ON"

The notion of switching on the interaction adiabatically represents a mathematical device that generates exact eigenstates of the interacting system from those of the

noninteracting system. Since we presumably know all about the noninteracting system, for example, the ground state, the excited states, etc., this procedure lets us follow the development of each eigenstate as the interaction between the particles is switched on. Specifically, we introduce a new time-dependent hamiltonian

$$\hat{H} = \hat{H}_0 + e^{-\epsilon|t|} \hat{H}_1 \quad (6.35)$$

where  $\epsilon$  is a small positive quantity. At very large times, both in the past and in the future, the hamiltonian reduces to  $\hat{H}_0$ , which presents a soluble problem. At the time  $t = 0$ ,  $\hat{H}$  becomes the full hamiltonian of the interacting system. If  $\epsilon$  tends to zero at the end of the calculation, the perturbation is turned on and off infinitely slowly, or adiabatically, and any meaningful result must be *independent* of the quantity  $\epsilon$ .

The hamiltonian (6.35) presents a time-dependent problem that depends on the parameter  $\epsilon$ , and we shall seek a solution in the interaction picture. It is readily verified that Eqs. (6.17) and (6.23) remain correct even when  $\hat{H}_1$  is time dependent in the Schrödinger picture, and we immediately obtain

$$|\Psi_I(t)\rangle = \hat{U}_\epsilon(t, t_0) |\Psi_I(t_0)\rangle \quad (6.36)$$

where the time-development operator depends explicitly on  $\epsilon$  and is given by

$$\begin{aligned} \hat{U}_\epsilon(t, t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \frac{1}{n!} \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_n \\ &\times e^{-\epsilon(|t_1| + \cdots + |t_n|)} T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] \end{aligned} \quad (6.37)$$

Now let the time  $t_0$  approach  $-\infty$ ; Eq. (6.35) shows that  $\hat{H}$  then approaches  $\hat{H}_0$ . In this limit, the Schrödinger-picture state vector reduces to

$$|\Psi_S(t_0)\rangle = e^{-iE_0 t_0/\hbar} |\Phi_0\rangle \quad (6.38)$$

where  $|\Phi_0\rangle$  is some time-independent stationary eigenstate of the unperturbed hamiltonian  $\hat{H}_0$

$$\hat{H}_0 |\Phi_0\rangle = E_0 |\Phi_0\rangle \quad (6.39)$$

and the corresponding interaction-picture state vector becomes

$$|\Psi_I(t_0)\rangle = e^{iH_0 t_0/\hbar} |\Psi_S(t_0)\rangle = |\Phi_0\rangle \quad (6.40)$$

Thus  $|\Psi_I(t_0)\rangle$  becomes time independent as  $t_0 \rightarrow -\infty$ ; alternatively, the same conclusion follows from the equation

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = e^{-\epsilon|t|} \hat{H}_1(t) |\Psi_I(t)\rangle \rightarrow 0 \quad t \rightarrow \pm\infty \quad (6.41)$$

If there were no perturbation, these eigenstates in the interaction picture would remain constant in time, being the stationary-state solutions to the unperturbed

Schrödinger equation. As  $t$  increases from  $-\infty$ , however, the interaction is turned on, and Eq. (6.36) determines how the state vector develops in time, all the way to the time  $t = 0$ , when the interaction is at full strength. For finite times  $|t| \ll \epsilon^{-1}$ , all of our previous results remain valid, in particular Eqs. (6.32) and (6.34). We thus obtain the basic relation

$$|\Psi_H\rangle = |\Psi_I(0)\rangle = \hat{U}_\epsilon(0, -\infty)|\Phi_0\rangle \quad (6.42)$$

which expresses an exact eigenstate of the interacting system in terms of an eigenstate of  $\hat{H}_0$ .

We must now ask what happens in the limit  $\epsilon \rightarrow 0$ . Do we get finite meaningful results? This question is answered by the Gell-Mann and Low theorem, which is proved in the next section.

#### GELL-MANN AND LOW THEOREM ON THE GROUND STATE IN QUANTUM FIELD THEORY<sup>1</sup>

The Gell-Mann and Low theorem is easily stated: If the following quantity exists to all orders in perturbation theory,

$$\lim_{\epsilon \rightarrow 0} \frac{\hat{U}_\epsilon(0, -\infty)|\Phi_0\rangle}{\langle\Phi_0|\hat{U}_\epsilon(0, -\infty)|\Phi_0\rangle} \equiv \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} \quad (6.43)$$

then it is an eigenstate of  $\hat{H}$ ,

$$\hat{H} \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} \quad (6.44)$$

This prescription generates the eigenstate that *develops adiabatically* from  $|\Phi_0\rangle$  as the interaction is turned on. If  $|\Phi_0\rangle$  is the ground state of the noninteracting system, the corresponding eigenstate of  $\hat{H}$  is usually the interacting ground state, but this is by no means necessary. For example, the ground-state energy of some systems does not have a perturbation series in the coupling constant. (For another example see Prob. 7.5.) Multiply Eq. (6.44) from the left by the state  $\langle\Phi_0|$ ; since  $\hat{H}_0|\Phi_0\rangle = E_0|\Phi_0\rangle$ , we conclude

$$E - E_0 = \frac{\langle\Phi_0|\hat{H}_1|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} \quad (6.45)$$

An essential point of the theorem is that the numerator and the denominator of Eq. (6.43) *do not separately exist* as  $\epsilon \rightarrow 0$ . An equivalent statement is that Eq. (6.42) becomes meaningless in the limit  $\epsilon \rightarrow 0$ ; indeed, its phase diverges like  $\epsilon^{-1}$  in this limit. The denominator in Eq. (6.43) serves precisely to cancel this infinite phase [see Eq. (6.51) and subsequent discussion]. The theorem thus asserts that if the ratio in Eq. (6.43) exists, the eigenstate is well defined and has the eigenvalue given in Eq. (6.45). We proceed to the proof given by Gell-Mann and Low.

<sup>1</sup> M. Gell-Mann and F. Low, *Phys. Rev.*, **84**:350 (1951).

Consider the expression

$$(\hat{H}_0 - E_0)|\Psi_0(\epsilon)\rangle = (\hat{H}_0 - E_0)\hat{U}_\epsilon(0, -\infty)|\Phi_0\rangle = [\hat{H}_0, \hat{U}_\epsilon(0, -\infty)]|\Phi_0\rangle \quad (6.46)$$

We shall explicitly evaluate the commutator appearing on the right side. Consider the  $n$ th term in Eq. (6.37) for the operator  $\hat{U}_\epsilon$ , and pick an arbitrary time ordering of the  $n$  time indices. The associated commutator can be written identically as

$$\begin{aligned} [\hat{H}_0, \hat{H}_1(t_i)\hat{H}_1(t_j)\cdots\hat{H}_1(t_k)] &\equiv [\hat{H}_0, \hat{H}_1(t_i)]\hat{H}_1(t_j)\cdots\hat{H}_1(t_k) \\ &\quad + \hat{H}_1(t_i)[\hat{H}_0, \hat{H}_1(t_j)]\cdots\hat{H}_1(t_k) + \cdots \\ &\quad + \hat{H}_1(t_i)\hat{H}_1(t_j)\cdots[\hat{H}_0, \hat{H}_1(t_k)] \end{aligned}$$

Furthermore, Eq. (6.8) allows us to write

$$\frac{\hbar}{i}\frac{\partial\hat{H}_1(t)}{\partial t} = [\hat{H}_0, \hat{H}_1(t)] \quad (6.47)$$

In consequence, each of the commutators with  $\hat{H}_0$  yields a time derivative of the interaction hamiltonian,

$$\begin{aligned} &[\hat{H}_0, \hat{H}_1(t_i)\hat{H}_1(t_j)\cdots\hat{H}_1(t_k)] \\ &\quad = \frac{\hbar}{i}\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \cdots + \frac{\partial}{\partial t_n}\right)\hat{H}_1(t_i)\hat{H}_1(t_j)\cdots\hat{H}_1(t_k) \end{aligned}$$

for all possible time orderings. Equation (6.46) thus becomes

$$\begin{aligned} (\hat{H}_0 - E_0)|\Psi_0(\epsilon)\rangle &= -\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{n!}\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n \\ &\quad \times e^{\epsilon(t_1+\cdots+t_n)}\left(\sum_{i=1}^n\frac{\partial}{\partial t_i}\right)T[\hat{H}_1(t_1)\cdots\hat{H}_1(t_n)]|\Phi_0\rangle \quad (6.48) \end{aligned}$$

In deriving Eq. (6.48), all the time derivatives have been taken outside of the time-ordering symbol. The validity of this step can be seen from the identity

$$\left(\sum_{i=1}^n\frac{\partial}{\partial t_i}\right)\theta(t_p-t_q)\theta(t_q-t_r)\cdots\theta(t_u-t_v)\equiv 0$$

where  $p, q, r, \dots, u, v$  is any permutation of the indices  $1, 2, \dots, n$ . The differentiation is most easily evaluated with the representation

$$\theta(t) = \int_{-\infty}^t dt' \delta(t')$$

which immediately yields  $d\theta(t)/dt = \delta(t) = \delta(-t)$ . Thus the integrand of Eq. (6.48) may be rewritten as

$$T\left[\left(\sum_{i=1}^n \frac{\partial}{\partial t_i}\right) \hat{H}_1(t_1) \cdots \hat{H}_1(t_n)\right] = \left(\sum_{i=1}^n \frac{\partial}{\partial t_i}\right) T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)]$$

All the time-derivative terms in Eq. (6.48) make the same contribution to the integral, as shown by changing dummy variables; we therefore retain just one, say  $\partial/\partial t_1$ , and multiply by a factor  $n$ . Integrate by parts with respect to  $t_1$ : This procedure leads to two terms, one of which is simply the integrand evaluated at the end points, and the other arises from the derivative of the adiabatic factor. We therefore obtain

$$(\hat{H}_0 - E_0)|\Psi_0(\epsilon)\rangle = -\hat{H}_1|\Psi_0(\epsilon)\rangle + \epsilon i\hbar g \frac{\partial}{\partial g} |\Psi_0(\epsilon)\rangle \quad (6.49)$$

where  $\hat{H}_1$  is assumed proportional to a coupling constant  $g$  in order to write

$$\left(\frac{-i}{\hbar}\right)^{n-1} \frac{1}{(n-1)!} g^n = i\hbar g \frac{\partial}{\partial g} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} g^n$$

By this means, we obtain a series that reproduces the state vector  $|\Psi_0(\epsilon)\rangle$  again. Equation (6.49) is readily rewritten as

$$(\hat{H} - E_0)|\Psi_0(\epsilon)\rangle = i\hbar\epsilon g \frac{\partial}{\partial g} |\Psi_0(\epsilon)\rangle \quad (6.50)$$

Multiply this equation on the left by  $[\langle\Phi_0|\Psi_0(\epsilon)\rangle]^{-1}\langle\Phi_0|$ ; since  $(\partial/\partial g)\langle\Phi_0| = 0$ , we find

$$\frac{\langle\Phi_0|\hat{H}_1|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} = i\hbar\epsilon g \frac{\partial}{\partial g} \ln \langle\Phi_0|\Psi_0(\epsilon)\rangle \equiv E - E_0 = \Delta E \quad (6.51)$$

If  $\epsilon$  were allowed to vanish at this point, it would be tempting to conclude that  $\Delta E = 0$ , which is clearly absurd. In fact, the amplitude  $\langle\Phi_0|\Psi_0(\epsilon)\rangle$  must acquire an infinite phase proportional to  $i\epsilon^{-1}$  so that  $\epsilon \ln \langle\Phi_0|\Psi_0(\epsilon)\rangle$  remains finite as  $\epsilon \rightarrow 0$ .‡ Equation (6.50) may be manipulated to give

$$\left(\hat{H} - E_0 - i\hbar\epsilon g \frac{\partial}{\partial g}\right) \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} = \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} \left[ i\hbar\epsilon g \frac{\partial}{\partial g} \ln \langle\Phi_0|\Psi_0(\epsilon)\rangle \right]$$

and a combination with Eq. (6.51) finally yields

$$(\hat{H} - E) \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} = i\hbar\epsilon g \frac{\partial}{\partial g} \left[ \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} \right] \quad (6.52)$$

We are now in a position to let  $\epsilon$  go to zero. By assumption, the quantity in brackets on the right side of Eq. (6.52) is finite to all orders in perturbation

‡ See, for example, J. Hubbard, *Proc. Roy. Soc. (London)*, A240:539 (1957).

theory, that is, in  $g$ , and the derivative with respect to  $g$  cannot change this property. Since the right side is multiplied by  $\epsilon$ , it vanishes as  $\epsilon$  tends to zero, which proves the basic theorem

$$(\hat{H} - E) \lim_{\epsilon \rightarrow 0} \frac{|\Psi_0(\epsilon)\rangle}{\langle \Phi_0 | \Psi_0(\epsilon) \rangle} = 0 \quad (6.53)$$

This proof applies equally well to the quantity

$$\frac{\hat{U}_\epsilon(0, +\infty) |\Phi_0\rangle}{\langle \Phi_0 | \hat{U}_\epsilon(0, +\infty) | \Phi_0 \rangle} \quad (6.54)$$

where

$$\hat{U}_\epsilon(0, +\infty) = \hat{U}_\epsilon^\dagger(+\infty, 0)$$

Here the system “comes back” from  $t = +\infty$ , where the eigenstate is  $|\Phi_0\rangle$ . If the state that develops out of  $|\Phi_0\rangle$  is nondegenerate, then these two definitions must be the same. They could differ by a phase factor, but the common normalization condition

$$\langle \Phi_0 | \left[ \frac{|\Psi_0\rangle}{\langle \Phi_0 | \Psi_0 \rangle} \right] = 1 \quad (6.55)$$

precludes even this possibility. Thus, for a *nondegenerate* eigenstate of  $\hat{H}$

$$\lim_{\epsilon \rightarrow 0} \frac{\hat{U}_\epsilon(0, +\infty) |\Phi_0\rangle}{\langle \Phi_0 | \hat{U}_\epsilon(0, +\infty) | \Phi_0 \rangle} = \lim_{\epsilon \rightarrow 0} \frac{\hat{U}_\epsilon(0, -\infty) |\Phi_0\rangle}{\langle \Phi_0 | \hat{U}_\epsilon(0, -\infty) | \Phi_0 \rangle} \quad (6.56)$$

As noted before, the state obtained from the adiabatic switching procedure need not be the true ground state, even if  $|\Phi_0\rangle$  is the noninteracting ground state. The Gell-Mann and Low theorem merely asserts that it is an eigenstate; in addition, if it is a nondegenerate eigenstate, then both ways of constructing it [Eq. (6.56)] must yield the same result.

## 7 □ GREEN'S FUNCTIONS

This section introduces the concept of a Green's function<sup>1</sup> (or propagator, as it is sometimes called), which plays a fundamental role in our treatment of many-particle assemblies.

### DEFINITION

The single-particle Green's function is defined by the equation

$$iG_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') = \frac{\langle \Psi_0 | T[\hat{\psi}_{H\alpha}(\mathbf{x}t) \hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \quad (7.1)$$

<sup>1</sup> V. M. Galitskii and A. B. Migdal, *Sov. Phys.-JETP*, **7**:96 (1958); A. Klein and R. Prange, *Phys. Rev.*, **112**:994 (1958); P. C. Martin and J. Schwinger, *Phys. Rev.*, **115**:1342 (1959).

where  $|\Psi_0\rangle$  is the *Heisenberg ground state* of the interacting system satisfying

$$\hat{H}|\Psi_0\rangle = E|\Psi_0\rangle \quad (7.2)$$

and  $\hat{\psi}_{H\alpha}(xt)$  is a *Heisenberg operator* with the time dependence

$$\hat{\psi}_{H\alpha}(xt) = e^{iHt/\hbar} \hat{\psi}_\alpha(x) e^{-iHt/\hbar} \quad (7.3)$$

Here the indices  $\alpha$  and  $\beta$  label the components of the field operators;  $\alpha$  and  $\beta$  can take two values for spin- $\frac{1}{2}$  fermions, whereas there are no indices for spin-zero bosons, because such a system is described by a one-component field. The  $T$  product here represents a generalization of that in Eq. (6.22):

$$T[\hat{\psi}_{H\alpha}(xt) \hat{\psi}_{H\beta}^\dagger(x' t')] = \begin{cases} \hat{\psi}_{H\alpha}(xt) \hat{\psi}_{H\beta}^\dagger(x' t') & t > t' \\ \pm \hat{\psi}_{H\beta}^\dagger(x' t') \hat{\psi}_{H\alpha}(xt) & t' > t \end{cases} \quad (7.4)$$

where the upper (lower) sign refers to bosons (fermions). More generally, the  $T$  product of several operators orders them from right to left in ascending time order and adds a factor  $(-1)^P$ , where  $P$  is the number of interchanges of *fermion* operators from the original given order. This definition agrees with that in Eq. (6.22), because  $\hat{H}_1$  always contains an *even* number of fermion fields. Equation (7.1) may now be written explicitly as

$$iG_{\alpha\beta}(xt, x' t') = \begin{cases} \frac{\langle \Psi_0 | \hat{\psi}_{H\alpha}(xt) \hat{\psi}_{H\beta}^\dagger(x' t') | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} & t > t' \\ \pm \frac{\langle \Psi_0 | \hat{\psi}_{H\beta}^\dagger(x' t') \hat{\psi}_{H\alpha}(xt) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} & t' > t \end{cases} \quad (7.5)$$

The Green's function is an expectation value of field operators; as such, it is simply a function of the coordinate variables  $xt$  and  $x't'$ . If  $\hat{H}$  is time independent, then  $G$  depends only on the time difference  $t - t'$ , which follows immediately from Eqs. (7.2) and (7.3):

$$\begin{aligned} iG_{\alpha\beta}(xt, x' t') &= \begin{cases} e^{iE(t-t')/\hbar} \frac{\langle \Psi_0 | \hat{\psi}_\alpha(x) e^{-iH(t-t')/\hbar} \hat{\psi}_\beta^\dagger(x') | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} & t > t' \\ \pm e^{-iE(t-t')/\hbar} \frac{\langle \Psi_0 | \hat{\psi}_\beta^\dagger(x') e^{iH(t-t')/\hbar} \hat{\psi}_\alpha(x) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} & t' > t \end{cases} \quad (7.6) \end{aligned}$$

Here the factor  $\exp[\pm iE(t - t')/\hbar]$  is merely a complex  $c$  number and may be taken out of the matrix element; in contrast, the operator  $\hat{H}$  between the field operators must remain as written.

### RELATION TO OBSERVABLES

There are several reasons for studying the Green's functions. First, the Feynman rules for finding the contribution of  $n$ th order perturbation theory are simpler for  $G$  than for other combinations of field operators. This result is discussed in detail in Sec. 9. Second, although the ground-state expectation value in Eq. (7.1) implies the loss of much detailed information about the ground state, the single-particle Green's function still contains the observable properties of greatest interest:

1. The expectation value of any single-particle operator in the ground state of the system
2. The ground-state energy of the system
3. The excitation spectrum of the system

The first two points are demonstrated below, while the third follows from the Lehmann representation, which is discussed later in this section.

Consider the single-particle operator

$$\hat{J} = \int d^3x \hat{\mathcal{J}}(x)$$

where  $\hat{\mathcal{J}}(x)$  is the second-quantized density for the first-quantized operator  $J_{\beta\alpha}(x)$ :

$$\hat{\mathcal{J}}(x) = \sum_{\alpha\beta} \hat{\psi}_\beta^\dagger(x) J_{\beta\alpha}(x) \hat{\psi}_\alpha(x)$$

The ground-state expectation value of the operator density is given by

$$\begin{aligned} \langle \hat{\mathcal{J}}(x) \rangle &\equiv \frac{\langle \Psi_0 | \hat{\mathcal{J}}(x) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \\ &= \lim_{x' \rightarrow x} \sum_{\alpha\beta} J_{\beta\alpha}(x) \frac{\langle \Psi_0 | \hat{\psi}_\beta^\dagger(x') \hat{\psi}_\alpha(x) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \\ &= \pm i \lim_{t' \rightarrow t^+} \lim_{x' \rightarrow x} \sum_{\alpha\beta} J_{\beta\alpha}(x) G_{\alpha\beta}(xt, x't') \\ &= \pm i \lim_{t' \rightarrow t^+} \lim_{x' \rightarrow x} \text{tr}[J(x) G(xt, x't')] \end{aligned} \quad (7.7)$$

Here the operator  $J_{\beta\alpha}(x)$  must act before the limit  $x' \rightarrow x$  is performed because  $J$  may contain spatial derivatives, as in the momentum operator. Furthermore, the symbol  $t^+$  denotes a time infinitesimally later than  $t$ , which ensures that the field operators in the third line occur in the proper order [compare Eq. (7.5)]. Finally, the sum over spin indices may be recognized as a trace of the matrix product  $JG$ , which is here denoted by  $\text{tr}$ . For example, the number density  $\langle \hat{n}(x) \rangle$ , the spin density  $\langle \hat{\sigma}(x) \rangle$ , and the total kinetic energy  $\langle \hat{T} \rangle$  are readily found to be

$$\langle \hat{n}(x) \rangle = \pm i \text{tr} G(xt, xt^+) \quad (7.8)$$

$$\langle \hat{\theta}(\mathbf{x}) \rangle = \pm i \text{tr} [\hat{\theta} G(\mathbf{x}t, \mathbf{x}t^+)] \quad (7.9)$$

$$\langle \hat{T} \rangle = \pm i \int d^3x \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \left[ -\frac{\hbar^2 \nabla^2}{2m} \text{tr} G(\mathbf{x}t, \mathbf{x}'t^+) \right] \quad (7.10)$$

The interesting question now arises: Is it also possible to construct the potential energy

$$\langle \hat{V} \rangle = \frac{1}{2} \sum_{\alpha\alpha' \beta\beta'} \int d^3x \int d^3x' \frac{\langle \Psi_0 | \hat{\psi}_\alpha^\dagger(\mathbf{x}) \hat{\psi}_\beta^\dagger(\mathbf{x}') V(\mathbf{x}, \mathbf{x}')_{\alpha\alpha', \beta\beta'} \hat{\psi}_\beta(\mathbf{x}') \hat{\psi}_\alpha(\mathbf{x}) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \quad (7.11)$$

and thereby determine the total ground-state energy? Since Eq. (7.11) involves four field operators, we might expect to need the two-particle Green's function. The Schrödinger equation itself contains the potential energy, however, which allows us to find  $\langle \hat{V} \rangle$  in terms of the single-particle Green's function. Consider the Heisenberg field operator  $\hat{\psi}_{H\alpha}(\mathbf{x}t)$ , with the hamiltonian

$$\begin{aligned} \hat{H} = & \sum_\alpha \int d^3x \hat{\psi}_\alpha^\dagger(\mathbf{x}) T(\mathbf{x}) \hat{\psi}_\alpha(\mathbf{x}) \\ & + \frac{1}{2} \sum_{\alpha\alpha' \beta\beta'} \int d^3x \int d^3x' \hat{\psi}_\alpha^\dagger(\mathbf{x}) \hat{\psi}_\beta^\dagger(\mathbf{x}') V(\mathbf{x}, \mathbf{x}')_{\alpha\alpha', \beta\beta'} \hat{\psi}_\beta(\mathbf{x}') \hat{\psi}_{\alpha'}(\mathbf{x}) \end{aligned} \quad (7.12)$$

The identity of the particles in the assembly requires that the interaction be unchanged under particle interchange

$$V(\mathbf{x}, \mathbf{x}')_{\alpha\alpha', \beta\beta'} = V(\mathbf{x}', \mathbf{x})_{\beta\beta', \alpha\alpha'} \quad (7.13)$$

[More formally, such a term is the only kind that gives a nonvanishing contribution in Eq. (7.12).] The Heisenberg equation of motion [Eq. (6.29)] relates the time derivative of  $\hat{\psi}$  to the commutator of  $\hat{\psi}$  with  $\hat{H}$ .

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}_{H\alpha}(\mathbf{x}t) = e^{i\hbar t/\hbar} [\hat{\psi}_\alpha(\mathbf{x}), \hat{H}] e^{-i\hbar t/\hbar} \quad (7.14)$$

where

$$\begin{aligned} [\hat{\psi}_\alpha(\mathbf{x}), \hat{H}] = & \sum_\beta \int d^3z [\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta^\dagger(\mathbf{z}) T(\mathbf{z}) \hat{\psi}_\beta(\mathbf{z})] + \frac{1}{2} \sum_{\beta\beta' \gamma\gamma'} \int d^3z \int d^3z' \\ & \times [\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta^\dagger(\mathbf{z}) \hat{\psi}_\gamma^\dagger(\mathbf{z}') V(\mathbf{z}, \mathbf{z}')_{\beta\beta', \gamma\gamma'} \hat{\psi}_\gamma(\mathbf{z}') \hat{\psi}_\beta(\mathbf{z})] \end{aligned} \quad (7.15)$$

We now use the very important identity

$$\begin{aligned} [A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA \\ &= \{[A, B]C - B[C, A]\} \\ &\quad + \{A[B, C] - B[C, A]\} \end{aligned} \quad (7.16)$$

which allows us to express Eq. (7.15) in terms of either *commutators* or *anti-commutators*. For definiteness, consider the fermion case, since this is more

complicated. With the canonical anticommutation relations [Eq. (2.3)] the commutator is readily evaluated, and we find

$$\begin{aligned} [\hat{\psi}_\alpha(\mathbf{x}), \hat{H}] &= T(\mathbf{x}) \hat{\psi}_\alpha(\mathbf{x}) - \frac{1}{2} \sum_{\beta\beta'\gamma'} \int d^3z \hat{\psi}_\beta^\dagger(\mathbf{z}) V(\mathbf{z}, \mathbf{x})_{\beta\beta', \alpha\gamma'} \hat{\psi}_{\gamma'}(\mathbf{x}) \hat{\psi}_{\beta'}(\mathbf{z}) \\ &\quad + \frac{1}{2} \sum_{\beta'\gamma\gamma'} \int d^3z' \hat{\psi}_\gamma^\dagger(\mathbf{z}') V(\mathbf{x}, \mathbf{z}')_{\alpha\beta', \gamma\gamma'} \hat{\psi}_{\gamma'}(\mathbf{z}') \hat{\psi}_{\beta'}(\mathbf{x}) \end{aligned} \quad (7.17)$$

In the first potential-energy term, change the dummy variables  $\beta \rightarrow \gamma$ ,  $\beta' \rightarrow \gamma'$ ,  $\gamma' \rightarrow \beta'$ ,  $\mathbf{z} \rightarrow \mathbf{z}'$ . The symmetry of the potential [Eq. (7.13)] and the anti-commutativity of the fields  $\hat{\psi}$  then yield

$$[\hat{\psi}_\alpha(\mathbf{x}), \hat{H}] = T(\mathbf{x}) \hat{\psi}_\alpha(\mathbf{x}) + \sum_{\beta'\gamma\gamma'} \int d^3z' \hat{\psi}_\gamma^\dagger(\mathbf{z}') V(\mathbf{x}, \mathbf{z}')_{\alpha\beta', \gamma\gamma'} \hat{\psi}_{\gamma'}(\mathbf{z}') \hat{\psi}_{\beta'}(\mathbf{x}) \quad (7.18)$$

while the field equation (7.14) becomes

$$\begin{aligned} \left[ i\hbar \frac{\partial}{\partial t} - T(\mathbf{x}) \right] \hat{\psi}_{H\alpha}(\mathbf{x}t) \\ = \sum_{\beta'\gamma\gamma'} \int d^3z' \hat{\psi}_{H\gamma}^\dagger(\mathbf{z}'t) V(\mathbf{x}, \mathbf{z}')_{\alpha\beta', \gamma\gamma'} \hat{\psi}_{H\gamma'}(\mathbf{z}'t) \hat{\psi}_{H\beta'}(\mathbf{x}t) \end{aligned} \quad (7.19)$$

Equations (7.18) and (7.19) are also correct for bosons.

Multiply Eq. (7.19) by  $\hat{\psi}_{H\alpha}^\dagger(\mathbf{x}'t')$  on the left, and then take the ground-state expectation value

$$\begin{aligned} \left[ i\hbar \frac{\partial}{\partial t} - T(\mathbf{x}) \right] \frac{\langle \Psi_0 | \hat{\psi}_{H\alpha}^\dagger(\mathbf{x}'t') \hat{\psi}_{H\alpha}(\mathbf{x}t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} &= \sum_{\beta'\gamma\gamma'} \int d^3z' \\ \times \frac{\langle \Psi_0 | \hat{\psi}_{H\alpha}^\dagger(\mathbf{x}'t') \hat{\psi}_{H\gamma}^\dagger(\mathbf{z}'t) V(\mathbf{x}, \mathbf{z}')_{\alpha\beta', \gamma\gamma'} \hat{\psi}_{H\gamma'}(\mathbf{z}'t) \hat{\psi}_{H\beta'}(\mathbf{x}t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \end{aligned} \quad (7.20)$$

In the limit  $\mathbf{x}' \rightarrow \mathbf{x}$ ,  $t' \rightarrow t^+$ , the left side is equal to

$$\pm i \lim_{t' \rightarrow t^+} \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \left[ i\hbar \frac{\partial}{\partial t} - T(\mathbf{x}) \right] G_{\alpha\alpha}(\mathbf{x}t, \mathbf{x}'t') \quad (7.21)$$

We now sum over  $\alpha$  and integrate over  $\mathbf{x}$ , which finally yields [compare Eq. (7.11)]

$$\langle \hat{V} \rangle = \pm \frac{1}{2} i \int d^3x \lim_{t' \rightarrow t^+} \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \sum_\alpha \left[ i\hbar \frac{\partial}{\partial t} - T(\mathbf{x}) \right] G_{\alpha\alpha}(\mathbf{x}t, \mathbf{x}'t') \quad (7.22)$$

A combination of Eqs. (7.10) and (7.22) then expresses the total ground-state energy solely in terms of the single-particle Green's function.

$$\begin{aligned} E &= \langle \hat{T} + \hat{V} \rangle = \langle \hat{H} \rangle \\ &= \pm \frac{1}{2} i \int d^3x \lim_{t' \rightarrow t^+} \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \left[ i\hbar \frac{\partial}{\partial t} + T(\mathbf{x}) \right] \text{tr } G(\mathbf{x}t, \mathbf{x}'t') \\ &= \pm \frac{1}{2} i \int d^3x \lim_{t' \rightarrow t^+} \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \left[ i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m} \right] \text{tr } G(\mathbf{x}t, \mathbf{x}'t') \end{aligned} \quad (7.23)$$

These expressions assume a simpler form for a homogeneous system in a large box of volume  $V$ , where the single-particle Green's function may be written as [compare Eq. (3.1)]<sup>1</sup>

$$G_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') = \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi V} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} G_{\alpha\beta}(\mathbf{k}, \omega) \quad (7.24)$$

In the limit  $V \rightarrow \infty$ , the sum over wave vectors reduces to an integral [Eq. (3.26)]

$$G_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') = (2\pi)^{-4} \int d^3k \int_{-\infty}^{\infty} d\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} G_{\alpha\beta}(\mathbf{k}, \omega) \quad (7.25)$$

and a combination of Eqs. (7.8), (7.23), and (7.25) gives

$$N = \int d^3x \langle \hat{n}(\mathbf{x}) \rangle = \pm i \frac{V}{(2\pi)^4} \lim_{\eta \rightarrow 0^+} \int d^3k \int_{-\infty}^{\infty} d\omega e^{i\omega\eta} \text{tr } G(\mathbf{k}, \omega) \quad (7.26)$$

$$E = \pm \frac{1}{2} i \frac{V}{(2\pi)^4} \lim_{\eta \rightarrow 0^+} \int d^3k \int_{-\infty}^{\infty} d\omega e^{i\omega\eta} \left( \frac{\hbar^2 k^2}{2m} + \hbar\omega \right) \text{tr } G(\mathbf{k}, \omega) \quad (7.27)$$

Here the convergence factor

$$\lim_{t' \rightarrow t^+} e^{i\omega(t'-t)} \equiv \lim_{\eta \rightarrow 0^+} e^{i\omega\eta}$$

defines the appropriate contour in the complex  $\omega$  plane; henceforth, the limit  $\eta \rightarrow 0^+$  will be implicit whenever such a factor appears.

For some purposes, it would be more convenient to have the difference  $\hbar\omega - \hbar^2 k^2/2m$  appearing in Eq. (7.27). This result can be achieved with the following trick, apparently due to Pauli and since rediscovered many times.<sup>2</sup> The hamiltonian is written with a variable coupling constant  $\lambda$  as

$$\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{H}_1$$

then

$$\hat{H}(1) = \hat{H} \quad \text{and} \quad \hat{H}(0) = \hat{H}_0$$

and we attempt to solve the time-independent Schrödinger equation for an arbitrary value of  $\lambda$ :

$$\hat{H}(\lambda) |\Psi_0(\lambda)\rangle = E(\lambda) |\Psi_0(\lambda)\rangle \quad (7.28)$$

where the state vector is assumed normalized

$$\langle \Psi_0(\lambda) | \Psi_0(\lambda) \rangle = 1$$

<sup>1</sup> For a proof that  $G$  depends only on the coordinate difference  $\mathbf{x} - \mathbf{x}'$  in a uniform system, see the discussion preceding Eq. (7.53).

<sup>2</sup> See, for example, D. Pines, "The Many-Body Problem," p. 43, W. A. Benjamin, Inc., New York, 1961; T. D. Schultz, "Quantum Field Theory and the Many-Body Problem," p. 18, Gordon and Breach, Science Publishers, New York, 1964.

The scalar product of Eq. (7.28) with  $\langle \Psi_0(\lambda) |$  immediately yields

$$E(\lambda) = \langle \Psi_0(\lambda) | \hat{H}(\lambda) | \Psi_0(\lambda) \rangle$$

and its derivative with respect to the parameter  $\lambda$  reduces to

$$\begin{aligned} \frac{d}{d\lambda} E(\lambda) &= \left\langle \frac{d\Psi_0(\lambda)}{d\lambda} \middle| \hat{H}(\lambda) | \Psi_0(\lambda) \right\rangle + \langle \Psi_0(\lambda) | \hat{H}(\lambda) \left| \frac{d\Psi_0(\lambda)}{d\lambda} \right\rangle \\ &\quad + \langle \Psi_0(\lambda) | \frac{d\hat{H}(\lambda)}{d\lambda} | \Psi_0(\lambda) \rangle \\ &= E(\lambda) \frac{d}{d\lambda} \langle \Psi_0(\lambda) | \Psi_0(\lambda) \rangle + \langle \Psi_0(\lambda) | \hat{H}_1 | \Psi_0(\lambda) \rangle \\ &= \langle \Psi_0(\lambda) | \hat{H}_1 | \Psi_0(\lambda) \rangle \end{aligned} \quad (7.29)$$

where the normalization condition has been used in obtaining the last line. Integrate Eq. (7.29) with respect to  $\lambda$  from zero to one and note that  $E(0) = E_0$  and  $E(1) = E$

$$E - E_0 = \int_0^1 \frac{d\lambda}{\lambda} \langle \Psi_0(\lambda) | \lambda \hat{H}_1 | \Psi_0(\lambda) \rangle \quad (7.30)$$

The shift in the ground-state energy is here expressed solely in terms of the matrix element of the interaction  $\lambda \hat{H}_1$ . Unfortunately, this matrix element is required for *all values* of the coupling constant  $0 < \lambda \leq 1$ . In the usual situation, where  $\hat{H}_1$  represents the potential energy [Eq. (7.12)], a combination of Eqs. (7.22) and (7.30) gives

$$E - E_0 = \pm \frac{1}{2} i \int_0^1 \frac{d\lambda}{\lambda} \int d^3x \lim_{t' \rightarrow t^+} \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \left[ i\hbar \frac{\partial}{\partial t} - T(\mathbf{x}) \right] \text{tr } G^\lambda(\mathbf{x}t, \mathbf{x}'t') \quad (7.31)$$

with the corresponding expression for a uniform system

$$E - E_0 = \pm \frac{1}{2} i \frac{V}{(2\pi)^4} \int_0^1 \frac{d\lambda}{\lambda} \int d^3k \int_{-\infty}^{\infty} d\omega e^{i\omega\eta} \left( \hbar\omega - \frac{\hbar^2 k^2}{2m} \right) \text{tr } G^\lambda(\mathbf{k}, \omega) \quad (7.32)$$

#### EXAMPLE: FREE FERMIONS

As an example of the above formalism, consider the Green's function for a noninteracting homogeneous system of fermions. It is first convenient to perform a canonical transformation to *particles* and *holes*. In the definition of the field [compare Eqs. (2.1) and (3.1)]

$$\hat{\psi}(\mathbf{x}) = \sum_{\mathbf{k}\lambda} \psi_{\mathbf{k}\lambda}(\mathbf{x}) c_{\mathbf{k}\lambda} \quad (7.33)$$

we redefine the fermion operator  $c_{\mathbf{k}\lambda}$  as<sup>1</sup>

$$c_{\mathbf{k}\lambda} = \begin{cases} a_{\mathbf{k}\lambda} & k > k_F \quad \text{particles} \\ b_{-\mathbf{k}\lambda}^\dagger & k < k_F \quad \text{holes} \end{cases} \quad (7.34)$$

<sup>1</sup> The absence of a particle with momentum  $+k$  from the filled Fermi sea implies that the system possesses a momentum  $-k$ . For a proper interpretation of the spin of the hole state, see Sec. 56.

which is a canonical transformation that preserves the anticommutation rules

$$\{a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\} = \{b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger\} = \delta_{\mathbf{k}\mathbf{k}'} \quad (7.35)$$

and therefore leaves the physics unchanged. Here the  $a$ 's and  $b$ 's clearly anti-commute with each other because they refer to different modes. The  $a^\dagger$  and  $a$  operators create and destroy particles above the Fermi sea, while the  $b^\dagger$  and  $b$  operators create and destroy holes inside the Fermi sea, as is evident from Eq. (7.34). The fields may now be rewritten in terms of these new operators as

$$\hat{\psi}_S(\mathbf{x}) = \sum_{\mathbf{k}\lambda > k_F} \psi_{\mathbf{k}\lambda}(\mathbf{x}) a_{\mathbf{k}\lambda} + \sum_{\mathbf{k}\lambda < k_F} \psi_{\mathbf{k}\lambda}(\mathbf{x}) b_{-\mathbf{k}\lambda}^\dagger \quad (7.36)$$

$$\hat{\psi}_I(\mathbf{x}t) = \sum_{\mathbf{k}\lambda > k_F} \psi_{\mathbf{k}\lambda}(\mathbf{x}) e^{-i\omega_{\mathbf{k}} t} a_{\mathbf{k}\lambda} + \sum_{\mathbf{k}\lambda < k_F} \psi_{\mathbf{k}\lambda}(\mathbf{x}) e^{-i\omega_{\mathbf{k}} t} b_{-\mathbf{k}\lambda}^\dagger \quad (7.37)$$

where the first equation is in the Schrödinger picture and the second equation is in the interaction picture. Equations (7.36) and (7.37) differ only in that the interaction picture contains a complex time-dependent phase. Correspondingly, the hamiltonian becomes

$$\begin{aligned} \hat{H}_0 &= \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} c_{\mathbf{k}\lambda}^\dagger c_{\mathbf{k}\lambda} \\ &= \sum_{\substack{\mathbf{k}\lambda > k_F \\ (\text{particles})}} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} - \sum_{\substack{\mathbf{k}\lambda < k_F \\ (\text{holes})}} \hbar\omega_{\mathbf{k}} b_{\mathbf{k}\lambda}^\dagger b_{\mathbf{k}\lambda} + \sum_{\substack{\mathbf{k}\lambda < k_F \\ (\text{filled Fermi sea})}} \hbar\omega_{\mathbf{k}} \end{aligned} \quad (7.38)$$

In the absence of particles or holes, the energy is that of the filled Fermi sea. Creating a hole lowers the energy, whereas creating a particle raises the energy. If the total number of fermions is fixed, however, particles and holes necessarily occur in pairs. Each particle-hole pair then has a net positive energy, showing that the filled Fermi sea represents the ground state.

By definition, the noninteracting fermion Green's function is given by

$$iG_{\alpha\beta}^0(\mathbf{x}t, \mathbf{x}'t') = \langle \Phi_0 | T[\hat{\psi}_{I\alpha}(\mathbf{x}t) \hat{\psi}_{I\beta}^\dagger(\mathbf{x}'t')] | \Phi_0 \rangle \quad (7.39)$$

where the noninteracting ground state vector is assumed normalized, and the superscript zero indicates that this is a Green's function with no interactions. We now observe that the particle and hole destruction operators both annihilate the ground state

$$b_{\mathbf{k}\lambda} |\Phi_0\rangle = a_{\mathbf{k}\lambda} |\Phi_0\rangle = 0 \quad (7.40)$$

since there are no particles above or holes below the Fermi sea in the state  $|\Phi_0\rangle$ . Equation (7.40) shows the usefulness of the particle-hole notation. The remaining term for each time ordering is easily computed, and we find

$$\begin{aligned} iG_{\alpha\beta}^0(\mathbf{x}t, \mathbf{x}'t') &= \delta_{\alpha\beta} V^{-1} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega_{\mathbf{k}}(t-t')} \\ &\times [\theta(t-t') \theta(k-k_F) - \theta(t'-t) \theta(k_F-k)] \end{aligned} \quad (7.41)$$

where the factor  $\delta_{\alpha\beta}$  arises because the sum over spin states is complete. In the limit of an infinite volume, the summation over  $\mathbf{k}$  becomes an integration

$$\begin{aligned} iG_{\alpha\beta}^0(\mathbf{x}t, \mathbf{x}'t') &= \delta_{\alpha\beta}(2\pi)^{-3} \int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega_k(t-t')} \\ &\times [\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)] \end{aligned} \quad (7.42)$$

It is now useful to introduce an integral representation for the step function

$$\theta(t-t') = - \int_{-\infty}^{\infty} \frac{d\omega e^{-i\omega(t-t')}}{2\pi i \omega + i\eta} \quad (7.43)$$

Equation (7.43) is readily verified as follows: If  $t > t'$ , then the contour must be closed in the lower-half  $\omega$  plane, including the simple pole at  $\omega = -i\eta$  with residue  $-1$ . If  $t < t'$ , then the contour must be closed in the upper-half  $\omega$  plane and gives zero, because the integrand has no singularities for  $\text{Im } \omega > 0$ . Equation (7.43) may be combined with Eq. (7.42) to give

$$\begin{aligned} G_{\alpha\beta}^0(\mathbf{x}t, \mathbf{x}'t') &= (2\pi)^{-4} \int d^3k \int_{-\infty}^{\infty} d\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \\ &\times \delta_{\alpha\beta} \left[ \frac{\theta(k-k_F)}{\omega - \omega_k + i\eta} + \frac{\theta(k_F-k)}{\omega - \omega_k - i\eta} \right] \end{aligned} \quad (7.44)$$

which immediately yields

$$G_{\alpha\beta}^0(\mathbf{k}, \omega) = \delta_{\alpha\beta} \left[ \frac{\theta(k-k_F)}{\omega - \omega_k + i\eta} + \frac{\theta(k_F-k)}{\omega - \omega_k - i\eta} \right] \quad (7.45)$$

It is instructive to verify explicitly that Eq. (7.44) indeed reproduces Eq. (7.42), and also that Eq. (7.45) gives the correct value for  $\langle \hat{N} \rangle$  [Eq. (7.26)] and  $E \equiv E_0$  [Eq. (7.27)]. Equation (7.45) also can be derived directly by evaluating the Fourier transform of Eq. (7.42), in which case the  $\pm i\eta$  terms are required to render the time integrals convergent.

#### THE LEHMANN REPRESENTATION<sup>1</sup>

Certain features of the single-particle Green's function follow directly from fundamental quantum-mechanical principles and are therefore independent of the specific form of the interaction. This section is devoted to such general properties. Although our final expressions are *formally* applicable to both bosons and fermions, the existence of Bose condensation at  $T = 0$  introduces additional complications (see Chap. 6), and we shall consider only fermions in the next two subsections. The exact Green's function is given by

$$iG_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') = \langle \Psi_0 | T[\hat{\psi}_{H\alpha}(\mathbf{x}t) \hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')] | \Psi_0 \rangle \quad (7.46)$$

<sup>1</sup> H. Lehmann, *Nuovo Cimento*, 11:342 (1954). Our treatment follows that of V. M. Galitskii and A. B. Migdal, *loc. cit.* and A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, "Methods of Quantum Field Theory in Statistical Physics," sec. 7, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963.

where the ground state is assumed normalized [ $\langle \Psi_0 | \Psi_0 \rangle = 1$ ]. In general, the Heisenberg field operators and state vectors in this expression are very complicated. Nevertheless, it is possible to derive some interesting and general results. Insert a complete set of Heisenberg states between the field operators; these states are eigenstates of the full hamiltonian, and include all possible numbers of particles. The right side of Eq. (7.46) becomes

$$\begin{aligned} iG_{\alpha\beta}(xt, x' t') = & \sum_n [\theta(t - t') \langle \Psi_0 | \hat{\psi}_{H\alpha}(xt) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{H\beta}^\dagger(x' t') | \Psi_0 \rangle \\ & - \theta(t' - t) \langle \Psi_0 | \hat{\psi}_{H\beta}^\dagger(x' t') | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{H\alpha}(xt) | \Psi_0 \rangle] \end{aligned} \quad (7.47)$$

Each Heisenberg operator may be rewritten in the form

$$\hat{O}_H(t) = e^{iEt/\hbar} \hat{O}_S e^{-iEt/\hbar}$$

which allows us to make explicit the time dependence of these matrix elements

$$\begin{aligned} iG_{\alpha\beta}(xt, x' t') = & \sum_n [\theta(t - t') e^{-i(E_n - E)(t - t')/\hbar} \langle \Psi_0 | \hat{\psi}_\alpha(x) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_\beta^\dagger(x') | \Psi_0 \rangle \\ & - \theta(t' - t) e^{i(E_n - E)(t - t')/\hbar} \langle \Psi_0 | \hat{\psi}_\beta^\dagger(x') | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_\alpha(x) | \Psi_0 \rangle] \end{aligned} \quad (7.48)$$

As a preliminary step, we show that the states  $|\Psi_n\rangle$  contain  $N \pm 1$  particles if the state  $|\Psi_0\rangle$  contains  $N$  particles. The number operator has the form

$$\hat{N} = \sum_\alpha \int d^3x \hat{\psi}_\alpha^\dagger(x) \hat{\psi}_\alpha(x)$$

and its commutator with the field operator is easily evaluated (for both bosons and fermions) as

$$[\hat{N}, \hat{\psi}_\beta(z)] = -\hat{\psi}_\beta(z)$$

or, equivalently,

$$\hat{N}\hat{\psi}_\beta(z) = \hat{\psi}_\beta(z)(\hat{N} - 1)$$

Apply this last operator relation to the state  $|\Psi_0\rangle$ :

$$\hat{N}[\hat{\psi}_\beta(z)|\Psi_0\rangle] = (N - 1)[\hat{\psi}_\beta(z)|\Psi_0\rangle] \quad (7.49)$$

where we have noted that  $|\Psi_0\rangle$  is an eigenstate of the number operator with eigenvalue  $N$ . Thus the field  $\hat{\psi}$  acting on the state  $|\Psi_0\rangle$  yields an eigenstate of the number operator with one less particle. Similarly, the operator  $\hat{\psi}^\dagger$  increases the number of particles by one. Equation (7.48) thus contains one new feature that does not occur in the ordinary Schrödinger equation, for we must now consider assemblies with different numbers of particles.

Until this point, the discussion has been completely general, assuming only that  $\hat{H}$  is time independent. Although it is possible to continue this analysis without further restriction, we shall now consider only the simpler case of translational invariance. This implies that the momentum operator, which is the generator of spatial displacements, commutes with  $\hat{H}$ . It is natural to use the

plane-wave basis of Eq. (3.1) for such a system, and the momentum operator is given by

$$\hat{\mathbf{P}} \equiv \sum_{\alpha} \int d^3x \hat{\psi}_{\alpha}^{\dagger}(\mathbf{x}) (-i\hbar \nabla) \hat{\psi}_{\alpha}(\mathbf{x}) = \sum_{\mathbf{k}\lambda} \hbar \mathbf{k} c_{\mathbf{k}\lambda}^{\dagger} c_{\mathbf{k}\lambda} \quad (7.50)$$

The commutator of  $\hat{\mathbf{P}}$  with the field operator  $\hat{\psi}$  (for both bosons and fermions) is easily evaluated as

$$-i\hbar \nabla \hat{\psi}_{\alpha}(\mathbf{x}) = [\hat{\psi}_{\alpha}(\mathbf{x}), \hat{\mathbf{P}}] \quad (7.51)$$

which can also be rewritten in integral form:

$$\hat{\psi}_{\alpha}(\mathbf{x}) = e^{-i\hat{\mathbf{P}} \cdot \mathbf{x}/\hbar} \hat{\psi}_{\alpha}(0) e^{i\hat{\mathbf{P}} \cdot \mathbf{x}/\hbar} \quad (7.52)$$

Since  $\hat{\mathbf{P}}$  is a constant of the motion, the complete set of states also can be taken as eigenstates of momentum. We therefore extract the  $\mathbf{x}$  dependence of the matrix elements in Eq. (7.48):

$$\begin{aligned} iG_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') &= \sum_n [\theta(t - t') e^{-i(E_n - E)(t - t')/\hbar} e^{i\mathbf{P}_n \cdot (\mathbf{x} - \mathbf{x}')/\hbar} \\ &\times \langle \Psi_0 | \hat{\psi}_{\alpha}(0) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{\beta}^{\dagger}(0) | \Psi_0 \rangle - \theta(t' - t) e^{i(E_n - E)(t - t')/\hbar} e^{-i\mathbf{P}_n \cdot (\mathbf{x} - \mathbf{x}')/\hbar} \\ &\times \langle \Psi_0 | \hat{\psi}_{\beta}^{\dagger}(0) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{\alpha}(0) | \Psi_0 \rangle] \end{aligned} \quad (7.53)$$

where we have observed that  $\hat{\mathbf{P}}|\Psi_0\rangle = 0$ . Equation (7.53) makes explicit that  $G$  depends only on the variables  $\mathbf{x} - \mathbf{x}'$  and  $t - t'$ .‡ The corresponding Fourier transform is

$$\begin{aligned} G_{\alpha\beta}(\mathbf{k}, \omega) &= \int d^3(\mathbf{x} - \mathbf{x}') \int d(t - t') e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} e^{i\omega(t - t')} G_{\alpha\beta}(\mathbf{x}t, \mathbf{x}'t') \\ &= V \sum_n \delta_{\mathbf{k}, \mathbf{P}_n/\hbar} \frac{\langle \Psi_0 | \hat{\psi}_{\alpha}(0) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{\beta}^{\dagger}(0) | \Psi_0 \rangle}{\omega - \hbar^{-1}(E_n - E) + i\eta} \\ &\quad + V \sum_n \delta_{\mathbf{k}, -\mathbf{P}_n/\hbar} \frac{\langle \Psi_0 | \hat{\psi}_{\beta}^{\dagger}(0) | \Psi_n \rangle \langle \Psi_n | \hat{\psi}_{\alpha}(0) | \Psi_0 \rangle}{\omega + \hbar^{-1}(E_n - E) - i\eta} \end{aligned} \quad (7.54)$$

where the  $\pm i\eta$  is again necessary to ensure the convergence of the integral over  $t - t'$ . In the first (second) term, the contribution vanishes unless the momentum of the state  $|\Psi_n\rangle$  corresponds to a wavenumber  $\mathbf{k}(-\mathbf{k})$ , which can be used to restrict the intermediate states:

$$\begin{aligned} G_{\alpha\beta}(\mathbf{k}, \omega) &= V \sum_n \left[ \frac{\langle \Psi_0 | \hat{\psi}_{\alpha}(0) | n\mathbf{k} \rangle \langle n\mathbf{k} | \hat{\psi}_{\beta}^{\dagger}(0) | \Psi_0 \rangle}{\omega - \hbar^{-1}(E_n - E) + i\eta} \right. \\ &\quad \left. + \frac{\langle \Psi_0 | \hat{\psi}_{\beta}^{\dagger}(0) | n, -\mathbf{k} \rangle \langle n, -\mathbf{k} | \hat{\psi}_{\alpha}(0) | \Psi_0 \rangle}{\omega + \hbar^{-1}(E_n - E) - i\eta} \right] \end{aligned} \quad (7.55)$$

‡ For many problems it is more convenient to assume that the interacting particles move relative to a fixed frame of reference. For example, in the problem of interacting electrons in crystalline solids and atoms, the crystalline lattice and heavy atomic nucleus provide such fixed frames. In this case the  $\hat{\mathbf{P}}$  of the interacting particles no longer commutes with  $\hat{H}$ , and the Green's function may depend explicitly on  $\mathbf{x}$  and  $\mathbf{x}'$ . This more complicated situation is discussed in Chap. 15.

Thus the general principles of quantum mechanics enable us to exhibit the frequency dependence of the Green's function, because  $\omega$  now appears only in the denominator of this sum.

It is helpful to examine these denominators in a little more detail. In the first sum the intermediate state has  $N + 1$  particles, and the denominator may be written as

$$\begin{aligned}\omega - \hbar^{-1}[E_n(N+1) - E(N)] &= \omega - \hbar^{-1}[E_n(N+1) - E(N+1)] \\ &\quad - \hbar^{-1}[E(N+1) - E(N)]\end{aligned}\quad (7.56)$$

Now  $E(N+1) - E(N)$  is the change in ground-state energy as one extra particle is added to the system. Since the volume of the system is kept constant, this change in energy is just the chemical potential [compare Eq. (4.3)]. Furthermore, the quantity  $E_n(N+1) - E(N+1) \equiv \epsilon_n(N+1)$  is the *excitation* energy of the  $N + 1$  particle system; by definition,  $\epsilon_n(N+1)$  is greater than or equal to zero. Similarly, the denominator of the second term can be written

$$\begin{aligned}\omega + \hbar^{-1}[E_n(N-1) - E(N)] &= \omega - \hbar^{-1}[E(N) - E(N-1)] \\ &\quad + \hbar^{-1}[E_n(N-1) - E(N-1)] \\ &= \omega - \hbar^{-1}\mu + \hbar^{-1}\epsilon_n(N-1)\end{aligned}\quad (7.57)$$

since  $E(N) - E(N-1)$  is again the chemical potential  $\mu$ , apart from corrections of order  $N^{-1}$ . Indeed, the very definition of the thermodynamic limit ( $N \rightarrow \infty$ ,  $V \rightarrow \infty$ , but  $N/V$  constant) implies

$$\mu(N+1) = \mu(N) + O(N^{-1})\quad (7.58)$$

Although we shall not attempt to prove this relation in general, it is readily demonstrated for a free Fermi gas at zero temperature, where the Pauli principle further ensures that  $\mu = \epsilon_F^0$ . Equations (7.56) and (7.57) can now be combined with Eq. (7.55) to give the Lehmann representation

$$\begin{aligned}G_{\alpha\beta}(\mathbf{k}, \omega) &= \hbar V \sum_n \left[ \frac{\langle \Psi_0 | \hat{\psi}_\alpha(0) | n\mathbf{k} \rangle \langle n\mathbf{k} | \hat{\psi}_\beta^\dagger(0) | \Psi_0 \rangle}{\hbar\omega - \mu - \epsilon_{n\mathbf{k}}(N+1) + i\eta} \right. \\ &\quad \left. + \frac{\langle \Psi_0 | \hat{\psi}_\beta^\dagger(0) | n, -\mathbf{k} \rangle \langle n, -\mathbf{k} | \hat{\psi}_\alpha(0) | \Psi_0 \rangle}{\hbar\omega - \mu + \epsilon_{n, -\mathbf{k}}(N-1) - i\eta} \right]\end{aligned}\quad (7.59)$$

It is possible to simplify the matrix structure of  $G$  in the special case of spin- $\frac{1}{2}$ . Since  $G$  is a  $2 \times 2$  matrix, it can be expanded in the complete set consisting of the unit matrix and the three Pauli spin matrices  $\sigma$ . If there is no preferred direction in the problem, then  $G$  must be a scalar under spatial rotations. Since  $\mathbf{k}$  is the only vector available to combine with  $\sigma$ ,  $G$  necessarily takes the form

$$G(\mathbf{k}, \omega) = a\mathbf{I} + b\boldsymbol{\sigma} \cdot \mathbf{k}$$

where the invariance under rotations implies that  $a$  and  $b$  are functions of  $\mathbf{k}^2$  and  $\omega$ . If, in addition, the hamiltonian is invariant under spatial reflections, then  $G$  must also have this property; but  $\sigma \cdot \mathbf{k}$  is a pseudoscalar under spatial reflections, so that the coefficient  $b$  must vanish. Thus, if the hamiltonian and ground state are invariant under spatial rotations and reflections, the Green's function has the following matrix structure

$$G_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta} G(\mathbf{k}, \omega) = \delta_{\alpha\beta} G(|\mathbf{k}|, \omega) \quad (7.60)$$

proportional to the unit matrix.

It is instructive to use Eq. (7.59) to reproduce our previous expression for  $G^0(\mathbf{k}, \omega)$  [Eq. (7.45)] in a free Fermi system. For the first term of Eq. (7.59), the added particle must lie above the Fermi sea, and the matrix elements of the field operators become

$$\langle \Psi_0 | \hat{\psi}_\alpha(0) | n\mathbf{k} \rangle \langle n\mathbf{k} | \hat{\psi}_\beta^\dagger(0) | \Psi_0 \rangle \rightarrow V^{-1} \delta_{\alpha\beta} \theta(k - k_F) \quad (7.61)$$

In the denominator of this term, the excitation energy is the difference between the actual energy of the additional particle and the energy that it would have at the Fermi surface. Thus the energy difference is given by

$$E_{\mathbf{k}}(N+1) - E(N+1) \equiv \epsilon_{\mathbf{k}}(N+1) \rightarrow \epsilon_{\mathbf{k}}^0 - \epsilon_F^0 = \frac{\hbar^2(k^2 - k_F^2)}{2m}$$

The second term of Eq. (7.59) clearly corresponds to a hole below the Fermi surface, and the matrix elements of the field operators become

$$\langle \Psi_0 | \hat{\psi}_\beta^\dagger(0) | n, -\mathbf{k} \rangle \langle n, -\mathbf{k} | \hat{\psi}_\alpha(0) | \Psi_0 \rangle \rightarrow V^{-1} \delta_{\alpha\beta} \theta(k_F - k)$$

The ground state of the  $N-1$  particle system is reached by letting a particle from the Fermi surface come down and fill up the hole; hence the energy difference in the second denominator is given by

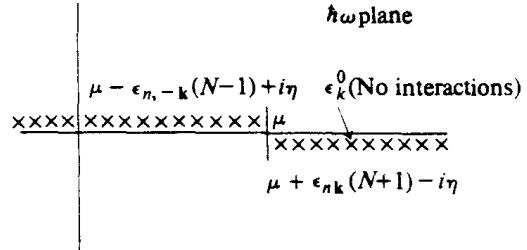
$$E_{-\mathbf{k}}(N-1) - E(N-1) \equiv \epsilon_{-\mathbf{k}}(N-1) \rightarrow \epsilon_F^0 - \epsilon_{-\mathbf{k}}^0 = \frac{\hbar^2(k_F^2 - k^2)}{2m}$$

Since  $\mu = \epsilon_F^0$  for a noninteracting system, we obtain Eq. (7.45).

As noted above, Eq. (7.59) exhibits the dependence of the *exact* Green's function on the frequency  $\omega$ , and it is interesting to consider the analytic properties of this function. The crucial observation is that *the function  $G(\mathbf{k}, \omega)$  is a meromorphic function of  $\hbar\omega$ , with simple poles at the exact excitation energies of the interacting system corresponding to a momentum  $\hbar\mathbf{k}$ .* For frequencies below  $\mu/\hbar$ , these singularities lie slightly above the real axis, and for frequencies above  $\mu/\hbar$ , these singularities lie slightly below the real axis (compare Fig. 7.1). In this way, the singularities of the Green's function immediately yield the energies of those excited states for which the numerator does not vanish. For an interacting system, the field operator connects the ground state with very many excited states of the system containing  $N \pm 1$  particles. For the non-

interacting system, however, the field operator connects only one state to the ground state, so that  $G^0(\mathbf{k}, \omega)$  has only a single pole, slightly below the real axis at  $\hbar\omega = \hbar^2 k^2/2m$  if  $k > k_F$  and slightly above the real axis at the same value of  $\hbar\omega$  if  $k < k_F$ .

It is clear from this discussion that the Green's function  $G$  is analytic in neither the upper nor the lower  $\omega$  plane. For contour integrations, however, it is useful to consider functions that are analytic in one half plane or the other.



**Fig. 7.1** Singularities of  $G(\mathbf{k}, \omega)$  in the complex  $\hbar\omega$  plane.

We therefore define a new pair of functions, known as *retarded* and *advanced* Green's functions

$$\begin{aligned} iG_{\alpha\beta}^R(\mathbf{x}t, \mathbf{x}'t') &= \langle \Psi_0 | \{\hat{\psi}_{H\alpha}(\mathbf{x}t), \hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')\} | \Psi_0 \rangle \theta(t - t') \\ iG_{\alpha\beta}^A(\mathbf{x}t, \mathbf{x}'t') &= -\langle \Psi_0 | \{\hat{\psi}_{H\alpha}(\mathbf{x}t), \hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')\} | \Psi_0 \rangle \theta(t' - t) \end{aligned} \quad (7.62)$$

where the braces denote an anticommutator. The analysis of these functions proceeds exactly as for the time-ordered Green's function. In a homogeneous system, we find the following Lehmann representation of their Fourier transforms:

$$\begin{aligned} G_{\alpha\beta}^{R,A}(\mathbf{k}, \omega) &= \hbar V \sum_n \left[ \frac{\langle \Psi_0 | \hat{\psi}_\alpha(0) | n\mathbf{k} \rangle \langle n\mathbf{k} | \hat{\psi}_\beta^\dagger(0) | \Psi_0 \rangle}{\hbar\omega - \mu - \epsilon_{n\mathbf{k}}(N+1) \pm i\eta} \right. \\ &\quad \left. + \frac{\langle \Psi_0 | \hat{\psi}_\beta^\dagger(0) | n, -\mathbf{k} \rangle \langle n, -\mathbf{k} | \hat{\psi}_\alpha(0) | \Psi_0 \rangle}{\hbar\omega - \mu + \epsilon_{n,-\mathbf{k}}(N-1) \pm i\eta} \right] \end{aligned} \quad (7.63)$$

Note that the Fourier transforms  $G^R(\mathbf{k}, \omega)$  and  $G^A(\mathbf{k}, \omega)$  are again meromorphic functions of  $\omega$ . All the poles of  $G^R(\mathbf{k}, \omega)$  lie in the lower half plane, so that  $G^R(\mathbf{k}, \omega)$  is analytic for  $\text{Im } \omega > 0$ ; in contrast, all the poles of  $G^A(\mathbf{k}, \omega)$  lie in the upper half plane, so that  $G^A(\mathbf{k}, \omega)$  is analytic for  $\text{Im } \omega < 0$ . For *real*  $\omega$ , these functions are simply related by

$$[G_{\alpha\beta}^R(\mathbf{k}, \omega)]^* = G_{\beta\alpha}^A(\mathbf{k}, \omega) \quad (7.64)$$

where the asterisk denotes complex conjugation. The retarded and advanced Green's functions differ from each other and from the time-ordered Green's function only in the convergence factors  $\pm i\eta$ , which are important near the singularities. If  $\omega$  is real and greater than  $\hbar^{-1}\mu$ , then the infinitesimal imaginary parts  $\pm i\eta$

in the second term of Eqs. (7.59) and (7.63) play no role. We therefore conclude that in this limited domain of the complex  $\omega$  plane

$$G_{\alpha\beta}^R(\mathbf{k}, \omega) = G_{\alpha\beta}(\mathbf{k}, \omega) \quad \hbar\omega \text{ real, } > \mu \quad (7.65a)$$

Similarly,

$$G_{\alpha\beta}^A(\mathbf{k}, \omega) = G_{\alpha\beta}(\mathbf{k}, \omega) \quad \hbar\omega \text{ real, } < \mu \quad (7.65b)$$

As noted previously,  $G_{\alpha\beta}$  is usually diagonal in the spin indices:  $G_{\alpha\beta} = G\delta_{\alpha\beta}$ . With the same assumptions, the retarded and advanced Green's functions are also diagonal, and we may solve for  $G$  as  $G = (2s+1)^{-1}\Sigma_\alpha G_{\alpha\alpha} \equiv (2s+1)^{-1}G_{\alpha\alpha}$  with the convention that repeated indices are to be summed.

If the spacing between adjacent energy levels is characterized by a typical value  $\Delta\epsilon$ , the discrete level structure can be resolved only over time scales long compared with  $\hbar/\Delta\epsilon$ . Conversely, if an observation lasts for a typical time  $\tau$ , then the corresponding energy resolution is of order  $\hbar/\tau$ . Since  $\Delta\epsilon$  becomes vanishingly small for a macroscopic sample, it generally satisfies the restriction  $\Delta\epsilon \ll \hbar/\tau$ , and we therefore detect only the *level density*, averaged over an energy interval  $\hbar/\tau$ . In the thermodynamic limit of a bulk system, it follows that the discrete variable  $n$  can be replaced by a continuous one. If  $dn$  denotes the number of levels in a small energy interval  $\epsilon < \epsilon_{n\mathbf{k}} < \epsilon + d\epsilon$ , then the summations in Eqs. (7.59) and (7.63) can be rewritten as

$$\begin{aligned} (2s+1)^{-1} V \sum_n |\langle n\mathbf{k} | \hat{\psi}_\alpha^\dagger(0) | \Psi_0 \rangle|^2 &\dots \\ &\approx (2s+1)^{-1} V \int dn |\langle n\mathbf{k} | \hat{\psi}_\alpha^\dagger(0) | \Psi_0 \rangle|^2 \dots \\ &= (2s+1)^{-1} V \int d\epsilon |\langle n\mathbf{k} | \hat{\psi}_\alpha^\dagger(0) | \Psi_0 \rangle|^2 \frac{dn}{d\epsilon} \dots \\ &\equiv \hbar^{-1} \int d\epsilon A(\mathbf{k}, \epsilon \hbar^{-1}) \dots \end{aligned} \quad (7.66a)$$

and

$$(2s+1)^{-1} V \sum_n |\langle n, -\mathbf{k} | \hat{\psi}_\alpha(0) | \Psi_0 \rangle|^2 \dots \equiv \hbar^{-1} \int d\epsilon B(\mathbf{k}, \epsilon \hbar^{-1}) \dots \quad (7.66b)$$

which define the positive-definite weight functions  $A(\mathbf{k}, \epsilon/\hbar)$  and  $B(\mathbf{k}, \epsilon/\hbar)$ . The corresponding Fourier transform of the single-particle Green's function becomes

$$G(\mathbf{k}, \omega) = \int_0^\infty d\omega' \left[ \frac{A(\mathbf{k}, \omega')}{\omega - \hbar^{-1}\mu - \omega' + i\eta} + \frac{B(\mathbf{k}, \omega')}{\omega - \hbar^{-1}\mu + \omega' - i\eta} \right] \quad (7.67)$$

which now has a *branch cut* in the complex  $\omega$  plane along the whole real axis. Thus the infinite-volume limit completely alters the analytic structure of  $G(\mathbf{k}, \omega)$ , because the discrete poles have merged to form a branch line. The same result describes a finite system whenever the individual levels cannot be resolved.

A similar analysis for the retarded and advanced Green's functions yields

$$G^{R,A}(\mathbf{k},\omega) = \int_0^\infty d\omega' \left[ \frac{A(\mathbf{k},\omega')}{\omega - \hbar^{-1}\mu - \omega' \pm i\eta} + \frac{B(\mathbf{k},\omega')}{\omega - \hbar^{-1}\mu + \omega' \pm i\eta} \right] \quad (7.68)$$

which shows that all three Green's functions can be constructed if  $A$  and  $B$  are known. In addition, the symbolic identity valid for real  $\omega$

$$\frac{1}{\omega \pm i\eta} = \mathcal{P} \frac{1}{\omega} \mp i\pi\delta(\omega) \quad (7.69)$$

shows that  $G^R$  and  $G^A$  satisfy dispersion relations

$$\text{Re } G^{R,A}(\mathbf{k},\omega) = \mp \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im } G^{R,A}(\mathbf{k},\omega')}{\omega - \omega'} \quad (7.70)$$

where  $\mathcal{P}$  denotes a Cauchy principal value. This equation also holds for finite systems, where  $\text{Im } G$  is a sum of delta functions.

These Green's functions all have a simple asymptotic behavior for large  $|\omega|$ . Consider the ground-state expectation value of the anticommutator

$$\langle \Psi_0 | \{\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta^\dagger(\mathbf{x}')\} | \Psi_0 \rangle = \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}') \quad (7.71)$$

An analysis similar to Eq. (7.53) shows that

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x}') &= (2s+1)^{-1} \sum_n [e^{i\mathbf{P}_n \cdot (\mathbf{x}-\mathbf{x}')/\hbar} |\langle \Psi_n | \hat{\psi}_\alpha^\dagger(0) | \Psi_0 \rangle|^2 \\ &\quad + e^{-i\mathbf{P}_n \cdot (\mathbf{x}-\mathbf{x}')/\hbar} |\langle \Psi_n | \hat{\psi}_\alpha(0) | \Psi_0 \rangle|^2] \end{aligned}$$

and its Fourier transform with respect to  $\mathbf{x} - \mathbf{x}'$  yields

$$\begin{aligned} 1 &= (2s+1)^{-1} V \sum_n [|\langle n\mathbf{k} | \hat{\psi}_\alpha^\dagger(0) | \Psi_0 \rangle|^2 + |\langle n, -\mathbf{k} | \hat{\psi}_\alpha(0) | \Psi_0 \rangle|^2] \\ &= \int_0^\infty d\omega [A(\mathbf{k},\omega) + B(\mathbf{k},\omega)] \end{aligned}$$

where the last line follows from Eq. (7.66). For  $|\omega| \rightarrow \infty$ , Eqs. (7.67) and (7.68) yield

$$\begin{aligned} G(\mathbf{k},\omega) &= G^R(\mathbf{k},\omega) = G^A(\mathbf{k},\omega) \sim \frac{1}{\omega} \int_0^\infty d\omega' [A(\mathbf{k},\omega') + B(\mathbf{k},\omega')] \\ &\sim \frac{1}{\omega} \quad |\omega| \rightarrow \infty \end{aligned} \quad (7.72)$$

which remains correct for an arbitrary interacting system.

#### PHYSICAL INTERPRETATION OF THE GREEN'S FUNCTION

To understand the physical interpretation of the single-particle Green's function, consider the interaction-picture state  $|\Psi_I(t')\rangle$ , and add a particle at the point  $(\mathbf{x}', t')$ :  $\hat{\psi}_{I\beta}^\dagger(\mathbf{x}', t')|\Psi_I(t')\rangle$ . Although this state is not in general an eigenstate of

the hamiltonian, it still propagates in time according to  $\hat{U}(t,t')\hat{\psi}_{I\beta}^\dagger(\mathbf{x}'t')|\Psi_I(t')\rangle$ . For  $t > t'$  what is the overlap of this state with the state  $\hat{\psi}_{Ix}^\dagger(\mathbf{x}t)|\Psi_I(t)\rangle$ ?

$$\begin{aligned} \langle\Psi_I(t)|\hat{\psi}_{Ix}(\mathbf{x}t)\hat{U}(t,t')\hat{\psi}_{I\beta}^\dagger(\mathbf{x}'t')|\Psi_I(t')\rangle \\ = \langle\Phi_0|\hat{U}(\infty,t)[\hat{U}(t,0)\hat{\psi}_{H\alpha}(\mathbf{x}t)\hat{U}(0,t)]\hat{U}(t,t') \\ \times [\hat{U}(t',0)\hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')\hat{U}(0,t')]|\hat{U}(t',-\infty)|\Phi_0\rangle \\ = \langle\Psi_0|\hat{\psi}_{H\alpha}(\mathbf{x}t)\hat{\psi}_{H\beta}^\dagger(\mathbf{x}'t')|\Psi_0\rangle \end{aligned}$$

where we have used the results of Sec. 6. This quantity is just the Green's function for  $t > t'$ , which therefore characterizes the propagation of a state containing an additional particle. In a similar way, if  $t < t'$ , the field operator first creates a hole at time  $t$ , and the system then propagates according to the full hamiltonian. These holes can be interpreted as particles going backward in time, as discussed in the famous papers of Feynman.<sup>1</sup> The probability amplitude at a later time  $t'$  for finding a single hole in the ground state of the interacting system is again just the Green's function for  $t < t'$ .

We shall now study how this propagation in time is related to the function  $G(\mathbf{k},\omega)$ , and, for definiteness, we shall consider only the usual case where the time scale is too short to resolve the separate energy levels.<sup>2</sup> By the definition of the Fourier transform, the time dependence is given by

$$G(\mathbf{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k},\omega) \quad (7.73)$$

If  $t > 0$ , the integral may be evaluated by deforming the contour into the lower half  $\omega$  plane. Since  $G(\mathbf{k},\omega)$  has a rather complicated analytic structure, it is convenient to separate Eq. (7.73) into two parts:

$$G(\mathbf{k},t) = \int_{-\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k},\omega) + \int_{\mu/\hbar}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k},\omega) \quad (7.74)$$

In the first term ( $\omega$  real and  $< \hbar^{-1}\mu$ ),  $G(\mathbf{k},\omega)$  coincides with the advanced Green's function  $G^A(\mathbf{k},\omega)$  [Eq. (7.65b)], and the integral thus becomes

$$\int_{-\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k},\omega) = \int_{-\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} G^A(\mathbf{k},\omega) \quad (7.75)$$

Now  $G^A(\mathbf{k},\omega)$  is analytic in the lower half plane, and the contour can be deformed from  $C_1$  to  $C'_1$  (Fig. 7.2a). Equation (7.72) shows that  $G^A$  (and  $G^R$ ) behaves

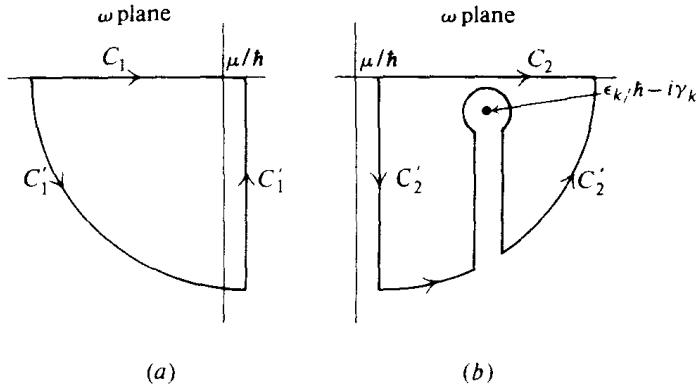
<sup>1</sup> R. P. Feynman, *Phys. Rev.*, **76**: 749 (1949); **76**: 769 (1949).

<sup>2</sup> Our argument follows that of V. M. Galitskii and A. B. Migdal, *loc. cit.* and of A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *loc. cit.*

like  $\omega^{-1}$  for  $|\omega| \rightarrow \infty$ ; Jordan's lemma<sup>1</sup> thus ensures that the contribution from the arc at infinity vanishes, and Eq. (7.75) reduces to

$$\int_{-\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k}, \omega) = \int_{\mu/\hbar - i\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} G^A(\mathbf{k}, \omega) \quad (7.76)$$

The second term of Eq. (7.74) can be treated similarly, because  $G(\mathbf{k}, \omega)$  coincides with  $G^R(\mathbf{k}, \omega)$  for real  $\omega > \mu/\hbar$ . There is one important new feature, however,



**Fig. 7.2** Contours used in evaluating  $G(\mathbf{k}, t)$  for  $t > 0$ .

because  $G^R(\mathbf{k}, \omega)$  is not analytic in the lower half  $\omega$  plane but instead has singularities. For definiteness we make a very elementary model of the interacting assembly and assume that  $G^R(\mathbf{k}, \omega)$  has a simple pole close to the real axis in the lower half plane at  $\omega = \hbar^{-1} \epsilon_k - i\gamma_k$  with residue  $a$ , where  $\epsilon_k > \mu$  and  $\epsilon_k - \mu \gg \hbar\gamma_k \geq 0$ . (If  $G^R$  has several poles, the same analysis applies to each one separately.) The contour  $C_2$  can be deformed to  $C_2'$  (Fig. 7.2b), and the large arc at infinity again makes no contribution; the second term of Eq. (7.74) then becomes

$$\int_{\mu/\hbar}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\mathbf{k}, \omega) = \int_{\mu/\hbar}^{\mu/\hbar - i\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\mathbf{k}, \omega) - ia e^{-i\epsilon_k t/\hbar} e^{-\gamma_k t} \quad (7.77)$$

A combination of Eqs. (7.76) and (7.77) yields

$$G(\mathbf{k}, t) = \int_{\mu/\hbar - i\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} [G^A(\mathbf{k}, \omega) - G^R(\mathbf{k}, \omega)] - ia e^{-i\epsilon_k t/\hbar} e^{-\gamma_k t} \quad (7.78)$$

If  $t$  is neither too large nor too small, the integral in Eq. (7.78) is negligible, and the state containing one additional particle propagates like an approximate eigenstate with a frequency  $\epsilon_k/\hbar$  and damping constant  $\gamma_k$ . More precisely, we shall now show that if

1.  $|t|(\epsilon_k - \mu) \gg \hbar$
2.  $|t| \gamma_k \lesssim 1$

<sup>1</sup> See, for example, E. G. Phillips, "Functions of a Complex Variable," p. 122, Interscience Publishers, Inc., New York, 1958.

then<sup>1</sup>

$$G(\mathbf{k}, t) \approx -ia e^{-i\epsilon_k t/\hbar} e^{-\gamma_k t} \quad (7.79)$$

Note that the condition  $\epsilon_k - \mu \gg \hbar\gamma_k$  is assumed implicitly, so that the pole must lie very close to the real axis. In this case, Eq. (7.79) shows that the real and imaginary parts of the poles of the analytic continuation of  $G^R(\mathbf{k}, \omega)$  into the lower half plane determine the frequency and lifetime of the excited states obtained by adding a particle to an interacting ground state. Equation (7.79) is readily proved by noting that the integrand in Eq. (7.78) is exponentially small as  $\text{Im } \omega$  becomes large and negative, so that the dominant contributions come from the region near the real axis. On the real axis, in the vicinity of the pole we have

$$\begin{aligned} G^R(\mathbf{k}, \omega) &\approx \frac{a}{\omega - \epsilon_k/\hbar + i\gamma_k} \\ G^A(\mathbf{k}, \omega) = [G^R(\mathbf{k}, \omega)]^* &\approx \frac{a}{\omega - \epsilon_k/\hbar - i\gamma_k} \end{aligned}$$

where the second relation follows from Eq. (7.64). These relations allow us to analytically continue  $G^R(\mathbf{k}, \omega)$  and  $G^A(\mathbf{k}, \omega)$  into the complex  $\omega$  plane, and the integral in Eq. (7.78) can therefore be written as

$$\begin{aligned} &\int_{\mu/\hbar-i\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} e^{-i\omega t} [G^A(\mathbf{k}, \omega) - G^R(\mathbf{k}, \omega)] \\ &\approx 2i\gamma_k a \int_{\mu/\hbar-i\infty}^{\mu/\hbar} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega - \hbar^{-1}\epsilon_k)^2 + \gamma_k^2} \\ &= -\frac{\gamma_k a e^{-i\mu t/\hbar}}{\pi} \int_0^\infty du \frac{e^{-ut}}{\gamma_k^2 + [\hbar^{-1}(\mu - \epsilon_k) - iu]^2} \\ &\approx -(\pi t)^{-1} \gamma_k a \hbar^2 (\mu - \epsilon_k)^{-2} e^{-i\mu t/\hbar} \ll -ia e^{-i\epsilon_k t/\hbar} e^{-\gamma_k t} \quad (7.80) \end{aligned}$$

where the third line is obtained with the substitution  $u = i(\omega - \hbar^{-1}\mu)$ . The final form follows by using assumptions 1 and 2, along with the condition  $\gamma_k \ll \hbar^{-1}(\epsilon_k - \mu)$ . Note that the last inequality in Eq. (7.80) fails if  $t$  is too large or too small. In a wholly analogous fashion, the poles of the analytic continuation of  $G^A(\mathbf{k}, \omega)$  into the upper half  $\omega$  plane determine the frequency and lifetime of the state obtained by creating a hole (destroying a particle) in the interacting ground state.

<sup>1</sup> The apparent exponential decay is slightly misleading because condition 2 restricts us to the region where  $e^{-\gamma t} \approx 1 - \gamma t$ .

### 8 WICK'S THEOREM<sup>1</sup>

The preceding section defined the single-particle Green's function and exhibited its relation to observable properties. This analysis in no way solves the fundamental many-body problem, however, and we must still calculate  $G$  for nontrivial physical systems. As our general method of attack, we shall evaluate the Green's function with perturbation theory. This procedure is most easily carried out in the interaction picture, where the various terms can be enumerated with a theorem of Wick, derived in this section. The remainder of this chapter (Sec. 9) is devoted to the diagrammatic analysis of the perturbation series.

The Green's function consists of a matrix element of Heisenberg operators in the exact interacting ground state. This form is inconvenient for perturbation theory, and we now prove a basic theorem that relates the matrix element of a Heisenberg operator  $\hat{O}_H(t)$  to the matrix element of the corresponding interaction operator  $\hat{O}_I(t)$ :

$$\frac{\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{1}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle} \langle \Phi_0 | \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^{\nu} \frac{1}{\nu!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_{\nu} e^{-\epsilon(|t_1| + \cdots + |t_{\nu}|)} T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_{\nu}) \hat{O}_I(t)] | \Phi_0 \rangle \quad (8.1)$$

Here the operator  $\hat{S}$  is defined by

$$\hat{S} = \hat{U}_{\epsilon}(\infty, -\infty) \quad (8.2)$$

The proof is as follows: The Gell-Mann and Low theorem expresses the ground state of the interacting system in the interaction picture

$$\frac{|\Psi_0\rangle}{\langle \Phi_0 | \Psi_0 \rangle} = \frac{\hat{U}_{\epsilon}(0, \pm\infty) |\Phi_0\rangle}{\langle \Phi_0 | \hat{U}_{\epsilon}(0, \pm\infty) |\Phi_0 \rangle}$$

The denominator on the left side of Eq. (8.1) can be calculated by writing  $\hat{U}_{\epsilon}(0, -\infty) |\Phi_0\rangle$  on the right and  $\hat{U}_{\epsilon}(0, \infty) |\Phi_0\rangle$  on the left

$$\begin{aligned} \frac{\langle \Psi_0 | \Psi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} &= \frac{\langle \Phi_0 | \hat{U}_{\epsilon}(0, \infty)^{\dagger} \hat{U}_{\epsilon}(0, -\infty) | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} \\ &= \frac{\langle \Phi_0 | \hat{U}_{\epsilon}(\infty, 0) \hat{U}_{\epsilon}(0, -\infty) | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} \\ &= \frac{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} \end{aligned} \quad (8.3)$$

where both Eqs. (6.15) and (6.16) have been used. In a similar way, the numerator on the left side of Eq. (8.1) becomes, with the aid of Eq. (6.31),

$$\begin{aligned} \frac{\langle \Phi_0 | \hat{U}_{\epsilon}(\infty, 0) \hat{U}_{\epsilon}(0, t) \hat{O}_I(t) \hat{U}_{\epsilon}(t, 0) \hat{U}_{\epsilon}(0, -\infty) | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} &= \frac{\langle \Phi_0 | \hat{U}_{\epsilon}(\infty, t) \hat{O}_I(t) \hat{U}_{\epsilon}(t, -\infty) | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} \end{aligned} \quad (8.4)$$

<sup>1</sup> G. C. Wick, *Phys. Rev.*, **80**:268 (1950).

The common denominators of Eqs. (8.3) and (8.4) cancel in forming the ratio, and we find

$$\frac{\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{\langle \Phi_0 | \hat{U}_\epsilon(\infty, t) \hat{O}_I(t) \hat{U}_\epsilon(t, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle} \quad (8.5)$$

The remaining problem is to rewrite the numerator of the right side of Eq. (8.5), containing the operator

$$\begin{aligned} & \hat{U}_\epsilon(\infty, t) \hat{O}_I(t) \hat{U}_\epsilon(t, -\infty) \\ &= \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_t^\infty dt_1 \cdots \int_t^\infty dt_n e^{-\epsilon(|t_1| + \cdots + |t_n|)} T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] \\ & \quad \times \hat{O}_I(t) \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^m \frac{1}{m!} \int_{-\infty}^t dt_1 \cdots \int_{-\infty}^t dt_m \\ & \quad \times e^{-\epsilon(|t_1| + \cdots + |t_m|)} T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_m)] \end{aligned} \quad (8.6)$$

where Eq. (6.37) has been used. The theorem will now be proved by demonstrating that the operator in the numerator on the right side of Eq. (8.1) is equal to Eq. (8.6). In the  $\nu$ th term of the sum in Eq. (8.1), divide the integration variables into  $n$  factors with  $t_i > t$  and  $m$  factors with  $t_i < t$ , where  $m + n = \nu$ . There are  $\nu!/m!n!$  ways to make this partition, and a summation over all values of  $m$  and  $n$  consistent with the restriction  $\nu = m + n$  completely enumerates the regions of integration in this  $\nu$ -fold multiple integral. The operator in Eq. (8.1) therefore becomes

$$\begin{aligned} & \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\nu, m+n} \frac{\nu!}{m!n!} \int_t^\infty dt_1 \cdots \int_t^\infty dt_n e^{-\epsilon(|t_1| + \cdots + |t_n|)} \\ & \quad \times T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] \hat{O}_I(t) \int_{-\infty}^t dt_1 \cdots \int_{-\infty}^t dt_m \\ & \quad \times e^{-\epsilon(|t_1| + \cdots + |t_m|)} T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_m)] \end{aligned} \quad (8.7)$$

The Kronecker delta here ensures that  $m + n = \nu$ , but it also can be used to perform the summation over  $\nu$ , which proves the theorem because Eq. (8.7) then reduces to Eq. (8.6).

In a similar manner, the expectation value of time-ordered Heisenberg operators may be written as

$$\begin{aligned} \frac{\langle \Psi_0 | T[\hat{O}_H(t) \hat{O}_H(t')] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} &= \frac{1}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle} \langle \Phi_0 | \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \\ & \quad \times \int_{-\infty}^\infty dt_1 \cdots \int_{-\infty}^\infty dt_\nu e^{-\epsilon(|t_1| + \cdots + |t_\nu|)} \\ & \quad \times T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_\nu) \hat{O}_I(t) \hat{O}_I(t')] | \Phi_0 \rangle \end{aligned} \quad (8.8)$$

This result depends on the observation

$$\begin{aligned} \langle \Phi_0 | \hat{U}_\epsilon(\infty, 0) [\hat{U}_\epsilon(0, t) \hat{O}_I(t) \hat{U}_\epsilon(t, 0)] [\hat{U}_\epsilon(0, t') \hat{O}_I(t') \hat{U}_\epsilon(t', 0)] \hat{U}_\epsilon(0, -\infty) | \Phi_0 \rangle \\ = \langle \Phi_0 | \hat{U}_\epsilon(\infty, t) \hat{O}_I(t) \hat{U}_\epsilon(t, t') \hat{O}_I(t') \hat{U}_\epsilon(t', -\infty) | \Phi_0 \rangle \end{aligned}$$

and we must therefore partition the integration variables into three distinct groups. Otherwise, the proof is identical with that of Eq. (8.1). Since Eqs. (8.1) and (8.8) both consist of ratios, the divergent phase factors cancel, and it is permissible to take the limit  $\epsilon \rightarrow 0$ . In this last form, these theorems are among the most useful results of quantum field theory.

As an interesting example, the exact Green's function may be written as

$$\begin{aligned} iG_{\alpha\beta}(x, y) &= \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^{\nu} \frac{1}{\nu!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_{\nu} \\ &\times \frac{\langle \Phi_0 | T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_{\nu}) \hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y)] | \Phi_0 \rangle}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle} \quad (8.9) \end{aligned}$$

where the notation  $x \equiv (\mathbf{x}, x_0) \equiv (\mathbf{x}, t_x)$  has been introduced. Here and henceforth, the subscript  $I$  will be omitted, since we shall consistently work in the interaction picture. It is also convenient to rewrite the interparticle potential in  $\hat{H}_1$  as

$$U(x_1, x_2) \equiv V(\mathbf{x}_1, \mathbf{x}_2) \delta(t_1 - t_2) \quad (8.10)$$

which allows us to write the integrations symmetrically.<sup>1</sup> For example, the numerator of Eq. (8.9), which we will denote by  $i\tilde{G}$ , becomes

$$\begin{aligned} i\tilde{G}_{\alpha\beta}(x, y) &\equiv iG_{\alpha\beta}^0(x, y) + \left( \frac{-i}{\hbar} \right) \sum_{\lambda\lambda'} \frac{1}{2} \int d^4x_1 d^4x'_1 U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} \\ &\times \langle \Phi_0 | T[\hat{\psi}_{\lambda}^{\dagger}(x_1) \hat{\psi}_{\mu}^{\dagger}(x'_1) \hat{\psi}_{\mu}(x'_1) \hat{\psi}_{\lambda'}(x_1) \hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y)] | \Phi_0 \rangle + \cdots \quad (8.11) \end{aligned}$$

where  $iG_{\alpha\beta}^0(x, y) = \langle \Phi_0 | T[\hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y)] | \Phi_0 \rangle$  refers to the noninteracting system. This expression shows that we must evaluate the expectation value in the non-interacting ground state of  $T$  products of creation and destruction operators of the form

$$\langle \Phi_0 | T[\hat{\psi}^{\dagger} \cdots \hat{\psi} \hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y)] | \Phi_0 \rangle \quad (8.12)$$

<sup>1</sup> The Green's function now assumes a covariant appearance and, indeed, is just that obtained in relativistic quantum electrodynamics, where the interaction of Eq. (8.10) is mediated by the exchange of virtual photons of the electromagnetic field. The only difference is that quantum electrodynamics involves the retarded electromagnetic interaction, whereas the present theory involves a static instantaneous potential proportional to a delta function  $\delta(t_1 - t_2)$ . It should be emphasized, however, that the *formalism* developed here applies equally well to relativistic quantum field theory, which is especially evident in Chap. 12, where we consider a nonrelativistic retarded interaction arising from phonon exchange.

It is clear that the creation and destruction operators must be paired or the expectation value vanishes; even in this lowest-order term, however, the straightforward approach of classifying all possible contributions by direct application of the commutation or anticommutation relations is very lengthy. Instead, we shall rely on Wick's theorem, which provides a general procedure for evaluating such matrix elements.

The essential idea is to move all destruction operators to the right, where they annihilate the noninteracting ground state. In so doing, we generate additional terms, proportional to the commutators or anticommutators of the operators involved in the interchanges of positions. For most purposes, it is more convenient to use the field operators directly rather than the operators  $\{c_k\}$  referring to a single mode. In most systems of interest,  $\hat{\psi}(x)$  can be uniquely separated into a destruction part  $\hat{\psi}^{(+)}(x)$  that annihilates the noninteracting ground state and a creation part  $\hat{\psi}^{(-)}(x)$ .<sup>‡</sup>

$$\hat{\psi}(x) = \hat{\psi}^{(+)}(x) + \hat{\psi}^{(-)}(x) \quad (8.13)$$

$$\hat{\psi}^{(+)}(x)|\Phi_0\rangle = 0 \quad (8.14)$$

Correspondingly, the adjoint operator becomes

$$\hat{\psi}^\dagger(x) = \hat{\psi}^{(+)\dagger}(x) + \hat{\psi}^{(-)\dagger}(x) \quad (8.15)$$

where

$$\hat{\psi}^{(-)\dagger}(x)|\Phi_0\rangle = 0 \quad (8.16)$$

Thus  $\hat{\psi}^{(+)}(x)$  and  $\hat{\psi}^{(-)\dagger}(x)$  are both destruction parts, while  $\hat{\psi}^{(-)}(x)$  and  $\hat{\psi}^{(+)\dagger}(x)$  are both creation parts. The notation is a vestige of the original application of Wick's theorem to relativistic quantum field theory, where (+) and (-) signs refer to a Lorentz-invariant decomposition into positive and negative frequency parts. For our purposes, however, they can be considered superscripts denoting destruction and creation parts. As an explicit example of this decomposition, consider the free fermion field, rewritten with the canonical transformation of Eq. (7.34):

$$\begin{aligned} \hat{\psi}(x) &= \sum_{\mathbf{k}\lambda > k_F} V^{-\frac{1}{2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)} \eta_\lambda a_{\mathbf{k}\lambda} + \sum_{\mathbf{k}\lambda < k_F} V^{-\frac{1}{2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)} \eta_\lambda b_{-\mathbf{k}\lambda}^\dagger \\ &\equiv \hat{\psi}^{(+)}(x) + \hat{\psi}^{(-)}(x) \end{aligned} \quad (8.17)$$

In this case, the symbols (+) and (-) may be interpreted as the sign of the frequencies of the field components *measured with respect to the Fermi energy*.

To present Wick's theorem in a concise and useful manner, it is necessary to introduce some new definitions.

1. *T product*: The *T* product of a collection of field operators has already been defined [Eq. (7.4)]. It orders the field operators with the latest time on

<sup>‡</sup> For a discussion of the special problems inherent in the treatment of condensed Bose systems, see Chap. 6.

the left and includes an additional factor of  $-1$  for each interchange of fermion operators. By definition

$$T(\hat{A}\hat{B}\hat{C}\hat{D} \dots) = (-1)^P T(\hat{C}\hat{A}\hat{D}\hat{B} \dots) \quad (8.18)$$

where  $P$  is the number of permutations of fermion operators needed to rearrange the product as given on the left side of Eq. (8.18) to agree with the order on the right side. It is clearly permissible to treat the boson fields as if they commute and the fermion fields as if they anticommute when reordering fields *within* a  $T$  product.

2. *Normal ordering*: This term represents a different ordering of a product of field operators, in which all the annihilation operators are placed to the right of all the creation operators, again including a factor of  $-1$  for every interchange of fermion operators. By definition

$$N(\hat{A}\hat{B}\hat{C}\hat{D} \dots) = (-1)^P N(\hat{C}\hat{A}\hat{D}\hat{B} \dots) \quad (8.19)$$

so that the fields *within* a normal-ordered product can again be treated as if they commute (bosons) or anticommute (fermions). For example, if we deal with fermion fields,

$$\begin{aligned} N[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] &= -\hat{\psi}^{(-)}(y) \hat{\psi}^{(+)}(x) \\ N[\hat{\psi}^{(+)}(x) \hat{\psi}^{(+)\dagger}(y)] &= -\hat{\psi}^{(+)\dagger}(y) \hat{\psi}^{(+)}(x) \end{aligned} \quad (8.20)$$

In both cases the creation part of the field is written to the left, and the factor  $-1$  reflects the single interchange of fermion operators. The reader is urged to write out several examples of each definition.

A normal-ordered product of field operators is especially convenient because *its expectation value in the unperturbed ground state*  $|\Phi_0\rangle$  *vanishes identically* [see Eqs. (8.14) and (8.16)]. This result remains true even if the product consists entirely of creation parts, as is clear from the adjoint of the equations defining the destruction parts. Thus the ground-state expectation value of a  $T$  product of operators [for example (8.12)] may be evaluated by reducing it to the corresponding  $N$  product; the fundamental problem is the enumeration of the additional terms introduced in the reduction. This process is simplified by noting that *both the  $T$  product and the  $N$  product are distributive*. For example,

$$\begin{aligned} N[(\hat{A} + \hat{B})(\hat{C} + \hat{D}) \dots] &= N(\hat{A}\hat{C} \dots) + N(\hat{A}\hat{D} \dots) + N(\hat{B}\hat{C} \dots) \\ &\quad + N(\hat{B}\hat{D} \dots) + \dots \end{aligned}$$

It is therefore sufficient to prove the theorems separately for creation or destruction parts.

3. *Contractions*: The contraction of two operators  $\hat{U}$  and  $\hat{V}$  is denoted  $\hat{U} \cdot \hat{V}$  and is equal to the difference between the  $T$  product and the  $N$  product.

$$\hat{U} \cdot \hat{V} \equiv T(\hat{U}\hat{V}) - N(\hat{U}\hat{V}) \quad (8.21)$$

It represents the additional term introduced by rearranging a time-ordered product into a normal-ordered product and is therefore different for different time orderings of the operators. As an example, all of the following contractions vanish

$$\hat{\psi}^{(+)} \cdot \hat{\psi}^{(-)} = \hat{\psi}^{(+)\dagger} \cdot \hat{\psi}^{(-)\dagger} = \hat{\psi}^{(+)\dagger} \cdot \hat{\psi}^{(-)} = \hat{\psi}^{(+)} \cdot \hat{\psi}^{(-)\dagger} = 0 \quad (8.22)$$

because the  $T$  product of these operators is identical with the  $N$  product of the same operators. To be more specific, consider the first pair of operators in Eq. (8.22). Their  $T$  product is given by

$$T[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] \equiv \begin{cases} \hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y) & t_x > t_y \\ \pm \hat{\psi}^{(-)}(y) \hat{\psi}^{(+)}(x) & t_y > t_x \end{cases} \quad (8.23)$$

where the  $\pm$  in the second line refers to bosons or fermions. But the field operator  $\hat{\psi}$  is a linear combination of interaction-picture operators of the form  $c_k e^{-i\omega_k t}$  [compare Eq. (6.10a)]. Thus, for either statistics, Eq. (8.23) may be rewritten as

$$T[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] = \pm \hat{\psi}^{(-)}(y) \hat{\psi}^{(+)}(x) \quad (8.24)$$

because  $\hat{\psi}^{(-)}$  and  $\hat{\psi}^{(+)}$  commute or anticommute *at any time*. Note that this result is true *only* in the interaction picture, where the operator properties are the same as in the Schrödinger picture. By the definition of a normal-ordered product, we have

$$N[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] \equiv \pm \hat{\psi}^{(-)}(y) \hat{\psi}^{(+)}(x) \quad (8.25)$$

and their contraction therefore vanishes

$$\hat{\psi}^{(+)}(x) \cdot \hat{\psi}^{(-)}(y) \equiv T[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] - N[\hat{\psi}^{(+)}(x) \hat{\psi}^{(-)}(y)] = 0 \quad (8.26)$$

The other contractions in Eq. (8.22) also vanish because all of the paired interaction-picture operators commute or anticommute with each other.

Equation (8.22) shows that *most contractions are zero*. In particular, a contraction of two creation parts or two destruction parts vanishes, and the only nonzero contractions are given by

$$\begin{aligned} \hat{\psi}^{(+)}(x) \cdot \hat{\psi}^{(+)\dagger}(y) &= \begin{cases} iG^0(x, y) & t_x > t_y \\ 0 & t_y > t_x \end{cases} \\ \hat{\psi}^{(-)}(x) \cdot \hat{\psi}^{(-)\dagger}(y) &= \begin{cases} 0 & t_x > t_y \\ iG^0(x, y) & t_y > t_x \end{cases} \end{aligned} \quad (8.27)$$

For fermions, this result is derived with the canonical anticommutation relations of the creation and destruction operators [Eq. (1.48)] and the definition of the free Green's function given in Eq. (7.41). A similar derivation applies for noncondensed bosons (see, for example, Chap. 12). Note that the contractions

are *c numbers* in the occupation-number Hilbert space, not operators. Equation (8.27) is more simply derived with the observation

$$\langle \Phi_0 | T(\hat{U}\hat{V}) | \Phi_0 \rangle = \langle \Phi_0 | \hat{U}^\cdot \hat{V}^\cdot | \Phi_0 \rangle + \langle \Phi_0 | N(\hat{U}\hat{V}) | \Phi_0 \rangle = \hat{U}^\cdot \hat{V}^\cdot \quad (8.28)$$

since  $\langle \Phi_0 | N(\hat{U}\hat{V}) | \Phi_0 \rangle$  vanishes by definition. The distributive properties then yield the contraction of the field operators themselves

$$\hat{\psi}_\alpha(x)^\cdot \hat{\psi}_\beta^\dagger(y)^\cdot = iG_{\alpha\beta}^0(x,y) \quad (8.29)$$

4. *A convention:* We introduce a further sign convention. Normal-ordered products of field operators with more than one contraction will have the contractions denoted by pairs of superscripts with single dots, double dots, etc. Two factors that are contracted must be brought together by rearranging the order of the operators within the normal product, always keeping the standard sign convention for interchange of operators. The contracted operators are then to be replaced by the value of the contraction given by Eq. (8.27). Since this contraction is now just a function of the coordinate variables, it can be taken outside of the normal-ordered product.

$$N(\hat{A}^\cdot \hat{B}\hat{C}^\cdot \hat{D} \cdots) = \pm N(\hat{A}^\cdot \hat{C}^\cdot \hat{B}\hat{D} \cdots) = \pm \hat{A}^\cdot \hat{C}^\cdot N(\hat{B}\hat{D} \cdots) \quad (8.30)$$

Finally, note that

$$\hat{U}^\cdot \hat{V}^\cdot = \pm \hat{V}^\cdot \hat{U}^\cdot \quad (8.31)$$

which follows from Eq. (8.21) and the definition of  $T$  product and normal-ordered product. It is now possible to state

5. *Wick's theorem:*

$$\begin{aligned} T(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}) &= N(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}) + N(\hat{U}^\cdot \hat{V}^\cdot \hat{W} \cdots \hat{X}\hat{Y}\hat{Z}) \\ &\quad + N(\hat{U}^\cdot \hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}) + \cdots \\ &\quad + N(\hat{U}^\cdot \hat{V}^\cdot \hat{W} \cdots \hat{X} \cdots \hat{Y} \cdots \hat{Z}) \\ &= N(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}) \\ &\quad + N(\text{sum over all possible pairs of} \\ &\quad \text{contractions}) \end{aligned} \quad (8.32)$$

The basic idea of the theorem is as follows: Consider a given time ordering, and start moving the creation parts to the left within this product of field operators. Each time a creation part fails to commute or anticommute, it generates an additional term, which is just the contraction. It is permissible to include *all* possible contractions, since the contraction vanishes if the creation part is already to the left of the destruction part (remember that *most contractions are zero*); hence the theorem clearly enumerates all the extra terms that occur in reordering a  $T$  product into a normal-ordered product.

To prove the theorem, we shall follow Wick's derivation and first prove the following.

6. *Basic lemma:* If  $N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y})$  is a normal-ordered product and  $\hat{Z}$  is a factor labeled with a time earlier than the times for  $\hat{U}, \hat{V} \cdots \hat{X}, \hat{Y}$ , then

$$\begin{aligned} N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y})\hat{Z} &= N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}') + N(\hat{U}\hat{V} \cdots \hat{X}'\hat{Y}\hat{Z}') \\ &\quad + \cdots + N(\hat{U}'\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}') + N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) \end{aligned} \quad (8.33)$$

Thus if a normal-ordered product is multiplied on the right with any operator at an earlier time, we obtain a sum of normal-ordered products containing the extra operator contracted in turn with all the operators standing in the original product, along with a term where the extra operator is included within the normal-ordered product. To prove the lemma, note the following points:

(a) If  $\hat{Z}$  is a destruction operator, then all the contractions vanish since  $T(\hat{A}\hat{Z}) = N(\hat{A}\hat{Z})$ . Thus, only the last term in Eq. (8.33) contributes and the lemma is proved.

(b) The operator product  $\hat{U}\hat{V} \cdots \hat{X}\hat{Y}$  can be assumed to be normal ordered, since otherwise the operators can be reordered on both sides of the equation. Our sign conventions ensure the same signature factor occurs in each term of Eq. (8.33) and therefore cancels identically.

(c) We can further assume that  $\hat{Z}$  is a creation operator, and  $\hat{U} \cdots \hat{Y}$  are all destruction operators. If the lemma is proved in this form, creation operators may be included by multiplying on the left; the additional contractions so introduced vanish identically and can therefore be added to the right side of Eq. (8.33) without changing the result.

Hence it is sufficient to prove Eq. (8.33) for  $\hat{Z}$  a creation operator and  $\hat{U} \cdots \hat{Y}$  destruction operators. The proof follows by induction. Equation (8.33) is evidently true for two operators by definition [Eq. (8.21)]

$$\hat{Y}\hat{Z} = T(\hat{Y}\hat{Z}) = \hat{Y}\hat{Z}' + N(\hat{Y}\hat{Z}) \quad (8.34)$$

We now assume it is true for  $n$  operators and prove it for  $n+1$  operators. Multiply the lemma (8.33) on the left by another destruction operator  $\hat{D}$  having a time later than that of  $\hat{Z}$ .

$$\begin{aligned} \hat{D}N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y})\hat{Z} &= N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}\hat{Y})\hat{Z} \\ &= N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}') + N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}'\hat{Y}\hat{Z}') + \cdots \\ &\quad + N(\hat{D}\hat{U}'\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}') + \hat{D}N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) \end{aligned} \quad (8.35)$$

Since  $\hat{U}, \hat{V} \cdots \hat{X}, \hat{Y}$  are all destruction operators and the contraction of  $\hat{Z}$  with any destruction operator is a  $c$  number,  $\hat{D}$  has been taken inside the normal ordering except for the very last term in Eq. (8.35), where  $\hat{Z}$  is still an operator.

Consider this last term, which, we assert, can be written as

$$\hat{D}N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) = N(\hat{D} \cdot \hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) + N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) \quad (8.36)$$

This equation is readily verified.

$$\begin{aligned} \hat{D}N(\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) &= (-1)^P \hat{D}\hat{Z}\hat{U}\hat{V} \cdots \hat{X}\hat{Y} \\ &= (-1)^P T(\hat{D}\hat{Z}) \hat{U}\hat{V} \cdots \hat{X}\hat{Y} \\ &= (-1)^P \hat{D} \cdot \hat{Z} \cdot \hat{U}\hat{V} \cdots \hat{X}\hat{Y} + (-1)^{P+Q} N(\hat{Z}\hat{D}) \hat{U}\hat{V} \cdots \hat{X}\hat{Y} \\ &= [(-1)^P]^2 \hat{D} \cdot \hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z} + [(-1)^{P+Q}]^2 N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) \\ &= N(\hat{D} \cdot \hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) + N(\hat{D}\hat{U}\hat{V} \cdots \hat{X}\hat{Y}\hat{Z}) \end{aligned} \quad (8.37)$$

In the second line  $\hat{Z}$  is moved to the left within the normal-ordered product, introducing a signature factor  $(-1)^P$ . The factors now appear in normal order, and the  $N$  product can be removed. Furthermore, the product  $\hat{D}\hat{Z}$  is already time ordered by assumption. The fourth line follows from the definition of a contraction, with a factor  $(-1)^Q$  arising from the interchange of  $\hat{D}$  and  $\hat{Z}$ . The last term in the fourth line is in normal order, because  $\hat{U}\hat{V} \cdots \hat{X}\hat{Y}$  are all destruction operators. The sign conventions then allow us to reorder the operators to obtain the final form, which proves the basic lemma (8.33).

The result can be generalized to normal-ordered products already containing contractions of field operators. Multiply both sides of Eq. (8.33) by the contraction of two more operators,  $\hat{R} \cdots \hat{S} \cdots$ , say, and then interchange the operators on *both sides*. Each term has the same overall sign change which cancels identically. Thus we can rewrite the basic lemma (8.33) as

$$\begin{aligned} N(\hat{U}\hat{V} \cdots \hat{X} \cdots \hat{Y})\hat{Z} &= N(\hat{U}\hat{V} \cdots \hat{X} \cdots \hat{Y} \cdots \hat{Y}\hat{Z}) + \cdots \\ &\quad + N(\hat{U} \cdot \hat{V} \cdots \hat{X} \cdots \hat{Y}\hat{Z}) + N(\hat{U}\hat{V} \cdots \hat{X} \cdots \hat{Y}\hat{Z}) \end{aligned} \quad (8.38)$$

*7. Proof of Wick's theorem:* Again the theorem will be proved by induction. It is obviously true for two operators, by the definition of a contraction

$$T(\hat{U}\hat{V}) = N(\hat{U}\hat{V}) + \hat{U} \cdot \hat{V} \quad (8.39)$$

Assume it is true for  $n$  factors, and multiply on the right by an operator  $\hat{\Omega}$  with a time *earlier* than that of any other factor.

$$\begin{aligned} T(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z})\hat{\Omega} &= T(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}\hat{\Omega}) \\ &= N(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z})\hat{\Omega} + N(\hat{U} \cdot \hat{V} \cdot \hat{W} \cdots \hat{X}\hat{Y}\hat{Z})\hat{\Omega} + \cdots \\ &= N(\hat{U}\hat{V}\hat{W} \cdots \hat{X}\hat{Y}\hat{Z}\hat{\Omega}) \\ &\quad + N(\text{sum over all possible pairs of contractions}) \end{aligned} \quad (8.40)$$

The operator  $\hat{\Omega}$  can be included in the  $T$  product because it is at a time earlier than any of those already in the  $T$  product. On the right side, we use our basic lemma (8.33) to introduce the operator  $\hat{\Omega}$  into the normal-ordered products. The restriction on the time of the operator  $\hat{\Omega}$  can now be removed by simultaneously reordering the operators in each term of Eq. (8.40). Again the sign conventions give the same overall sign on both sides of the equation, which therefore remains correct. Wick's theorem has now been proved under the assumption that the operators are either creation or destruction parts of the field. The  $T$  product and the normal-ordered product are both distributive, however, and Wick's theorem thus applies to the fields themselves.

It must be emphasized that Wick's theorem is an *operator identity* that remains true for an arbitrary matrix element. Its real use, however, is for a ground-state average  $\langle \Phi_0 | \dots | \Phi_0 \rangle$ , where all uncontracted normal-ordered products vanish. In particular, the exact Green's function [Eq. (8.9)] consists of all possible fully contracted terms.

## 9□DIAGRAMMATIC ANALYSIS OF PERTURBATION THEORY

Wick's theorem allows us to evaluate the exact Green's function (8.9) as a perturbation expansion involving only wholly contracted field operators in the interaction picture. These contractions are just the free-field Green's functions  $G^0$  [Eq. (8.29)], and  $G$  is thereby expressed in a series containing  $U$  and  $G^0$ . This expansion can be analyzed directly in coordinate space, or (for a uniform system) in momentum space. As noted previously, the zero-temperature theory for condensed bosons requires a special treatment (Chap. 6), and we shall consider only fermions in this section.

### FEYNMAN DIAGRAMS IN COORDINATE SPACE

As an example of the utility of Wick's theorem, we shall calculate the first-order contributions in Eq. (8.11). The expectation value of all the terms containing normal-ordered products of operators vanishes in the noninteracting ground state  $|\Phi_0\rangle$ , leaving only the fully contracted products of field operators. Wick's theorem then requires us to sum over all possible contractions, and Eq. (8.29) shows that the only nonvanishing contraction is between a field  $\psi_\alpha$  and an adjoint field  $\hat{\psi}_\beta^\dagger$ . In this way, the first-order term of Eq. (8.11) becomes

$$\begin{aligned} i\tilde{G}_{\alpha\beta}^{(1)}(x, y) = & \frac{-i}{\hbar} \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^4x_1 d^4x'_1 U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} \\ & \{ iG_{\alpha\beta}^0(x, y) [iG_{\mu'\mu}^0(x'_1, x'_1) iG_{\lambda'\lambda}^0(x_1, x_1) - iG_{\mu'\lambda}^0(x'_1, x_1) iG_{\lambda'\mu}^0(x_1, x'_1)] \\ & \quad {}_{(A)} \quad \quad \quad {}_{(B)} \\ & + iG_{\alpha\lambda}^0(x, x_1) [iG_{\lambda'\mu}^0(x_1, x'_1) iG_{\mu'\beta}^0(x'_1, y) - iG_{\lambda'\beta}^0(x_1, y) iG_{\mu'\mu}^0(x'_1, x'_1)] \\ & \quad {}_{(C)} \quad \quad \quad {}_{(D)} \\ & + iG_{\alpha\mu}^0(x, x'_1) [iG_{\mu'\lambda}^0(x'_1, x_1) iG_{\lambda'\beta}^0(x_1, y) - iG_{\mu'\beta}^0(x'_1, y) iG_{\lambda'\lambda}^0(x_1, x_1)] \} \quad (9.1) \\ & \quad {}_{(E)} \quad \quad \quad {}_{(F)} \end{aligned}$$

The reader is urged to obtain Eq. (9.1) directly from Eq. (8.11) by enumerating all nonvanishing contributions for all possible time orderings. This procedure is very complicated, even in the first order, and Wick's theorem clearly provides a very powerful and simple tool.

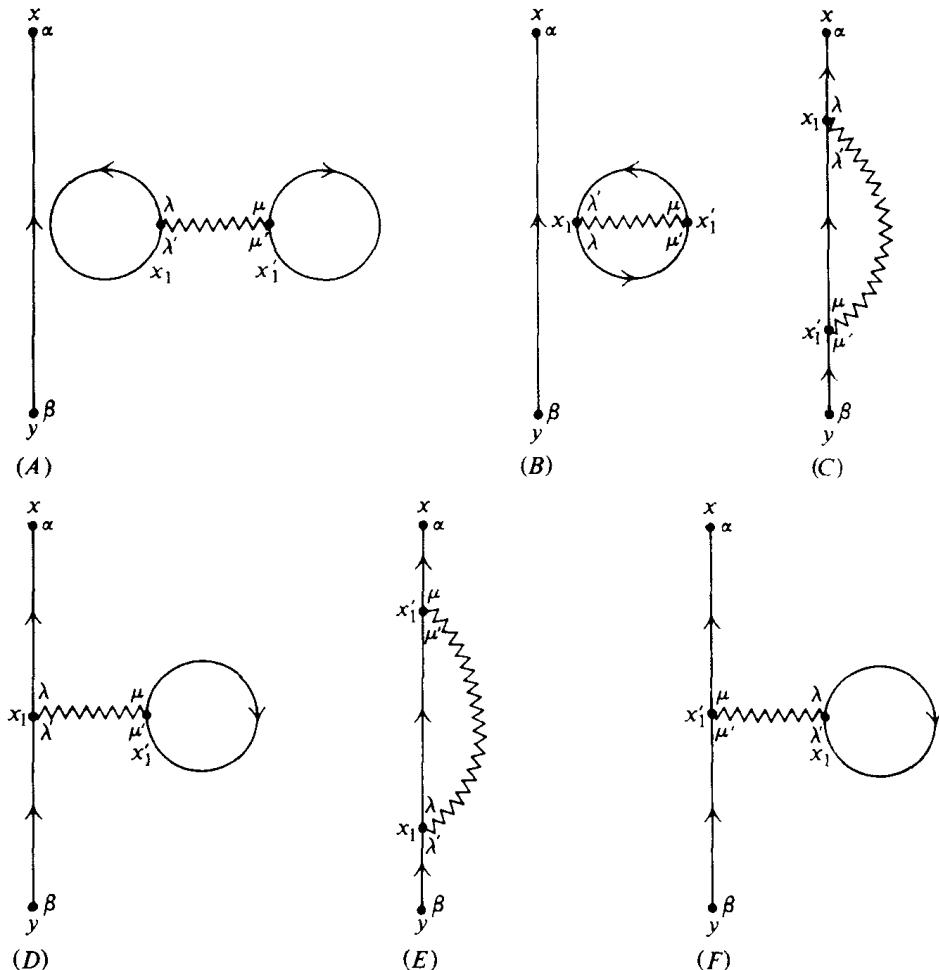


Fig. 9.1 First-order contributions to  $\tilde{G}_{\alpha\beta}(x, y)$ .

We can now associate a picture with each of the terms appearing in expression (9.1), as illustrated in Fig. 9.1. The Green's function  $G^0$  is denoted by a straight line with an arrow running from the second argument to the first, while the interaction potential is denoted by a wavy line. These diagrams appearing in the perturbation analysis of  $G$  form a convenient way of classifying the terms obtained with Wick's theorem. They are known as *Feynman diagrams* because the first diagrammatic expansion of this form was developed by Feynman in his

work on quantum electrodynamics.<sup>1</sup> The precise relation with quantum field theory was first demonstrated by Dyson.<sup>2</sup>

The analytic expression in Eq. (9.1) and the corresponding diagrams (Fig. 9.1) have several interesting features.

1. The terms  $A$ ,  $B$ ,  $D$ , and  $F$  contain a Green's function with both arguments at the same time, which is indicated by a solid line closed on itself. By the definition [Eq. (7.41)], the expression  $iG_{\alpha\beta}^0(x,x)$  is ambiguous, and it is necessary to decide how to interpret it. This quantity represents a contraction of  $\hat{\psi}$  and  $\hat{\psi}^\dagger$ , but the time-ordered product is undefined at equal times. Such a term, however, arises from a contraction of two fields *within* the interaction hamiltonian  $\hat{H}_I$ , where they appear in the form  $\hat{\psi}_\beta^\dagger(x)\hat{\psi}_\alpha(x)$  with the adjoint field always occurring to the *left* of the field. In consequence, the Green's function at equal times must be interpreted as

$$\begin{aligned} iG_{\alpha\beta}^0(x,x) &= \lim_{t' \rightarrow t^+} \langle \Phi_0 | T[\hat{\psi}_\alpha(xt) \hat{\psi}_\beta^\dagger(xt')] | \Phi_0 \rangle \\ &= -\langle \Phi_0 | \hat{\psi}_\beta^\dagger(x) \hat{\psi}_\alpha(x) | \Phi_0 \rangle \\ &= -(2s+1)^{-1} \delta_{\alpha\beta} n^0(\mathbf{x}) \\ &= -\frac{\delta_{\alpha\beta} N}{(2s+1)V} \quad \text{uniform system} \end{aligned} \quad (9.2)$$

for a system of spin- $s$  fermions. Here  $n^0(\mathbf{x})$  is the particle density in the unperturbed ground state [compare Eq. (7.8)] and need not be identical with  $n(\mathbf{x})$  in the interacting system because the interaction may redistribute the particles. For a uniform system, however,  $n^0 = n = N/V$ , because the interaction does not change the *total* number of particles. The terms  $D$  and  $F$  thus represent the lowest-order direct interaction with all the particles that make up the non-interacting ground state (filled Fermi sea), while the terms  $C$  and  $E$  provide the corresponding lowest-order exchange interaction. Here the terms "direct" and "exchange" arise from the original antisymmetrized Slater determinants, as discussed below Eq. (3.37).

2. The terms  $A$  and  $B$  are *disconnected diagrams*, containing subunits that are not connected to the rest of the diagram by any lines. Equation (9.1) shows that such terms typically have Green's functions and interactions whose arguments close on themselves. As a result, the contribution of this subunit can be factored out of the expression for  $\tilde{G}$ . Thus, in the terms  $A$  and  $B$  above,  $iG_{\alpha\beta}^0(x,y)$  represents one factor and the integral represents another factor. To *first order in the interaction*, we assert that Eq. (8.11) can be rewritten as shown in Fig. 9.2. Each diagram in this figure denotes a well-defined integral, given in Eq. (9.1). The validity of Fig. 9.2 is readily verified by expanding the product and retaining only the first-order terms, which are just those in Fig. 9.1. The

<sup>1</sup> R. P. Feynman, *loc. cit.*

<sup>2</sup> F. J. Dyson, *Phys. Rev.*, **75**:486 (1949); **75**:1736 (1949).

$$i\tilde{G}_{\alpha\beta}(x,y) = \left[ \begin{array}{c} \text{Diagram: two vertical lines with arrows, a wavy line between them, and a loop attached to the right line.} \\ + \end{array} + \text{Diagram: two vertical lines with arrows, a wavy line between them, and a loop attached to the left line.} + \dots \right] \times \left[ 1 + \text{Diagram: one loop with a wavy line attached to it.} + \text{Diagram: two loops with a wavy line between them.} + \dots \right]$$

Fig. 9.2 Factorization of first-order contributions to  $\tilde{G}_{\alpha\beta}(x,y)$ .

additional terms of second order in the interaction are here unimportant because the present calculation is consistent only to *first* order in the interaction.

The denominator  $\langle \Phi_0 | \hat{S} | \Phi_0 \rangle = \langle \Phi_0 | \hat{U}(\infty, -\infty) | \Phi_0 \rangle$  in Eq. (8.9) has been ignored to this point, and we shall now evaluate it to first order in the interaction potential. The operator  $\hat{U}(\infty, -\infty)$  is the same as that in the numerator of Eq. (8.9), except that the operators  $\hat{\psi}_\alpha(x)\hat{\psi}_\beta^\dagger(y)$  must be deleted. Thus the denominator can also be evaluated with Wick's theorem, and only the fully contracted terms contribute. The resulting calculation evidently yields the terms shown in Fig. 9.3, where each diagram again stands for a well-defined

Fig. 9.3 Disconnected diagrams in the  $\langle \Phi_0 | \hat{S} | \Phi_0 \rangle = 1 + \text{Diagram} + \text{Diagram} + \dots$  denominator of  $G_{\alpha\beta}(x,y)$ .

integral. These integrals are precisely the same as those appearing in the terms  $A$  and  $B$  of Eq. (9.1). We therefore conclude that *the contribution of the denominator in Eq. (8.9) exactly cancels the contribution of the disconnected diagrams in the numerator*. This important result has so far been verified only to lowest order in the interaction, but we shall now prove it to all orders.<sup>1</sup>

A disconnected diagram closes on itself; consequently, its contribution to  $G_{\alpha\beta}(x,y)$  factors. Thus the  $n$ th-order term of the numerator of Eq. (8.9) can be written as

$$\begin{aligned} iG_{\alpha\beta}^{(n)}(x,y) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^{n+m} \delta_{n,m} \frac{1}{n!m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \\ &\quad \times \langle \Phi_0 | T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_m) \hat{\psi}_\alpha(x) \hat{\psi}_\beta^\dagger(y)] | \Phi_0 \rangle_{\text{connected}} \\ &\quad \times \int_{-\infty}^{\infty} dt_{m+1} \cdots \int_{-\infty}^{\infty} dt_n \langle \Phi_0 | T[\hat{H}_1(t_{m+1}) \cdots \hat{H}_1(t_n)] | \Phi_0 \rangle \end{aligned} \quad (9.3)$$

which can be seen by applying Wick's theorem on both sides of this expression. The second factor, containing  $n$  integrations, in general consists of many disconnected parts. The factor  $n!/n!m!$  represents the number of ways that the  $n$

<sup>1</sup> Here we follow the proof given by A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *op. cit.*, sec. 8.

operators  $\hat{H}_1(t_i)$  can be partitioned into two groups, and, as noted before,  $\hat{H}_1$  can be moved inside the  $T$  product with no additional changes of sign. Equation (9.3) must now be summed over all  $\nu$ , which is trivially performed with the Kronecker delta, and the numerator of Eq. (8.9) becomes

$$\begin{aligned} i\tilde{G}_{\alpha\beta}(x,y) &= \sum_{m=0}^{\infty} \left(\frac{-i}{\hbar}\right)^m \frac{1}{m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \\ &\quad \times \langle \Phi_0 | T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_m)] \hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y) ] | \Phi_0 \rangle_{\text{connected}} \\ &\quad \times \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \\ &\quad \times \langle \Phi_0 | T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] | \Phi_0 \rangle \end{aligned} \quad (9.4)$$

The first factor is the sum of all *connected* diagrams, while the second is identical with the denominator  $\langle \Phi_0 | \hat{S} | \Phi_0 \rangle$ . We therefore obtain the fundamental formula

$$\begin{aligned} iG_{\alpha\beta}(x,y) &= \sum_{m=0}^{\infty} \left(\frac{-i}{\hbar}\right)^m \frac{1}{m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \\ &\quad \times \langle \Phi_0 | T[\hat{H}_1(t_1) \cdots \hat{H}_1(t_m)] \hat{\psi}_{\alpha}(x) \hat{\psi}_{\beta}^{\dagger}(y) ] | \Phi_0 \rangle_{\text{connected}} \end{aligned} \quad (9.5)$$

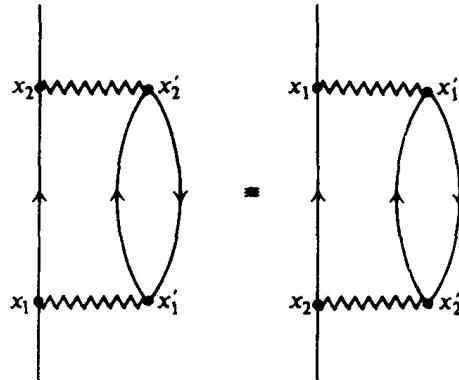
which expresses the factorization of disconnected diagrams. A related “linked-cluster” expansion for the ground-state energy was first conjectured by Brueckner,<sup>1</sup> who verified the expansion to fourth order in the interaction potential; the proof to all orders was then given by Goldstone<sup>2</sup> with the techniques of quantum field theory. Equation (9.5) is important because it allows us to ignore all diagrams that contain parts not connected to the fermion line running from  $y$  to  $x$ .

The expansion of  $G_{\alpha\beta}(x,y)$  into *connected* diagrams is wholly equivalent to the original perturbation series. These are the celebrated Feynman diagrams, and we shall now derive the precise rules that relate the diagrams to the terms of the perturbation series. It must be emphasized, however, that the detailed structure of the Feynman rules depends on the form of the interaction hamiltonian  $\hat{H}_1$ , and the present derivation applies *only* to a system of identical particles interacting through a two-body potential.

3. For any given diagram, there is an identical contribution from all similar diagrams that differ merely in the permutation of the labels  $1 \cdots m$  in the interaction hamiltonian  $\hat{H}_1$ . For example, the two diagrams in Fig. 9.4 have the same numerical value because they differ merely in the labeling of the dummy integration variables. In addition, they have the same sign because

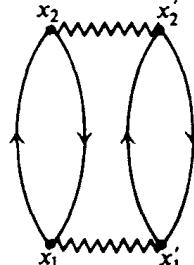
<sup>1</sup> K. A. Brueckner, *Phys. Rev.*, **100**:36 (1955).

<sup>2</sup> J. Goldstone, *Proc. Roy. Soc. (London)*, **A239**:267 (1957).



**Fig. 9.4** Typical permutation of  $\hat{H}_1$  in connected diagrams.

$\hat{H}_1$  contains an even number of fermion fields and may therefore be moved at will within the  $T$  product. In  $m$ th order there are  $m!$  possible interchanges of this type corresponding to the  $m!$  ways of choosing the interaction hamiltonian  $\hat{H}_1$  in applying Wick's theorem. All of these terms make the same contribution to the Green's function, so that we can count each diagram just once and cancel the factor  $(m!)^{-1}$  in Eq. (9.5). Note that this result is true only for the *connected* diagrams, where the external points  $x$  and  $y$  are fixed. In contrast, the *disconnected* diagram shown in Fig. 9.5 represents only a single term. This result is



**Fig. 9.5** Typical disconnected diagram.

easily seen by expanding  $\langle \Phi_0 | \hat{S} | \Phi_0 \rangle$  with Wick's theorem. There is only one way to contract all of the fields, and the diagram obtained by the interchange  $x_1 x'_1 \leftrightarrow x_2 x'_2$  does not correspond to a new and different analytic term. This distinction between connected and disconnected diagrams is one of the basic reasons for studying the Green's function; the fixed external points greatly simplify the counting of diagrams in perturbation theory.

We therefore find the following rule for the  $n$ th-order contribution to the single-particle Green's function  $G_{\alpha\beta}(x, y)$ :

- (a) Draw all *topologically distinct* connected diagrams with  $n$  interaction lines  $U$  and  $2n + 1$  directed Green's functions  $G^0$ .

This procedure can be simplified with the observation that a fermion line either *closes on itself* or *runs continuously* from  $y$  to  $x$ . Each of these diagrams

represents all the  $n!$  different possibilities of choosing among the set of variables  $(x_1 x'_1) \cdot \dots \cdot (x_n x'_n)$ . If there is a question as to the precise meaning of topologically distinct diagrams, Wick's theorem can always be used to verify the enumeration.

4. In our first-order example [Eq. (9.1)] we note that the terms  $C$  and  $E$  are equal, as are the terms  $D$  and  $F$ ; they differ only in that  $x$  and  $x'$  (and the corresponding matrix indices) are interchanged, whereas the potential is symmetric under this substitution [Eq. (7.13)]. It is therefore sufficient to retain just one diagram of each type, simultaneously omitting the factor  $\frac{1}{2}$  in front of Eq. (9.1), which reflects the factor  $\frac{1}{2}$  in the interaction potential [Eq. (2.4)].<sup>1</sup>

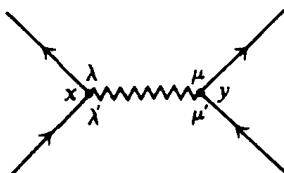


Fig. 9.6 Matrix indices for  $U(x,y)_{\lambda\lambda',\mu\mu'}$ .

We therefore obtain the additional rules:

- (b) Label each vertex with a four-dimensional space-time point  $x_i$ .
- (c) Each solid line represents a Green's function  $G_{\alpha\beta}^0(x,y)$  running from  $y$  to  $x$ .
- (d) Each wavy line represents an interaction

$$U(x,y)_{\lambda\lambda',\mu\mu'} = V(x,y)_{\lambda\lambda',\mu\mu'} \delta(t_x - t_y)$$

where the association of matrix indices is shown in Fig. 9.6.

- (e) Integrate all internal variables over space and time.

5. We note that the summations appearing in the subscript indices on the Green's functions and interaction potentials in Eq. (9.1) are precisely in the form of a matrix product that runs along the fermion line. Thus we state the rule:

- (f) There is a spin matrix product along each continuous fermion line, including the potentials at each vertex.

6. The overall sign of the various contributions appearing in Eq. (9.1) or the diagrams appearing in Fig. 9.1 is determined as follows. Every time a fermion line closes on itself, the term acquires an extra minus sign. This is seen by noting that the fields contracted into a closed loop can be arranged in the order  $[\hat{\psi}^\dagger(1)\hat{\psi}(1)] [\hat{\psi}^\dagger(2)\hat{\psi}(2)] \dots [ \hat{\psi}(N)]$  with no change in sign. An odd number of interchanges of fermion operators is now needed to move the last field operator over to its proper position at the left. Thus we obtain the rule:

- (g) Affix a sign factor  $(-1)^F$  to each term, where  $F$  is the number of closed fermion loops in the diagram.

<sup>1</sup> Note that this result again applies only to connected diagrams, as is evident from Fig. 9.1A and B.

7. The  $n$ th-order term of Eq. (9.5) has an explicit numerical factor  $(-i/\hbar)^n$ , while the  $2n + 1$  contractions of field operators contribute an additional factor  $i^{2n+1}$  [see Eq. (8.29)]. We therefore obtain the rule:

(h) To compute  $G(x, y)$  assign a factor  $(-i)(-i/\hbar)^n(i)^{2n+1} = (i/\hbar)^n$  to each  $n$ th-order term.

Finally, the earlier discussion of Eq. (9.2) yields the rule:

(i) A Green's function with equal time variables must be interpreted as

$$G_{\alpha\beta}^0(xt, x't')$$

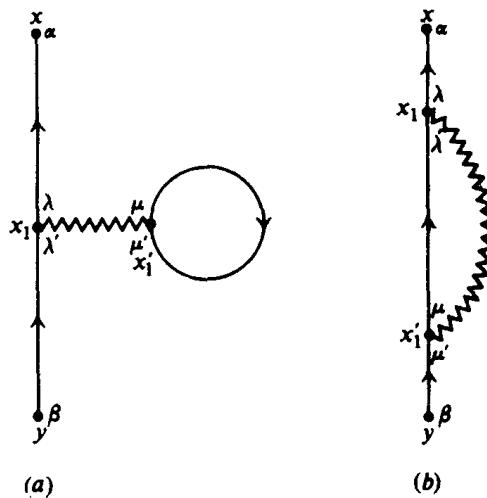


Fig. 9.7 All first-order Feynman diagrams for  $G_{\alpha\beta}(x, y)$ .

(a)

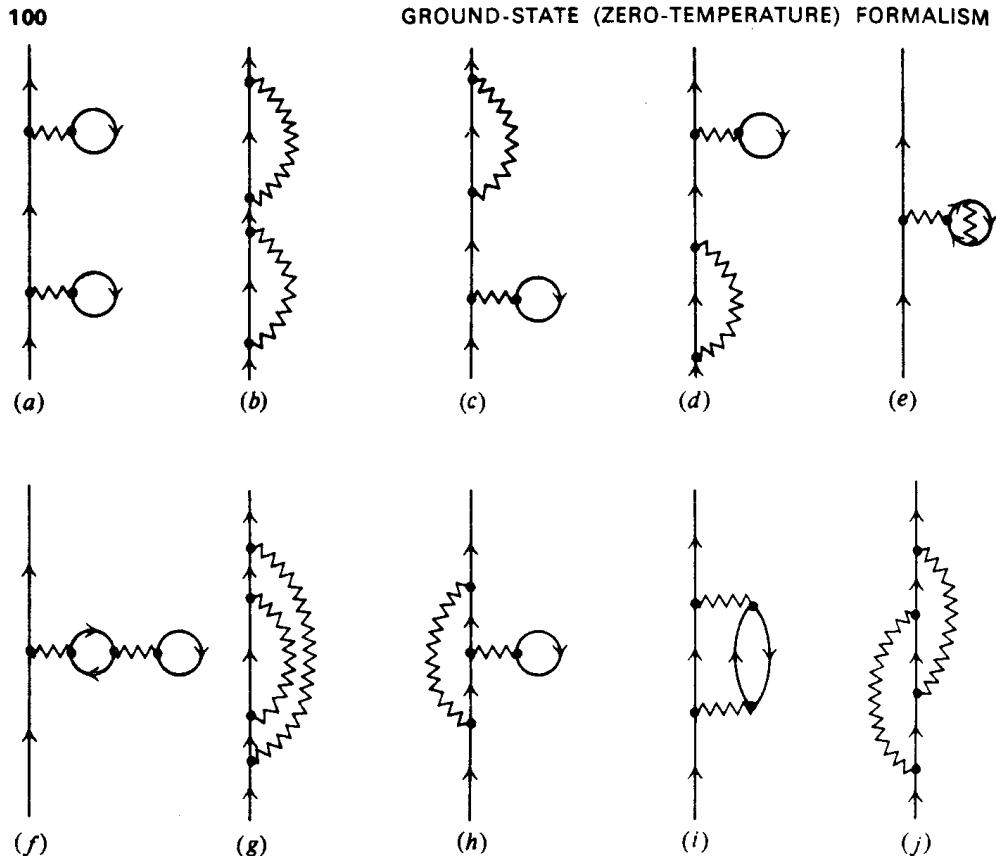
(b)

The foregoing arguments provide a unique prescription for drawing all Feynman diagrams that contribute to  $G(x, y)$  in coordinate space. Each diagram corresponds to an analytic expression that can now be written down explicitly with the Feynman rules. The calculation of  $G$  thus becomes a relatively automatic process.

As an example of the Feynman rules, we shall now write out the complete first-order contribution to  $G_{\alpha\beta}(x, y)$ , shown in Fig. 9.7,

$$\begin{aligned} G_{\alpha\beta}^{(1)}(x, y) &= i\hbar^{-1} \int d^4x_1 \int d^4x'_1 \{ (-1) G_{\alpha\lambda}^0(x, x_1) U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} G_{\lambda'\beta}^0(x_1, y) \\ &\quad \times G_{\mu'\mu}^0(x'_1, x_1) + G_{\alpha\lambda}^0(x, x_1) U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} G_{\lambda'\mu}^0(x_1, x'_1) G_{\mu'\beta}^0(x'_1, y) \} \end{aligned} \quad (9.6)$$

Here and henceforth, an implicit summation is to be carried out over all repeated spin indices. The corresponding second-order contribution  $G^{(2)}(x, y)$  requires more work, and we merely assert that there are 10 second-order Feynman diagrams (Fig. 9.8). The reader is urged to convince himself that these diagrams exhaust the class of second-order topologically distinct connected diagrams, and to write down the analytic expression associated with each term.



**Fig. 9.8** All second-order Feynman diagrams for  $G_{\alpha\beta}(x, y)$ .

#### FEYNMAN DIAGRAMS IN MOMENTUM SPACE

In principle, the Feynman rules enable us to write down the exact Green's function to arbitrary order, but the actual evaluation of the terms can lead to formidable problems because *each* noninteracting Green's function  $G^0(x, y)$  consists of two disjoint pieces. Thus even the first-order contribution [Eq. (9.6)] must be split into many separate pieces according to the relative values of the time variables. In contrast, the Fourier transform  $G^0(x, y, \omega)$  with respect to time has a simple form, and it is convenient to incorporate this into the calculations. Although it is possible to consider a mixed representation  $G_{\alpha\beta}(x, x', \omega)$ , which would apply to spatially inhomogeneous systems with a time-independent hamiltonian, we shall now restrict the discussion to uniform and isotropic systems, where the exact Green's function takes the form  $\delta_{\alpha\beta}G(x - y)$ . The spatial and temporal invariance then allows a full Fourier representation, and we write

$$G_{\alpha\beta}(x, y) = (2\pi)^{-4} \int d^4k e^{ik \cdot (x-y)} G_{\alpha\beta}(k) \quad (9.7a)$$

$$G_{\alpha\beta}^0(x, y) = (2\pi)^{-4} \int d^4k e^{ik \cdot (x-y)} G_{\alpha\beta}^0(k) \quad (9.7b)$$

where the limit  $V \rightarrow \infty$  has already been taken. Here a convenient four-dimensional notation has been introduced

$$d^4k \equiv d^3k d\omega \quad k \cdot x \equiv \mathbf{k} \cdot \mathbf{x} - \omega t \quad (9.8)$$

In addition, we assume that the interaction depends only on the coordinate difference

$$U(x, x') = V(\mathbf{x} - \mathbf{x}') \delta(t - t') \quad (9.9)$$

It may then be written as

$$\begin{aligned} U(x, x')_{\alpha\alpha', \beta\beta'} &= (2\pi)^{-4} \int d^4k e^{ik \cdot (x-x')} U(k)_{\alpha\alpha', \beta\beta'} \\ &= (2\pi)^{-3} \int d^3k e^{ik \cdot (\mathbf{x}-\mathbf{x}')} V(\mathbf{k})_{\alpha\alpha', \beta\beta'} \delta(t - t') \end{aligned} \quad (9.10)$$

where

$$U(k)_{\alpha\alpha', \beta\beta'} = V(\mathbf{k})_{\alpha\alpha', \beta\beta'} \quad (9.11a)$$

$$= \int d^3x e^{-ik \cdot x} V(\mathbf{x})_{\alpha\alpha', \beta\beta'} \quad (9.11b)$$

is the spatial Fourier transform of the interparticle potential.

As an example of the transformation to momentum space, consider the diagram shown in Fig. 9.7b

$$\begin{aligned} G_{\alpha\beta}^{(1b)}(x, y) &= i\hbar^{-1} \int d^4x_1 d^4x'_1 (2\pi)^{-16} \int d^4k d^4p d^4p_1 d^4q \\ &\quad \times G_{\alpha\lambda}^0(k) U(q)_{\lambda\lambda', \mu\mu'} G_{\lambda'\mu}^0(p) G_{\mu'\beta}^0(p_1) \\ &\quad \times e^{ik \cdot (x-x_1)} e^{iq \cdot (x_1-x'_1)} e^{ip \cdot (x_1-x'_1)} e^{ip_1 \cdot (x'_1-y)} \\ &= i\hbar^{-1} (2\pi)^{-8} \int d^4k d^4p d^4p_1 d^4q G_{\alpha\lambda}^0(k) U(q)_{\lambda\lambda', \mu\mu'} \\ &\quad \times G_{\lambda'\mu}^0(p) G_{\mu'\beta}^0(p_1) e^{ik \cdot x} e^{-ip_1 \cdot y} \delta^{(4)}(p+q-k) \delta^{(4)}(p_1-q-p) \\ &= (2\pi)^{-4} \int d^4k e^{ik \cdot (x-y)} [i\hbar^{-1} G_{\alpha\lambda}^0(k) (2\pi)^{-4} \int d^4p \\ &\quad \times U(k-p)_{\lambda\lambda', \mu\mu'} G_{\lambda'\mu}^0(p) G_{\mu'\beta}^0(k)] \end{aligned} \quad (9.12)$$

where the four-dimensional Dirac delta function has the usual integral representation

$$\delta^{(4)}(p) = (2\pi)^{-4} \int d^4x e^{ip \cdot x}$$

Note that Eq. (9.12) indeed has the expected form, and comparison with Eq. (9.7a) identifies the quantity in square brackets as the corresponding contribution to  $G_{\alpha\beta}(k) \equiv G_{\alpha\beta}(\mathbf{k}, \omega)$ .

This approach is readily generalized. Consider the typical internal vertex shown in Fig. 9.9. In accordance with our definitions of Fourier transforms in Eqs. (9.10) and (9.7b), we can also assign a conventional direction  $x' \rightarrow x$  to the interaction  $U(x - x')$ . [This convention cannot alter the problem since the potential is symmetric  $U(x - x')_{\lambda\lambda', \mu\mu'} = U(x' - x)_{\mu\mu', \lambda\lambda'}$ .] The coordinate  $x$

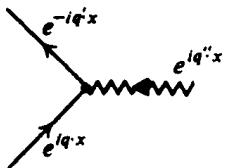
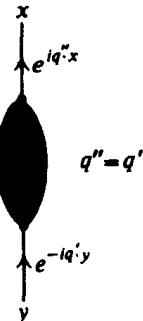


Fig. 9.9 Typical internal vertex in a Feynman diagram.

now appears *only* in the plane-wave exponential, and there is a factor  $e^{+iq \cdot x}$  for each incoming line and  $e^{-iq \cdot x}$  for each outgoing line. The integration over  $x$  therefore yields

$$\int d^4x e^{i(q-q'+q'')\cdot x} = (2\pi)^4 \delta^{(4)}(q - q' + q'') \quad (9.13)$$

which conserves energy and momentum at each internal vertex. The only remaining question is the end points, where the typical structure is shown in Fig. 9.10.

Fig. 9.10 Typical structure of Feynman diagrams for  $G_{\alpha\beta}(x - y)$ .

The translational invariance ensures that  $q' = q''$ , as seen explicitly in Eq. (9.12); the remaining factor  $e^{iq'(x-y)}$  is just that needed in the definition of the Fourier transform of  $G_{\alpha\beta}(q')$ .

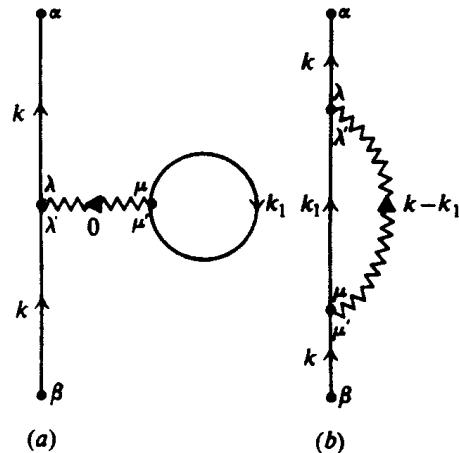
We can now state the Feynman rules for the  $n$ th-order contribution to  $G_{\alpha\beta}(\mathbf{k}, \omega) \equiv G_{\alpha\beta}(k)$ :

1. Draw all topologically distinct connected diagrams with  $n$  interaction lines and  $2n + 1$  directed Green's functions.
2. Assign a direction to each interaction line; associate a directed four-momentum with each line and conserve four-momentum at each vertex.
3. Each Green's function corresponds to a factor

$$G_{\alpha\beta}^0(\mathbf{k}, \omega) = \delta_{\alpha\beta} G^0(\mathbf{k}, \omega) = \delta_{\alpha\beta} \left[ \frac{\theta(|\mathbf{k}| - k_F)}{\omega - \omega_k + i\eta} + \frac{\theta(k_F - |\mathbf{k}|)}{\omega - \omega_k - i\eta} \right] \quad (9.14)$$

4. Each interaction corresponds to a factor  $U(q)_{\lambda\lambda', \mu\mu'} = V(\mathbf{q})_{\lambda\lambda', \mu\mu'}$  where the matrix indices are associated with the fermion lines as in Fig. 9.11.
5. Perform a spin summation along each continuous particle line including the potential at each vertex.
6. Integrate over the  $n$  independent internal four-momenta.

7. Affix a factor  $(i/\hbar)^n (2\pi)^{-4n} (-1)^F$  where  $F$  is the number of closed fermion loops.
8. Any single-particle line that forms a closed loop as in Fig. 9.11a or that is linked by the same interaction line as in Fig. 9.11b is interpreted as  $e^{i\omega\eta} G_{\alpha\beta}(\mathbf{k}, \omega)$ , where  $\eta \rightarrow 0^+$  at the end of the calculation.



**Fig. 9.11** All first-order Feynman diagrams for  $G_{\alpha\beta}(k)$ .

As an example of the Feynman rules in momentum space, we compute the first-order contribution  $G_{\alpha\beta}^{(1)}(\mathbf{k}, \omega)$ , shown in Fig. 9.11. Although the topological structure is identical with the corresponding diagrams in coordinate space (Fig. 9.7), the labeling and interpretation are naturally quite different. In Fig. 9.11a, the four-vector associated with the interaction vanishes because of the conservation requirement at each end. A straightforward identification yields

$$\begin{aligned}
 G_{\alpha\beta}^{(1)}(k) &= i\hbar^{-1}(-1)(2\pi)^{-4} \int d^4 k_1 G_{\alpha\lambda}^0(k) U(0)_{\lambda\lambda', \mu\mu'} G_{\lambda'\beta}^0(k) G_{\mu'\mu}^0(k_1) e^{i\omega_1\eta} \\
 &\quad + i\hbar^{-1}(2\pi)^{-4} \int d^4 k_1 G_{\alpha\lambda}^0(k) U(k - k_1)_{\lambda\lambda', \mu\mu'} \\
 &\quad \times G_{\lambda'\mu}^0(k_1) G_{\mu'\beta}^0(k) e^{i\omega_1\eta} \\
 &= i\hbar^{-1} G^0(k) \{ (2\pi)^{-4} \int d^4 k_1 [-U(0)_{\alpha\beta, \mu\mu'} G^0(k_1) e^{i\omega_1\eta} \\
 &\quad + U(k - k_1)_{\alpha\mu, \mu\beta} G^0(k_1) e^{i\omega_1\eta}] \} G^0(k) \quad (9.15)
 \end{aligned}$$

where the spin summation has been simplified with the Kronecker delta for each factor  $G^0$ . Here, and subsequently, we use the conventions that

$$U(0) \equiv U(k = 0) \quad (9.16a)$$

$$V(0) \equiv V(\mathbf{k} = 0) \quad (9.16b)$$

To make further progress, we shall consider spin- $\frac{1}{2}$  particles with two distinct possibilities for the interaction potential.

1. If the interaction is spin independent, then it has the form  $\mathbf{1}(1)\mathbf{1}(2)$  in spin space, namely, the unit spin matrix with respect to both particles:

$$U(q)_{\alpha\beta,\lambda\mu} = U(q)\delta_{\alpha\beta}\delta_{\lambda\mu} \quad (9.17)$$

The matrix elements then become

$$U_{\alpha\beta,\mu\mu} = 2U\delta_{\alpha\beta} \quad U_{\alpha\mu,\mu\beta} = U\delta_{\alpha\beta} \quad (9.18)$$

2. If the interaction is spin dependent of the form  $\sigma(1)\cdot\sigma(2)$ , then

$$U(q)_{\alpha\beta,\lambda\mu} = U(q)\sigma(1)_{\alpha\beta}\cdot\sigma(2)_{\lambda\mu} \quad (9.19)$$

and the relevant quantities are

$$\sigma_{\alpha\beta}\cdot\sigma_{\mu\mu} = 0 \quad \sigma_{\alpha\mu}\cdot\sigma_{\mu\beta} = [(\sigma)^2]_{\alpha\beta} = 3\delta_{\alpha\beta} \quad (9.20)$$

These results have been obtained with the observations  $\text{tr}\sigma = 0$  and  $\text{tr}\mathbf{1} = 2$ . For interactions of the form

$$V(\mathbf{x}_1 - \mathbf{x}_2) = V_0(|\mathbf{x}_1 - \mathbf{x}_2|)\mathbf{1}(1)\mathbf{1}(2) + V_1(|\mathbf{x}_1 - \mathbf{x}_2|)\sigma(1)\cdot\sigma(2) \quad (9.21)$$

Eqs. (9.15) to (9.20) show that  $G^{(1)}$  is indeed diagonal in the matrix indices:

$$G_{\alpha\beta}^{(1)} = \delta_{\alpha\beta} G^{(1)}.$$

The exact Green's function can always be written in the form

$$G(k) = G^0(k) + G^0(k)\Sigma(k)G^0(k) \quad (9.22)$$

which defines the self-energy  $\Sigma(k)$ . The first term is just the zero-order contribution, and the structure of the second term follows from that of Fig. 9.10. The same structure occurs in Eq. (9.15), which thus identifies the first-order self-energy as

$$\begin{aligned} \hbar\Sigma^{(1)}(k) &= i(2\pi)^{-4} \int d^4k_1 [-2V_0(0) + V_0(\mathbf{k} - \mathbf{k}_1) \\ &\quad + 3V_1(\mathbf{k} - \mathbf{k}_1)] G^0(k_1) e^{i\omega_1\eta} \end{aligned} \quad (9.23)$$

The frequency integral can now be performed explicitly with Eq. (9.14)

$$\int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{i\omega_1\eta} \left[ \frac{\theta(|\mathbf{k}_1| - k_F)}{\omega_1 - \omega_{\mathbf{k}_1} + i\eta} + \frac{\theta(k_F - |\mathbf{k}_1|)}{\omega_1 - \omega_{\mathbf{k}_1} - i\eta} \right] = i\theta(k_F - |\mathbf{k}|)$$

where the convergence factor requires us to close the contour in the upper-half plane. The momentum integral in the first term of Eq. (9.23) then gives the particle density  $n = N/V$  [compare Eq. (3.27)], and we find

$$\begin{aligned} \hbar\Sigma^{(1)}(k) &\equiv \hbar\Sigma^{(1)}(\mathbf{k}) \\ &= nV_0(0) - (2\pi)^{-3} \int d^3k' [V_0(\mathbf{k} - \mathbf{k}') + 3V_1(\mathbf{k} - \mathbf{k}')] \theta(k_F - k') \end{aligned} \quad (9.24)$$

Note that the first-order self-energy is frequency independent. The two terms appearing in Eq. (9.24) have the following physical interpretation. The first term represents the Born approximation for forward scattering from the particles in the medium (Fig. 9.11a), and the second represents the exchange scattering with the particles in the medium, again in Born approximation (Fig. 9.11b).

#### DYSON'S EQUATIONS<sup>1</sup>

We shall now classify the various contributions in an arbitrary Feynman diagram. This procedure yields Dyson's equations, which summarize the Feynman-Dyson perturbation theory in a particularly compact form.

1. *Self-energy insertion*: Our graphical analysis makes clear that the exact Green's function consists of the unperturbed Green's function plus all connected

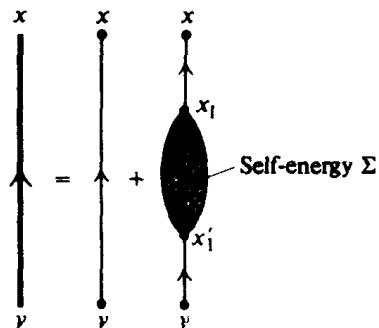


Fig. 9.12 General structure of  $G_{\alpha\beta}(x,y)$ .

terms with a free Green's function at each end. This structure is shown in Fig. 9.12, where the heavy line denotes  $G$  and the light line denotes  $G^0$ . The corresponding analytic expression is given by

$$G_{\alpha\beta}(x,y) = G^0_{\alpha\beta}(x,y) + \int d^4x_1 \int d^4x'_1 G^0_{\alpha\lambda}(x,x_1) \Sigma(x_1,x'_1)_{\lambda\mu} G^0_{\mu\beta}(x'_1,y) \quad (9.25)$$

which defines the *self-energy*  $\Sigma(x_1,x'_1)_{\lambda\mu}$ . A self-energy insertion is defined as any part of a diagram that is connected to the rest of the diagram by two particle lines (one in and one out).

We next introduce the concept of a *proper* self-energy insertion, which is a self-energy insertion that cannot be separated into two pieces by cutting a single particle line. For example, Figs. 9.8a, 9.8b, 9.8c, and 9.8d all contain improper self-energy insertions, while the remaining terms of Fig. 9.8 contain only *proper* self-energy insertions. By definition, the proper self-energy is the sum of all proper self-energy insertions, and will be denoted  $\Sigma^*(x_1,x'_1)_{\alpha\beta}$ . It follows from

<sup>1</sup> F. J. Dyson, *loc. cit.* (A discussion of the vertex part and the complete set of Dyson's equations is presented in Chap. 12 of this book.)

these definitions that the self-energy consists of a sum of all possible repetitions of the proper self-energy.

$$\begin{aligned}\Sigma(x_1, x_1') = \Sigma^*(x_1, x_1') + \int d^4x_2 d^4x'_2 \Sigma^*(x_1, x_2) G^0(x_2, x_2') \Sigma^*(x_2', x_1') \\ + \int d^4x_2 d^4x'_2 \int d^4x_3 d^4x'_3 \Sigma^*(x_1, x_2) G^0(x_2, x_3') \Sigma^*(x_3', x_1') \\ \times \Sigma^*(x_2', x_3) G^0(x_3, x_3') \Sigma^*(x_3', x_1') + \dots\end{aligned}\quad (9.26)$$

Here each quantity denotes a matrix in the spinor indices, and the indices are therefore suppressed. The structure of Eq. (9.26) is shown in Fig. 9.13. Corre-

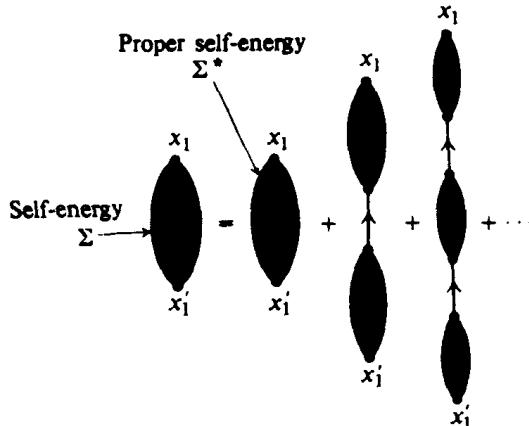


Fig. 9.13 Relation between self-energy  $\Sigma$  and proper self-energy  $\Sigma^*$ .

spondingly, the single-particle Green's function [Eq. (9.25)] becomes (Fig. 9.14)

$$\begin{aligned}G(x, y) = G^0(x, y) + \int d^4x_1 d^4x'_1 G^0(x, x_1) \Sigma^*(x_1, x'_1) G^0(x'_1, y) \\ + \int d^4x_1 d^4x'_1 \int d^4x_2 d^4x'_2 G^0(x, x_1) \Sigma^*(x_1, x'_1) \\ \times G^0(x'_1, x_2) \Sigma^*(x_2, x'_2) G^0(x'_2, y) + \dots\end{aligned}\quad (9.27)$$

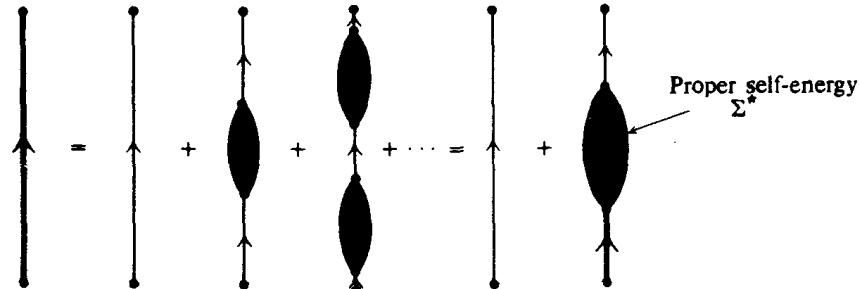


Fig. 9.14 Dyson's equation for  $G_{\alpha\beta}(x, y)$ .

which can be summed formally to yield an *integral equation* (Dyson's equation) for the exact  $G$ .

$$G_{\alpha\beta}(x, y) = G_{\alpha\beta}^0(x, y) + \int d^4x_1 d^4x'_1 G_{\alpha\lambda}^0(x, x_1) \Sigma^*(x_1, x'_1)_{\lambda\mu} G_{\mu\beta}(x'_1, y)\quad (9.28)$$

The validity of Eq. (9.28) can be verified by iterating the right side, which reproduces Eq. (9.27) term by term.

Dyson's equation naturally becomes much simpler if the interaction is invariant under translations and the system is spatially uniform. In this case the quantities appearing in Eq. (9.27) depend only on the coordinate differences, and it is possible to introduce four-dimensional Fourier transforms in these differences. With the definition

$$\Sigma^*(x, y)_{\alpha\beta} = (2\pi)^{-4} \int d^4k e^{i\mathbf{k}\cdot(x-y)} \Sigma^*(k)_{\alpha\beta} \quad (9.29)$$

and Eq. (9.7), the space-time integrations in Eq. (9.28) are readily evaluated, and we find an *algebraic* equation in momentum space [compare Eq. (9.22)]

$$G_{\alpha\beta}(k) = G_{\alpha\beta}^0(k) + G_{\alpha\lambda}^0(k) \Sigma^*(k)_{\lambda\mu} G_{\mu\beta}(k) \quad (9.30)$$

In the usual case,  $G$ ,  $G^0$ , and  $\Sigma^*$  are all diagonal in the matrix indices, and Dyson's equation can then be solved explicitly as

$$G(k) = \frac{1}{[G^0(k)]^{-1} - \Sigma^*(k)} \quad (9.31)$$

The inverse of  $G^0$  is given by

$$[G^0(k)]^{-1} \equiv [G^0(\mathbf{k}, \omega)]^{-1} = \omega - \omega_{\mathbf{k}} \equiv \omega - \hbar^{-1} \epsilon_{\mathbf{k}}^0 \quad (9.32)$$

because the  $\pm i\eta$  in Eq. (9.14) is now irrelevant, and we find

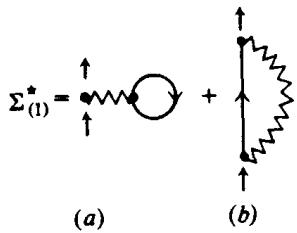
$$G_{\alpha\beta}(k) \equiv G_{\alpha\beta}(\mathbf{k}, \omega) = \frac{1}{\omega - \hbar^{-1} \epsilon_{\mathbf{k}}^0 - \Sigma^*(\mathbf{k}, \omega)} \delta_{\alpha\beta} \quad (9.33)$$

In the general case, this expression must be replaced by an inverse matrix that solves the *matrix* equation (9.30). As shown in Sec. 7, the singularities of the exact Green's function  $G(\mathbf{k}, \omega)$ , considered as a function of  $\omega$ , determine both the excitation energies  $\epsilon_{\mathbf{k}}$  of the system and their damping  $\gamma_{\mathbf{k}}$ . Furthermore, the Lehmann representation ensures that for real  $\omega$

$$\begin{aligned} \text{Im } \Sigma^*(\mathbf{k}, \omega) &\geq 0 & \omega < \mu/\hbar \\ \text{Im } \Sigma^*(\mathbf{k}, \omega) &\leq 0 & \omega > \mu/\hbar \end{aligned} \quad (9.34)$$

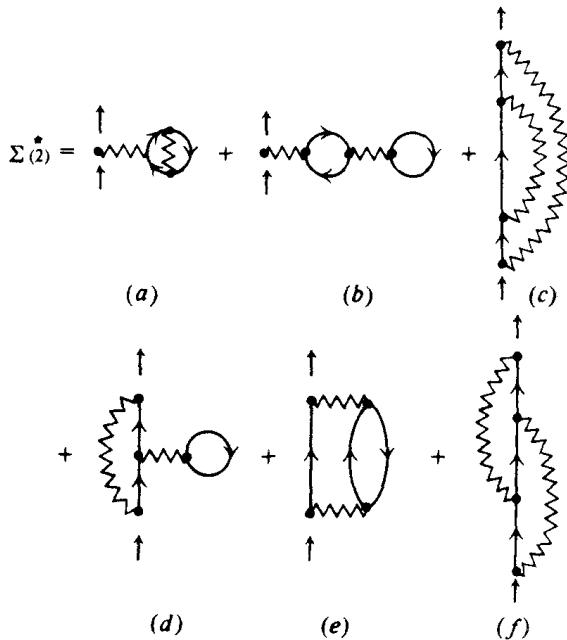
so that the chemical potential can be determined as the point where  $\text{Im } \Sigma^*(\mathbf{k}, \omega)$  changes sign.

As an example of the present analysis, we shall consider all the first- and second-order diagrams, shown in Figs. 9.7 and 9.8. It is evident that both first-order terms represent proper self-energy insertions; as a result the first-order proper self-energy  $\Sigma_{(1)}^*$  is given by the diagrams in Fig. 9.15. Here the small arrows at the ends specify how the Green's functions are to be connected, and the diagrams can be interpreted either in coordinate space or in momentum space. The situation is considerably more complicated in second order. In particular, the diagrams in Fig. 9.8a to d represent all possible second-order iterations of  $\Sigma_{(1)}^*$ , and therefore correspond to improper self-energy insertions.

Fig. 9.15 First-order proper self-energy  $\Sigma_{(1)}^*$ .

On the other hand, the remaining terms (Fig. 9.8e to j) all contain proper self-energy insertions, and we now exhibit *all* contributions to  $\Sigma_{(2)}^*$  in Fig. 9.16.

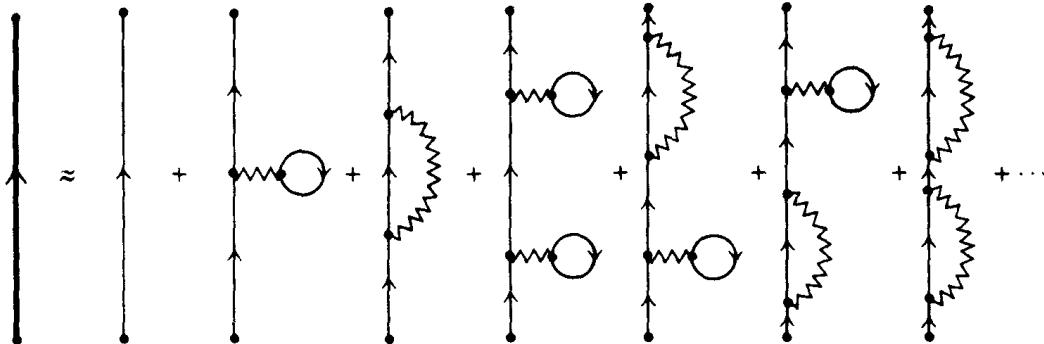
A particularly simple approximation is to write  $\Sigma^*(\mathbf{k}, \omega) \approx \Sigma_{(1)}^*(\mathbf{k}, \omega) \equiv \Sigma_{(1)}^*(\mathbf{k})$  [see Eq. (9.24)] in the solution of Dyson's equation (9.33). This approximation corresponds to summing an *infinite* class of diagrams containing arbitrary

Fig. 9.16 Second-order proper self-energy  $\Sigma_{(2)}^*$ .

iterations of  $\Sigma_{(1)}^*$  (Fig. 9.17). The poles of the approximate Green's function occur at the energy

$$\begin{aligned}\epsilon_{\mathbf{k}}^{(1)} &= \epsilon_{\mathbf{k}}^0 + \hbar \Sigma_{(1)}^*(\mathbf{k}) \\ &= \frac{\hbar^2 k^2}{2m} + nV_0(0) - (2\pi)^{-3} \int d^3 k' [V_0(\mathbf{k} - \mathbf{k}') + 3V_1(\mathbf{k} - \mathbf{k}')] \theta(k_f - k')\end{aligned}\quad (9.35)$$

which determines the energy  $\epsilon_{\mathbf{k}}^{(1)}$  of a state with momentum  $\hbar\mathbf{k}$  containing an additional particle. Here the term  $nV_0(0)$  is a *constant* energy shift; it arises from the "tadpole" diagram Fig. 9.15a and represents the forward scattering

Fig. 9.17 Approximate  $G$  obtained with the substitution  $\Sigma^* \approx \Sigma_{(1)}^*$ .

off all the other particles. The integral term depends on  $\mathbf{k}$  and arises from Fig. 9.15b. In the present (first-order) approximation, the proper self-energy is *real*, and the system propagates forever without damping. This example clearly demonstrates the power of Dyson's equation, because *any* approximation for  $\Sigma^*$  generates an *infinite-order* approximate series for the Green's function. Dyson's equation thus enables us to sum an infinite class of perturbation terms in a compact form.

The explicit solution for  $G$  [Eq. (9.33)] allows us to rewrite the ground-state energy of a uniform system [Eqs. (7.27) and (7.32)] in a particularly simple form. Consider Eq. (7.27) for spin- $s$  fermions with  $\Sigma^*$  and  $G$  diagonal in the matrix indices. A combination with Eq. (9.33) yields

$$\begin{aligned} E &= -iV(2s+1) \int \frac{d^4k}{(2\pi)^4} e^{i\omega\eta} \frac{1}{2\hbar} \left[ \frac{\hbar\omega + \epsilon_k^0}{\hbar\omega - \epsilon_k^0 - \hbar\Sigma^*(\mathbf{k}, \omega)} \right] \\ &= -iV(2s+1)(2\pi)^{-4} \int d^4k e^{i\omega\eta} \{ [\epsilon_k^0 + \frac{1}{2}\hbar\Sigma^*(\mathbf{k}, \omega)] G(\mathbf{k}, \omega) + \frac{1}{2}\hbar \} \\ &= -iV(2s+1)(2\pi)^{-4} \int d^4k e^{i\omega\eta} [\epsilon_k^0 + \frac{1}{2}\hbar\Sigma^*(\mathbf{k}, \omega)] G(\mathbf{k}, \omega) \end{aligned} \quad (9.36)$$

where the last line is obtained with the limiting procedure

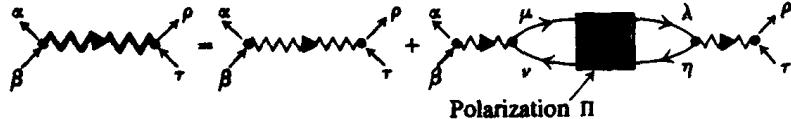
$$\begin{aligned} \lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\eta} &\equiv \lim_{\eta \rightarrow 0^+} \left[ \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} e^{-\epsilon|\omega|} e^{i\omega\eta} \frac{d\omega}{2\pi} \right] \\ &= \lim_{\eta \rightarrow 0^+} \left[ \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \frac{\epsilon}{\eta^2 + \epsilon^2} \right] = 0 \end{aligned} \quad (9.37)$$

It is readily verified that this is the correct limiting process by applying Eq. (9.36) to a noninteracting Fermi system. In the same way, Eq. (7.32) can be rewritten as

$$\begin{aligned} E - E_0 &= -\frac{1}{2}iV(2s+1) \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4k}{(2\pi)^4} e^{i\omega\eta} \hbar \left[ \frac{\hbar\omega - \epsilon_k^0}{\hbar\omega - \epsilon_k^0 - \hbar\Sigma^{*\lambda}(\mathbf{k}, \omega)} \right] \\ &= -\frac{1}{2}iV(2s+1)(2\pi)^{-4} \int_0^1 d\lambda \lambda^{-1} \int d^4k e^{i\omega\eta} \hbar \Sigma^{*\lambda}(\mathbf{k}, \omega) G^\lambda(\mathbf{k}, \omega) \end{aligned} \quad (9.38)$$

where both  $\Sigma^{*\lambda}$  and  $G^\lambda$  must be evaluated for all  $\lambda$  between 0 and 1.

2. *Polarization insertion:* A similar analysis can be carried out for the interaction between two particles, which always consists of the lowest-order interaction plus a series of connected diagrams with lowest-order interactions coming in and out (Fig. 9.18). We can evidently write an integral equation for

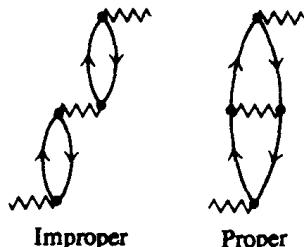


**Fig. 9.18** General structure of the effective interaction  $U_{\alpha\beta,\rho\tau}$ .

the exact interaction; this equation again becomes simpler for a uniform system, where it is possible to work in momentum space. If  $U(q)_{\alpha\beta,\rho\tau}$  and  $U_0(q)_{\alpha\beta,\rho\tau}$  denote the exact and lowest-order interactions, the corresponding equation takes the form

$$U(q)_{\alpha\beta,\rho\tau} = U_0(q)_{\alpha\beta,\rho\tau} + U_0(q)_{\alpha\beta,\mu\nu} \Pi_{\mu\nu,\eta\lambda}(q) U_0(q)_{\eta\lambda,\rho\tau} \quad (9.39)$$

which defines the *polarization insertion*  $\Pi_{\mu\nu,\eta\lambda}(q)$ . It is also convenient to introduce the concept of a proper polarization  $\Pi^*$ , which is a polarization part that cannot be separated into two polarization parts by cutting a single interaction line (Fig. 9.19). Equation (9.39) can then be rewritten as an equation between



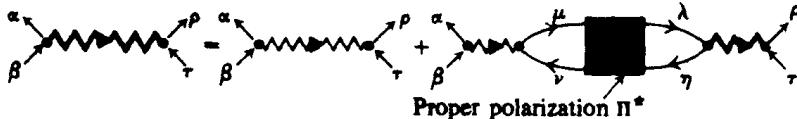
**Fig. 9.19** Typical improper and proper polarization insertions.

the exact interaction and the proper polarization (Fig. 9.20). For a homogeneous system this equation becomes an algebraic equation

$$U(q)_{\alpha\beta,\rho\tau} = U_0(q)_{\alpha\beta,\rho\tau} + U_0(q)_{\alpha\beta,\mu\nu} \Pi_{\mu\nu,\eta\lambda}^*(q) U(q)_{\eta\lambda,\rho\tau} \quad (9.40)$$

In general, Eqs. (9.39) and (9.40) have a complicated matrix structure, and we shall usually consider only spin-independent potentials

$$U_0(q)_{\alpha\beta,\rho\tau} = U_0(q) \delta_{\alpha\beta} \delta_{\rho\tau} \quad (9.41)$$



**Fig. 9.20** Dyson's equation for  $U_{\alpha\beta,\rho\tau}$ .

It then follows immediately that the exact interaction has the same structure

$$U(q)_{\alpha\beta,\rho\tau} = U(q) \delta_{\alpha\beta} \delta_{\rho\tau} \quad (9.42)$$

where the function  $U(q)$  is determined by the simpler equations

$$U(q) = U_0(q) + U_0(q) \Pi(q) U_0(q) \quad (9.43a)$$

$$U(q) = U_0(q) + U_0(q) \Pi^*(q) U(q) \quad (9.43b)$$

Here we have introduced the abbreviations

$$\Pi(q) \equiv \Pi_{\alpha\alpha,\lambda\lambda}(q) \quad (9.44a)$$

$$\Pi^*(q) \equiv \Pi_{\alpha\alpha,\lambda\lambda}^*(q) \quad (9.44b)$$

and a direct solution of Eq. (9.43b) yields

$$U(q) = \frac{U_0(q)}{1 - \Pi^*(q) U_0(q)} \quad (9.45)$$

This result can be used to define a generalized dielectric function  $\kappa(q)$

$$U(q) = \frac{U_0(q)}{\kappa(q)} \quad (9.46)$$

which characterizes the modification of the lowest-order interaction by the polarization of the medium. Comparison of Eqs. (9.45) and (9.46) yields

$$\kappa(q) = 1 - U_0(q) \Pi^*(q) \quad (9.47)$$

#### GOLDSTONE'S THEOREM

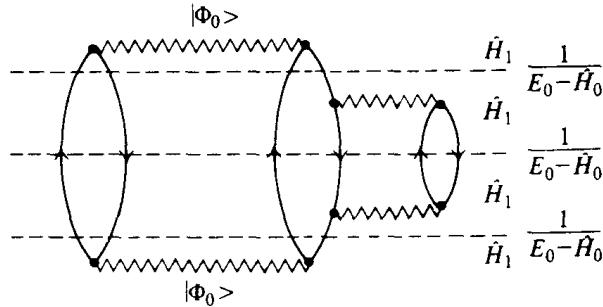
The application of quantum field theory to the many-body problem was initiated by Goldstone in 1957.<sup>1</sup> He proved the cancellation of the disconnected diagrams to all orders, and derived the following expression for the energy shift of the ground state

$$E - E_0 = \langle \Phi_0 | \hat{H}_1 \sum_{n=0}^{\infty} \left( \frac{1}{E_0 - \hat{H}_0} \hat{H}_1 \right)^n | \Phi_0 \rangle_{\text{connected}} \quad (9.48)$$

where  $\hat{H}_0$  and  $\hat{H}_1$  are the time-independent operators in the Schrödinger representation. This result can be interpreted by inserting a complete set of eigenstates of  $\hat{H}_0$  between each interaction  $\hat{H}_1$ . The  $\hat{H}_0$  in the denominator can then be replaced by the corresponding eigenvalue. All matrix elements of the operator in Eq. (9.48) that start from the ground state  $|\Phi_0\rangle$  and end with the ground state  $|\Phi_0\rangle$  are to be included. We can visualize these matrix elements in the following way: the operator  $\hat{H}_1$  acting on the state  $|\Phi_0\rangle$  creates two particles and two holes. This state then propagates with  $(E_0 - \hat{H}_0)^{-1}$ , and the next  $\hat{H}_1$

<sup>1</sup> J. Goldstone, *loc. cit.*

can then create more particles and holes or scatter the existing particles or holes. The resulting intermediate state again propagates with  $(E_0 - \hat{H}_0)^{-1}$ , and so on. The final  $\hat{H}_1$  must then return the system to the ground state  $|\Phi_0\rangle$ . A typical process may be pictured as shown in Fig. 9.21, where an arrow running upward



**Fig. 9.21** Typical Goldstone diagram in the expansion of  $E - E_0$ .

represents the presence of a particle, an arrow running downward represents the presence of a hole, and a horizontal wavy line represents the application of an  $\hat{H}_1$ . Thus the sequence of events starts at the bottom of the diagram and proceeds upward. These diagrams are known as *Goldstone diagrams* and merely keep track of all the matrix elements that contribute in evaluating Eq. (9.48). The subscript “connected” means that only those diagrams that are connected to the final interaction are to be included. In particular, the state  $|\Phi_0\rangle$ , which has no particles or holes present, can never occur as an intermediate state in Eq. (9.48), for the resulting matrix element would necessarily consist of disconnected parts.

Goldstone’s theorem (9.48) is an exact restatement (to all orders) of the familiar time-independent perturbation expression for the ground-state energy. This equivalence is readily verified in the first few terms by inserting a complete set of eigenstates of  $\hat{H}_0$  between each interaction  $\hat{H}_1$ .

$$E - E_0 = \langle \Phi_0 | \hat{H}_1 | \Phi_0 \rangle + \sum_{n \neq 0} \frac{\langle \Phi_0 | \hat{H}_1 | \Phi_n \rangle \langle \Phi_n | \hat{H}_1 | \Phi_0 \rangle}{E_0 - E_n} + \dots \quad (9.49)$$

The corresponding Goldstone diagrams for a homogeneous medium (see Prob. 3.13) are shown in Fig. 9.22. The first two diagrams represent the usual direct and exchange contributions in  $\langle \Phi_0 | \hat{H}_1 | \Phi_0 \rangle$ .

In applying Goldstone’s theorem to a uniform system, we observe that the *momentum* will be conserved at every interaction because the matrix elements in  $\hat{H}_1$  involve an integration over all space. Furthermore, the particles in the intermediate states have *physical* unperturbed energies  $\epsilon_{\mathbf{q}}^0$  related to their momentum  $\mathbf{q}$ , and the virtual nature of the intermediate state is summarized in the energy denominators. In contrast, the Feynman-Dyson perturbation theory for the

Green's function, from which we can also compute  $E - E_0$ , conserves *both energy and momentum* at every vertex, but the intermediate particles can propagate with any frequency  $\omega$ , independent of  $\mathbf{q}$ . For this reason, the Feynman-Dyson approach has the advantage of being manifestly covariant, which is essential in any relativistic theory. Nevertheless, the two approaches merely represent two different ways of grouping and interpreting the terms in the perturbation expansion, and all physical results must be identical.

$$E - E_0 = \text{(Diagram 1)} + \text{(Diagram 2)} + \text{(Diagram 3)} + \text{(Diagram 4)} + \dots$$

**Fig. 9.22** All first- and second-order Goldstone diagrams for  $E - E_0$  in a uniform system.

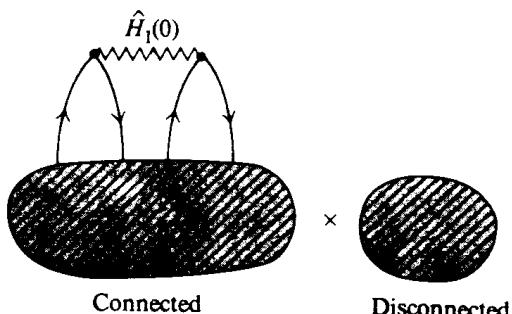
We now prove Goldstone's theorem [Eq. (9.48)]. If the ground state of the interacting system is obtained adiabatically from that of the noninteracting system, the Gell-Mann and Low theorem [Eq. (6.45)] expresses the energy shift of the ground state as

$$E - E_0 = \frac{\langle \Phi_0 | \hat{H}_1 \hat{U}(0, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | \hat{U}(0, -\infty) | \Phi_0 \rangle} \quad (9.50)$$

The numerator can be evaluated by writing

$$\begin{aligned} \langle \Phi_0 | \hat{H}_1 \hat{U}(0, -\infty) | \Phi_0 \rangle &= \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^{\nu} \frac{1}{\nu!} \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_{\nu} \\ &\quad \times \langle \Phi_0 | T[\hat{H}_1 \hat{H}_1(t_1) \cdots \hat{H}_1(t_{\nu})] | \Phi_0 \rangle \end{aligned} \quad (9.51)$$

Here the factor  $\hat{H}_1$  appearing on the left has been incorporated in the  $T$  product since  $\hat{H}_1 \equiv \hat{H}_1(0)$  corresponds to a later time than all the other factors in the integrand. Use Wick's theorem to evaluate all the contractions that contribute to the matrix element in Eq. (9.51). The factor  $\hat{H}_1(0)$  provides a fixed external point that enables us to distinguish between connected and disconnected diagrams; a connected diagram is one that is contracted into  $\hat{H}_1(0)$ . The distinction is illustrated in Fig. 9.23. Suppose that there are  $n$  connected  $\hat{H}_1$ 's



**Fig. 9.23** Typical connected and disconnected Goldstone diagrams.

and  $m$  disconnected  $\hat{H}_1$ 's where  $\nu = n + m$ ; this partition can be performed in  $\nu!/n!m!$  ways. The summation over  $\nu$  in Eq. (9.51) can therefore be rewritten

$$\begin{aligned} & \sum_n \sum_m \left(\frac{-i}{\hbar}\right)^{n+m} \frac{\nu!}{n!m!\nu!} \frac{1}{\nu!} \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n \\ & \times \langle \Phi_0 | T[\hat{H}_1 \hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] | \Phi_0 \rangle_C \int_{-\infty}^0 dt_{n+1} \cdots \int_{-\infty}^0 dt_{n+m} \\ & \times \langle \Phi_0 | T[\hat{H}_1(t_{n+1}) \cdots \hat{H}_1(t_{n+m})] | \Phi_0 \rangle \end{aligned} \quad (9.52)$$

just as in Eq. (9.4). (For simplicity, we now use a subscript  $C$  to indicate connected.) The summation over  $m$  reproduces the denominator of Eq. (9.50), and we thus obtain

$$\begin{aligned} E - E_0 = & \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n \\ & \times \langle \Phi_0 | T[\hat{H}_1 \hat{H}_1(t_1) \cdots \hat{H}_1(t_n)] | \Phi_0 \rangle_C \end{aligned} \quad (9.53)$$

which demonstrates the cancellation of the disconnected diagrams in this expression.

We now proceed to *carry out the time integrations in Eq. (9.53) explicitly*. Consider the  $n$ th-order contribution and insert the relation between  $\hat{H}_1(t)$  and  $\hat{H}_1$  from Eq. (6.5)

$$\begin{aligned} [E - E_0]^{(n)} = & \left(\frac{-i}{\hbar}\right)^n \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n e^{\epsilon(t_1+t_2+\cdots+t_n)} \\ & \times \langle \Phi_0 | \hat{H}_1 e^{iB_0 t_1/\hbar} \hat{H}_1 e^{-iB_0 t_1/\hbar} e^{iB_0 t_2/\hbar} \hat{H}_1 e^{-iB_0 t_2/\hbar} \cdots \\ & \hat{H}_1 e^{-iB_0 t_{n-1}/\hbar} e^{iB_0 t_n/\hbar} \hat{H}_1 e^{-iB_0 t_n/\hbar} | \Phi_0 \rangle_C \end{aligned}$$

Here we have observed that all  $n!$  possible time orderings make identical contributions (see Sec. 6) and therefore work with one definite time ordering of the operators in this matrix element. The adiabatic damping factor has also been explicitly restored. Change variables to *relative times*

$$\begin{aligned} x_1 &= t_1 & t_1 &= x_1 \\ x_2 &= t_2 - t_1 & t_2 &= x_2 + x_1 \\ x_3 &= t_3 - t_2 & \text{or} & t_3 = x_3 + x_2 + x_1 \\ &\vdots & &\vdots \\ &\vdots & &\vdots \\ x_n &= t_n - t_{n-1} & t_n &= x_n + x_{n-1} + \cdots + x_1 \end{aligned}$$

and use

$$\hat{H}_0 |\Phi_0\rangle = E_0 |\Phi_0\rangle$$

This transformation yields

$$\begin{aligned} [E - E_0]^{(n)} &= \left(\frac{-i}{\hbar}\right)^n \langle \Phi_0 | \hat{H}_1 \int_{-\infty}^0 e^{i\epsilon x_1} e^{i(H_0-E_0)x_1/\hbar} dx_1 \hat{H}_1 \\ &\quad \times \int_{-\infty}^0 e^{i(n-1)\epsilon x_2} e^{i(H_0-E_0)x_2/\hbar} dx_2 \hat{H}_1 \cdots \\ &\quad \int_{-\infty}^0 e^{\epsilon x_n} e^{i(H_0-E_0)x_n/\hbar} dx_n \hat{H}_1 | \Phi_0 \rangle_c \end{aligned}$$

The integrations can now all be carried out explicitly, and we find

$$\begin{aligned} [E - E_0]^{(n)} &= \langle \Phi_0 | \hat{H}_1 \frac{1}{E_0 - \hat{H}_0 + i\epsilon n \hbar} \hat{H}_1 \frac{1}{E_0 - \hat{H}_0 + i\epsilon(n-1) \hbar} \hat{H}_1 \cdots \\ &\quad \hat{H}_1 \frac{1}{E_0 - \hat{H}_0 + i\epsilon \hbar} \hat{H}_1 | \Phi_0 \rangle_c \end{aligned}$$

This result immediately yields Goldstone's theorem Eq. (9.48) because the limitation to connected diagrams ensures that  $|\Phi_0\rangle$  cannot appear as an intermediate state, and the state  $|\Phi_0\rangle$  is nondegenerate. It follows that  $E_0 - \hat{H}_0 + i\epsilon \hbar$  can never vanish, so that the convergence factor  $+i\epsilon$  becomes irrelevant, and we can use the propagator  $[E_0 - \hat{H}_0]^{-1}$ , as in Eq. (9.48).

This formal proof can be made more concrete by explicitly considering all  $n$ th-order Feynman diagrams that contribute to Eq. (9.53). Each diagram consists of unperturbed Green's functions  $G^0$ , which evidently contain both particle and hole propagation [compare Eq. (7.41)]. These diagrams can be grouped into sets containing  $n!$  equivalent diagrams that differ only by permuting the time variables. The symmetry of the integrand again allows the replacement

$$\frac{1}{n!} \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n = \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n$$

With the choice of a definite time ordering, each of the  $n!$  diagrams now represents a distinct process. The integral over relative times ( $0 \geq x_i \geq -\infty$ ) then yields  $n!$  distinct Goldstone diagrams corresponding to the  $n!$  possible time orderings of the original Feynman diagram. Thus the set of all possible time-ordered connected Feynman diagrams gives the complete set of connected Goldstone diagrams. The Feynman-Dyson and Goldstone approaches are clearly equivalent to every order in perturbation theory, but the *Feynman-Dyson analysis has the fundamental advantage of combining many terms of time-independent perturbation theory into a single Feynman diagram*. We may note that a similar analysis applies to *any* ground-state expectation value of Heisenberg field operators, for example, the single-particle Green's function  $G(x,y,\omega)$ , which is the Fourier transform of Eq. (9.5). If the integration over all times is carried out explicitly, the resulting perturbation expansion may be classified according to the intermediate states, just as in Fig. 9.21. In this way we can obtain a unique correspondence between a given Feynman diagram and a set of Goldstone diagrams (or diagrams of time-independent perturbation theory).

Goldstone's theorem (9.48) was originally stimulated by Brueckner's theory of strongly interacting Fermi systems. This Brueckner-Goldstone approach has formed the basis for extensive work on the ground-state properties of nuclear matter,<sup>1</sup> He<sup>3</sup>,<sup>2</sup> and atoms.<sup>3</sup>

### PROBLEMS

**3.1.** Show that when  $t < t_0$  the integral equation for  $\hat{U}(t, t_0)$  can be written as

$$\hat{U}(t, t_0) = 1 + \frac{i}{\hbar} \int_t^{t_0} dt' \hat{H}_1(t') \hat{U}(t', t_0)$$

Hence show that

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \left( \frac{i}{\hbar} \right)^n \frac{1}{n!} \int_t^{t_0} dt_1 \cdots \int_t^{t_0} dt_n \tilde{T}[\hat{H}_1(t_1) \cdots \hat{H}_1(t_n)]$$

where  $\tilde{T}$  denotes the anti-time-ordering (latest times to the right). Derive this result from Eqs. (6.16) and (6.23).

**3.2.** One of the most useful relations in quantum field theory is

$$e^{iS} \hat{O} e^{-iS} = \hat{O} + i[\hat{S}, \hat{O}] + \frac{i^2}{2!} [\hat{S}, [\hat{S}, \hat{O}]] + \frac{i^3}{3!} [\hat{S}, [\hat{S}, [\hat{S}, \hat{O}]]] + \cdots$$

Verify this result to the order indicated. Evaluate the commutators explicitly and re-sum the series to derive Eqs. (6.10) from Eqs. (6.7) and (6.9).

**3.3.** Define the two-particle Green's function by

$$\begin{aligned} G_{\alpha\beta;\gamma\delta}(\mathbf{x}_1 t_1, \mathbf{x}_2 t_2; \mathbf{x}'_1 t'_1, \mathbf{x}'_2 t'_2) \\ = (-i)^2 \frac{\langle \Psi_0 | T[\hat{\psi}_{\alpha}(\mathbf{x}_1 t_1) \hat{\psi}_{\beta}(\mathbf{x}_2 t_2) \hat{\psi}_{\delta}^{\dagger}(\mathbf{x}'_2 t'_2) \hat{\psi}_{\gamma}^{\dagger}(\mathbf{x}'_1 t'_1)] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \end{aligned}$$

Prove that the expectation value of the two-body interaction in the exact ground state is given by

$$\langle \hat{V} \rangle = -\frac{1}{2} \int d^3x \int d^3x' V(\mathbf{x}, \mathbf{x}')_{\mu'\lambda', \mu\lambda} G_{\lambda\lambda'; \mu\mu'}(\mathbf{x}' t, \mathbf{x} t'; \mathbf{x}' t^+, \mathbf{x} t^+)$$

<sup>1</sup> K. A. Brueckner, C. A. Levinson, and H. M. Mahmoud, *Phys. Rev.*, **95**:217 (1954); H. A. Bethe, *Phys. Rev.*, **103**:1353 (1956); K. A. Brueckner and J. L. Gammel, *Phys. Rev.*, **109**:1023 (1958); K. A. Brueckner, Theory of Nuclear Structure, in C. DeWitt (ed.), "The Many-Body Problem," p. 47, John Wiley and Sons, Inc., New York, 1959; H. A. Bethe, B. H. Brandow, and A. G. Petschek, *Phys. Rev.*, **129**:225 (1963); see also Chap. 11.

<sup>2</sup> K. A. Brueckner and J. L. Gammel, *Phys. Rev.*, **109**:1040 (1958); T. W. Burkhardt, *Ann. Phys. (N.Y.)*, **47**:516 (1968); E. Østgaard, *Phys. Rev.*, **170**:257 (1968).

<sup>3</sup> See, for example, H. P. Kelly, Correlation Structure in Atoms, in K. A. Brueckner (ed.), "Advances in Theoretical Physics," vol. 2, p. 75, Academic Press Inc., New York, 1968. A review of this topic is also given in Correlation Effects in Atoms and Molecules, R. Lefebvre and C. Moser (eds.), "Advances in Chemical Physics," vol. XIV, Interscience Publishers, New York, 1969.

**3.4.** Consider a many-body system in the presence of an external potential  $U(\mathbf{x})$  with a spin-independent interaction potential  $V(\mathbf{x} - \mathbf{x}')$ . Show that the exact one-particle Green's function obeys the equation of motion

$$\begin{aligned} & \left[ i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2 \nabla_1^2}{2m} - U(\mathbf{x}_1) \right] G_{\alpha\beta}(\mathbf{x}_1 t_1, \mathbf{x}'_1 t'_1) \\ & \mp i \int d^3x_2 V(\mathbf{x}_1 - \mathbf{x}_2) G_{\alpha\gamma; \beta\gamma}(\mathbf{x}_1 t_1, \mathbf{x}_2 t_1; \mathbf{x}'_1 t'_1, \mathbf{x}_2 t'_1) \\ & = \hbar \delta(\mathbf{x}_1 - \mathbf{x}'_1) \delta(t_1 - t'_1) \delta_{\alpha\beta} \end{aligned}$$

where the upper (lower) sign refers to bosons (fermions) and the two-particle Green's function is defined in Prob. 3.3.

**3.5.** Use Eqs. (3.29) and (3.30) to verify Eq. (7.58) for an ideal Fermi gas, and show that  $\mu = \epsilon_F^0$ .

**3.6.** Consider the function

$$F_\alpha(z) = \frac{2}{\pi\alpha} \sum_{n=0}^{\infty} \frac{1}{n^2 + z/\alpha^2}$$

and discuss its analytic structure in the complex  $z$  plane.

(a) Show that the series can be summed to give  $F_\alpha(z) = z^{-\frac{1}{2}} \coth(\pi z^{\frac{1}{2}}/\alpha) + (\alpha/\pi z)$ , which has the *same* analytic structure.

(b) Examine the limit  $\alpha \rightarrow 0$  and compare with the discussion of Eq. (7.67).

**3.7.** (a) If  $\int_{-\infty}^{\infty} dx |\rho(x)| < \infty$ , show that  $f(z) \equiv \int_{-\infty}^{\infty} dx \rho(x)(z-x)^{-1}$  is bounded and analytic for  $\text{Im } z \neq 0$ . Prove that  $f(z)$  is discontinuous across the real axis whenever  $\rho(x) \neq 0$ , and thus  $f(z)$  has a branch cut in this region.

(b) Assume the following simple form  $\rho(x) = \gamma(\gamma^2 + x^2)^{-1}$ . Evaluate  $f(z)$  explicitly for  $\text{Im } z > 0$  and find its analytic continuation to  $\text{Im } z < 0$ .

(c) Repeat part (b) for  $\text{Im } z < 0$ . Compare and discuss.

**3.8.** Derive the Lehmann representation for  $D(\mathbf{k}, \omega)$ , which is the Fourier transform of

$$iD(x, y) \equiv \frac{\langle \Psi_0 | T[\tilde{n}_H(x) \tilde{n}_H(y)] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}$$

with the *density fluctuation operator* defined by

$$\tilde{n}(\mathbf{x}) \equiv \hat{\psi}_\alpha^\dagger(\mathbf{x}) \hat{\psi}_\alpha(\mathbf{x}) - \frac{\langle \Psi_0 | \hat{\psi}_\alpha^\dagger(\mathbf{x}) \hat{\psi}_\alpha(\mathbf{x}) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}$$

Show that  $D(\mathbf{k}, \omega)$  is a meromorphic function with poles in the second and fourth quadrant of the complex  $\omega$  plane. Introduce the corresponding retarded and advanced functions, and construct a Lehmann representation for their Fourier transforms. Discuss the analytic properties and derive the *dispersion relations* analogous to Eq. (7.70).

**3.9.** Make the canonical transformation to particles and holes for fermions  $c_{\mathbf{k}\lambda} = \theta(k - k_F) a_{\mathbf{k}\lambda} + \theta(k_F - k) b_{-\mathbf{k}\lambda}^\dagger$ . By applying Wick's theorem, prove the relation

$$\begin{aligned} c_1^\dagger c_2^\dagger c_4 c_3 &= N(c_1^\dagger c_2^\dagger c_4 c_3) + \theta(k_F - k_2) [\delta_{24} N(c_1^\dagger c_3) - \delta_{23} N(c_1^\dagger c_4)] \\ &\quad + \theta(k_F - k_1) [\delta_{13} N(c_2^\dagger c_4) - \delta_{14} N(c_2^\dagger c_3)] \\ &\quad + \theta(k_F - k_1) \theta(k_F - k_2) [\delta_{13} \delta_{24} - \delta_{14} \delta_{23}] \end{aligned}$$

where the normal-ordered products on the right side now refer to the new particle and hole operators, and the subscripts indicate the quantum numbers  $(\mathbf{k}, \lambda)$ .

**3.10.** Verify the cancellation of disconnected diagrams [Eqs. (8.11), (9.3), and (9.4)] explicitly to second order in the interaction potential.

**3.11.** Consider a system of noninteracting spin- $\frac{1}{2}$  fermions in an external static potential with a hamiltonian  $\hat{H}^{\text{ex}} = \int d^3x \hat{\psi}_\alpha^\dagger(\mathbf{x}) V_{\alpha\beta}(\mathbf{x}) \hat{\psi}_\beta(\mathbf{x})$ .

(a) Use Wick's theorem to find the Feynman rules for the single-particle Green's function in the presence of the external potential.

(b) Show that Dyson's equation becomes

$$G_{\alpha\beta}^{\text{ex}}(x, y) = G_{\alpha\beta}^0(x - y) + \hbar^{-1} \int d^3z G_{\alpha\lambda}^0(x - z) V_{\lambda\lambda'}(\mathbf{z}) G_{\lambda'\beta}^{\text{ex}}(z, y)$$

(c) Express the ground-state energy in a form analogous to Eqs. (7.23) and (7.31). What happens if the particles also interact?

**3.12.** Consider a uniform system of spin- $\frac{1}{2}$  fermions with spin-independent interactions.

(a) Use the Feynman rules in momentum space to write out the second-order contributions to the proper self-energy; evaluate the frequency integrals (some of them will vanish).

(b) Hence show that the second-order contribution to the ground-state energy can be written

$$\begin{aligned} \frac{E^{(2)}}{V} &= 2m\hbar^{-2} \int \dots \int (2\pi)^{-9} d^3k d^3p d^3l d^3n \delta^{(3)}(\mathbf{k} + \mathbf{p} - \mathbf{l} - \mathbf{n}) \\ &\quad \times [2V(\mathbf{l} - \mathbf{k})^2 - V(\mathbf{l} - \mathbf{k}) V(\mathbf{p} - \mathbf{l})] \theta(k_F - p) \theta(k_F - k) \\ &\quad \times \theta(n - k_F) \theta(l - k_F) (p^2 + k^2 - l^2 - n^2 + i\eta)^{-1} \end{aligned}$$

(c) Specialize to an electron gas and rederive the results of Prob. 1.4.

**3.13.** Derive the expression for  $E^{(2)}$  given in Prob. 3.12 from Goldstone's theorem (9.48). From this result, give the rules for evaluating those Goldstone diagrams shown in Fig. 9.22.

**3.14.** Use Eq. (9.33) to show that the energy  $\epsilon_k$  and damping  $|\gamma_k|$  of long-lived single-particle excitations are given by

$$\epsilon_k = \epsilon_k^0 + \text{Re } \hbar \Sigma^\star(\mathbf{k}, \epsilon_k/\hbar)$$

$$\gamma_k = \left[ 1 - \frac{\partial \text{Re } \Sigma^\star(\mathbf{k}, \omega)}{\partial \omega} \Big|_{\epsilon_k/\hbar} \right]^{-1} \text{Im } \Sigma^\star(\mathbf{k}, \epsilon_k/\hbar)$$

**3.15.** Consider a uniform system of spin- $\frac{1}{2}$  fermions with the spin-dependent interaction potential of Eq. (9.21), and assume that  $\Pi_{\mu\nu, \kappa\lambda}^\star(q)$  may be approximated by  $\frac{1}{2}\Pi^0(q)\delta_{\nu\kappa}\delta_{\lambda\mu}$ .

(a) Solve Eq. (9.40) to find

$$U(q)_{\alpha\beta, \rho\tau} = \frac{V_0(q)\delta_{\alpha\beta}\delta_{\rho\tau}}{1 - V_0(q)\Pi^0(q)} + \frac{V_1(q)\sigma_{\alpha\beta}\cdot\sigma_{\rho\tau}}{1 - V_1(q)\Pi^0(q)}$$

(b) Combine Eqs. (9.39) and (9.40) to obtain Dyson's equation for  $\Pi$  in terms of  $\Pi^\star$  and  $U_0$ . Solve this equation with the above approximation for  $\Pi^\star$ , and prove that

$$\Pi_{\mu\nu, \eta\lambda}(q) = \frac{1}{2}\Pi^0(q)\delta_{\nu\eta}\delta_{\lambda\mu} + \frac{1}{2}\Pi^0(q)U(q)_{\nu\mu, \lambda\eta}\frac{1}{2}\Pi^0(q)$$

where  $U(q)$  is taken from (a).