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Distributed Time-Difference-of-Arrival (TDOA)-based Localization of a Moving Target

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***Abstract*— Localization and tracking of a moving target has been established as a key problem in wireless sensor networks, with many algorithms being proposed in this area. In particular, time-difference of arrival (TDOA) localization is considered to be a cost-effective and accurate localization technique. However, traditional TDOA algorithms rely on a central node that produces an estimate of the target's location by gathering measurements from all other nodes in the network. In this work, we solve the problem by distributing the estimation among all agents in the network, avoiding problems posed by the centralized approach, such as single-node failure. Each agent in the network runs its own extended Kalman filter (EKF) in order to estimate the target's position, while a neighbor-based averaging procedure is proposed to facilitate the consensus of agents' estimates. This approach does not require each node to fully observe the process, i.e., some nodes in the network may have an insufficient number of neighbors to accurately estimate the target's position on their own. We show that the estimation error is bounded, with a numerical example illustrating the performance of the proposed algorithm.**

I. INTRODUCTION

Over the past two decades, wireless sensor networks, often enabled by mobile robots, have received increasing attention due to their potential application to a number of diverse areas [1], such as environmental monitoring, space exploration, military applications, target tracking, and health care. For target tracking and localization, estimating the position of a moving target is a key problem. Time-difference-of-arrival (TDOA) localization algorithms are widely used for precise localization of a target, and have been implemented in many fields such as wireless ranging radar systems and cellular positioning systems [2]. This work is motivated by tracking the movement of a fish using acoustic telemetry, where an acoustic tag has been surgically implanted into the fish. The acoustic tag emits a signal periodically, which could be detected by special acoustic receivers deployed either at fixed locations or on mobile robots. If multiple receivers detect the same signal, it is possible to infer the fish's location using the detection times at these receivers. In TDOA–based localization, a reference node is chosen and the time of arrival (TOA) of the emitted signal for all other nodes in the network are subtracted from the reference node's TOA, generating TDOA measurements at the reference node. Assuming that the propagation speed of the signal is known, the TDOA measurements can be converted to range-difference measurements, which are then used to estimate the location of the target [3], [4].

Several potential issues arise when implementing such a centralized approach, including single-node failure, i.e., failure of the reference node would obstruct the entire network from achieving its goal, along with high traffic on the reference node's communication channel. Therefore, it is desirable to employ a distributed scheme for target localization, where each node in the network independently estimates the target's position without the need for a centralized node. Some of the previous work done in distributed TDOA localization includes the work presented in [5], where the authors propose dividing the network into clusters, with each cluster head taking on the role of a reference node for that cluster. The problems posed by such approach are similar to that of the centralized scheme, where the failure of the cluster head prevents the entire cluster from achieving its goal. In [6], the authors combine Taylor-series estimation algorithm with a gossip algorithm for distributed localization.

In our proposed scheme, each node assumes itself as the reference node, and generates TDOA measurements from its neighbors. However, this technique would require each node to have a sufficient number of neighbors (2 or 3 neighbors for a target moving in a 2D- or 3D-space, respectively) in order to estimate the target's position accurately. To overcome this challenge, we propose the use of weighted averaging between the nodes of the network, allowing nodes with insufficient neighbors to estimate the target's position accurately. We show that the estimation error is bounded, and illustrate the algorithm's performance through simulation. Our proposed algorithm consists of two steps: in the first step, each agent generates TDOA measurements from its neighbors and runs an extended Kalman filter (EKF) for estimating the target's position. In the second step, each neighbor computes a weighted average of the estimates and error covariance matrices between itself and its neighbors.

The remainder of this paper is organized as follows. In Section II, we formally describe the problem of distributed localization of a moving target using TDOA measurements and introduce our proposed approach. In Section III, we show that the maximum estimation error of all agents is bounded in mean square by invoking the stochastic stability lemma. Section IV consists of a simulation example. Finally, some concluding remarks are given in Section V.

Throughout this paper, $\|\cdot\|$ denotes the Euclidian norm of real vectors or the spectral norm of real matrices, $E\{x\}$ is the expectation value of x, and $E\{x|y\}$, the expectation value of x conditioned on y. Moreover, $\mathbb{R}^q$ denotes the real

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q–dimensional vector space, I_n denotes the $n \times n$ identity matrix, and $A \otimes B$ denotes the Kronecker product of matrices A and B.

II. Problem Formulation

We consider a target moving in a d-dimensional space according to the following model

$$x_{k+1} = Ax_k + Bw_k \tag{1}$$

Here, $x \in \mathbb{R}^{2d}$ is the state vector of the moving target containing the position and the velocity of the target, i.e., $x_k = \begin{bmatrix} s_k & \dot{s}_k \end{bmatrix}^T$, where $s_k = \begin{bmatrix} {}^x p_k & {}^y p_k \end{bmatrix}^T$ (or $s_k = \begin{bmatrix} {}^x p_k & {}^y p_k & {}^z p_k \end{bmatrix}^T$) is the 2D (respectively, 3D) position of the target at time k. The variable $w_k \in \mathbb{R}^{2d}$ denotes the process noise, which is assumed to be an uncorrelated, zero-mean, white Gaussian noise with

$$E\{w_k w_l^T\} = Q\delta_{kl} \tag{2}$$

where δ_{kl} is the Kronecker delta.

We also consider a network of N identical agents, represented by an connected, undirected graph, where the position of the i-th agent at time k is given by $\mu_k^i = \begin{bmatrix} {}^x p_k^i & {}^y p_k^i \end{bmatrix}^T$ (or $\mu_k^i = \begin{bmatrix} {}^x p_k^i & {}^y p_k^i & {}^z p_k^i \end{bmatrix}^T$). Any two agents in the network are considered *neighbors* if they can exchange information. Each agent measures the TDOA of a signal between itself and its neighbors. When multiplied by the propagation speed of the signal, the measurement model for agent i can be expressed as

$$z_k^i = h^i(x_k) + v_k^i \tag{3}$$

where,

$$h^i(x_k) = \begin{bmatrix} h_1^i(x_k) \\ h_2^i(x_k) \\ \vdots \\ h_{|\mathcal{N}_i|}^i(x_k) \end{bmatrix} = \begin{bmatrix} \|s_k - \mu_k^i\| - \|s_k - \mu_k^{i,1}\| \\ \|s_k - \mu_k^i\| - \|s_k - \mu_k^{i,2}\| \\ \vdots \\ \|s_k - \mu_k^i\| - \|s_k - \mu_k^{i,|\mathcal{N}_i|}\| \end{bmatrix} \tag{4}$$

Here we use $\mathcal{N}_i$ to donate the set of all neighbors of agent i, and $\mu_k^{i,j}$ to denote the position of the j-th neighbor of agent i at time k. Additionally, $v_k \in \mathbb{R}^{|\mathcal{N}_i|}$ is the measurement noise, which is assumed to be uncorrelated, zero-mean, white Gaussian noise with

$$E\{v_k^i {v_l^j}^T\} = R^i \delta_{kl}\delta_{ij} \tag{5}$$

The measurement function in (4) can be linearized, yielding (for convenience, assuming 2D space):

$$H_k^i = \begin{bmatrix} \frac{\partial h_1^i}{\partial\, {}^x p_k} & \frac{\partial h_1^i}{\partial\, {}^y p_k} & 0 & 0 \\ \frac{\partial h_2^i}{\partial\, {}^x p_k} & \frac{\partial h_2^i}{\partial\, {}^y p_k} & 0 & 0 \\ \vdots & & & \\ \frac{\partial h_{|\mathcal{N}_i|}^i}{\partial\, {}^x p_k} & \frac{\partial h_{|\mathcal{N}_i|}^i}{\partial\, {}^y p_k} & 0 & 0 \end{bmatrix} \tag{6}$$

where

$$\frac{\partial h_j^i}{\partial\, {}^x p_k} = \frac{{}^x p_k - {}^x p_k^i}{\|s_k - \mu_k^i\|} - \frac{{}^x p_k - {}^x p_k^{i,j}}{\|s_k - \mu_k^{i,j}\|} \tag{7}$$

$$\frac{\partial h_j^i}{\partial\, {}^y p_k} = \frac{{}^y p_k - {}^y p_k^i}{\|s_k - \mu_k^i\|} - \frac{{}^y p_k - {}^y p_k^{i,j}}{\|s_k - \mu_k^{i,j}\|} \tag{8}$$

Here, ${}^x p_k^{i,j}$ and ${}^y p_k^{i,j}$ are the x- and y-coordinates of the j-th neighbor of agent i in the 2D plane at time index k.

Each agent in the group implements an averaged extended Kalman filter (EKF) to estimate the target's state. Namely, each agent utilizes a two-step recursion consisting of time-update and measurement-update with linearization between these two steps, followed by weighted averaging between each agent and its neighbors. These steps are summarized below.

1) Time-update:

$$\hat{x}_{k|k-1}^i = A\bar{x}_{k-1}^i \tag{9}$$

$$P_{k|k-1}^i = A\bar{P}_{k-1}^i A^T + BQB^T \tag{10}$$

2) Measurement-update:

$$K_k^i = P_{k|k-1}^i {H_k^i}^T \left(H_k^i P_{k|k-1}^i {H_k^i}^T + R^i \right)^{-1} \tag{11}$$

$$\hat{x}_{k|k}^i = \hat{x}_{k|k-1}^i + K_k^i \left(z_k^i - h^i(\hat{x}_{k|k-1}^i) \right) \tag{12}$$

$$P_{k|k}^i = (I - K_k^i H_k^i) P_{k|k-1}^i (I - K_k^i H_k^i)^T + K_k^i R^i {K_k^i}^T \tag{13}$$

3) Weighted Averaging:

$$\bar{x}_k^i = \sum_{j=1}^N w_{ij} \hat{x}_{k|k}^j \tag{14}$$

$$\bar{P}_k^i = \sum_{j=1}^N w_{ij} P_{k|k}^j \tag{15}$$

Here $\hat{x}_{k|l}^i$, $P_{k|l}^i$, with $l = k$ or $k-1$ representing the i-th agent's estimate of the target's state and its covariance, respectively, at time k after the l-th measurement was taken. Additionally, the weights w_{ij} in (14)–(15) assign larger weights for agents with a bigger number of neighbors,

$$w_{ij} = \begin{cases} 0 & , j \quad \notin \mathcal{N}_i \cup \{i\} \\ \frac{|\mathcal{N}_j|}{\sum_{l \in \mathcal{N}_i \cup \{i\}} |\mathcal{N}_l|} & , j \quad \in \mathcal{N}_i \cup \{i\} \end{cases} \tag{16}$$

Other approaches can be found in [7] and [8] in the context of consensus filters, where the author reduced the Kalman filter problem into two separate dynamical consensus problems. The problem we investigate in this paper is estimating the the target's state by all agents in the network, including agents that do not have sufficient neighbors to observe the process on their own. This mission is different from sensor network consensus, where the main mission of the network is to reach an agreement over the parameters being monitored.

III. Stochastic Stability of Averaged EKF

Analyzing the stability of the extended Kalman filter has been investigated in [9] and [10]. In this section we utilize a similar framework to the one presented in [9] for analyzing the stochastic stability of the EKF under the weighted averaging scheme in (9)–(15) when implemented by each agent in the network. For examination of the stability properties of the averaged EKF, a one-step formulation in terms of the *a priori* variables of (9)–(15) is used. In particular, denoting $\hat{x}^i_{k|k-1}$ by $\hat{x}^i_k$, and substituting (14) and (12) into (9), we obtain a recursive form of the i–th agent estimate:

$$\hat{x}^i_{k+1} = \sum_{j=1}^{N} w_{ij}\left[A\hat{x}^j_k + AK^j_k\left(z^j_k - h^j(\hat{x}^j_k)\right)\right] \tag{17}$$

Similarly, denoting $P^i_{k|k-1}$ by P^i_k, and substituting (15) and (13) into (10), we obtain a recursive form of the i–th agent's error covariance matrix:

$$\begin{aligned} P^i_{k+1} &= \sum_{j=1}^{N} w_{ij}\left[A(I_n - K^j_k H^j_k)P^j_k(I_n - K^j_k H^j_k)^T A^T\right] \\ &\quad + \sum_{j=1}^{N} w_{ij}\left[AK^j_k R^j {K^j_k}^T A^T\right] + BQB^T \end{aligned} \tag{18}$$

For notation consistency, we redefine K^i_k as

$$K^i_k = P^i_k {H^i_k}^T\left(H^i_k P^i_k {H^i_k}^T + R^i\right)^{-1} \tag{19}$$

In the averaged EKF formulation, we note the use of the linearized measurement function, H^i_k. It is possible to expand the measurement function h^i_k via

$$h^i(x_k) - h^i(\hat{x}^i_k) = H^i_k(x_k - \hat{x}^i_k) + \chi^i(x_k, \hat{x}^i_k) \tag{20}$$

where

$$\chi^i(x_k, \hat{x}^i_k) = \begin{bmatrix} \chi^i_1(x_k, \hat{x}^i_k) \\ \chi^i_2(x_k, \hat{x}^i_k) \\ \vdots \\ \chi^i_{|\mathcal{N}_i|}(x_k, \hat{x}^i_k) \end{bmatrix} \tag{21}$$

and

$$\begin{aligned} \chi^i_j(x_k, \hat{x}_k) =& \|s_k - \mu^i_k\| - \frac{\partial h^i_j}{\partial\, {}^y p_k}\left({}^y p_k - {}^y\hat{p}_k\right) \\ & - \|s_k - \mu^{i,j}_k\| - \frac{\partial h^i_j}{\partial\, {}^x p_k}\left({}^x p_k - {}^x\hat{p}_k\right) \end{aligned} \tag{22}$$

The i-th agent's estimation error is defined as

$$\tilde{x}^i_k = x_k - \hat{x}^i_k \tag{23}$$

which can be obtained by subtracting (17) from (1), yielding

$$\tilde{x}^i_{k+1} = \sum_{j=1}^{N} w_{ij}\left[A(I - K^j_k H^j_k)\tilde{x}^j_k + \rho^j_k + \psi^j_k\right] \tag{24}$$

where

$$\begin{aligned} \rho^i_k &= -AK^i_k\chi^i(x_k, \hat{x}^i_k) \\ \psi^i_k &= Bw_k - AK^i_k v^i_k \end{aligned} \tag{25, 26}$$

A. Error Bounds for the Averaged EKF

The basis of the convergence analysis is the stochastic stability lemma [11], [12], which is given as follows:

Lemma 3.1: If there exist real numbers $\bar{v}, \underline{v}, \mu > 0$ and $0 < \alpha \leq 1$, and there is a stochastic process $V_k(\zeta_k)$ with the following properties:

$$\underline{v}\|\zeta_k\|^2 \leq V_k(\zeta_k) \leq \bar{v}\|\zeta_k\|^2 \tag{27}$$

$$E\{V_{k+1}(\zeta_{k+1})|\zeta_k\} - V_k(\zeta_k) \leq \mu - \alpha V_k(\zeta_k) \tag{28}$$

then the random process ζ_k is exponentially bounded in mean square with probability one, as in

$$E\{\|\zeta_k\|^2\} \leq \frac{\bar{v}}{\underline{v}}E\{\|\zeta_0\|^2\}(1-\alpha)^k + \frac{\mu}{\underline{v}}\sum_{l=1}^{k-1}(1-\alpha)^l \tag{29}$$

for every $k \geq 0$. Moreover, the stochastic process is bounded with probability one.

Proof: The proof of this lemma is provided in [12] and [9]. ■

Let $\tilde{\boldsymbol{x}}_k = \begin{bmatrix}\tilde{x}^{1^T}_k & \ldots & \tilde{x}^{N^T}_k\end{bmatrix}^T$ denote the network-wide estimation error of all agents in the network. Using (24), we can obtain the dynamics for the network-wide estimation error

$$\tilde{\boldsymbol{x}}_k = (W\otimes A)(I_{2dN} - K_k H_k)\tilde{\boldsymbol{x}}_k + (W\otimes I_{2d})\boldsymbol{\rho}_k + (W\otimes I_{2d})\boldsymbol{\psi}_k \tag{30}$$

Here, $K_k = \text{blkdiag}\begin{bmatrix}K^1_k & \ldots & K^N_k\end{bmatrix}$ and $H_k = \text{blkdiag}\begin{bmatrix}H^1_k & \ldots & H^N_k\end{bmatrix}$ are block diagonal matrices for the filtering gain and linearized measurement matrices, respectively, while W is a stochastic matrix representing the averaging weights, $[W]_{ij} = w_{ij}$. The terms $\boldsymbol{\rho}_k$ and $\boldsymbol{\psi}_k$ are the concatenations of the terms ρ^i_k and ψ^i_k, respectively. From (25) and (26), we can write

$$\begin{aligned} \boldsymbol{\rho_k} &= -(I_N \otimes A)K_k\boldsymbol{\chi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_k) \\ \boldsymbol{\psi}_k &= (I_N \otimes B)\boldsymbol{w}_k - (I_N \otimes A)K_k\boldsymbol{v}_k \end{aligned} \tag{31, 32}$$

where $\boldsymbol{w}_k = 1_N \otimes w_k$, $\boldsymbol{x}_k = 1_N \otimes x_k$, $\boldsymbol{v}_k = \begin{bmatrix}v^{1^T}_k & \ldots & v^{N^T}_k\end{bmatrix}^T$, and $\boldsymbol{\chi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_k) = \begin{bmatrix}\chi^1(x_k, \hat{x}^1_k)^T & \ldots & \chi^N(x_k, \hat{x}^N_k)^T\end{bmatrix}^T$.

Similarly, we define the network-wide error covariance, $P_k = \text{blkdiag}\begin{bmatrix}P^1_k & \ldots & P^N_k\end{bmatrix}$. Noting that (18) is equivalent to

$$P^i_{k+1} = \sum_{j=1}^{N} w_{ij}\left[AP^j_k A^T - AP^j_k {H^j_k}^T {K^j_k}^T A^T\right] + BQB \tag{33}$$

we get

$$\begin{aligned} P_{k+1} &= (\sqrt{W}\otimes A)P_k(\sqrt{W}\otimes A)^T \\ &\quad - (\sqrt{W}\otimes A)P_k H^T_k K^T_k(\sqrt{W}\otimes A)^T \\ &\quad + (\sqrt{W}\sqrt{W}^T \otimes BQB) \end{aligned} \tag{34}$$

Furthermore, we restrict P_k to have a block diagonal structure in order to capture the distributed nature of this problem. The network-wide gain matrix K_k can then be expressed as

$$K_k = P_k H^T_k(H_k P_k H^T_k + R)^{-1} \tag{35}$$

where the matrix $R = \text{blkdiag}\begin{bmatrix} R^1 & \ldots & R^N \end{bmatrix}$ is the network-wide measurement noise covariance matrix.

Before stating the main result of this section, we present the following lemmas that will aid in analyzing the averaged EKF.

Lemma 3.2: Consider the system given by (1), (3) and the averaged EKF in (17)–(19). Let $\mathcal{N} \triangleq \sum_{i=1}^{N} \mathcal{N}_i$, and assume that there exist positive real numbers $\bar{a}, \bar{h}, \bar{p}, \underline{p}, \delta, \sigma_w, \sigma_v > 0$ such that the following inequalities hold for all $k \geq 0$:

$$\|A\| \leq \quad \bar{a} \tag{36}$$

$$BB^T \leq \quad \delta I \tag{37}$$

$$\|H_k\| \leq \quad \bar{h} \tag{38}$$

$$\underline{p} I \leq P_k \leq \quad \bar{p} I \tag{39}$$

$$Q \geq \quad \sigma_w^2 I_{2d} \tag{40}$$

$$R \geq \quad \sigma_v^2 I_{|\mathcal{N}|} \tag{41}$$

then there exists a real number $0 < \alpha < 1$ such that the inequality

$$\begin{aligned}&(I_{2dN} - K_k H_k)^T (W \otimes A)^T P_{k+1}^{-1} \\ &\times (W \otimes A)(I_{2dN} - K_k H_k) \leq (1-\alpha) P_k^{-1}\end{aligned} \tag{42}$$

with K_k given by (35), holds for $k \geq 0$.

Proof: Rearranging the terms in (34)

$$\begin{aligned}P_{k+1} =& (\sqrt{W} \otimes A)\left[(I_{2dN} - K_k H_k) P_k (I_{2dN} - K_k H_k)^T\right. \\ &+ K_k H_k P_k (I_{2dN} - K_k H_k)^T \\ &\left.+ (I_N \otimes BQB^T)\right](\sqrt{W} \otimes A)^T\end{aligned} \tag{43}$$

Now, we consider the term $K_k H_k P_k (I_{2dN} - K_k H_k)^T$. With (35), it can be verified that

$$\begin{aligned}&P_k (I_{2dN} - K_k H_k)^T \\ &\quad = P_k - P_k H_k^T (H_k P_k H_k^T + R)^{-1} H_k P_k\end{aligned} \tag{44}$$

which is symmetric, and applying the matrix inversion lemma, we get

$$P_k (I_{2dN} - K_k H_k)^T = (P_k^{-1} H_k^T R^{-1} H_k)^{-1} > 0 \tag{45}$$

because $P_k^{-1} > 0$. Additionally, from (35), and since $P_k > 0$ and $R > 0$,

$$K_k H_k = P_k H_k^T (H_k P_k H_k^T + R)^{-1} H_k \geq 0 \tag{46}$$

From (45) and (46), it can be seen that

$$K_k H_k P_k (I_{2dN} - K_k H_k)^T \geq 0 \tag{47}$$

Therefore, from (43) and (47), the inequality

$$\begin{aligned}P_{k+1} \geq& (\sqrt{W} \otimes A)\left[(I_{2dN} - K_k H_k) P_k (I_{2dN} - K_k H_k)^T\right. \\ &\left.+ (I_N \otimes BQB^T)\right](\sqrt{W} \otimes A)^T\end{aligned} \tag{48}$$

holds, and can be rewritten as

$$\begin{aligned}&P_{k+1} \geq (\sqrt{W} \otimes A)(I_{2dN} - K_k H_k) \times \\ &\left[P_k + (I_{2dN} - K_k H_k)^{-1}(I_N \otimes BQB^T)(I_{2dN} - K_k H_k)^{-T}\right] \\ &\times (I_{2dN} - K_k H_k)^T (\sqrt{W} \otimes A)^T\end{aligned} \tag{49}$$

From (35), and the assumed bounds (36)–(41), it is easy to see that

$$\|K_k\| \leq \frac{\bar{p}\bar{h}}{\sigma_v^2} \tag{50}$$

which leads to

$$\begin{aligned}&P_{k+1} \geq (\sqrt{W} \otimes A)(I_{2dN} - K_k H_k) \times \\ &\left[P_k + \frac{\delta \sigma_w^2}{\left(1 + \frac{\bar{p}\bar{h}}{\sigma_v^2}\right)^2} I_{2dN}\right](I_{2dN} - K_k H_k)^T (\sqrt{W} \otimes A)^T\end{aligned}$$

Taking the inverse of both sides, and multiplying from the left by $(I_{2dN} - K_k H_k)^T (\sqrt{W} \otimes A)^T$ and from the right by $(\sqrt{W} \otimes A)(I_{2dN} - K_k H_k)$ we get

$$\begin{aligned}&(I_{2dN} - K_k H_k)^T (\sqrt{W} \otimes A)^T P_{k+1}^{-1} (\sqrt{W} \otimes A) \\ &\times (I_{2dN} - K_k H_k) \leq \left[1 + \frac{\delta \sigma_w^2}{\bar{p}\left(1 + \frac{\bar{p}\bar{h}}{\sigma_v^2}\right)^2}\right]^{-1} P_k^{-1}\end{aligned} \tag{51}$$

Since W is stochastic,

$$\begin{aligned}&(I_{2dN} - K_k H_k)^T (W \otimes A)^T P_{k+1}^{-1} \\ &\times (W \otimes A)(I_{2dN} - K_k H_k) \leq (I_{2dN} - K_k H_k)^T \\ &\times (\sqrt{W} \otimes A)^T P_{k+1}^{-1} (\sqrt{W} \otimes A)(I_{2dN} - K_k H_k) \\ &\leq (1-\alpha) P_k^{-1}\end{aligned} \tag{52}$$

i.e., inequality (42) with

$$1 - \alpha = \left(1 + \frac{\delta \sigma_w^2}{\bar{p}\left(1 + \frac{\bar{p}\bar{h}}{\sigma_v^2}\right)^2}\right)^{-1} \tag{53}$$

■

It is worth noting that conditions (36)–(38) and (40)–(41) are easy to satisfy, since we assume knowledge of the system's matrices. Condition (39) is specifically addressed in Section III-B.

Lemma 3.3: Let the assumptions in Lemma 3.2 hold, and assume that there exist positive real numbers $\kappa_\chi, \epsilon_\chi > 0$, such that the nonlinear function $\boldsymbol{\chi}(\xi_1, \xi_2)$ in (31) is bounded by

$$\|\boldsymbol{\chi}(\xi_1, \xi_2)\| \leq \kappa_\chi \|\xi_1 - \xi_2\|^2 \tag{54}$$

for $\xi_1, \xi_2 \in \mathbb{R}^{2dN}$ with $\|\xi_1 - \xi_2\| < \epsilon_\chi$, then there is a positive real number $\kappa_\rho > 0$ such that

$$\begin{aligned}&\boldsymbol{\rho}_k^T (W \otimes I_n)^T P_{k+1}^{-1} \\ &\times [2(W \otimes A)(I_{2dN} - K_k H_k)\tilde{\boldsymbol{x}}_k + (W \otimes I_{2d})\boldsymbol{\rho}_k] \leq \kappa_\rho \|\tilde{\boldsymbol{x}}_k\|^3\end{aligned} \tag{55}$$

holds for $\|\tilde{\boldsymbol{x}}_k\| \leq \epsilon_\chi$.

Proof: From (31) and (50), a bound can be established on $\|\boldsymbol{\rho}_k\|1$

$$\|\boldsymbol{\rho}_k\| \leq \frac{\bar{a}\bar{p}\bar{h}}{\sigma_v^2} \|\boldsymbol{\chi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_k)\| \tag{56}$$

For $\|\tilde{\boldsymbol{x}}_k\| = \|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k\| \le \epsilon_\chi$, using (54), one gets

$$\|\boldsymbol{\rho}_k\| \le \frac{\bar{a}\bar{p}\bar{h}}{\sigma_v^2}\kappa_\chi \|\tilde{\boldsymbol{x}}_k\|^2 \triangleq \kappa' \|\tilde{\boldsymbol{x}}_k\|^2 \le \kappa' \epsilon_\chi \|\tilde{x}_k^i\| \tag{57}$$

which in turn leads to

$$\begin{aligned}&\boldsymbol{\rho}_k^T (W \otimes I_n)^T P_{k+1}^{-1} \\ &\times [2(W \otimes A)(I_{2dN} - K_k H_k)\tilde{\boldsymbol{x}}_k + (W \otimes I_{2d})\boldsymbol{\rho}_k] \\ &\le \kappa' \frac{1}{\underline{p}} \left[2\bar{a}\left(1 + \frac{\bar{p}\bar{h}^2}{\sigma_v^2}\right) + \kappa' \epsilon_\chi\right] \|\tilde{\boldsymbol{x}}_k\|^3\end{aligned} \tag{58}$$

i.e., inequality (55) with

$$\kappa_\rho = \kappa' \frac{1}{\underline{p}} \left[2\bar{a}\left(1 + \frac{\bar{p}\bar{h}^2}{\sigma_v^2}\right) + \kappa' \epsilon_\chi\right] \tag{59}$$

■

Lemma 3.4: Let the assumptions in Lemma 3.2 hold, then there is a positive real number $\kappa_\psi > 0$ such that

$$E\{\boldsymbol{\psi}_k{}^T P_{k+1}^{-1} \boldsymbol{\psi}_k\} \le \kappa_\psi \tag{60}$$

for all $k \ge 0$.

Proof: Using the definition of $\boldsymbol{\psi}_k$ given by (32), and since $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are uncorrelated, we can expand the expectation expression in (60) via

$$\begin{aligned}E\{\boldsymbol{\psi}_k{}^T P_{k+1}^{-1} \boldsymbol{\psi}_k\} &= E\{\boldsymbol{w}_k^T (I_N \otimes B)^T P_{k+1}^{-1} (I_N \otimes B)\boldsymbol{w}_k\} \\ &+ E\{\boldsymbol{v}_k^T K_k^T (I_N \otimes A)^T P_{k+1}^{-1} (I_N \otimes A) K_k \boldsymbol{v}_k\}\end{aligned} \tag{61}$$

From the bound in (36)–(41) and (50), equation (61) can be converted to an inequality via

$$\begin{aligned}E\{\boldsymbol{\psi}_k{}^T P_{k+1}^{-1} \boldsymbol{\psi}_k\} \le \frac{1}{\underline{p}} E\{\boldsymbol{w}_k^T (I_N \otimes B)^T (I_N \otimes B)\boldsymbol{w}_k\} \\ + \frac{1}{\underline{p}}\left(\frac{\bar{a}\bar{p}\bar{h}}{\sigma_v^2}\right)^2 E\{\boldsymbol{v}_k{}^T \boldsymbol{v}_k\}\end{aligned} \tag{62}$$

Because both sides of (62) are scalars, we can take the trace of the right-hand side without changing its value

$$\begin{aligned}E\{\boldsymbol{\psi}_k{}^T P_{k+1}^{-1} \boldsymbol{\psi}_k\} &\le \frac{1}{\underline{p}} \mathrm{tr}\left((I_N \otimes B) E\{\boldsymbol{w}_k \boldsymbol{w}_k^T\} (I_N \otimes B)^T\right) \\ &+ \frac{1}{\underline{p}}\left(\frac{\bar{a}\bar{p}\bar{h}}{\sigma_v^2}\right)^2 \mathrm{tr}\left(E\{\boldsymbol{v}_k \boldsymbol{v}_k{}^T\}\right) \\ &= \frac{2dN\delta\sigma_w^2}{\underline{p}} + \frac{\mathcal{N}\bar{a}^2\bar{p}^2\bar{h}^2}{\underline{p}\sigma_v^2}\end{aligned} \tag{63}$$

Defining

$$\kappa_\psi \triangleq \frac{2dN\delta\sigma_w^2}{\underline{p}} + \frac{\mathcal{N}\bar{a}^2\bar{p}^2\bar{h}^2}{\underline{p}\sigma_v^2} \tag{64}$$

leads to $E\{\boldsymbol{\psi}_k{}^T P_{k+1}^{-1} \boldsymbol{\psi}_k\} \le \kappa_\psi$. ■

With the above results, we can now present the following theorem regarding the stochastic stability of the averaged EKF.

Theorem 3.5: Consider the movement model for the target given by (1) and the nonlinear measurement model for each agent given by (3), along with the averaged EKF in (9)–(15), and let the assumptions of Lemmas 3.2 and 3.3 hold. then the network-wide estimation error $\tilde{\boldsymbol{x}}_k$ given by (30) is exponentially bounded in mean square with probability one.

Proof: We choose

$$V_k(\tilde{\boldsymbol{x}}_k) = \tilde{\boldsymbol{x}}_k P_k^{-1} \tilde{\boldsymbol{x}}_k \tag{65}$$

From (39), we have

$$\frac{1}{\bar{p}} \le V_k(\tilde{\boldsymbol{x}}_k) \le \frac{1}{\underline{p}} \tag{66}$$

which satisfies the first condition of Lemma 3.1 with $\underline{v} = \frac{1}{\bar{p}}$ and $\bar{v} = \frac{1}{\underline{p}}$. What remains is to establish an upper bound on $E\{V_{k+1}(\tilde{\boldsymbol{x}}_{k+1})|\tilde{\boldsymbol{x}}_k\}$. Inserting (30) into (65) yields

$$\begin{aligned}V_{k+1}(\tilde{\boldsymbol{x}}_{k+1}) = [&(W \otimes A)(I_{2dN} - K_k H_k)\tilde{\boldsymbol{x}}_k + \\ &(W \otimes I_{2d})\boldsymbol{\rho}_k + (W \otimes I_{2d})\boldsymbol{\psi}_k]^T P_k^{-1} \\ &\times [(W \otimes A)(I_{2dN} - K_k H_k)\tilde{\boldsymbol{x}}_k \\ &+ (W \otimes I_{2d})\boldsymbol{\rho}_k + (W \otimes I_{2d})\boldsymbol{\psi}_k]\end{aligned} \tag{67}$$

Taking the conditional expectation $E\{V_{k+1}(\tilde{\boldsymbol{x}}_{k+1})|\tilde{\boldsymbol{x}}_k\}$ and applying Lemmas 3.2, 3.3, and 3.4 for $\|\tilde{\boldsymbol{x}}_k\| \le \epsilon_\chi$ we obtain

$$\begin{aligned}E\{V_{k+1}(\tilde{\boldsymbol{x}}_{k+1})|\tilde{\boldsymbol{x}}_k\} &\le (1-\alpha)V_k(\tilde{\boldsymbol{x}}_k) \\ &+ \kappa_\rho \epsilon_\chi \|\tilde{\boldsymbol{x}}_k\| + \kappa_\psi\end{aligned} \tag{68}$$

Defining $\epsilon = \min\left(\epsilon_\chi, \frac{\alpha}{2\bar{p}\kappa_\rho}\right)$, then for $\|\tilde{\boldsymbol{x}}_{\boldsymbol{k}}\| \le \epsilon$

$$\kappa_\rho \|\tilde{\boldsymbol{x}}_{\boldsymbol{k}}\| \|\tilde{\boldsymbol{x}}_{\boldsymbol{k}}\|^2 \le \frac{\alpha}{2\bar{p}} \|\tilde{\boldsymbol{x}}_{\boldsymbol{k}}\|^2 \le \frac{\alpha}{2} V_k(\tilde{\boldsymbol{x}}_k) \tag{69}$$

Applying inequality (69) to (68) yields

$$E\{V_{k+1}(\tilde{\boldsymbol{x}}_{k+1}) - V_k(\tilde{\boldsymbol{x}}_k) \le -\frac{\alpha}{2} V_k(\tilde{\boldsymbol{x}}_k) + \kappa_\psi \tag{70}$$

which satisfies the second condition of Lemma 3.1, ensuring that the network-wide estimation error $\tilde{\boldsymbol{x}}_k$ remains bounded in mean square. ■

B. Boundedness of Averaged Covariance Matrix

The boundedness of the covariance matrix in inequality (39) under the proposed method must be addressed. To that end, we utilize an operator originally presented in [13].

Lemma 3.6: Let $\boldsymbol{P}$ denote the set of matrices $\{P^1, \ldots, P^N\}$ and $\boldsymbol{M}$ denote the set of matrices $\{M^1, \ldots, M^N\}$. Define the following operator

$$f_{n+1}^i(\boldsymbol{M}, \boldsymbol{P}) = \sum_{j=1}^N w_{ij} M^j f_n^j(\boldsymbol{M}, \boldsymbol{P}) M^{j^T} \tag{71}$$

with

$$f_1^i(\boldsymbol{M}, \boldsymbol{P}) = \sum_{j=1}^N w_{ij} M^j P^j M^{j^T} \tag{72}$$

If there exist positive-definite matrices $\boldsymbol{X} = \{X^1, \ldots, X^N\}$ such that $X^i > f_1^i(\boldsymbol{M}, \boldsymbol{X})$, then the following statements hold for all $i = 1, \ldots, N$:

1) For all bounded positive semi-definite matrices $P^i \ge 0$, $\lim_{n\to\infty} f_n^i(\boldsymbol{M}, \boldsymbol{P}) = 0$.

2) For all bounded positive semi-definite matrices $P_0^i \geq 0$, and $U^i \geq 0$, the sequences $P_{k+1}^i = f_1^i(\boldsymbol{M}, \boldsymbol{P}_k) + U^i$ are bounded.

Proof:

1) Since $P^i \geq 0$ and $X^i > 0$, then there exists a constant $\alpha > 0$ such that $P^i \leq \alpha X^i, \forall i$. Then it is clear that $f_n^i(\boldsymbol{M}, \boldsymbol{P}) \leq \alpha f_n^i(\boldsymbol{M}, \boldsymbol{X})$. Additionally, since $f_1^i(\boldsymbol{M}, \boldsymbol{X}) < X^i$, then there exists a constant $r \in (0,1)$ such that $f_1^i(\boldsymbol{M}, \boldsymbol{X}) \leq rX^i$. Then, $f_2^i(\boldsymbol{M}, \boldsymbol{X}) \leq r f_1^i(\boldsymbol{M}, \boldsymbol{X}) \leq r^2 X^i$, and sequentially, $f_n^i(\boldsymbol{M}, \boldsymbol{X}) \leq r^n X^i$. Therefore, $f_n^i(\boldsymbol{M}, \boldsymbol{P}) \leq \alpha r^n X^i$ and $\lim_{n\to\infty} f_n^i(\boldsymbol{M}, \boldsymbol{P}) = 0$.

2) From the definition of P_{k+1}^i, we have

$$
\begin{aligned}
P_2^i =& \sum_{j=1}^{N} w_{ij} M^j P_1^j {M^j}^T + U^i \\
=& \sum_{j=1}^{N} w_{ij} M^j f_1^j(\boldsymbol{M}, \boldsymbol{P}_0) {M^j}^T \\
&+ \sum_{j=1}^{N} w_{ij} M^j U^j {M^j}^T + U^i \\
=& f_2^i(\boldsymbol{M}, \boldsymbol{P}_0) + f_1^i(\boldsymbol{M}, \boldsymbol{U}) + U^i
\end{aligned}
$$

and by induction, it can be shown that

$$
P_{k+1}^i = f_{k+1}^i(\boldsymbol{M}, \boldsymbol{P_0}) + \sum_{l=1}^{k} f_l^i(\boldsymbol{M}, \boldsymbol{U}) + U^i
$$

From the proof of Part 1, there exist constants $\alpha, \mu > 0$ and $0 < r < 1$ such that $f_k^i(\boldsymbol{M}, \boldsymbol{P}_0) \leq \alpha r^k X^i$ and $U^i \leq \mu X^i$, which is used to show that

$$
P_k^i \leq \left(\alpha + \frac{\mu}{1-r}\right) X^i
$$

■

Corollary 3.7: For given matrices $A, B, Q, \boldsymbol{R} = \{R^1, \ldots, R^N\}$, and time varying matrices $\boldsymbol{H}_k = \{H_k^1, \ldots, H_k^N\}$, define the operator

$$
\begin{aligned}
g_k^i(\boldsymbol{P}) =& \sum_{j=1}^{N} w_{ij} \left[A(I - K_k^j H_k^j) P^j (I - K_k^j H_k^j)^T A^T\right] \\
&+ \sum_{j=1}^{N} w_{ij} \left[A K_k^j R^j {K_k^j}^T A^T\right] + BQB^T
\end{aligned}
\tag{73}
$$

where

$$
K_k^i = P^i {H_k^i}^T \left(H_k^i P^i {H_k^i}^T + R^i\right)^{-1}
$$

If there exist positive-definite matrices $\boldsymbol{X} = \{X^1, \ldots, X^N\}$ such that $X^i > f_1^i(A(I - \boldsymbol{K}_k \boldsymbol{H}_k), \boldsymbol{X})$ for all $k \geq 0$, where $f_1^i(\cdot, \cdot)$ is defined in Lemma 3.6, then for all positive semi-definite matrices $P_0^i \geq 0$, the sequences $P_{k+1}^i = g_k^i(\boldsymbol{P}_k)$ are bounded.

Proof: This corollary can be proven by directly applying the results in Lemma 3.6, defining $M^i = A(I - K_k^i H_k^i)$ and $U^i = \sum_{j=1}^{N} w_{ij} \left[A K_k^j R^j {K_k^j}^T A^T\right] + BQB^T$. ■

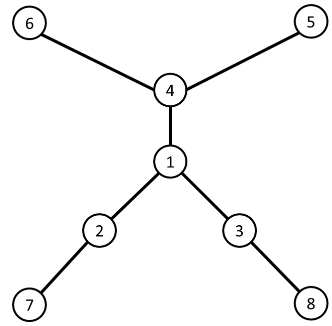

Fig. 1. Network topology showing communication links between 8 agents.

Examining (18), it can be seen that the error covariance matrix under the averaged scheme takes the form $P_{k+1}^i = g_k^i(\boldsymbol{P}_k)$. Therefore, by Corollary 3.7, P_k^i remains bounded under the proposed scheme.

IV. Simulation Results

In this section, we simulate the proposed algorithm on a network of 8 agents estimating the position of a moving target in a 3D environment. The communication links between the agents are dictated by the network topology represented in Fig. 1. Spatially, the agents are located at $(0,0,20)$, $(-20,-20,-20)$, $(20,-20,-20)$, $(0,20,-20)$, $(30,30,0)$, $(-30,30,0)$, $(-30,-30,0)$, and $(30,-30,0)$, respectively. The initial position of the moving target is set to $(20,20,20)$.

The A and B matrices of the target's movement model in (1) were obtained from discretizing the following continuous-time model

$$
\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} w
$$

with a step size of 0.1 seconds. The process noise w and the measurement noise v^i are white Gaussian noise with variance $\sigma_w = 0.35$, and $\sigma_v = 3$, respectively.

It is clear from the network topology that only agents 1 and 4 have a sufficient number of neighbors to accurately estimate the target's position independently. On the other hand, all other agents cannot fully observe the process, and would not be able to estimate the target's position without the aid of the network.

All agents perform the proposed estimation scheme initialized with $\hat{x}_0^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, and $P_0^i = I_6$. The simulation environment is shown in Fig. 2, where the circle represents the location of the target. The squares in Fig. 2 represent the position of each agent, and their estimates of the target's location are shown with their corresponding lines. The straight lines between the agents represent the communication links of the network. Fig. 3 depicts the norm of the covariance matrix for each of the agents, and it can be easily seen that the covariance matrix remains bounded under the proposed method. Finally, Fig 4 shows the norm of

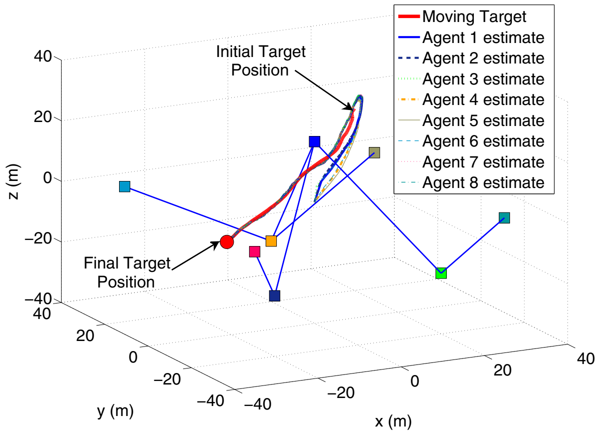

Fig. 2. Simulation of proposed method with a network of 8 agents.

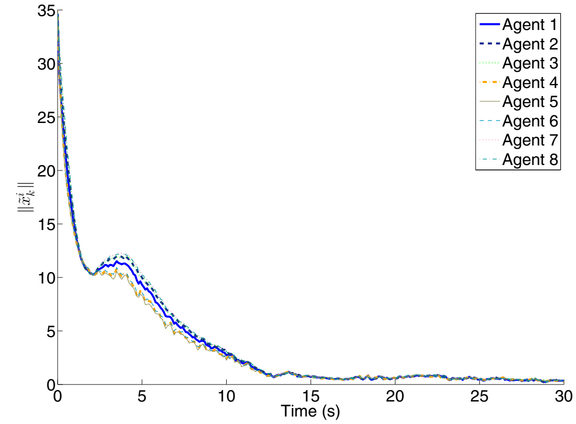

Fig. 4. Norm of estimation error for each agent in the network.

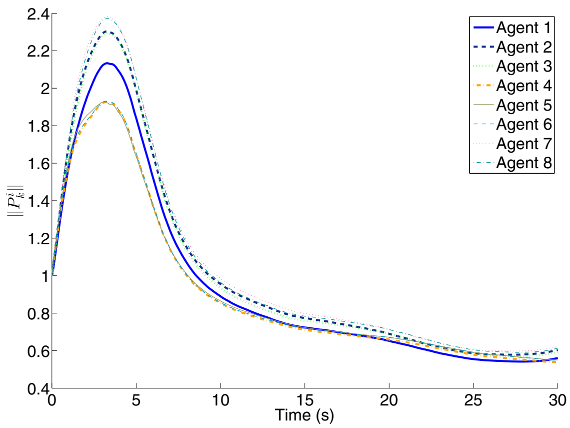

Fig. 3. Norm of covariance matrix for each agent in the network.

the estimation error of all agents decreasing as the network estimates the target's location.

V. Conclusion

In this paper, we discuss the problem of distributed localization of a moving target by a network of agents using TDOA measurements. A weighted-average EKF is proposed to eliminate the need of all agent's in the network having a sufficient number of neighbors to observe the process. We establish the boundedness of the estimation error under certain conditions (Corollary 3.7). Admittedly these conditions cannot be verified readily. Intuitively, it is conjectured that these conditions are correlated with the network topology; confirming this conjecture is part of our future work. In addition, while this work focuses on the case of fixed sensor locations, it is desirable to consider the case where the sensors will be mobile robots. Therefore, future work will expand analysis to the cases of intermittent observations and switching network topologies along with the development of a cooperative control law in order to localize and track a moving target.

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