Legendre Functions

Paolo Maccallini

January 2026

Index

1 Legendre polynomials: definitions and properties 2
2 Legendre's differential equation 7
3 Legendre functions of the second kind 16
4 Orthogonality of Legendre polynomials 20
5 Applications in physics 26
6 References 37

1 Legendre polynomials: definitions and properties

DEF. 1 (*Legendre polynomials*). These are polynomials defined in $[-1,1]$ by the sum

Eq. 1 $$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{E\left(\frac{n}{2}\right)}(-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}, \quad E\left(\frac{n}{2}\right) = \begin{cases} \frac{n}{2} \\ \frac{n-1}{2} \end{cases}$$

It can be verified that the expanded form of this sum is

Eq. 2 $$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left(x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 24(2n-1)(2n-3)(2n-5)}x^{n-6} + \dots\right)$$

The multiplicative factor is often reported in a different form: in fact, we have

$$\frac{(2n)!}{2^n(n!)^2} = \frac{2n(2n-1)(2n-2)\dots 1}{2n(2n-2)(2n-4)\dots 1\cdot n!} = \frac{(2n-1)(2n-3)(2n-5)\dots 1}{n!} = \frac{(2n-1)!!}{n!}$$

NOTE 1 (*first polynomials*). The first polynomials can be easily calculated:

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2-1) \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \quad P_5(x) = \frac{1}{8}(63x^5-70x^3+15)$$

PROP. 1 (*Rodrigues' formula*). It can be shown that:

Eq. 3 $$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$$

Proof. Let us consider the following integrals:

$$\int_0^x \xi^{n-2k}d\xi = \frac{x^{n+1-2k}}{n+1-2k} = \frac{(n-2k)!}{(n+1-2k)!}x^{n+1-2k}$$

$$\int_0^x\int_0^{\xi_1} \xi^{n-2k}d\xi_1 d\xi_2 = \frac{x^{n+2-2k}}{(n+2-2k)(n+1-2k)} = \frac{(n-2k)!}{(n+2-2k)!}x^{n+2-2k}$$

$$\int_0^x\int_0^{\xi_2}\int_0^{\xi_1} \xi^{n-2k}d\xi_1 d\xi_2\, d\xi_3 = \frac{x^{n+3-2k}}{(n+3-2k)(n+2-2k)(n+1-2k)}$$
$$= \frac{(n-2k)!}{(n+3-2k)!}x^{n+3-2k}$$

...

More generally, we have

$$\int_0^x \int_0^{\xi_{n-1}} \dots \int_0^{\xi_1} \xi^{n-2k} d\xi_1 d\xi_2 \dots d\xi_n = \frac{(n-2k)!}{(2n-2k)!} x^{2n-2k}$$

Therefore, by integration *n*-times the polynomial $P_n(x)$ we have

$$\int_0^x \int_0^{\xi_{n-1}} \dots \int_0^{\xi_1} P_n(\xi) d\xi_1 d\xi_2 \dots d\xi_n = \frac{1}{2^n} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-2k)!} \frac{(n+2k)!}{(2n-2k)!} x^{2n-2k} =$$

$$= \frac{1}{2^n} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{x^{2n-2k}}{k!\,(n-k)!} = \frac{1}{n!\,2^n} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{n!}{k!\,(n-k)!} x^{2n-2k} = \frac{1}{n!\,2^n} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \binom{n}{k} x^{2n-2k}$$

For *n* even, we have

$$\sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \binom{n}{k} x^{2n-2k} = \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n}{k} x^{2^{n-k}} =$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2^{n-k}} - \sum_{k=\frac{n}{2}+1}^{n} (-1)^k \binom{n}{k} x^{2^{n-k}} = (1-x^2)^n - \sum_{k=\frac{n}{2}+1}^{n} (-1)^k \binom{n}{k} x^{2^{n-k}}$$

By derivation, we have

$$P_n(\xi) = \frac{d^n}{dx^n} \left(\int_0^x \int_0^{\xi_{n-1}} \dots \int_0^{\xi_1} P_n(\xi) d\xi_1 d\xi_2 \dots d\xi_n \right) =$$

$$= \frac{1}{n!\,2^n} \frac{d^n \left(\sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \binom{n}{k} x^{2n-2k} \right)}{dx^n} = \frac{1}{n!\,2^n} \frac{d^n \left((1-x^2)^n - \sum_{k=\frac{n}{2}+1}^{n} (-1)^k \binom{n}{k} x^{2^{n-k}} \right)}{dx^n} =$$

$$= \frac{1}{n!\,2^n} \frac{d^n (1-x^2)^n}{dx^n} - \frac{d^n \left(\sum_{k=\frac{n}{2}+1}^{n} (-1)^k \binom{n}{k} x^{2^{n-k}} \right)}{dx^n}$$

We note now that the highest exponent of *x* within the sum is $2n - 2\frac{n}{2} - 2 = 2n - n - 2 = n - 2$. Therefore, the derivative of the sum is zero, and our theorem is proven for *n* even. For *n* odds, the proof is similar ■

PROP. 2 (*generating function*). It can be shown that:

Eq. 4 $\qquad (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

Proof. Omitted ■

PROP. 3 (*Recursion formula*). It can be shown that

Eq. 5 $$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Proof. By derivation with respect to *t* of Eq. 4, we have

$$\frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1} \Rightarrow \frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}} = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

By applying again Eq. 4:

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1} \Rightarrow$$

$$\Rightarrow x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1} - 2x \sum_{n=0}^{\infty} P_n(x) n t^n + \sum_{n=0}^{\infty} P_n(x) n t^{n+1}$$

Note now that

$$\sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{m=1}^{\infty} P_{m-1}(x) t^m = \sum_{m=0}^{\infty} P_{m-1}(x) t^m$$

$$\sum_{n=0}^{\infty} P_n(x) n t^{n-1} = \sum_{m=-1}^{\infty} P_{m+1}(x)(m+1) t^m = \sum_{m=0}^{\infty} P_{m+1}(x)(m+1) t^m$$

$$\sum_{n=0}^{\infty} P_n(x) n t^{n+1} = \sum_{m=1}^{\infty} P_{m-1}(x)(m-1) t^m = \sum_{m=0}^{\infty} P_{m-1}(x)(m-1) t^m$$

Therefore, we have found:

$$x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_{n-1}(x) t^n =$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1) t^n - 2x \sum_{n=0}^{\infty} P_n(x) n t^n + \sum_{n=0}^{\infty} P_{n-1}(x)(n-1) t^n \Rightarrow$$

$$\Rightarrow x P_n(x) - P_{n-1}(x) = P_{n+1}(x)(n+1) - 2x P_n(x) n + P_{n-1}(x)(n-1) \Rightarrow$$

$$\Rightarrow x P_n(x) + 2x P_n(x) n = P_{n+1}(x)(n+1) + P_{n-1}(x)(n-1) + P_{n-1}(x) \Rightarrow$$

$$\Rightarrow (2n+1) x P_n(x) = P_{n+1}(x)(n+1) + P_{n-1}(x) n$$

This is what we wanted to prove ■

PROP. 4 (*extreme values*). It can be shown that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

Proof. We use Eq. 4:

$$x = 1 \rightarrow (1 - 2t + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(1)t^n \Leftrightarrow (1 - t)^{-1} = \sum_{n=0}^{\infty} P_n(1)t^n$$

Since $(1 - t)^{-1} = \sum_{n=0}^{\infty} t^n$ when $|t| < 1$, it follows that $P_n(1) = 1$. We also have

$$x = -1 \rightarrow (1 - 2t + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-1)t^n \Leftrightarrow (1 + t)^{-1} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

We note that

$$(1 + t)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} t^n = \sum_{n=0}^{\infty} \frac{(-1)!}{n!\,(-1-n)!} t^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)(-1-1)(-1-2)\ldots(-1-n+1)(-1-n)!}{n!\,(-1-n)!} t^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 4 \cdot \ldots \cdot n}{n!} t^n = \sum_{n=0}^{\infty} (-1)^n t^n$$

Therefore, we have found $\sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)t^n$ which leads to $P_n(-1) = (-1)^n$ ■

Example 1 (*application of the recursion formula).* The formula we proved in PROP. 3 can be used to derive all the polynomials from $P_0(x)$ and $P_1(x)$. For $P_2(x)$ and $P_3(x)$ we have

$$P_2(x) = \frac{3}{2} x P_1(x) - \frac{1}{2} P_0(x) = \frac{3}{2} x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{5}{3} x \frac{1}{2}(3x^2 - 1) - \frac{2}{3} x = \left(\frac{15}{6} x^3 - \frac{5}{6} x - \frac{2}{3} x\right) = \frac{15}{6} x^3 - \left(\frac{5+4}{6}\right) x = \frac{5}{2} x^3 - \frac{3}{2} x$$

And so forth. The recursion formula can be used in a recursive script for polynomial visualisation, as shown below.

CODE 1 (*Plotting Polynomials*). The following is a Python script that calculates the first *N* Legendre polynomials, given $P_0(x)$ and $P_1(x)$, using the recursive formula. Then it plots them (Figure 1). It stores their values in the columns of a data frame (I used the package *pandas*). It calculates $P_{n+1}(x)$ from $P_n(x)$ and $P_{n-1}(x)$ recursively within a loop.

```
# file name: Recursive_Legendre_Poly.py
#
# This script derives the values of the first n (specified by the user)
# Legendre Polynomials and plot them
#
```

```
import os # functions for interacting with the operating system
import pandas as pd # dataframes similar to R
import matplotlib.pyplot as plt # a plotting library
import math # mathematical functions
import numpy as np # arrays and matrices
#
# Change the directory to the folder of the script
#
script_dir = os.path.dirname(os.path.abspath(__file__))
print(f"script_dir: {script_dir}")
os.chdir(script_dir)
#
# Number of Legendre Polynomials to calculate, and number of values of x
#
N = 6 # first n+1 Legendre Polynomials (n>1)
r = 100 # number of values for x between 0 and 1
#
# Legendre Polynomials are stored in a data frame such that column n is P_n(x)
#
mydata = pd.DataFrame(0.0, index=range(r), columns=range(N))
#
# Store P_0 and P_1, 1 and x respectively
#
for i in range(r):
    mydata.at[i,0] = 1.0    # P_0(x) = 1
mydata[1] = np.linspace(-1, 1, r) # P_1(x) = x
#
# Calculate values from P_2 to P_n using the recursive relation
#
x = mydata[1] # x values
for n in range(1,N-1):
    print(f"Calculating P_{n+1}(x)")
    for i in range(r):
        P1 = mydata.at[i,n-1] # P_(n-1)(x)
        P2 = mydata.at[i,n] # P_(n)(x)
        P3 = ((2*n+1)*x[i]*P2 - n*P1)/(n+1) # P_(n+1)(x)
        mydata.at[i,n+1] = P3 # store P_(n+1)(x)
#
# Plot
#
plt.figure(figsize=(8, 5))
for n in range(0,N):
    Pn = mydata[n]
    plt.plot(x, Pn, label=f'$P_{n}(x)$')
plt.xlabel('abscissa (x)')
plt.ylabel('$P_n(x)$')
plt.title('Legendre Polynomials $P_n(x)$')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.savefig('Legendre_Polynomials.jpg', dpi=300)
plt.close() # Close the current figure
```

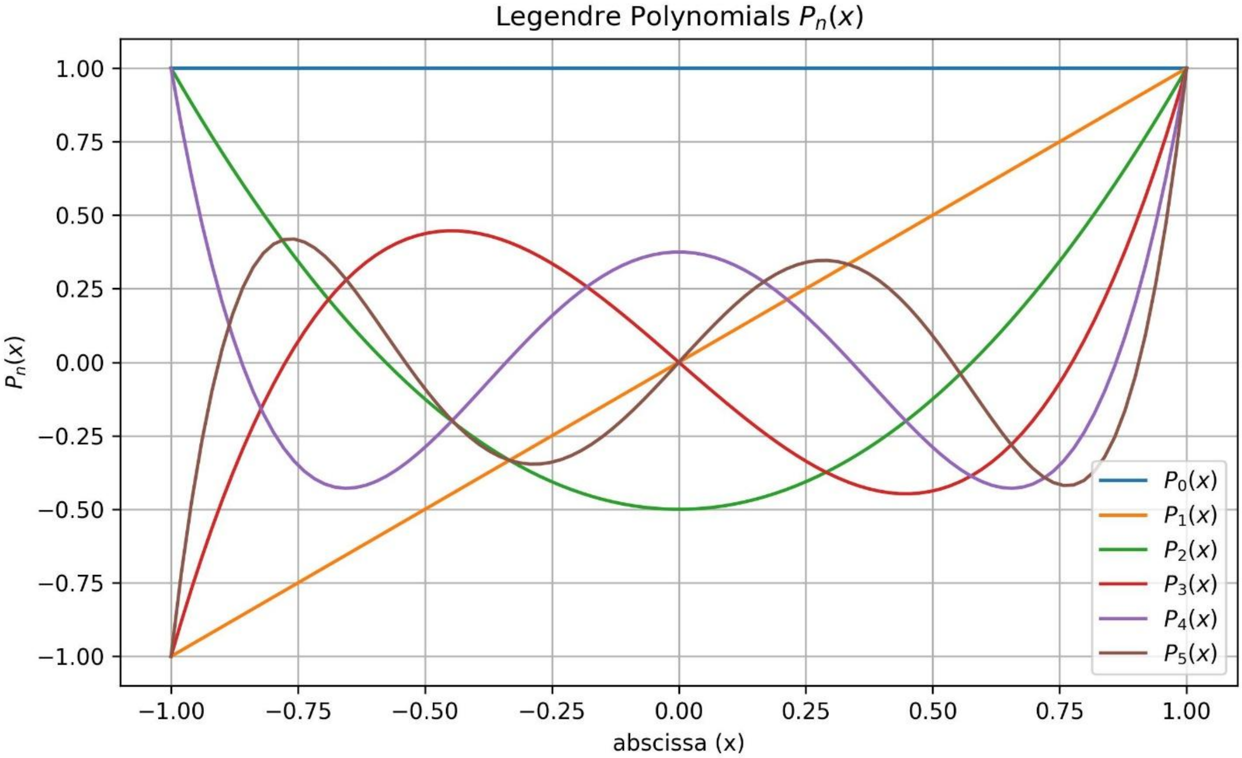

Figure 1. First six Legendre Polynomials.

2 Legendre's differential equation

DEF. 2 (*Associated Legendre's differential equation*). It is the following linear ordinary differential equation of the second order:

Eq. 6 $$\frac{d\left((1-x^2)y'\right)}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y = 0 \Leftrightarrow (1-x^2)y'' - 2xy' + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y = 0$$

DEF. 3 (*Legendre's differential equation*). For $m = 0$, the associated Legendre's equation is called simply the Legendre equation:

Eq. 7 $$\frac{d\left((1-x^2)y'\right)}{dx} + n(n+1)y = 0 \Leftrightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

PROP. 5 (*Legendre and Laplace*). The associated Legendre equation emerges from the solution of Laplace's equation in spherical coordinates by separation of variables.

Proof. The Laplace's equation in polar coordinates is

Eq. 8 $$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) = 0$$

where θ is the polar angle (from the positive z-axis), and ϕ is the azimuthal angle (plane *xy*). We search for a solution of the type

$$f(\rho, \phi, \theta) = R(\rho)F(\phi)T(\theta)$$

Therefore, Laplace's equation becomes

$$\frac{F(\phi)T(\theta)}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 R'(\rho)) + \frac{R(\rho)T(\theta)}{\rho^2\sin^2\theta}F''(\phi) + \frac{R(\rho)F(\phi)}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, T'(\theta)) = 0$$

We multiply by $\frac{\rho^2}{R(\rho)F(\phi)T(\theta)}$ and obtain:

$$\frac{1}{R(\rho)}\frac{\partial(\rho^2 R'(\rho))}{\partial\rho} = -\frac{1}{\sin^2\theta}\frac{F''(\phi)}{F(\phi)} - \frac{1}{T(\theta)\sin\theta}\frac{\partial(\sin\theta\, T'(\theta))}{\partial\theta}$$

Since the left-hand side of the equation depends only on ρ, while the right-hand side depends only on θ and ϕ, they must both be constant:

$$\begin{cases} \dfrac{1}{R(\rho)}\dfrac{\partial(\rho^2 R'(\rho))}{\partial\rho} = \alpha \\ \dfrac{1}{\sin^2\theta}\dfrac{F''(\phi)}{F(\phi)} + \dfrac{1}{T(\theta)\sin\theta}\dfrac{\partial(\sin\theta\, T'(\theta))}{\partial\theta} = -\alpha \end{cases}$$

The second equation can be written:

$$\frac{F''(\phi)}{F(\phi)} + \frac{\sin\theta}{T(\theta)}\frac{\partial(\sin\theta\, T'(\theta))}{\partial\theta} = -\alpha\sin^2\theta \Rightarrow \frac{\sin\theta}{T(\theta)}\frac{\partial(\sin\theta\, T'(\theta))}{\partial\theta} + \alpha\sin^2\theta = -\frac{F''(\phi)}{F(\phi)}$$

Again, the two hands of the equation must be constant, and we have

Eq. 9
$$\begin{cases} \frac{\partial(\rho^2 R'(\rho))}{\partial\rho} - \alpha R(\rho) = 0 \\ \frac{\sin\theta}{T(\theta)}\frac{\partial(\sin\theta T'(\theta))}{\partial\theta} + \alpha\sin^2\theta = \beta \\ F''(\phi) + \beta F(\phi) = 0 \end{cases}$$

We now make the position $x = \cos\theta$ and we have

$$T'(\theta) = \frac{dT(\theta)}{d\theta} = \frac{dT(x)}{dx}\frac{dx}{d\theta} = -\frac{dT(x)}{dx}\sin\theta$$

Therefore, the second equation in Eq. 9 can be written:

$$-\frac{\sin\theta}{T(\theta)}\frac{d\left(\frac{dT(x)}{dx}\sin^2\theta\right)}{dx}\frac{dx}{d\theta} + \alpha\sin^2\theta = \beta \Leftrightarrow \frac{\sin^2\theta}{T(\theta)}\frac{d\left(\frac{dT(x)}{dx}\sin^2\theta\right)}{dx} + \alpha\sin^2\theta = \beta \Leftrightarrow$$

$$\Leftrightarrow \frac{d\left(\frac{dT(x)}{dx}(1-x^2)\right)}{dx} + \left(\alpha - \frac{\beta}{1-x^2}\right)T(\theta) = 0$$

If we now chose $\alpha = n(n+1)\ and\ \beta = m^2$ we have

Eq. 10
$$\begin{cases} \frac{\partial(\rho^2 R'(\rho))}{\partial \rho} - n(n+1)R(\rho) = 0 \\ \frac{d\left(\frac{dT(x)}{dx}(1-x^2)\right)}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right)T(\theta) = 0 \\ F''(\phi) + m^2 F(\phi) = 0 \end{cases}, \quad x = \cos\theta$$

The second equation is Eq. 6 and this proves the theorem ■

PROP. 6 (*Legendre and Laplace*). When a problem has axial symmetry with respect to z, then Laplace equation in spherical coordinates is equivalent to the following two linear ODEs:

Eq. 11
$$\begin{cases} \frac{\partial(\rho^2 R'(\rho))}{\partial \rho} - n(n+1)R(\rho) = 0 \\ \frac{d\left(\frac{dT(x)}{dx}(1-x^2)\right)}{dx} + n(n+1)T(\theta) = 0 \end{cases}, \quad x = \cos\theta$$

The second equation is the Legendre equation in Eq. 7.

Proof. If the problem is independent from ϕ, then $\beta = 0$ in Eq. 9, and $m = 0$ in Eq. 10 ■

DEF. 4 (*Euler's differential equation*). It is also called Cauchy's differential equation:

Eq. 12
$$x^2 y'' + (2k+1)xy' + \beta^2 y = 0$$

PROP. 7 (*Euler's differential equation*). The general integral of Eq. 12 can be written

Eq. 13
$$y = C_1 x^{-k+\sqrt{k^2-\beta^2}} + C_2 x^{-k-\sqrt{k^2-\beta^2}}$$

Proof. We search for solutions of the kind $y = x^p$ and we have

$$x^2 p(p-1)x^{p-2} + (2k+1)xpx^{p-1} + \beta^2 x^p = 0 \Leftrightarrow$$

$$\Leftrightarrow p(p-1) + (2k+1)p + \beta^2 = 0 \Leftrightarrow p^2 + 2kp + \beta^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow p = -k \pm \sqrt{k^2 - \beta^2}$$

Hence, two solutions are $y = x^{-k\pm\sqrt{k^2-\beta^2}}$ ■

NOTE 2. We recognise that the first ODE in Eq. 10 is an example of Euler's equation:

$$\frac{\partial(\rho^2 R'(\rho))}{\partial \rho} - n(n+1)R(\rho) = 0 \Leftrightarrow \rho^2 R''(\rho) + 2\rho R'(\rho) - n(n+1)R(\rho) = 0$$

This is equivalent to Eq. 12, where $2k + 1 = 2 \Leftrightarrow k = \frac{1}{2}$ and $\beta^2 = -n(n+1)$, therefore the general integral is

$$R(\rho) = C_1 \rho^{-\frac{1}{2}+\sqrt{\frac{1}{4}+n(n+1)}} + C_2 \rho^{-\frac{1}{2}-\sqrt{\frac{1}{4}+n(n+1)}} = C_1 \rho^{-\frac{1}{2}+\frac{\sqrt{1+4n^2+4n}}{2}} + C_2 \rho^{-\frac{1}{2}-\frac{\sqrt{1+4n^2+4n}}{2}} =$$

$$= C_1 \rho^{-\frac{1}{2}+\frac{1+2n}{2}} + C_2 \rho^{-\frac{1}{2}-\frac{1+2n}{2}} = C_1 \rho^n + C_2 \rho^{-(n+1)}$$

We also note that the third ODE in Eq. 10 has the general integral given by

$$F(\phi) = C_1 + C_2 e^{-m^2 \phi}$$

as can be easily proven. Therefore, Eq. 10 can be written:

Eq. 14
$$\begin{cases} R(\rho) = C_1 \rho^n + C_2 \rho^{-(n+1)} \\ \dfrac{d\left(\frac{dT(x)}{dx}(1-x^2)\right)}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right) T(\theta) = 0 \\ F(\phi) = C_1 + C_2 e^{-m^2 \phi} \end{cases}, \quad x = \cos\theta$$

while Eq. 11 is equivalent to

Eq. 15
$$\begin{cases} R(\rho) = C_1 \rho^n + C_2 \rho^{-(n+1)} \\ \dfrac{d\left(\frac{dT(x)}{dx}(1-x^2)\right)}{dx} + n(n+1) T(x) = 0 \end{cases}, \quad x = \cos\theta$$

PROP. 8 (*LEMMA*). It can be shown that

Eq. 16
$$\prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] = (-1)^{\frac{k}{2}} \frac{(n+(k-1))!!}{(n-1)!!} \frac{n!!}{(n-k)!!} \quad k \ even$$

Eq. 17
$$\prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)] = (-1)^{\frac{k-1}{2}} \frac{(n+(k-1))!!}{n!!} \frac{(n-1)!!}{(n-k)!!} \quad k \ odds$$

Proof. We note that

$$(n+b)(n-a) = n^2 + bn - an - ab = n^2 + n(b-a) - ab = n(n+b-a) - ab$$

For $b = h-1$ and $a = h-2$ we have

$$(n + (h-1\,))(n - (h-2)) = n(n+1) - (h-1)(h-2) \Leftrightarrow$$

$$\Leftrightarrow (h-1)(h-2) - n(n+1) = (-1)(n + (h-1\,))(n - (h-2))$$

Therefore, for k even, we have

$$\prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] = \prod_{\substack{h=2 \\ even}}^{k} (-1)(n + (h-1))(n - (h-2)) =$$

$$= (-1)^{\frac{k}{2}} (n+1)(n+3) \ldots \big(n + (k-3)\big)\big(n + (k-1)\big) \cdot n(n-2)(n-4) \ldots \big(n - (k-2)\big) =$$

$$= (-1)^{\frac{k}{2}} \frac{\big(n + (k-1)\big)!!}{(n-1)!!} \frac{n!!}{(n-k)!!}$$

Analogously, for k odds we have

$$\prod_{\substack{h=3\\odds}}^{k}[(h-1)(h-2)-n(n+1)]=\prod_{\substack{h=3\\odds}}^{k}(-1)(n+(h-1))(n-(h-2))=$$

$$=(-1)^{\frac{k-1}{2}}(n+2)(n+4)\dots\big(n+(k-1)\big)\cdot(n-1)(n-3)\dots\big(n-(k-4)\big)\big(n-(k-2)\big)=$$

$$=(-1)^{\frac{k-1}{2}}\frac{\big(n+(k-1)\big)!!}{n!!}\frac{(n-1)!!}{(n-k)!!}$$

And the proposition is proven ■

PROP. 9 (*Solution of Legendre differential equation*). The Legendre polynomial $P_n(x)$ is a solution of the Legendre differential equation (Eq. 7).

Proof. We search for a solution of Eq. 7 in the form $y=\sum_{k=-\infty}^{\infty}c_k x^{k+\beta}$ with $c_k=0$ for $k<0$ (Frobenius' method). The first and the second derivatives of this solution are, respectively

$$y'=\sum_{k=-\infty}^{\infty}c_k(k+\beta)x^{k+\beta-1},\quad y''=\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta-2}$$

Therefore, we can write:

$$-2xy'=-2\sum_{k=-\infty}^{\infty}c_k(k+\beta)x^{k+\beta}$$

$$(1-x^2)y''=\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta-2}-\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta}$$

By substituting y, $-2xy'$, and $(1-x^2)y''$ in Eq. 7 we have

$$\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta-2}-\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta}-$$

$$-2\sum_{k=-\infty}^{\infty}c_k(k+\beta)x^{k+\beta}+n(n+1)\sum_{k=-\infty}^{\infty}c_k x^{k+\beta}=0$$

Since

$$\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta-2}=\sum_{h=-\infty}^{\infty}c_{h+2}(h+\beta+2)(h+\beta+1)x^{h+\beta}$$

we can write

$$\sum_{k=-\infty}^{\infty}c_{k+2}(k+\beta+2)(k+\beta+1)x^{k+\beta}-\sum_{k=-\infty}^{\infty}c_k(k+\beta)(k+\beta-1)x^{k+\beta}-$$

$$-2\sum_{k=-\infty}^{\infty} c_k(k+\beta)x^{k+\beta} + n(n+1)\sum_{k=-\infty}^{\infty} c_k x^{k+\beta} = 0 \Leftrightarrow$$

Eq. 18 $c_{k+2}(k+\beta+2)(k+\beta+1) + [n(n+1) - (k+\beta)(k+\beta-1) - 2(k+\beta)]c_k = 0$

This must be true for any value of k. In particular, for $k = -2$ we have $c_k = c_{-2} = 0$, and the above relation becomes

$$c_0\beta(\beta-1) = 0$$

For $c_0 \neq 0$ this implies $\beta = 0$ or $\beta = 1$. For $\beta = 0$, Eq. 18 becomes

$$c_{k+2}(k+2)(k+1) + [n(n+1) - k(k-1) - 2k]c_k = 0 \Leftrightarrow$$

$$c_{k+2} = -\frac{n(n+1) - k(k-1) - 2k}{(k+2)(k+1)}c_k = -\frac{n(n+1) - k^2 - k}{(k+2)(k+1)}c_k \Leftrightarrow$$

Eq. 19 $\quad c_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)}c_k, \quad k = 0, 1, 2, \dots$

In order to explore these coefficients, let's see the first values they assume:

$$c_2 = -\frac{n(n+1)}{2 \cdot 1}c_0$$

$$c_3 = \frac{1 \cdot 2 - n(n+1)}{3 \cdot 2}c_1$$

$$c_4 = \frac{2 \cdot 3 - n(n+1)}{4 \cdot 3}c_2 = -\frac{n(n+1)}{2 \cdot 1}\frac{2 \cdot 3 - n(n+1)}{4 \cdot 3}c_0$$

$$c_5 = \frac{3 \cdot 4 - n(n+1)}{5 \cdot 4}c_3 = \frac{1 \cdot 2 - n(n+1)}{3 \cdot 2}\frac{3 \cdot 4 - n(n+1)}{5 \cdot 4}c_1$$

$$c_6 = \frac{4 \cdot 5 - n(n+1)}{6 \cdot 5}c_4 = -\frac{n(n+1)}{2 \cdot 1}\frac{2 \cdot 3 - n(n+1)}{4 \cdot 3}\frac{4 \cdot 5 - n(n+1)}{6 \cdot 5}c_0$$

$$c_7 = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)}c_5 = \frac{1 \cdot 2 - n(n+1)}{3 \cdot 2}\frac{3 \cdot 4 - n(n+1)}{5 \cdot 4}\frac{5 \cdot 4 - n(n+1)}{7 \cdot 6}c_1$$

$$c_8 = \frac{7 \cdot 6 - n(n+1)}{9 \cdot 7}c_6 = -\frac{n(n+1)}{2 \cdot 1}\frac{2 \cdot 3 - n(n+1)}{4 \cdot 3}\frac{4 \cdot 5 - n(n+1)}{6 \cdot 5}\frac{7 \cdot 6 - n(n+1)}{9 \cdot 7}c_0$$

and so forth. This suggests a general expression for c_k:

Eq. 20 $\quad \begin{cases} c_k = \frac{c_0}{k!}\prod_{h=2}^{k}[(h-1)(h-2) - n(n+1)] & k, h \; even \\ c_k = \frac{c_1}{k!}\prod_{h=3}^{k}[(h-1)(h-2) - n(n+1)] & k, h \; odds \end{cases}$

Now, we can write the solution $y = \sum_{k=-\infty}^{\infty} c_k x^{k+\beta} = \sum_{k=-\infty}^{\infty} c_k x^k$ as follows

Eq. 21 $$y = c_0 + \sum_{k\ even} \frac{c_0 x^k}{k!} \prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] + c_1 x + \sum_{k\ odds} \frac{c_1 x^k}{k!} \prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)]$$

For $h = n + 1$ we have $(h-1)(h-2) - n(n+1) = n(n-1) - n(n+1) = 0$. Therefore, for n even, we have

$$\sum_{k\ even} \frac{c_0 x^k}{k!} \prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] = \sum_{\substack{k=2 \\ even}}^{n} \frac{c_0 x^k}{k!} \prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)]$$

While the other sum remains unlimited. On the other hand, for n odds we have

$$\sum_{k\ odds} \frac{c_1 x^k}{k!} \prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)] = \sum_{\substack{k=3 \\ odds}}^{n} \frac{c_1 x^k}{k!} \prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)]$$

and the other sum remains unlimited. If we consider only solutions of the Legendre differential equation given by a limited sum, we can write them with the following two sums, one for n even and the other for n odd:

Eq. 22 $$\begin{cases} y_n = c_0 + \sum_{\substack{k=2 \\ even}}^{n} \frac{c_0 x^k}{k!} \prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] & n\ even \\ y_n = c_1 x + \sum_{\substack{k=3 \\ odds}}^{n} \frac{c_1 x^k}{k!} \prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)] & n\ odd \end{cases}$$

with c_0, c_1 arbitrary constants. We now write in an extended form these polynomials for the first values of n. For $n = 0$, $y_0 = c_0$ and for $n = 1$, $y_1 = c_1 x$. For $n = 2$ we have

$$y_2 = c_0 + \frac{c_0 x^2}{2!} [(2-1)(2-2) - 2(2+1)] = c_0 - \frac{c_0 x^2}{2!} 6 = c_0 (1 - 3x^2) = -2 c_0 P_2(x)$$

For $n = 3$:

$$y_3 = c_1 x + \frac{c_1 x^3}{3!} [(3-1)(3-2) - 3(3+1)] = c_1 x - 10 \frac{c_1 x^3}{3!} =$$

$$= -c_1 \frac{2}{3} \frac{1}{2} (5x^3 - 3x) = -\frac{2}{3} c_1 P_3(x)$$

For $n = 4$:

$$y_4 = c_0 + \sum_{\substack{k=2 \\ even}}^{4} \frac{c_0 x^k}{k!} \prod_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - 20] = c_0 - 20 \frac{c_0 x^2}{2} + 14 \cdot 20 \frac{c_0 x^4}{4!} =$$

$$= c_0 \frac{8}{3}\frac{1}{8}(3 - 30x^2 + 35x^4) = \frac{8}{3} c_0 P_4(x)$$

For $n = 5$:

$$y_5 = c_1 x + \frac{c_1 x^3}{6}[(3-1)(3-2) - 30] + \frac{c_1 x^5}{5!} \prod_{\substack{h=3 \\ odds}}^{5} [(h-1)(h-2) - 30] =$$

$$= c_1 \left(x + \frac{14}{3}x^3 + \frac{21}{5}x^5 \right) = c_1 \frac{8}{15}\frac{1}{8}(15x + 70x^3 + 63x^5) = c_1 \frac{8}{15} P_5(x)$$

Therefore, we have shown so far that y_n is the corresponding Legendre polynomial (see NOTE 1) multiplied by a constant that is a function of n, as summarised in the table below.

n	0	1	2	3	4	5
c_0	1	-	$-\frac{1}{2}$	-	$\frac{3}{8}$	-
c_1	-	1	-	$-\frac{3}{2}$	-	$\frac{15}{8}$
y_n	P_0	P_1	P_2	P_3	P_4	P_5

It can be proved that, in fact, this is the case for any value of n by using the results. Let us go through this. Consider Eq. 22 for n even, and let us substitute Eq. 16 in it:

$$y_n = c_0 + \sum_{\substack{k=2 \\ even}}^{n} \frac{c_0 x^k}{k!} (-1)^{\frac{k}{2}} \frac{\big(n + (k-1)\big)!!}{(n-1)!!} \frac{n!!}{(n-k)!!} =$$

$$= \sum_{\substack{k=0 \\ even}}^{n} \frac{c_0 x^k}{k!} (-1)^{\frac{k}{2}} \frac{\big(n + (k-1)\big)!!}{(n-1)!!} \frac{n!!}{(n-k)!!}$$

We now consider the substitution $k = n - 2p$, and we have

$$y_n = \sum_{p=0}^{\frac{n}{2}} \frac{c_0 x^{n-2p}}{(n-2p\,)!} (-1)^{\frac{n}{2}-p} \frac{\big(n + (n - 2p\, - 1)\big)!!}{(n-1)!!} \frac{n!!}{(n - n + 2p\,)!!} =$$

$$= \sum_{p=0}^{\frac{n}{2}} c_0 x^{n-2p} (-1)^{\frac{n}{2}-p} \frac{(2n - 2p\, - 1)!!}{(n-1)!!\,(n-2p\,)!} \frac{n!!}{(2p\,)!!}$$

We note now that $(2p\,)!! = 2^p p!$ and also that

Eq. 23 $\qquad (2n - 2p\, - 1)!! = \frac{(2n-2p\,-1)!!(2n-2p\,)!!}{(2n-2p\,)!!} = \frac{(2n-2p\,)!}{(2n-2p\,)!!} = \frac{(2n-2p\,)!}{2^{n-p}(n-p\,)!}$

Therefore, we have

$$y_n = c_0 \sum_{p=0}^{\frac{n}{2}} x^{n-2p} \, (-1)^{\frac{n}{2}-p} \frac{1}{(n-1)!! \, (n-2p\,)!} \frac{(2n-2p\,)!}{2^{n-p}(n-p\,)!} \frac{n!!}{2^p p!} =$$

$$= c_0 \frac{n!!}{(n-1)!!} \frac{1}{2^n} \sum_{p=0}^{\frac{n}{2}} x^{n-2p} \, (-1)^{\frac{n}{2}-p} \frac{(2n-2p\,)!}{p! \, (n-p\,)! \, (n-2p\,)!}$$

We note also that $(-1)^{\frac{n}{2}-p} = (-1)^{\frac{n}{2}}(-1)^{-p} = (-1)^{-p} = (-1)^p$. By comparison with Eq. 1, we have

$$y_n = c_0 \frac{n!!}{(n-1)!!} \frac{1}{2^n} \sum_{p=0}^{\frac{n}{2}} x^{n-2p} \, (-1)^p \frac{(2n-2p\,)!}{p! \, (n-p\,)! \, (n-2p\,)!} = \frac{n!!}{(n-1)!!} P_n(x) \quad n \; even$$

Now, let us consider Eq. 22 for *n* odds and let us substitute Eq. 17 in it:

$$y_n = c_1 x + \sum_{\substack{k=3 \\ odds}}^{n} \frac{c_1 x^k}{k!} \prod_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)] =$$

$$= c_1 x + \sum_{\substack{k=3 \\ odds}}^{n} \frac{c_1 x^k}{k!} (-1)^{\frac{k-1}{2}} \frac{\big(n+(k-1)\big)!!}{n!!} \frac{(n-1)!!}{(n-k)!!} =$$

$$= c_1 \sum_{\substack{k=1 \\ odds}}^{n} \frac{x^k}{k!} (-1)^{\frac{k-1}{2}} \frac{\big(n+(k-1)\big)!!}{n!!} \frac{(n-1)!!}{(n-k)!!}$$

We now consider the substitution $k = n - 2p$, and we have

$$y_n = c_1 \sum_{p=0}^{\frac{n-1}{2}} \frac{x^{n-2p}}{(n-2p\,)!} (-1)^{\frac{n-2p-1}{2}} \frac{\big(n+(n-2p\ -1)\big)!!}{n!!} \frac{(n-1)!!}{(n-n+2p\,)!!} =$$

$$= c_1 \sum_{p=0}^{\frac{n-1}{2}} x^{n-2p} \, (-1)^{\frac{n-2p-1}{2}} \frac{(2n-2p\ -1)!!}{(2p)!! \, (n-2p\,)!} \frac{(n-1)!!}{n!!} =$$

$$= c_1 \frac{(n-1)!!}{n!!} \sum_{p=0}^{\frac{n-1}{2}} x^{n-2p} \, (-1)^{\frac{n-2p-1}{2}} \frac{(2n-2p\ -1)!!}{(2p)!! \, (n-2p\,)!}$$

Since $(2p\,)!! = 2^p p!$ and given Eq. 23, we have

$$y_n = c_1 \frac{(n-1)!!}{n!!} \sum_{p=0}^{\frac{n-1}{2}} x^{n-2p} \, (-1)^{\frac{n-2p-1}{2}} \frac{1}{2^p p! \, (n-2p\,)!} \frac{(2n-2p\,)!}{2^{n-p}(n-p\,)!} =$$

$$= c_1 \frac{(n-1)!!}{n!!} \frac{1}{2^n} \sum_{p=0}^{\frac{n-1}{2}} x^{n-2p} \, (-1)^{\frac{n-2p-1}{2}} \frac{(2n-2p\,)!}{p! \, (n-2p\,)! \, (n-p\,)!}$$

Since $(-1)^{\frac{n-2p-1}{2}} = (-1)^{\frac{n-1}{2}}(-1)^{-p} = (-1)^{-p} = (-1)^p$, we have found

$$y_n = c_1 \frac{(n-1)!!}{n!!} \frac{1}{2^n} \sum_{p=0}^{\frac{n-1}{2}} x^{n-2p} (-1)^p \frac{(2n-2p)!}{p!(n-p)!(n-2p)!} = c_1 \frac{(n-1)!!}{n!!} P_n(x) \quad n\ odds$$

This proves that the Legendre polynomial $P_n(x)$ is a solution to the Legendre differential equation ■

NOTE 3 (*alternative expression*). It is worth noting that Eq. 22 is an alternative expression of the Legendre polynomials.

3 Legendre functions of the second kind

NOTE 4 (*infinite polynomials*). We have seen that the limited sums in Eq. 21 are the Legendre polynomials. There are two unlimited sums included in Eq. 21, one for *n* even and the other for *n* odd. In particular, we have

$$\begin{cases} y_n = c_0 + \sum\limits_{\substack{k=2 \\ even}}^{\infty} \dfrac{c_0 x^k}{k!} \prod\limits_{\substack{h=2 \\ even}}^{k} [(h-1)(h-2) - n(n+1)] & n\ odd \\ y_n = c_1 x + \sum\limits_{\substack{k=3 \\ odds}}^{\infty} \dfrac{c_1 x^k}{k!} \prod\limits_{\substack{h=3 \\ odds}}^{k} [(h-1)(h-2) - n(n+1)] & n\ even \end{cases}$$

By considering the results in PROP. 8 we have

Eq. 24
$$\begin{cases} y_n = c_0 \frac{n!!}{(n-1)!!} \sum_{\substack{k=0 \\ even}}^{\infty} \frac{x^k}{k!} (-1)^{\frac{k}{2}} \frac{(n+(k-1))!!}{(n-k)!!} & n\ odd \\ y_n = c_1 \frac{(n-1)!!}{n!!} \sum_{\substack{k=1 \\ odds}}^{\infty} \frac{x^k}{k!} (-1)^{\frac{k-1}{2}} \frac{(n+(k-1))!!}{(n-k)!!} & n\ even \end{cases}$$

DEF. 5 (*Legendre functions of the second kind*). The unlimited polynomials in Eq. 24 are defined as Legendre functions of the second kind. For n odds we consider the substitution $k = 2h$, while for *n* even we make the position $k = 2h + 1$:

Eq. 25
$$\begin{cases} Q_n(x) = c_0 \frac{n!!}{(n-1)!!} \sum_{h=0}^{\infty} (-1)^h x^{2h} \frac{(n+2h-1)!!}{(2h)!(n-2h)!!} & n\ odds \\ Q_n(x) = c_1 \frac{(n-1)!!}{n!!} \sum_{h=0}^{\infty} (-1)^h x^{2h+1} \frac{(n+2h)!!}{(2h+1)!(n-2h-1)!!} & n\ even \end{cases}$$

If we expand the sums, for the first terms, for *n* odd we have

$$Q_n(x) = c_0 \left(1 - \frac{(n+1)n}{2!} x^2 + \frac{(n+3)(n-2)(n+1)n}{4!} x^4 - \frac{(n+5)(n-4)(n+3)(n-2)(n+1)n}{6!} x^6 + \cdots \right.$$

For *n* even, we have

$$Q_n(x) = c_1 \left(x - \frac{(n+2)(n-1)}{3!} x^3 + \frac{(n+4)(n-3)(n+2)(n-1)}{5!} x^5 - \frac{(n+6)(n-5)(n+4)(n-3)(n+2)(n-1)}{7!} x^7 + \cdots \right.$$

The two constants c_0, c_1 are defined as follows:

Eq. 26 $$c_0=\frac{(-1)^{\frac{n+1}{2}}2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!!},\quad c_1=\frac{(-1)^{\frac{n}{2}}2^{n}\left[\left(\frac{n}{2}\right)!\right]^2}{n!}$$

PROP. 10 (*Recursion formula*). It can be shown that:

Eq. 27 $$Q_{n+1}(x)=\frac{2n+1}{n+1}xQ_n(x)-\frac{n}{n+1}Q_{n-1}(x)$$

which is formally identical to Eq. 5. In fact, the values in Eq. 26 are chosen so that this recursion formula holds.

Proof. Omitted ■

NOTE 5 (*first polynomials*). We now use the recursion formula to calculate the first values of $Q_n(x)$ from $Q_0(x)$ and $Q_1(x)$. For $n=0$, we use the formula for *n* even, and we have

$$c_1=\frac{(-1)^{\frac{n}{2}}2^n\left[\left(\frac{n}{2}\right)!\right]^2}{n!}=1\Rightarrow$$

$$\Rightarrow Q_0(x)=x-\frac{2(-1)}{3!}x^3+\frac{4(-3)2(-1)}{5!}x^5-\frac{6(-5)4(-3)2(-1)}{7!}x^7+\cdots=$$

$$=x+\frac{2!}{3!}x^3+\frac{4!}{5!}x^5+\frac{6!}{7!}x^7+\cdots=x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\cdots=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

By comparison with the well-known Taylor expansion

Eq. 28 $$\ln\left(\frac{1+x}{1-x}\right)=2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

we conclude that

Eq. 29 $$Q_0(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Now, we move on $Q_1(x)$, using the formula for *n* odds:

$$c_0=\frac{(-1)^{\frac{n+1}{2}}2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!!}=-1\Rightarrow$$

$$\Rightarrow Q_1(x)=-1+\frac{2}{2!}x^2-\frac{4(-1)2}{4!}x^4+\frac{6(-3)4(-1)2}{6!}x^6-\cdots=-1+\frac{2}{2!}x^2+\frac{4\cdot 2}{4!}x^4+\frac{6\cdot 3\cdot 4\cdot 2}{6!}x^6-\cdots=$$

$$Q_1(x)=-1+\frac{1}{1!}x^2+\frac{2!}{3!}x^4+\frac{4!}{5!}x^6+\cdots=-1+\frac{1}{1}x^2+\frac{1}{3}x^4+\frac{1}{5}x^6+\cdots=$$

$$=-1+x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots=-1+x\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\right.$$

With the same line of reasoning used for $Q_0(x)$ we can conclude that

Eq. 30 $$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

It is now possible to derive the following expressions of $Q_n(x)$ by means of the recursion formula (Eq. 27). For $n = 2$, we find

$$Q_2(x) = \frac{3}{2}xQ_1(x) - \frac{1}{2}Q_0(x) = \frac{3}{2}x\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right) - \frac{1}{2}\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right) \Rightarrow$$

Eq. 31 $$Q_2(x) = \left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x$$

We note, from the expressions of $Q_0(x), Q_1(x), Q_2(x)$ that these functions diverge for $x = \pm 1$ because of the presence of $\ln\left(\frac{1+x}{1-x}\right)$.

CODE 2 (*plotting functions of the second kind*). We now plot $Q_n(x)$ for the first few values of *n* using a Python script that implements the recursion formula (Eq. 27) and the expressions of $Q_0(x), Q_1(x)$ calculated in NOTE 5. The script is similar to CODE 1, and it generates Figure 2. Note how the Legendre functions of the second type diverge at ± 1.

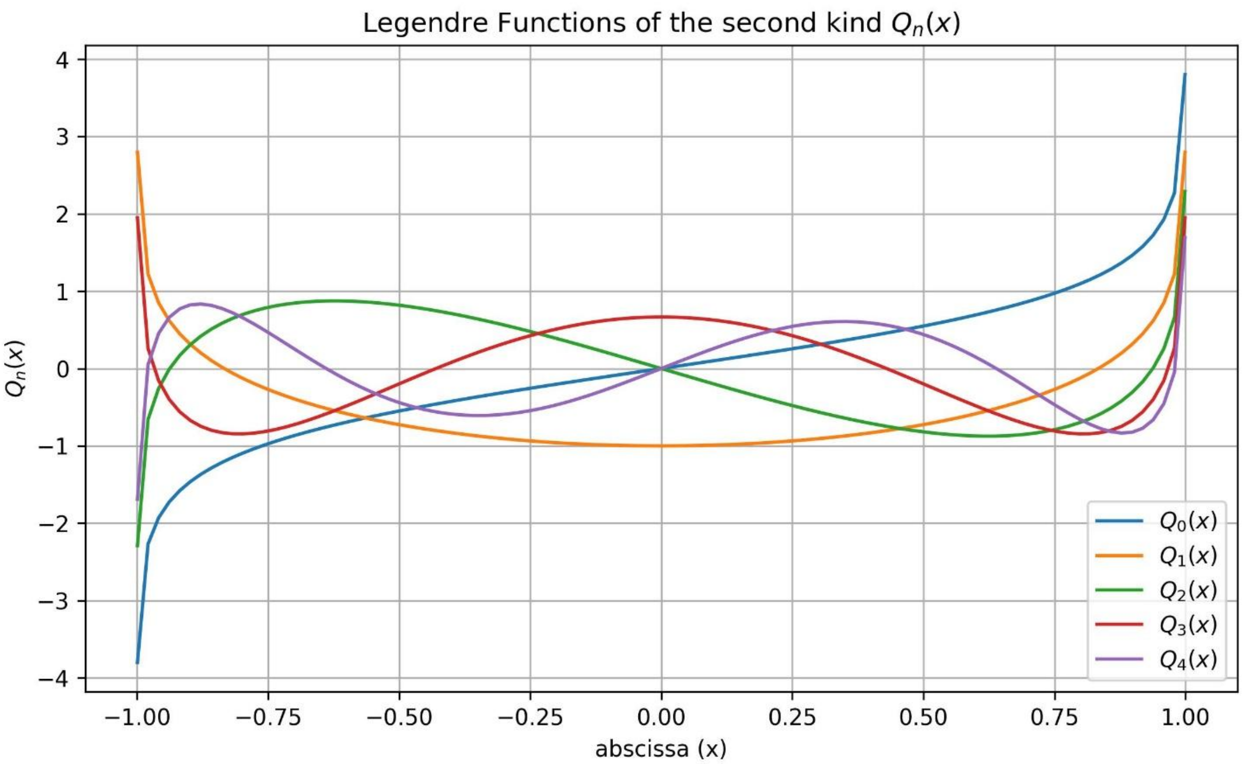

Figure 2. First five Legendre functions of the second kind.

```
# file name: Recursive_Legendre_Poly_2.py
#
# This script derives the values of the first n (specified by the user)
# Legendre Functions of the second kind and plot them
#
import os # functions for interacting with the operating system
import pandas as pd # dataframes similar to R
```

```
import matplotlib.pyplot as plt # a plotting library
import math # mathematical functions
import numpy as np # arrays and matrices
#
# Change the drectory, to the folder of the script
#
script_dir = os.path.dirname(os.path.abspath(__file__))
print(f"script_dir: {script_dir}")
os.chdir(script_dir)
#
# Number of Legendre Functions of the second kind to calculate, and number of values of x
#
N = 5 # first n+1 Legendre Functions of the second kind (n>1)
r = 100 # number of values for x between -1 and 1
eps = 1.0e-3 # small number to avoid singularities at x = +/- 1
x = np.linspace(-1+eps,1-eps, r) # x values between -1 and 1
#
# Legendre Functions of the second kind are stored in a data frame such that column n is Q_n(x)
#
mydata = pd.DataFrame(0.0, index=range(r), columns=range(N))
#
# Store Q_0 and Q_1, 1 and x respectively
#
for i in range(r):
    mydata.at[i,0] = 0.5*math.log((1+x[i])/(1-x[i])) # Q_0(x)
    mydata.at[i,1] = x[i]*mydata.at[i,0]-1 # Q_1(x)
#
# Calculate values from Q_2 to Q_n using the recursive relation
#
for n in range(1,N):
    print(f"Calculating Q_{n+1}(x)")
    for i in range(r):
        Q1 = mydata.at[i,n-1] # Q_(n-1)(x)
        Q2 = mydata.at[i,n] # Q_(n)(x)
        Q3 = ((2*n+1)*x[i]*Q2 - n*Q1)/(n+1) # Q_(n+1)(x)
        mydata.at[i,n+1] = Q3 # store Q_(n+1)(x)
#
# Plot
#
plt.figure(figsize=(8, 5))
for n in range(0,N):
    Pn = mydata[n]
    plt.plot(x, Pn, label=f'$Q_{n}(x)$')
plt.xlabel('abscissa (x)')
plt.ylabel('$Q_n(x)$')
plt.title('Legendre Functions of the second kind $Q_n(x)$')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.savefig('Legendre_Functions of the second kind.jpg', dpi=300)
plt.close() # Close the current figure
```

PROP. 11 (*Solution of Legendre differential equation*). With the introduction of the Legendre function of the second kind, we can now write the general integral of the Legendre differential equation (Eq. 21) in terms of $P_n(x)$ and $Q_n(x)$:

Eq. 32 $$y = K_1 P_n(x) + K_2 Q_n(x)$$

4 Orthogonality of Legendre polynomials

PROP. 12 (*Orthogonality*). Legendre polynomials are orthogonal in $[-1,1]$:

Eq. 33 $\int_{-1}^{1} P_m(x)P_n(x)dx = 0\,, \quad m \neq n$

Proof. We know that Legendre polynomials are solutions of Legendre's equation, therefore, we can write:

$$\begin{cases}(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0\\(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0\end{cases}$$

We multiply the first equation by $P_m(x)$ and the second by $P_n(x)$, then we subtract the second equation from the first one:

$$(1-x^2)(P_n''P_m - P_m''P_n) - 2x(P_n'P_m - P_m'P_n) + \big(n(n+1) - m(m+1)\big)P_nP_m = 0$$

We recognise now that $P_n''P_m - P_m''P_n = \frac{d}{dx}(P_n'P_m - P_nP_m')$, therefore, we have

$$(1-x^2)\frac{d}{dx}(P_n'P_m - P_nP_m') - 2x(P_n'P_m - P_m'P_n) + \big(n(n+1) - m(m+1)\big)P_nP_m = 0 \Rightarrow$$

$$\Rightarrow \frac{d[(P_n'P_m - P_nP_m')(1-x^2)]}{dx} = \big(m(m+1) - n(n+1)\big)P_nP_m$$

We now integrate between -1 and 1 both members, and we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{\big((P_n'P_m - P_nP_m')(1-x^2)\big)_{-1}^{1}}{m(m+1) - n(n+1)} = 0$$

This proves the theorem ■

PROP. 13. When $m = n$ we have

Eq. 34 $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$

Proof. From the generating function (Eq. 4), we have

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_m(x)P_n(x)t^{n+m}$$

If we integrate both hands of the equation, by also considering the orthogonality of the Legendre polynomials, we find that:

$$\int_{-1}^{1}\frac{1}{1-2xt+t^2}dx = \int_{-1}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_m(x)P_n(x)t^{n+m}\,dx =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^{1} P_m(x) P_n(x) t^{n+m} dx = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x) dx$$

Let us now consider the substitution $\xi = 1 + t^2 - 2xt \Leftrightarrow x = -\frac{\xi - 1 - t^2}{2t}$ in the integral of the left hand:

$$\int_{-1}^{1} \frac{1}{1 - 2xt + t^2} dx = -\frac{1}{2t} \int_{1+t^2+2t}^{1+t^2-2t} \frac{1}{\xi} d\xi = -\frac{1}{2t} \ln(\xi) \Big|_{1+t^2+2t}^{1+t^2-2t} =$$

$$= -\frac{1}{2t} \ln\left(\frac{(1-t)^2}{(1+t)^2}\right) = \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$$

By the expansion in Eq. 28, we have $\frac{1}{t} \ln\left(\frac{1+t}{1-t}\right) = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}$, therefore we can write

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x) dx = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} \Rightarrow \int_{-1}^{1} P_n^2(x) dx = \frac{2}{2n+1}$$

This proves the theorem ■

PROP. 14 (*Expansion by Legendre polynomials*). If $f(x)$ and its derivative have a finite number of discontinuities in $[-1,1]$, then wherever they are continuous functions, we have

Eq. 35 $$\begin{cases} f(x) = \sum_{k=0}^{\infty} A_k P_k(x) \\ A_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx \end{cases}$$

while in the points of discontinuity, we have $\frac{f(x-0)+f(x+0)}{2} = \sum_{k=0}^{\infty} A_k P_k(x)$.

Proof. We derive here the expression of the generalised Fourier coefficients A_k assuming that the expansion holds:

$$f(x) = \sum_{k=0}^{\infty} A_k P_k(x) \Rightarrow \int_{-1}^{1} f(x) P_m(x) dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_k(x) P_m(x) dx = A_m \int_{-1}^{1} P_m^2(x) dx$$

Since $\int_{-1}^{1} P_m^2(x) dx = \frac{2}{2m+1}$ (Eq. 34), we find $A_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx$ ■

Example 2 (*first expansion*). Let us search now for the expansion in terms of Legendre polynomials of the following function:

Eq. 36 $$f(x) = \begin{cases} 1 & x \in]0,1] \\ 0 & x \in [-1,0[\end{cases}$$

The calculation of the Fourier generalised coefficients is straightforward:

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx = \frac{2n+1}{2} \int_{0}^{1} P_n(x) dx =$$

$$= \frac{2n+1}{2^{n+1}} \sum_{k=0}^{E\left(\frac{n}{2}\right)} \left((-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-2k)!} \int_0^1 x^{n-2k} dx \right)$$

where we used Eq. 1. Since $\int_0^1 x^{n-2k} dx = \left.\frac{x^{n-2k+1}}{n-2k+1}\right|_0^1 = \frac{1}{n-2k+1}$, the generalized Fourier coefficients are

Eq. 37 $\qquad A_n = \frac{2n+1}{2^{n+1}} \sum_{k=0}^{E\left(\frac{n}{2}\right)} \left((-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k+1)!} \right), \quad n = 0, 1, 2, \dots$

By applying this formula, we have

Eq. 38 $\qquad f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \frac{11}{32} P_5(x) - \cdots$

In Figure 3, the approximation of $f(x)$, using the first 8 addends of the expansion (including the null ones).

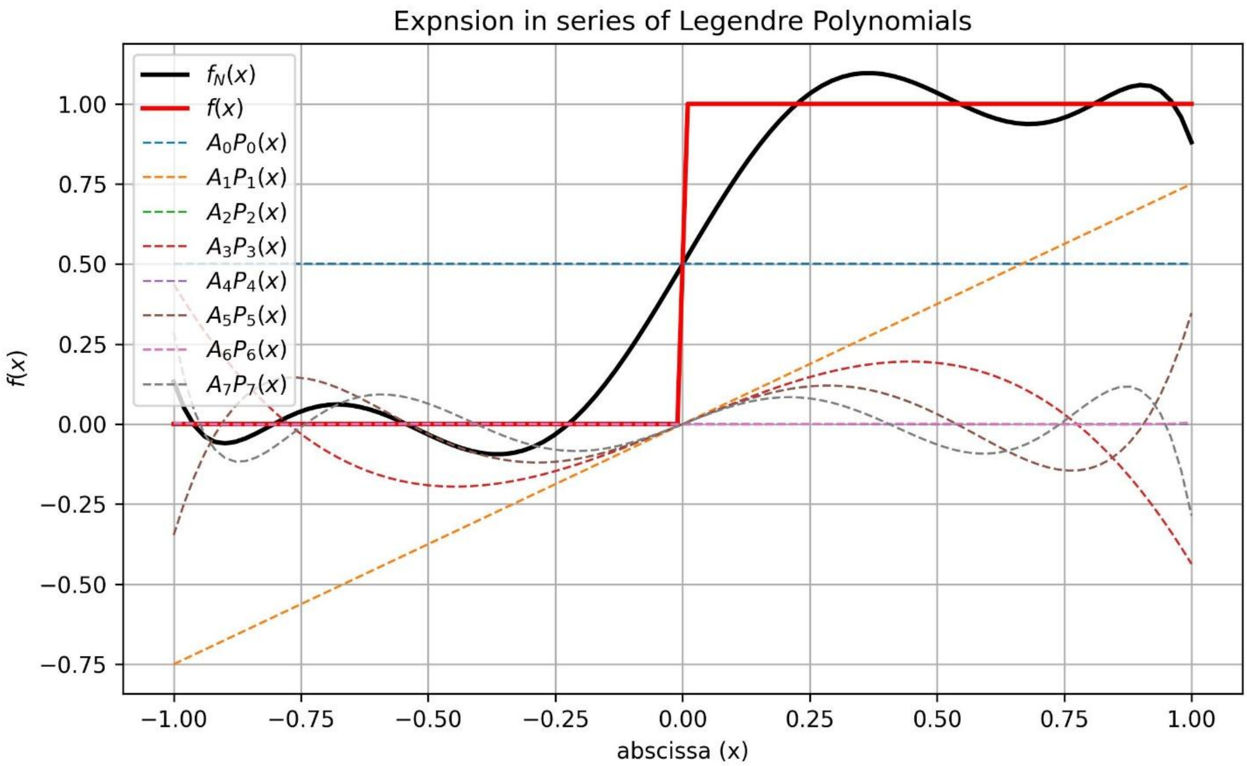

Figure 3. Expansion of the step function (Eq. 35) using the first 8 addends of ***Error! Reference source not found.***.

CODE 3 (*first expansion*). The following Python script calculates the approximation of Eq. 38 using the first 9 Legendre polynomials. It calculates the generalized Fourier coefficients by numeric integration of $A_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$. The function to be approximated (the step function, in this case) is defined within a Python function, therefore this script can be easily adapted to any other function. It plots Figure 3.

```
# file name: Recursive_Legendre_Expansion_1.py
#
```

```
# Derive and plot the expansion in Legendre polynomials of the step function
#
import os # functions for interacting with the operating system
import pandas as pd # dataframes similar to R
import matplotlib.pyplot as plt # a plotting library
import math # mathematical functions
import numpy as np # arrays and matrices
#
# Change the drectory, to the folder of the script
#
script_dir = os.path.dirname(os.path.abspath(__file__))
print(f"script_dir: {script_dir}")
os.chdir(script_dir)
#
# Number of Legendre Polynomials to calculate, and number of values of x
#
N = 8 # first n+1 Legendre Polynomials (n>1)
r = 100 # number of values for x between -1 and 1
#
# Legendre Polynomials are stored in a data frame such that column n is P_n(x)
#
mydata = pd.DataFrame(0.0, index=range(r), columns=range(N))
#
# Store P_0 and P_1, 1 and x respectively
#
for i in range(r):
    mydata.at[i,0] = 1.0    # P_0(x) = 1
mydata[1] = np.linspace(-1, 1, r) # P_1(x) = x
#
# Calculate values from P_2 to P_n using the recursive relation
#
x = mydata[1] # x values
for n in range(1,N-1):
    print(f"Calculating P_{n+1}(x)")
    for i in range(r):
        P1 = mydata.at[i,n-1] # P_(n-1)(x)
        P2 = mydata.at[i,n] # P_(n)(x)
        P3 = ((2*n+1)*x[i]*P2 - n*P1)/(n+1) # P_(n+1)(x)
        mydata.at[i,n+1] = P3 # store P_(n+1)(x)
#
# Function f(x) to be expanded
#
def get_f(x):
    """Function to be expanded in Legendre Polynomials"""
    myfunction = pd.DataFrame(0.0, index=range(r), columns=range(1))
    for i in range(r):
        if x[i] < 0.0:
            myfunction.at[i,0] = 0.0
        else:
            myfunction.at[i,0] = 1.0
    return myfunction
#
# Calculate generalized Fourier coefficients a_n
#
mycoeff = pd.DataFrame(0.0, index=range(N), columns=range(1))
for i in range(N):
    Pn = mydata[i] # P_n(x)
    integrand = pd.DataFrame(0.0, index=range(r), columns=range(1))
```

```
    for j in range(r):
        integrand.at[j,0] = Pn[j]*get_f(x).at[j,0] # f(x)*P_n(x)
    mycoeff.at[i,0] = (2*i+1)/2 * np.trapezoid(integrand[0], x) # integral using trapezoidal rule
    print(f"a_{i} = {mycoeff.at[i,0]}")
#
# Build the interpolated function f_N(x)
#
f_N = pd.DataFrame(0.0, index=range(r), columns=range(1))
for i in range(N):
    Pn = mydata[i] # P_n(x)
    An = mycoeff.at[i,0] # a_n
    for j in range(r):
        f_N.at[j,0] += An*Pn[j] # f_N(x) += a_n*P_n(x)
#
# Plot
#
plt.figure(figsize=(8, 5))
plt.plot(x, f_N[0], label='$f_N(x)$', color='black', linewidth=2) # interpolation
plt.plot(x, get_f(x)[0], label='$f(x)$', color='red', linewidth=2) # original function
for n in range(0,N):
    Pn = mydata[n]
    An = mycoeff.at[n,0]
    plt.plot(x, An*Pn, label=f'$A_{n}P_{n}(x)$', linewidth=1, linestyle='--') # components
plt.xlabel('abscissa (x)')
plt.ylabel('$f(x)$')
plt.title('Expnsion in series of Legendre Polynomials')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.savefig('Legendre_Expansion_1.jpg', dpi=300)
plt.close() # Close the current figure
```

Example 3 (*another expansion*). Let us search now for the expansion in terms of Legendre polynomials of the following function:

Eq. 39 $$g(x) = \begin{cases} 0 & x \in \,]0,1] \\ -1 & x \in [-1,0[\end{cases}$$

We note that $g(x) = f(x) - 1$, with $f(x)$ given by Eq. 36, therefore, we have

Eq. 40 $$g(x) = -\frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) - \cdots$$

Example 4 (*another expansion*). Let us search now for the expansion in terms of Legendre polynomials of the following function:

Eq. 41 $$v(x) = \begin{cases} 1 & x \in \,]0,1] \\ -1 & x \in [-1,0[\end{cases}$$

Since $v(x) = f(x) + g(x)$ (see Eq. 36 and Eq. 40) we have

Eq. 42 $$v(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \cdots$$

More generally, we have the following coefficients:

Eq. 43 $$A_n = \frac{2n+1}{2^n} \sum_{k=0}^{E\left(\frac{n}{2}\right)} \left((-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k+1)!} \right), \quad n = 1, 2, \ldots$$

I used a script similar to CODE 3 to plot the approximation of $v(x)$ in Figure 4.

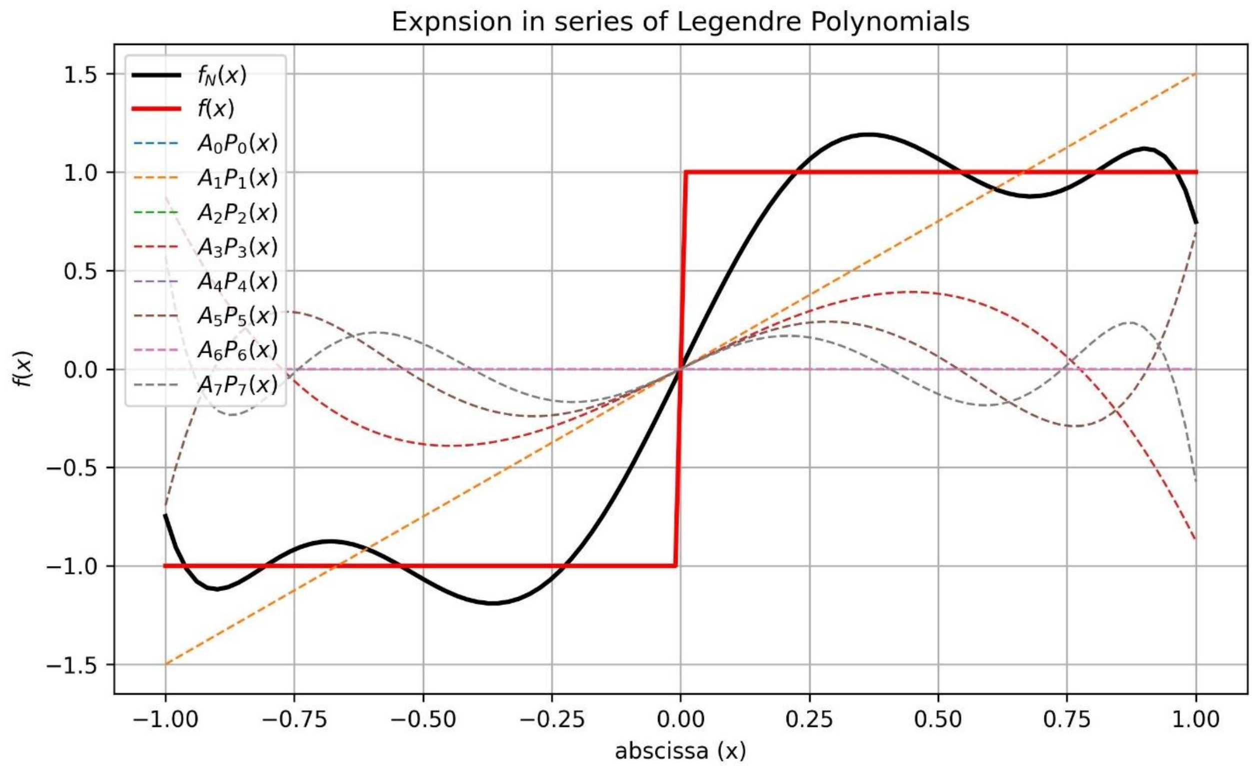

Figure 4. Expansion of the step function (Eq. 41) using the first 8 addends.

Example 5 (*power of x*). Let us search now for the expansion in terms of Legendre polynomials of the following function $f(x) = x^n$. For $n = 1$ we have $f(x) = P_1(x)$ and more generally we can say that $f(x) = \sum_{k=1}^{n} A_k P_k(x)$, because otherwise we would have a polynomial of higher order on the right-hand side. For $n = 2$ we have

$$x^2 = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x)$$

Since $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, we have

$$x^2 = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) = A_0 + A_1 x + \frac{3}{2}x^2 A_2 - \frac{1}{2}A_2 = A_0 - \frac{A_2}{2} + A_1 x + \frac{3}{2}A_2 x^2$$

It must also be true that:

$$\begin{cases} A_0 - \dfrac{A_2}{2} = 0 \\ A_1 = 0 \\ \dfrac{3}{2}A_2 = 1 \end{cases} \Rightarrow \begin{cases} A_0 = \dfrac{1}{3} \\ A_1 = 0 \\ A_2 = \dfrac{2}{3} \end{cases}$$

Therefore, we have found:

Eq. 44 $\qquad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$

To calculate the expansion of x^3, we note that the recursion formula (PROP. 3) can be written

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$$

Therefore, by multiplying by *x* both hands of Eq. 44 we have

$$x^3 = \frac{1}{3}xP_0(x) + \frac{2}{3}xP_2(x) = \frac{1}{3}P_1(x) + \frac{2}{3}\left(\frac{3}{5}P_3(x) + \frac{2}{5}P_1(x)\right) =$$

$$= \frac{1}{3}P_1(x) + \frac{2}{5}P_3(x) + \frac{4}{15}P_1(x) = \left(\frac{1}{3} + \frac{4}{15}\right)P_1(x) + \frac{2}{5}P_3(x) \Rightarrow$$

Eq. 45 $\qquad x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$

For x^4, we can apply the same method, and we find

Eq. 46 $\qquad x^4 = \frac{1}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x)$

More generally, is it possible to show that

Eq. 47 $\qquad x^n = \sum_{k=0}^{E\left(\frac{h}{2}\right)} 2^{n-2k}(2n - 4k + 1)\frac{(n-k)!n!}{k!(2n-2k+1)!}P_{n-2k}(x)$

5 Applications in physics

Example 6 (*gravitational potential of a ring*). Let us consider a ring with a radius *a* and a constant linear density of mass of σ (Figure 5). We are interested in calculating the gravitational potential generated by the ring.

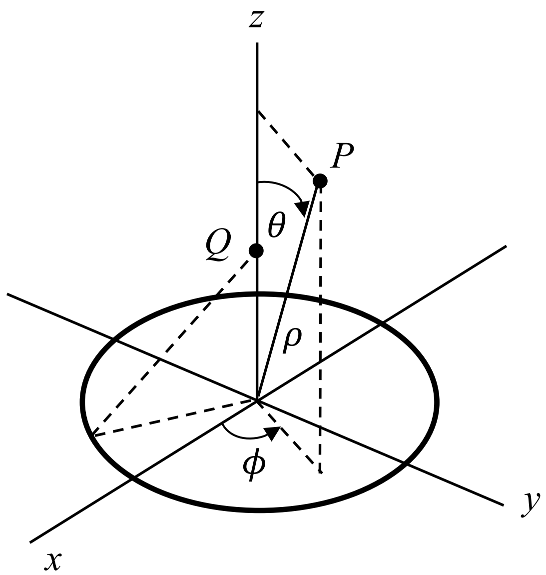

Figure 5. Reference for Example 6.

We know, from Mechanics, that the Laplacian of the gravitational potential V is zero in space that is not occupied by mass. Therefore, we must solve the differential equation in Eq. 8, which, because of the symmetry of the problem with respect to z, can be written:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

According to Eq. 15, this differential equation is equivalent to the following system:

$$\begin{cases} R(\rho)=C_1\rho^n+C_2\rho^{-(n+1)} \\ \dfrac{d\left(\dfrac{dT(\xi)}{d\xi}(1-\xi^2)\right)}{d\xi}+n(n+1)T(\xi)=0 \end{cases},\quad \xi=\cos\theta$$

According to PROP. 11, the general integral of the second equation is

$$T(\xi)=K_1P_n(\xi)+K_2Q_n(\xi)$$

I conclusion, the gravitational potential of the ring is

$$V(\rho,\theta,\phi)=R(\rho)T(\cos\theta)=(K_1P_n(\cos\theta)+K_2Q_n(\cos\theta))\big(C_1\rho^n+C_2\rho^{-(n+1)}\big)$$

We note that $V(\rho,\theta,\phi)$ is limited, therefore K_2 must be zero ($Q_n(\cos\theta)$ is unlimited). Therefore, we have

Eq. 48 $$V(\rho,\theta,\phi)=\big(C_1\rho^n+C_2\rho^{-(n+1)}\big)P_n(\cos\theta)$$

Now, let us consider the case $\rho\le a$: it must be $C_2=0$, otherwise the potential would be unlimited on the z-axis. Therefore, the general integral for $\rho\le a$ is

Eq. 49 $$V(\rho,\theta,\phi)=\textstyle\sum_{n=0}^{\infty}A_n\rho^nP_n(\cos\theta),\quad \rho\le a$$

What can we use as boundary conditions? We note that the calculation of the gravitational potential of the points on the z-axis (like point Q in Figure 5) is straightforward:

$$V(Q)=-G\frac{2\pi\sigma a}{\sqrt{a^2+z_Q^2}}\Rightarrow V(\rho,\phi,\theta=0)=-G\frac{2\pi\sigma a}{\sqrt{a^2+\rho^2}}$$

We now ask that $V(\rho,\theta,\phi)$ equals $G\frac{2\pi\sigma a}{\sqrt{a^2+\rho^2}}$ for $\theta=0$, remembering that $P_n(\cos 0)=P_n(1)=1$ (PROP. 4):

$$\sum_{n=0}^{\infty}A_n\rho^n=-G\frac{2\pi\sigma a}{\sqrt{a^2+\rho^2}}=-G\frac{2\pi\sigma}{\sqrt{1+\left(\frac{\rho}{a}\right)^2}}\Rightarrow\left(1+\left(\frac{\rho}{a}\right)^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{A_n}{2\pi\sigma G}\rho^n$$

On the other hand, the binomial formula gives

$$\left(1+\left(\frac{\rho}{a}\right)^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{\rho}{a}\right)^{2n}$$

And this means that, by comparison, $A_{2n} = -\frac{2\pi\sigma G}{a^{2n}}\binom{-\frac{1}{2}}{n}$. Therefore, Eq. 49 becomes

Eq. 50 $$V(\rho,\theta,\phi) = -2\pi\sigma G \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{\rho^{2n}}{a^{2n}} P_{2n}(\cos\theta), \quad \rho < a$$

For $\rho > a$, we must assume $C_1 = 0$ in Eq. 48, otherwise the potential would tend to infinity for ρ that goes to infinite. So, we have the general integral:

Eq. 51 $$V(\rho,\theta,\phi) = \sum_{n=0}^{\infty} B_n \rho^{-(n+1)} P_n(\cos\theta), \quad \rho > a$$

As a boundary condition, we now ask that Eq. 50 and Eq. 51 give the same result for $\rho \to a$:

$$-2\pi\sigma G \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} P_{2n}(\cos\theta) = \sum_{n=0}^{\infty} \frac{B_{2n}}{a^{2n+1}} P_{2n}(\cos\theta) \Leftrightarrow B_{2n} = -2\pi\sigma G \binom{-\frac{1}{2}}{n} a^{2n+1}$$

We can now write the complete solution for the gravitational potential:

Eq. 52 $$\begin{cases} V(\rho,\theta,\phi) = -2\pi\sigma G \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{\rho^{2n}}{a^{2n}} P_{2n}(\cos\theta), & \rho < a \\ V(\rho,\theta,\phi) = -2\pi\sigma G \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{a^{2n+1}}{\rho^{2n+1}} P_{2n}(\cos\theta), & \rho > a \end{cases}$$

We note now that

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\dots}{n!\left(-\frac{1}{2}-n\right)!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!} =$$

$$= (-1)^n \frac{\left(\frac{1}{2}\right)\left(\frac{1+2}{2}\right)\left(\frac{1+4}{2}\right)\left(\frac{1+6}{2}\right)\dots\left(\frac{1+2n-2}{2}\right)}{n!} = \frac{(-1)^n}{2^n}\frac{1\cdot 3\cdot 5\cdot 7\cdot \dots \cdot (2n-1)}{n!} =$$

$$= \frac{(-1)^n}{2^n}\frac{(2n-1)!!}{n!}$$

By multiplying the numerator and denominator by $(2n)!! = 2^n n!$ we have:

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n}{2^n}\frac{(2n-1)!!\,(2n)!!}{n!\,(2n)!!} = \frac{(-1)^n}{2^n}\frac{(2n)!}{n!\,(2n)!!} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

The potential in Eq. 52 can therefore be written:

Eq. 53 $$\begin{cases} V(\rho,\theta,\phi) = -2\pi\sigma G \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\frac{\rho^{2n}}{a^{2n}}P_{2n}(\cos\theta), & \rho \le a \\ V(\rho,\theta,\phi) = -2\pi\sigma G \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\frac{a^{2n+1}}{\rho^{2n+1}}P_{2n}(\cos\theta), & \rho > a \end{cases}$$

If we expand the first addends of this sum for $\rho \le a$, we have

$$\frac{|V(\rho,\theta,\phi)|}{2\pi\sigma G} = P_0(\cos\theta) - \frac{1}{2}\frac{\rho^2}{a^2}P_2(\cos\theta) + \frac{3}{8}\frac{\rho^4}{a^4}P_{2n}(\cos\theta) - \frac{5}{16}\frac{\rho^6}{a^6}P_6(\cos\theta) + \cdots$$

On the other hand, for $\rho > a$ we have

$$\frac{|V(\rho,\theta,\phi)|}{2\pi\sigma G} = \frac{a}{\rho}P_0(\cos\theta) - \frac{1}{2}\frac{a^3}{\rho^3}P_2(\cos\theta) + \frac{3}{8}\frac{a^5}{\rho^5}P_{2n}(\cos\theta) - \frac{5}{16}\frac{a^7}{\rho^7}P_6(\cos\theta) + \cdots$$

The gravitational potential and the corresponding equipotential curves on the plane *y-z* are plotted in Figure 6 (in absolute value). Note the singularity of the gravitational potential in the space occupied by the ring.

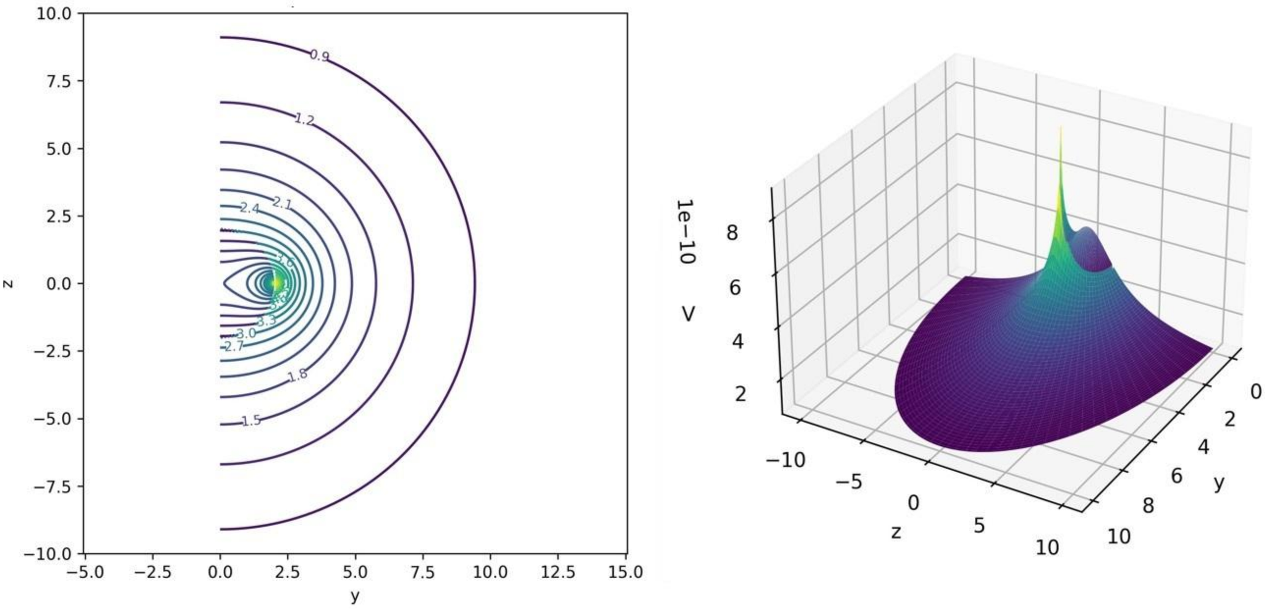

Figure 6. On the left are the equipotential curves on the plane y-z for the gravitational potential of the ring (right). Radius of the ring $a = 2m$ and a linear density $\sigma = 1\ kg/m$. We used the first 100 polynomials of the series. This is the absolute value of the potential.

CODE 4 (*first expansion*). This Python script uses Eq. 53 to calculate the gravitational potential of a ring of radius *a* specified by the user as a function of the points of the plane *y-z*. It also derives the equipotential curves on the same plane. To compute the first $2N$ Legendre polynomials (with N specified by the user), it uses pieces of CODE 1.

```
# file name: Recursive_Legendre_Gravity.py
#
# This script plots the gravitational potential of a ring mass
#
import os # functions for interacting with the operating system
import pandas as pd # data frames similar to R
import matplotlib.pyplot as plt # a plotting library
```

```
from mpl_toolkits.mplot3d import Axes3D  # noqa: F401 (needed for 3D)
import math # mathematical functions
import numpy as np # arrays and matrices
#
# Change the drectory, to the folder of the script
#
script_dir = os.path.dirname(os.path.abspath(__file__))
print(f"script_dir: {script_dir}")
os.chdir(script_dir)
#
# Parameters of the ring
#
a = 2.0 # radius of the ring in meters
sigma = 1.0 # mass per unit length of the ring in kg/m
G = 6.67430e-11 # gravitational constant in m^3 kg^-1 s^-2
#
# Number of Legendre Polynomials to calculate, and values of xi = cos(theta)
#
N = 50 # first n+1 Legendre Polynomials (n>1)
r = 100 # number of values for theta between 0 and 2*pi, and for rho between 0 and a
theta = np.linspace(0, math.pi, r) # theta values between 0 and 2*pi
xi = np.cos(theta) # xi = cos(theta)
#
# A function that computes a multiplicative coefficient for the potential
#
def Coeff(n):
    """Compute the multiplicative coefficient for the potential"""
    Mul = ((-1)**n)*math.factorial(2*n)/(2**(2*n)*(math.factorial(n))**2)
    return Mul
#
# Legendre Polynomials are stored in a data frame such that column n is P_n(x)
#
mydata = pd.DataFrame(0.0, index=range(r), columns=range(2*N))
#
# Store P_0 and P_1
#
for i in range(r):
    mydata.at[i,0] = 1.0    # P_0(x) = 1
#
for i in range(r):
    mydata.at[i,1] = xi[i]         # P_1(x) = xi
#
# Calculate values from P_2 to P_n using the recursive relation
#
for n in range(1,2*N-1):
    print(f"Calculating P_{n+1}(x)")
    for i in range(r):
        P1 = mydata.at[i,n-1] # P_(n-1)(x)
        P2 = mydata.at[i,n] # P_(n)(x)
        P3 = ((2*n+1)*xi[i]*P2 - n*P1)/(n+1) # P_(n+1)(x)
        mydata.at[i,n+1] = P3 # store P_(n+1)(x)
#
# Define the gravitational potential of a ring on plane y-z, as a function of theta and rho
#
# Case rho <= a
#
rho1 = np.linspace(0, a, r)
V1 = np.zeros((r,r)) # potential initialization
```

```
for i in range(r):
    for j in range(r):
        for n in range(N):
            V1[i,j] += 2.0*math.pi*G*sigma*Coeff(n)*mydata.at[j,2*n]*(rho1[i]/a)**(2*n)
#
# Case rho > a
#
rho2 = np.linspace(a, 5*a, r)
V2 = np.zeros((r,r)) # potential initialization
for i in range(r):
    for j in range(r):
        for n in range(N):
            V2[i,j] += 2.0*math.pi*G*sigma*Coeff(n)*mydata.at[j,2*n]*(a/rho2[i])**(2*n+1)
#
# Plot the potential as a surface in 3D on plane Y-Z
#
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
#
# Set mask for singularity at rho=a, theta=pi/2
#
delta_Rho = a/50
delta_Theta = math.pi/12
#
# Plot V1
#
Theta, Rho = np.meshgrid(theta, rho1) # grid for theta (column) and rho (row)
# mask = ((np.abs(Rho-a)<delta_Rho)&(np.abs(Theta-math.pi/2)<delta_Theta)) # mask for singularity at rho=a, theta=pi/2
V1[mask] = np.nan # set singularity to NaN
Z = Rho * np.cos(Theta) # convert to cartesian coordinates
Y = Rho * np.sin(Theta) # convert to cartesian coordinates
surf = ax.plot_surface(Y,Z,V1, cmap='viridis')
#
# Plot V2
#
Theta, Rho = np.meshgrid(theta, rho2) # grid for theta (column) and rho (row)
# mask = ((np.abs(Rho-a)<delta_Rho)&(np.abs(Theta-math.pi/2)<delta_Theta)) # mask for singularity at rho=a, theta=pi/2
V2[mask] = np.nan # set singularity to NaN
Z = Rho * np.cos(Theta) # convert to cartesian coordinates
Y = Rho * np.sin(Theta) # convert to cartesian coordinates
surf = ax.plot_surface(Y,Z,V2, cmap='viridis')
#
# Axes, labels, view angle
#
ax.set_xlabel('y')
ax.set_ylabel('z')
ax.set_zlabel('V')
ax.view_init(elev=30, azim=30) # azim is rotation around vertical axis
plt.savefig('Ring_potential.jpg', dpi=300)
plt.close() # Close the current figure
#
# Plot equipotential curves on plane Y-Z
#
plt.figure(figsize=(6, 6))
#
# V1
```

```
#
Theta, Rho = np.meshgrid(theta, rho1)
Y = Rho * np.sin(Theta)
Z = Rho * np.cos(Theta)
levels = 20  # number of equipotential curves
cs = plt.contour(Y, Z, V1,levels=levels,cmap='viridis')
#
# V2
#
Theta, Rho = np.meshgrid(theta, rho2)
Y = Rho * np.sin(Theta)
Z = Rho * np.cos(Theta)
levels = 20  # number of equipotential curves
cs = plt.contour(Y, Z, V2,levels=levels,cmap='viridis')
#
# Labels and title
#
plt.clabel(cs, inline=True, fontsize=8)
plt.xlabel('y')
plt.ylabel('z')
plt.title('equipotential curves')
plt.axis('equal')
plt.savefig('Ring_equipotential_curves.jpg', dpi=300)
plt.close() # Close the current figure
```

NOTE 6 (*alternative calculation*). We can also derive the gravitational potential from the expression derived from the gravitational field (the potential is the differential form of the gravitational field).

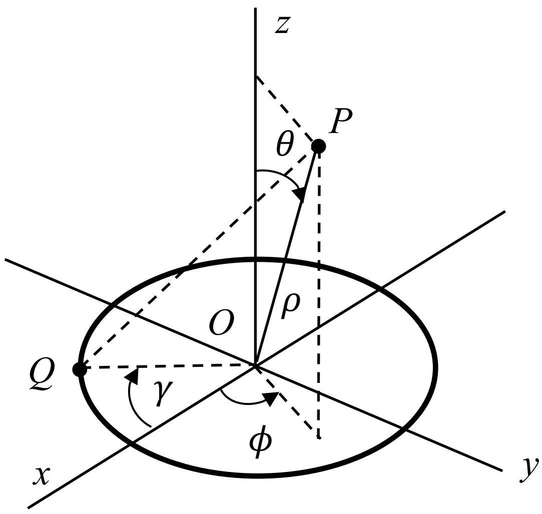

Figure 7. Reference for the direct calculation of the gravitational potential in P.

If we consider Figure 7 we can write the contribution to the gravitational potential in P due to the elementary mass in Q as follows:

$$dV(\rho,\phi,\theta) = -G\sigma\frac{d\gamma}{|\overrightarrow{QP}|} = -G\sigma\frac{d\gamma}{|\overrightarrow{QO}+\overrightarrow{OP}|}$$

We also know that

$$\overrightarrow{QO}+\overrightarrow{OP}=-\begin{bmatrix}a\cos\gamma\\a\sin\gamma\\0\end{bmatrix}+\begin{bmatrix}\rho\sin\theta\cos\phi\\\rho\sin\theta\sin\phi\\\rho\cos\theta\end{bmatrix}=\begin{bmatrix}\rho\sin\theta\cos\phi-a\cos\gamma\\\rho\sin\theta\sin\phi-a\sin\gamma\\\rho\cos\theta\end{bmatrix}$$

Therefore, we have

$$\left|\overrightarrow{QO}+\overrightarrow{OP}\right|^2=\rho^2\sin^2\theta\cos^2\phi+a^2\cos^2\gamma-2a\rho\sin\theta\cos\phi\cos\gamma+$$

$$+\rho^2\sin^2\theta\sin^2\phi+a^2\sin^2\gamma-2a\rho\sin\theta\sin\phi\sin\gamma+\rho^2\cos^2\theta=$$

$$=\rho^2+a^2-2a\rho\sin\theta\,(\cos\phi\cos\gamma+\sin\phi\sin\gamma)$$

Since the potential must have symmetry with respect to *z,* we can chose an arbitrary ϕ, zero for instance, and we have

$$dV(\rho,\theta)=-G\sigma a\frac{d\gamma}{\sqrt{\rho^2+a^2-2a\rho\sin\theta\cos\gamma}}=-\frac{G\sigma a}{\sqrt{\rho^2+a^2}}\frac{d\gamma}{\sqrt{1-\frac{2a\rho}{\rho^2+a^2}\sin\theta\cos\gamma}}$$

Then, by integrating for Q that moves across the ring, we have

$$V(\rho,\theta)=-G\sigma a\int_0^{2\pi}\frac{d\gamma}{\sqrt{\rho^2+a^2-2a\rho\sin\theta\cos\gamma}}=-2G\sigma a\int_0^{\pi}\frac{d\gamma}{\sqrt{\rho^2+a^2-2a\rho\sin\theta\cos\gamma}}$$

We know from trigonometry that $2\cos^2\frac{\gamma}{2}-1=\cos\gamma$, therefore we have

$$V(\rho,\theta)=-2G\sigma a\int_0^{\pi}\frac{d\gamma}{\sqrt{\rho^2+a^2-2a\rho\sin\theta\left(2\cos^2\frac{\gamma}{2}-1\right)}}=$$

$$=-2G\sigma a\int_0^{\pi}\frac{d\gamma}{\sqrt{\rho^2+a^2+2a\rho\sin\theta-4a\rho\sin\theta\cos^2\frac{\gamma}{2}}}=$$

$$=-\frac{2G\sigma a}{\sqrt{\rho^2+a^2+2a\rho\sin\theta}}\int_0^{\pi}\frac{d\gamma}{\sqrt{1-\frac{4a\rho\sin\theta}{\rho^2+a^2+2a\rho\sin\theta}\cos^2\frac{\gamma}{2}}}$$

We operate the substitution $\omega=\frac{\gamma}{2}$ and the integral becomes:

$$V(\rho,\theta)=-\frac{4G\sigma a}{\sqrt{\rho^2+a^2+2a\rho\sin\theta}}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{1-\frac{4a\rho\sin\theta}{\rho^2+a^2+2a\rho\sin\theta}\cos^2\omega}}$$

We consider the substitution $\omega=\frac{\pi}{2}-\beta$ and we have

$$V(\rho,\theta) = -\frac{4G\sigma a}{\sqrt{\rho^2 + a^2 + 2a\rho\sin\theta}}\int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - \frac{4a\rho\sin\theta}{\rho^2 + a^2 + 2a\rho\sin\theta}\sin^2\beta}}$$

Now, we observe that with the position $\frac{\rho}{a} = \tan\alpha$ we have

$$\frac{4\frac{\rho}{a}\sin\theta}{\frac{\rho^2}{a^2} + 1 + 2\frac{\rho}{a}\sin\theta} \le \frac{4\frac{\rho}{a}\sin\theta}{\frac{\rho^2}{a^2} + 1} \le \frac{4\frac{\rho}{a}}{\frac{\rho^2}{a^2} + 1} = \frac{2\tan\alpha}{\tan^2\alpha + 1} = \sin\alpha \le 1$$

This means that the integral is a complete elliptic function of the first kind, and we can express the gravitational potential of the ring in a compact way:

Eq. 54 $$V(\rho,\theta) = -\frac{4G\sigma a}{\sqrt{\rho^2+a^2+2a\rho\sin\theta}} K\left(k = \sqrt{\frac{4a\rho\sin\theta}{\rho^2+a^2+2a\rho\sin\theta}}\right)$$

Example 7 (*thermic field*). We have a sphere of radius a such that the upper half of its surface $\left(\theta \in \left[0,\frac{\pi}{2}\right[\right)$ is kept at temperature T_0, while the lower half is at temperature $-T_0$. WE want to derive the thermic field inside the sphere, at equilibrium. The differential problem is

Eq. 55 $$\begin{cases} \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial f_T}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f_T}{\partial\theta}\right) = 0 \\ f_T(\rho = a,\phi,\theta) = T_0 \quad \in \left[0,\frac{\pi}{2}\right[\\ f_T(\rho = a,\phi,\theta) = -T_0 \quad \in \left]\frac{\pi}{2},\pi\right] \end{cases}$$

where f_T is the thermic field. We note that this problem does not depend from ϕ (it is symmetric with respect to *z*, use Figure 5). Also, the thermic field is limited, therefore we discard the solutions based on the Legendre functions of the second kind. In conclusion, the solution of this problem is of the following type:

$$f_T(\rho,\theta) = P_n(\cos\theta)\left(C_1\rho^n + \frac{C_2}{\rho^{1+n}}\right)$$

We note that, since the thermic field cannot go to infinity for ρ that goes to zero, it must be $C_2 = 0$. Then we can write the general integral as follows:

$$f_T(\rho,\theta) = \sum_{n=0}^{\infty} A_n\rho^n P_n(\cos\theta)$$

For $\rho = a$, the boundary conditions can be written

$$\sum_{n=0}^{\infty}\frac{A_n a^n}{T_0}P_n(\cos\theta) = \begin{cases} 1 & \xi > 0 \\ -1 & \xi < 0 \end{cases}$$

We have already studied this case in Example 4 and by using these results we have

$$A_n = \frac{2n+1}{2^n}\frac{T_0}{a^n}\sum_{k=0}^{E\left(\frac{n}{2}\right)}\left((-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-2k+1)!}\right), \quad n = 1, 2, \dots$$

In conclusion, the thermic field is

Eq. 56 $$f_T(\rho,\theta) = T_0 \sum_{n=1}^{\infty} \frac{2n+1}{2^n}\frac{\rho^n}{a^n}\sum_{k=0}^{E\left(\frac{n}{2}\right)}\left((-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k+1)!}\right) P_n(\cos\theta)$$

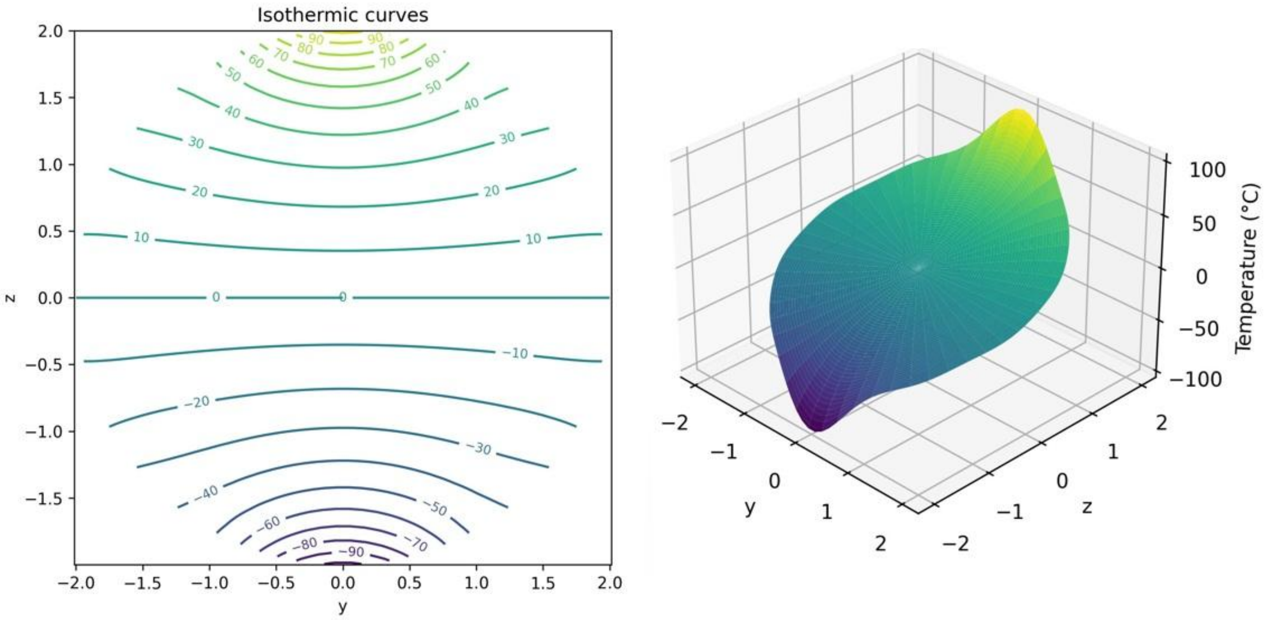

Figure 8. On the left are the isothermal curves on the plane y-z for the sphere of Example 7 (right). Radius of the sphere $a = 2m$. We used the first 10 polynomials of the series. Plotted by CODE 5.

CODE 5 (thermic field). This python script calculates and plots the thermic field for Example 7 by using the first N terms of Eq. 56.

```
# file name: Recursive_Legendre_Temperature.py
#
# This script plots the thermic profile inside a sphere with one hemisphere at T0 and
# the other hemisphere at -T0, using the recursive definition of Legendre Polynomials.
#
import os # functions for interacting with the operating system
import pandas as pd # dataframes similar to R
import matplotlib.pyplot as plt # a plotting library
from mpl_toolkits.mplot3d import Axes3D  # noqa: F401 (needed for 3D)
import math # mathematical functions
import numpy as np # arrays and matrices
#
# Change the drectory, to the folder of the script
#
script_dir = os.path.dirname(os.path.abspath(__file__))
print(f"script_dir: {script_dir}")
os.chdir(script_dir)
#
# Parameters of the sphere
#
```

```
a = 2.0 # radius of the ring in meters
T0 = 100 # temperature of one hemisphere in Celsius
#
# Number of Legendre Polynomials to calculate, and values of xi = cos(theta)
#
N = 10 # first n+1 Legendre Polynomials (n>1)
r = 100 # number of values for theta between 0 and 2*pi, and for rho between 0 and a
theta = np.linspace(0, 2*np.pi, r) # theta values between 0 and 2*pi
xi = np.cos(theta) # xi = cos(theta)
#
# A function that computes a multiplicative coefficient for the potential
#
def Coeff(n):
    """Compute the multiplicative coefficient for the potential"""
    #
    Mul = 0.0
    for k in range(0, (n//2)+1):
        Mul += ((-1)**k)*math.factorial(2*n-2*k)/(math.factorial(k)*math.factorial(n-k)*math.factorial(n-2*k+1))
    Mul = ((2*n+1.0)/2.0**(n+1))*Mul
    #
    return Mul
#
# Legendre Polynomials are stored in a data frame such that column n is P_n(x)
#
mydata = pd.DataFrame(0.0, index=range(r), columns=range(N))
#
# Store P_0 and P_1
#
for i in range(r):
    mydata.at[i,0] = 1.0 # P_0(x) = 1
#
for i in range(r):
    mydata.at[i,1] = xi[i] # P_1(x) = xi
#
# Calculate values from P_2 to P_n using the recursive relation
#
for n in range(1,N-1):
    print(f"Calculating P_{n+1}(x)")
    for i in range(r):
        P1 = mydata.at[i,n-1] # P_(n-1)(x)
        P2 = mydata.at[i,n] # P_(n)(x)
        P3 = ((2*n+1)*xi[i]*P2 - n*P1)/(n+1) # P_(n+1)(x)
        mydata.at[i,n+1] = P3 # store P_(n+1)(x)
#
# Define the thermic field on plane y-z, as a function of theta and rho
#
rho = np.linspace(0, a, r)
T = np.zeros((r,r)) # temperature initialization
for i in range(r):
    for j in range(r):
        for n in range(1,N):
            T[i,j] += Coeff(n)*T0*Coeff(n)*mydata.at[j,n]*(rho[i]/a)**n
#
# Plot the temperature as a surface in 3D on plane Y-Z
#
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
Theta, Rho = np.meshgrid(theta, rho)
```

```
Z = Rho * np.cos(Theta) # convert to cartesian coordinates
Y = Rho * np.sin(Theta) # convert to cartesian coordinates
surf = ax.plot_surface(Y,Z,T,cmap='viridis')
#
# Axes, labels, view angle
#
ax.set_xlabel('y')
ax.set_ylabel('z')
ax.set_zlabel('Temperature (°C)')
ax.view_init(elev=30, azim=-45) # azim is rotation around vertical axis
plt.savefig('Temperature.jpg', dpi=300)
plt.close() # Close the current figure
#
# Plot isothermic curves on plane Y-Z
#
plt.figure(figsize=(6, 6))
Theta, Rho = np.meshgrid(theta, rho)
Y = Rho * np.sin(Theta)
Z = Rho * np.cos(Theta)
levels = 20  # number of isothermic curves
cs = plt.contour(Y, Z, T,levels=levels,cmap='viridis')
#
# Labels and title
#
plt.clabel(cs, inline=True, fontsize=8)
plt.xlabel('y')
plt.ylabel('z')
plt.title('Isothermic curves')
plt.axis('equal')
plt.savefig('Sphere_isothermic_curves.jpg', dpi=300)
plt.close() # Close the current figure
```

6 References

This is mainly a rearrangement of the material covered in chapter 7 of the Italian edition of [1]. I found Eq. 1 in [2] and Eq. 47 in [3].

1. Spiegel MR, “Fourier Analysis“, McGraw-Hill, 1974
2. Gradshteyn IS et Ryzhik IM, “Table of integrals, series, and products”, Academic Press, 1980
3. “Ettore Majorana: Notes on Theoretical Physics”, Kluwer Academic Publishers, 2003