QUANTITATIVE FINANCIAL MODELLING

THE GREEKS

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Abstract

This paper will discuss the theoretical and computational issues of the Greeks of an European option, more precisely the role and the meaning of the Greeks, the use and the practical computational issues of delta hedging. Before discussing the Greek letters, it is necessary to assume an option pricing model from which the Greeks will be calculated. This paper will use and discuss the Black–Scholes–Merton model for European options. In the appendix, plots of the Greeks can be found. It is shown the theoretical relationship between each Greek and the changes in the price of the option.

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1 Black and Scholes Option pricing model

The Black-Scholes option pricing model is a widely-used mathematical model for valuing options. It was first introduced in 1973 by Fischer Black and Myron Scholes, and has since become a widely-accepted tool for pricing options in the financial markets. The Black-Scholes model takes into account the time remaining until the option expires (time-to-maturity) T, the underlying asset's price S, the option's strike price K, the volatility of the underlying asset σ, the risk-free interest rate r, and the option's type (call or put) to determine its theoretical value.
One of the key assumptions of the Black-Scholes model is that the underlying asset follows a geometric Brownian motion, which means that its price follows a random walk with a constant drift and volatility. This assumption allows for the use of well-known statistical distributions, such as the standard log-normal distribution, to model the asset's price and calculate the option's value.
With this model, it is possible to derive a closed pricing formula for a vanilla European call [put] option maturing at some future date T.

Definition A call [put] option is the right to buy [sell] the underlying asset S_t at a preset strike price K at or within a fixed expiration date, depending on the type of the option (respectively, European or American).

Here below are shown the formulas for a call and a put option:

$$Call = S_0 N(d_1) - Ke^{-rT} N(d_2) \qquad Put = N(-d_2)Ke^{-rT} - S_0 N(-d_1)$$

Where:

- $d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$
- $d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$
- $N(x)^1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$

Moreover, $N(d_2)$ simply represents the probability that a call option will be exercised in a risk-less world. On the other hand, $N(d_1)$ is not easy to interpret. To this aim, the expression $S_0 N(d_1) e^{rT}$ put in evidence the expected stock price at time T in a risk-free world when stock price and the strike price are equal. As it is mentioned by Hull J. C. [5], the strike price K will be paid if the stock price S will be greater than K and, as mentioned before, it has probability $N(d_2)$.

As the main purpose of the document, once we defined the preliminary concepts, the next section will focus the analysis on the computation and on the discussion about Greeks.

[1]CDF: Cumulative Distribution Function of a Gaussian distribution

2 Greeks

Risk management is one of the main activity of a financial institution that sells an option to a client in the over-the-counter markets (OTC). If it occurs that one option is traded on an exchange or in the OTC market and it is the same sold by the financial institution, a risk-neutral exposure can be obtained if the institution buys again the same option (Sinclair E., 2013 [11]). The problem happens when the option has been modified by the client and the institution will not be able to hedge its position anymore or simply it will be more difficult. The Greeks present an alternative approach to this problem. Each Greek letter represents a particular measure of a different risk dimension in an option position and the aim of a trader is to manage the Greeks so that all risks are considered acceptable.
As introduced at the beginning of the analysis, to calculate a Greek, it is necessary to take in consideration of an option pricing model that in this paper, as well as in the more general cases, will correspond to the Black-Scholes one. When computing Greek letters, traders normally set the volatility equal to the current implied volatility. Implied volatility, if put into the Black-Scholes option pricing formula, gives the market-observed price of an option and it corresponds to the market's expectation of the volatility of a stock over the life of the said option.
Suppose we observe today a price of $C_0^\star$ for a European call option; the implied volatility σ^{imp} is the quantity that solves the following equation:

$$C_0^\star = C^{BS}(S_0, K, t = 0, T, r, \sigma^{imp})$$

When the volatility is considered equal to the implied one, the model gives the option price at a particular time as an exact function of the price of the underlying asset, the implied volatility, interest rates, and dividends (even if in our case, we assume that the option will not pay any dividends). The only way the option price can change in a short time period it occurs if and only if one of these variables change (Orlandi G. et al., 2017 [7]). A trader naturally feels confident if the risks of changes in all these variables have been adequately hedged.

2.1 Delta

The first Greek that this paper will discuss is the Delta (Δ). The Delta of a European call option is given by the first partial derivative of the pricing formula with respect to the value of the risk factor.
More specifically, we get:

$$\Delta_{call} = \frac{\partial C}{\partial S} = N(d_1) \quad \delta_{put} = \frac{\partial P}{\partial S} = -N(-d_1)$$

From a financial point of view, we should define the Delta as the number of units of the stock a trader should hold for each option sold to create a less risky portfolio (Hentsche L, 2003 [4]).

In sum, the option Delta is simply the ratio of a change in the price of an option relative to a change in the price of the underlying ([4]).

For instance, suppose that the Delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount.

A call Delta moves in [0, 1] and a put Delta in [-1, 0], which implies:

- If the share price increases, call Deltas get closer to 1, and put Deltas get closer to 0.
- If the share price decreases, call Deltas get closer to 0, and put Deltas get closer to -1.

Moreover, since this measure considers the marginal changes, it is additive across different strike prices if the underlying is always the same.
Concerning the options, the Δ is positive for long call and short put, negative for long put and short call. As a general rule, options that are in-the-money (ITM) have Deltas greater than 0.50. Options that are out-of-the-money (OTM) have Deltas less than 0.50. Finally, options that are at-the-money (ATM) have Deltas that are about 0.50 (Riaz A., 2019 [9]).
In practice many traders believe that, although mathematically incorrect, the Delta is a statistical approximation of the likelihood of the option expiring in-the-money. Under the definition pointed out by Orlando G. et al. [7], an option with a 0.75 Delta would have a 75% chance of being in-the-money at expiration.

2.1.1 Delta Hedging

Delta hedging is an option strategy that tries to limit risks in financial assets. It uses financial instruments or market strategies to offset the risk of any adverse price movements. Put another way, investors hedge one investment by making a trade in another. Delta hedging is a hedging strategy that involves the use of the Greek Delta. This section will explain this concept.

Suppose that, the price of a certain stock is \$100 and the option price is \$10. Imagine an investor who has sold call options to buy 2000 shares of a stock. If Δ = 60% as the previous assumption, the investor's position could be hedged by buying
$0.6 \cdot 2000 = 1200$ shares. The gain (loss) on the stock position would then tend to offset the loss (gain) on the option position.
For example, if the stock price goes up by \$1 (producing a gain of \$1200 on the shares purchased), the option price will tend to go up by $0.6 \cdot \$1 = \$0.60$ (producing a loss of \$1,200 on the options written); while, if the stock price goes down by \$1 (producing a loss of \$1200 on the shares purchased), the option price will tend to go down by \$0.60 (producing a gain of \$1200 on the options written). In this example, the Delta of the trader's short position in 2000 options is:

$$0.6 \cdot (-2000) = -1200$$

This means that the trader loses 1200 ΔS on the option position when the stock price increases by ΔS. The Delta of one share of the stock is 1.0, so that the long position in 1200 shares has a Delta of +1200. The Delta of the trader's overall position in this example is, therefore, zero. The Delta of the stock position offsets the Delta of the option position. A position with a Delta of zero is referred to as **delta neutral** (Hentschel L., 2003 [4]).

It is important to realize that, since the Delta of an option does not remain constant, the trader's position remains Delta hedged for only a short period of time. Notice that the hedge has to be adjusted periodically. This is known as re-balancing which means assuming continuously long and short position. A procedure such as this, where the hedge is adjusted on a regular basis, is referred to as dynamic hedging. On the contrary, as explained by Hull J.C. [5], it can be contrasted with static hedging, where a hedge is set up initially and never adjusted.

2.1.2 Possible computational issues of Delta Hedging

The Black-Scholes model requires five inputs, however only volatility cannot be directly observed by the market and must be estimated. This poses a natural question of how to best estimate the volatility. Aside from implied volatility there is also historical volatility that shows how volatile a stock has been based on price movements that have occurred in the past (Canina L. et al., 1993 [2]). Although option traders in practice may study historical volatility to make informed decisions as to the value of options traded on a stock, it is not a direct function of option prices (Derman E., 1996 [3]). Most of times, traders consider the theoretical value to be between the bid and the ask prices (Leoni P., 2014 [6]). On occasion, however, a trader will calculate implied volatility for the bid, the ask, the last trade price, or, sometimes, another value altogether. Different calculations of implied volatility leads to different pricing results which can affect the results of the Delta hedging strategy (Passarelli D., 2012 [8]).

Although outside of the scope of this paper it is possible to prove and show that the profit and loss of a Delta hedge in fact depends on the difference between realized and implied volatility of the hedging time period.

$$\text{Cumulative PL}_{\Delta t} \approx \frac{1}{2}\sum_{t=1}^{N}\Gamma_{t-1}\cdot S_{t-1}^{2}\left[\left(\frac{\Delta S_t}{S_{t-1}}\right)^2 - \left(\sigma\cdot\sqrt{\Delta t}\right)^2\right] \tag{1}$$

If we assume $r \approx 0$, then we can express the Θ as follows as seen in the previous equation:

$$\Theta \approx -\frac{1}{2}\Gamma S^2\sigma^2$$

Thus we can conclude that Delta hedging depends also largely on the relationship between theta and gamma as well that generally have opposite sign (Derman E., 1996 [3]).

2.2 Gamma

Gamma (Γ) is the rate of change of an option's Delta given a change in the price of the underlying asset. Gamma is conventionally stated in terms of Deltas per dollar move (Hentsche L., 2003 [4]). It is described by

$$\Gamma_{call} = \frac{\partial^2 C}{\partial S^2} = \frac{\mathbf{N}'(d_1)}{S\sigma\sqrt{T}} = \frac{\partial^2 P}{\partial S^2} = \Gamma_{put}$$

In addition to the Delta, Gamma can be also used to calculate the changes of the option due to changes in the underlying stock price. As it could be known, given the twice differentiable function f described here as follow.

$$\Delta f(x) \approx \Delta_c \cdot \Delta s + \frac{1}{2} \cdot \Gamma_c \cdot (\Delta s)^2$$

The above mentioned expression indicates that Gamma puts in evidence that the option value is not a linear function of the underlying stock price.
The Gamma of a long position is always positive. For an *at-the-money* option (European or American, call or put, is not relevant), Gamma increases as the time-to-maturity decreases. A short position *at-the-money* options have the maximum Gammas, which means that the value of the position of the option owner is highly sensitive to variations in the stock price (Riaz A., 2019 [9]).

As an illustration, have a look to a simple example of Gamma. In the following table, note how each option's Gamma relates to the option's new Delta after $\$1$ changes in the share price:

Type	**Delta**	**Gamma**	**New Delta (↑ +$1)**	**New Delta (↓ -$1)**
Call	+0.50	+0.05	**+0.55**	+0.45
Call	+0.20	+0.02	**+0.22**	+0.18
Put	-0.35	+0.03	-0.32	-0.38
Put	-0.35	+0.10	-0.45	-0.65

In this example, the highlighted numbers represent a growth in the option's exposure. Moreover, this table proves how Gamma can be analysed:

- If the share price increases of $\$1$, to estimate the option's new Delta, add the option's Gamma to this latter;
- If the share price decrease of $\$1$, to estimate the option's new Delta, subtract the option's Gamma to this latter.

If Gamma is small, Delta changes slowly, and adjustments to keep a portfolio $\Delta - neutral$ need to be made only infrequently. Anyway, if Gamma is highly negative or positive, Delta is very sensitive to the price of the underlying asset. It is then enough risky to hold a Δ-neutral portfolio unchanged for a long period (Bjork T., 2020 [1]).

2.3 Theta

Theta (Θ) of a portfolio composed by options is the rate of change of the value of the option price, with respect to the time to expiration, i.e. it is referred to the time decay of the portfolio.
The Θ_{call} and Θ_{put} formulas are as follows:

$$\Theta_{call} = \frac{-S\sigma N'(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2) \qquad \Theta_{put} = \frac{-S\sigma N'(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Where $N'(d_1)$ is the Probability Density Function for a standardised Normal distribution. In these formulas, time is measured in years. Usually, when Theta is quoted, time is measured in days, and it represents the change in the portfolio value when 1 day passes, ceteris paribus. In sum, it is possible to measure either "*per calendar day*" ($\frac{\Theta}{365}$) or "*per trading day*" ($\frac{\Theta}{252}$).

Theta is typically higher for short-dated options, especially near-the-money, as there is more urgency for the underlying to move in the money before expiration (Orlando G. et al., 2017 [7]). Theta can be a good measure or a bad one, depending on the position. Long options carry with them negative Theta; short options carry positive Theta. Theta hurts long option positions; whereas it helps short option positions (Hull J.C., 2022 [5]).

For instance, take an 80-strike call (K = 80) with a theoretical value of 3.16 on a stock at \$82 a share. The 32-day 80 call has a Theta of 0.03. If a trader owned one of these calls, the trader's position would theoretically lose 0.03, or \$0.03, as the time until expiration change from 32 to 31 days. This trader has a negative Theta position. A trader short one of these calls would have an overnight theoretical profit of \$0.03 attributed to Theta. This trader would have a positive Theta (Orlando G. et al., 2017 [7]). Theta affects put traders as well. Using all the same modeling inputs, the 32-day 80-strike put would have a theta of 0.02. A put holder would theoretically lose \$0.02 a day, and a put writer would theoretically make \$0.02 (Hentschel L., 2003 [4]).

The difference in Theta between calls and puts solely depends on the individual stock's cost of carry. Thus, an underlying asset without a dividend or a dividend yield that is implied (i.e., stock index future) will have call and put Thetas that are equal. When the cost of carry for a stock is positive (i.e., dividend yield is less than the interest rate), Theta for the call is higher than Theta for the put. When the cost of carry for the stock is negative (i.e., dividend yield is greater than the interest rate), Theta for the call is lower than that for the put (Riaz A., 2019 [9]).

Differently from the Delta, as highlighted by Hentschel L. [4], Theta is not a hedge parameter because there is uncertainty about the future stock price, but there is no uncertainty about the passage of time. In the light of the above, it makes sense to hedge against changes in the price of the underlying asset, but it does not make any sense to hedge against the passage of time. In spite of this, traders regard to Theta as a useful descriptive statistic

for a portfolio.
When Gamma is positive, Theta tends to be negative. The portfolio falls down in value if there is no change in stock price S, but increases in value if there is a large positive or negative change in S. When Gamma is negative, Theta tends to be positive. The reverse holds as well.

2.4 Vega

Vega (ν) is used to describe the sensitivity of the price of a financial derivative, to changes in the volatility of the underlying asset. The Black-Scholes model assumes that the volatility of the asset underpinning an option is constant. This means that the inferred volatilities of all options on the asset are constant and equal to this assumed volatility. Even if, what happens in reality is that the volatility of an asset changes over time (Shover L., 2013 [10]). As a result, the value of an option is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

Definition Vega of an option is the rate of change in its value with respect to the volatility of the beginning asset.
In the light of that, Vega call and put coincides and are defined as follow:

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\sqrt{T}N'(d1)$$

The stock price's relationship to the strike price is a major determining factor of an option's Vega. Implied volatility affects only the time value portion of an option. Because ATM options have the greatest amount of time value, they will naturally have higher Vegas. In-the-money and out-of-the-money options have lower Vega values than those of the ATM options (Bjork T., 2020 [1]).
As time passes, there is less time premium in the option that can be affected by changes in IV. Consequently, Vega gets smaller as expiration approaches (Sinclair E., 2013 [11]).

Here, for instance, imagine that you are an options trader and you are considering buying a call option on a stock with a current price of S = \$100. The option has a strike price of K = \$110 and will expires in three months (T = 3). The option's implied volatility σ^{imp} is 20%.
You can use the option's Vega to estimate the potential impact of changes in the stock's implied volatility on the option's price. For example, if the option's Vega is 0.50 and the implied volatility increases to 25%, the option's price is likely to increase by 0.50 · (25% - 20%) = \$0.25. Conversely, if the implied volatility decreases to 15%, the option's price is likely to decrease by 0.50 · (20% - 15%) = \$0.25.
This information can help to make informed decisions about whether to buy or sell the option and how to manage the risk.

Calculating Vega from the Black–Scholes–Merton model and its extensions may seem strange because one of the assumptions underlying the model is that volatility is constant.

It would be theoretically more correct to calculate Vega from a model in which volatility is assumed to be stochastic. However, traders prefer the simpler approach of measuring Vega in terms of potential movements in the Black–Scholes–Merton implied volatility.

2.5 Rho

Rho (ρ) is used to represent the sensitivity of an option's price to changes in the risk-free interest rate r. For both a European call and put option, we show:

$$\rho_{call} = KTe^{-rT}N(d_2) \qquad \rho_{put} = -KTe^{-rT}N(-d_2)$$

Where $N(d_2)$ has already been above defined.

It measures how the value of a portfolio changes as the risk-free interest rate increases or decreases in value, with all of the other components constant. Once again, concerning European options, the risk less interest rate r is considered for a maturity T equal to the option's expiration.
When the interest rate rises by 1%, the value of the call increases by the amount of its ρ. The same holds for a put (Hull J.C., 2022 [5]).
One should take into consideration the cost of capital. Since this effect is applicable to cost over time, interest rate changes impact longer term options much more than near-term options. The higher the price of the stock and the longer time until expiration generally means a greater sensitivity to changes in interest rates which equates to higher absolute Rho values (Hull J.C., 2022 [5]).

For example, consider a call option on a non-dividend-paying stock where the stock price is \$49, the strike price is \$50, the risk-free interest rate is 5%, the time to maturity is 20 weeks, and the implied volatility is 20%. In this case, S_0 = 49, K = 50, r = 0.05, σ = 0.2, and T = 0.3846.
The option's Rho ρ is

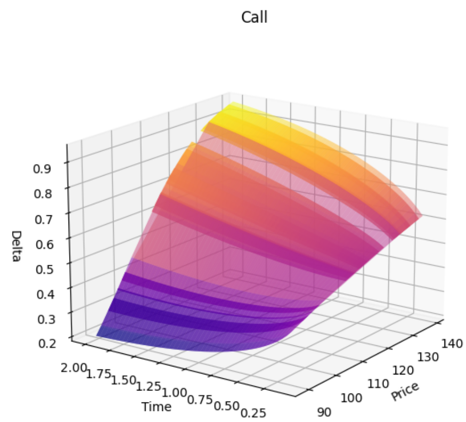
$$KTe^{-rT}N(d_2) = 8.91$$

This means that a 1% increase in the risk-free interest rate (from 5% to 6%) increases the price of the option by 0.01 $\cdot$ 8.91 = 0.0891.

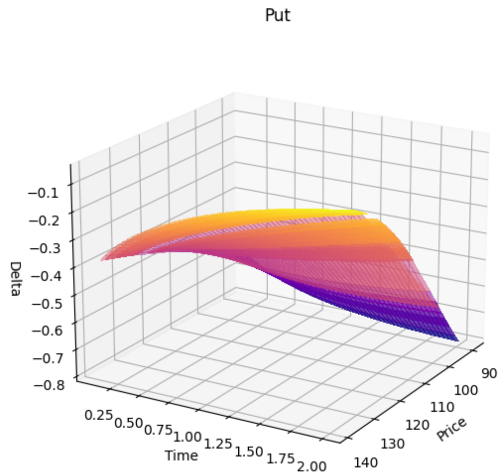
To conclude, the next section will show graphically, in a 3-dimensional representation, all the Greek letters presented so far.

3 Appendix

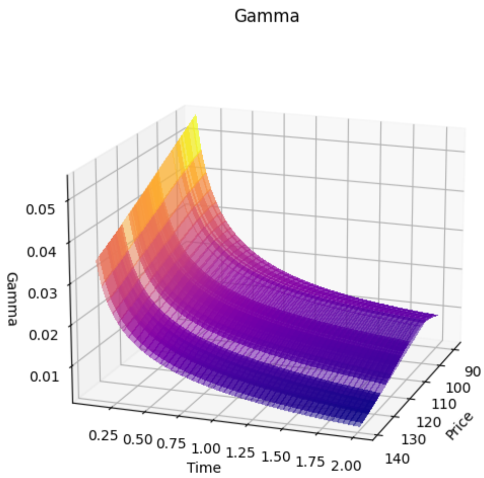
The plots were calculated with a volatility σ of 0.25, and 0 interest rate r.



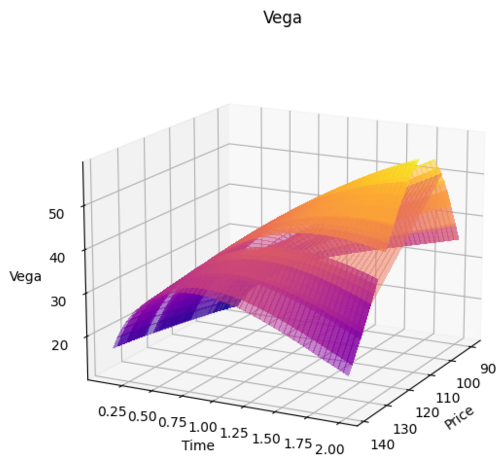
(a) Delta Call



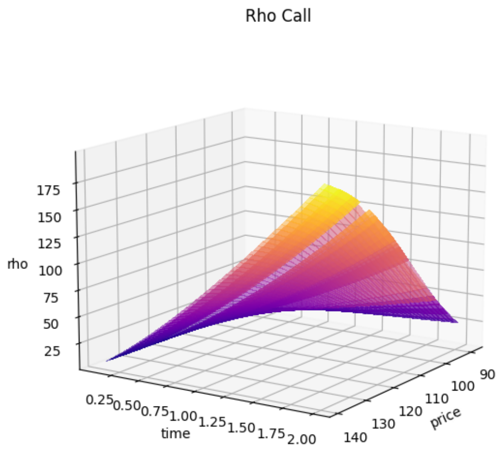
(b) Delta Put



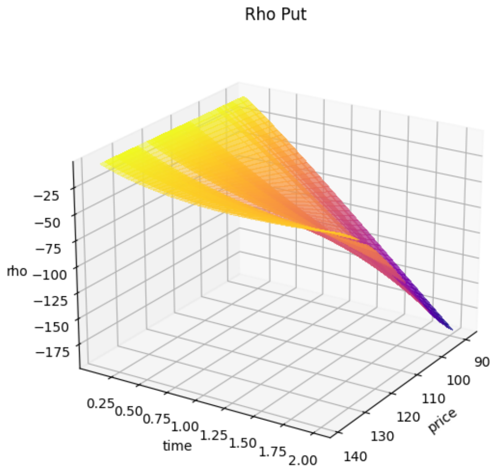
(c) Gamma



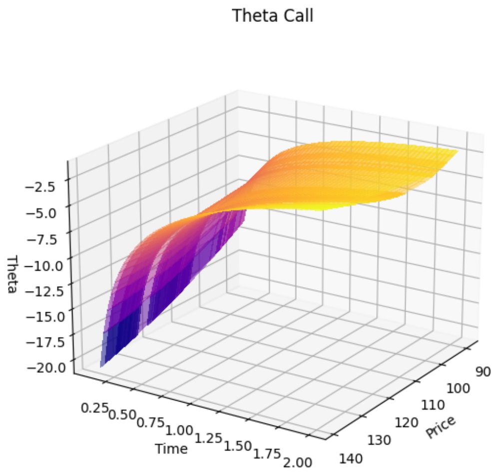
(d) Vega



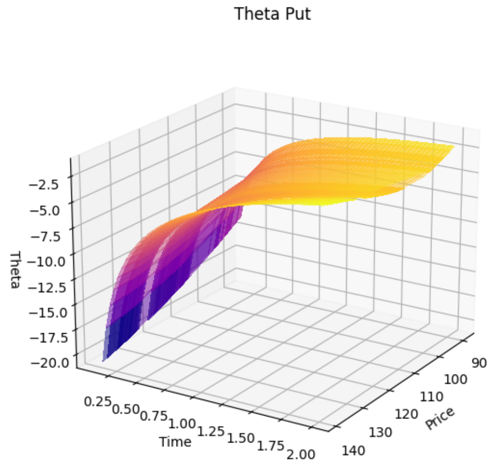
(e) Rho Call



(f) Rho Put



(g) Theta Call



(h) Theta Put

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