Financial Optimization and Asset Management

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Contents

1 Introduction

The diversification problem and the being risk adverse of the investor have always been one of the main issues in portfolio theory. The principle theory behind the diversification concept is that investors should hold portfolios and focus on the relationship between the individual securities within the portfolio. Through the concepts presented in the Mean-Variance theory, investors can draw practical guides into constructing investment portfolios that maximize their expected return based on a given level of risk.

In this paper, we aim to develop a brief analysis and application with 10 different assets prices with weekly observations that go from December 8^{th} 1997 to December 28^{th} 1998. The examination has been developed with the Weakened Mean Variance Markowitz model, highlighting the no short selling case. Subsequently to the first part of the examination, in the modelling computation we have divided the dataset in two parts: in-sample (35 observations) and out-of-sample (21 observations). The first ones have been used to observe the model while the second ones have been implemented to test the previous applied model. Furthermore, in the second part of the analysis we have inserted an additional constraint, defined as the Cardinal Constraint. This will lead us to a slight different resolution of the problem, changing the number of the assets considered in the portfolio composition. The last analysis is a comparison with the Sharpe ratio between the in-sample and the out-of-sample including and not the Cardinality Constraint.

2 Portfolio theory

In 1952, inspired by Bruno De Finetti with an insurance application, Harry Markowitz developed the so-called *Mean Variance model.* The foundation of the model has been formulated by considering some basic investor's needs. In particular, it considers two profiles: the **profit maximizer**, an investor whose philosophy is "the more the better", which means that only great amount of money with no risk is preferred, and the **price-taker**, for which the assets' prices can not be influenced by the investor: this implies that the wealth only depends on the selection of the fractions of assets constituting the portfolio (X_1,...,X_n). In reality what happens is that, even if a risk adverse investor is willing to have a maximum level of returns associated with zero risk, to realize a well diversified portfolio he/she should combine the maximization of the expected returns together with the minimization of risks.

Since the risk adverse rational investor always aims to pursue two different contrasted tasks (the maximization of the return and the minimization of risk), it is necessary to realize a model that considers the two requests at the same time. The starting point should be the measure of risk and return of assets with uncertain behavior.

2.1 Modern Portfolio Theory

Considering a market with n risky assets, it is possible to reduce the risk of portfolio by investing in more than one stock. In the light of that, the key idea is to obtain a diversified portfolio, i.e. a portfolio that includes assets with higher risk and assets with lower risk (investor expects higher premiums from the higher risky assets). The aim of the model is to build the so-called "Efficient Frontier" of non-dominated portfolio, which allows traders to visualise which, among the chosen assets, perform better, so that they can make a comparison among all the possible alternatives of investment. For this purpose, the main risk measures used are standard deviation and expected returns of the portfolio.

As soon as we approach ourselves to the research and the computation of the Efficient Frontier (EF), we might recall the concept of *non-dominated portfolio.* They are those with the minimum possible risk for a given expected return or those with a pre-fixed level of expected return for a given minimum level of risk. One of the most important example of non-dominated portfolio is the portfolio Minimum Variance (MV), which can be obtained as follows:

$$\begin{aligned} &\min_{x} \sigma_{P(x)} \\ s.t.: \; &P(x) = (x_1, x_2, \ldots, x_n) \end{aligned} \tag{1}$$

Where the single points on the EF can be found to solve the constrained optimization problem. Exploiting the construction of the Efficient Frontier, we are able to distinguish two different cases: allowed short selling (SS) and not allowed short selling (no-SS). The EF assumes two different representation when considering or not the short selling: in the first case (SS) the frontier is unlimited (*torpedo-shaped*) but in the second case it is obtained as a limited curve (*umbrella-shaped*).

3 Markowitz mean-variance optimization model

In order to proceed with the analysis, we might consider and overview of the models not considering the SS. We start fixing μ as the generic value for $E(R_P)$ and from that we are able to state that a point P_μ is on the opportunity frontier if, among all the points with $E(R_p)$=μ, it is the one that minimizes the variance (frontier point). The problem of finding P_μ is formulated as a constrained optimization problem in the decision variables $X_1, X_2, X_3, \ldots, X_n$ (portfolio fractions). Here, the Efficient Frontier is limited and characterized by the inclusion of the non negativity constraints on the decision variables (portfolio fractions), which can only be positive, if some budget is invested in asset j, or equal to 0, if asset j is not included in the portfolio.

$$
\begin{gathered}
\min_X \sum_{j=1}^{n} \sigma_j^2 X_j^2 + 2\sum_{i=1}^{n}\sum_{j>1} \sigma_{ij} X_i X_j \\
\sum_{j=1}^{n} X_j = 1 \\
\sum_{j=1}^{n} E(R_j) X_j = \mu \\
X_j \geq 0
\end{gathered}
\tag{2}
$$

Specifically, we can define the minimum expected return portfolio as the correspondent asset with the minimum expected return (P_{ERmin} with μ_{ERmin} and $\sigma^2{}_{ERmin}$, which not necessarily corresponds to the MV portfolio) and the maximum expected return portfolio as the correspondent asset with maximum expected return (P_{ERmax} with μ_{ERmax} and $\sigma^2{}_{ERmax}$). The portfolio P_{ERmin} and P_{ERmax} correspond to the tips of the EF with no SS. Worth of noting is that all the frontier portfolios have expected return value between μ_{ERmin} and μ_{ERmax}, providing the range of the possible expected return values of a portfolio obtainable in the considered market.

Going on with the procedure, it is important to proceed with the EF approximation procedure as follows:

- Define the Opportunity set;
- Find P_{ERmin} and P_{ERmax};
- Fix in the range [μ_{ERmin}, μ_{ERmax}] a finite number of values μ;
- For each fixed value of μ solve the MwV with no short selling.

The Efficient Frontier will be described and approximated at its best when it has the largest number of values fixed for μ possible. To compute the Minimum Variance portfolio, in this case, we have to remove the constraint on the portfolio expected return value.

In order to follow the aim of the analysis, we highlight a slight difference in the model chosen. In our specific case the portfolio's expected return should be at least equal to μ, defining the final formulation as:

$$\begin{gathered}\min_X \sum_{j=1}^{n} \sigma_j^2 X_j^2 + 2\sum_{i=1}^{n}\sum_{j>1} \sigma_{ij} X_i X_j \\ \sum_{j=1}^{n} X_j = 1 \\ \sum_{j=1}^{n} E(R_j) X_j \geq \mu \\ X_j \geq 0 \end{gathered} \tag{3}$$

It is a relaxed version of the model and, worth of noting is that when the "standard" model is feasible, the solution of the relaxed model is exactly equivalent to solving the standard one.

3.1 Time windows

Subsequently to the MwV description, we adopt a portfolio selection strategy which includes two different time windows defined as *in-sample* and *out-of-sample* for the periods chosen.

For the general portfolio selection we observe the price variation of the given assets over the *learning time interval* referred to the in-sample to select the best portfolio asset composition, after that we re-estimate the model testing the performance of the selected portfolio in the *subsequent time interval*, referred to the out-of-sample. Through this approach, we are able to observe the historical assets' returns and select the best portfolio to hold in the future. The in-sample section represents the past, henceforth the known, whilst the out-of-sample serves for the unknown at the time of the portfolio selection.

3.2 Additional constraints

In the mathematical programming models, we can also have the opportunity to add other constraints. In this particular case, we can decide how many and which assets go in the selected portfolio by fixing a precise or a maximum portfolio cardinality. This type of model is constructed through the addition of binary variables Y_i with i=1,2,3,...,n. The Y_i will be equal to 1 if the specified asset i is included in the portfolio and, 0 otherwise. We are now able to rewrite the model adding this constraint.

$$\begin{gathered} \min_X \sum_{j=1}^{n} \sigma_j^2 X_j^2 + 2\sum_{i=1}^{n}\sum_{j>1} \sigma_{ij} X_i X_j \\ \sum_{j=1}^{n} X_j = 1 \\ \sum_{j=1}^{n} E(R_j) X_j = \mu \\ \sum_{i=1}^{n} Y_i \leq k \\ X_i \leq Y_i \\ X_j \geq 0 \\ Y_i \; binary \end{gathered} \tag{4}$$

It is important to higlight that the X_i and Y_i are dependent, infact if $X_i > 0$ then $Y_i = 1$ whilst if $Y_i = 0$ then $X_i = 0$ necessarily.

3.3 Sharpe Ratio

To compare the return of the investments we rely on the Sharpe Ratio. Introduced by William F. Sharpe in 1996 as an outgrowth of the Capital Asset Pricing Model, this ratio helps to compare the return of the made investment with its risk. It divides a portfolio's excess returns by a measure of its volatility to assess risk-adjusted performance. The mathematical formulation is obtained as:

$$Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p} \tag{5}$$

The numerator is the difference over time between realized returns and the risk-free rate of return. The denominator, instead, is the standard deviation of returns over the same period of time, as measure of volatility and risk.

4 Application

In this paper, the conducted analysis has been developed with 56 prices obtained from the weekly observation of 10 assets from December 8^{th} 1997 to December 28^{th} 1998. From the prices given, we have been able to obtain the return for each data with the implementation of the following formula:

$$Ret = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{6}$$

After that, in the Excel sheet 1.1-1.2, we have computed the expected returns and the standard deviation for all the assets in the given market and the matrix of variance and covariance between the return of any two assets.

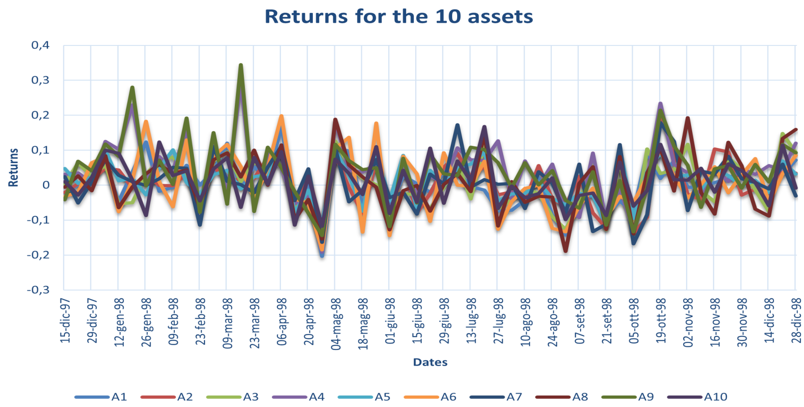

Figure 1: Returns time series

4.1 In-sample analysis

As previously introduced, our model foresees a data division. The first 35 represent the training set of the model which relies on the past and known values. Here, in the Excel sheet 2.1, we have found the maximum possible expected return value (ER_{max}) and the minimum possible variance value (ER_{min} or MV). To calculate the ER_{max}, it has been obtained as follows:

$$\begin{gathered} \max_X \sum_{j=1}^{n} E(R_j) X_j \\ \sum_{j=1}^{n} X_j = 1 \end{gathered} \tag{7}$$

It can also been expressed in its matrix form as:

$$\begin{gathered} \max ER^T X \\ e^T X = 1 \end{gathered} \tag{8}$$

On the other hand, to calculate the ER_{min}, which corresponds to the MV portfolio, the formulation has been the following:

$$\begin{gathered} \min_X \sum_{j=1}^{n} \sigma_j^2 X_j^2 + \sum_{i=1}^{n} \sum_{i \neq j} \sigma_{ij} X_i X_j \\ \sum_{j=1}^{n} X_j = 1 \end{gathered} \tag{9}$$

In the same way defined for the ER_{max}, the MV has its matrix form too:

$$\begin{gathered} \min X^T V X \\ e^T X = 1 \end{gathered} \tag{10}$$

For the results obtained in this first step, we can say that for the ER_{min} the portfolio variance is around 0,26% and its expected return is 1,51%. On the contrary, for the ER_{max} the portfolio variance shows an higher value, around 0,85% and its expected return is even higher being equal to 4,79%. Subsequently, we have applied the MwV model with no SS fixing the expected return value μ_1 as:

$$\mu_1 = \frac{ER_{max} + ER_{min}}{2} \tag{11}$$

Obtaining this μ_1 value, we have defined the optimal portfolio, which is characterized by the $E(P^{\mu_1}) = 3,14935\%$, $\sigma^2 = 0,37828\%$ and the weighted fractions for each asset in the portfolio as the table shows below:

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**
Weights	0%	0%	0%	37,295%	53,449%	0%	0%	0%	6,785%	2,471%
Sum	100%									

Table 1: P^{μ_1} - 35 observations

According to the above defined table, we can state that only 4 out of 10 asset have been chosen for the portfolio composition; highlighting the best selection for the investor. In the next section, in the Excel sheet 2.2, we have determined the $P^{\mu_1 card}$.

Recalling what has been defined in the theory section, when adding the Cardinality Constraint, we are inserting a new binary variable dependent on X. We are requested to associate a specific value to k, which in this case specifies that the number of asset inserted in the portfolio must be equal to 5.

Estimating again the model, the minimum variance and the maximum expected return stay the same given that in these calculations there is no additional constraint. After that, with the implementation of the cardinality constraint, we will include 5 asset in the portfolio composition instead of 4, that was the amount suggested in the previous computation. We notice that the results for the portfolio variance and portfolio expected return are almost similar to what we have obtained in the previous case given the redistribution of the investment percentages. This outcome might be justified by that, including or not, the fifth asset, it would not change the reached level of the variance minimization.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**
Weights	0%	0%	0%	37,300%	53,504%	0%	0%	0%	6,782%	2,415%
Sum	100%									

Table 2: $P^{\mu_1 card}$ - 35 observations

4.2 Out-of-sample analysis

To test the model defined in the previous paragraph, we select the remaining 21 rows of the dataset. Basically, we have proceed in the same way as we have done before in order to have some evidence to compare the results. The analysis begins lacking the additional constraint in the first model computed. We have calculated again the minimum variance portfolio whose results for the portfolio variance and its expected returns are respectively: 0,23138% and -0,21018%. On the contrary, the solution for the maximum expected return portfolio are, in the same order, equal to: 0,62820% and 2,32518%. This difference highlighted in the results is given by the new data observed from those of the previous paragraph.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**
Weights	0%	0%	0%	37,585%	37,420%	0%	0%	0%	0%	24,995%
Sum	100%									

Table 3: P^{μ_1} - 21 observations

From these outcomes, we are taking in consideration only 3 asset for the whole portfolio. This has brought a redistribution whose heaviest evidence is seen in the asset A10. Going on, we have added again the Cardinality Constraint keeping the k value equal to 5. Once again, the inclusion of five asset has been respected but the investment percentage has been only distributed between 3 asset for the portfolio composition, remaining identical to the previous one. Lastly, we aim to focus the analysis on the Sharpe ratio comparison. As proof of the results commented above, for the two cases in-sample and the out-of-sample the Sharpe ratio is the same. For that relying on the 35 observation it is equal to 8,3255 and to 3,3864 for the last 21 observation. According to the Sharpe ratio theory, the higher the result the more attractive the portfolio will be. It is important to recall that the expected return is fixed and the result relies on the variance. This proves that to low variance values correspond high Sharpe ratio which indicates a good investing strategy.

In conclusion, considering the results obtained, it is possible to state that the portfolio built with the in-sample results to be the most convenient to invest in. This might be explained by that, in general, the more observations we have, the more the variance value tends to decrement, contrary to the expected return which, instead, increments. This outcome might be led by a more precise analysis on which the investor can rely on. The graph attached below can help us to recap all the analysis developed in this paper, showing the different results for each model for the portfolio composition.

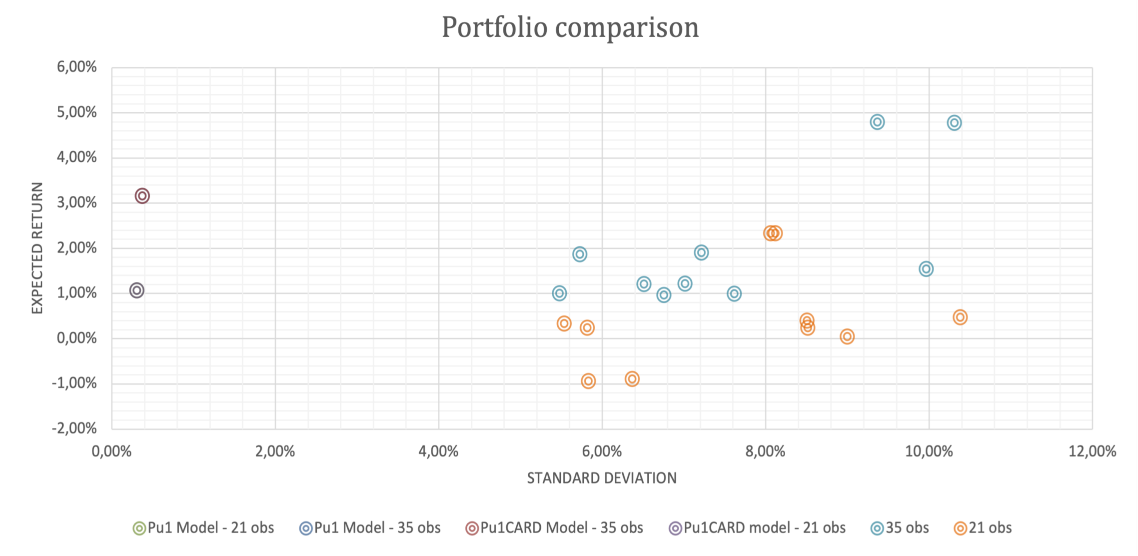

Figure 2: Portfolio comparison