LEMMA Let $\mathcal{J}$ be a projective geometry (without forms).
Let $A \subseteq \mathcal{J}$ be a subspace of $\mathcal{J}$.
Let V be the vector space inducing $\mathcal{J}$.
Let U be the subspace of V inducing A. Write $V = U \oplus W$.
Then,

$$\underbrace{\mathrm{Aut}(\mathcal{J}/A)}_{G} \cong \underbrace{\mathrm{Hom}(\overline{W}, \overline{U})}_{N} \rtimes \underbrace{PGL(\overline{W})}_{H}.$$

subspace of $\mathcal{J}$ induced by W

proof:

WNTS: $N \triangleleft G$, $N \cap H = \{e\}$, and $G = NH$.

Let $\bar{v} \in \mathcal{J}$. Let $v \in V$ induce $\bar{v}$. $v = u^{\in U} + w^{\in W}$ uniquely (and so $\bar{v} = \bar{u} + \bar{w}$)

Any $\bar{\varphi} \in \mathrm{Aut}(\mathcal{J}/A)$ is induced by a linear map $\varphi \in \mathrm{Aut}(V/U)$ so

$$\varphi(v) = \varphi(u+w) = \varphi(u) + \varphi(w) = u + \underbrace{\varphi(w)}_{u' + w'}$$

Let $\varphi_1 : W \longrightarrow U \quad w \longmapsto u'$

$\varphi_2 : W \longrightarrow W \quad w \longmapsto w'$

Note that $\varphi_1 \in \mathrm{Hom}(W, U)$ $\varphi_2 \in GL(W)$ since φ is linear.
$\bar{\varphi}$ is obtained from φ by quotienting by scalars.

Not sure how to view N as a normal subgroup.

LEMMA 3.4.3 Let M be an internally finite infinite structure. Let J_1, J_2 be a pair of basic pure projective geometries (with no forms) s.t.

- they are A-definable (and fully embedded over A) for A alg closed;
- $J_1 \underset{A}{\perp} J_2$
- $(J_1, J_2; A_1, A_2)$ is canonically embedded in M
  (where $A_i := J_i \cap A$)

Let $J = J_1 \cup J_2$. Then, the group G induced on J by the internal automorphism group of M (fixing A?) contains $\mathrm{Aut}(J; A_J)^{(\infty)}$ (understood internally)

just commutator subgroup of $\mathrm{Aut}(J; A_J)$

<u>POINT</u>: $G \leq \mathrm{Aut}(J; A_J)$ but it still contains $\mathrm{Aut}(J; A_J)^{(\infty)}$

<u>Proof</u>: write V_i for the vector space inducing J_i, and $V_i = U_i \oplus W_i$ as in Lemma where $U_i = V_i \cap \mathrm{acl}^{eq}(A)$.

FROM Lemma + $J_1 \underset{A}{\perp} J_2$, $\mathrm{Aut}(J; A_J) = \prod_{i \in \{1,2\}} \overbrace{\mathrm{Hom}(\overline{W}_i, \overline{U}_i) \rtimes PGL(\overline{W}_i)}^{\mathrm{Aut}(J_i / A_i) \text{ by Lemma}}$

FACT For $N \rtimes H$, $(N \rtimes H)^{(1)} = [N;N][N;H] \rtimes H^{(1)}$, so

$$\mathrm{Aut}(J;A_J)^{(\infty)} = \prod_{i \in \{0,1\}} \mathrm{Hom}(\overline{W_i}, \overline{U_i}) \rtimes PSL(W_i)$$

Since $J_1 \underset{A}{\perp} J_2$, we also have $J_1/A_1 \underset{A}{\perp} J_2/A_2$ by Lemma 2.4.11 (using full embeddedness)

Since there is no form $J_i/A_i \cong \overline{W_i}$ (by definition of localization)

$\mathrm{Aut}(M/A)$ induces on $(J_1/A_1, J_2/A_2)$ $PGL(W_1) \times PGL(W_2)$
([CH] say $PSL(W_1) \times PSL(W_2)$?)

Since $G \leq \mathrm{Aut}(J;A_J)$, consider the natural projection
$\phi : G \longrightarrow \mathrm{Aut}(J_2;A_2)$. Let $H_1 := \mathrm{Ker}\,\phi$.
$H_1 \trianglelefteq G$ and $PSL(W_1) \leq H_1$ (since these get sent to 0)

So $PSL(W_1) \leq H_1^{(\infty)} \trianglelefteq \mathrm{Aut}(J_1;A_1)$
since it projects trivially on $\mathrm{Aut}(J_2/A_2)$

By inspection, any perfect normal subgroup of $\mathrm{Hom}(\overline{W_1}, \overline{U_1}) \rtimes PGL(W_1)$ contains $\mathrm{Hom}(\overline{W_1}, \overline{U_1}) \rtimes PSL(W_1)$.

Hence $\mathrm{Aut}(J;A_J)^{(\infty)} \leq G$. $\square$