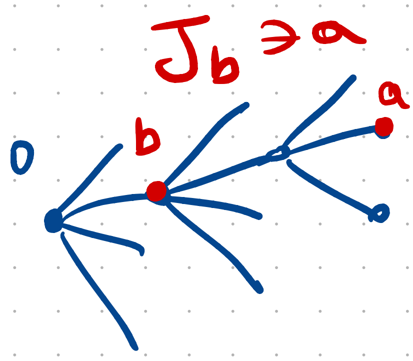
Recall:

M is coordinatised by Lie geometries if there is a 0-definable tree with domain M, finite height, unique 0-def root s.t.



for each $a \in M$ above the root, either

- a is algebraic over its immediate predecessor,
- $\exists b < a$ and b-definable projective J_b s.t. J_b is fully embedded over M and either
  1. $a \in J_b$; or
  2. (case with affine or quadratic geometries)

$$J_a \perp J_b \iff J_a \underset{(a,b)}{\perp} J_b$$

maybe we mean this?

Let J_a and J_b be defined over a and b respectively. We say

$J_a \underset{C}{\perp} J_b$ if every C-definable relation on $J_a \cup J_b$ is a Boolean combination of rectangles $R_a \times R_b$ for R_i $acl^{eq}(C)$-def in J_i.

IS THIS THE RIGHT NOTION of orthogonality for J_a and J_b defined over parameters not necessarily in C?

Lemma 2.4.3 (ALBERTO) Let $A \subseteq M$, J_1, J_2 normal geometries fully embedded over A in M. Then, we have:

- $J_1 \underset{A}{\perp} J_2$ OR
- J_1 and J_2 are A-linked: there is an A-definable bijection between J_1 and J_2.

A theme today will be:
How do we deal with the case of $J_a \not\perp J_b$?

We should NOT get $\emptyset$-linkedness (or (ab)-linkedness) but the proofs are clearly doing some clever moves from the assumptions there.

§ 2.5 CANONICAL PROJECTIVE GEOMETRIES

LEMMA 2.4.8 [AMENDED] Let D and I be stably embedded. Let $\{J_a \mid a \in I\}$ be a collection of uniformly a-definable stably embedded definable subsets of $\mathcal{M}$.
Let $\mathrm{acl}^{eq}_D(\emptyset) = \mathrm{dcl}^{eq}_D(\emptyset)$.
Suppose $D \perp_a J_a$ and $D \perp I$. Then,

$$D \perp \underbrace{I \cup \bigcup_{a \in I} J_a}$$

and this is stably embedded.

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Let J_b be a b-definable weak projective Lie geometry in $\mathcal{M}$. J is a **CANONICAL PROJECTIVE GEOMETRY** if

1. J_b is fully embedded over b;
2. $b' \equiv b \quad b' \neq b \Rightarrow J_b \perp J_{b'}$

LEMMA 2.5.2 Let P_b be a b-definable projective geometry fully embedded in Lie-coordinatizable $\mathcal{M}$. Then, there is a canonical projective geometry in $\mathcal{M}^{eq}$ non-orthogonal to P_b over a finite set.

LEMMA 2.5.2 (PART 1) Suppose $\mathcal{M}$ is coordinatised by Lie geometries & ω-categorical. Suppose $P_{b'}$ is a b'-definable basic projective geometry fully embedded in $\mathcal{M}$ (over b').

Then, $P_{b'} \not\perp_a J_a$ where J_a is one of the coordinatising geometries*

* = we also include conjugates of coordinatising geoms

<u>Proof</u>

By induction on $h \geq 0$.

<u>CLAIM</u>: $\exists I_h$ 0-definable and stably embedded s.t.

$$N_h \subseteq I_h \quad \text{and} \quad P_{b'} \perp I_h$$

tree up to height h

<u>BC</u>: $h=0$ is trivial $I_h = \{0\}$ $P_{b'} \perp 0$

<u>IS</u>: By ω-categoricity, there are finitely many types of geometries appearing at height $h+1$.

Let a_i be representative of a type at height $h+1$.

for $b < a_i$ $a_i \in J_b^i$ b-def and stably embedded.

By ω-cat $tp(b)$ is isolated

$$J_c^i = \underbrace{\phi(M, c)}_{\text{is s.t. } \phi(x,b) \text{ defines } J_b^i} \wedge c \models tp(b)$$

Now $\{J_c^i \mid c \in I_n\}$ is unif def with each member stably emb over c and s.t. $P_{b'} \perp_c J_c^i$.

By Lemma 2.4.8

$$P_{b'} \perp \underbrace{I_n \cup \bigcup_{c \in I_n} J_c^i}_{\text{is stably embedded}}$$

(from Dugald: P_1, P_2, Q 0-def & stably emb
$Q \perp P_1$ + $Q \perp P_2 \Rightarrow Q \perp P_1 \cup P_2$)

$$P_{b'} \perp \underbrace{\bigcup_{i=1}^{n} \Big(I_n \cup \bigcup_{c \in I_n} J_c^i \Big)}_{I_{n+1}}$$

Note $N_{n+1} \subseteq I_{n+1}$

This concludes the induction.
At the height of the tree we get

$$P_{b'} \perp M$$

※ = is not going to be a Boolean combo of $acl^{eq}(\emptyset)$-def rectangles $\blacksquare$

Lemma 2.5.3 (PART 2) Suppose P_b is a coordinatising geometry for $\mathcal{M}$ s.t. $P_b \perp J_c$ for any $c < b$.
Let $P_{b'}$ be a conjugate of P_b s.t.

$$P_b \underset{C}{\not\perp} P_{b'}$$

for some finite C. Then,

$$P_b \underset{(b,b')}{\not\perp} P_{b'}$$

(in particular, P_b and $P_{b'}$ are (bb')-linked)

$$P_b \underset{C}{\not\perp} P_{b'}$$

Consider the conjugates $P_{b'}$ of P_b. If $P_b, P_{b'}$ are nonorthogonal over a finite set, then the appropriate localization of $P_{b'}$ is orthogonal to the coordinatizing geometries Q for b over any set over which $P_{b'}$ and Q are defined. It follows by induction that $P_{b'} \cap acl(b', b_1, \ldots, b_i) = \emptyset$ for all $i \leq n$; notice that the induction step is vacuous when b_i is algebraic over its predecessor. For $i = n$ we have $acl(b, b') \cap P_{b'} = \emptyset$ and similarly $acl(b, b') \cap P_b = \emptyset$. Thus the nonorthogonality gives a unique (b, b')-definable bijection between P_b and $P_{b'}$, preserving the unoriented weak structure, and also, by an explicit hypothesis, preserving the Witt defect in the quadratic case.

$\exists!$ (b,b')-def bij $\parallel$

$$P_b \to P_{b'}$$

<u>IDEA</u> ?

$$P_b \not\perp_C P_{b'} \quad + \quad P_b \perp Q_{b_i} \quad \text{for } b_i < b$$

? $\Longrightarrow$ $$P_{b'} \perp_{\bigcirc\, b' b_i} Q_{b_i}$$

something?

LEMMA 2.4.11 $\Longrightarrow$ $$\mathrm{Loc}(P_{b'} / b' b_i) \perp_{b' b_i} Q_{b_i}$$

? proof by induction

$\Longrightarrow$ $$\mathrm{ocl}(b' b) \cap P_{b'} = \emptyset$$
$$\mathrm{ocl}(b' b) \cap P_b = \emptyset$$

? $\Longrightarrow$ $$P_b \not\perp_{(b, b')} P_{b'}$$

LEMMA 2.5.2 (PART 3) Let P_b be a b-definable projective geometries fully embedded in ω-categorical $\mathcal{M}$.
Suppose that for any two conjugates of P_b, $P_{b'}$ and $P_{b''}$ we have

$$P_{b'} \underset{C}{\not\perp} P_{b''} \text{ over some finite } C \Rightarrow P_{b'} \underset{(b'b'')}{\not\perp} P_{b''}.$$

Then, there is a canonical projective geometry $Q_c \subseteq \mathcal{M}^{eq}$ s.t. $c \in dcl^{eq}(b)$ and

$$Q_c \underset{b}{\not\perp} P_b$$

<u>Proof</u>:

WRITE $b' \sim b''$ iff $P_{b'} \not\perp P_{b''}$

<u>CLAIM</u>: $\sim$ is a 0-def equiv. rel on $tp(b) \times tp(b)$

<u>proof</u>: $\emptyset$-def comes from non-orth being preserved by automs.

$P_{b'} \not\perp P_{b''}$ + $P_{b''} \not\perp P_{b'''}$

$\Rightarrow$ a unique (b', b'') bij $P_{b'} \to P_{b''}$ + unique (b'', b''') bij
$(b' b'' b''')$-def bij $P_{b'} \to P_{b'''}$

$$P_{b'} \underset{b'b''b'''}{\not\perp} P_{b'''}$$

$$\Rightarrow P_{b'} \underset{b'b'''}{\not\perp} P_{b'''}$$

P & Q are 0-def and stably emb + $P \perp Q$
$\Rightarrow P \underset{B}{\perp} Q$ for any B

$\boxtimes$

CLAIM 2: $b' \sim b'' \Rightarrow \exists!$ $(b' b'')$-def isomorphs preserving weak structure $P_{b'} \to P_{b''}$.

proof: Essentially Lemma 2.4.6 & 2.4.7 to give that weak structure is preserved. ▨

Let $c = b/\sim \in M^{eq}$ $\qquad \cancel{P(x,b)} =: P_b$

$x \approx y$ iff $\exists z z'$ $P(x, z) \wedge P(y, z')$

$\wedge \bigvee_i \varphi_i (x, y, z, z')$

different options for the unique (z,z')-def bijection.

$x \in P_b$ $\quad x \approx y$ iff $\exists b' \sim b$ $\quad y \in P_{b'}$
(and $P_b \not\perp P_{b'}$)

So consider

$$Q_c^w := (\exists z P(x,z) \wedge z/\sim = c) / \approx$$

Since the bijections preserve weak structure

Q_c^w has an induced weak struct isom to P_b.

Q_c be the associated basic proj geom.

- $P_b \underset{b}{\not\perp} Q_c$ due to a b-def bijection
- Q_c is fully embedded over c (should be inherited from P_b)
- For $c \equiv c'$ $c \neq c'$ $Q_c \perp Q_{c'}$

Essentially this is because each c is associated to a $\sim$-equiv class