Separation of RAAGs based on trees

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In this piece I will present some of the work I did during my summer research project under the supervision of Dr Richard Wade between July and September 2018. The purpose of the project was that of investigating the separation profile of Right Angled Artin Groups, focusing in particular on the case of RAAGs based on trees. The main theorem that I will present shows exactly how the separation profile of RAAGs based on trees depends on the diameter of the base tree. In Section 1, I will give some general information on RAAGs following mainly the more extensive reviews in [4] and [10]. Hence, I will move on to defining the separation profile of a graph and of finitely generated groups following [2]. In Section 3, I will review some previously known results which yield the separation profile of RAAGs based on trees of diameter not larger than 2 and reduce the problem of the separation profile of RAAGs based on trees to a single case. Section 4 will be dedicated to the proof of the theorem that the separation profile of the RAAG based on the diameter 3 segment is $n^{\frac{2}{3}}$. After proving a general result about the existence of a centerpoint, I will present my proof for the upper bound of this RAAG. This is an original result of mine. I will conclude with the proof that the lower bound is the same as the upper bound. This latter proof was found by Dr David Hume at the end of my research project.

1 Right Angled Artin Groups

1.1 RAAGs and minimal words

Right Angled Artin Groups are groups defined on a base graph in the following way:

Definition 1 (Right Angled Artin Group)**.** Let Γ be a simple graph with finite vertex set $V\Gamma$ and finite edge set $E\Gamma$. The **Right Angled Artin Group** based on

Γ, A_Γ is given by the following finite group presentation:

$$A_\Gamma := \langle v \in V\Gamma \mid [v,w] \text{ where } \{v,w\} \in E\Gamma \rangle$$

Hence, given a **defining graph** Γ, the RAAG based on it will have as generators the vertices of Γ, and any two vertices forming an edge will commute.
Note that I take the defining graph to be simple, that is without loops or multiple edges, in order to avoid redundancy.

I will often talk of elements of RAAGs in terms of words representing them with letters from the set of vertices of the defining graph. Now, since different words may represent the same element, it is useful to introduce the notion of a set of words of minimal length representing the same element. Let $w = a_1^{n_1}a_2^{n_2}...a_m^{n_m}$ be a word with $a_i \in V\Gamma$ for each i. We call each $a_i^{n_i}$ a **syllable**.

Definition 2 (*Min(g)*)**.** Let A_Γ be a RAAG. Let $g \in A_\Gamma$. We call ***Min(g)*** the set of words with minimal number of syllables representing g formed by letters from $V\Gamma$.

Now, an important result regarding how an arbitrary word w representing some element g of a RAAG can be transformed into a word in $Min(g)$ is given in [6, p.302, Th.14.4]:

Theorem 1 (Same words in a RAAG)**.** *Let A_Γ be a RAAG. Let w be a word representing $g \in A_\Gamma$. Then, we can transform w into any element of $Min(g)$ by a finite sequence of the following moves on syllables:*

1. *Remove a_i^j when $j = 0$.*
2. *Replace $a_i^j a_{i+1}^k$ with a_i^{j+k} if $a_i = a_{i+1}$*
3. *Replace $a_i^j a_{i+1}^k$ with $a_{i+1}^k a_i^j$ if $[a_i, a_{i+1}] = 1$.*

Furthermore, in such sequence the number of syllables never increases.

1.2 CAT(0) Cube Complexes and hyperplanes

For each RAAG A_Γ we can construct the associated **Salvetti Complex** $\mathcal{S}_\Gamma$. This is built in the following way [4]:
Begin with a wedge of circles, one for each vertex of Γ, and label them accordingly. For each edge, say $\{v,w\}$, attach a 2-torus gluing a 2-cell along the path $vwv^{-1}w^{-1}$. More generally, for each n-clique[1] in Γ, attach an n-torus in the

[1]Recall that an n-clique is a set of vertices whose induced subgraph is complete.

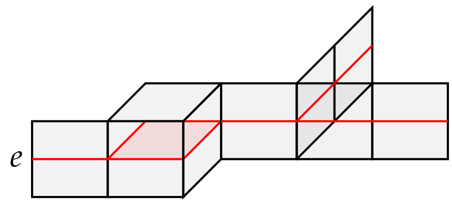


Figure 1: An example of a cube complex (in gray) and a hyperplane dual to the edge e (in red).

following way: consider the n-torus as an n-dimensional hypercube in which opposite faces are identified. Thus, glue the faces of this cube to the corresponding $n-1$-tori according to the previously established labelings.
Note that by construction the 2-skeleton of the Salvetti Complex for A_Γ is the Cayley 2-complex associated with it. Hence, the fundamental group of $\mathcal{S}_\Gamma$ is A_Γ. Furthermore, the universal cover of the Salvetti Complex $\tilde{\mathcal{S}}_\Gamma$ has 1-skeleton corresponding to the Cayley graph of A_Γ with respect to the generating set $V\Gamma$.

We know from [4] that the universal cover of the Salvetti Complex $\tilde{\mathcal{S}}_\Gamma$ with the ℓ^2 metric is a CAT(0) cube complex. This result will be useful later since I will employ various features of CAT(0) cube complexes in my upper bound proof. The definition of a CAT(0) cube complex would probably require an excessive detour into geometric aspects that are unnecessary for the purposes of this piece. Thus, for the definition of a CAT(0) cube complex, the reader is advised to read [10, p.5-6]. Here I'll simply provide the definition of a cube complex and describe some standard properties of CAT(0) cube complexes.

Definition 3 (Cube Complex)**.** A **cube complex** is a quotient space $X = C/\mathcal{F}$ where C is a disjoint union of unit euclidean cubes of various dimensions, and $\mathcal{F}$ is a collection of isometries between faces of the cubes in C.

An important notion that I'll be employing later is that of hyperplanes in CAT(0) cube complexes. Like standard Euclidean hyperplanes, hyperplanes in a CAT(0) cube complex separate the space in two disconnected components.

Definition 4 (midcube)**.** Let C be an n-dimensional unit cube. A **midcube** of C is an $n-1$-dimensional unit cube passing through the barycentre of C and parallel to one of its faces.

Now, let X be a CAT(0) cube complex.We define an equivalence relation $\tilde{\square}$ on the edges of X given by the transitive closure of the relation $\square$: given two edges e and f of X, $e\square f$ iff e and f are opposite edges of some square in X. Write $[e]$ for the equivalence class of e with respect to $\tilde{\square}$.

Hence, we define an hyperplane dual to an edge e as follows:

Definition 5 (hyperplane)**.** Let X be a CAT(0) cube complex. Let e be some edge in X. The **hyperplane** $\mathcal{H}$ dual to $[e]$ is the space given by the set of all midcubes intersecting edges in $[e]$.

We know from [10, Th.1.1] that given a hyperplane $\mathcal{H}$ in a CAT(0) cube complex X, $X \setminus \mathcal{H}$ consists of two disconnected components. We call these components **half spaces**.

2 Separation Profile

In this section I will introduce the concept of the separation profile of an infinite graph and explain how it extends also to finitely generated groups. In doing so, I will follow mainly the definitions given in [2]. Given an infinite graph X the basic idea behind the separation profile is that it is a function which measures how easily can finite subgraphs of X be cut into small enough pieces. Because of this we need to define first what it means to cut a finite graph in this way:

Definition 6 (Cut Set, Cut Size)**.** Let Γ be a finite graph with n vertices. A **cut set** of Γ is a set of vertices S such that any connected component of $\Gamma \setminus S$ has at most $\frac{n}{2}$ vertices. The **cut size** of Γ, $Cut(\Gamma)$, is given by:

$$Cut(\Gamma) := min\{|S| \text{ such that } S \text{ is a cut set for } \Gamma\}$$

Definition 7 (Separation)**.** Let X be a (possibly infinite) graph. The **separation profile** of X is the function $sep_X : \mathbb{N} \to \mathbb{N}$ given by:

$$sep_X(n) := max\{Cut(\Gamma) \mid \Gamma \leq X, |\Gamma| = n\}$$

That is, the cut size of a finite graph measures what is the smallest number of vertices one needs to remove from a finite graph so that the remaining connected components have size smaller than half the size of the original graph. The separation profile measures what is the greatest cut size for a subgraph of size n of a given graph.

In order to apply the notion of separation to a finitely generated group, we first need to introduce the notion of quasi-isometry:

Definition 8 (Quasi-Isometry)**.** Let (X, d_x) and (Y, d_y) be metric spaces. Then, X and Y are **quasi-isometric** iff there is some constant $k \geq 1$ and a map $f : X \to Y$ such that for all $x_1, x_2 \in X$:

$$\frac{1}{k} d_X(x_1, x_2) - k \leq d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2) - k$$

Now, let X and Y be graphs with the standard metric on them. Then, we know from [2, p.4] that if X and Y are quasi-isometric, then they have the same

separation profile. Furthermore, we know from [6, p.132, Th.7.3] that given a finitely generated group G and finite generating sets S and S' for G, the Cayley graphs $\Gamma(G, S)$ and $\Gamma(G, S')$ are quasi-isomorphic.
Hence, given a finitely generated group G and a finitely generating set S for G, we can define the **separation profile of** G, $sep_G : \mathbb{N} \to \mathbb{N}$ as the separation profile of the Cayley Graph $\Gamma(G, S)$.

Finally, separation profiles are generally considered under the equivalence relation $\approx$ defined below:

Definition 9 ($\leq, \approx$)**.** Let $f, g : \mathbb{N} \to \mathbb{N}$. Then we write $f \leq g$ iff there is a constant $C \in \mathbb{N}$ such that:

$$f(n) \leq Cg(n) + C \text{ for all } n \in \mathbb{N}$$

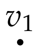
In this case we say that g has **order** greater than f. Furthermore, we write $f \approx g$ iff $f \leq g$ and $g \leq f$. In this case we say that f and g have the same order.

Some times (such as in the lower bound proof), I'll apply the notion of order also to the cut size of a graph since we'll be considering the cut size of a graph Γ of size n for arbitrary n.

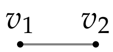
3 Separation of RAAGs Based on Trees

In this section I will introduce some previously known results about quasi-isometries of RAAGs based on trees, and about the separation profile for some such RAAGs.

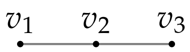
Firstly, we can classify RAAGs based on trees according to the diameter of their defining tree. In particular, we know that if the diameter of T is at most 2, then, A_T is either $\mathbb{Z}$ or $\mathbb{Z}^2$ or of the form $F \times \mathbb{Z}$ where F is a free group:



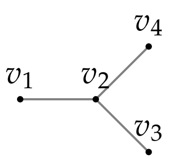
$$A_T = \langle v_1 \rangle = \mathbb{Z}$$



$$A_T = \langle v_1, v_2 \mid [v_1, v_2] \rangle \cong \mathbb{Z}^2$$



$$A_T = \langle v_1, v_2, v_3 \mid [v_1, v_2], [v_2, v_3] \rangle \cong F_2 \times \mathbb{Z}$$



$$A_T = \langle v_1, v_2, v_3 \mid [v_1, v_2], [v_2, v_3], [v_2, v_4] \rangle \cong F_3 \times \mathbb{Z}$$

$\vdots$

We know from [2, Prop.4.1] that $sep_{\mathbb{Z}^d} \approx n^{\frac{(d-1)}{d}}$ for $d \geq 1$. Hence, $sep_{\mathbb{Z}} \approx 1$, and $sep_{\mathbb{Z}^2} \approx \sqrt{n}$. For the case of $\mathbb{Z}$, one could have also simply noted that any finitely generated group whose Cayley graph with respect to some generators is a tree has separation 1. Finally, since the Cayley graph of any free group is a tree, we have from [3, Lemma 7.2] that $sep_{F\times\mathbb{Z}} = \sqrt{n}$ for free F. Thus, we know that if T is a tree of diameter 2, then $sep_{A_T} \approx \sqrt{n}$.

Finally, we know from [1, p.220] that all RAAGs based on trees of diameter at least 3 are quasi-isomorphic. Thus, the only case of a RAAG based on a tree for which we don't know its separation is that of the segment with four points:

$$A_T = \langle a_1, a_2 \mid [a_1, a_2], [a_2, a_3], [a_3, a_4] \rangle$$

In the rest of this piece I will focus on proving the separation profile of A_T. From now on, I will take A_T to refer to this group if not specified. By finding its separation profile, we'll know the separation profile of any RAAG based on a tree.

4 Separation of A_T

In this section I shall prove that $sep_{A_T} \approx n^{\frac{2}{3}}$. In doing so, I will begin by proving that $sep_{A_T} \preceq n^{\frac{2}{3}}$. This is an original result of mine. Secondly, I will present a proof that $n^{\frac{2}{3}} \preceq sep_{A_T}$ which was found by Dr. David Hume after the end of my research project. Hence, we obtain the following theorem:

Theorem 2 (Separation of A_T)**.** *Let A_T be the RAAG given by the group presentation $\langle a_1, a_2, a_3, a_4 | [a_1, a_2], [a_2, a_3], [a_3, a_4] \rangle$.Then, we have that:*

$$sep_{A_T}(n) \approx n^{\frac{2}{3}} \tag{1}$$

More generally, we now know the separation of any RAAG based on a tree:

Theorem 3 (Separation of RAAGs based on trees)**.** *Let T be a tree and A_T the RAAG based on it. If T has diameter 0, then $sep_{A_T}(n) \approx 1$. If T has diameter 1 or 2, then $sep_{A_T}(n) \approx \sqrt{n}$. If T has diameter ≥ 3, then $sep_{A_T} \approx n^{\frac{2}{3}}$.*

4.1 Upper Bound

Proposition 1 (Upper Bound)**.** *Let A_T be the RAAG given by the group presentation $\langle a_1, a_2, a_3, a_4 | [a_1, a_2], [a_2, a_3], [a_3, a_4] \rangle$.Then, we have that:*

$$sep_{A_T} \preceq n^{\frac{2}{3}} \tag{2}$$

4.1.1 Existence of a centerpoint

In order to prove the proposition above, I shall use the fact that any subgraph Γ of the Cayley graph of A_T has a centerpoint. That is, a point x such that any half space in Γ containing it will contain at least $\frac{n}{2}$ vertices. The existence of such a point is a consequence of Helly's theorem for median spaces and the fact that the Cayley graph of A_T is a median space.

Let (X, d) be a metric space. We say that a point c is **between** a and b iff $d(a, c) + d(c, b) = d(a, b)$. A **median space** is a metric space (X, d) such that for each triplet of points x, y, z there exists a unique median. That is a point m such that:

- $d(x, m) + d(m, y) = d(x, y)$
- $d(x, m) + d(m, z) = d(x, z)$
- $d(y, m) + d(m, z) = d(y, z)$

Finally, recall that $S \subset X$ is **convex** iff for any two points $x, y \in S$, S contains all points between them.

What matters for our proof is that for median spaces a particular version of Helly's Theorem[2] holds:

Theorem 4 (Helly's Theorem for Median Spaces). *Let (X, d) be a median space. Let $C_1, \ldots, C_k$ be convex sets in X such that $C_i \cap C_j \neq \emptyset$ for each $i, j \in \{1, ..k\}$. Then $\bigcap_{i=1}^{k} C_i \neq \emptyset$.*

Proof. We can prove the result by induction. The base cases for $k = 1$ and $k = 2$ are trivial. Consider the case for $k = 3$. Let C_1, C_2, C_3 be convex. Let $x \in C_1 \cap C_2$, $y \in C_1 \cap C_3$, $z \in C_2 \cap C_3$. Now, let m be their median point. Then, since m is between x and y, and C_1 is convex $m \in C_1$. Similarly $m \in C_2$ and $m \in C_3$. Thus, $m \in \bigcap_{i=1}^{3} C_i \neq \emptyset$.

The structure for the proof of the induction step follows closely that of this case. Suppose the claim of the theorem holds for k. Let $C_1, \ldots, C_{k+1}$ be convex sets of X such that $C_i \cap C_j \neq \emptyset$ for each $i, j \in \{1, ..k+1\}$. Now, we know from the

[2]The standard version of Helly's Theorem regards Euclidean spaces. If a reader is interested in a proof of the Euclidean version of Helly's Theorem and of the consequent Centerpoint Theorem, they could read [7, Ch.4]. However, the proofs are considerably easier for the case of median spaces and CAT(0) cube complexes respectively. [9, p.9] has a different proof that in a CAT(0) cube complex a finite set of pairwise intersecting half spaces has non-empty intersection.

inductive hyporthesis that $B_1 := \bigcap_{i=1}^{k} C_i \neq \emptyset$, $B_2 := \bigcap_{i=2}^{k+1} C_i \neq \emptyset$ and $B_3 := \bigcap_{i\neq 3}^{k+1} C_i \neq \emptyset$. Thus, let $x_1 \in B_1$, $x_2 \in B_2$ and $x_3 \in B_3$. Let m be their median. Now, m is between x_2 and x_3 which are both in C_{k+1}, and so, by convexity of C_{k+1}, $m \in C_{k+1}$. For any $i \neq 1$, $x_1, x_2 \in C_i$, and so $m \in C_i$ by convexity. Finally, $x_1, x_3 \in C_1$, and so by convexity $m \in C_1$.

Thus, $m \in \bigcap_{i=1}^{k+1} C_i \neq \emptyset$. □

Now, CAT(0) cube complexes with the ℓ^1 metric are median spaces [5][p.1]. We further know as a consequence of [11] that half spaces are convex.[3]

Thus, we can now prove that given a finite set of points in a CAT(0) cube complex, they will have a centerpoint.

Theorem 5 (Existence of a Centerpoint)**.** *Let X be a CAT(0) cube complex. Let $S \subseteq X$ be a finite set of points of X. Then, there is a point c such that any closed half-space containing c has at least $\frac{n}{2}$ points of S.*

Proof. Consider the family of open half spaces containing more than $\frac{n}{2}$ points of S. Note that since S is finite, only finitely many of them differ in terms of which points of S they contain. Thus, for H'_j, $j \in J$ half-spaces containing the same elements of S, let H_J be their intersection. This will still be convex since half-spaces are convex and the intersection of convex sets is convex. Hence, we can consider a finite family of convex sets $H_1, ..., H_k$. Now, any two H_i's will have non-empty intersection (since each of them contains more than half of the points of S). By Helly's Theorem for median spaces, $\bigcap_{i=1}^{k} H_i \neq \emptyset$. Let $c \in \bigcap_{i=1}^{k} H_i \neq \emptyset$. Let H be any closed half-space going through c. Now, the open half-space complementary to H must have less than $\frac{n}{2}$ points of S since it doesn't contain c. Hence, H contains more than $\frac{n}{2}$ points of S.

[3]More specifically, let $\mathcal{H}$ be the hyperplane dual to $[e]$. Give the vertices of the edges crossed by $\mathcal{H}$ an orientation as follows: call one vertex of e $\partial_0 e$ and the other $\partial_1 e$. Now, for each edge f crossed by $\mathcal{H}$ label $\partial_0 f$ the vertex in the same connected component of $\partial_0 e$ and label the other vertex $\partial_1 f$. Now, let $\partial_0\mathcal{H} := \{v \in [e] \mid v \in \partial_0 e\}$ and similarly define $\partial_1\mathcal{H}$. What [11, Thm 4.13] says is that given vertices $v, w \in \partial_0\mathcal{H}$, any geodesic between them is entirely contained in $\partial_0\mathcal{H}$. Hence, given the ℓ^1 metric on a CAT(0) cube complex, and two vertices in its 1-skeleton in the same half space, any geodesic between them is entirely contained within the same half space. For, suppose that some geodesic between them passes through the other half space. Then, it needs to cross $\mathcal{H}$ at some point, and in particular a subpath of that geodesic must be a path from an element of $\partial_0\mathcal{H}$ to another of its elements crossing the hyperplane. But, by [11, Thm 4.13] that cannot be a shortest path between them, and so we would have a contradiction. Hence, given two points in the same half space, the line between them in the ℓ^1 metric must be contained entirely in that half space. Thus, half spaces in CAT(0) cube complexes are convex.

Thus, any half-space containing c will have at least $\frac{n}{2}$ points by construction. $\square$

Now, we know from [4][p.10] that the universal cover of the Salvetti Complex associated with A_T is a $CAT(0)$ cube complex. Thus, since we can view the Cayley graph of A_T as the 1-skeleton of the universal cover of the associated Salvetti Complex, the Cayley graph of A_T and Γ are $CAT(0)$ cube complexes. Hence, Γ has a centerpoint c with respect to its vertices.

4.1.2 The Upper Bound proof

Proof of Proposition 2. Let Γ be a finite subgraph with n vertices of X, the Cayley graph of A_T with respect to $\{a_1, a_2, a_3, a_4\}$. Let c be a centerpoint of Γ, whose existence is proved in Theorem 5. Consider Γ translated so that c is the vertex standing for the identity element of A_T. We will construct a cut set S for Γ. Firstly, I investigate sets of vertices that separate Γ in a way similar to hyperplanes. Sets of this type will build the cut set for Γ. Secondly, I consider some external cuts which remove half-spaces distant from the origin. Finally, I find some internal cuts which separate the remaining connected components of Γ. I will also label each connected component of $\Gamma \setminus S$ in order to show that each of them has size $\leq n/2$.

Hyperplanes and cosets Let $g \in A_T$. Note that the hyperplane $\mathcal{H}$ dual to the edge $[g, ga_2]$ (in the universal cover of the Salvetti Complex) is a tree which is a copy of the Cayley graph of F_2. Furthermore, any path from ga_2 to ga_2^{-1} must pass through $\mathcal{H}$. Hence, any such path must contain an edge of the form $[gw, gwa_2]$ where $w \in \langle a_1, a_3 \rangle$. Therefore, any connected component of $\Gamma \setminus g\langle a_1, a_3 \rangle$ is entirely contained in one of the two half-spaces[4] formed by $\mathcal{H}$. Furthermore, one half-space contains ga_2, and the other ga_2^{-1}.

Similarly, any connected component of $\Gamma \setminus g\langle a_2, a_4 \rangle$ is entirely contained in one of the two half-spaces formed by the hyperplane dual to $[g, ga_3]$. One half-space will contain ga_3, and the other ga_3^{-1}.
The hyperplane dual to $[g, ga_1]$ is a copy of the Cayley graph $\Gamma(\mathbb{Z}, 1)$. Hence, by the same argument as above, $g\langle a_2 \rangle$ disconnects ga_1 from ga_1^{-1}. In particular, let $\mathcal{S}_1$ be the half-space formed by the hyperplane dual to $[g, ga_1]$ containing ga_1 and let $\mathcal{S}_2$ be the half-space formed by the hyperplane dual to ga_1^{-1} containing ga_1^{-1}. Now, $g\langle a_2 \rangle$ disconnects X such that the connected component containing

[4]Note that any hyperplane I will consider contains no vertex of Γ. Because of this, given one of these hyperplanes, the two closed half-spaces formed by it won't share any vertex of Γ. Hence, for the purposes of this proof we can take all the half-spaces being considered to be closed.

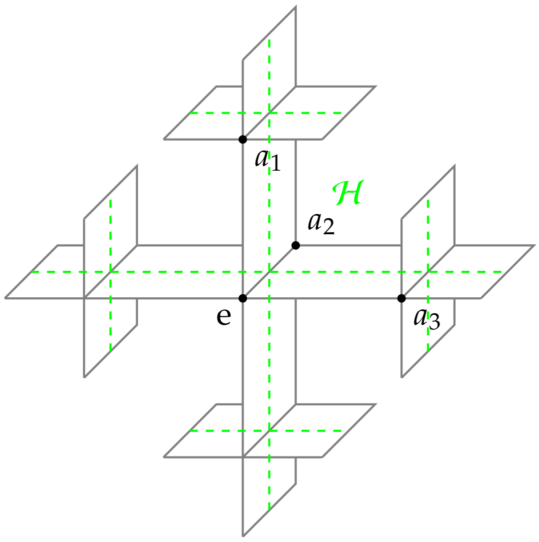


Figure 2: Here we can see $\mathcal{H}$, the hyperplane dual to a_2 on a fragment of $\tilde{\mathcal{S}}_T$. In this picture I'm not drawing any of the vertices with some a_4 in its minimal word since $\mathcal{H}$ doesn't meet any of them.

ga_1 is entirely contained in $\mathcal{S}_1$, and the connected component containing ga_1^{-1} is entirely contained in $\mathcal{S}_2$. Similarly, $g\langle a_3\rangle$ disconnects X such that the connected component of $X \setminus g\langle a_3\rangle$ containing ga_4 is contained in the half-space dual to $[g, ga_4]$ containing ga_4 and that containing ga_4^{-1} is contained in the half-space dual to $[g, ga_4^{-1}]$ containing ga_4.

External cuts Let $G_k := \Gamma \cap a_2^k\langle a_1, a_3\rangle \cup a_2^{-k}\langle a_1, a_3\rangle$. Note that by the Pidgeonhole Principle we can choose $0 \le k_2 \le n^{\frac{1}{3}}$ such that $|G_{k_2}| \le n^{\frac{2}{3}}$. From the previous paragraph we know that $a_2^{k_2}\langle a_1, a_3\rangle$ disconnects X so that any connected component of $X \setminus a_2^{k_2}\langle a_1, a_3\rangle$ is entirely contained in one of the two half-spaces formed by the hyperplane dual to $[a_2^{k_2}, a_2^{k_2+1}]$. Call $\mathcal{S}_1$ the half-space containing $a_2^{k_2-1}$, and $\mathcal{S}_2$ that containing $a_2^{k_2+1}$. Now, $a_2^{k_2-1}$ is connected to 1 without crossing our hyperplane (just consider the path with word $a_2^{k_2-1}$). Thus, by our choice of origin as the centerpoint of Γ, $|\mathcal{S}_1| \ge \frac{n}{2}$ and $\mathcal{S}_2 < \frac{n}{2}$. Similarly, for $a_2^{-k_2}\langle a_1, a_3\rangle$, let $\mathcal{S}_3$ be the half-space containing $a_2^{-k_2+1}$, and $\mathcal{S}_4$ that containing $a_2^{-k_2-1}$. Again, by our choice of origin, $|\mathcal{S}_3| \ge \frac{n}{2}$ and $\mathcal{S}_4 < \frac{n}{2}$. Thus, any connected component of $\Gamma \setminus G_{k_2}$ with more than $n/2$ vertices must be in $\Gamma \cap \mathcal{S}_1 \cap \mathcal{S}_3$.

Similarly, let $F_k := \Gamma \cap a_3^k\langle a_2, a_4\rangle \cup a_3^{-k}\langle a_2, a_4\rangle$, choose k_3 as above, and note that any connected component of $\Gamma \setminus\ F_{k_3}$ with more than $n/2$ vertices must be in $\Gamma \cap \mathcal{S}'_1 \cap \mathcal{S}'_3$ where the $\mathcal{S}'_i$'s are chosen in the same way as the previous case. Hence, any connected component of $\Gamma \setminus (G_{k_2} \cup F_{k_3})$ with more than $n/2$ vertices must be contained in $\Gamma \cap \mathcal{S}_1 \cap \mathcal{S}_3 \cap \mathcal{S}'_1 \cap \mathcal{S}'_3$.

labelling We shall further label the connected components outside of $\Gamma \cap \mathcal{S}_1 \cap \mathcal{S}_3 \cap \mathcal{S}'_1 \cap \mathcal{S}'_3$ in order to later prove that every element of Γ is in some connected component of size smaller than $n/2$.

In order to do so, I shall prove the following lemma:

Lemma 1 (minimal words starting in a_2^l)**.** *Let h be a vertex of Γ with a minimal word beginning with a_2^l, for $l > k_2$. Then, in $\Gamma \setminus a_2^{k_2}\langle a_1, a_3\rangle$, h is in the same connected component as $a_2^{k_2+1}$.*

Proof. Suppose by contradiction that $h = a_2^l w$ isn't in the same connected component of $a_2^{k_2+1}$. Then, the path represented by w starting at a_2^l crosses $a_2^{k_2}\langle a_1, a_3\rangle$ at some point $a_2^{k_2}v$ where v is a word in $\langle a_1, a_3\rangle$. Thus, let $w = w'w''$, where $a_2^l w'$ is a word representing $a_2^{k_2}v$. Now, from Theorem 1, we know that we can obtain $a_2^l w'$ from $a_2^{k_2}v$ by a decreasing sequence of words through moves of type 1,2,3 [5]. However, v contains no instance of a_2 and $k_2 < l$. So, at least one move must increase the number of syllables adding a_2's. Contradiction. □

Similarly, the following hold: any minimal word beginning with a_2^{-l}, for $l > k_2$ is in the same connected component of $\Gamma \setminus a_2^{-k_2}\langle a_1, a_3\rangle$ as $a_2^{-k_2-1}$; any minimal word beginning with a_3^m, for $m > k_3$ is in the same connected component of $\Gamma \setminus a_3^{k_3}\langle a_2, a_4\rangle$ as $a_3^{k_3+1}$; any minimal word beginning with a_3^{-m}, for $m > k_3$ is in the same connected component of $\Gamma \setminus a_m^{-k_3}\langle a_2, a_4\rangle$ as $a_3^{-k_3-1}$.

Internal cuts Now, consider $\Gamma' := \Gamma \cap \mathcal{S}_1 \cap \mathcal{S}_3 \cap \mathcal{S}'_1 \cap \mathcal{S}'_3$.

Let $C := \{g \in \Gamma | g = a_2^l a_3^m, -k_2 < l < k_2, -k_3 < m < k_3\}$. Clearly, $|C| \leq 2n^{\frac{1}{3}}.2n^{\frac{1}{3}} = 4n^{\frac{2}{3}}$.

Lemma 2 (minimal words starting in $a_3^m a_2^l a_1^k$)**.** *Let h be a vertex of Γ' with a minimal word representation starting in $a_3^m a_2^l a_1^k$ for $k \neq 0$. Then, in $\Gamma' \setminus C$, h is in the same connected component of $a_3^m a_2^l a_1$.*

Proof. Proof by contradiction. Let h have a minimal word representation of the form $a_3^m a_2^l a_1^k w$. Suppose h is not in the same component of $a_3^m a_2^l a_1$. Then, the

[5] To see this note that a subword of a minimal word is still minimal

path given by w starting at $a_3^m a_2^l a_1$ must cross C at some point. Let $w = w'w''$, where $a_3^m a_2^l a_1^k w'$ is a word representation for $a_3^p a_2^r$, the point of C intersected by our path.

Now, a subword of a minimal word is still minimal [since otherwise, we could have a representation with fewer syllables of the original minimal word]. Thus, $a_3^m a_2^l a_1^k w'$ is minimal and represents the same element as $a_3^p a_2^r$. But this latter word has fewer syllables and so $a_3^m a_2^l a_1 w'$ cannot be minimal. Contradiction. □

Note that similar results hold for a_1^{-1}, a_4, and a_4^{-1}. That is, if h has a minimal word representation starting in $a_3^m a_2^l a_1^{-k}$, then it is in the same component of $\Gamma' \setminus C$ as $a_3^m a_2^l a_1^{-1}$. If h has a minimal word representation starting in $a_3^m a_2^l a_4^k$, then it is in the same component of $\Gamma' \setminus C$ as $a_3^m a_2^l a_4$. If h has a minimal word representation starting in $a_3^m a_2^l a_4^{-k}$, then it is in the same component of $\Gamma' \setminus C$ as $a_3^m a_2^l a_4^{-1}$.

Now, note that any element of $\Gamma' \setminus C$ must have a minimal word representation starting with one of $a_3^m a_2^l a_1$, $a_3^m a_2^l a_1^{-1}$, $a_3^m a_2^l a_4$, or $a_3^m a_2^l a_4^{-1}$ for $m, l \in \mathbb{Z}$, otherwise they would be in C. Hence, every connected component of $\Gamma' \setminus C$ can be labelled by $a_3^m a_2^l a_j^i$, for $m, l \in \mathbb{Z}$ $j \in \{1, 4\}$, $i \in \{1, -1\}$, depending on the elements they contain.

Cutset Thus, we can choose our cutset to be $S := G_{k_2} \cup F_{k_3} \cup C$, which has size $\leq 6n^{\frac{2}{3}}$. We shall prove that any connected component of $\Gamma \setminus S$ has $< \frac{n}{2}$ elements. Hence, $sep_{A_T}(n) \leq n^{\frac{2}{3}}$.

Let $g \in \Gamma$. Note that g must have a minimal word beginning with a word in the following list [with $l, m \in \mathbb{Z}$]:

1. a_2^l, for $l \geq k_2$
2. $a_2^- l$, for $l \geq k_2$
3. a_3^m, for $m \geq k_3$
4. $a_3^- m$, for $m \geq k_3$
5. $a_3^m a_2^l a_1$, for $|l| < k_2, |m| < k_3$
6. $a_3^m a_2^l a_1^{-1}$, for $|l| < k_2, |m| < k_3$
7. $a_3^m a_2^l a_4$, for $|l| < k_2, |m| < k_3$

8. $a_3^m a_2^l a_4^{-1}$, for $|l| < k_2, |m| < k_3$

9. $a_3^m a_2^l$, for $|l| < k_2, |m| < k_3$

This is the case since in our group $[a_1, a_2]$, $[a_2, a_3]$, and $[a_3, a_4]$; Hence, given any minimal word starting with some different set of initial letters, we can perform moves of type 1-3 until it becomes a minimal word in our list.

Let g be of type 1. For $l = k_2$, $g \in G_{k_2}$. Otherwise, by Lemma1, g is in the same connected component of $a_2^{k_2+1}$ with respect to $\Gamma \setminus G_{k_2}$. Furthermore, such component has $< \frac{n}{2}$ elements.
Let g be of type 5. We shall prove that any element in the connected component containing g is contained in the half-space formed by the hyperplane dual to $[a_3^m a_2^l, a_3^m a_2^l a_1]$ containing $a_3^m a_2^l a_1$.
Let h be any element of the other half-space. Then, any path from g to h must cross $a_3^m \langle a_2 \rangle$ (call it $\mathcal{S}$). Now, any path going through $a_3^m a_2^r$ for $-k_2 < r < k_2$ crosses C. Any path crossing $a_3^m a_2^{r'}$ for $|r'| > k_2$ must cross first G_{k_2} since it has a minimal word representation starting in $a_2^{r'}$. Thus, there is no path from g to h without crossing some element of our cut set. Hence, the connected component containing g is entirely contained in $\mathcal{S}$. Now, since the other half-space contains 1, the connected component containing g has less than $n/2$ elements.
Clearly, elements of type 9 are in C. All the other cases are analogous to one of those described above.

Hence, we have proven that each element of $\Gamma \setminus S$ is in some connected component with less than $n/2$ vertices. And so, S is a cut set of size $\leq 6n^{\frac{2}{3}}$ for Γ. Thus,

$$sep_{A_T} \preceq n^{\frac{2}{3}}$$

$\square$

4.1.3 Lower Bound Proof

In order to prove that $n^{\frac{2}{3}}$ is also a lower bound for the separation profile of A_T, we need to find a subgraph of the Cayley graph of A_T with n vertices with no cut set of size smaller than $n^{\frac{2}{3}}$. In this subsection I will present one such subgraph. The proof I'm presenting here was found by Dr David Hume and it uses a path-based argument which is commonly used in lower bound proofs for the separation profile. An interesting aspect of this subgraph is that it has a very different appeareance from graphs previously known to have cut size of $n^{\frac{2}{3}}$ such as the n-ball in $\mathbb{Z}^3$.

Before giving the appropriate subgraph and computing their minimal cut size, I shall introduce a useful lemma presented in one of Dr David Hume's research group meetings. Let Γ be a graph. Recall that the **boundary** of a set of vertices $B \subseteq V\Gamma$ is the set of vertices w in $\Gamma \setminus B$ such that for some vertex $v \in B$, $\{v, w\}$ forms an edge of Γ. We write ∂B to denote the boundary of B.

Lemma 3. *Let Γ be a finite graph with n vertices. Let S be a cut set such that $S \leq n/4$. Then, there are $C, D \subseteq V\Gamma$ such that $n/4 \leq |C|, |D| \leq n/2$, $\partial C, \partial D \subseteq S$.*

Proof. List all of the connected components of $\Gamma \setminus S$ as $\Gamma_1, \Gamma_2, ..., \Gamma_k$ such that for each i, $|\Gamma_i| \geq |\Gamma_{i+1}|$. Each Γ_i has size $\leq n/2$ since S is a cut set. Now, if $|\Gamma_1| > n/4$, let $C = \Gamma_1$. Otherwise, consider $\Gamma_1 \cup \Gamma_2$ and repeat the process. More generally, assuming $|\Gamma_1 \cup ... \cup \Gamma_m| < n/4$, if $|\Gamma_1 \cup ... \cup \Gamma_{m+1}| > n/4$, take $C = \Gamma_1 \cup ... \cup \Gamma_{m+1}$. Note that we can't have $|\Gamma_1 \cup ... \cup \Gamma_m| < n/4$ and $|\Gamma_1 \cup ... \cup \Gamma_{m+1}| > n/2$ since that would mean that $\Gamma_{m+1} > n/4$. But then, since $|\Gamma_i| \geq |\Gamma_{i+1}|$, $\Gamma_m > n/4$ and so, $|\Gamma_1 \cup ... \cup \Gamma_m| > n/4$. Now, the process must eventually terminate for the choice of C since there are more than $\frac{3}{4}n$ vertices in $\Gamma_1 \cup ... \cup \Gamma_k$. For D repeat the same process starting from Γ_{i+1} where Γ_i is the last connected component added to C. The process must terminate also for the choice of D since there are more than $n/2$ vertices in $\Gamma_{i+1} \cup ... \cup \Gamma_k$ □

Now, let me give the subgraph of a Cayley graph of A_T with no cutsets of size $\leq n^{\frac{2}{3}}$. Fix $n \in \mathbb{N}$. Let $G := \{g \in A_T \mid g = a_2^k a_3^l f^m$ for $f \in \{a_1, a_4\}, 0 \leq k, l, m < n^{\frac{1}{3}}\}$. Thus, $|G| = n$. Let Γ be the Cayley graph of G with respect to $\{a_1, a_2, a_3, a_4\}$. This is a subgraph of the Cayley graph of A_T. We shall prove that Γ has cut-size $\geq n^{\frac{2}{3}}$.

Lemma 4. *Let Γ be the graph described above. Then, $Cut(\Gamma) \geq n^{\frac{2}{3}}$.*

Proof. Let S be a cut set for Γ of size $\leq n/4$. Note that by Proposition 2 there are cut sets of size $\approx n^{\frac{2}{3}}$ and for large enough n , $n^{\frac{2}{3}} \leq n/4$. Thus, for n large enough such S exists. Thus, by Lemma 3, there are $C, D \subseteq V\Gamma$ such that $n/4 \leq |C|, |D| \leq n/2$ and $\partial C, \partial D \subseteq S$.

Now, choose two random vertices of Γ $a_2^k a_3^l f^m$ and $a_2^{k'} a_3^{l'} f'^{m'}$. We shall construct a path from the first to the second through describing an algorithm. Begin with $a_2^k a_3^l f^m$. For $f = a_1$ remove each a_1 until you get $a_2^k a_3^l$ and add a_4's until you get $a_2^k a_3^l a_4^m$, otherwise, skip this step. At this point add or remove a_3's until you get the word $a_2^k a_3^{l'} a_4^m$; remove all of the a_4's and add m' times a_1's instead. Now, add or remove a_2's until you get $a_2^{k'} a_3^{l'} a_1^{m'}$. Now, if $f' = a_4$ remove all of the a_1's and add a_4's instead, otherwise, skip this step.[6]

[6]Thus, for example, for $a_2^2 a_3^4 a_1^3$ and $a_2^5 a_3^3 a_1$ our algorithm describes the path $a_1^{-3} a_4^3 a_3^{-1} a_4^{-3} a_1 a_2$.

Now, let $z \in \Gamma$. We shall prove that there are at most $8n^{\frac{4}{3}}$ paths of the type described above going through z. For $z = a_2^a a_3^b a_1^c$ note that the paths going through z are all those from some $x = a_2^k a_3^l f^m$ and $y = a_2^{k'} a_3^{l'} f'^{m'}$ such that:

- $a = k, b = l, f = a_1, c \leq m$ [and so we have $(2m)n$ choices for x and y since we can choose any k', l', m' and f' and there are only $n^{\frac{1}{3}}$ for each of k', l', m'].
- $a = k, b = l', f = a_1, c \leq m'$ [still $(2m)n$ choices as above]
- $a = k', b = l', f = a_4, c \leq m'$ [still $(2m)n$ choices as above]

The case is analogous for $z = a_2^a a_3^b a_4^c$. Hence, the largest number of paths going through some z is $8n^{\frac{4}{3}}$ for z of the form $a_2^a a_3^b$.

Now, since each path going from C to D must pass through S [since they are disjoint and with boundary $\subseteq S$], there are are at most $8n^{\frac{4}{3}}|S|$ paths of the type described above from C to D. Furthermore, since given any fixed $x \in C$ and $y \in D$ there is exactly one such path, there are exactly $|C||D|$ many of them from C to D. Hence, we have that $8n^{\frac{4}{3}}|S| \geq |C||D|$. And so, since $n/4 \leq |C|, |D|$:

$$|S| \geq \frac{|C||D|}{8n^{\frac{4}{3}}} \geq \frac{n^2}{16(8n^{\frac{4}{3}})} = \frac{1}{128} n^{\frac{2}{3}} \tag{3}$$

Thus, we have that $S \geq n^{\frac{2}{3}}$, and so $Cut(\Gamma) \geq n^{\frac{2}{3}}$. □

Thus, since for n large enough we can find a subgraph of Γ with cut size $\geq \frac{2}{3}$ we have found a lower bound for the separation profile of A_T:

Proposition 2 (Lower Bound). *Let A_T be the RAAG given by the group presentation $\langle a_1, a_2, a_3, a_4 | [a_1, a_2], [a_2, a_3], [a_3, a_4] \rangle$.Then, we have that:*

$$n^{\frac{2}{3}} \leq sep_{A_T} \tag{4}$$

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