

Extra exercises are marked with a **. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

ZOMBIE EXERCISE 1. Let F be an infinite field. Show that there is an elementary extension $F \preceq K$ containing a transcendental element t over F . Considering $\text{acl}(t)$, show that there is no ω -categorical theory extending the theory of infinite fields.

EXERCISE 2. Consider $\mathcal{Q}_3 := (\mathbb{Q}; <, P_0, P_1; (c_i)_{i < \omega})$ where P_0 and P_1 are unary predicates partitioning \mathbb{Q} into dense subsets and the c_i name an increasing sequence of elements of P_0 . Show that the theory \mathcal{Q}_3 has exactly three countable models up to isomorphism. Show that for each $n > 3$ there is a complete theory with exactly n countable models up to isomorphism.

EXERCISE 3. Work in a countable relational language \mathcal{L} . Let \mathcal{C} be a class of finite \mathcal{L} -structures closed under isomorphism and substructures. Show that the following are equivalent:

- \mathcal{C} has the amalgamation property;
- \mathcal{C} has the 1-point amalgamation property: Let $A, B_0, B_1 \in \mathcal{C}$ and $f_i : A \rightarrow B_i$ be embeddings with $|B_i \setminus A| = 1$ for $i \in \{0, 1\}$. Then, there is some $D \in \mathcal{C}$ and embeddings $g_i : B_i \rightarrow D$ for $i \in \{0, 1\}$ such that $g_0 \circ f_0 = g_1 \circ f_1$.

EXERCISE 4. Work in a countable relational language \mathcal{L} with no 0-ary relation (and no constant symbol). Let \mathcal{C} be a class of finite \mathcal{L} -structures closed under isomorphisms, substructures and with the amalgamation property. Deduce that \mathcal{C} also has the joint embedding property.

EXERCISE 5. Let $\mathcal{L}_{\mathbb{Q}^{\geq 0}}$ be a language with a binary relation d_q for each $q \in \mathbb{Q}^{\geq 0}$. Consider the class \mathcal{K} of finite metric spaces $(M; (d_q)_{q \in \mathbb{Q}^{\geq 0}})$ for which the distance between any two points is always rational and for each $q \in \mathbb{Q}^{\geq 0}$, d_q is interpreted as the binary relation holding of two points if and only if they are at distance q . Show that \mathcal{K} is a Fraïssé class. The associated Fraïssé limit $\mathbf{U}_{\mathbb{Q}}$ is called the **rational Urysohn space**. Is this structure ω -categorical?

Definition 1. The **exponent** of a group G is the least common multiple of the orders of its elements (or ∞ if there is no such least common multiple).

Fact 2. An Abelian group of finite exponent is a direct sum of finite cyclic groups.

EXERCISE 6. Show that an ω -categorical group has finite exponent. Deduce that an infinite Abelian group is ω -categorical if and only if it has finite exponent. Describe an axiomatisation of the different complete ω -categorical theories of (infinite) Abelian groups.

Definition 3. Let G be a group. We say that $H \leq G$ is a **characteristic subgroup** of G if and only if for all $\phi \in \text{Aut}(G)$, $\phi(H) \leq H$. We say that G is **characteristically simple** if it has no proper nontrivial characteristic subgroups.

** **EXERCISE 7.** Note that if H is a characteristic subgroup of G every automorphism of G induces an automorphism of G/H . Let G be an ω -categorical countable group.

1. Show that G has finitely many characteristic subgroups;
2. Show that if H is a characteristic subgroup of G such that H and G/H are both infinite, then both H and G/H are ω -categorical;
3. Show that G has a finite characteristic series

$$G_0 = \{1\} < G_1 < \cdots < G_m = G,$$

where for each $i < m$, G_{i+1}/G_i is characteristically simple and either finite or ω -categorical.