FINITE STRUCTURES

A **SIMPLE LIE GEOMETRY** L is either

① a WEAK linear geometry of any type other than Polar or quadratic; (we are not naming field)

② The projectivisation of something in ①;

③ The affine or quadratic part of a geometry.

A **COORDINATIZING STRUCTURE** of type (e, K) and dimension d is a structure $\mathcal{C}$ with transitive automorphism group and an $\mathrm{Aut}(\mathcal{C})$-invariant equivalence relation with $e < \infty$ many classes s.t. each class is associated with a simple Lie geometry L of dimension d over a finite field K (we may associate to $\mathcal{C}$ also the type of the geometry).

$\mathcal{C}$ is **PROPER** if each equivalence class of $\mathcal{C}$ as a geometry is canonically embedded in $\mathcal{C}$ (i.e. the automorphism group induced on each class is dense in the automorphism group of the associated geometry and equal to it if d is finite)

$\mathcal{C}$ finite dimensional is **SEMIPROPER** if $\mathrm{Aut}(\mathcal{C})$ induces on each class a subgroup $G \leq \mathrm{Aut}(L)$ containing $\mathrm{Aut}(L)^{(\infty)}$ (so close enough to $\mathrm{Aut}(L)$, and in particular a classical group).

$$\text{,,}P\Gamma L(K,d)^{(\infty)}$$

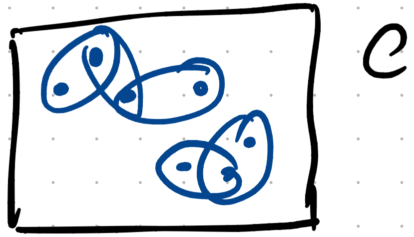
$$PSL(K,d) \leq PGL(K,d) \leq P\Gamma L(K,d)$$

EXAMPLES:

- Basic projective space of dim d over K is a semiproper coord. structure of type $(1, K)$ and dim d with L the weak proj. space
- A weak polar geom. is a proper coord. structure with $e = 2$.

Let $\mathcal{C}$ be a coordinatising structure of type (e, K) and dim d.
Let τ be a type over $\emptyset$ of a finite algebraically closed subset $A \subseteq \mathcal{C}$. The GRASSMANIAN $\Gamma(\mathcal{C}, \tau)$ is the set of realisations of τ in $\mathcal{C}$ with the structure induced by $\mathcal{C}$.
We say it is of type (e, K, τ) and dimension d.

EXAMPLE: $\mathcal{C}$ = a degenerate space of size $d = \{1, \dots, d\}$.
τ_k = type of a k-element subset of $\mathcal{C}$.



$\Gamma(\mathcal{C}, \tau_k) \cong J(d, k)$ the Johnson graph
domain: k-element subsets of d
Relations: $S_i(x, y)$ for $i \leq k$ saying $|x \cap y| = i$.

FACT 3.3.3 [KLM] $\forall k\ \exists n_k$ s.t. for any primitive structure M of order at least n_k, if M has at most k 4-types, then M is isomorphic to a semiproper Grassmanian of type (e, K, τ) with $e, |K|, |\tau| \leq k$.

- KLM does this for 5. See [M. 1997] for 4. False for 3.

Internally finite structures

$L :=$ a finite language (also works with L a recursive set).
$T :=$ an L-theory

Coding L-formulas in ω (by Gödel numbering) we can view L and T as 0-definable sets in $(V, \in)$

Work in non-standard extension $(V^*, \in) \succneqq (V, \in)$ with non-standard natural numbers.

We will work with an internally finite structure M which internally is an L-structure
(i.e. $(V^*, \in) \vdash$ "M is an L-structure")
but externally is an L_0-structure (where $L \subseteq L_0$)

M^* is the structure with the same domain as M whose atomic relation symbols are the names of the relations with finitely many variables defined on M by L_0-formulas (say L^* is the language of M^*)

So, we have:

$(M; L_0) \rightsquigarrow V^*$ believes this is an L-structure

$\downarrow$ reduct (loose L_0-forms with inf. many (but internally fin many) variables
loose variables with non-standard indices)

$(M^*, L^*) \rightsquigarrow V^*$ does not see L^* as a language.

$\downarrow$ reduct (we can view $L \subseteq L^* \subseteq L_0$ since its atomic symbols
were coded by natural numbers)

(M, L)

IDEA: (M^*, L^*) has a lot more information than (M, L)

EXAMPLE:

For new, let M_n: path $\underset{1}{\circ}\!-\!\circ \cdots \circ\!-\!\underset{n}{\circ}$ in language $L = L_n = \{E\}$

Let $\mathcal{U}$ be a non-principal ultrafilter on ω.

Now, consider each (M_n, L_n) inside $\mathbb{N}_n$ (copy of $\mathbb{N}$) in a copy V_n of V

$V^* := \prod V_n / \mathcal{U}$ $\qquad M = \prod M_n / \mathcal{U}$

Now,

$m \in \langle m_n \mid n \in \omega \rangle / \mathcal{U}$ is internally an L-formula if

$\{ n \in \omega \mid V_n \models$ "m_n is an L-formula" $\} \in \mathcal{U}$ $\leftarrow$ gives us L_0

$V^* \models$ "$M \models m$" iff $\{ n \in \omega \mid V_n \models (M_n \models m_n) \} \in \mathcal{U}$

$\uparrow$ L_0-formula

① we get L_0-formulas in non-standardly many variables

$$\left\langle \bigwedge_{i<j<n} x_i \neq x_j \;\middle|\; n \in \omega \right\rangle / \mathcal{U}$$

② we get L_0 formulas describing "x and y are at distance n" for n non-standard:

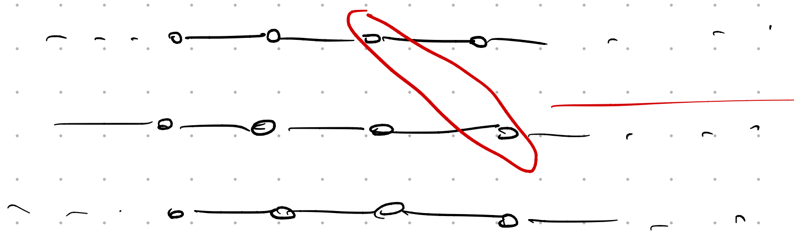
$$\langle D_n(x,y) \mid n \in \omega \rangle / \mathcal{U}$$

D_n: x and y are at distance n

③ we still have L-formulas inside of L_0 (identified with constant sequences).

When moving from L_0 to L^* we kill formulas of the form ①
but preserve formulas of the form ② and ③

So, $\mathrm{Aut}(M, L_0) \leq \mathrm{Aut}(M^*, L^*) \leq \mathrm{Aut}(M, L)$



M^* keeps track of non-standard distances between points

Why use this setting?

Let M be an L-structure

(a) M is κ-quasifinite if in a non-standard extension $(V^*, \in) \succ (V, \in)$ it is elementarily equivalent to an internally finite model with finitely many internal κ-types.

Issue: M may not even be a structure in V^*

I think they mean: there is an int. finite model M_0 with fin. many internal κ-types s.t. externally we see $M \equiv M_0|_L$

(b) M is quasifinite if in a non-standard $(V^*, \in) \succ (V, \in)$ it is elementarily equivalent (in the original language L) to an internally finite L_0-structure with finitely many internal κ-types for all κ.

(c) M is STRONGLY quasifinite if in a non-standard $(V^*, \in) \succ (V, \in)$ it is elementarily equivalent (in original language L) to an internally finite L_0-structure with a finite number of internal k-types, which coincide with k-types for all k.

At the end of this chapter we will have proved:

M is Lie coordinatizable iff M is strongly quasifinite

We also have finitary re-phrasings:

(b) $\Rightarrow$ (b') by contrapositive

(b') $\Rightarrow$ $\mathcal{M}$ is pseudofinite $\Rightarrow$ (b)

(a') A structure $\mathcal{M}$ is k-quasifinite if and only if there is a finite number N such that for an arbitrary sentence φ true in $\mathcal{M}$, there is a finite structure $\mathcal{N}$ satisfying φ in which there are at most N formulas in k free variables.

(b') A structure $\mathcal{M}$ is quasifinite if and only if there is a function $\nu : \mathbb{N} \to \mathbb{N}$ such that for any n and an arbitrary sentence φ true in $\mathcal{M}$, there is a finite structure $\mathcal{N}$ satisfying φ in which there are at most $\nu(k)$ formulas in k free variables for $k \leq n$.

(c') A structure $\mathcal{M}$ is STRONGLY quasifinite if there is a function $\nu : \mathbb{N} \longrightarrow \{\text{finite sets of } L\text{-formulas}\}$ s.t. for any n and arbitrary sentence φ true in $\mathcal{M}$, there is a finite structure $\mathcal{N}$ satisfying φ and s.t. the formulas in $\mathcal{N}$ in k-many variables are contained in $\nu(k)$.

<u>Note</u>: quasifinite $\Rightarrow$ pseudofinite and ω-categorical