FINITE STRUCTURES WITH FEW TYPES

Goal. Prove Lemmas 2.5.3, 2.5.4 and 2.5.5. To do this, I will first need to recall some of the BASIC NOTIONS from the previous sessions. I'll say a bit more than what will be needed in the statements/proofs – this will hopefully be useful, as it's been a while since our last meeting.

Things in blue are are remarks and additions that *I* feel like should be made.

Things in red are additions to statements or questions that I have.

1. Reminders

1.1. General background.

Definition 1. Let $\mathcal{M} \subseteq \mathcal{N}$ be structures with $\mathcal{M}$ definable in $\mathcal{N}$, with canonical parameter $a \in \mathcal{N}^{\mathsf{eq}}$, and $A \subseteq \mathcal{N}$. We say that:

- $\mathcal{M}$ is *stably embedded over A in* $\mathcal{N}$ if every $\mathcal{N}$-definable subset of $\mathcal{M}$ (in the sense of $\mathcal{N}$) is $\mathcal{M}A$-definable (again, in the sense of $\mathcal{N}$).
- $\mathcal{M}$ is *canonically embedded over A in* $\mathcal{N}$ if every aA-definable subset of $\mathcal{M}$ (in the sense of $\mathcal{N}$) is 0-definable (in the sense of $\mathcal{M}$).
- $\mathcal{M}$ is *fully embedded over A in* $\mathcal{N}$ if it is both stably and canonically embedded over A in $\mathcal{N}$.

Remark 2. Suppose that $\mathcal{M}$ is fully embedded over c in $\mathcal{N}$ and that $c' \in \mathsf{dcl}(c)$. Then $\mathcal{M}$ is fully embedded over cc' in $\mathcal{N}$.

Definition 3. We say that a structure $\mathcal{M}$ is *coordinatised by Lie geometries* if it carries a ($\emptyset$-definable) tree structure of finite height with unique $\emptyset$-definable root, such that the following properties hold:

- (Coordinatisation). For each $a \in \mathcal{M}$ above the root, either a is algebraic over its immediate predecessor in the tree ordering or there exists some $b < a$ and a b-definable projective geometry J_b fully embedded (over b) in $\mathcal{M}$ such that:

Date: April 5, 2024 – The cross-referencing may be a bit off, and there will certainly be mistakes everywhere, I'm sorry!

(1) $a \in J_b$; or

(2) There is $c \in \mathcal{M}$ with $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) with vector part J_c such that $a \in A_c$ and the projectivisation of J_c is J_b,

- (ORIENTATION) If $a, b \in \mathcal{M}$ have the same type and are associated with coordinatising quadratic geometries J_a, J_b in $\mathcal{M}$, respectively, then every definable map between J_a and J_b that preserves all structure of J_a and J_b also preserves the Witt defect.

Lemma 4 (Lemma 2.4.3, Alberto)**.** *Let J_a and J_b be* normal geometries[1] *which are a-definable and b-definable, respectively in a structure $\mathcal{M}$. Suppose that J_a and J_b are fully embedded over ab in $\mathcal{M}$. Then one of the following occurs:*

(1) J_a and J_b are orthogonal *over ab; Notation: $J_a \perp_{ab} J_b$.*[2]

(2) J_a and J_b are ab-linked.

Recall:

$$\begin{aligned} J_a \perp_{ab} J_b :\Longleftrightarrow\ & \text{every } ab\text{-definable relation on } J_a \cup J_b \\ & \text{is a Boolean combination of rectangles} \\ & R_a \times R_b, \text{ where } R_a \subseteq J_a \text{ is } \mathsf{acl}(a)\text{-definable} \\ & R_b \subseteq J_b \text{ is } \mathsf{acl}(b)\text{-definable.} \end{aligned}$$

Remark 5.

- $J_a \not\perp_a J_a$ (because of the equality relation, of course).
- Unpacking $A \not\perp_C B$: There is a C-definable relation $R \subseteq AB$ which is not a Boolean combination of rectangles $R_A \times R_B$ where R_A is $\mathsf{acl}(C)$-definable and R_B is $\mathsf{acl}(B)$-definable.
- If $J_a \not\perp_{a,b} J_b$, and J_a is fully embedded (over a), J_b is fully embedded (over b), then Lemma 2.4.3 *does not* give us a linkage. To get this it suffices to show that

$$\mathsf{acl}(a, b) \cap J_a = \mathsf{acl}(a, b) \cap J_b = \emptyset,$$

WHY?

[1]This is a definition to give the proof an axiomatic flavour. For our purposes, projective geometries are normal.

[2]I'll keep the a, b in the base of $\perp$, if missing it should be assumed that we are talking about orthogonality over defining parameters.

Lemma 6 (Lemma 2.4.6, Nick). *Let J_1 and J_2 be basic linear geometries canonically embedded and definable over A in a structure $\mathcal{M}$. Any A-definable bijection between their projectivisations lifts to an A-definable isomorphism of (unoriented) weak geometries.*

Discussion. When I was presenting Lemma 2.4.7 (essentially the same as above, with the additional assumption of *pseudofinitenes* and the stronger conclusion that we can preserve the orientation) I said that the proof given by Nick for Lemma 2.4.6 should allow us to preserve the orientation. I still don't know why it does not work, but it seems like they don't think it works, as they make explicit discussions about this later on. Help?

Definition 7. Let P be a projective Lie geometry, arising from a linear Lie geometry L. Suppose that P is fully embedded in a structure $\mathcal{M}$. Let $A \subseteq \mathcal{M}$ be a finite set. The *localisation of P over A*, P/A is defined as follows: Let

$$L_A^\perp := \{v \in L : \forall w \in \mathsf{acl}(A) \cap L.\ \beta(v, w) = 0\}$$

(if the bilinear form β is around and we're not in the polar case – otherwise make the "obvious" adjustments). Then

$$P/A \ := {}^{(L_A^\perp/(L_A^\perp \cap \mathsf{acl}(A)))\setminus \mathsf{acl}(\emptyset)}\Big/_{\mathsf{acl}(x)=\mathsf{acl}(y)}.$$

Remark 8.

- A definable map between localisations of geometries induces a definable map between the geometries (in eq).
- Let P_b be a projective geometry fully embedded in $\mathcal{M}$ over b and suppose that $C \subseteq \mathsf{acl}(b)$ is a finite subset. Then the localisation of P_b over C is simply P_b.

Lemma 9. *Let $\mathcal{M}$ be a structure and P, Q basic projective geometries defined and fully embedded over a set $A \subseteq \mathcal{M}$. Suppose that $P \perp_A Q$ and let $\hat{P}, \hat{Q}$ be localisations of P, Q respectively which are both defined over a set B. Then $\hat{P} \perp_B \hat{Q}$.*

1.2. **Canonical Projective Geometries.**

Definition 10. Let $\mathcal{M}$ be a Lie coordinatisable structure and J_b a b-definable projective geometry in $\mathcal{M}$. We say that J_b is a *canonical projective* geometry (in $\mathcal{M}$) if:

(1) J_b is fully embedded over b; and

(2) If $\mathsf{tp}(b) = \mathsf{tp}(b')$ and $b \neq b'$ then $J_b \perp_{bb'} J_{b'}$.

We can find canonical projective geometries:

Lemma 11 (Lemma 2.5.2, Paolo)**.** *Let $\mathcal{M}$ be a Lie coordinatisable structure and P_b a b-definable projective geometry fully embedded (over b?) in $\mathcal{M}$. Then, there is a canonical projective geometry J_c in $\mathcal{M}^{\mathsf{eq}}$ which is non-orthogonal to P_b over a finite set.*

2. New Things

Lemma 12 (Lemma 2.5.3)**.** *Let $\mathcal{M}$ be a Lie coordinatisable structure. Let:*

- *P_b be a b-definable projective geometry fully embedded in $\mathcal{M}$.*
- *J_c be a canonical projective geometry with canonical parameter c.*

Suppose that:

$$P_b \text{ is non-orthogonal to } J_c \text{ (i.e. } P_b \not\perp_{bc} J_c\text{)}.$$

Then:

(1) $c \in \mathsf{dcl}(b)$.

(2) $P_b \subseteq \mathsf{dcl}(b, J_c)$.

Proof.

(1) For the first point, since $\mathcal{M}$ is $\aleph_0$-categorical, it suffices to show that the set $\{c\}$ is $\mathsf{Aut}(\mathcal{M}/b)$-invariant. Thus, it suffices to show that c does not have any other b-conjugates. Let $c' = \sigma(c)$, for somr $c \in \mathsf{Aut}(\mathcal{M}/b)$. Since $P_b \not\perp_{bc} J_c$ we have that $P_b \not\perp_{bc'} J_{c'}$ (since after moving the relation on $P_b \cup J_c$ that wasn't a Boolean combination of rectangles by an automorphism fixing b it remains not a Boolean combination or rectangles). But then $J_c \not\perp_{cc'} J_{c'}$ WHY? and since J_c is a canonical projective geometry we must have that $c = c'$.

(2) For the second point, we consider the localisations of J_c and P_b over $\{b, c\}$. There is a (b, c)-definable bijection between them WHY? and then we can finish off with Remark 5.

□

Lemma 13 (Lemma 2.5.4)**.** *Let $\mathcal{M}$ be a Lie coordinatisable structure and J_c, $J_{c'}$ be c-definable and c′-definable canonical projective geometries, respectively (not necessarily conjugate). Suppose that:*

$$J_c \text{ and } J'_c \text{ are non-orthogonal (i.e. } J_c \not\perp_{c,c'} J_{c'}\text{)}.$$

Then:

(1) $\mathsf{dcl}(c) = \mathsf{dcl}(c')$.

(2) There is a unique (b, b')*-definable bijection between them, which is an isomorphism of weak, unoriented geometries.*

Proof.

(1) The first point is immediate from the previous lemma, since we can apply it twice to get $c \in \mathsf{dcl}(c')$ and $c \in \mathsf{dcl}(c')$.

(2) But now, we have $J_c \not\perp_{cc'} J_{c'}$, but since $\mathsf{dcl}(c) = \mathsf{dcl}(c')$ by Remark 5 and Lemma 4 we have that J_c and J_c' are (c, c')-linked. Thus by Lemma 6 there is a bijection between them which is an isomorphism of weak unoriented geometries.

$\square$

Lemma 14 (Lemma 2.5.5)**.** *Let* $\mathcal{M}$ *be a Lie coordinatisable structure and* J_c *be a canonical projective geometry (with canonical parameter* c*), in* $\mathcal{M}$*. There is a coordinatising geometry* P_b *and a definable unoriented weak isomorphism of* P_b *and* J_c*.*

Moreover, if J_c *us a projective* quadratic *geometry then we may choose* b *so that if we orient* J_c *according to this isomorphism then the orientation is independent of the choice of* b *within its type over* c *(i.e. it is the same for all* c*-conjugates of* b*).*

Proof. Let P_b be a coordinatising geometry non-orthogonal to J_c and chosen so that b is as low in the tree as possible, subject to the non-orthogonality assumption.

Why should such a P_b exist? Arguing as in Lemma 2.5.2 we see that if J_c is orthogonal to each of the coordinatising geometries (over their defining parameters), then it must be orthogonal to the whole tree (eventually using Lemma 2.4.8 (Dugald), and thus it ends up being orthogonal to itself, which is never the case. Paolo covered this argument (in the proof of Lemma 2.5.2), but let's cover it again, briefly:

The minimality condition thus guarantees us that:

$$\mathsf{acl}(b) \cap J_c = \emptyset.$$

Why? Let P_b be a coordinatising geometry, minimal subject to non-orthogonality. This means that b appears in a branch $(b_1, \ldots, b_n)$ of our coordinatising tree and for all $i < n$ we have that:

$$P_{b_i} \perp_{b_i, c} J_c$$

(ignore all the acl-steps in the tree. – everything will be vacuous in these cases). We prove by induction on $i \leq n$ that

$$\mathsf{acl}(b_1, \dots, b_i) \cap J_c = \emptyset.$$

For the base step, we have $P_{b_1} \perp_{b_1,c} J_c$. By definition, this means that every b_1c-definable relation on $P_{b_1} \cup J_c$ is a Boolean combination of rectangles $R_1 \times R_2$ where $R_1 \subseteq P_{b_1}$ is $\mathsf{acl}(b_1)$-definable and $R_2 \subseteq J_c$ is $\mathsf{acl}(c)$-definable.

Suppose $D \subseteq J_c$ is a finite b_1c-definable set. Then, viewing D as a subset of $J_c \cup P_{b_1}$ we know that we can write this as a Boolean combination of rectangles. In particular, D is $\mathsf{acl}(c)$-definable. So $D \subseteq \mathsf{acl}(\mathsf{acl}(c)) \cap J_c = \mathsf{acl}(c) \cap J_c = \emptyset$. We continue inductively, considering also localisations over the previous sets.

Since J_c a canonical projective geometry (with canonical parameter c) and we have a projective geometry P_b fully embedded in $\mathcal{M}$ we may apply Lemma 2.5.3 we get that:

$$c \in \mathsf{dcl}(b)$$

In particular, we have that $\mathsf{acl}(b, c) \cap J_c = \emptyset$. We also have that $\mathsf{acl}(b, c) \cap P_b = \emptyset$. Thus non-orthogonality gives us a (unique) definable weak unoriented isomorphism (by Lemma 2.4.3) and 6.

This finishes the first part of the Lemma.

For the moreover part, we observe that by the (Orientation) condition we must have that conjugates of b over c have compatible orientations (i.e. they are determined solely by the isomorphism of weak unoriented geometries). □