Normal Geometries

Alberto Miguel Gómez

9 February 2024

1 Normal Geometries: Definition and Examples

Definition 1.1 (2.4.1)**.** We call a structure J a **normal geometry** if (in every elementary extension) it satisfies the following properties:

(i) $\mathrm{acl}(\varnothing) = \varnothing$ and, for all $a \in J$, $\mathrm{acl}(a) = \{a\}$.

(ii) (Exchange) If $a \in \mathrm{acl}(Ba') \setminus \mathrm{acl}(B)$, then $a' \in \mathrm{acl}(Ba)$.

(iii) If $a \in J^{\mathrm{eq}}$, then there is some $B \subseteq J$ such that $\mathrm{acl}(a) = \mathrm{acl}(B)$.

(iv) Let $\varnothing \neq J_0 \subseteq J$ be 0-definable and $a, a' \in J$. If $\mathrm{tp}(a/J_0) = \mathrm{tp}(a'/J_0)$, then $a = a'$.

We say a normal geometry is **reduced** if, in addition, it satisfies the following:

(v) $\mathrm{acl}^{\mathrm{eq}}(\varnothing) = \mathrm{dcl}^{\mathrm{eq}}(\varnothing)$.

Remark 1.2. In the definition, (i) and (ii) say that (J, acl) forms a *geometry* (in the pregeometry sense). Moreover, (iii) is an equivalent condition to *geometric elimination of imaginaries*, as it was introduced in Paolo's talk, *viz.*, for any $e \in J^{\mathrm{eq}}$,

$$e \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(e) \cap J).$$

This is, in fact, the version of (iii) that we use throughout the proof.

Example 1.3 (2.4.5)**.** Let $J = (M, E)$ be a structure with an equivalence relation E with two infinite classes. It is easy to check that J is a normal geometry. But J is not reduced: for any $a \in M$, $a/E \in \mathrm{acl}^{\mathrm{eq}}(\varnothing)$, but $\mathrm{dcl}^{\mathrm{eq}}(\varnothing) = \varnothing$.

Lemma 1.4 (2.4.2)**.** *(1) Projective geometries are normal geometries.*

(2) Basic projective geometries are reduced normal geometries.

Proof. (1) Apart from the polar and quadratic cases, (i) and (ii) are checked as usual for projective geometries (note that (i) follows precisely because we are taking projectivisations). For the remaining cases, the key is that "the various parts of the geometry do not interact pointwise" (p. 31). So, e.g., if we consider the quadratic geometry and take a form $q \in Q$, then $\mathrm{acl}(q)$ will be the "linear span" of q inside Q, and thus $\mathrm{acl}(q) \cap V = \varnothing$. Similarly, in the polar case, for $v \in V$ and $w \in W$, as we may assume V and W to be disjoint, once again their corresponding linear spans remain inside the appropriate vector space. So they reduce to the previous cases.

To also satisfy the condition $\mathrm{acl}(\varnothing) = \varnothing$, we may, in each case, choose a model for the geometry where the underlying field K is encoded in M^{eq}.

(iii) is precisely Lemma 2.3.3 (cf. Nick's talk). In particular, by Remark 2.3.7, these are all examples of structures having geometric but not weak elimination of imaginaries.

Note that (iv) is trivial if we are in either the degenerate or the pure vector space cases. By Corollary 2.2.9, the 0-definable linear forms must come either from the inner product or, in the polar case, from the dual. Hence, if we are in the inner product or the orthogonal case, then nontrivial 0-definable subsets must be the sets of values of a hermitian form β or a quadratic form q, respectively, on a projective point.

Thus, if a and a' are projective points with the same type over J_0 (a 0-definable subset of the projective space), then they lift (using the quotient map) to points in the linear space having the same type over the preimage of J_0. The claim to check now is that these preimages actually generate V, so that the preimages of a and a' must be equal, and therefore $a = a'$.

In the polar case, we might have that both V and W are 0-definable (this is the case, e.g., when it is basic). But, by the aforementioned Corollary, its definable linear functions are those coming from the respective duals of V and W. If $\lambda, \lambda' \in V^*$ are such that $\lambda \equiv_V \lambda'$, then we can check that $\lambda = \lambda'$. Similarly for forms in W^*.

In the quadratic case, we might have that both V and Q are 0-definable. Again, vectors with the same type over V and forms with the same type over Q can be dealt with as in previous cases. If $q_1, q_2 \in Q$ are such that $q_1 \equiv_V q_2$, then the argument as in the polar case shows again that $q_1 = q_2$. Finally, if $v, v' \in V$ are such that $v \equiv_Q v'$, then $v +_Q q = v' +_Q q$ for any $q \in Q$, and since V acts regularly on Q, this implies that $\lambda_v = \lambda_{v'}$. Since β is non-degenerate, this means that $v = v'$. Using the quotient map to the projectivisation, we see that this also applies to projective points.

(2) Let V now be a basic linear geometry, let J be its projectivisation, and J_s its semiprojectivisation. Let $a \in \mathrm{acl}^{\mathrm{eq}}_J(\varnothing)$. We want to show that $a \in \mathrm{dcl}^{\mathrm{eq}}_J(\varnothing)$. Since we have a surjective, finite-to-one map $J_s \to J$, pick a lift of a in J_s, say $\hat{a}$. Then $\hat{a} \in \mathrm{acl}^{\mathrm{eq}}_{J_s}(\varnothing)$, and, by Lemma 2.3.6, $\hat{a} \in \mathrm{dcl}^{\mathrm{eq}}_{J_s}(\mathrm{acl}^{\mathrm{eq}}_{J_s}(\hat{a}) \cap J_s)$. But note that

$$\mathrm{acl}^{\mathrm{eq}}_{J_s}(\hat{a}) \cap J_s \subseteq \mathrm{acl}^{\mathrm{eq}}_{J_s}(\varnothing) \cap J_s = \mathrm{acl}_{J_s}(\varnothing) = \varnothing.$$

Therefore, $\mathrm{acl}^{\mathrm{eq}}_{J_s}(\hat{a}) \cap J_s = \varnothing$, and thus $\hat{a} \in \mathrm{dcl}^{\mathrm{eq}}_{J_s}(\varnothing)$. Since automorphisms of J lift to automorphisms of J_s, it follows that $a \in \mathrm{dcl}^{\mathrm{eq}}_J(\varnothing)$, as required. $\square$

2 Normal Geometries and Orthogonality

Lemma 2.1 (2.4.3)**.** *Let J_1, J_2 be normal geometries which are fully embedded and* 0*-definable in a structure $\mathcal{M}$. Then one of the following holds:*

(i) J_1 and J_2 are ***orthogonal****, i.e., every* 0*-definable relation on $J_1 \cup J_2$ is a finite Boolean combination of sets of the form $R_1 \times R_2$ with R_i an $\mathrm{acl}^{\mathrm{eq}}_{J_i}(\varnothing)$-definable relation on J_i.*

(ii) J_1 and J_2 are 0*-**linked**, i.e., there is a* 0*-definable bijection $J_1 \to J_2$.*

Proof. We adapt the earlier proof of this result from [Hru93]. Assume that (i) fails, i.e., $J_1 \not\perp J_2$. We want to construct a 0-definable bijection as in (ii). The strategy is the following: first, we find a pair (a_1, a_2) of elements in J_1 and J_2, respectively, which we can use to find a 0-definable bijection F between 0-definable subsets $D_1 \subseteq J_1$ and $D_2 \subseteq J_2$. Once we have this F, since 0-definable sets determine types over F by (iv), we can extend it to a 0-definable bijection between J_1 and J_2, as required. We divide this into several steps, but first let us clarify our notation: Throughout the proof, we write $\mathrm{acl}(x)$ for $\mathrm{acl}_{\mathcal{M}}(x)$; algebraic closure within the normal geometries will be explicitly indicated by a subscript when necessary.

Claim 1. If $b \in J_1^{n_1}$, $T \subseteq J_1^{n_1} \times J_2^{n_2}$ is 0-definable, and $T(b) := \{b' \in J_2^{n_2} : (b, b') \in T\}$ is not $\mathrm{acl}^{\mathrm{eq}}_{J_2}(\varnothing)$-definable, then $\mathrm{acl}^{\mathrm{eq}}(b) \cap J_2 \neq \varnothing$.

Proof. Since J_2 is stably embedded, $T(b)$ is J_2-definable. Let $c \in J_2^{\mathrm{eq}}$ be the canonical parameter for $T(b)$. As $T(b)$ is c-definable but not $\mathrm{acl}^{\mathrm{eq}}_{J_2}(\varnothing)$-definable, we have $c \notin \mathrm{acl}^{\mathrm{eq}}_{J_2}(\varnothing)$. By (iii) from the definition of a normal geometry (cf. Remark 1.2), since $c \in J_2^{\mathrm{eq}}$, we have $c \in \mathrm{acl}^{\mathrm{eq}}_{J_2}(\mathrm{acl}^{\mathrm{eq}}_{J_2}(c) \cap J_2)$. Therefore, $\mathrm{acl}^{\mathrm{eq}}(c) \cap J_2 \neq \varnothing$. Since c is the canonical parameter of a b-definable set, it follows that $\mathrm{acl}^{\mathrm{eq}}(b) \cap J_2 \neq \varnothing$, as required. $\square_{\text{Claim 1}}$

Claim 2. There exist $a_1 \in J_1$ and $a_2 \in J_2$ such that $a_1 \in \mathrm{acl}(a_2)$.

Proof. Let $R \subseteq J_1^{n_1} \times J_2^{n_2}$ be a 0-definable relation which is not a finite union of rectangles $A_1 \times A_2$, where each $A_i \subseteq J_i^{n_i}$ is $\mathrm{acl}^{\mathrm{eq}}_{J_i}(\varnothing)$-definable. In particular, by compactness, we can find some $b_1 \in J_1^{n_1}$ such that $R(b_1) := \{b_2 \in J_2^{n_2} : (b_1, b_2) \in R\}$ is not $\mathrm{acl}^{\mathrm{eq}}_{J_2}(\varnothing)$-definable. Hence, by Claim 1, $\mathrm{acl}^{\mathrm{eq}}(b_1) \cap J_2 \neq \varnothing$.

Pick $a_2 \in \mathrm{acl}^{\mathrm{eq}}(b_1) \cap J_2$. Let $S(a_2)$ be the locus of b_1 over a_2. Since a_2 is algebraic over b_1, it follows that a_2 is algebraic over $S(a_2)$. This then implies that $S(a_2)$ is not $\mathrm{acl}^{\mathrm{eq}}_{J_1}(\varnothing)$-definable. So another application of Claim 1 gives us that $\mathrm{acl}^{\mathrm{eq}}(a_2) \cap J_1 \neq \varnothing$. Thus, we can pick $a_1 \in \mathrm{acl}^{\mathrm{eq}}(a_2) \cap J_1$. This works. $\square_{\text{Claim 2}}$

Claim 3. For a_1 and a_2 as in Claim 2, $\mathrm{dcl}(a_1) = \mathrm{dcl}(a_2)$.

Proof. Observe that $\mathrm{acl}(a_1) \cap J_2 \subseteq \mathrm{acl}(a_2) \cap J_2 = \mathrm{acl}_{J_2}(a_2) = \{a_2\}$ by (i) from the definition of a normal geometry. Moreover, the same argument as in the last paragraph of Claim 2 shows that $\mathrm{acl}^{\mathrm{eq}}(a_1) \cap J_2 \neq \varnothing$. In particular, if $a_2' \equiv_{a_1} a_2$, this implies that $a_2' = a_2$. Thus, $a_2 \in \mathrm{dcl}(a_1)$. A parallel argument shows that $a_1 \in \mathrm{dcl}(a_2)$. Hence, $\mathrm{dcl}(a_1) = \mathrm{dcl}(a_2)$, as required. $\square_{\text{Claim 3}}$

Therefore, there exist 0-definable $\varnothing \neq D_i \subseteq J_i$ for $i = 1, 2$ and a 0-definable bijection $F\colon D_1 \to D_2$ (you can think of this as coming from the locus of a_1 and a_2). This bijection is unique: if f, g are two 0-definable bijections $D_1 \to D_2$, then $f^{-1}g$ is a 0-definable permutation of D_2. Since D_2 is 0-definable, it follows that $f^{-1}g = \mathrm{id}_{D_2}$.

Claim 4. For any $a_1 \in J_1$, there is a unique $a_2 \in J_2$ such that $\mathrm{acl}(a_1) = \mathrm{acl}(a_2)$.

Proof. Let $a_1 \in J_1$. Since D_1 is 0-definable, by (iv), a_1 is determined by its type over D_1. So, by compactness, there is $\varphi(x, b) \in \mathrm{tp}(a_1/D_1)$ such that a_1 is determined by $C(a_1) := \{b' \in D_1 : \varphi(a_1, b')\}$. Then we redo the argument from Claim 2 applied to $F[C(a_1)]$ to find the unique a_2 with $\mathrm{acl}(a_1) = \mathrm{acl}(a_2)$. $\square_{\text{Claim 4}}$

Similarly, for any $a_2 \in J_2$, there is a unique $a_1 \in J_1$ with $\mathrm{acl}(a_1) = \mathrm{acl}(a_2)$, and so, $\mathrm{dcl}(a_1) = \mathrm{dcl}(a_2)$. Thus, we can extend F to a 0-definable bijection $J_1 \to J_2$, as required. $\square$

The point of the above result is that "in either case, the induced structure on $J_1 \cup J_2$ is completely determined." ([Hru93, p. 177])

Remark 2.2 (2.4.4)**.** Under the same hypothesis as in the above lemma, if we further assume that J_1 and J_2 are *reduced*, then we can strengthen the condition in (i) to the following:

(i') J_1 and J_2 are **strictly orthogonal**, i.e., every 0-definable relation on $J_1 \cup J_2$ is a finite Boolean combination of sets of the form $R_1 \times R_2$ with R_i a 0-*definable* relation on J_i.

This is because, since J_1 and J_2 are reduced, we have $\mathrm{acl}^{\mathrm{eq}}_{J_i}(\varnothing) = \mathrm{dcl}^{\mathrm{eq}}_{J_i}(\varnothing)$, and clearly $\mathrm{dcl}^{\mathrm{eq}}_{J_i}(\varnothing)$-definable relations are 0-definable.

Example 2.3 (2.4.5 (again))**.** Let $J_1 = (M_1, E_1)$ and $J_2 = (M_2, E_2)$ be two structures as in Example 1.3. Consider the structure $J_1 \times J_2$ after adding a bijection $J_1/E_1 \to J_2/E_2$. We claim that, inside $J_1 \times J_2$, J_1 and J_2 are orthogonal but not strictly orthogonal. Indeed, note that the preimages of each of the E-classes (which are nontrivial proper subsets of $J_1 \cup J_2$) are $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$-definable (since each of the representatives in the sort of $(J_1 \times J_2)/E$ is in $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ by Example 1.3) but not 0-definable.

References

[Hru93] Ehud Hrushovski, *Finite Structures with Few Types*, Finite and Infinite Combinatorics in Sets and Logic (N. W. Sauer, R. E. Woodrow, and B. Sands, eds.), Springer Netherlands, 1993, pp. 175–187.