Homogeneity of μ-envelopes

10/5/24

Three definitions

1) Suppose (V, A) is an affine space (i.e. V is a vector space and $V \curvearrowright A$ regularly), which is defined over B. We say A is **free over B** if there is no projective geometry J defined over B with $A \subseteq \mathrm{acl}(B, J)$.

2) An element a of A, or $\mathrm{tp}(a/B)$, is said to be **affinely isolated** over B if (V, A) is free over B.

3) Suppose A, A' are affine spaces defined over B. They are **almost orthogonal** if there is no pair a, a' with $a \in A$, $a' \in A'$, $\mathrm{acl}(a, B) = \mathrm{acl}(a', B)$.

Uniqueness of Parallel Lines (?)

Suppose (V, A), (V', A') are almost orthogonal affine spaces defined and free over the algebraically closed set B, with PV and PV' complete 1-types over B. Let J be a projective geometry defined over B, not of quadratic type, and stably embedded in M. For $a \in A$, $a' \in A'$, $c \in J \setminus B$, the triple (a, a', c) is algebraically independent.

Proof

A, A' almost orthogonal $\Longrightarrow$ a, a' indep. $/B$

free over B $\Longrightarrow$ a, c are indep $/B$

$+$ a', c are indep $/B$.

If two of A, A', J are orthogonal,

then $\Longrightarrow$ (a, a', c) is indep $/B$

so when all of A, A', J are

non-orthogonal. $\Longrightarrow$ can identify

PV with part of J.

Consider the structure $\underline{J \cup A}$.

A is definable over $J \cup \{a\}$ (and B) hence $J \cup A$ is stably embedded in M (since J is).

Suppose towards contradiction that $rk(aa'c/B) = 2$. Take $\underline{a_1, c_1}$ independent copy of a, c over Ba'.

Note that $2 = rk(ac/B) > rk(ac/Ba') = 1$ so if $rk(aca_1c_1/B) = 4$ then we have $rk(ac/Ba_1c_1) = 2$ but $a' \in acl(Ba_1c_1)$

So we get

$$\mathrm{rk}\left(ac/Ba_1c_1a'\right) = 2 \quad \text{hence}$$

$$\mathrm{rk}\left(ac\,a_1c_1a'/B\right) = 5, \quad \text{a contradiction.}$$

Thus, we get

$$\mathrm{rk}\left(\underbrace{ac\,a_1c_1}/B\right) = 3.$$

But this is now a computation only about $J \cup A$, which is stably embedded and modular. Thus, there is some $d \in \underline{(J \cup A)} - B$ such that $d \in \mathrm{acl}(acB) \cap \mathrm{acl}(a_1c_1B)$, hence $d \in \mathrm{acl}(a',B)$. Thus,

$$\mathrm{acl}(d,B) = \mathrm{acl}(a',B).$$

If $d \in A$, contradicts $\underline{A, A' \text{ almost orthogonal}}$.

If $\underline{d \in J}$, contradicts $\underline{\text{free over } B}$. $\blacksquare$

Lemma Let M be Lie coordinatized.

Let A be an affine space defined and free over $B = acl(B)$.

Suppose $B \subseteq B' = acl(B)$, B' finite, and J a canonical projective geometry associated with A. Assume

1. $J \cap B' \subseteq B$.
2. $J \cap B$ is nondegenerate (if there's a form around)
3. If J is a quadratic space, then the Q-sort of J meets B.

Then A either meets B' or is free over B'!

Proof "A need not remain a geometry over B' but will split into a finite number of affine pregeometries over B'!" (?)

The proof is by induction on the coordinatization tree. Suppose $B' = acl(B, a')$, where a' is in a B-definable affine, projective, or quadratic space; call it A'.

Assume A does not meet B', but some "affine part" $A_0 \subseteq A$ relative to B' is contained in $acl(\underbrace{B, a'}_{B'}, J_{b'})$ where $J_{b'}$ is a B'-definable projective space. But

$$J_{b'} \subseteq acl(B, J, b') \text{ so}$$

$$A_0 \subseteq acl(\underbrace{B, a'}_{b'}, J)$$

and $A \cap acl(J, B) = \emptyset$.

Hence A, A' are non-orthogonal, and

$A' \cap acl(J, B) = \emptyset$. Hence, by (3),

A' is affine and free over B.

Now if A, A' are not almost orthogonal over B, then B' meets A (recall $B' = \mathrm{acl}(B')$).

So A, A' are almost orthogonal over B.

Now we are in the situation of the Uniqueness of Parallel Lines Lemma.

Pick $a \in A$. We have $a \in \mathrm{acl}(B, a', J)$ and (J, A) is modular.

$\Rightarrow \exists c \in J \cap \mathrm{acl}(Baa')$ with

$a \in \mathrm{acl}(Ba'c)$. (?)

Then $c \notin B$ and c is not in the quadratic part of J (if it exists).

Now we localize J at B to get

$$J_B \quad (\text{i.e. } J_B = (J \cap B)^{\perp} / \mathrm{rad}((J \cap B)^{\perp})).$$

By (3), this is not a quadratic geometry.

By (2), $J \subseteq \mathrm{cl}(B \cup J_B)$.

"normally our B, J would break up into a number of pregeometries, at least one $((J \cap B)^{\perp})$ sitting over the localization, while some of the cosets would be affine pregeometries. However, since $J \cap B$ is nondegenerate, all elements of J lie in translations by elements of B of $(J \cap B)^{\perp}$."

Replacing c by an element of J_B with the same algebraic closure as c

$\Longrightarrow$ a, a', c are algebraically indep, a contradiction.

by uniqueness of parallel lines.

$\blacksquare$

Main Theorem (Lemma 3.2.4)

Let $\mathcal{M}$ be an adequate regular expansion of a LC'd structure, μ a dimension function, and let E, E' be μ-envelopes. If $f: A \xrightarrow{\sim} A'$ is partial elementary with $A \subseteq E$, $A' \subseteq E'$, then f extends to an elementary map $\tilde{f}: E \to E'$. In particular, μ-envelopes are unique and homogeneous.

Proof

- Reduce to E, E' finite by back-and-forth.

- Reduce to $A = \mathrm{acl}(A)$ and to extending f to $\mathrm{acl}(A \cup \{b\})$ for some $b \in E - A$.

Divide into two cases:

1./ There is a standard system of geometries J and some $a \in A$ such that $J_a \cap E$ is not contained in A.

Expand $J_a \rightsquigarrow J^{\circ}_{a^*}$ basic projective defined over $a^* = \mathrm{acl}(a)$.

Take covering finite dimensional linear spaces L, L'

$$\begin{tikzcd} L \arrow[d] & L' \arrow[d] \\ J^{\circ}_{a^*} \cap E \arrow[r, "f"] & J^{\circ}_{f(a^*)} \cap E' \end{tikzcd}$$

Lift to $\hat{f}$, some isomorphism $L \to L'$,

$$\begin{tikzcd} L \arrow[r, "\hat{f}"] \arrow[d] & L' \arrow[d] \\ J^0_{a^*} \cap E \arrow[r, "f"] & J^0_{f(a^*)} \cap E \end{tikzcd}$$

and if can be arranged that this square commutes by Witt's lemma.

$A = acl(A)$ + weak EI + stable embedding

$$\Rightarrow \quad tp(L/L \cap A) \vdash tp(A/L).$$

$$tp(L'/L \cap A) \vdash tp(A'/L')$$

$\Rightarrow$ f is partial elementary.

Case 2 is Not Case 1. And Not Today...