Reminders (everyone)

Def^n Let $\mathcal{M} \subseteq \mathcal{N}$ be structures; $\mathcal{M}$ definable in $\mathcal{N}$ with canonical parameter $a \in \mathcal{N}^{eq}$, and $A \subseteq N$. We say that:

1) $\mathcal{M}$ is <u>canonically embedded over A</u> in $\mathcal{N}$ if the 0-definable subsets of $\mathcal{M}$ are aA-definable in $\mathcal{N}$.

2) $\mathcal{M}$ is <u>stably embedded over A</u> in $\mathcal{N}$ if the $\mathcal{N}$-definable subsets of $\mathcal{M}$ are MA-definable (uniformly...).

3) $\mathcal{M}$ is <u>fully embedded over A</u> if it is both canonically embedded over A in $\mathcal{N}$ and stably embedded over A in $\mathcal{N}$.

A lot of words for saying $\mathcal{M}$ is <u>1/2/3</u> embedded in the $\mathcal{L}(A)$-structure $\mathcal{N}$.

Reminders (Alberto)

Lemma (2.4.3, 2.4.4). Let $\mathcal{M}$ be a structure and $A \subseteq M$. Let J_1, J_2 be normal (resp. reduced normal) geometries fully embedded over A in $\mathcal{M}$. Then, one of the following occurs:

1. J_1 and J_2 are <u>orthogonal</u> (resp. strictly orthogonal) <u>over A</u>: Every A-definable subset of $J_1 \cup J_2$ is a Boolean combination of $acl(A)$-definable (resp. A-definable) rectangles.
2. J_1 and J_2 are <u>A-linked</u>: There is an A-definable bijection between J_1 and J_2.

Lemma (2 4 2) If J is a projective (resp. basic projective) geometry then J is normal (resp. reduced normal).

The point of the next lemmas is that when two (fully embedded) projective spaces are nonorthogonal then the linkage lifts to one between the linear spaces.

Reminders (Nick)

Lemma (2.4.6). Let J_1 and J_2 be basic linear geometries canonically embedded in the structure $\mathcal{M}$. Suppose that in $\mathcal{M}$ there is a 0-definable bijection

$$f : PJ_1 \longrightarrow PJ_2$$

between their projectivisations Then, there is a 0-definable bijection.

$$\hat{f} \cdot J_1 \longrightarrow J_2$$

which is an isomorphism of unoriented weak geometries and which induces f.

Unoriented: Forget about the Witt defect in the quadratic case

Weak: Forms take values on a K-line, so isomorphisms are similarities rather than isometries.

Lemma (2.4.7) Let $\mathcal{M}$ be a pseudofinite structure and J_1, J_2 basic linear geometries canonically embedded and definable (over $\emptyset$). In $\mathcal{M}$ suppose that we have a 0-definable bijection

$$f : PJ_1 \longrightarrow PJ_2 \quad \text{(a posteriori an iso.)}$$

Then there is a 0-definable bijection $\hat{f} : J_1 \to J_2$ inducing f and such that $\hat{f}$ is an isomorphism of weak linear geometries.

What do we have to do?

$$[q, q'] \in \tau[K] \iff [\hat{f}(q), \hat{f}(q')] \in \tau[K]$$

$$\tau(X) = X^2 + X$$

$$q\left(\sqrt{q+q'}\right) \qquad \alpha^{3/2} q \sqrt{q+q'}$$

In the proof of Lemma 2.4.6 we produced a map $\hat{f}$ s.t.

$$B_{J_2}(\hat{f}(v), \hat{f}(w)) = \underset{\in K}{\alpha}\, B_{J_1}(v, w)$$

From this we deduce that

$$\hat{f}(q)\left(f(v)\right) = \alpha\, q(v)$$

So really it is enough to show that $\alpha = 1$, i.e. that $\hat{f}$ is an isometry.

<u>Recall</u> we may adjust $\hat{f}$ by constants in K

so once we know that

$$\beta_{J_2}\left(\hat{f}(v), \hat{f}(w)\right) = \alpha\, \beta_{J_1}(v, w)$$

Take $\hat{f}'(v) = \alpha^{-1/2} \hat{f}(v)$ this produces an isometry, so preserves the symplectic structure exactly and hence preserves the Witt defect.

Localisations

<u>Def.</u> Let $\mathcal{M}$ be a structure, $A \subseteq M$ and P a projective geometry (wlog fully embedded) in $\mathcal{M}$. We construct the <u>localisation of P over A</u> P/A as follows:

1. Say $L = (V, \dots)$ is the linear geometry whose projectivisation is P
2. $L_A^{\perp} = \{ v \in V : \forall x \in V \cap acl(A) \ \beta(x,v) = 0 \}$

   In the degenerate case and pure vector space case $\beta(x,y) = 0$ $L_A^{\perp} = L$

   In the polar case $L = (V, W, \dots)$

   $$L_A^{\perp} = \{ v \in V : \forall x \in W \cap acl(A) \ \beta(v,x) = 0 \} \cup \{ w \in W : \forall x \in V \cap acl(A) \ \beta(x,w) = 0 \}$$
3. $L_A^{\perp} / \underbrace{L_A^{\perp} \cap acl(A)}$ the quotient space

   $\hookrightarrow$ $acl(A)$ is just linear span
4. $P/A = Proj\left(L_A^{\perp} / L_A^{\perp} \cap acl(A) \right)$

If L supports a quadratic geometry then so does $L_A^\perp$ the quadratic forms which vanish on $L_A^\perp \cap acl(A)$

so P/A is still a quadratic space.

L is an orthogonal space over a field of characteristic 2. $A = \{v\} \subseteq L$

$$L_A^\perp = v^\perp \text{ and } L_A^\perp \cap acl(A) = \langle v \rangle$$

$\bar{L} = v^\perp / \langle v \rangle$

- Case 1 v is q-isotropic $(q(v) = 0)$ and then the quadratic form on L descends to a quadratic form on $\bar{L}$ and thus we still get an orthogonal space.
- Case 2 v is not q-isotropic: for every non-singular plane H containing $\langle v \rangle$ $q|_{H^\perp}$ gives a different quadratic form on $\bar{L}$

  $\bar{Q}$ is the space of all of these P/A is a quadratic space.

The point is that the localisation of a basic projective geometry is going to be a basic projective geometry usually of the same type

<u>Lemma</u> (2.4.11). Let $\mathcal{M}$ be a structure; $A \subseteq M$ and P, Q (basic) projective geometries which are:

1. definable over A
2. fully embedded over A
3. orthogonal over A

Let $B \subseteq M$ and $\hat{P}, \hat{Q}$ localisations of P and Q resp which are definable over B. Then $\hat{P}$ and $\hat{Q}$ are orthogonal over B.

<u>Proof</u>

<u>Claim</u> If P is definable and fully embedded over A $\hat{P}$ is a localisation of P definable over B then $\hat{P}$ is fully embedded over AB.

- Canonically embedded : 0-def. relation on $\hat{P}$ is an $\mathcal{M}$-definable relation on P, since P is stably embedded over A this is a PA-definable subset of P then we get AB-definable subset of $\hat{P}$.

- Stably embedded. $\mathcal{M}$-def. subset of $\hat{P}$, $\mathcal{M}$-def. subset of P, PA-definable subset of P and $\hat{P}A$-def. subset of $\hat{P}$.

Assume $A \subseteq B$; $\hat{P}$, $\hat{Q}$ non-orthogonal over B.
By Lemma 2.4.3 (+ projective geometries are normal)
$\hat{P}$ and $\hat{Q}$ are B-linked and the B-definable bijection between them is unique and thus it is definable over $A \cup (acl(B) \cap (P \cup Q))$ ⇝ $\hat{P}$, $\hat{Q}$ are definable over this
so we may use it to get an A-definable relation between P and Q $(P \not\perp_A Q)$.

$B \subseteq A$ if we assume $\hat{P} \not\perp_B \hat{Q}$ then we get an A-definable bijection between them and this induces an A-definable relation between P & Q.