Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

ZOMBIE EXERCISE 1. Let F be an infinite field. Show that there is an elementary extension $F \preceq K$ containing a transcendental element t over F. Considering $\operatorname{acl}(t)$, show that there is no ω-categorical theory extending the theory of infinite fields.

EXERCISE 2. For $n > 2$, consider $\mathcal{Q}_n := (\mathbb{Q}; <, P_0, \dots, P_{n-3}; (c_i)_{i<\omega})$, where the P_i are unary predicates partitioning $\mathbb{Q}$ into dense subsets and the c_i name an increasing sequence of elements of P_0. Show that the theory $\mathcal{Q}_n$ has exactly n countable models up to isomorphism.

EXERCISE 3. Work in a countable relational language $\mathcal{L}$. Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures closed under isomorphism and substructures. Show that the following are equivalent:

- $\mathcal{C}$ has the amalgamation property;
- $\mathcal{C}$ has the 1-point amalgamation property: Let $A, B_0, B_1 \in \mathcal{C}$ and $f_i : A \to B_i$ be embeddings with $|B_i \setminus A| = 1$ for $i \in \{0, 1\}$. Then, there is some $D \in \mathcal{C}$ and embeddings $g_i : B_i \to D$ for $i \in \{0, 1\}$ such that $g_0 \circ f_0 = g_1 \circ f_1$.

EXERCISE 4. Work in a countable relational language $\mathcal{L}$ with no 0-ary relation (and no constant symbol). Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures closed under isomorphisms, substructures and with the amalgamation property. Deduce that $\mathcal{C}$ also has the joint embedding property.

EXERCISE 5. Let $\mathcal{L}_{\mathbb{Q}^{\geq 0}}$ be a language with a binary relation d_q for each $q \in \mathbb{Q}^{\geq 0}$. Consider the class $\mathcal{K}$ of finite metric spaces $(M; (d_q)_{q \in \mathbb{Q}^{\geq 0}})$ for which the distance between any two points is always rational and for each $q \in \mathbb{Q}^{\geq 0}$, d_q is interpreted as the binary relation holding of two points if and only if they are at distance q. Show that $\mathcal{K}$ is a Fraïssé class. The associated Fraïssé limit $\mathbb{U}_{\mathbb{Q}}$ is called the **rational Urysohn space**. Is this structure ω-categorical?

Definition 1. The **exponent** of a group G is the least common multiple of the orders of its elements (or ∞ if there is no such least common multiple).

Fact 2. *An Abelian group of finite exponent is a direct sum of finite cyclic groups.*

EXERCISE 6. Show that an ω-categorical group has finite exponent. Deduce that an infinite Abelian group is ω-categorical if and only if it has finite exponent. Describe an axiomatisation of the different complete ω-categorical theories of (infinite) Abelian groups.

Definition 3. Let G be a group. We say that $H \leq G$ is a **characteristic subgroup of** G if and only if for all $\phi \in \operatorname{Aut}(G)$, $\phi(H) \leq H$. We say that G is **characteristically simple** if it has no proper nontrivial characteristic subgroups.

$\star\star$ **EXERCISE 7.** Note that if H is a characteristic subgroup of G every automorphism of G induces an automorphism of G/H. Let G be an ω-categorical countable group.

1. Show that G has finitely many characteristic subgroups;
2. Show that if H is a characteristic subgroup of G such that H and G/H are both infinite, then both H and G/H are ω-categorical;
3. Show that G has a finite characteristic series

$$G_0 = \{1\} < G_1 < \cdots < G_m = G,$$

where for each $i < m$, G_{i+1}/G_i is characteristically simple and either finite or ω-categorical.