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JSL 2005.

P, Q 0-def sets,
B set of param.

Write $P \perp_B Q$ if ~~for~~ any formula
$\phi(\bar{x}, \bar{y})$ over B $(\bar{x} \in P^l, \bar{y} \in Q^m)$
is a finite ~~union~~ b.c. of rectangles $R \times R'$

Remark: Such a b.c. reduces to a union,
~~&~~ we can R, R' to be acl(B)-definable.

Various Statements:

(1) If $P \perp Q$ both stably embedded,
then $P \perp_B Q$ for any params. B.

(1′) If $P \perp Q$, both stab emb.,
then $P \cup Q$ is stably embedded.

(1) $\iff$ (1′) (Lemma 2.5)

= Thm 3.2 of paper

They imply

. .

(2) If P_1, P_2, Q are 0-def stably embedded and $Q \perp P_1$, $Q \perp P_2$ then $Q \perp P_1 \cup P_2$. (Cor 3.4)

This is basically Lemma 2.4.8 of CH, but they omitted 'stably embedded'}

<u>Example</u>: Generic $R \subseteq Q \times P_1 \times P_2$

Then $Q \perp P_1$, $Q \perp P_2$

but $Q \perp P_1 \times P_2$ so (2) fails without <u>stable embeddedness</u>.

<u>Lemma 3.1</u> There is no infinite group G s.t. G acts oligomorphically on itself by conjugation.

<u>Pf</u>: Suppose we have such G, countable.

Then G is ω-categorical (R-N)

& G has finitely many normal subgps.

G has finitely many 0-def (char) subgroups,

so consider a max chain

$G = G_0 > G_1 > \ldots > G_n = 1$ of characteristic subgps.

Can assume $|G : G_1|$ infinite.

So G/G_1 is infinite, ω-categ, characteristically simple gp, with finitely many normal subgps.

Apps, Wilson: Such G/G_1 is one of

- el ab. $~~C_p \oplus \cdots \oplus~~$ C_p^{ω}
- Boolean power of a finite simple gp F.

$= \{$continuous function $f : C \to F\}$

C: Cantor set

inf many normal subgps:

- perfect p-gp.

Again, ∞ many normal subgps.

Theorem 3.2 Suppose $P \perp Q$ each stably embedded

Then for any $B \subseteq M$, $P \perp_B Q$.

We'll actually prove (1') where the conclusion is $P \cup Q$ is stably embedded.

Pf: Suppose $P \cup Q$ is not stably embedded.

So So there's a relation $\phi(\bar{x}, \bar{y})$ ($\bar{x} \in P^l$, $y \in Q^m$) with canonical parameter c, with ϕ_c not $(P \cup Q)$-definable.

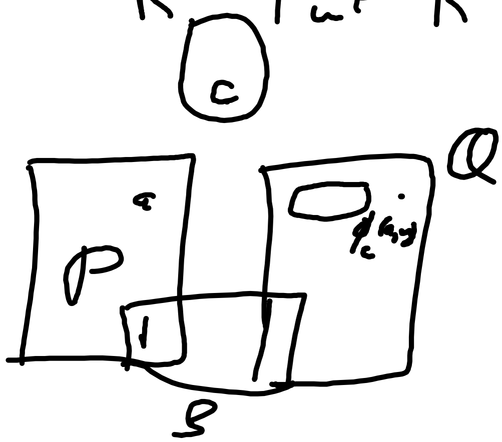
Choose base $B \subseteq P \cup Q$ s.t

$$tp(c/B) \Rightarrow tp(c/B \cup P)$$

and $tp(c/B) \Rightarrow tp(c/B \cup Q)$

(take union of a chain to get B)

Put $R = tp(c/B)$



For $a \in P^l$, $\phi_c(a,y) \cap Q^m$ is Q-definable (Q stab emb.)

with canonical param $f_c(a) \in Q^{eq}$

Define E.R. on P^l:

$$E x x' \iff f_c(x) = f_c(x').$$

E is c-definable E.R. on P^l, so is B-definable.

f_c gives a definable bijection $U \to V$

where $U \subseteq P^{eq}$ $\quad V \subseteq Q^{eq}$

(parametrised by c)

Put $S = U \cup V \cup R$.

Have $rest_V : Aut(S/P) \to Aut(V/P)$ isom.

$rest_U : Aut(S/Q) \to Aut(U/Q)$ isom.

U is V-internal via f_c and vice versa.

<u>Obtain:</u> $Aut(U/Q)$, $Aut(V/P)$ are ∞-definable grps.

∎ (Hrushovski 2002 'Computing the Galois gp of a linear differential equation', Theorem B)

Also $\mathrm{Aut}(U/Q) \subseteq \mathrm{dcl}(P)$ as $U \subseteq P^{eq}$.
$\mathrm{Aut}(V/P) \subseteq \mathrm{dcl}(Q)$.

So $G_P := \mathrm{Aut}(S/P) \cong \mathrm{Aut}(V/P) \subseteq \mathrm{dcl}(Q)$
$G_Q := \mathrm{Aut}(S/Q) \cong \mathrm{Aut}(U/Q) \subseteq \mathrm{dcl}(P)$.

Claim: Both G_P and G_Q act regularly on R.

(Transitive as R is an orbit $/B$, and by $tp(c/B)$ determines $tp(c/P)$.)

If $g \in G_P$ fixes c, P, then
g fixes pointwise $V \subseteq \mathrm{dcl}(Pc)$. Forcing $g = 1$.

If $g \in G_P$, $h \in G_Q$ then
$$[g,h] = g^{-1}\underbrace{h^{-1}gh}_{\text{fix } P} \in \mathrm{Aut}(S/P \cup Q) = 1.$$

So G_P and G_Q commute.

Each $c \in R$ gives an isomorphism
$$\alpha_c : G_P \to G_Q$$
$$\alpha_c(g) = h \quad \text{where } g(c) = h^{-1}(c).$$

α_c is $\bar{c}$-definable and induces a bijection
$$(G_P)^n/\sim \;\longrightarrow\; (G_Q)^n/\sim \quad \text{any } n.$$

This bijection does not depend on c.

Suppose $\alpha_c(\bar{g}) = \bar{h}$ $\qquad \alpha_c(g_i) = h_i$

$g_i(c) = h_i^{-1}(c)$: each i.

Suppose $\alpha_{c'}(\bar{g}) = \bar{h}'$. $\qquad g_i(c') = h_i'^{-1}(c')$.

Pick $k \in G_Q$ with $k(c) = c'$

(G_Q trans. on R)

Then $k^{-1}h_i'^{-1}k(c) = k^{-1}h_i'^{-1}(c') = k^{-1}g_i(c')$

$= g_i k^{-1}(c') = g_i(c) = h_i^{-1}(c)$

So by regularity of G_Q on R, $k^{-1}h_i'^{-1}k = h_i^{-1}$.

So $k^{-1}\bar{h}'k = \bar{h}$.

So the bijection $G_P^n/_\sim \longrightarrow G_Q^n/_\sim$

is B-definable.

As $P \perp Q$, this forces that

$G_P^n/_\sim$, $G_Q^n/_\sim$ are finite.

In particular, G_P has finitely many orbits on G_P^n by conjugation, contrary to 3.1.