Model Theory II

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1 Why Model Theory II?

A major theme of model theory is that certain combinatorial properties of definable sets in a first order theory yield a lot of structural information about it. They can imply that a theory has a good notion of independence, or dimension, like in vector spaces, or help us understand the behaviour of groups or geometries interpretable in it. Sometimes, these properties can determine important algebraic features. This course will focus mainly on the best model theoretic properties: stability, and its strengthenings ω-stability and superstability.

Modern model theory begins with the work of Morley [14], and subsequently Shelah [15] on the spectrum problem: what are the possible behaviours of the function $I(\aleph_\alpha, T)$, counting the number of non-isomorphic models of T of cardinality $\aleph_\alpha$? Morley showed that, for countable T, if $I(\aleph_\alpha, T) = 1$ for some uncountable cardinal, then this is the case for all uncountable cardinals. Shelah studied for which theories we can define a system of invariants under which sufficiently large models of T can be classified. In this process he defined various properties that shaped the development of model theory. Two of the major results of [15] are that $I(\aleph_\alpha, T)$ is non-decreasing for uncountable cardinals and the Main Gap Theorem: for $\alpha > 0$, either T has the maximum number of models in each uncountable cardinality, $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$, or it satisfies few model theoretic properties, including superstability, and it is bounded above by $\beth_{\omega_1}(|\alpha|)$.

Over the years, a very sophisticated theory of stability was developed, with fruitful generalisations to NIP [16], simple [9], and NSOP$_1$ theories [7]. A map of the universe with more model theoretic properties and many examples is given on the website forkinganddividing.com. Model theory has fruitful applications and interactions in algebraic geometry [3], differential algebra [13], group theory [5, 1], number theory [11], and combinatorics [12]. Some classical notions from model theory have been independently discovered many times, such as VC-dimension in probability and combinatorics [10], and PAC learning and Littlestone dimension in machine learning [6]. My own research at TU Wien focuses on the computational complexity of problems in model theoretic structures [2], and on interactions between model theory and probability [4]. Overall, model theory is a highly versatile subject with many beautiful results, some of which we will cover in this course. I hope you will enjoy it!

2 The monster model

This first lecture will require some additional knowledge of set theory and cardinal arithmetics. If you are not familiar with set theory, Appendix A in [17] should contain most relevant facts. One of the advantages of the construction of the monster model in this lecture is preventing us from keeping track of issues of cardinal arithmetics later in the course.

Almost every article or book in model theory begins with the convention that we are working in a monster model $\mathbb{M}$. These are very large models of T distinguished by being highly saturated, strongly homogeneous, and universal in the following sense:

Definition 2.1. Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is:

- **κ-saturated** if it realises types (in finitely many variables) over sets of parameters of cardinality $< \kappa$;
- **κ-universal** if every model of T of cardinality $< \kappa$ elementarily embeds into $\mathcal{M}$;
- **κ-homogeneous** if for all $A \subseteq M$ of size $< \kappa$ and $a \in M$, every elementary map $f : A \to \mathcal{M}$ can be extended to an elementary map $A \cup \{a\} \to M$;
- **strongly κ-homogeneous** if for all $A \subseteq M$ of size $< \kappa$, any elementary map $f : A \to M$ can be extended to an automorphism of $\mathcal{M}$.

We say that $\mathcal{M}$ is **saturated** if it is $|M|$-saturated.

Remark 2.2. Recall that $\mathcal{M}$ is κ-saturated if and only if it is κ-saturated over 1-types, i.e. if it realises 1-types over sets of parameters of cardinality $< \kappa$;

In the following exercises we work with $|\mathcal{L}| \leq \kappa$:

Exercise 2.3. Prove that if $\mathcal{M}$ is κ-saturated then it is κ-homogeneous.

Exercise 2.4. Show that if $\mathcal{M}$ is κ-saturated, then it is κ^+-universal.

Exercise 2.5. (a) Show that if $\mathcal{M}$ is $|M|$-homogeneous, then it is strongly $|M|$-homogeneous. (b) For each cardinal κ, give an example of a κ-saturated structure $\mathcal{M}$ which is not strongly ω-homogeneous.

Exercise 2.6. Show that $\mathcal{M}$ is κ-saturated if and only if it is κ-homogeneous and κ^+-universal.

Ideally, we would like to work with a large saturated model since these models are universal, homogeneous and strongly homogeneous. We will see that for this we will need additional set theoretic assumptions.

Remark 2.7. Let $(X, <)$ be a linear order. We say that $Y \subseteq X$ is **cofinal** with X if for each $x \in X$ there is some $y \in Y$ with $x \leq y$. The **cofinality** of X, $\mathrm{cf}(X)$ is the smallest cardinality of a cofinal subset of X. We say that an infinite cardinal is κ is **regular** if $\mathrm{cf}(\kappa) = \kappa$. Any successor cardinal κ^+ is a regular cardinal, and so is ω.

Theorem 2.8. *Let $|\mathcal{L}| \leq \kappa$ and $\mathcal{M}$ be a model of cardinality $\leq 2^\kappa$. Then, $\mathcal{M}$ has an elementary extension $\mathcal{N}$ which is κ^+-saturated and of size $\leq 2^\kappa$.*

Proof. We build an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\kappa^+}$ such that

- $|\mathcal{M}_\lambda| \leq 2^\kappa$ for each $\lambda < \kappa$;
- $\mathcal{M}_{\lambda^+}$ realises all types over subsets of $\mathcal{M}_\lambda$ of cardinality $\leq \kappa$;

Firstly we show such chain exists and then we will show that its union, $\mathcal{N}$ satisfied the requirements of the theorem. We show how to perform the successor step of this construction. Since by inductive hypothesis $|\mathcal{M}_\lambda| \leq 2^\kappa$, $\mathcal{M}_\lambda$ has 2^κ subsets of size $\leq \kappa$. Since $|\mathcal{L}| \leq \kappa$, over each $B \subseteq \mathcal{M}_\lambda$ with $|B| \leq \kappa$, there are at most 2^κ-many 1-types. In particular, there are at most 2^κ-many 1-types over sets of size $\leq \kappa$. All of these can be realised in a model $\mathcal{M}_{\lambda^+}$ of cardinality $\leq 2^\kappa$.

We show that $\mathcal{N} := \bigcup_{\lambda<\kappa^+} \mathcal{M}_\lambda$ is κ^+-saturated and of size $\leq 2^\kappa$. For κ^+-saturation, consider $B \subseteq \mathcal{N}$ of size $< \kappa^+$. Since κ^+ is regular there must be some $\lambda \leq \kappa^+$ such that $B \subseteq \mathcal{M}_\lambda$ (otherwise, there would be a cofinal subset with κ^+ of cardinality $\leq |B| < \kappa^+$). Hence, all 1-types over B are realised in $\mathcal{M}_{\lambda^+}$. Finally, for the cardinality,

$$|\mathcal{N}| \leq \bigcup_{\lambda<\kappa^+} 2^\kappa \leq 2^\kappa,$$

where the last inequality holds since we are taking a union of sets of size $\leq \alpha$ over ordinals $< \alpha$, where α is an infinite cardinal. □

Definition 2.9 (Some set theoretic jargon)**.** Let κ be an infinite cardinal. A cardinal α is called a **strong limit cardinal** if for all cardinals $\beta < \alpha$, we have $2^\beta < \alpha$. A regular strong limit cardinal is called a **strongly inaccessible cardinal**.

Remark 2.10. It is easy to construct strong limit cardinals within ZFC. Moreover, the **global continuum hypothesis**, (GCH) implies that every limit cardinal is a strong limit cardinal. However, ZFC is consistent with there being no strongly inaccessible cardinals apart from ω.

Corollary 2.11. *Let $|\mathcal{L}| \leq \kappa$ and T be an $\mathcal{L}$-theory (with infinite models).*

(a) Assuming (GCH)*, T has a saturated model in each regular cardinal $\nu > \kappa$;*

(b) T has a saturated model in each strongly inaccessible cardinal $\nu > \kappa$.

Proof. (Omitted from lecture) The ideas are essentially the same of the previous proof. (a) If ν is a successor cardinal the argument is immediate. For a limit cardinal, one can use an analogue of the argument below. (b) Starting from a model $\mathcal{M}_0$ of cardinality κ, using Theorem 2.8, we build an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\nu}$, where $\mathcal{M}_{\lambda^+}$ is λ^+-saturated and of cardinality $\leq 2^\lambda$, and take $\mathcal{N}$ to be the union of this chain. Since ν is a strong limit cardinal,

$$|\mathcal{N}| \leq \bigcup_{\lambda<\nu} 2^\lambda \leq \nu.$$

Note that a β-saturated model must be of cardinality $\geq \beta$. Hence, $|\mathcal{N}| \geq \lambda^+$ for each $\lambda < \nu$, meaning that $|\mathcal{N}| = \nu$. Finally, we need to show saturation. Take $A \subset \mathcal{N}$ of cardinality $< \nu$. Since ν is regular, a set of cardinality $|A|$ cannot be cofinal with it, meaning that there is some $|A| \leq \lambda < \nu$ such that $\mathcal{M}_\lambda$ entirely contains $|A|$. Since $\mathcal{M}_\lambda$ is λ^+ saturated, it realised all types over A, and so does $\mathcal{N}$. □

Example 2.12.

- $(\mathbb{C}; 0, 1; +, \cdot)$ is a saturated model of the theory of algebraically closed fields;
- In general, one can prove that stable theories have saturated models of arbitrarily large cardinalities;
- If the continuum hypothesis is false, the theory of $(\mathbb{N}; 0, 1; +, \cdot)$ has no saturated models of cardinality κ for each $\aleph_0 < \kappa < 2^{\aleph_0}$.

Convention 2.13. From now on we will work with a **monster model** $\mathbb{M}$. which is κ-saturated, κ-universal and strongly κ-homogeneous for κ a cardinal larger than all of the cardinalities of models and sets of parameters that we want to consider. Thus, all models $\mathcal{M}, \mathcal{N}, \ldots$ we will consider will be elementarily embedded into this monster model, all sets of parameters $A, B, \ldots$ will be subsets of the monster model of cardinality $< \kappa$, and a set of formulas will be consistent if it is realised in $\mathbb{M}$. Finally, for a formula ϕ or a type p, we write $\models \phi$ (or $\models p$) if $\mathbb{M} \models \phi$ (respectively, $\mathbb{M} \models p$).

Remark 2.14. There are several ways to achieve the above:

- Assume that strongly inaccessible cardinals exist and work in a sufficiently large one. *We will adopt this approach* since it allows us to move quickly to do more model theory;
- Work in BGC (Bernays-Gödel+Global Choice) set theory. This is a conservative extension of ZFC which allows working with classes. In this framework we can build the monster model as a class-size union of chains.
- Work with a special model (see Definition 2.16) of cardinality $\nu = \beth_\kappa(\aleph_0)$. This will be ν^+-universal and strongly κ-homogeneous (add so κ-saturated by Exercise 2.6). This framework has the advantage of allowing us to work entirely within ZFC. If you are not comfortable with strongly inaccessible cardinals, you are welcome to read Subsection 2.1 and work with a large enough special model instead.

Lemma 2.15. *Let X be a definable subset of $\mathbb{M}$ and A a set of parameters (i.e. a set of size $< \kappa$ inside of $\mathbb{M}$). Then, the following are equivalent:*

(a) X is definable over A;

(b) X is $\mathrm{Aut}(\mathbb{M}/A)$-invariant (i.e. invariant under automorphisms of $\mathbb{M}$ fixing A pointwise).

Proof. Exercise. I will include the proof in a later version of the notes for completeness. □

2.1 Aside: special models

An issue with our definition of monster model being a saturated model of size a strongly inaccessible cardinal is that it makes it less transparent that our results are provable in ZFC. A more cautious reader might want to work with special models. An even more set theoretically oriented reader, might be interested in the approach of [8], which partially justifies the standard model theoretic practice of assuming we are working with a saturated model of large enough cardinality.

Definition 2.16. An infinite structure $\mathcal{M}$ of cardinality κ is **special** if it is the union of an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\kappa}$, where the λ are *cardinals* of size $< \kappa$ and each $\mathcal{M}_\lambda$ is κ^+-saturated.

$\star\star$ **Exercise 2.17.** Let $|\mathcal{L}| \leq \kappa$. Show that the following hold:

(a) If $\mathcal{M}$ is saturated then it is special;

(b) A special structure of regular cardinality is saturated;

(c) Suppose that $\lambda < \nu$ implies $2^\lambda \leq \nu$. Then, T has a special model of cardinality ν;

(d) A special structure of cardinality κ is κ^+-universal and strongly $\mathrm{cf}(\kappa)$-homogeneous.

Definition 2.18. For every cardinal μ, the **beth function** is defined as

$$\beth_\alpha(\mu) = \begin{cases} \mu & \text{if } \alpha = 0, \\ 2^{\beth_\beta(\mu)}, & \text{if } \alpha = \beta + 1, \\ \sup_{\beta<\alpha} \beth_\beta(\mu) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Remark 2.19. We have that $\mathrm{cf}(\beth_\kappa(\aleph_0)) = \kappa$, meaning that a special model of cardinality $\nu = \beth_\kappa(\aleph_0)$ is strongly κ-homogeneous, ν^+-universal, and κ-saturated. There is no harm in working with a special model of such cardinality as the monster model (except from having to prove exercise 2.17).

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