Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $\mathcal{L}_{\text{ring}}$ be the language of rings. For p prime, we denote by ACF_p the theory of algebraically closed fields of characteristic p. Similarly, ACF_0 denotes the theory of algebraically closed fields of characteristic 0.

EXERCISE 1. Let ϕ be an $\mathcal{L}_{\text{ring}}$-sentence. Prove that the following are equivalent:

- $\mathrm{ACF}_0 \vDash \phi$;
- for all sufficiently large primes p, $\mathrm{ACF}_p \vDash \phi$;
- there are arbitrarily large primes p such that $\mathrm{ACF}_p \vDash \phi$.

Deduce that ACF_0 is not finitely axiomatizable.

Definition 2. Let K be a field. We say that a map $f : K^n \to K^n$ is a **polynomial map** if it is of the form

$$f(x_1, \dots, x_n) = (p_1(x_1, \dots, x_n), \dots, p_n(x_1, \dots, x_n)),$$

where $p_i \in K[x_1, \dots, x_n]$ for each $i \leq n$.

The following theorem was first proven using model theory (indeed, you only need Exercise 1 and the fact that ACF_p is complete for each prime p):

$\star\star$ **EXERCISE 2.** Prove the Ax-Grothendieck Theorem: let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If f is one to one, then f is onto. [Hint: for $d \in \mathbb{N}$, there is an $\mathcal{L}_{\text{ring}}$-sentence Φ_d expressing that for all polynomial maps f such that every polynomial p_i in it has degree $\leq d$, if f is one-to-one, then it is onto.]

Definition 3. Let $\mathcal{L}_{\text{gr}}$ consist of a single binary relation E and T_{gr} be the theory of undirected graphs without loops. For $n, m \in \mathbb{N}$, the Alice restaurant axiom $A_{n,m}$ is the following $\mathcal{L}_{\text{gr}}$ sentence:

$$\forall x_1, \dots, x_n, y_1, \dots, y_m \Big(\bigwedge_{i,j} x_i \neq y_j \to (\exists z \bigwedge_{i \leq n} E(z, x_i) \wedge \bigwedge_{j \leq m} (\neg E(z, y_j) \wedge z \neq y_j)) \Big).$$

Let T_{rg} be obtained by $T_{\text{gr}} \cup \{A_{n,m} | n, m \in \mathbb{N}\}$. We call T_{rg} the **theory of the random graph**.

Definition 4. We say that an $\mathcal{L}$-formula ϕ is **quantifier-free** if it does not contain any quantifier.

Definition 5. We say that an $\mathcal{L}$-theory T has **quantifier elimination** if every $\mathcal{L}$-sentence is equivalent, modulo T, to a quantifier-free $\mathcal{L}$-formula. That is, for every $\mathcal{L}$-formula $\phi(\overline{x})$ with free variables $\overline{x}$, there is a quantifier-free $\mathcal{L}$-formula $\psi(\overline{x})$ such that

$$T \vdash \forall \overline{x} (\phi(\overline{x}) \leftrightarrow \psi(\overline{x})) \ .$$

EXERCISE 3. Show that the theory of the random graph is ω-categorical. Deduce that it is complete and with quantifier elimination.

The following is really an exercise in probability, but it is the reason for the name of the random graph.

$\star\star$ **EXERCISE 4.** Let $0 < p < 1$. Take n vertices and for each pair of distinct vertices choose independently at random with probability p whether they form an edge. Let $G(n, p)$ be the graph obtained in this manner. Show that for each $k, l \in \mathbb{N}$,

$$\mathbb{P}(G(n, p) \vDash A_{k,l}) \to 1 \text{ as } n \to \infty.$$

Prove that for any $\mathcal{L}_{\text{gr}}$-sentence ϕ, $T_{rg} \vDash \phi$ if and only if

$$\mathbb{P}(G(n, p) \vDash \phi) \to 1 \text{ as } n \to \infty.$$

EXERCISE 5. Show that the following theories do NOT have quantifier elimination:

- $\mathrm{Th}(\mathbb{N}; <)$;
- $\mathrm{Th}(\mathbb{Z}; +)$;
- $\mathrm{Th}(\mathbb{R}; 0, 1, +, \cdot, -)$;
- $\mathrm{Th}(\mathbb{Q}; 0, 1, +, \cdot, -, <)$.