Orthogonality Revisited

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Let us start by recalling the set-up from last week. Working inside a non-standard model of set theory, we let $\mathcal{M}$ stand for an internally finite structure with internal language $\mathcal{L}_0$. We write $\mathcal{M}^*$ for the reduct of $\mathcal{M}$ to $\mathcal{L}^*$, which is the language obtained from $\mathcal{L}_0$ by restricting to formulae with a standard finite number of variables. Recall also the following result from Dugald's talk:

Lemma 0.1 (3.4.2)**.** *Let $\mathcal{M}$ be internally finite and J a (standard finite) disjoint union of $\varnothing$-definable (in $\mathcal{M}$) basic projective simple Lie geometries with no extra structure. Let $G := \mathrm{Aut}(J)$ and $G_1 := \mathrm{Aut}(J)^{(\infty)}$, understood internally. Let $H := \mathrm{Aut}(M)^J$. TFAE:*

(i) J is canonically embedded in $\mathcal{M}^*$.

(ii) $G_1 \leq H$.

Our goal is to obtain a similar dichotomy between orthogonality and linkage for geometries definable in $\mathcal{M}$. This will also allow us to reformulate this criterion in the affine case. Recall that "orthogonal" in the present context means the following:

Definition 0.2. Let $\mathcal{M}$ be an internally finite structure. We say J_1 and J_2 are **orthogonal** over A in $\mathcal{M}$ if $(J_1, J_2; \mathrm{acl}(A) \cap J_1, \mathrm{acl}(A) \cap J_2)$ is canonically embedded in $\mathcal{M}$.

The following lemma has been proven by Paolo:

Lemma 0.3 (3.4.3)**.** *Let $\mathcal{M}$ be an internally finite infinite structure. Let J_1 and J_2 be basic pure projective geometries defined and orthogonal over $A = \mathrm{acl}(A)$. Let $J := J_1 \cup J_2$ and $A_J := A \cap J$. Then the permutation group G induced on J by the internal automorphism group of $\mathcal{M}$ fixing A contains $\mathrm{Aut}(J; A_J)^{(\infty)}$.*

Remark 0.4 (3.4.4)**.** In the above lemma, $(J; A_J)$ is to be understood as a "pointed geometry," that is, as the expansion of the geometry J by adding constants for all elements of A_J. The authors note that Lemma 3.4.2 (0.1 above) from Dugald's talk can also be proven for pointed geometries.

1 Joint orthogonality

The first step is to define a notion of "joint orthogonality":

Definition 1.1 (3.4.5)**.** We say that a collection S_i of A-definable sets is **jointly orthogonal** over A in $\mathcal{M}$ if the disjoint union of the structures $(S_i; \mathrm{acl}(A) \cap S_i)$ is canonically embedded in $\mathcal{M}$.

The first observation to make is that we can, under some specific circumstances, increase the size of the base over which sets are jointly orthogonal:

Lemma 1.2 (3.4.6)**.** *Let J_i be structures A-definable in $\mathcal{M}$ with weak elimination of imaginaries, and let $B \subseteq J := \bigcup_i J_i$. Then the J_i are jointly orthogonal over A in $\mathcal{M}$ iff they are jointly orthogonal over $A \cup B$ in $\mathcal{M}$.*

Proof. ($\Rightarrow$) Suppose that the J_i's are jointly orthogonal over A. We want to show that $\bigcup_i (J_i; \mathrm{acl}(AB) \cap J_i)$ is canonically embedded in $\mathcal{M}$. So let R be an $A \cup (\bigcup_i \mathrm{acl}(AB) \cap J_i)$-definable relation on J (note that we can write A since each J_i is A-definable). In particular, by quantifying the parameters away, we can view R as the "specialization" of a $\varnothing$-definable relation S on J. Since $\bigcup_i (J_i; \mathrm{acl}(A) \cap J_i)$ is canonically embedded in $\mathcal{M}$, we can write S as a Boolean combination of products of $(\mathrm{acl}(A) \cap J_i)$-definable relations on J_i. Since we only need to add parameters from $\mathrm{acl}(B) \cap J_i$, we can therefore write R as a Boolean combination of products of $(\mathrm{acl}(AB) \cap J_i)$-definable relations on J_i. Therefore, $\bigcup_i (J_i; \mathrm{acl}(AB) \cap J_i)$ is canonically embedded in $\mathcal{M}$.

($\Leftarrow$) Suppose the J_i's are jointly orthogonal over $A \cup B$. We want to show that $\bigcup_i (J_i; \mathrm{acl}(A) \cap J_i)$ is canonically embedded in $\mathcal{M}$. So let R be an A-definable relation on J. Since J is already canonically embedded in $\mathcal{M}$ over $A \cup B$, R is $\bigcup_i (\mathrm{acl}(AB) \cap J_i)$-definable. Consider R as an element of J^{eq} and let $e := \mathrm{acl}(R) \cap J$. Then, since J has weak elimination of imaginaries, $R \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(R) \cap J) = \mathrm{dcl}^{\mathrm{eq}}(e)$, and since R is A-definable, $e \subseteq \mathrm{acl}(A) \cap J$. Therefore, R is $\bigcup_i (\mathrm{acl}(A) \cap J_i)$-definable, and thus $\bigcup_i (J_i; \mathrm{acl}(A) \cap J_i)$ is canonically embedded in $\mathcal{M}$. $\square$

It is easy to see that, for pairs of geometries J_i and J_j, being orthogonal and jointly orthogonal over A coincide. The following lemma shows that, when we move to consider larger collections of geometries, joint orthogonality can still be determined by looking at what happens at the level of pairs:

Lemma 1.3 (3.4.7)**.** *Let $\mathcal{M}$ be an internally finite structure. Let J_i, for $i \in I$, be fully*[1] *embedded projective Lie geometries in $\mathcal{M}^*$ such that (i) J_i is A-definable in $\mathcal{M}^*$ for all $i \in I$, and (ii) J_i and J_j are orthogonal over A in $\mathcal{M}^*$ for all $i \neq j \in I$. Then they are jointly orthogonal over A in $\mathcal{M}^*$.*

[1] The original text only requires canonical embeddedness, but Paolo has noted that the use of 3.4.6 seems to require full embeddedness.

Proof. Let $A_i := \mathrm{acl}(A) \cap J_i$. By (ii), $(J_i \cup J_j; A_i \cup A_j)$ is canonically embedded in $\mathcal{M}^*$. We now define finite subsets $A_i \subseteq B_i \subseteq J_i$ by:

$$B_i := \begin{cases} A_i & \text{if } J_i \text{ is pure projective,} \\ \text{a non-deg. subspace with a quadratic point} & \text{otherwise.} \end{cases}$$

Let $B := A \cup \bigcup_i B_i$.

Claim. J_i and J_j are orthogonal over B in $\mathcal{M}^*$ for all $i \neq j \in I$.

Proof. If J_i and J_j are pure projective, there is nothing to do. Otherwise, note that, as $B_i \cup B_j \subseteq J_i \cup J_j$, the left-to-right implication from Lemma 3.4.6 (1.2) applies, so that J_i and J_j are orthogonal over B in $\mathcal{M}^*$.[2] □

By Lemma 3.4.6 (1.2), it suffices to show that the J_i are jointly orthogonal over B ($= A \cup B$). If $B_i \neq A_i$, we may replace J_i by its localization J_i/B_i, as they are "definably equivalent" over B_i. Since B_i contains a quadratic point[3], the localization J_i/B_i will be of the same type as J_i (see, e.g., 2.4.10).

Let H be the group of permutations induced by $\mathrm{Aut}(M/A)$ on $\bigcup_i (J_i; B_i)$. Let $G_i := \mathrm{Aut}(J_i; B_i)^{(\infty)}$. By the claim, for each $i \neq j$, $(J_i, J_j; B_i, B_j)$ is canonically embedded in $\mathcal{M}^*$, so that, by Lemma 3.4.3, $\mathrm{Aut}(M/A)^{J_i \cup J_j} \supseteq \mathrm{Aut}(J_i, J_j; B_i, B_j)^{(\infty)} = G_i \times G_j$. In particular, it follows that $H \supseteq \prod_i G_i$. Therefore, by Lemma 3.4.2 (0.1) from Dugald's talk, the J_i's are jointly orthogonal over B, as required. □

2 Back to simple Lie geometries

Recall that we showed Lemma 2.4.3, which told us that any two normal geometries which are fully embedded and 0-definable in a larger structure $\mathcal{M}$ are either orthogonal or 0-linked. Moreover, we had previously shown in 2.4.2 that basic projective geometries are (reduced) normal, so that the above dichotomy applies to them. The proof of this dichotomy back then was done axiomatically from the definition of a normal geometry.

We will now (partially) recover this result appealing instead to the results from finite group theory that Paolo covered in his talk. To do so, we restrict ourselves to basic *simple* projective Lie geometries. The advantage is that we only require canonical instead of full embeddedness, but the price we pay is that we lose the dichotomy: an exceptional case appears.

Lemma 2.1 (3.4.8)**.** *Let $\mathcal{M}$ be an internally finite structure. Let J_1, J_2 be 0-definable basic simple projective Lie geometries canonically embedded in $\mathcal{M}^*$. Exactly one of the following holds in $\mathcal{M}^*$:*

[2] Note that the proof of this direction does not require weak elimination of imaginaries for the J_i's, so we do not get in trouble with 2.3.7.

[3] I think this might mean an element of J_i with an isotropic representative (?).

(i) J_1 and J_2 are orthogonal (over $\varnothing$).

(ii) There is a 0-definable bijection between J_1 and J_2.

(iii) J_1 and J_2 are of pure projective type and there is a 0-definable duality[4] between them making the pair (J_1, J_2) a polar space.

Proof. We work internally throughout. Let S be the permutation group induced on $J := J_1 \cup J_2$ by (internal) automorphisms of $\mathcal{M}$. For $i = 1, 2$, let G_i be the (internal) automorphism group of the geometry J_i. Let $S_1 := S \cap (G_1 \times G_2)^{(\infty)}$. In particular, we have projections $S \twoheadrightarrow G_i$, which induce projections $\pi_i \colon S^{(\infty)} \twoheadrightarrow G_i^{(\infty)}$. Since $S^{(\infty)}$ is simple, we must have either $\ker \pi_i = \{1\}$ or $G_i^{(\infty)}$.

Case 1: There is $i \in \{1, 2\}$ such that $\ker \pi_i = S^{(\infty)}$.

Then $S \cong G_1 \times G_2$, which implies that $S_1 = (G_1 \times G_2)^{(\infty)}$. Therefore, $(G_1 \times G_2)^{(\infty)} = S_1^{(\infty)} \leq S$, which entails by Lemma 3.4.2 (0.1) that J is canonically embedded in $\mathcal{M}^*$. As $J = J_1 \cup J_2$, by definition this means that J_1 and J_2 are orthogonal over $\varnothing$.

Case 2: $\ker \pi_i = \{1\}$ for $i = 1, 2$.

Then the π_i are group isomorphisms, and thus we obtain an isomorphism $G_1^{(\infty)} \to G_2^{(\infty)}$. In particular, G_1 and G_2 have isomorphic internal socles, so that by Fact 3.3.4(2),[5] J_1 and J_2 are isomorphic as weak geometries.[6] Identifying them along this isomorphism, we can then view S_1 as the graph of some automorphism of the resulting weak geometry (call it J_*).

Subcase 1: J_* is not of pure projective type (i.e., it has a form).

By Fact 3.3.4(1)(iii) from Paolo's talk, it follows that the automorphism whose graph S_1 encodes is an inner automorphism of the geometry J_*. Say that this inner automorphism corresponds to an isomorphism $h \colon J_1 \to J_2$ of the original geometries. Then, since $S_1 \lhd S$, h is S-invariant, and hence 0-definable.

Subcase 2: J_* is of pure projective type (i.e., it is the projectivisation of a pure vector space).

Then the automorphism that S_1 is a graph of "may be the composition of an inner automorphism [i.e., those maps coming from $\mathrm{P\Gamma L}(V)$ for V the underlying vector space] and a graph automorphism [i.e., the map on J_* induced by the inverse transpose automorphism]." This follows from the exceptional case in Fact 3.3.4(1)(iii). Thus, we can view S_1 as an isomorphism[7] $J_1^* \to J_2$. Since J_2 is 0-definable, this yields an interpretation of J_1^* in $\mathcal{M}$. In particular, we can view the isomorphism as a map $J_1 \times J_2 \to K$, which thus gives the pair (J_1, J_2) the structure of a polar space. The isomorphism is again 0-definable. $\square$

[4] I.e., a map $J_1^* \to J_2$.

[5] In the book, they write Fact 3.3.11, which seems to be a typo.

[6] This assumes that we are ignoring the degenerate case.

[7] This uses the definition of the inverse transpose automorphism; cf. [Wil09, §3.3.4].

References

[Wil09] Robert Wilson, *The finite simple groups*, vol. 251, Springer, 2009.