The next lecture is entirely dedicated to exercises.

EXERCISE 1. Let $\phi(x;y)$ and $\psi(x,z)$ be stable $\mathcal{L}$-formulas. Show the following:

1. $\phi^{\text{opp}}(y;x) := \phi(x;y)$ is stable;
2. $\neg\phi(x;y)$ is stable;
3. $\theta(x;yz) := \phi(x;y) \wedge \psi(x;z)$ is stable;
4. $\theta(x;yz) := \phi(x;y) \vee \psi(x;z)$ is stable;
5. For $y = uv$ and $c \in \mathbb{M}^{|v|}$, $\phi(x;u,c)$ is stable.

EXERCISE 2. Show that the following are equivalent:

1. T is stable (in the sense of being κ-stable for some infinite κ);
2. every $\mathcal{L}$-formula $\phi(x;y)$ is stable for T;
3. T is κ-stable for all κ such that $\kappa^{|T|} = \kappa$.

Definition 1. For κ an infinite cardinal, let

$$\text{ded}(\kappa) := \sup\{|I| \ : I \text{ is a linear ordering with a dense subset of size } \kappa\}.$$

It is easy to see that $\kappa < \text{ded}(\kappa) \leq 2^\kappa$.

Definition 2. Let T be a countable theory. Write f_T : Card $\to$ Card for the function on cardinals given by

$$f_T(\kappa) := \sup\{|S_n(M)| \ : \mathcal{M} \models T, |M| = \kappa, n \in \omega\}.$$

[It is easy to see that if we fixed n in the definition above, we would still get f_T.]

EXERCISE 3. Prove that if T is unstable, then $f_T(\kappa) \geq \text{ded}(\kappa)$ for all cardinals $\kappa \geq |T|$.

Recall the following definitions and lemmas from the Model Theory course:

Definition 3. Let I be an infinite linear order and A a set of parameters. We say that $(a_i|i \in I)$ is **indiscernible** over A if for every $\mathcal{L}(A)$-formula $\phi(x_1,\ldots,x_n)$ and $i_1 < \cdots < i_n, j_1 < \cdots < j_n$ from I, we have that

$$\models \phi(a_{i_1},\ldots,a_{i_n}) \leftrightarrow \phi(a_{j_1},\ldots,a_{j_n}). \tag{1}$$

We say that the sequence is **totally indiscernible** over A if the condition 1 holds for any $\{i_1,\ldots,i_n\}, \{j_1,\ldots,j_n\}$ from I of size n.

For a sequence $(a_i|i \in I)$, its **EM-type** (i.e. Ehrenfeucht-Motowski type) over A is given by

$$\text{EM}(a_i|i \in I) := \{\phi(x_1,\ldots,x_n) \in \mathcal{L}(A) \mid \models \phi(a_{i_1},\ldots,a_{i_n}) \text{ for all } i_1 < \cdots < i_n, n < \omega\}.$$

Lemma 4 (Extracting indiscernible sequences)**.** *Let A be a set of parameters and $(b_i|i \in I)$ a infinite sequence. Let J be a linear order. Then, there is a sequence $(a_j|j \in J)$ which is indiscernible over A and realising the same EM-type as $(b_i|i \in I)$.*

EXERCISE 4.

- Show that T is stable if and only if there is an infinite sequence $(a_i|i < \omega)$ and a formula $\phi(x,y)$ such that $\models \phi(a_i,a_j)$ if and only if $i < j$;
- Show that if T is unstable there is an indiscernible sequence which is not totally indiscernible;

- Show that if T is stable then every indiscernible sequence is totally indiscernible. [Hint: Say that we have an indiscernible sequence $(a_i | i < \omega)$ which is not totally indiscernible. Show that there is a formula $\phi(x_1, \ldots, x_n)$ such that for some transposition τ switching only two consecutive variables

$$\vdash \phi(a_1, \ldots, a_n) \wedge \neg\phi(a_{\tau(1)}, \ldots, a_{\tau(n)}).$$

Use this formula to find an unstable formula in T.]

Theorem 5 (Erdös-Makkai)**.** *Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ with $|B| < |\mathcal{F}|$. Then, there are sequences $(b_i | i < \omega)$ of elements of B and $(S_i | i < \omega)$ of elements of $\mathcal{F}$ such that for all $i, j \in \omega$, we have that*

- *EITHER $b_i \in S_j$ if and only if $j < i$;*
- *OR $b_i \in S_j$ if and only if $i < j$.*

EXERCISE 5 (Proof of Erdös-Makkai)**.** Note that there is $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| = |B|$ and for all $B_0, B_1 \subseteq B$ finite, if there is some $S \in \mathcal{F}$ such that $B_0 \subseteq S, B_1 \subseteq B \setminus S$, then there is some $S' \in \mathcal{F}$ with $B_0 \subseteq S', B_1 \subseteq B \setminus S'$. Note that there is $S^\star \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$. Now, prove Erdös-Makkai. [Hint: you need to construct appropriate sequences in $S^\star$, $B \setminus S^\star$ and $\mathcal{F}'$, and then use Ramsey's theorem.]