1. The Cherlin-Hrushovski rank

Definable means "definable *with parameters*".

Definition 1.1. Let $D \subseteq \mathcal{M}$ be a definable set. We define *Cherlin-Hrushovski rank* of D, denoted $\mathsf{CH\text{-}rk}(D)$. First, we inductively define $\mathsf{CH\text{-}rk}(D) \geq n$ for $n \in \mathbb{N}$:

(1) $\mathsf{CH\text{-}rk}(D) \geq 0$ if D is non-empty.

(2) $\mathsf{CH\text{-}rk}(D) > 0$ if D is infinite.
[In particular, if $\mathsf{CH\text{-}rk}(D) = 0$ then D is finite.]

(3) For $n \in \mathbb{N}$, $\mathsf{CH\text{-}rk}(D) \geq n+1$ if there exist definable sets D_1, D_2 and definable functions $\pi : D_1 \to D$ and $f : D_1 \to D_2$ such that:
  (a) For all $d \in D$ we have $\mathsf{CH\text{-}rk}\left(\pi^{-1}(d)\right) = 0$.
  [In particular, π is surjective and finite-to-one.]
  (b) $\mathsf{CH\text{-}rk}(D_2) > 0$.
  [In particular, D_2 is infinite.]
  (c) For all $d \in D_2$ we have $\mathsf{CH\text{-}rk}(f^{-1}(d)) \geq n$.
  [In particular, if $n \geq 0$, then f is also surjective.]

As is usual, $\mathsf{CH\text{-}rk}(D) = n$ if $\mathsf{CH\text{-}rk}(D) \geq n$ and $\mathsf{CH\text{-}rk}(D) \ngeq n+1$. If $\mathsf{CH\text{-}rk}(D) \geq n$ for all $n \in \mathbb{N}$ then we write $\mathsf{CH\text{-}rk}(D) = \infty$.

Remarks 1.2.

(1) If $\mathcal{M}$ is not $\aleph_0$-categorical (in particular $\aleph_0$-saturated), then $\mathsf{CH\text{-}rk}$ should be defined in a saturated model. This will not be an issue here, as we will only work in the $\aleph_0$-categorical context.

(2) The definition above can be applied either to $\mathcal{M}$ or to $\mathcal{M}^{\mathsf{eq}}$. When the distinction becomes significant, the following terminology will be used:
  - $\mathsf{CH\text{-}rk}$ calculated in $\mathcal{M}$ will be referred to as *pre-rank*;
  - $\mathsf{CH\text{-}rk}$ calculated in $\mathcal{M}^{\mathsf{eq}}$ will be referred to simply as *rank* (somehow indicating that $\mathcal{M}^{\mathsf{eq}}$ is the right context to carry out these calculations).

To begin to understand this definition, we can imagine the simplifying scenario where we enforce that $D_1 = D$ and $\pi = \mathsf{id}$. With this additional assumption, the definition simply says that $\mathsf{CH\text{-}rk}(D) \geq n$ if there is a definable set D_2 and a definable function $f : D \to D_2$ whose fibres all have rank at least n. Equivalently, (again, assuming that $D_1 = D$ and $\pi = \mathsf{id}$) $\mathsf{CH\text{-}rk}(D) \geq n+1$ means that there is a *uniformly definable partition*[A partition of of the form $\{\phi(x,t) : t \in E\}$, where E is a definable set.] of D into infinitely many parts of rank at least n.

The definition weakens this a little bit, since $\mathsf{CH\text{-}rk}(D) \geq n$ if there is a *definable finite cover* D' of D [A definable surjective map $\pi : D' \to D$ such that $\pi^{-1}(d)$ is finite for all $d \in D$.] with $\mathsf{CH\text{-}rk}(D') \geq n$.

The following example, from [Sim22], illustrates why this may be necessary:

Example 1.3. Let $\mathcal{M}$ be an infinite set with no extra structure (so we are working in the language of pure equality). And let $D \subseteq \mathcal{M}^{\mathsf{eq}}$ be the set of subsets of M of size 2 (naturally a definable set in $\mathcal{M}^{\mathsf{eq}}$, as it is the quotient of the definable set $\{(a,b) : a \neq b\}$ by the definable equivalence relation $(a,b) \sim (c,d) \iff \{a,b\} = \{c,d\}$). Then:

Claim 1.3.1. *$\mathsf{CH\text{-}rk}(D) \geq 2$.*

Proof of Claim. First, we show that $\mathsf{CH\text{-}rk}(D) \geq 2$. Let $D_1 = \{(a,b) \in M^2 : a \neq b\}$, $D_2 = M$, $f : D_1 \to D_2$ the projection onto the first coordinate and $\pi_\sim : D_1 \to D$ the "selector function" for classes of the equivalence relation $\sim$ (from above). Since D_2 is infinite and so is $f^{-1}(d)$ for all $d \in D_2$, this inequality follows. ◀
[In fact, $\mathsf{CH\text{-}rk}(D) = 2$, but one inequality suffices to make the point I want to make, and this note is already getting out of hand.]

Claim 1.3.2. *Let D' be an infinite definable subset of $\mathcal{M}^{\mathsf{eq}}$ and $f : D \to D'$ a definable map. Then f has finite fibres.*

Proof of Claim. [The intuition is that we cannot definably choose an element from each member of D.]
Fix an infinite definable subset D' of $\mathcal{M}^{\mathsf{eq}}$ and suppose toward a contradiction that there is a definable map $f : D \to D'$ with infinite fibres. Observe that if for some $d \neq d' \in D'$ there are $\{a, b\} \in f^{-1}(d)$ and $\{a', b'\} \in f^{-1}(d')$ such that $|\{a, b\} \cap \{a', b'\}| = 1$, then, as all elements in the same fibre are conjugates, this means that for all $\{a, b\} \in f^{-1}(d)$ and all $\{a', b'\} \in f^{-1}(d')$ we have that $|\{a, b\} \cap \{a', b'\}| = 1$. This is easily seen to be impossible.

So any two distinct fibres of f must consist of pairs of elements of M with trivial intersection. But, of course, this means that f can have only one fibre, which is impossible. ◀
[The point of the second claim is that if we enforce in the definition that $D_1 = D$ and $\pi = \mathsf{id}$, then the rank of D would be 1.]

We now give two more examples with computations of $\mathsf{CH\text{-}rk}$. These are lifted from [Wol20].

Example 1.4. Let $\mathcal{L} = \{E_1, E_2\}$ be a language with two binary relation symbols and $\mathcal{M}$ an $\mathcal{L}$-structure in which E_1, E_2 are both equivalence relations with infinitely many infinite classes, so that each E_1-class is refined by infinitely many E_2-classes.

- Let D be an E_2-class. By definition $\mathsf{CH\text{-}rk}(D) \geq 0$, since D is infinite. Of course, there is no partition of D into infinitely many infinite sets
- The rank of an E_1 class in $\mathcal{M}$ is 2.
- The rank of $\mathcal{M}$ is 3.

In each case, it is easy to see why the rank is at least the given value. The other direction requires a bit more care.

The next example is also a continuation from the end of David's talk:

Example 1.5 ([CH02, Example 2.1.11])**.** Let $\mathcal{L} = \{0, +\}$ and p a fixed prime. Let $A = \bigoplus_{n \in \omega} \mathbb{Z}/p^2\mathbb{Z}$, i.e.

$$\mathsf{dom}(\mathcal{M}) = \{(a_i)_{i \in \omega} : a_i \in \mathbb{Z}/p^2\mathbb{Z}, \text{ and } a_i = 0 \text{ for all but finitely many } i \in \omega\}.$$

This is, of course, a countable $\aleph_0$-categorical structure.

Recall that, for $a \in A$ we write pa for $\underbrace{a + \cdots + a}_{p \text{ times}}$. We write $A[p]$ for the subgroup of A consisting of p-th powers, or, equivalently of all the elements of A of order p:

$$A[p] := \{a \in A : pa = 0\}.$$

This is an $\mathbb{F}_p$-vector space of infinite dimension, and we shall denote its projectivisation by J_0, so

$$J_0 = A[p] \setminus \{0\}/\sim,$$

where $a \sim b$ if, and only if $a = rb$ for some $r \in \mathbb{F}_p$. For $c \in A$ we will write A_c for the set $\{a \in A : pa = c\}$. We Lie coordinatised this structures, resulting in a tree of height 4. The calculations in [Wol20] give us:

$$\mathsf{CH\text{-}rk}(A) = 2.$$

(More precisely, $\mathsf{CH\text{-}rk}(A_c) = 1$, for all $c \in A$ and $\mathsf{CH\text{-}rk}(J_0) = 1$.)

The examples above are special cases of Corollary 1.27

1.1. **The very basics.** We start by listing some of the most basic properties of $\mathsf{CH\text{-}rk}$.

Lemma 1.6 (Lemma 2.2.2(1))**.** *Let $D \subseteq \mathcal{M}$ be a definable set. Then $\mathsf{CH\text{-}rk}(D) = 0$ if, and only if D is finite.*

Proof. It suffices to show that if D is finite, then $\mathsf{CH\text{-}rk}(D) \ngeq 1$. Suppose not. Then, by definition, there is a definable set D_1, an infinite definable set D_2, a surjective map $f : D_1 \to D_2$ (since $\mathsf{CH\text{-}rk}\left(f^{-1}(d)\right) \geq 0$, for all $d \in D_2$) and a surjective finite-to-one map $\pi : D_1 \to D$. But, it is clear that D_1 is also infinite, and hence D must be infinite, a contradiction. $\square$

Lemma 1.7 (Lemma 2.2.2(2))**.** *Let A, B be definable sets. If $A \subseteq B$ then $\mathsf{CH\text{-}rk}(A) \leq \mathsf{CH\text{-}rk}(B)$.*

Proof. Formally we prove, by induction on $n \in \mathbb{N}$, that, for definable sets A, B, if $A \subseteq B$ and $\mathsf{CH\text{-}rk}(A) \geq n$ then $\mathsf{CH\text{-}rk}(B) \geq n$. The base case ($n = 0$) is trivial. For the inductive step, suppose that the result holds for definable sets of rank at least n and $\mathsf{CH\text{-}rk}(A) \geq n + 1$. By definition, there exist definable sets D_1, D_2 and definable functions $\pi : D_1 \to A$, and $f : D_1 \to D_2$, such that: D_2 is infinite; $\mathsf{CH\text{-}rk}\left(\pi^{-1}(d)\right) = 0$ for all $d \in D_1$; and $\mathsf{CH\text{-}rk}\left(f^{-1}(d)\right) \geq n$, for all $d \in D_2$.

Without loss of generality, we may assume that D_1 and B are disjoint (e.g. by appending a new fixed coordinate to all elements of D_1). Let $D_1' = D_1 \sqcup (B \setminus A)$, pick a point $d_\star \in D_2$ and define:

$$\begin{aligned} g : D_1' &\to D_2 \\ d &\mapsto \begin{cases} f(d) & \text{if } d \in D_1 \\ d_\star & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \pi' : D_1' &\to B \\ d &\mapsto \begin{cases} \pi(d) & \text{if } d \in D_1 \\ d & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, for each $d \in D_2$ we have $g^{-1}(d) \subseteq f^{-1}(d)$. Since $\mathsf{CH\text{-}rk}\left(f^{-1}(d)\right) \geq n$, by inductive hypothesis, $\mathsf{CH\text{-}rk}\left(g^{-1}(d)\right) \geq n$. Thus, by definition, we have that $\mathsf{CH\text{-}rk}(B) \geq n + 1$, as required. $\square$

Lemma 1.8 (Remark in the proof of Lemma 2.2.2)**.** *Let A and B be definable sets. Then:*

$$\mathsf{CH\text{-}rk}(A \cup B) = \max\{\mathsf{CH\text{-}rk}(A), \mathsf{CH\text{-}rk}(B)\}.$$

Proof. It is clear form Lemma 1.7 that $\max\{\mathsf{CH\text{-}rk}(A), \mathsf{CH\text{-}rk}(B)\} \leq \mathsf{CH\text{-}rk}(A \cup B)$.

For the inequality $\mathsf{CH\text{-}rk}(A \cup B) \leq \max\{\mathsf{CH\text{-}rk}(A), \mathsf{CH\text{-}rk}(B)\}$, we start by observing that it suffices to prove it when A and B are disjoint. Indeed, once we have proved it for two disjoint sets, we can immediately generalise it by induction to unions of n disjoint sets. Then, clearly:

$$\begin{aligned} \mathsf{CH\text{-}rk}(A \cup B) &\leq \max\{\mathsf{CH\text{-}rk}(A \setminus (A \cap B)), \mathsf{CH\text{-}rk}(B \setminus (A \cap B)), \mathsf{CH\text{-}rk}(A \cap B)\} \\ &\leq \max\{\mathsf{CH\text{-}rk}(A), \mathsf{CH\text{-}rk}(B)\}, \end{aligned}$$

where the last inequality follows immediately from Lemma 1.7.

Let $n \in \mathbb{N}$ be arbitrary and suppose that A and B are disjoint sets and $\mathsf{CH\text{-}rk}(A \cup B) \geq n + 1$ (the case $n = 0$ is trivial). The result will follow almost immediately from the definition and the pigeonhole principle. Explicitly, if $\mathsf{CH\text{-}rk}(A \cup B) \geq n + 1$, then, there are definable sets D_1, D_2 and surjective definable maps $\pi : D_1 \to D$ and $f : D_1 \to D_2$ such that D_2 is infinite, $\pi^{-1}(d)$ is finite, for all $d \in D$ and

$\mathsf{CH\text{-}rk}(f^{-1}(d)) \geq n$, for all $d \in D_2$. We may partition D_2 into two disjoint definable sets, as follows:

$$D_2 = \{d \in D_2 : \pi(f^{-1}(d)) \in A\} \sqcup \{d \in D_2 : \pi(f^{-1}(d)) \in B\}.$$

At most one of these sets is finite, by the pigeonhole principle, and thus at least one of $\mathsf{CH\text{-}rk}(A)$ or $\mathsf{CH\text{-}rk}(B)$ must be greater than $n+1$. $\square$

1.2. **Ranks of elements.**

Definition 1.9. Let $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$. The *Cherlin-Hrushovski rank of a over B*, denoted $\mathsf{CH\text{-}rk}(a/B)$, is:

$$\mathsf{CH\text{-}rk}(a/B) := \min\{\mathsf{CH\text{-}rk}(D) : D \in \mathsf{tp}(a/B)\}.$$

Remark 1.10. Let $\mathcal{M}$ be an $\aleph_0$-categorical structure $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$ a finite subset. Then, there is a smallest B-definable subset of $\mathcal{M}$ containing a.
[By $\aleph_0$-categoricity of $\mathcal{M}$ and finiteness of B there are only finitely many B-definable subsets, up to equivalence. Finite lattices have minimal elements.]

Thus, the following is well-defined:

Definition 1.11. Let $\mathcal{M}$ be an $\aleph_0$-categorical structure $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$ a finite subset. The *locus of a over B* is the smallest B-definable subset containing a.

In particular:

Remark 1.12. Let $\mathcal{M}$ be $\aleph_0$-categorical, $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$ a finite set. Then $\mathsf{CH\text{-}rk}(a/B)$ is precisely the rank of the locus of a over B.

We may now translate the results of the previous subsection, in the context of ranks of elements:

Lemma 1.13 (Lemma 2.2.2(1') and (2")). *Let $a \in \mathcal{M}$ and $B, B_1, B_2 \subseteq \mathcal{M}$. Then:*

(1) $\mathsf{CH\text{-}rk}(a/B) = 0$ *if, and only if* $a \in \mathsf{acl}(B)$.
(2) *If* $B_1 \subseteq B_2$ *then* $\mathsf{CH\text{-}rk}(a/B_2) \leq \mathsf{CH\text{-}rk}(a/B_1)$.
[In the book, they say that this amounts to $D_1 \subseteq D_2 \implies \mathsf{CH\text{-}rk}(D_1) \leq \mathsf{CH\text{-}rk}(D_2)$. I think it's just from the definition. Say $\mathsf{CH\text{-}rk}(a/B_1) = \mathsf{CH\text{-}rk}(D)$ for some B_1-definable set D. Then D is also B_2-definable, so $\mathsf{CH\text{-}rk}(a/B_2) \leq \mathsf{CH\text{-}rk}(D) = \mathsf{CH\text{-}rk}(a/B_1)$.]

Lemma 1.14 (Extension Property, Lemma 2.2.2(2')). *Let D be a non-empty B-definable set. Then, there is a complete type over B containing D and having the same rank.*

Proof. The lemma follows easily from the fact that:

$$\mathsf{CH\text{-}rk}(A \cup B) = \max\{\mathsf{CH\text{-}rk}(A), \mathsf{CH\text{-}rk}(B)\}.$$

Let's actually go through the details:

Let D be B-definable. To fix notation, say D is given by $\phi(x, b_0)$ for some $\mathcal{L}$-formula $\phi(x, y)$ and some $b_0 \in B$. Now, let $\pi(x)$ be the following partial type:

$$\pi(x) = \{\psi(x, b) \in \mathcal{L}(B) : \mathsf{CH\text{-}rk}(\phi(x, b_0) \wedge \neg\psi(x, b)) < \mathsf{CH\text{-}rk}(\phi(x, b_0))\}.$$

Clearly $\phi(x, b_0) \in \pi(x)$. We claim that $\pi(x)$ is finitely consistent. Indeed, suppose that $\psi_1(x, b), \psi_2(x, b) \in \pi(x)$, and assume toward a contradiction that $\psi_1(x, b) \wedge \psi_2(x, b)$ is empty. By definition, then, $\mathsf{CH\text{-}rk}(\psi_1(x, b) \wedge \psi_2(x, b)) = 0$, but:

$$\mathsf{CH\text{-}rk}(\phi(x, b_0)) = \max \left\{ \begin{matrix} \mathsf{CH\text{-}rk}(\phi(x, b_0) \wedge \neg\psi_1(x, b)), \\ \mathsf{CH\text{-}rk}(\phi(x, b_0) \wedge \neg\psi_2(x, b)), \\ \mathsf{CH\text{-}rk}(\phi(x, b_0) \wedge \psi_1(x, b) \wedge \psi_2(x, b)) \end{matrix} \right\},$$

and by assumption, the RHS above is less than $\mathsf{CH\text{-}rk}(\phi(x,b_0))$, a contradiction. So, let $p(x)$ extend $\pi(x)$ to a complete type over B.

To finish the proof, suppose that for some formula $\psi(x,b) \in p(x)$ we have that $\mathsf{CH\text{-}rk}(\psi(x,b)) < \mathsf{CH\text{-}rk}(\phi(x,b_0))$. Then $\mathsf{CH\text{-}rk}(\psi(x,b) \wedge \phi(x,b_0)) < \mathsf{CH\text{-}rk}(\phi(x,b_0))$, so $\neg\psi(x,b) \in p(x)$, a contradiction. □

1.3. **The not so very basics.** The next lemma is a useful tool for computing ranks. We will repeatedly use it in the proof of *'additivity'* (Proposition 1.17):

Lemma 1.15 (Lemma 2.2.3)**.** *Let $\mathcal{M}$ be $\aleph_0$-categorical. Then, the following are equivalent:*

(1) $\mathsf{CH\text{-}rk}(a/b) \geq n+1$.
(2) There are a', c with $a' \in \mathsf{acl}(abc) \setminus \mathsf{acl}(bc)$, and $\mathsf{CH\text{-}rk}(a/a'bc) \geq n$

Remark 1.16. Observe, for instance, that by the lemma above, the following are equivalent:

(1) $\mathsf{CH\text{-}rk}(a) = 1$.
(2) There are a', c with $a' \in \mathsf{acl}(ac) \setminus \mathsf{acl}(c)$ such that $a \in D$, for some $a'c$-definable D, and for all a', c with $a' \in \mathsf{acl}(ac) \setminus \mathsf{acl}(c)$ we have $a \in \mathsf{acl}(a'c)$.

Proof. For the entirety of the proof, let D be the locus of a over b.
[The point being that if $a, \alpha \in D$ then $a \equiv_b \alpha$. Indeed, suppose toward a contradiction that $\alpha \in D$ and there is some b-definable set D' such that $\alpha \in D'$, but $a \notin D'$. Then, the locus of a should have been $D \cap (\neg D')$. Thus by the locus of a over b we essentially mean the formula isolating $\mathsf{tp}(a/b)$.]

(1) ⇒ (2) . By definition, $\mathsf{CH\text{-}rk}(a/B) = \mathsf{CH\text{-}rk}(D)$. Let D_1, D_2, π and f witness that $\mathsf{CH\text{-}rk}(D) \geq n+1$. Let $c \in \mathcal{M}$ be a finite tuple such that D_1, D_2, π and f are all c-definable. By $\aleph_0$-categoricity, it follows that $D_2 \setminus \mathsf{acl}(bc) \neq \emptyset$, since D_2 is infinite.
[The algebraic closure of finite sets in $\aleph_0$-categorical structures is finte.]
So, we may pick some $a' \in D_2 \setminus \mathsf{acl}(bc)$. We know that $\mathsf{CH\text{-}rk}(f^{-1}(a')) \geq n$, by assumption, and since $f^{-1}(a')$ is $a'bc$-definable, by the Extension Property (Lemma 1.14) and ω-saturation, we can find some $a_1 \in f^{-1}(a')$ such that $\mathsf{CH\text{-}rk}(a_1/a'bc) = \mathsf{CH\text{-}rk}(f^{-1}(a'))$. Now, to finish the proof, let $a_0 = \pi(a_1)$. It is easy to see that $a' \in \mathsf{acl}(a_0bc)$, since the fibre of π above a_0 is an algebraic set.

Claim 1.16.1. $\mathsf{CH\text{-}rk}(a_0/a'bc) \geq n$*.*

Proof of Claim. We know that $\mathsf{CH\text{-}rk}(a_1/a'bc) \geq n$. Let A_1 be the locus of a_1 over $a'bc$ and A_0 the locus of a_2 over $a'bc$. Clearly π is a finite-to-one map from A_1 to A_0, so the claim follows. ◀

Now, to finish the proof, since $a_0 \in D$, we have that $\mathsf{tp}(a_0/b) = \mathsf{tp}(a/b)$. By strong ω-homogeneity there is an automorphism σ taking a_0 to a. But then we are done, after replacing a' by $\sigma(a')$ and c by $\sigma(c)$.

(2) ⇒ (1) Suppose that there are a' and c as in (2). Let D_1 be the set:

$$\{(x,y) : \mathsf{tp}(xy/bc) = \mathsf{tp}(aa'/bc)\}$$

[This is definable, because $\mathcal{M}$ is $\aleph_0$-categorical, and types over finite sets are isolated.]
Let f be the projection of D_1 onto the second coordinate and take D_2 to be $f(D_1)$.

Claim 1.16.2. *Let π be the projection of D_1 onto the first coordinate. Then $\pi(D_1) = D$.*

Proof of Claim. Recall that D is the locus of a over b. Suppose first that $\alpha \in D$. In particular, this means that $\mathsf{tp}(a/b) = \mathsf{tp}(\alpha/b)$. By strong ω-homogeneity, there is an automorphism $\sigma \in \mathsf{Aut}(\mathcal{M}/bc)$ taking a to α. Taking $\alpha' = \sigma(a')$ gives us $\mathsf{tp}(\alpha\alpha') = \mathsf{tp}(aa'/bc)$, so $\alpha \in \pi(D_1)$. Conversely, suppose that $(\alpha, \alpha') \in D_1$, then, we claim that $\alpha \in D$. Since $\alpha \in D_1$, we have $a \equiv_b \alpha$, and thus $\alpha \in D_1$. ◀
[The point of the assumption $a' \in \mathsf{acl}(abc) \setminus \mathsf{acl}(bc)$ is that the set D_2 is infinite and that the fibre in D_1 above any $d \in D$ is finite.]

To finish the proof, it remains to show that for all $d \in D_2$ we have $\mathsf{CH\text{-}rk}\left(f^{-1}(d)\right) \geq n$.

Claim 1.16.3. $\mathsf{CH\text{-}rk}\left(f^{-1}(a')\right) \geq n$

Proof of Claim. By assumption, we have $\mathsf{CH\text{-}rk}(a/a'bc) \geq n$. So, by definition, the locus of a over $a'bc$ has rank at least n. ◀

Finally, observe that all fibres of f are conjugates under automorphisms, so we are done.

□

Proposition 1.17 (Lemma 2.2.4)**.** *Let $\mathcal{M}$ be $\aleph_0$-categorical. If* $\mathsf{CH\text{-}rk}(a/bc)$ *and* $\mathsf{CH\text{-}rk}(b/c)$ *are finite then:*

$$\mathsf{CH\text{-}rk}(ab/c) = \mathsf{CH\text{-}rk}(a/bc) + \mathsf{CH\text{-}rk}(b/c).$$

Proof. The proof is by induction on the $n = \mathsf{CH\text{-}rk}(a/bc) + \mathsf{CH\text{-}rk}(b/c)$. The base case is trivial. Indeed, if $\mathsf{CH\text{-}rk}(a/bc) + \mathsf{CH\text{-}rk}(b/c) = 0$, then $\mathsf{CH\text{-}rk}(a/bc) = 0$, so $a \in \mathsf{acl}(bc)$ and $\mathsf{CH\text{-}rk}(b/c) = 0$, so $b \in \mathsf{acl}(c)$. Combining the two conclusions, we obtain that $a \in \mathsf{acl}(b)$, so $a, b \in \mathsf{acl}(c)$ and thus $\mathsf{CH\text{-}rk}(ab/c) = 0$.

Now, for the inductive step. First, we shall show the inequality $\mathsf{CH\text{-}rk}(ab/c) \leq n$. To this end, pick some d such that $\mathsf{acl}(abcd) \setminus \mathsf{acl}(abc) \neq \emptyset$ and let $e \in \mathsf{acl}(abcd) \setminus \mathsf{acl}(cd)$. By Lemma 1.15, it suffices to show that $\mathsf{CH\text{-}rk}(ab/cde) < n$. Explicitly, our inductive hypothesis will give us the following:

(IH) If $\mathsf{CH\text{-}rk}(a/bcde) + \mathsf{CH\text{-}rk}(b/cde) < n$ then:

$$\mathsf{CH\text{-}rk}(ab/cde) = \mathsf{CH\text{-}rk}(a/bcde) + \mathsf{CH\text{-}rk}(b/cde).$$

At this point, there are two cases to consider:

- Case 1: If $e \notin \mathsf{acl}(bcd)$, then $e \in \mathsf{acl}(abcd) \setminus \mathsf{acl}(bcd)$. In this case, we claim that $\mathsf{CH\text{-}rk}(a/bcde) < \mathsf{CH\text{-}rk}(a/bc) \leq n$. Indeed, assume toward a contradiction that $\mathsf{CH\text{-}rk}(a/bcde) = \mathsf{CH\text{-}rk}(a/bc)$. Now, applying Lemma 1.15, we see that we have found d, e with $e \in \mathsf{acl}(abcd) \setminus \mathsf{acl}(bcd)$ and $\mathsf{CH\text{-}rk}(a/bcde) \geq n$. Thus $\mathsf{CH\text{-}rk}(a/bc) \geq \mathsf{CH\text{-}rk}(a/bc) + 1$, which is, of course, a contradiction. Thus $\mathsf{CH\text{-}rk}(a/bcde) < \mathsf{CH\text{-}rk}(a/bc)$. We always have that $\mathsf{CH\text{-}rk}(b/cde) \leq \mathsf{CH\text{-}rk}(b/cd)$, so

$$\mathsf{CH\text{-}rk}(a/bcde) + \mathsf{CH\text{-}rk}(b/cde) < \mathsf{CH\text{-}rk}(a/bc) + \mathsf{CH\text{-}rk}(b/c) = n,$$

  and, by (IH) we obtain:

$$\mathsf{CH\text{-}rk}(ab/cde) = \mathsf{CH\text{-}rk}(a/bcde) + \mathsf{CH\text{-}rk}(b/cde) < n,$$

  as required.
- Case 2: If $e \in \mathsf{acl}(bcd)$, then $e \in \mathsf{acl}(bcd) \setminus \mathsf{acl}(cd)$. In this case, we claim that $\mathsf{CH\text{-}rk}(b/cde) < \mathsf{CH\text{-}rk}(b/cd)$. The argument is similar. If not, then we have that $\mathsf{CH\text{-}rk}(b/cde) = \mathsf{CH\text{-}rk}(b/cd)$, and again, we have found d, e such that $e \in \mathsf{acl}(bcd) \setminus \mathsf{acl}(cd)$. So again by Lemma 1.15 we can conclude that $\mathsf{CH\text{-}rk}(b/cd) \geq \mathsf{CH\text{-}rk}(b/cd) + 1$, which is a contradiction. We again conclude by (IH) exactly as in the previous case.

It remains to show the inequality $\mathsf{CH\text{-}rk}(ab/c) \geq n$. Observe, first, that if $\mathsf{CH\text{-}rk}(b/c) = 0$ then:

$$\mathsf{CH\text{-}rk}(ab/c) \geq \mathsf{CH\text{-}rk}(a/c) \geq \mathsf{CH\text{-}rk}(a/bc) = n,$$

and we are done.
[The first inequality holds because the locus of ab over c certainly contains the locus of a over c.]
Thus, we may assume that $\mathsf{CH\text{-}rk}(b/c) > 0$. By Lemma 1.15 it suffices to find some b', d such that $b í n \mathsf{acl}(bcd) \setminus \mathsf{acl}(cd)$ and $\mathsf{CH\text{-}rk}(ab/b'cd) \geq n-1$, for then we have $\mathsf{CH\text{-}rk}(ab/c) \geq n$.

To this end, pick b', d such that $b' \in \mathsf{acl}(bcd) \setminus \mathsf{acl}(cd)$ and $\mathsf{CH\text{-}rk}(b/b'cd) = \mathsf{CH\text{-}rk}(b/c) - 1$.
[Such elements exist precisely by Lemma 1.15.]

To finish the proof, by Lemma 1.15 it suffices to show that $\mathsf{CH\text{-}rk}(ab/b'cd) \geq n-1$.

By the Extension Property (Lemma 1.14) we may assume that $\mathsf{CH\text{-}rk}(a/bb'cd) = \mathsf{CH\text{-}rk}(a/bc)$.
[The point is that the locus D of a over bc is, of course, definable over $\{b, b', c, d\}$, thus, by ω-saturation we can find an element $a_\star \in D$, such that $\mathsf{CH\text{-}rk}(a_\star/bb'cd) = \mathsf{CH\text{-}rk}(D) = \mathsf{CH\text{-}rk}(a/bc)$. By strong ω-homogeneity, we can find an automorphism σ which fixes b and c sending $a_\star$ to a. Replacing d and b', by their images under σ the claimed equality holds. Of course, since σ fixes b and c, the previous conditions on b' and d are still true.]
Thus:

$$\mathsf{CH\text{-}rk}(a/bb'cd) + \mathsf{CH\text{-}rk}(b/b'cd) = n - 1,$$

so, in particular, by (IH) we have that:

$$\mathsf{CH\text{-}rk}(ab/b'cd) = \mathsf{CH\text{-}rk}(a/bb'cd) + \mathsf{CH\text{-}rk}(b/b'cd) < n,$$

and the result follows. □

Corollary 1.18 (Corollary 2.2.5)**.** *If* $\mathsf{CH\text{-}rk}(D) = 1$, *then* acl *defines a pregeometry on* D.

Proof. For notational convenience, throughout this proof we will write $\mathsf{acl}(-)$ to mean $\mathsf{acl}(-) \cap D$. We must show that for all singletons $a, b \in D$ and finite tuples c from D we have that:

$$a \in \mathsf{acl}(bc) \setminus \mathsf{acl}(c) \implies b \in \mathsf{acl}(ac).$$

Let a, b and c satisfy the antecedent of the implication above. We know that

$$\mathsf{CH\text{-}rk}(ab/c) = \mathsf{CH\text{-}rk}(b/ac) + \mathsf{CH\text{-}rk}(a/c)$$

and we wish to show that $\mathsf{CH\text{-}rk}(b/ac) = 0$. Thus, it suffices to show that $\mathsf{CH\text{-}rk}(ab/c) = \mathsf{CH\text{-}rk}(a/c)$. Since $a \notin \mathsf{acl}(c)$ we must have that $\mathsf{CH\text{-}rk}(a/c) \geq 1$, and since $\mathsf{CH\text{-}rk}(a) = 1$, we must have that $\mathsf{CH\text{-}rk}(a/c) = 1$. Similarly, $\mathsf{CH\text{-}rk}(ab/c) = 1$, and we are done. □

Definition 1.19. We say that a and b are *independent over* C, written $a \mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}_C b$ if:

$$\mathsf{CH\text{-}rk}(a/bC) = \mathsf{CH\text{-}rk}(a/C) + \mathsf{CH\text{-}rk}(b/C).$$

Remark 1.20. Equivalently, $a \mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}_C b$ if, and only if, $\mathsf{CH\text{-}rk}(a/bC) = \mathsf{CH\text{-}rk}(a/C)$.

We collect here some of the properties of $\mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}$:

Lemma 1.21 (Lemma 2.2.7)**.** *The independence relation* $\mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}$ *satisfies the following properties:*

(1) Symmetry*:* $a \mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}_E b \iff b \mathop{\smile\!\!\!\!\!|}^{\mathsf{ch}}_E a$.

(2) Monotonicity: $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} bc \implies a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} b$.
(3) Base Monotonicity: *For* $E \subseteq F \subseteq G$ *we have* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} F \implies a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{B} F$.
(4) Transitivity:
(5) *If* $a \in \mathsf{acl}(bC)$ *then* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{C} b \iff a \in \mathsf{acl}(C)$.

Remark 1.22. The conjunction of properties (2) – (4) above are equivalent to the following equivalence:

$$a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} bc \iff a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{Ec} b \text{ and } a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} c$$

Proof.

(1) Symmetry: Suppose that $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} b$. Then, by definition, we have that:
$$\mathsf{CH\text{-}rk}(a/bC) = \mathsf{CH\text{-}rk}(a/C)$$
By Proposition 1.17 we know that:
$$\mathsf{CH\text{-}rk}(a/bC) + \mathsf{CH\text{-}rk}(b/C) = \mathsf{CH\text{-}rk}(b/aC) + \mathsf{CH\text{-}rk}(a/C).$$
and thus $\mathsf{CH\text{-}rk}(b/C) = \mathsf{CH\text{-}rk}(b/aC)$.

(2) Monotonicity, Base Monotonicity, and Transitivity: We show the equivalent equivalence. Suppose first that $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} bc$. By definition, we have that:
$$\mathsf{CH\text{-}rk}(a/bcE) = \mathsf{CH\text{-}rk}(a/E)$$
We must show that $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} cb$, i.e. that $\mathsf{CH\text{-}rk}(a/bcE) = \mathsf{CH\text{-}rk}(a/cE)$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{E} c$, i.e. that $\mathsf{CH\text{-}rk}(a/cE) = \mathsf{CH\text{-}rk}(a/E)$. Both of these follow immediately, since:
$$\mathsf{CH\text{-}rk}(a/bcE) \le \mathsf{CH\text{-}rk}(a/cE) \le \mathsf{CH\text{-}rk}(a/E),$$
and
$$\mathsf{CH\text{-}rk}(a/E) \ge \mathsf{CH\text{-}rk}(a/cE) \ge \mathsf{CH\text{-}rk}(a/bcE).$$
Conversely, by symmetry we have that $\mathsf{CH\text{-}rk}(b/acE) = \mathsf{CH\text{-}rk}(b/cE)$ and $\mathsf{CH\text{-}rk}(c/aE) = \mathsf{CH\text{-}rk}(c/E)$. Then:
$$\begin{aligned}\mathsf{CH\text{-}rk}(abc/E) &= \mathsf{CH\text{-}rk}(b/acE) + \mathsf{CH\text{-}rk}(c/aE) + \mathsf{CH\text{-}rk}(a/E)\\ &= \mathsf{CH\text{-}rk}(a/E) + \mathsf{CH\text{-}rk}(b/E) + \mathsf{CH\text{-}rk}(c/E),\end{aligned}$$
and we are done.

(3) Suppose that $a \in \mathsf{acl}(bC)$, so $\mathsf{CH\text{-}rk}(a/bC) = 0$. First, we show that if $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{C} b$ then $a \in \mathsf{acl}(C)$. Indeed, if $\mathsf{CH\text{-}rk}(a/bC) = \mathsf{CH\text{-}rk}(a/C)$, it follows that $\mathsf{CH\text{-}rk}(a/C) = 0$, so $a \in \mathsf{acl}(C)$. Conversely, suppose that $a \in \mathsf{acl}(C)$. Then $\mathsf{CH\text{-}rk}(a/C) = 0$, so $\mathsf{CH\text{-}rk}(a/C) = 0 = \mathsf{CH\text{-}rk}(a/bC)$ and thus $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathsf{ch}}_{C} b$.

□

1.4. **… And geometries.** Brief reminder (from Paolo's talk):

Definition 1.23 ((Weak) Linear Geometries)**.** A *weak linear geometry* is one of the following six (types) of structures.

(1) A Degenerate Space
(2) A Pure Vector Space
(3) A Polar Space
(4) An Inner Product Space
(5) An Orthogonal Space
(6) A Quadratic Space

A *linear geometry* is a weak linear geometry $\mathcal{M}$ expanded by a set of algebraic constants in $\mathcal{M}^{\mathsf{eq}}$, i.e. expanded by adding to the language a subset of $\mathsf{acl}^{\mathsf{eq}}(\emptyset)$.

These are the building blocks of the following kinds of geometries:

Definition 1.24.

(1) An *unoriented weak linear geometry* a weak linear geometry of type (1)-(5) or a reduct of a QUADRATIC SPACE in which we have forgotten the Witt defect function.
(2) A *basic linear geometry* is a linear geometry with the elements of K and L named by constants, and in the case of a POLAR SPACE, the two vector spaces V and W named by unary predicates.
(3) A *projective geometry* is a structure obtained by a linear geometry by factoring out the equivalence relation $\mathsf{acl}(x) = \mathsf{acl}(y)$.
(4) An *affine geometry* is a pair (J, A), consisting of a linear geometry J with underlying vector space V (one of the two vector spaces in the POLAR SPACE case), and a definable subset A on which V acts definably and regularly, where J carries its given structure and A.

Lemma 1.25 (Lemma 2.2.10)**.** *The linear, affine, and projective geometries are all of* pre-*rank 1.*

Proof. "Do them one-by-one, using QE, if you don't trust their proof." □

Definition 1.26. A structure $\mathcal{M}$ is *Lie coordinatised* if it admits a tree structure $<$ of finite height (where $<$ is an invariant partial order) with a unique 0-definable root such that:

(1) *Coordinatisation*: For all $a \in M$ above the root, one of the following holds:
 (A) a is algebraic over its $<$-predecessor.
 OR
 (B) There is $b < a$ and a b-definable projective geometry J_b, fully embedded in $\mathcal{M}$ such that either:
  (i) $a \in J_b$, or
  (ii) There is some $c \in M$ such that $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) such that $a \in A_c$ and the projectivisation of J_c is J_b.
(2) *Orientation*: If $a, b \in M$ have the same type over $\emptyset$ and are associated with coorinatising quadratic geometries J_a, J_b, then any definable map between them which preserves everything other than w also preserves w.

We say that $\mathcal{M}$ is *Lie coordinatisable* if it is interpretable with a structure $\mathcal{N}$ with finitely many 1-types over $\emptyset$ which is Lie coordinatised.

Corollary 1.27 (Corollary 2.2.11)**.** *If $\mathcal{M}$ is Lie coordinatisable then $\mathcal{M}$ has finite rank, at most the height of the coordinisation tree.*

Proof. [Idea:Every time we meet a geometry the rank goes up by 1. The algebraic steps don't increase the rank.] □

Corollary 1.28. *Let J be a linear, projective, or affine geometry and a, b are finite tuples from J. If $\mathsf{acl}(a) \cap \mathsf{acl}(b) = C$ then $a \mathop{\smile\!\!\!\!\!\!|}^{\mathsf{ch}}_{C} b$. So J is* one-based*.*

2. MORE CHERLIN-HRUSHOVSKI RANKS

Definition 2.1. Induction

- $\mathsf{CH\text{-}rk}_0(D) = \begin{cases} 0 & \text{if } D \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$
- Let $\alpha \in \mathbf{Ord}$, and assume that $\mathsf{CH\text{-}rk}_\beta(D)$ has been defined for all $\beta < \alpha$. Then:
 (1) $\mathsf{CH\text{-}rk}_\alpha(D) \geq 0$ if D is non-empty.

(2) $\mathsf{CH\text{-}rk}_\alpha(D) > 0$ if $\mathsf{CH\text{-}rk}_\beta(D) = \infty$ for all $\beta < \alpha$.
(3) For $n \in \mathbb{N}$, $\mathsf{CH\text{-}rk}_\alpha(D) \geq n+1$ if there exist definable sets D_1, D_2 and definable functions $\pi : D_1 \to D$ and $f : D_1 \to D_2$ such that:
  (a) For all $d \in D$ we have $\mathsf{CH\text{-}rk}_\alpha\left(\pi^{-1}(d)\right) = 0$.
  (b) $\mathsf{CH\text{-}rk}_\alpha(D_2) > 0$.
  (c) For all $d \in D_2$ we have $\mathsf{CH\text{-}rk}_\alpha(f^{-1}(d)) \geq n$.

TODO: Look at their pseudofinite example...

3. An alternative definition

In this section, we follow [Sim22, Section 2.4] rather closely. Diversions from that source will be indicated in red.

By a *uniformly definable family* $(X_t : t \in E)$ we mean that E is a definable set and there is a formula $\phi(x, y)$ such that, for each $t \in E$, the set X_t is precisely $\phi(x, t)$. Such a family is *weakly k-inconsistent*, for $k \in \mathbb{N}$, if whenever $X_{t_1}, \dots, X_{t_k}$ are pairwise distinct members of the family then $\bigcap_{i \leq k} X_{t_i} = \emptyset$.
[A uniformly definable family $(X_t : t \in E)$ is called *k-inconsistent*, for $k \in \mathbb{N}$ if whenever $t_1, \dots, t_k$ are pairwise distinct then $\bigcap_{i \leq k} X_{t_i} = \emptyset$.]

Definition 3.1 ([Sim22, Definition 2.2])**.** Let D be a definable subset of $\mathcal{M}$ (not $\mathcal{M}^{\mathsf{eq}}$). We define $\text{þ-}\mathsf{rk}(D) \geq n$, by induction on $n \in \mathbb{N}$:

- $\text{þ-}\mathsf{rk}(D) \geq 0$ if D is consistent.
- $\text{þ-}\mathsf{rk}(D) > 0$ if D is infinite.
- $\text{þ-}\mathsf{rk}(D) \geq n+1$ if there is a uniformly definable weakly k-inconsistent family $(X_t : t \in E)$ of subsets of D, containing infinitely many pairwise distinct sets, such that $\text{þ-}\mathsf{rk}(X_t) \geq n$ for all $t \in E$.

Theorem 3.2. *For any structure $\mathcal{M}$ and any definable set $D \subseteq M^k$ not $\mathcal{M}^{\mathsf{eq}}$ we have that:*

$$\text{þ-}\mathsf{rk}(D) = \mathsf{CH\text{-}rk}(D).$$

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