Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

EXERCISE 1. Show that a sequence of elements in $(\mathbb{Q}, <)$ is indiscernible if and only if it is either constant, strictly increasing or strictly decreasing.

Definition 1. Let $\mathcal{L}$ be a relational language. Let A, B be $\mathcal{L}$-structures. We write $\binom{B}{A}$ for the set of substructures of B isomorphic to A. Given structures S, M, L and $c < \omega$, we write

$$L \to (M)^S_c,$$

if for every colouring of $\chi : \binom{L}{S} \to c$ of the substructures of L isomorphic to S with c many colours, there is a substructure $M' \in \binom{L}{M}$ such that $\binom{M'}{S}$ is monochromatic.

We say that a hereditary class $\mathcal{C}$ of $\mathcal{L}$-structures is **Ramsey** if for every $S, M \in \mathcal{C}$ and for every $c < \omega$ there exists an $\mathcal{L} \in \mathcal{C}$ such that $L \to (M)^S_c$.

EXERCISE 2. Deduce that the class of finite structures with equality is Ramsey from the infinite Ramsey theorem. Is the class of finite linear orders Ramsey? Is the class of finite graphs Ramsey?

Definition 2. For a set A, a function $f : A^n \to A$ is **essentially unary** if there some $i \leq n$ and a unary function $g : A \to A$ such that for all $(a_1, \dots, a_n) \in A$, $f(a_1, \dots, a_n) = g(a_i)$.

$\star\star$ **EXERCISE 3.** Let $\nabla = \{(x, y) \in \mathbb{N}^2 | x < y\}$. Let $f : \mathbb{N}^2 \to \mathbb{N}$ and $X, Y \subseteq \mathbb{N}$ be infinite. Then there are infinite $X' \subseteq X, Y' \subseteq Y$ such that the restriction of f to $(X \times Y) \cap \nabla$ is of one of the following kinds: injective, or constant, or essentially unary with respect to a unary injection.

Definition 3. Let $\mathcal{L}$ be a language. A **Skolem theory**, Skolem$(\mathcal{L})$ is a theory in a language $\mathcal{L}_{\text{Sko}} \supseteq \mathcal{L}$ with the following properties:

- Skolem$(\mathcal{L})$ has quantifier elimination;
- Skolem$(\mathcal{L})$ is universal;
- every $\mathcal{L}$-structure can be expanded to a model of Skolem$(\mathcal{L})$;
- $|\mathcal{L}_{\text{Sko}}| \leq \max(|\mathcal{L}|, \aleph_0)$.

EXERCISE 4. Show that every language $\mathcal{L}$ has a Skolem theory.

EXERCISE 5. Let T be an $\mathcal{L}$-theory and Skolem$(\mathcal{L})$ a Skolem theory for $\mathcal{L}$. Show that all substructures in every model of $T \cup$ Skolem$(\mathcal{L})$ are elementary.

Definition 4. Let T be a Skolem theory in the language $\mathcal{L}^*$, $\mathcal{M} \models T$ and $X \subseteq M$. Let $\mathcal{H}(X)$ be the $\mathcal{L}^*$-substructure of $\mathcal{M}$ generated by X. By Exercise 5, $\mathcal{H}(X) \preceq \mathcal{M}$. We call $\mathcal{H}(X)$ the **Skolem hull** of X.

EXERCISE 6. Let T^* be a Skolem theory, $\mathcal{M} \models T^*$ and $A = (a_i | i \in I)$ be an indiscernible sequence in M, where I is an ordered set. Suppose that $\tau : A \to A$ is a permutation preserving the order given by I. Show that there is an automorphism $\sigma : \mathcal{H}(A) \to \mathcal{H}(A)$ extending τ.

$\star\star$ **EXERCISE 7.** Let κ be some cardinal number and consider $I := \kappa \times \mathbb{Q}$ with the lexicographical ordering. Show that I has 2^κ order-preserving permutations.

Let T be an $\mathcal{L}$-theory and $\kappa \geq \max(|\mathcal{L}|, \aleph_0)$. Deduce from Exercise 6 that there is a model $\mathcal{N} \models T$ with 2^κ automorphisms.