TODAY WE PROVE

LEMMA 2.3.19 Lie coordinatizable structures are ω-categorical.

M is coordinatised by Lie geometries if there is an 0-definable tree with domain M, finite height, unique 0-definable root s.t.

For each $a \in M$ above the root either

- a is algebraic over its immediate predecessor
- $\exists\, b < a$ and b-definable projective geometry J_b s.t. J_b is fully embedded in M and either

  ① $a \in J_b$; or

  ~~② $\exists\, c \in M$ $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) with vector part J_c s.t. $a \in A_c$ and the projectivization of J_c is J_b.~~

- ~~The ORIENTATION CONDITION for quadratic geometries~~ is respected.

for simplicity, we run our proof for structures with a coordinatization avoiding ②.
We may think of M as COORDINATIZED by PROJECTIVE GEOMETRIES.

M is **Lie coordinatizable** if it is bi-interpretable with a Lie coordinatized structure with finitely many 1-types over $\emptyset$.

Otherwise an infinite set with infinitely many constant would be counted as Lie coordin.

NOTATIONAL CORNER:

Say M is coordinatised by Lie geometries.

Say a is at height h and contained in J_b for $b < a$.
We say that J_b is at height h. (even if b is at height $< h$).

We write N_h for the 0-definable set given by the elements of M at height $\leq h$.

EXAMPLE (4.2.1 in Wolf 2020)

Let $(M; E_1, E_2)$ be the structure with two equivalence relations

where • E_1 has infinitely many infinite classes;

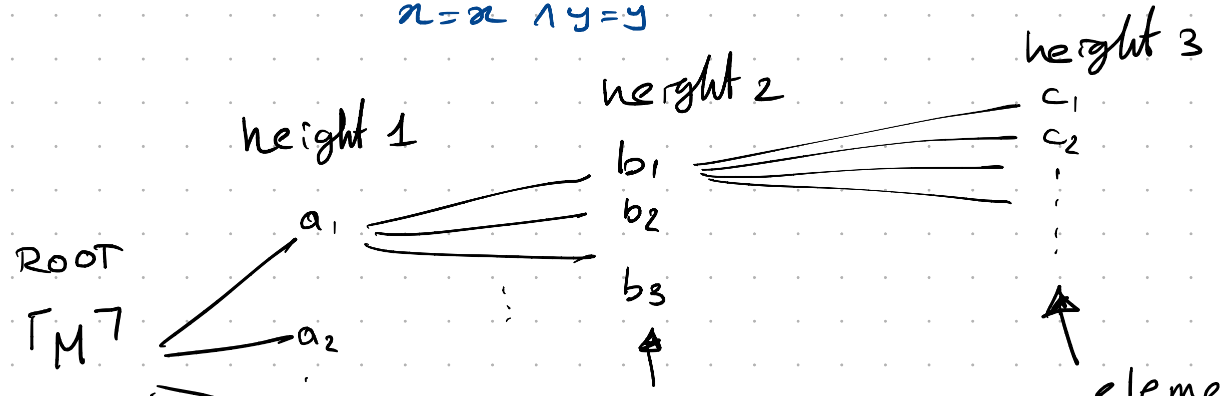
• E_2 refines each E_1 class into inf. many infinite classes.

look at $(M, \ulcorner M \urcorner, M/E_1, M/E_2; E_1, E_2) \subseteq M^{eq}$ (then we get bi-interpretation)

sort for $M/\approx$ where $x \approx y$ iff $x = x \wedge y = y$

a/E_1 lies in $\ulcorner M \urcorner$-def degenerate set M/E_1

a/E_2 at height 2 lies in a/E_1-definable degenerate set $(a/E_1)/E_2$ (Note M/E_2 is <u>NOT</u> fully embedded)



Reps of E_1-classes

$N_1 = (\ulcorner M \urcorner, M/E_1)$

representatives of E_2-classes refining the E_1 class represented by a_1

$N_2 = (\ulcorner M \urcorner, M/E_1, M/E_2)$

elements in M in the class b_1

$N_3(M, \ulcorner M \urcorner, M/E_1, M/E_2)$

Note M is Lie coordinatizable, but not coordinatized by Lie geometries.

J has GEI iff $\forall e \in J^{eq}$ $e \in \mathrm{acl}^{eq}(\mathrm{acl}^{eq}(e) \cap J)$.

well known: WEI $\Rightarrow$ GEI

GEI Lemma: Suppose J is a-definable and fully embedded in M. Suppose J has GEI. Let $j \in J$, $A \subseteq M$, $B := Aa$. Then

$$tp(j/B) \dashv\vdash tp^{J}(j/\mathrm{acl}^{eq}_{J}(\underbrace{\mathrm{acl}(B) \cap J}_{D})).$$

Proof: Let $X \subseteq J$ be B-definable. Let e be the canonical parameter for X in J^{eq}.

By fully embedded, every automorphism of M fixing X is also an automorphism of J fixing X. So $e \in \mathrm{acl}^{eq}(B)$ in M. By GEI,

$e \in \mathrm{acl}^{eq}_{J}(\mathrm{acl}^{eq}_{J}(e) \cap J) \subseteq \mathrm{acl}^{eq}_{J}(\mathrm{acl}^{eq}_{M}(B) \cap J) = \mathrm{acl}^{eq}_{J}(D)$.

So X is definable over $\mathrm{acl}^{eq}_{J}(D)$.

$j \not\equiv_B j' \Rightarrow$ there is B-definable X witnessing

X is also $\mathrm{acl}^{eq}_{J}(D)$-def in J $\Longrightarrow$ $j \not\equiv_{\mathrm{acl}^{eq}_{J}(D)} j'$ in J. ∎

LEMMA: PROJECTIVE GEOMETRIES have GEI.

PROOF STRUCTURE

CLAIM 1: $\forall h\ \forall n$, N_h realises finitely many n-types over $\emptyset$.

CLAIM 1 gives ω-categoricity once we get to $h = \text{height}(M, <)$.

Proof by INDUCTION on h:

BC1: ✓ **IH1**: $\forall n$ N_h realises fin many n-types over $\emptyset$

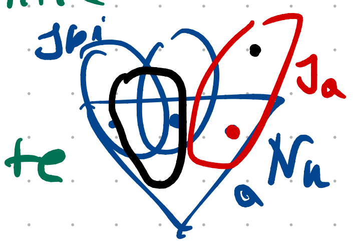
IS1: NTP N_{h+1} realises fin many n-types over $\emptyset$ for each n.

geometries at height $h+1$

Let $N = N_h \cup (J_{b_1} \cup \dots \cup J_{b_r})$. Let J be a-def at height $h+1$.

So $a \in N_h$.

CLAIM 2: J realises finitely many 1-types over finite subsets of $N \cup \{a\}$



CLAIM 3: For $A \subseteq N$ finite $\text{acl}(Aa) \cap J$ is finite

PROVE claim 3 ← use IH1

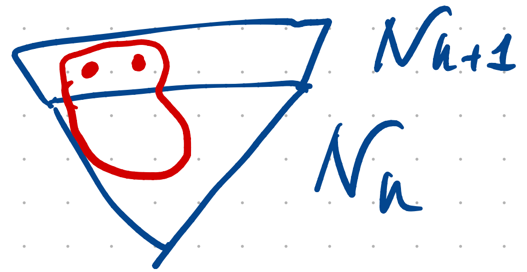
prove CLAIM 3 ⇒ CLAIM 2 ← USE GEI

prove IS1:

by induction on K = # elements in n-tuple outside of N_h.

BC2 = **IH1**.

IS2: Use CLAIM 2



For CLAIM 3 $\Rightarrow$ CLAIM 2:

A well known fact on ω-categorical structures:

LEMMA (EVANS - TSANKOV 2013) Let J be ω-categorical. Let $D \subseteq J$ be finite. Then there are finitely many 1-types over $\mathrm{acl}^{eq}(D)$.

Proof of CLAIM 3 $\Rightarrow$ CLAIM 2: Suppose for $A \subseteq N$ finite

$\underbrace{\mathrm{acl}(Aa) \cap J}_{D}$ is finite. By GEI LEMMA, for $j \in J$

$$tp_M(j / Aa) \dashv\vdash tp_J(j / \mathrm{acl}^{eq}_J(D)).$$

By LEMMA above there are only finitely many RHS-types.

So CLAIM 2 holds. $\blacksquare$

CLAIM 3: For $A \subseteq \mathcal{N}$ finite $\mathrm{acl}(Aa) \cap J$ is finite.

Proof: Let $C = Aa$.

Suppose by contradiction $\mathrm{acl}(C) \cap J$ is infinite.

Then, there are arbitrarily large finite dimensional C-definable subspaces V of J $\left(\{ j \in J \mid M \models \varphi_i(j, C) \} \text{ for } \varphi_i(x, c) \cap J(x) \text{ algebraic} \right)$.

Note that since $\mathcal{N}_h$ is 0-definable, $\mathcal{N}$ is B-definable where $B = \{b_1 \dots b_r\}$ $\mathcal{N} = \mathcal{N}_h \cup \bigcup_{i=1}^{r} J_{b_i}$.

CASE 1: $r = 0$. So $\mathcal{N} = \mathcal{N}_h$.

Fix $V = \varphi(J, C)$. Let

$n_1 = |\{ tp(c g C) \mid g \in Aut(M/a) \}|$

$n_1 \leq \underbrace{\# \text{ of } 2|C|\text{-types over } \emptyset}$

This is finite by IH1 (we are assuming that $\mathcal{N}_h$ satisfies only finitely many n-types for each n).

For fixed V, C-definable finite dimensional subset of J,

$$n_2 = |\{ tp(V\,gV) \mid g \in Aut(M/a)\}|$$

<u>CLAIM</u>: $n_2 \leq n_1$

Suppose $C\,gC \equiv C\,g'C$ for $g, g' \in Aut(M/a)$. Then, $gC \equiv_C g'C$

So $\exists\, \alpha \in Aut(M/C)$ $\alpha\, gC = g'C$. Since V is C-definable,

$\alpha(V\,gV) = V\,g'V \Rightarrow V\,gV \equiv V\,g'V$.

So $C\,gC \equiv C\,g'C \Rightarrow V\,gV \equiv V\,g'V$ and so $n_2 \leq n_1$ $\blacksquare$

Now, J is fully embedded over a in $\mathcal{M}$. So any automorphism $\sigma \in Aut(J)$ extends to an automorphism $g \in Aut(M/a)$. Moving V by automorphisms, we can choose $V \cap gV$ to take any dimension $\leq \dim V$.

In particular,

$$\dim V \leq n_2 \leq n_1.$$

But by assumption, we can take $\dim V$ to be arbitrarily large

$\times$ since n_1 is a fixed bound.

CASE 2: $r > 0$. So $N = N_h \cup \bigcup_{i=1}^{r} J_{b_i}$ is B-definable for $B = \{b_1 \dots b_r\}$

Since $B \subseteq N_h$, $\underbrace{\mathrm{acl}(Ba) \cap J}$ is finite by CASE1.

We call it V_0. We run essentially the same argument.

Suppose by contradiction $\mathrm{acl}(C) \cap J$ has finite dimensional subspaces of arbitrarily large dim. Take V to be such a space and

$$n_1 = |\{ tp(C \, g C) \mid g \in \mathrm{Aut}(M / Ba)\}|$$

$$n_2 = |\{ tp(V \, g V) \mid g \in \mathrm{Aut}(M / Ba)\}|$$

Again $n_2 \leq n_1$ for n_1 bounded.

Now, since $g \in \mathrm{Aut}(M/Ba)$, we only count automorphisms fixing V_0.

Let $V_0' = V \cap V_0$.

Then, $n_2 \geq \dim(V) - \dim(V_0') \geq \dim(V) - \dim(V_0)$

Again, this yields that n_2 is unbounded as we increase $\dim(V)$.

$\times\!\!\times$ So we have proven CLAIM 3

$\blacksquare$

PROOF of IS1: Recall, we want to prove that

N_{h+1} realises only finitely many n-types over $\emptyset$ for each n.

Proof by induction on $k = \#$ elements in n-tuple outside of N_h.

BC2: $k=0$ so we are counting the types over $\emptyset$ realised in N by n-tuples $(a_1 \dots a_n)$ s.t. $a_i \in N_h$ for each i (recall N_h is 0-def)

By IH1 there are finitely many such types

IH2: suppose claim holds up to $k-1$ for all n.

IS2: suppose we look at n-types of tuples $(c_1 \dots c_k\, a_1 \dots a_{n-k})$ for $c_1 \dots c_k$ from height $h+1$.

Let $J_{b_1} \dots J_{b_k}$ be the associated b_i-definable geometries.

(or b_i-definable algebraic closures).

CLAIM: Suppose that $b_1 \dots b_k\, a_1 \dots a_{n-k} \equiv b_1' \dots b_k'\, a_1' \dots a_{n-k}'$.

Then, the n-types $(d_1 \dots d_k\, a_1 \dots a_{n-k})$ for $d_i \in J_{b_i}$ are the same as the n-types $(d_1' \dots d_k'\, a_1' \dots a_{n-k}')$ for $d_i' \in J_{b_i'}$

(Note: $b_i \equiv b_i' \Rightarrow$ both define isomorphic geometries containing elements at height $h+1$)

This is just moving stuff by automorphisms ✓ ☒

By IH1 there are only finitely many n-types $(b_1 \ldots b_k a_1 \ldots a_{n-k})$

So, by CLAIM, we need to fix representatives for these n-types $(\bar{b}^s \bar{a}^s \mid s \leq S)$ and show that each gives rise to at most finitely many n-types of the form $(\underbrace{c_1 \ldots c_k}_{c_i \in J_{b_i}^s} a_1^s \ldots a_{n-k}^s)$.

By IH2: N_{h+1} realises only finitely many $(n-1)$-types $(c_2 \ldots c_k a_1 \ldots a_{n-k})$ with c_i at height $h+1$.

So, we only have finitely many types $(\underbrace{c_2 \ldots c_k}_{c_i \in J_{b_i}^s}, a_1^s \ldots a_{n-k}^s)$.

Finally, recall from CLAIM 2:

$J_{b_1}^s$ realises only finitely many 1-types over finite subsets of $N_h \cup \bigcup_{i=2}^{k} J_{b_i}^s \cup \{a\}$.

so only finitely many possible types of c_1 over $c_2 \ldots c_k a_1^s \ldots a_{n-k}^s$ and so only finitely many n-types of the form $(c_1 \ldots c_k a_1^s \ldots a_{n-k}^s)$ for $c_i \in J_{b_i}^s$.

So N_{h+1} realises only finitely many n-types over $\emptyset$ with k-many elts at height $h+1$.

This concludes IS2, and so proves IS1 and the Theorem. $\square$