

Homogeneity of μ -envelopes

10/5/24

Three definitions

- 1) Suppose (V, A) is an affine space (i.e. V is a vector space and $V \twoheadrightarrow A$ regularly), which is defined over B . We say A is free over B if there is no projective geometry J defined over B with $A \subseteq \text{acl}(B, J)$.
- 2) An element a of A , or $\text{tp}(a/B)$, is said to be affinely isolated over B if (V, A) is free over B .
- 3) Suppose A, A' are affine spaces defined over B . They are almost orthogonal if there is no pair a, a' with $a \in A, a' \in A'$, $\text{acl}(a, B) = \text{acl}(a', B)$.

Uniqueness of Parallel Lines (?)

Suppose (V, A) , (V', A') are almost orthogonal affine spaces defined and free over the algebraically closed set B , with PV and PV' complete I -types over B . Let J be a projective geometry defined over B , not of quadratic type, and stably embedded in M . For $a \in A$, $a' \in A'$, $c \in J \setminus B$, the triple (a, a', c) is algebraically independent.

Proof

A, A' almost orthogonal $\Rightarrow a, a'$ indep./ B

free over $B \Rightarrow a, c$ are indep./ B
+ a', c are indep./ B .

If two of A, A', J are orthogonal,

then $\Rightarrow (a, a', c)$ is indep./ B

so when all of A, A', J are

non-orthogonal. \Rightarrow can identify

PV with part of J .

Consider the structure $\underline{J \cup A}$.

A is definable over $J \cup \{a\}$

(and B) hence $J \cup A$

is stably embedded in M

(since J is).

Suppose towards contradiction
that $\text{rk}(aa'c/B) = 2$. Take $\overline{a_1, c_1}$
independent copy of a, c over Ba' .

Note that $2 = \text{rk}(a^c/B) > \text{rk}(a^c/Ba') = 1$

so if $\text{rk}(\overline{a^c a_1, c_1}/B) = 4$ then we have

$\text{rk}(a^c/B_{a_1, c_1}) = 2$ but $a' \in \text{acl}(B_{a_1, c_1})$

So we get

$$\text{rk}(a^c / B_{a,c}, a') = 2 \text{ hence}$$

$$\text{rk}(a^c a, c, a' / B) = 5, \text{ a contradiction.}$$

Thus, we get

$$\text{rk}(\underline{a^c a, c} / B) = 3. \leftarrow$$

But this is now a computation only about JVA , which is stably embedded and modular. Thus, there is some $d \in \underline{(JVA) - B}$ such that $d \in \text{acl}(a^c B) \cap \text{acl}(a, c, B)$, hence $d \in \text{acl}(a', B)$. Thus,

$$\text{acl}(d, B) = \text{acl}(a', B).$$

If $d \in A$, contradicts A, A' almost orthogonal.

If $d \in J$, contradicts free over B .

□

Lemma Let M be Lie coordinatized.

Let A be an affine space ^{defined and free} over $B = \text{acl}(B)$.

Suppose $B \subseteq B' = \text{acl}(B)$, B' finite,

and J a canonical projective geometry associated with A . Assume

1. $J \cap B' \subseteq B$.

2. $J \cap B$ is nondegenerate (if there's a form around)

3. If J is a quadratic space, then the Q -set of J meets B .

Then A either meets B' or is free over B' .

Proof " A need not remain a geometry over B' but will split into a finite number of affine pregeometries over B' . (?)

The proof is by induction on the coordinatization tree. Suppose $\underline{B'} = \text{acl}(B, a')$, where a' is in a B -definable affine, projective, or quadratic space; call it A' .

Assume A does not meet B' , but some "affine part" $\underline{A_0 \subseteq A}$ relative to B' is contained in $\text{acl}(\underline{B}, \underline{a'}, \underline{J_{b'}})$ where $J_{b'}$ is a B' -definable projective space. But

$$\underline{J_{b'}} \subseteq \text{acl}(B, J, b') \text{ so}$$

$$A_0 \subseteq \text{acl}(\underline{B}, \underline{a'}, \underline{J})$$

$$\text{and } A \cap \text{acl}(J, B) = \emptyset.$$

Hence A, A' are non-orthogonal, and

$$A' \cap \text{acl}(J, B) \neq \emptyset. \text{ Hence, by (3),}$$

A' is a line and free over B .

Now if A, A' are not almost orthogonal over B , then B' meets A (recall $B' = \text{acl}(B')$).

So A, A' are almost orthogonal over B .

Now we are in the situation of the Uniqueness of Parallel Lines Lemma.

Pick $a \in A$. We have $a \in \text{acl}(B, a', J)$ and (J, A) is modular.
 $\Rightarrow \exists c \in J \cap \text{acl}(B, a')$ with
 $a \in \text{acl}(B, a', c)$. (?)

Then $c \notin B$ and c is not in the quadratic part of J (if it exists).

Now we localize J at B to get

$$J_B \quad (\text{i.e. } J_B = (J \cap B)^\perp / \text{rad}(J \cap B)^\perp)$$

By (3), this is not a quadratic geometry.

By (2), $J \subseteq \text{rad}(B \cup J_B)$.

"normally over B , J would break up into a number of pregeometries, at least one $((J \cap B)^\perp)$ sitting over the localization, while some of the cosets would be affine pregeometries. However, since $J \cap B$ is nondegenerate, all elements of J lie in translations by elements of B of $(J \cap B)^\perp$."

Replacing c by an element of J_B
with the same algebraic closure as c

$\Rightarrow a, a', c$ are algebraically indep, or

contradiction.



by uniqueness,

of parallel
lines.



Main Theorem (Lemma 3.2.4)

Let M be an adequate regular expansion of a LC'd structure, μ a dimension function, and let E, E' be μ -envelopes. If $f: A \xrightarrow{\sim} A'$ is partial elementary with $A \subseteq E, A' \subseteq E'$, then f extends to an elementary map $\tilde{f}: E \rightarrow E'$. In particular, μ -envelopes are unique and homogeneous.

Proof

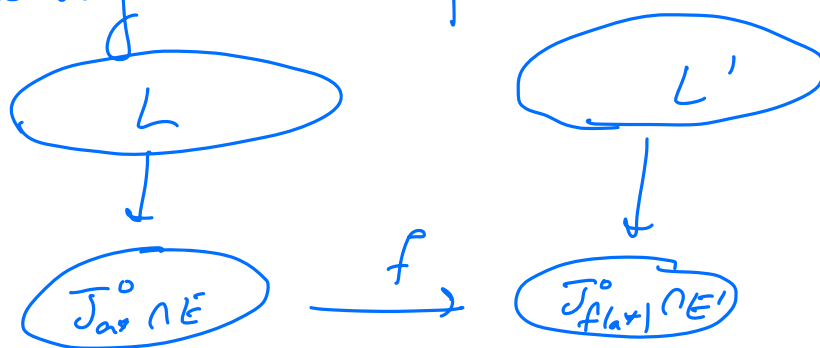
- Reduce to E, E' finite by back-and-forth.
- Reduce to $A = \text{acl}(A)$ and to extending f to $\text{acl}(A \cup \{b\})$ for some $b \in E - A$.

Divide into two cases:

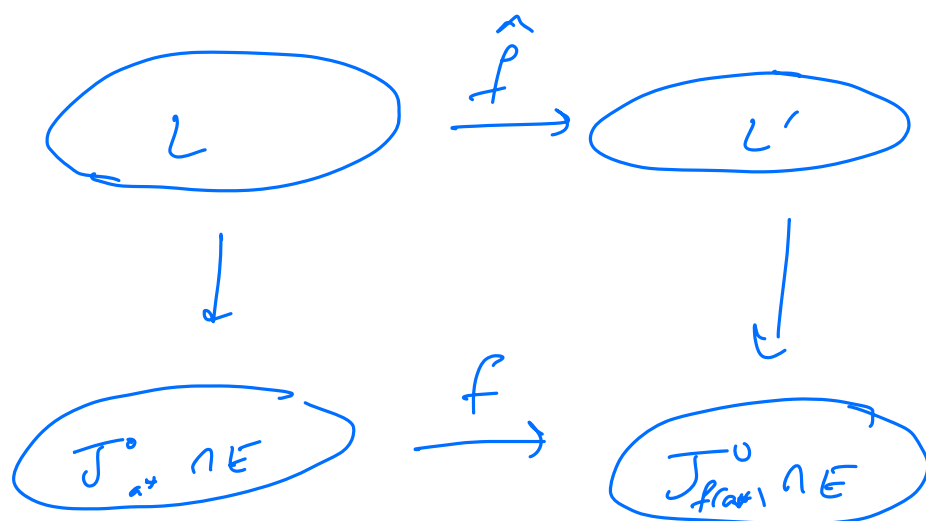
1. There is a standard system of geometries J and some $a \in A$ such that $J_a \cap E$ is not contained in A .

Expand $J_a \rightsquigarrow J_{a^*}^0$ basic projective defined over $a^* = \text{acl}(a)$.

finite dimensional
Take corresponding linear spaces L, L'



Lift to \hat{f} , some isomorphism $L \rightarrow L'$



and it can be arranged that this square commutes by Witt's lemma.

$A = \text{acl}(A) + \text{weak EI} + \text{stable embedding}$

$$\Rightarrow \text{tp}(L/L \cap A) \vdash \text{tp}(A/L).$$

$$\text{tp}(L'/L \cap A) \vdash \text{tp}(A'/L')$$

$\Rightarrow f$ is partial elementary.

Case 2 is Not Case 1. And Not Today...