β SYMPLECTIC non-deg. + skew-symmetric + sesquilinear

In $char(K) = 2$ $\beta(x,x) = 0$

β UNITARY non-deg + hermitian symm (wrt σ) + sesquilinear.

INNER PRODUCT SPACE (V, K, L, β)

$\beta : V \times V \longrightarrow L$ — 1-dim K-space

β is sesq. wrt σ

- $\sigma = 1$ (so β is BIL) β symplectic. $\sim Sp(w, q)$
- σ non-trivial β unitary. $\sim GU(w, q)$

technical corner: We want to view structures where forms are scalar multiples of each other as isomorphic. (V, K, L, β) $(V, K, L, \alpha\cdot\beta)$

$v \in L \xmapsto{f} \alpha \cdot v$ $f(\beta(u,v)) = \alpha\,\beta(u,v)$

$f : V \longrightarrow V$ $v \longmapsto \sqrt{\alpha}\cdot v$

β is ORTHOGONAL symm + non-deg + bilinear

In char 2 odd $\beta(x,x) = 0$

For β orth. we say that $q : V \to K$ is an associated

QUADRATIC FORM

$$① \quad q(\lambda x) = \lambda^2 q(x)$$

$$② \quad q(x+y) = q(x) + q(y) + \beta(x,y)$$

If $\mathrm{char}(K) \neq 2$, we can recover $q(x) = \frac{1}{2}\beta(x,x)$ ⊛

$\beta(x,x) = q(2x) - q(x) - q(x) = 4q(x) - 2q(x) = 2q(x)$

we can always recover β from q by ②

ORTHOGONAL SPACE (V, K, L, q) $\quad q : V \to L$

q is a quadratic form whose associated β is orthogonal.

$\sim$ infinite dim version of orthogonal groups $O_w(q)$

What about $\mathrm{char}(K) = 2$?

there are multiple quadratic forms compatible with orth. β.

Q' the set of quadratic forms assoc. with β.

$q_1, q_2 \in Q'$ $\quad q_1 - q_2$ is the square of a linear functional

$\sqrt{q_1(x) + q_2(x)}$ $\quad \sqrt{a} + \sqrt{b} = \sqrt{a+b}$

is a linear functional.

V^* acts regularly on Q' $\quad (f, q) \longmapsto q + f^2$

$g : V \longrightarrow V^*$ $\quad v \longmapsto \underbrace{\beta(v, \cdot)}$ λ_v

identifies V with a subspace of V^*.

there is a free (but not nec. transitive) action of V on Q'.

Q is an orbit of this action of V on Q'.

$\beta_Q : Q \times V \longrightarrow K \qquad \beta_Q(q, v) = q(v)$

$+_Q : V \times Q \longrightarrow Q$ is reg. action of V on Q

$$v +_Q q = q + \lambda_v^2$$

$-_Q : Q \times Q \longrightarrow V \qquad (q_1, q_2) \longmapsto v_R$ where $q_1 - q_2 = \lambda_v^2$

two formed spaces are equivalent when there is an isomorphism between them.

In the even dim case we have up to equivalence two quadratic forms induced by β.

$w : Q \longrightarrow \{0, 1\} \subseteq K$

with defect — in dim $2n$ case

the maximal $\underbrace{q\text{-isotropic}}_{W \text{ s.t. } q|_W = 0}$ subspace of V. dim n / dim $n-1$

with defect $n - \dim(W)$

$\sim O_{2n}^+(q) \qquad O_{2n}^-(q)$

In the infinite dimensional case, all $q \in Q$ induced by β are equivalent.

$q_1, q_2 \in Q$, $\sqrt{q_1 + q_2} = \lambda_v$ for $v \in V$

identify v with λ_v $q(\underbrace{\sqrt{q_1+q_2}}_{\in V}) \in K$

- $q_1(\sqrt{q_1+q_2}) = q_2(\sqrt{q_1+q_2})$

- write $[q_1, q_2] = q_1(\sqrt{q_1+q_2})$

Let $q_1, q_2, q_3 \in Q$ $v = \sqrt{q_1+q_2}$ $w = \sqrt{q_1+q_3}$ $\alpha = \beta(v,w)$

$$[q_1, q_2] + [q_1, q_3] + [q_2, q_3] = \tau(\alpha)$$

Artin-Schreier poly $\tau(x) = x^2 + x$.

$$[q_1, q_2] \in \tau(K)$$

- this is an equivalence rel.

$$\tau(\alpha) + \tau(\beta) + [q_1, q_3] = \tau(\gamma) \qquad [q_1, q_3] = \alpha^2 + \beta^2 + \gamma^2 + \alpha + \beta + \gamma$$

$$= (\alpha+\beta+\gamma)^2 + \alpha + \beta + \gamma$$

- this equivalence relation has two classes.

$W: Q \to \{0,1\} \subseteq K$ name each class.

QUADRATIC GEOMETRY

$$(V, Q, K, \beta_V, +_Q, -_Q, \beta_Q, \omega)$$

only one isom type

POLAR SPACE

$$(\underbrace{V \cup W}_{\text{one equiv. rel. with two classes.}}, K, L, \beta)$$

a non-degenerate bilinear pairing $\beta : V \times W \longrightarrow L$

$$\beta : (V \times W) \cup (W \times V) \longrightarrow L \quad \text{symmetric}$$

Notation corner: $W = V^*$ and $V = W^*$.

In finite dim $\beta : V \times V^* \longrightarrow L \quad \beta(v, f) = f(v)$.

one can give V the weak topology induced by W.
weakest TVS topology $\beta(\cdot, w) \longrightarrow$ is continuous.
and vice-versa.

CONTINUOUS DUAL V' of V (TVS) is the space of all cts lin fns $\phi : V \to K$

A WEAK LINEAR GEOMETRY is a structure of one of the above six types.

A LINEAR GEOMETRY is an expansion of a weak one by a set of $\underbrace{\text{algebraic constants in } M^{eq}}_{\in\ \mathrm{acl}^{eq}(\emptyset)\ ?}$.

UNORIENTED WLG any of above but forget ω in the quadratic case.

A BASIC LG is a LG in which elts of K and L have been named and in the polar case we name the two spaces.

PROJECTIVE GEOMETRY from a LG factoring out eq rel

$ord(x) = ord(y)$.

SEMIPROJECTIVE GEOMETRY from BLG by factoring out the rel. $x^Z = y^Z$ for Z the center of the aut group (the set of scalar transf respecting any additional structure).

- In symplectic case $\beta(\lambda u, \lambda v) = \beta(u,v)$

$$\lambda^2 \beta(u,v) \quad \text{so} \quad \lambda = \pm 1$$

V acting regularly and definably on def A & AFFINE V-space.

J a L G. V (one of V or W in polar)

AFFINE GEOMETRY (J, A) is a structure where J carries its given str. and A carries the action of V but has no further struc.

Ex: V act regularly on itself $\oplus : V \times A^{\not\,V} \longrightarrow A$

$(v, a) \longmapsto v + a$

$(V, K, L, \beta; A)$

for $a_1, a_2 \in A$ $\exists!\, v \in V$ s.t. $v \oplus a_1 = a_2$.

def function $\ominus : A \times A \longrightarrow V$

$(a_1, a_2) \longmapsto v$

$\beta(x, a_1 \ominus a_2)$ for a_1 a_2 affine...

$\boxtimes$