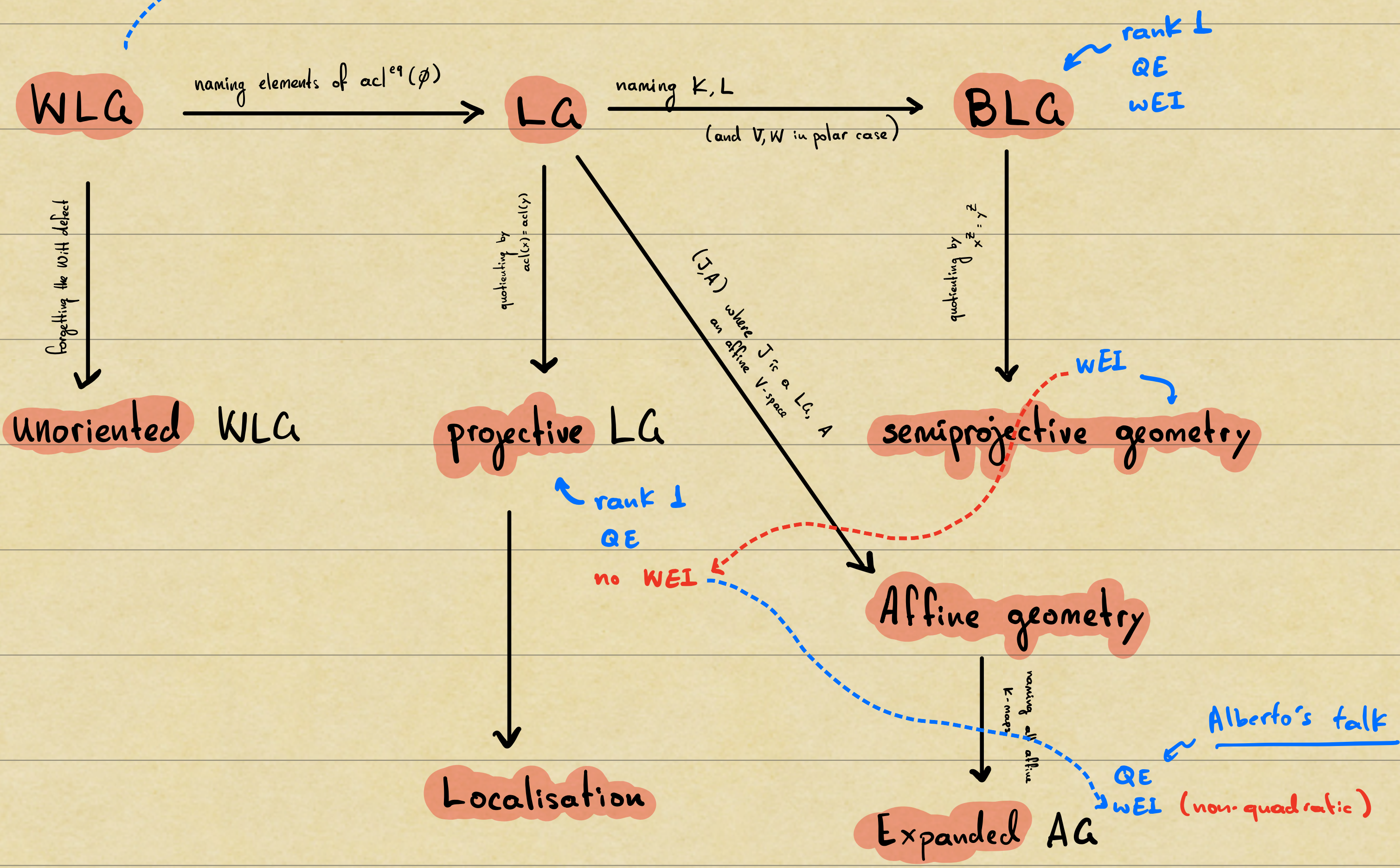
Previously on... Previously on

Finite Structures with Few Types

Types of WEAK LINEAR GEOMETRIES:

(i) degenerate space
(ii) pure vector space
(iii) inner product space
(iv) orthogonal space
(v) polar space
(vi) quadratic geometry



2.1.3. Coordinatisation

Def. Let $\mathcal{M} \subseteq \mathcal{N}$ be structures. Suppose that $\mathcal{M}$ is definable in $\mathcal{N}$ with canonical parameter $a \in \mathcal{N}^{eq}$ and let $A \subseteq N$. We say that:

(i) $\mathcal{M}$ is stably embedded in $\mathcal{N}$ over A if every N-definable subset of M is Aa-definable (def. in the sense of $\mathcal{N}$). uniformly

(ii) $\mathcal{M}$ is canonically embedded in $\mathcal{N}$ over A if every aA-def subset of $\mathcal{M}$ (in the sense of $\mathcal{N}$) is $\emptyset$-def. in the sense of $\mathcal{M}$.

(iii) $\mathcal{M}$ is fully embedded in $\mathcal{N}$ over A if its both stably and canonically embedded in $\mathcal{N}$ over A.

Def. A structure $\mathcal{M}$ is coordinatised by Lie geometries (Lie coordinatised) if it has a $\emptyset$-definable tree structure $<$ of finite height of finite height with a unique $\emptyset$-definable root such that:

1. Coordinatisation: For all $a \in \mathcal{M}$ above the root one of the following holds:

   (0) a is algebraic over its $<$-predecessor.

   OR there is $b < a$ and a b-definable projective geometry J_b fully embedded in $\mathcal{M}$ such that either:

$\oplus$ new $\mathbb{Z}/p^2\mathbb{Z}$ David's talk

(i) $a \in J_b$; or

(ii) there is $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) such that $a \in A_c$ and the projectivisation of J_c is J_b.

2. <u>Orientation</u>: If $a, b \in \mathcal{M}$ have the same type over $\emptyset$ and are associated with coordinatising geometries J_a and J_b then any definable map between J_a and J_b which preserves everything but ω also preserves ω.

If we ignore condition 2 then we say $\mathcal{M}$ is weakly Lie coordinatised.

Back to <u>RANKS</u>

if it has finitely many 1-types & is bi-interpretable with a Lie coordinatised structure.

at most the height of the tree.

<u>Theorem.</u> If $\mathcal{M}$ is Lie coordinatisable then it has finite rank.

Perhaps on Lie coordinatized structures: I remember the definition being very hard to parse. In his talk on 2.1.3, David Evans had a really interesting non-trivial example of a Lie coordinatized structure towards the end.

In my talk on 2.3.19 on the omega-categoricity of Lie coordinatized structures I gave the example of a finitely refining equivalence relation.

Daniel Wolf's paper should also go through some things more formally than the definitions in the book.

Back to <u>IMAGINARY ELEMENTS</u>

<u>Lemma.</u> The following are equivalent for a $\emptyset$-definable $D \subseteq \mathcal{M}$:

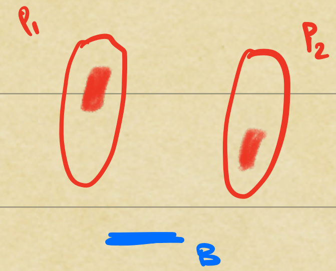
1. D is stably embedded in $\mathcal{M}$ and admits weak EI. $a \in \mathcal{M}^{eq}$ $\quad dcl^{eq}(acl^{eq}(a) \cap M)$

2. For any $a \in \mathcal{M}^{eq}$, $tp(a / acl^{eq}(a) \cap D) \vdash tp(a/D)$.

Using this lemma together with weak EI for basic linear geometries and expanded affine geometries we may prove:

<u>Theorem.</u> If $\mathcal{M}$ is Lie coordinatisable then it is $\aleph_0$-categorical.

2.4. ORTHOGONALITY

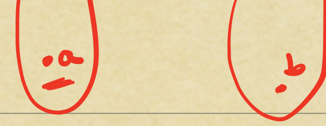


<u>Def.</u> Let T be a first-order L-theory with distinguished sorts P_1 and P_2. Let $\mathcal{M} \models T$ and $B \subseteq M$ a finite set. We say that P_1 and P_2 are orthogonal over B if every $L(B)$-definable subset of $P_1^{n_1} \times P_2^{n_2}$ is a finite union of $\operatorname{acl}(B)$-definable rectangles $R_1 \times R_2$ with $R_i \subseteq P_i^{n_i}$ $(i=1,2)$. We denote this by

$$P_1 \underset{B}{\perp} P_2 .$$

If, on the other hand there is a B-definable bijection between P_1 and P_2 then we say that P_1 and P_2 are B-linked.

When $B = \emptyset$ we don't even mention it.



Remarks. Clearly, if $P_1 \underset{B}{\perp} P_2$ (and P_i are infinite) then they are not B-linked.

(i) $\boxed{P_1 \underset{B}{\perp} P_2}$ if, and only if, $\forall a \in P_1^n \ \forall b \in P_2^m \ tp(a/acl(B)) \vdash tp(a/acl(B)b)$.

(ii) Suppose P_1, P_2 are orthogonal, $\emptyset$-definable and stably embedded in $\mathcal{M}$. Then:

$$P_1 \cup P_2 \text{ is stably embedded in } \mathcal{M} \text{ iff } \forall B \subseteq_{fin} \mathcal{M} \ \ P_1 \underset{B}{\perp} P_2$$

2.4.8

Theorem. Let P, Q be orthogonal (A) $\emptyset$-definable (A) subsets each stably embedded in a (A) structure $\mathcal{M}$. Then for any $B \subseteq M$, $P \underset{AB}{\perp} Q$ (i.e. $P \cup Q$ is s.e.). More generally if:

(i) I, D are stably embedded

(ii) $\{J_a : a \in I\}$ is a uniformly def. family of stably embedded subsets

(iii) $D \underset{a}{\perp} J_a$ and $D \perp I$

Then:

(a) $\mathcal{J} := I \cup \left(\bigcup_{a \in I} J_a \right)$ is stably embedded.

(b) $D \perp \mathcal{J}$.

2.4.8

Def. A normal geometry is a structure $\mathcal{J}$ with the following properties: (uniformly in all elementary ext.)

(i) $acl(a) = \{a\}$ for all $a \in \mathcal{J}$

(ii) **Exchange**: If $a \in \operatorname{acl}(Bc) \setminus \operatorname{acl}(B)$ then $c \in \operatorname{acl}(Ba)$.

(iii) **Geometric elimination of imaginaries**: For all $a \in J^{eq}$ we have that

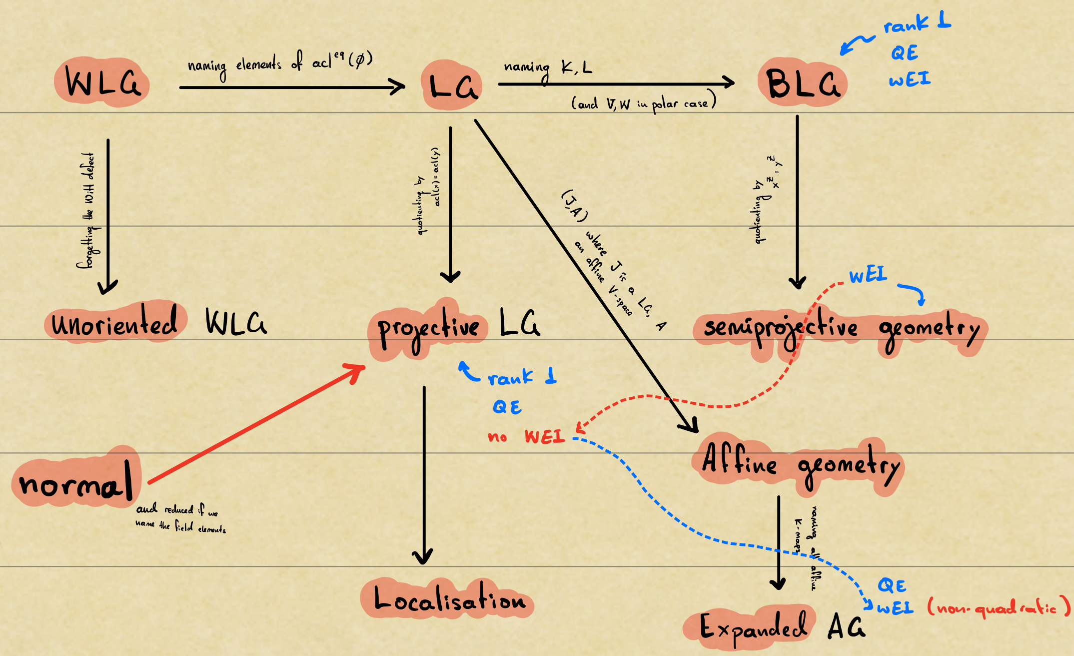
$$a \in \operatorname{acl}^{eq}(\operatorname{acl}^{eq}(a) \cap J).$$

(iv) For any $\emptyset$-definable non-empty $J_0 \subseteq J$: if $a \equiv_{J_0} b$ then $a = b$.

It is called reduced if in addition:

(v) $\operatorname{acl}^{eq}(\emptyset) = \operatorname{dcl}^{eq}(\emptyset)$.

Theorem. Projective geometries are normal. Basic projective geometries are normal and reduced.



Why do we care?

Theorem. Suppose that P_1 and P_2 are $\emptyset$-definable fully embedded normal geometries in some structure $\mathcal{M}$. Then, one of the following holds:

(i) $P_1 \perp P_2$

(ii) P_1 and P_2 are linked.

$\rightsquigarrow$

Theorem. Suppose that P_1 and P_2 are normal geometries which are a_1- and a_2- definable in a structure $\mathcal{M}$. Let $B = \{a_1, a_2\}$ and suppose that P_1 and P_2 are fully embedded over B. Then one of the following holds:

(i) $P_1 \underset{B}{\perp} P_2$

(ii) P_1 and P_2 are B-linked.

this lemma also makes sense over parameters

Lemma. Let J_1 and J_2 be basic linear geometries canonically embedded in a structure $\mathcal{M}$.

Suppose that in $\mathcal{M}$ there is a $\emptyset$-definable bijection

$$f : PJ_1 \to PJ_2 \quad \text{(i.e. they are linked)}$$

between their projectivisations.

Then, there is a $\emptyset$-definable bijection

$$\hat{f} : J_1 \longrightarrow J_2$$

which induces f and such that $\hat{f}$ is an isomorphism of weak geometries.

__Lemma.__ Let $\mathcal{M}$ be a structure and P, Q basic projective geometries defined and fully embedded over $A \subseteq M$. Suppose that $P \perp_A Q$. Let $\hat{P}, \hat{Q}$ be localisations of P and Q respectively both definable over some set B. Then $\hat{P} \perp_B \hat{Q}$.

2.5. Canonical Projective Geometries

6. Canonical projective geometries

One of the less attractive properties of a geometry carrying an inner product is that there are definable subspaces of arbitrarily large finite codimension, and thus a number of closely related geometries which can be associated with the original one. The following notion distinguishes the "master" geometry from its "offspring",

Throughout this section (and, in fact, for the rest of the talk) we work inside a Lie coordinatised structure $\mathcal{M}$.

Def. Let J_b be a b-definable weak projective geometry in some structure $\mathcal{M}$. We call J_b a canonical projective geometry (in $\mathcal{M}$), if:

canonical parameter

(i) J_b is fully embedded over b in $\mathcal{M}$.

(ii) Whenever $b' \equiv b$ we have that $J_b \perp J_{b'}$

Proposition (Existence). Let P_b be a b-definable projective geometry fully embedded in $\mathcal{M}$. Then, there is a canonical projective geometry $J \in \mathcal{M}^{eq}$ such that

$$J \not\perp_A P_b$$

for some finite set $A \subseteq M$.

b-def. b'-def { with b, b' the canonical parameters

Lemma (Parameters). Let P_b and $P_{b'}$ be canonical projective geometries in $\mathcal{M}$, not necessarily conjugate. If $P_b \not\perp P_{b'}$ then:

(i) $\mathrm{dcl}(b) = \mathrm{dcl}(b')$

(ii) there is a unique bb'-definable ~~isomorphism~~ bijection between them, which is an isomorphism of weak (unoriented) geometries. Moreover, we can assign compatible orientations to the quadratic geometries of $\mathcal{M}$.

3.1. Envelopes

Still $\mathcal{M}$ is Lie coordinatised.

Def. A standard system of geometries is a $\emptyset$-definable function

$$J : A \longrightarrow \mathcal{M}^{eq}$$

where A is the locus of a complete type (over $\emptyset$) and for all $a \in A$, $J_a := J(a)$ is a canonical projective geometry.

Let $J : A \longrightarrow \mathcal{M}^{eq}$, $J' : A' \longrightarrow \mathcal{M}^{eq}$ be standard systems of geometries. If for some $a \in A$, $a' \in A'$ we have $J_a \not\perp J_{a'}$ then there is a 1-to-1 identification between J and J' (using linkage & the fact that A, A' are loci of complete types).

Def. In the situation above, we say that J and J' are equivalent.

Def. We say that $\mathcal{M}'$ is a regular expansion of $\mathcal{M}$ if it is obtained by adjoining to $\mathcal{M}$ finitely many sorts of $\mathcal{M}^{eq}$ (with their induced structure).

Remark. A regular expansion of a Lie coordinatised structure is Lie coordinatisable.

Def. A regular expansion is called adequate if it contains a copy of each canonical projective geometry which is non-orthogonal to a coordinatising geometry.

From now on we shall assume that $\mathcal{M}$ is an adequate regular expansion of a Lie coordinatised structure.

(i) - (vi)

Def. An approximation to a geometry of a given type is a geometry of the same type of finite or countable dimension.

Def. A dimension function is a function μ defined on equivalence classes of standard systems of geometries with values isomorphism types of approximations to projective geometries of the given type.

... What?

really any $[J]_\sim$

Any $\underline{J} = (J_a : a \in A)$ picks out a type of geometry and $\mu(J)$ is an approximation to said geometry.

Def. Let μ be a dimension function. A μ-envelope is a subset E such that:

(i) E is algebraically closed in $\mathcal{M}$ (not $\mathcal{M}^{eq}$).

(ii) For any $c \in M \setminus E$ there is a standard system $J : A \to \mathcal{M}^{eq}$ and some $b \in A \cap E$ such that

$$\operatorname{acl}(Ec) \cap J_b \supsetneq \operatorname{acl}(E) \cap J_b$$

(iii) For $J : A \to \mathcal{M}^{eq}$ a standard system of geometries and $b \in A \cap E$.

$$J_b \cap E$$

is a geometry of the right type with $\dim(J_b \cap E) = \mu(J)$. In this case we write $\dim_J(E)$ for $\dim(J_b \cap E)$.

Theorem. Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatised structure. Let μ be a dimension function.

(i) Existence: μ-envelopes exist.

(ii) Finiteness: If for every standard system J in $\mathcal{M}$, $\mu(J)$ is finite then every μ-envelope is finite.

(iii) Homogeneity & Uniqueness: Let E, E' be μ-envelopes and $A \subseteq E, B \subseteq E'$ if $f : A \to B$ is elementary (in $\mathcal{M}$) then it extends to an elementary $\hat{f} : E \to E'$. In particular μ-envelopes are homogeneous and unique.

<u>Corollary</u>. Whether or not $E \subseteq M$ is a μ-envelope does not depend on the choice of coordinatisation of $\mathcal{M}$.

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