Quadratic Geometries $(V, Q ; \ldots, w)$ — Witt defect

V: β; Q: quad. forms

$w : Q \to \{0,1\}$

$\bar{w} = 1 + w$

Lemma 2.5.7 In a Lie coordinatized $\mathcal{M}$ the quadratic geometries can be assigned compatible orientations, in the sense that in non-orthog. geometries the orientations are identified by the canonical weak unoriented isomorphism between appropriate localizations. This can be done 0-definably.

Clarify: 'assigned'

- If J_a is a quad. geom. in $\mathcal{M}$ with Witt w; let $\bar{J}_a$ be the same thing with Witt $\bar{w}$.

For every $b \equiv a$ replace J_b with $\bar{J}_b$ to obtain $\bar{\mathcal{M}}$.
This is interdef. with $\mathcal{M}$ & still Lie coord. 2.5.7 says that by doing this repeatedly, obtain $\bar{\mathcal{M}}$ satisfying the extra condition.

Def 2.5.6. 1. A standard system of geometries (for $\mathcal{M}$) is a 0-def. function $J : A \to \mathcal{M}^{eq}$ where A is a complete type over $\emptyset$ & $(J(a) : a \in A)$ is a family of canonical proj. geometries — J_a

2. Standard systems $J : A \to \mathcal{M}^{eq}$ & $J' : A' \to \mathcal{M}^{eq}$

are equivalent if there exists $a \in A$ & $a' \in A'$ s.t. $J(a)$ & $J(a')$ are non-orthogonal. In this case there is a 0-def. bijection $\alpha : A \to A'$ given by $\alpha(ga) = ga'$ (for $g \in Aut(M)$).
[essentially 2.5.4].

<u>Proof of 2.5.7</u>: Consider standard systems of projective quadratic geometries under equivalence. For each eq. class choose a representative. Use these to reorient the coordinatizing proj. quad. geoms P_b in M:

there is a unique (chosen) canonical proj. quad. geom. J_c non-orthogonal to P_b. By canonicity $c \in dcl(b)$ (2.5.3). There is a canonical weak def. iso between P_b and the localization of J_c at $A = acl(b) \cap J_c$.
Let (V, Q) be the linear q. geom. assoc. to J_c & $B = acl(b) \cap (V, Q)$.
<u>Note</u>: $B \cap Q = \emptyset$, otherwise the loc. of J_c at B is an orthog. geometry, so not iso. to P_b (everything is fully embedded over b). So $B \leq V$.

(1) Use the given ω on J_c to define a ω_B on the localization (V_B, Q_B)

(2) Transfer w_B to give a Witt defect on P_δ.

(3) Check the compatibility condition.

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<u>Localization</u>

$$J = (\overset{\beta}{V}, \overset{w}{Q}) \quad \text{oriented quadratic.} \quad B \leq V$$

$$J_B = (V_B, Q_B)$$

$$V_B = B^\perp / \underbrace{B^\perp \cap B}_{\mathrm{rad}(B)} \qquad \text{form } \beta_B \text{ induced by } \beta.$$

$$Q_B = \{ q|_{B^\perp} : q \in Q \quad q(\mathrm{rad}(B)) = 0 \}$$

V_B acts by $+$ on Q_B.

<u>Def</u> Let $B = \mathrm{rad}(B) \oplus B_0$. So B_0 is a non-deg. f.d. space. If $q \in Q$ then $q|_{B_0}$ so it has a well defined Witt defect $w_0(q|_{B_0})$

Define $w_B(q|_{B^\perp}) = w(q) + w_0(q|_{B_0})$.

<u>induced</u> <u>W. defect</u> on Q_B.

<u>Check</u>: 1) Well-defined. If $q, q' \in Q$ have same restr. to $B^\perp$ ($\&$ $q(\mathrm{rad}\, B) = q'(\mathrm{rad}\, B) = 0$) then write $q' = v + q$. Then $v \in (B^\perp)^\perp = B$

Write $v = r + v_0$ $\quad r \in \mathrm{rad}B$ $\&$ $v_0 \in B_0$.

$[q, q'] = q(v) = q(r + v_0) = q(r) + q(v_0) + \beta(r, v_0) = q(v_0)$

Let $q_0 = q|_{B_0}$ $\&$ $q_0' = q'|_{B_0}$

then $[q_0, q_0'] = q(v_0)$

(as for $u \in B_0$ $\quad q'(u) = q(u) + (\beta(v_0 + r, u))^2$
$$= q(u) + (\beta(v_0, u))^2 \Big)$$

if $[q, q'] \in \tau[K]$ then $\omega(q) = \omega(q')$ ✓
& $\omega_0(q_0) = \omega_0(q_0')$

Similarly if $[q, q'] \notin \tau[K]$ ... ✓. #

2) Indep. of B_0.

3) It's a Witt defect!

4) Suppose $B \le C \le B^\perp$ (say with $\mathrm{rad}(C) = \mathrm{rad}(B)$).

Can localize (V_B, Q_B, w_B) over C.

What we obtain is (V_C, Q_C) & it can be shown that the W. induced from w_B is equal to w_C.

<u>Compatibility</u> Suppose P_b & $\tilde{P}_a$ are coordinatizing proj. quad. geometries which are non-orth. They are non-orthog to a unique (chosen) canonical $J_c = J = (P, V, Q)$ & have orientations induced from localizations of J_c over, say, B & A.

For some X there is an X-def. weak iso.

$\alpha : (P_b)_X \to (\tilde{P}_a)_X$.

Want to show : α preserves induced ω.
on these localizations .