§ 3.3 FINITE STRUCTURES

SOME GROUP THEORY

$G' := [G, G] = \{x^{-1}y^{-1}xy \mid x, y \in G\}$ is the COMMUTATOR of G

$G^{(n)} = (G^{(n-1)})'$ From this we can get the derived series

$$\dots \triangleleft G^{(2)} \triangleleft G' \triangleleft G$$

For finite groups, the series terminates at $G^{(\infty)}$

G is PERFECT if $G = G^{(\infty)}$. Note: simple groups are perfect

G is SOLUBLE if $G^{(\infty)} = \{1\}$

$$\operatorname{soc}(G) = \prod_{\substack{H \triangleleft G \\ \text{minimal}}} H$$

G is ALMOST SIMPLE if there is a non-Abelian simple group S s.t. $S \leq G \leq \operatorname{Aut}(S)$

FACT 1: If G is almost simple, $\operatorname{soc}(G) = S$.

Schreier conjecture / FACT 2: For S simple $\operatorname{Aut}(S)/S$ is soluble.

CLASSICAL GROUPS [see Kleidman & Liebeck 1990]

The subgroup structure of finite classical groups

$\Gamma L(V,K)$ = non-singular semilinear transformations = $\mathrm{Aut}(V;K)$.

Consider a space with a form (V,K,κ) where

case **L**inear $\quad \kappa \equiv 0$

case **S**ymplectic $\quad \kappa = \beta$ a non-degenerate symplectic form $\quad \ell = 2$

case **O**rthogonal $\quad \kappa = Q$ a non-degenerate quadratic form $\quad \ell = 1$

case **U**nitary $\quad \kappa = \beta$ a non-degenerate unitary form. $\quad \ell = 2$

$I(V,K,\kappa) :=$ group of κ-isometries

$S(V,K,\kappa) := I(V,K,\kappa) \cap SL(V,K) :=$ group of special κ-isometries

$\Delta(V,K,\kappa) :=$ group of κ-similarities: $g \in GL(V,K)$ s.t.

$\exists \lambda \in K^{\times} \ \forall v \in V^{\ell} \quad \kappa(gv) = \lambda \kappa(v)$

$\Gamma(V,K,\kappa) :=$ group of κ-semisimilarities: $g \in \Gamma L(V,K)$ s.t.

$\exists \lambda \in K^{\times}$ and $\alpha \in \mathrm{Aut}(K)$ s.t. $\quad \kappa(gv) = \lambda \kappa(v)^{\alpha}$

FACT: $S \leq I \leq \Delta \leq \Gamma$

We also define $A(V, K, \kappa)$ and $\Omega(V, K, \kappa)$ as follows ($n = \dim V$, $K = \mathbb{F}_q$)

$$A = \begin{cases} \Gamma\langle\iota\rangle & \text{in case } \mathbf{L} \text{ for } n \geq 3 \\ \Gamma & \text{otherwise} \end{cases}$$

↳ "inverse transpose" automorphism*

$$\Omega = \begin{cases} \text{the unique subgroup of } S \text{ of index 2} & \text{in case } \mathbf{O} \\ S & \text{otherwise} \end{cases}$$

A CLASSICAL GROUP is any group G satisfying

$$\Omega \leq G \leq A \quad \text{OR} \quad \overline{\Omega} \leq G \leq \overline{A}$$

projective versions quotienting out by scalars

in one of the cases $\mathbf{L}, \mathbf{S}, \mathbf{O}, \underline{\underline{\mathbf{U}}}$.

*Inverse transpose of an invertible matrix A is $(A^{-1})^T$. It can be extended to an automorphism of ΓL of order 2 (for $n \geq 3$), so $\Gamma\langle\iota\rangle \cong \Gamma . 2$ for $n \geq 3$ in case $\mathbf{L}$.

Table 2.1.B			
case	X	notation	terminology
$\mathbf{L} = \mathbf{L}^+$	$\Omega = S$	$SL_n(q) = SL_n^+(q)$	linear groups
	$I = \Delta$	$GL_n(q) = GL_n^+(q)$	
	Γ	$\Gamma L_n(q)$	
	$A = \Gamma\langle\iota\rangle$		
$\mathbf{U} = \mathbf{L}^-$	$\Omega = S$	$SU_n(q) = SL_n^-(q)$	unitary groups
	I	$GU_n(q) = GL_n^-(q)$	
	Δ		
	$A = \Gamma$	$\Gamma U_n(q)$	
$\mathbf{S}$	$\Omega = S = I$	$Sp_n(q)$	symplectic groups
	Δ	$GSp_n(q)$	
	$\Gamma = A$	$\Gamma Sp_n(q)$	
$\mathbf{O}^\epsilon$, $\epsilon \in \{\pm, \circ\}$	Ω	$\Omega_n^\epsilon(q)$ *	orthogonal groups
	S	$SO_n^\epsilon(q)$	
	I	$O_n^\epsilon(q)$	
	Δ	$GO_n^\epsilon(q)$	
	$\Gamma = A$	$\Gamma O_n^\epsilon(q)$	

* In case $\mathbf{O}^\circ$ we actually write $Y_n(q)$ instead of $Y_n^\circ(q)$ for $Y \in \{\Omega, SO, O, GO, \Gamma O\}$.

THEOREM 3 (2.1.3 in K&L) For $n \geq 2, 3, 4, 7$ in L, U, S, O resp. $\overline{\Omega}$ is non-Abelian simple, except for $L_2(2), L_2(3), U_3(2)$ and $Sp_4(2)$.

THEOREM 4 (2.1.4 in K&L) n as above and $\overline{\Omega}$ simple. Then, $\overline{A} = \mathrm{Aut}(\overline{\Omega})$ except when $\Omega = Sp_4(q)$ for q even and $\Omega = \Omega_8^+ q$.

THEOREM 5 (2.9.2 in K&L + p.51) Ω is soluble if and only if $n=1$ or $\Omega \approx SL_2(2), SL_2(3), Sp_2(2), Sp_2(3), SU_2(2), SU_2(3), \Omega_3(3), \Omega_4^+(2), \Omega_4^+(3)$.

Moreover, if Ω is insoluble it is perfect, except when $\Omega \approx Sp_4(2)$

THEOREM 6 (+ Prop 2.9.1 K&L) For $n \geq 7$, $\overline{\Omega}$ is always simple and distinct geometries give rise to non-isomorphic simple groups.

FACT 7 For n large enough $\Gamma^{(\infty)} = \Omega$.

FINITE STRUCTURES

A SIMPLE LIE GEOMETRY L is either

(i) a **WEAK** linear geometry of any type other than Polar or quadratic;
(ii) The projectivisation of something in (i);
(iii) The affine or quadratic part of a geometry.

In (iii), the linear part can be considered as encoded in $L^{\subset q}$.
I think we need to think of these as canonical wrt their automorphism groups (which are of the form Γ).

FACT 3.3.4 Let k be an integer. There is $d = d_k$ s.t. for any finite basic projective simple Lie geometry L of dimension $\geq d$,

(i) $\overset{\overline{\Omega}}{G} = \mathrm{soc}(\mathrm{Aut}(L))$ is simple non-Abelian (THEOREM 3) and $\underbrace{\mathrm{Aut}(L)/G}_{\overline{\Gamma}/\overline{\Omega}}$ is soluble of class ≤ 2 (i.e. $H^{(n)} = \{1\}$ for $n \leq 2$).

(usually $G = \mathrm{Aut}(L)'$ (cf Rem 3.3.5))

(ii) G and $\mathrm{Aut}(L)$ have same orbits on L^k;

(iii) For L as a WEAK GEOMETRY, $\underbrace{\mathrm{Aut}(L) = \mathrm{Aut}(G)}_{\overline{\Gamma} = \mathrm{Aut}(\overline{\Omega})}$ (THEOREM 4)

EXCEPT: L is a pure v-space. Then, $|\mathrm{Aut}(G) : \mathrm{Aut}(L)| = 2$ and $\mathrm{Aut}(G)$ is realised geometrically by the automorphism group of the weak polar geometry (L, L^*).

[This was the exception of the inverse transpose automorphism]

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If J_1 and J_2 are non-degenerate basic projective geometries non-quadratic of large enough dimension and $\mathrm{soc}(J_1) \cong \mathrm{soc}(J_2)$ then they are isomorphic as weak geometries (cf. Theorem 6)

A COORDINATIZING STRUCTURE of type (e, K) and dimension d is a structure $\mathcal{C}$ with transitive automorphism group and an $Aut(\mathcal{C})$-invariant equivalence relation with $e < \infty$ many classes s.t. each class is associated with a simple Lie geometry L of dimension d over a finite field K (we may associate to $\mathcal{C}$ also the type of the geometry).

$\mathcal{C}$ is PROPER if each equivalence class of $\mathcal{C}$ as a geometry is canonically embedded in $\mathcal{C}$ (i.e. the automorphism group induced on each class is dense in the automorphism group of the associated geometry and equal to it if d is finite)

$\mathcal{C}$ finite dimensional is SEMIPROPER if $Aut(\mathcal{C})$ induces on each class a subgroup $G \leq Aut(L)$ containing $Aut(L)^{(\infty)}$ (so close enough to $Aut(L)$, and in particular a classical group).

EXAMPLE: The basic projective space of dim d over K is a semiproper coordinatising structure of type $(1, K)$ and dim d associated with the weak projective space.

- A weak polar geometry is a proper coordinatising structure with $e = 2$.

Let $\mathcal{C}$ be a coordinatising structure of type (e, K) and dim d. Let τ be a type over $\emptyset$ of a finite algebraically closed subset $A \subseteq \mathcal{C}$. The GRASSMANIAN $\Gamma(\mathcal{C}, \tau)$ is the set of realisations of τ in $\mathcal{C}$ with the structure induced by $\mathcal{C}$. We say it is of type (e, K, τ) and dimension d.

Example: $\mathcal{C}$ = degenerate space of size $d = \{1, \dots, d\}$
τ_k = type of a k-element subset of $\{1, \dots, d\}$.
$\Gamma(\mathcal{C}, \tau_k) \cong J(d, k)$, the **Johnson graph** on k-element subsets of d with relations $S_i(x, y)$ for $i \le k$ to denote $|x \cap y| = i$.

FACT 3.3.3 [KLM] $\forall k \; \exists n_k$ s.t. for any primitive structure M of order at least n_k, if M has at most k 4-types, then M is isomorphic to a semiproper Grassmanian of type (e, K, τ) with $e, |K|, |\tau| \le k$.

- KLM does this for 5. See [M. 1997] for 4. False for 3.

SOME GROUP COHOMOLOGY [cf. Aschbacher Ch 6]

V a vector space over K, $\pi: G \to Aut(V)$ a representation of G on V.

A 1-COCYCLE is a map $\gamma : G \to V$ s.t. $\forall\, g, h \in G$

$$\gamma(gh) = \gamma(g)^{h} + \gamma(h)$$

* apply $\pi(h)$

$\Gamma(G,V)$, the space of 1-cocycles from G to V is a group with

$$(\gamma + \delta)(g) = \gamma(g) + \delta(g).$$

A 1-COBOUNDARY is, for $v \in V$ a map $v\alpha : G \to V$ where $v\alpha(g) = v - v^{g}$.

1-coboundaries form a normal subgroup $V\alpha$ of $\Gamma(G,V)$.

$H^1(G,V) = \Gamma(G,V)/V\alpha$ is the FIRST cohomology group of π.

In our context V will be an FG-module and so $H^1(G,V)$ is a vector space over F.

FACT 3.3.6 For any finite basic simple linear geometry V of dimension at least 5, if $G = (\mathrm{Aut}\, V)^{(\infty)}$ acts on an affine space A over V as to induce its standard action on V, then,

EITHER G fixes a point of A;

OR $G = Sp_{2n}(q)$ for $\mathbb{F}_q$ of char 2 acting on its natural module V and $G \curvearrowright A$ is definably equivalent to its action on Q, the space of quadratic maps associated to the symplectic form of V.

Proof:

Fix $a \in A$. $f_a : G \to V$ given by $f_a(g) = a^g \ominus a$ is a 1-cocycle.

Consider $H^1(G, V)$. For $a, b \in A$, f_a and f_b are in the same class wrt $V\alpha$ since $f_a - f_b = (a \ominus b)^g - (a \ominus b)$

If $H^1(G,V)$ is trivial, then f_a is always a 1-coboundary.

So $\exists v \in V$ s.t. $\forall g \in G$ $v^g - v = a^g \ominus a$.

From this we get that $(a \oplus -v)^g = a \oplus -v$ and so we found a fixed point for $G \curvearrowright A$.

There are tables computing $H^1(G, V)$ for these groups [Jones & Parshall '76]
We get that $H^1(G, V)$ is trivial, except for the case of $Sp_{2n}(q)$, where $H^1(G, V) \cong K$.

Now, the affine space B given by the regular action of V on itself is 0.
For $\alpha \in K^\times$, the αQ give rise to 1-cocycles in distinct cosets, (again via $q^g - q$), where αQ is the space of quadratic forms inducing $\alpha\beta$.

So A must give rise to the same coset as αQ for some $\alpha \in K^\times$

We can view αQ as Q by replacing the action $q \longmapsto q + \lambda_v^2$ by $q \longmapsto q + \lambda^2_{\alpha^{\frac{1}{2}} v}$.

In particular αQ (and so A) is definably isomorphic to Q by rescaling of the regular action.

$\square$

Think can encode field on orbits on 4-tuples by cross-ratio.