

§ 2.1.2 GEOMETRIES

① DEGENERATE SPACE - pure set with equality

② PURE VECTOR SPACE (V, K)

scalar mult $K \times V \rightarrow V$

σ be an alt of K with $\sigma^2 = 1$

A SESQUILINEAR FORM wrt σ is $\beta: V \times V \rightarrow K$ s.t.

$$\beta(x+y, z) = \beta(x, z) + \beta(y, z)$$

$$\beta(\lambda x, y) = \lambda \beta(x, y)$$

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$$\beta(x, \lambda y) = \lambda^\sigma \beta(x, y)$$

$\sigma = 1$ we say β is BIILINEAR

symmetry $\beta(x, y) = \beta(y, x)$

skew-symmetry $\beta(x, y) = -\beta(y, x)$

hermitian symmetry $\beta(x, y) = \beta(y, x)^\sigma$

$\beta(x, y) = 0$ iff $\beta(y, x) = 0$

β is NON-DEGENERATE if $\forall x (\forall y \beta(x, y) = 0 \rightarrow x = 0)$

β SYMPLECTIC non-deg. + skew-symmetric + sesquilinear

$$\text{In char}(K) = 2 \quad \beta(\alpha, \alpha) = 0$$

β UNITARY non-deg + hermitian symm (wrt σ) + sesquilinear.

INNER PRODUCT SPACE (V, K, L, β)

$$\beta: V \times V \longrightarrow L$$

\hookrightarrow 1-dm K -space

$\beta \rightsquigarrow$ sesq. wrt σ

- $\sigma = 1$ (so β is BIL) β symplectic. $\sim \mathrm{Sp}(w, q)$

- σ non-trivial β unitary. $\sim \mathrm{GU}(w, q)$

technical corner: We want to view structures where forms are scalar multiples of each other as isomorphic. $(V, K, L, \beta) \sim (V, K, L, \alpha \cdot \beta)$

$\sqrt{\alpha} \xrightarrow{f} \alpha \cdot v \quad f(\beta(u, v)) = \alpha \beta(u, v)$

$$f: V \rightarrow V \quad v \mapsto \sqrt{\alpha} \cdot v$$

β is ORTHOGONAL symm + non-deg + bilinear

In char 2 odd $\beta(x, x) = 0$

For β orth. we say that $q: V \rightarrow K$ is an associated QUADRATIC FORM

$$\textcircled{1} \quad q(\lambda x) = \lambda^2 q(x)$$

$$\textcircled{2} \quad q(x+y) = q(x) + q(y) + \beta(x, y)$$

If $\text{char}(K) \neq 2$, we can recover $q(x) = \frac{1}{2} \beta(x, x)$ $\textcircled{\ast}$

$$\beta(x, x) = q(2x) - q(x) - q(x) = 4q(x) - 2q(x) = 2q(x)$$

We can always recover β from q by $\textcircled{2}$

ORTHOGONAL SPACE (V, K, L, q)

q is a quadratic form whose associated β is orthogonal.

\sim infinite dim version of orthogonal groups $O_w(q)$

$$\beta_Q: Q \times V \rightarrow K$$

$$\beta_Q(q, v) = q(v)$$

$+_Q: V \times Q \rightarrow Q$ is reg. action of V on Q

$$v +_Q q = q + \lambda^2 v$$

$$-_Q: Q \times Q \rightarrow V \quad (q_1, q_2) \mapsto v_R \text{ where } q_1 - q_2 = \lambda v$$

two formed spaces are equivalent when there is an isomorphism between them.

In the even dim case we have up to equivalence two quadratic forms induced by β .

$$\omega: Q \longrightarrow \{0, 1\} \subseteq K$$

with defect — In dim $2n$ case

the maximal q -isotropic subspaces of V .
w s.t. $q|w=0$

dim n with defect $n - \dim(W)$

dim $n-1$

$$\sim O_{2n}^+(q) \quad O_{2n}^-(q)$$

In the infinite dimensional case, all $q \in Q$ induced by β are equivalent.

$$q_1, q_2 \in Q, \sqrt{q_1 + q_2} = \lambda v \text{ for } v \in V$$

identify v with λv $q(\sqrt{q_1 + q_2}) \in K$

$$- q_1(\sqrt{q_1 + q_2}) = q_2(\sqrt{q_1 + q_2}) \notin V$$

$$- \text{write } [q_1, q_2] = q_1(\sqrt{q_1 + q_2})$$

$$\text{Let } q_1, q_2, q_3 \in Q \quad v = \sqrt{q_1 + q_2} \quad w = \sqrt{q_1 + q_3} \quad \alpha = \beta(v, w)$$

$$[q_1, q_2] + [q_1, q_3] + [q_2, q_3] = \gamma(\alpha)$$

↑ Artin-Schreier poly

$$[q_1, q_2] \in \gamma(K)$$

$$\gamma(\alpha) = \alpha^2 + \alpha.$$

- this is an equivalence rel.

$$\gamma(\alpha) + \gamma(\beta) + [q_1, q_3] = \gamma(\gamma) \quad [q_1, q_3] = \alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \gamma \\ = (\alpha + \beta + \gamma)^2 + \alpha\beta + \gamma$$

- this equivalence relation has two classes.

$$w: Q \rightarrow \{0, 1\} \subseteq K \text{ name each class.}$$

QUADRATIC GEOMETRY

$(V, Q, K, \beta_V, +Q, -Q, \beta_Q, \omega)$

only one form
type

POLAR SPACE

$(\underbrace{V \cup W}_{\text{one equiv. rel. with two classes}}, K, L, \beta)$

a non-degenerate bilinear pairing
 $\beta : V \times W \rightarrow L$

$\beta : (V \times W) \cup (W \times V) \rightarrow L$ symmetric

Notation corner: $W = V^*$ and $V = W^*$.

In finite dim $\beta : V \times V^* \rightarrow L$ $\beta(v, f) = f(v)$.

one can give V the weak topology induced by W .] and vice-versa.

[weakest TVS topology
 $\beta(\cdot, \omega) \rightarrow$ is continuous.

CONTINUOUS DUAL V' of V (TVS) is the space of all ctg lin fns $\phi : V \rightarrow K$

A WEAK LINEAR GEOMETRY is a structure of one of the above six types.

A LINEAR GEOMETRY is an expansion of a weak one by a set of algebraic constants in M^9 .

$$\in \text{od}^{\text{alg}}(\emptyset) ?$$

UNORIENTED WLG any of above but forget W in the quadratic case.

A BASIC LG is a LG in which elts of K and L have been named and in the polar case we name the two spaces.

PROJECTIVE GEOMETRY \wr from a LG factoring out eq rel $oel(x) = oel(y)$.

SEMI PROJECTIVE GEOMETRY from BLG by
 factoring out the rel. $x^2 = y^2$ for Z the center of
 the aut group (the set of scalar transf respecting any additional
 structure).

- = In symplectic case $\beta(\lambda u, \lambda v) = \beta(u, v)$
 $\lambda^2 \beta(u, v) \quad \text{so} \quad \lambda = \pm 1$

V acting regularly and definably on def $A \triangleleft$ AFFINE V -space.

J a LG. V (one of V or W in polar)

AFFINE GEOMETRY (J, A) is a structure where J carries its given str. and A carries the action of V but has no further struc.

Ex: V act regularly on itself $\oplus: V \times A^{\times} \xrightarrow{V} A$
 $(v, a) \mapsto v+a$
 $(V, K, L, \beta; A)$

for $a_1, a_2 \in A \quad \exists! v \in V$ s.t. $v \oplus a_1 = a_2$.

def function $\ominus: A \times A \rightarrow V$
 $(a_1, a_2) \mapsto v$

$\beta(n, a_1 \ominus a_2)$ for a_1, a_2 affine...

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