FSFT — May 9, 2025

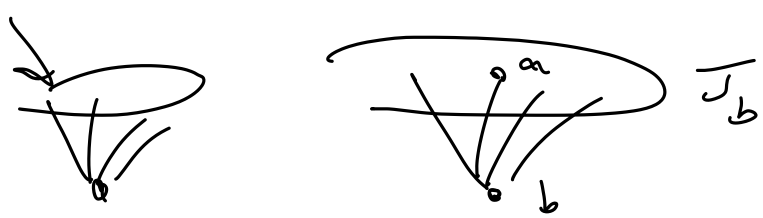
Def

$\mathcal{M}$ is **locally Lie coordinatized** if it has nonstd finite order, has finitely many 1-types, carries a tree structure of finite height whose root is 0-def'l, and has a collection $\mathcal{J}$ of pairs (b, J) with $b \in \mathcal{M}$, $J \subseteq \mathcal{M}$ a b-definable component of a b-def'l basic semiproj, linear, or affine geometry satisfying

tree order

1. If a is not the root, there is $b < a$ st either $a \in acl(b)$ or there's a pair (b, J_b) with $a \in J_b$.

alg.



2. If $(b, J_b) \in \mathcal{J}$ w/ J semiprojective or linear then J is canonically embedded in $\mathcal{M}$.

3. Affine spaces are preceded in the tree by their linear versions.

Lemma 3.5.7

Let $\mathcal{M}$ be internally finite, J semiprojective or linear, B-def'l, and let $C \subseteq J$ be finite with the localization $(J/_C)$ canonically embedded in $(\mathcal{M}^*; B)$. Assume that C is non-degenerate if J involves a form and, otherwise, if J is pure projective then assume that in $\mathcal{M}^*$ the definable dual of the linear model is trivial.

Then, the group G induced on J

by the internal automorphism group $\mathrm{Aut}(M;B)$ contains $(\mathrm{Aut}(J;C))^{(\infty)}$.

iterated derived subgroup.

$\mathrm{Aut}(M;B) \curvearrowright J$

$$\begin{tikzcd} J \arrow[d] \\ J/C \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Aut}(M;B) \arrow[d] \\ \mathrm{Aut}(J) \end{tikzcd}$$

Claim: image contains $(\mathrm{Aut}(J;C))^{(\infty)}$.

$$\begin{tikzcd} \mathrm{Aut}(M;B) \arrow[d] \\ \mathrm{Aut}(J/C) \end{tikzcd} \curvearrowright J/C$$

J/C canonically embedded.

First case: J has a form.

We have assumed that C is nondegenerate so $J = C \oplus (C)^{\perp}$. So $\mathrm{Aut}(J;C)$ and $\mathrm{Aut}(J/C)$ are really both

$\mathrm{Aut}((C)^{+})$.

Lemma 3.4.2 (Mis-cited in Chatzidakis Hrushovski)

J is canonically embedded in $(M; C)$ iff $\mathrm{Aut}(J; C)$ contains

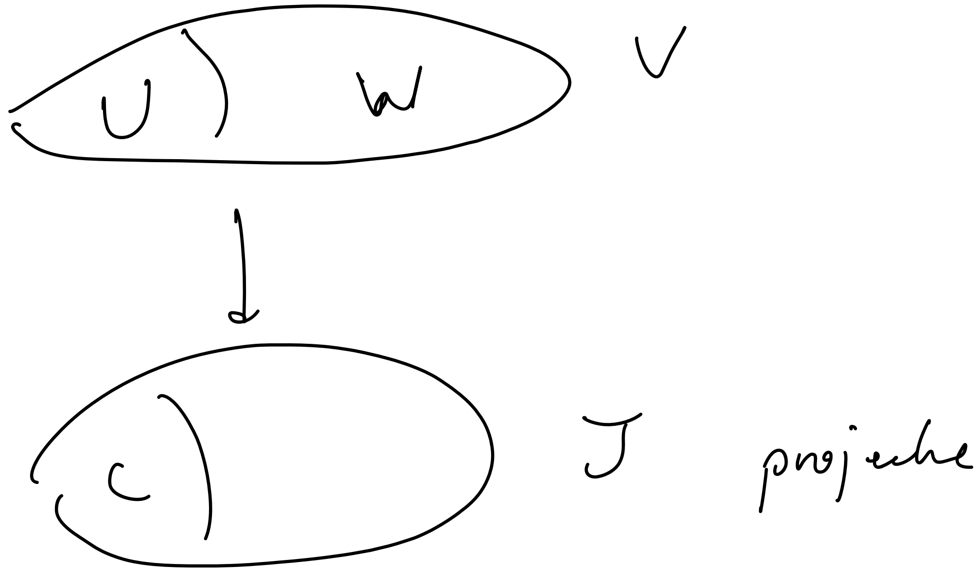
$$\left(\mathrm{Aut}(J; C)\right)^{(\infty)} = \left(\mathrm{Aut}(J/C)\right)^{(\infty)}.$$

$\Longrightarrow$ Case with tori is a consequence of this earlier lemma.

Case 2: Pure projection case.

Take V to be the linear model of J.

Let $V = W \oplus U$ where U covers C.



$\mathrm{Aut}(V;U)$ wrt a basis of V starting with a basis for U and ending with a basis for W, each element of $\mathrm{Aut}(V;U)$ looks like a matrix

$$\left(\begin{array}{c|c} I_n & A \\ \hline 0 & B \end{array}\right)$$

additive

$A \in \mathrm{Hom}(W, U)$

$B \in GL(W)$

$$\left(\begin{array}{c|c} I_n & A \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} I_n & C \\ \hline 0 & D \end{array}\right) = \left(\begin{array}{c|c} I_n & C+AD \\ \hline 0 & BD \end{array}\right)$$

Clarifies the semi-direct product

From this it follows

$$\left(\begin{array}{c|c} I & A \\ \hline 0 & B \end{array}\right)^{-1} = \left(\begin{array}{c|c} I & -AB^{-1} \\ \hline 0 & B^{-1} \end{array}\right)$$

$$Aut(J;U)^{(\infty)} = \left\{ \left(\begin{array}{c|c} I & ? \\ \hline & GL(W)^{(\infty)} \end{array}\right) \right\}$$

$$\leq Hom(W,U) \rtimes SL(W)$$

since $GL(W)^{(\infty)} = GL'(W$

$= SL(W)$.

Wilson's Finite Simple Groups

Know that $\mathrm{Aut}(\mathcal{M}; B)$ induces at least $SL(W)$ on J/C so know that the subgroup of $\mathrm{Hom}(W, U) \rtimes SL(W)$ inducing G is $H \rtimes SL(W)$ with H an $SL(W)$-invariant subgroup of $\mathrm{Hom}(W, U)$. Then H will be of the form $\mathrm{Hom}(W, U_0)$ for some subspace $U_0 \leq U$. Follows that $P(W \oplus U_0)$ is the unique minimal $G^{(\infty)}$-invariant subspace of J. (using description of $G^{(\infty)} = \mathrm{Hom}(W, U_0) \rtimes SL(W)$) so G-invariant.

But in the pure projective case, there can be no definable subspace of finite codimension. (recall U and thus U_0 were fin dim). So $U = U_0$ and thus

$H = \mathrm{Hom}(W', U) \rtimes SL(W)$ so we

get what we want. $\blacksquare$

Lemma 3.5.8

Let M be an internally finite locally Lie coordinatized structure wrt coordinate systems in $\mathcal{J}$ and suppose that

1. Whenever $J_b \in \mathcal{J}$ is projective w/ linear model V the definable dual is trivial.

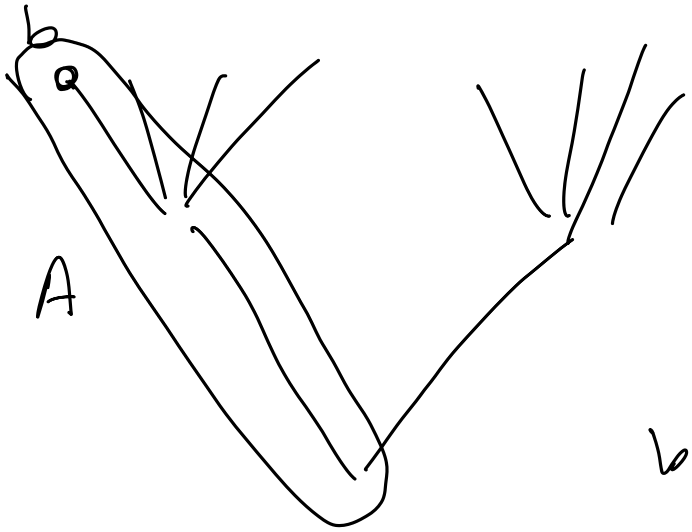
2. Whenever $J_b \in \mathcal{J}$ is symplectic of char 2 then there are no def'l quadratic forms compatible with the form.

<u>Then</u>, for any finite subset $A \subseteq M$ closed downwards in the tree structure, we have

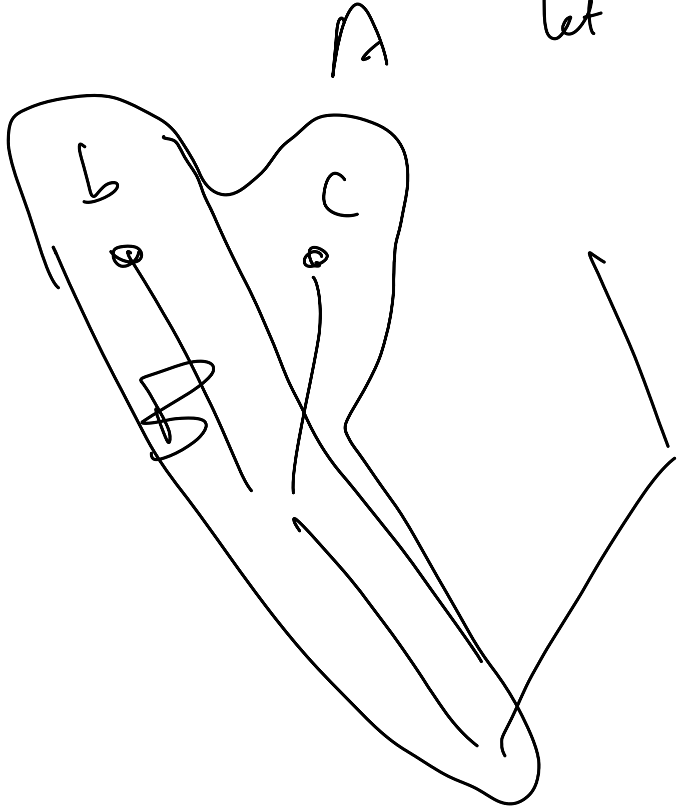
3. For $b \in A$, if J_b is nonaffine then for some finite $C \subseteq J_b$ the structure $(J; C)$ is canonically embedded in M^{*} over A.

4. If $J_1, J_2 \in \mathcal{J}$ nonaffine w/ defining parameters in A, if $C_i = \mathrm{acl}_{M^*}(A) \cap J_i$ then either $(J_1; C_1)$ $(J_2; C_2)$ are orthogonal over A or else there is an A-def'l bijection between J_1/C_1 and J_2/C_2.

Start of the proof. Do induction on $|A|$. We will do (3). Let A, b be given. If A is the branch below b, we're done by definition of locally Lie coordinatized.



WMA then that A contains some element not on the branch below b, so let c be maximal such.



$B = A - \{c\}$.

Can apply induction to B, b.

$\rightsquigarrow$ Arguments.