Weak linear geometry : K finite field ; V K-space $\cdot : K \times V \to V$; L 1-dim K space

1. Degenerate $(V ; =)$
2. Pure v.space $(V, K ; (+,-,0_V) ; (+,-,\cdot,0,1) ; \cdot)$
3. Polar space $(V \cup W, K, L ; \beta)$ $\beta : V \times W \to L$ non-degenerate pairing (symmetrized).
4. Inner product space $(V, K, L ; \beta)$ $\beta : V \times V \to L$ non-deg. sesquilinear form wrt σ, $\sigma^2 = 1$. $\sigma = 1$: β symplectic
$\sigma \neq 1$: hermitian (unitary).
5. Orthogonal $(V, K, L ; q)$ $q : V \to L$ quadratic form
assoc. bilinear form β non-deg.
6. Quadratic geometry. $(V\ Q, K ; \beta_V, +_Q, -_Q, \beta_Q, \omega)$
char $K = 2$; β_V symplectic ; Q V-orbit on q-forms $\rightsquigarrow \beta_V$;
ω Witt defect.

BASIC linear geometry : elts. of K & L are named ; in (3) name V, W.

Remark : If $\dim V = \aleph_0$ (= $\dim W$ in Case (3)) each of these strc. is ω-categorical.

(2.2.8) Lemma. Suppose J is a (inf. dimensional) basic linear geometry. (in the indicated language). Then J has QE

Pf: Use b+f. Suppose $\bar{a}, \bar{b}$ are tuples in J with same q.f. type & $c \in J$. Find $d \in J$ with $\text{qftp}(\bar{a}^\wedge c) = \text{qftp}(\bar{b}^\wedge d)$.

Case 0 Degenerate. Ex.

Case 1 J 'classical' $J = (V ; \beta, q)$.
(2), (4), (5)

WLOG $\bar{a}, \bar{b}$ enumerate K-subspaces of V ; A, B. $c \notin A$
The q.f. type to be realised by d (over B) is of the form

$$\begin{cases} x \notin B \\ \beta(b, x) = \lambda(b) & (b \in B, \text{ some } \lambda \in B^*) \\ q(x) = \alpha & (\text{some } \alpha \in K) \end{cases}$$

Clear if (2). So assume β is non-degenerate.
Enlarge B so that $B^\perp \cap B = \{0\}$. (extend λ arbitrarily).

So $V = B \perp B^\perp$. As $\lambda \in B^*$ so there is $b' \in B$ st $\lambda(b) = \beta(b, b')$ for all $b \in B$.

Replacing x by $x - b'$ we can assume $\lambda(b) = 0$ $\forall b \in B$. ie. $x \in B^\perp$. The only condition left is $q(x) = \alpha$.

But if $q \neq 0$ then $q(B^\perp \setminus \{0\}) = K$ //.
(as $\dim B^\perp \geq 3$)

[Every elt of $K^\times$ is a square so only issue is to find a non-zero $v \in B^\perp$ with $q(v) = 0$. No anisotropic orthog. space of dimension > 2.] //.

<u>Case 2</u> Polar space (3) Ex.

<u>Case 3</u> <u>Quadratic geometry</u>. If $\bar{a}$ (and $\therefore$ $\bar{b}$) intersects Q non-trivially, this reduces to the orthogonal case (5). So suppose $\bar{a}, \bar{b}$ are in V. If $c \in V$, this is the symplectic case (4). We can assume β is non-deg. on A and B (subspaces enumerated by $\bar{a}, \bar{b}$).

Assume $c \in Q$. The type to be realized by d is a q.f. $d \in Q$ with $d|B$ & $\omega(d)$ fixed.

All possibilities can be realized: all possibilities for $d|B$ can be achieved as $B^* \underset{\beta}{\cong} B$. Need to show extensions of $d|B$ in Q include forms with both defects.

$$v + q = q + \lambda_v^2 \qquad [q, v+q] = q(v)$$

$$\tau(K) = \{\alpha^2 + \alpha : \alpha \in K\}$$

$v \in B^\perp$ $\qquad q(B^\perp) = K$, so any Witt defect arises. #

<u>Remarks</u> (1) In (6) also get QE without ω for inf. dim. case.

(2) In f.d. case + [4].[5] QE is equivalent to Witt's Theorem (Aschbacher §20).

§2.1.3 Coordinatization

(2.1.9) Def. Let $M \subseteq N$ be structures with M (parameter) definable in N.
Let $a \in N^{eq}$ represent the set M (canonical parameter).

1. M is canonically embedded in N if the 0-def. relations in M are precisely the relations in M which are a-definable in N.
2. M is stably embedded in N if every N-definable relation in M is M-definable, uniformly.
3. Fully embedded: both.

Remark: ① If N is ω-cat, M fully embedded $\Leftrightarrow$ $Aut(N/a)|M = Aut(M)$.

② In the polar space, the v. space V is canonically, but not stably embedded.

(2.1.10) Def. A structure $\mathcal{M}$ is coordinatized by Lie geometries (Lie coordinat-ized) if it has a tree structure $<$ of finite height (invariant p.o.) with a unique $\emptyset$-def. root such that:

1. (Coordinatization) For all $a \in \mathcal{M}$ above the root one of the following holds:

(0) a is algebraic over its $<$-predecessor;

OR there is $b < a$ & a b-def. proj. geometry J_b fully embedded in $\mathcal{M}$ st. either

(i) $a \in J_b$; or

(ii) there is $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) st. $a \in A_c$ and the projectivization of J_c is J_b.

2. (Orientation) If $a, b \in \mathcal{M}$ have the same type (over $\emptyset$) & are associated with coordinatizing quadratic geometries J_a, J_b, then any definable map between them which preserves everything other than ω also preserves ω.

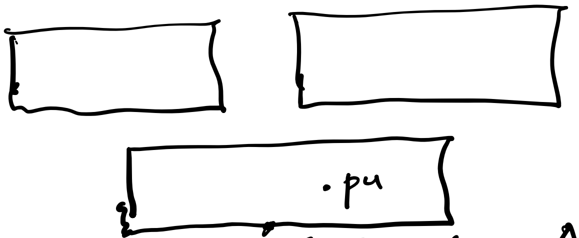
(2.1.12) Def. A str. $\mathcal{N}$ is Lie coordinatizable if it is biinterpretable with a structure $\mathcal{M}$ with finitely many 1-types (over $\emptyset$) which is Lie coordinatized.

Weak Lie coordinatized / able: without the orientation condition.

THM 2 ((1) $\Leftrightarrow$ (5)) $\mathcal{M}$ smoothly approximable $\Leftrightarrow$ $\mathcal{M}$ Lie coordinatizable.

THM 3 $\mathcal{N}$ is a reduct of a s.a. str. $\Leftrightarrow$ $\mathcal{N}$ is weak Lie coordinatizable.

Example 2.1.11 Let $A = \left(\bigoplus_{i \in \omega} (\mathbb{Z}/p^2\mathbb{Z}), + \right)$ p prime.



$u + A[p] = \{ v \in A : pv = pu \}$

$pA = A[p] = \{ a \in A : pa = 0 \}$

$px = c$

Coordinatization

$u + A[p]$

$A[p] \setminus \{0\}$

Root: 0 $\quad J_0$

$A[p] \setminus \{0\}$

scalars

$c/\sim$ $\quad c$ $\quad (A[p], A_c)$

$A_c = \{ a \in A : pa = c \}$

Finite Proj. Finite Affine.