Previously on... March 2025

Finite Structures with Few Types

2.1.2. Geometries

Def. A weak linear geometry is a structure of one of the following six types:

(i) A degenerate space — V a pure set in the language of equality.

(ii) A pure vector space — (V, K) where:

- K is a finite field;
- V is a K-vector space.

(iii) An inner product space — $(V, K, L; \beta)$ where:

- V and K are as above;
- L is a 1-dimensional K-vector space (a K-line);
- $\beta : V \times V \longrightarrow L$ is a non-degenerate sesquilinear form wrt a fixed involution $\sigma \in \mathrm{Aut}(K)$ such that:
  - If $\sigma = 1$ then β is symplectic
  - If $\sigma \neq 1$ then β is hermitian

(iv) An orthogonal space — $(V, K, L; q)$ where:

- V, K, L are as before.

- $q : V \to L$ is a quadratic form whose associated bilinear form

$$\beta(x,y) = q(x+y) - q(x) - q(y)$$

is non-degenerate.

(v) A polar space — $(V \cup W, K, L; \beta)$ where:

- V, K, L are as before, W is also a K-vector space
- $\beta : V \times W \longrightarrow L$ is a non-degenerate bilinear form.

(vi) A quadratic geometry — $(V, Q, K; \beta_V, +_Q, -_Q, \beta_Q, \omega)$ where:

- V, K as before but now $\mathrm{char}(K) = 2$;
- Q is a set of quadratic forms $q : V \to K$ with associated bilinear form β_V such that $V \curvearrowright Q$ regularly by translation;
- $+_Q : V \times Q \to Q$, $\quad -_Q : Q \times Q \longrightarrow V$, $\quad \beta_Q : Q \times V \longrightarrow K$;
- $\omega : Q \longrightarrow \mathbb{F}_2$ is the Witt defect.

So a WLG is one of:
(i) degenerate space
(ii) pure vector space
(iii) inner product space
(iv) orthogonal space
(v) polar space
(vi) quadratic geometry

the details are covered in Paolo's talk from ADD DATE!

it is called unoriented if in (vi) we drop the Witt defect

Def. A linear geometry is a weak linear geometry expanded by constants for a subset of $\mathrm{acl}^{eq}(\emptyset)$.

Def. A basic linear geometry is a linear geometry with the elements of K and L named by constants.

Def. A projective linear geometry is obtained from a linear geometry by factoring out the equivalence relation:

$$x \sim y :\Longleftrightarrow \operatorname{acl}(x) = \operatorname{acl}(y)$$

Def. A semiprojective geometry is obtained from a basic linear geometry by factoring out the equivalence relation

$$x \sim y :\Longleftrightarrow x^Z = y^Z$$

where Z is the centre of the automorphism group.

Def. Let M be a structure, $V \subseteq M$ a definable vector space and $A \subseteq M$ a definable set. We say that A is an affine V-space if $V \curvearrowright A$ definably and regularly. If J is a linear geometry with underlying vector space V then an affine geometry is a structure (J, A) in which J carries its given structure, A is an affine V-space and A carries no further structure.

Summary

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{WLG} \arrow[r, "\text{naming elements of } acl^{eq}(\emptyset)"] \arrow[d, "\text{forgetting the Witt defect}"'] & \text{LG} \arrow[r, "\text{naming } K, L", "\text{(and } V, W \text{ in polar case)}"'] \arrow[d, "\text{quotienting by } acl(x) = acl(y)"'] \arrow[dr, "{(J, A) \text{ where } J \text{ is a LG, } A \text{ an affine } V\text{-space}}"] & \text{BLG} \arrow[d, "\text{quotienting by } x^Z = y^Z"'] \\
\text{Unoriented WLG} & \text{projective LG} & \text{semiprojective geometry} \\
 & & \text{Affine geometry}
\end{tikzcd}$$

Theorem. If J is an infinite dimensional BLG then J has QE (in the specified language).

↪ Projectivising really means projectivising (i.e. acl is just linear span)

Remark. If $J = (V, \ldots)$ is a BLG with $\dim(V) = \aleph_0$ ($= \dim(W)$ in the polar case) then J is $\aleph_0$-categorical.

2.2.1. RANK

either saturated or $\aleph_0$-categorical

Def. Let $D \subseteq \mathcal{M}^{eq}$ be a definable set. We define the rank of D, $rk(D) \in \omega \cup \{\infty\}$ inductively, as follows:

(i) $rk(D) \geq 0$ if $D \neq \emptyset$

(ii) $rk(D) > 0$ if D is infinite

(iii) $rk(D) \geq n+1$ if there are definable $D_1, D_2 \subseteq M$, $\pi : D_1 \to D$ and $f : D_1 \to D_2$ such that

(i) $rk(\pi^{-1}(d)) = 0 \quad \forall d \in D$ (i.e. D_1 is a finite cover of D)

(ii) $rk(D_2) > 0$ (i.e. D_2 is infinite)

(iii) $rk(f^{-1}(d)) \geq n \quad \forall d \in D_2$

When we compute the rank in $\mathcal{M}$ rather than $\mathcal{M}^{eq}$ we call it pre-rank.

Def. Let $\mathcal{M}$ be $\aleph_0$-categorical $B \subseteq M$ finite and $p \in S_n(B)$. Then

$$rk(p) := \min\{rk(D) : D \in p\}$$

We write $rk(a/B)$ to mean $rk(p)$ where $p = tp(a/B)$.

Basic properties of rank

1. $rk(D) = 0$ iff D is finite. (i.e. $rk(a/B) = 0$ iff $a \in acl(B)$)

2. $rk(D_1 \cup D_2) = \max\{rk(D_1), rk(D_2)\}$ (if $A \subseteq B$ then $rk(a/B) \leq rk(a/A)$)

3. Extension property If D is B-definable then there is $p \in S(B)$ such that $D \in p$ and $rk(p) = rk(D)$.

4. $rk(a/b) \geq n+1$ admits an acl characterisation (Lemma 2.2.3)

<u>Theorem</u> (Additivity). If $rk(a/bc)$ and $rk(b/c)$ are finite then so is $rk(ab/c)$.

In fact:

$$rk(ab/c) = rk(a/bc) + rk(b/c).$$

$\rightsquigarrow$ If $rk(D) = 1$ then acl defines a pregeometry on D.

<u>Def.</u> We say that a and b are independent over C if

$$rk(a/bC) = rk(a/C).$$

Basic Properties of independence

To make the statements below shorter let us write $a \mathop{\perp}^{rk}_{C} b$ for $rk(a/bC) = rk(a/C)$

1. Symmetry: $a \mathop{\perp}^{rk}_{C} b \iff b \mathop{\perp}^{rk}_{C} b$.
2. Monotonicity & Transitivity: $a \mathop{\perp}^{rk}_{E} bc \iff a \mathop{\perp}^{rk}_{Ec} b$ and $a \mathop{\perp}^{rk}_{E} c$.
3. If $a \in acl(bC)$ then $a \mathop{\perp}^{rk}_{C} b \iff a \in acl(C)$.

By quantifier elimination we get:

<u>Theorem</u>. The linear affine and projective geometries are all of (pre-) rank 1.

$\rightsquigarrow$ If J is such a geometry then for any $a, b \in J$ we have $a \mathop{\perp}^{rk}_{acl(a) \cap acl(b)} b$.

<u>2.3.</u> IMAGINARY ELEMENTS

<u>Def.</u> We say that $\mathcal{M}$ has weak elimination of imaginaries if for all $a \in \mathcal{M}^{eq}$ we have:

$$a \in dcl^{eq}(acl^{eq}(a) \cap M).$$

Making use of the extension property and the additivity the rank:

Lemma. Let J be a linear, projective or affine geometry, $a \in J^{eq}$. Then

$$\operatorname{acl}^{eq}(a) = \operatorname{acl}^{eq}\left(\operatorname{acl}^{eq}(a) \cap J\right).$$

geometric EI (see later)

Theorem. Let J be a basic linear or semiprojective geometry. Then J has weak elimination of imaginaries.

Remark. Projective geometries need not have weak elimination of imaginaries.

Def. Let V be a K-vector space and A an affine V-space (in some structure $\mathcal{M}$). A K-affine map is a map $\lambda : A \longrightarrow K$ such that

$$\lambda\left(\sum{}' \alpha_i (a_i)\right) = \sum \alpha_i (\lambda(a_i))$$

for scalars $\alpha_i \in K$ with $\sum' \alpha_i = 1$. We write A^* for the set of $\mathcal{M}$-definable affine K-maps on A.

Theorem. Let J be a basic non-quadratic linear geometry and (J, A) an affine geometry. Then (J, A, A^*) has QE and weak EI.

(1u) 2.4. Normal Geometries

uniformly in all elementary ext.

Def. A **normal geometry** is a structure J with the following properties:

(i) $\mathrm{acl}(a) = \{a\}$ for all $a \in J$

(ii) Exchange: If $a \in \mathrm{acl}(Bc) \setminus \mathrm{acl}(B)$ then $c \in \mathrm{acl}(Ba)$.

(iii) Geometric elimination of imaginaries: For all $a \in J^{eq}$ we have that

$$a \in \mathrm{acl}^{eq}(\mathrm{acl}^{eq}(a) \cap J).$$

(iv) For any $\emptyset$-definable non-empty $J_0 \subseteq J$: if $a \equiv_{J_0} b$ then $a = b$.

It is called **reduced** if in addition:

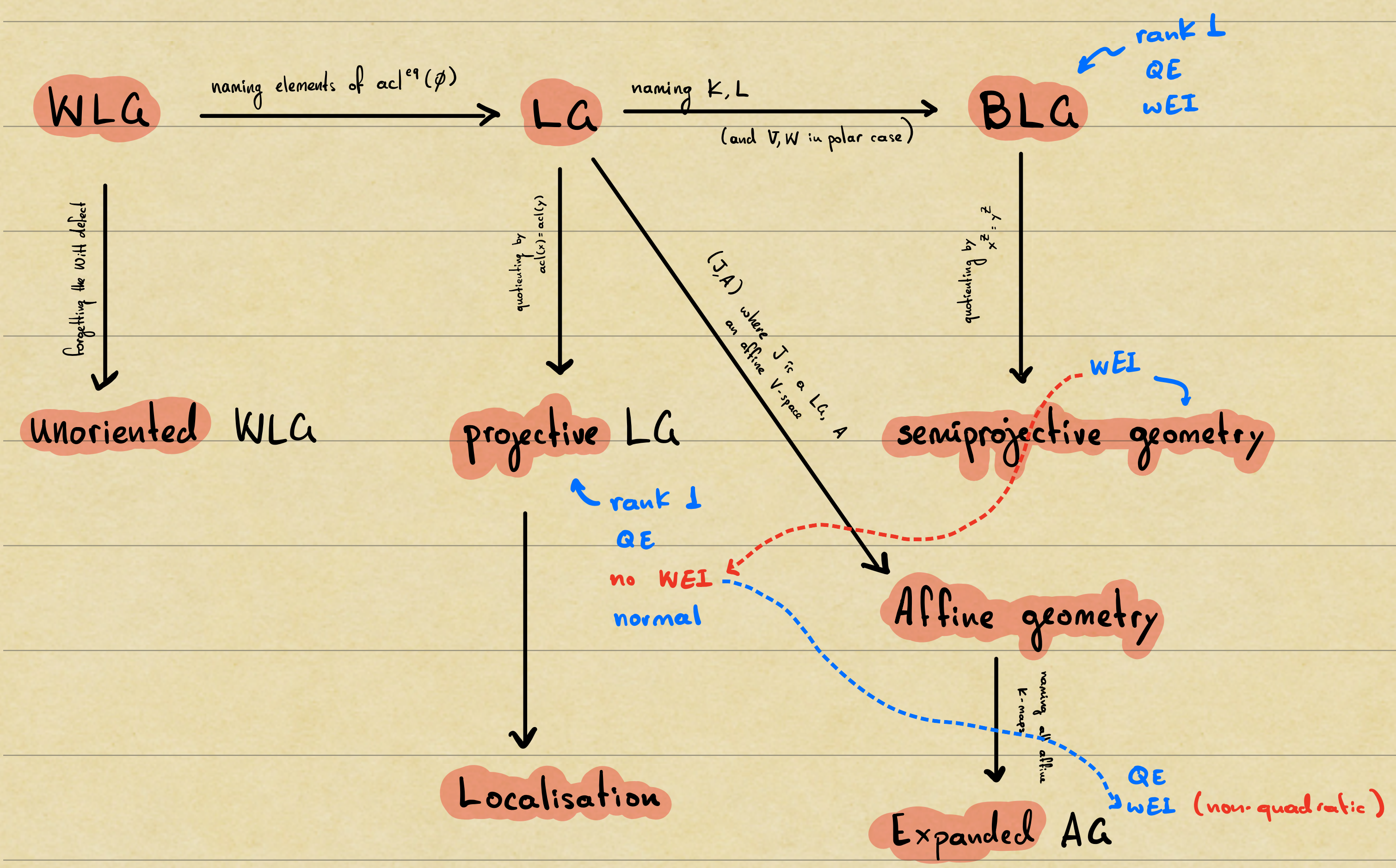
(v) $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$.

Theorem. Projective geometries are normal. Basic projective geometries are normal and reduced.

Def. Let P be a projective geometry living in some structure $\mathcal{M}$ and $A \subseteq \mathcal{M}$ a finite set. The localisation of P over A, P/A is the geometry obtained from the associated linear geometry L as follows:

- $L_A = \mathrm{acl}(A) \cap L$.
- $L_A^{\perp} = \{ v \in V : \forall x \in L_A \ \beta(x,v) = 0 \}$;
- $\mathrm{Proj}(L_A / L_A^{\perp})$.

Summary II



2.1.3. Coordinatisation

Def. Let $\mathcal{M} \subseteq \mathcal{N}$ be structures. Suppose that $\mathcal{M}$ is definable in $\mathcal{N}$ with canonical parameter $a \in \mathcal{N}^{eq}$ and let $A \subseteq N$. We say that:

(i) $\mathcal{M}$ is stably embedded in $\mathcal{N}$ over A if every N-definable (in the sense of $\mathcal{N}$) subset of $\mathcal{M}$ is uniformly Aa-definable (in the sense of $\mathcal{N}$).

(ii) $\mathcal{M}$ is canonically embedded in $\mathcal{N}$ over A if every aA-definable subset of $\mathcal{M}$ (in the sense of $\mathcal{N}$) is $\emptyset$-definable in the sense of $\mathcal{M}$

(iii) $\mathcal{M}$ is fully embedded in $\mathcal{N}$ over A if it is both stably and canonically embedded in $\mathcal{N}$ over A.

Remark. If $\mathcal{N}$ is $\aleph_0$-categorical then $\mathcal{M}$ is fully embedded in $\mathcal{N}$ (meaning over $\emptyset$) if, and only if:

$$\mathrm{Aut}(\mathcal{N}/a)\big|_{\mathcal{M}} = \mathrm{Aut}(\mathcal{M}),$$

where $a \in \mathcal{N}^{eq}$ is the canonical parameter of $\mathcal{M}$.

Def. A structure $\mathcal{M}$ is coordinatised by Lie geometries (Lie coordinatised) if it has a $\emptyset$-definable tree structure $<$ of finite height of finite height with a unique $\emptyset$-definable root such that:

1. <u>Coordinatisation</u>: For all $a \in \mathcal{M}$ above the root one of the following holds:

   (0) a is algebraic over its $<$-predecessor.

   <u>OR</u> there is $b < a$ and a b-definable projective geometry J_b fully embedded in $\mathcal{M}$ such that either:

   (i) $a \in J_b$; or

   (ii) there is $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) such that $a \in A_c$ and the projectivisation of J_c is J_b.

2. <u>Orientation</u>: If $a, b \in \mathcal{M}$ have the same type over $\emptyset$ and are associated with coordinatising geometries J_a and J_b then any definable map between J_a and J_b which preserves everything but ω also preserves ω.

If we ignore condition 2 then we say $\mathcal{M}$ is weakly Lie coordinatised.

over $\emptyset$

Def. A structure $\mathcal{M}$ is (weakly) Lie coordinatisable if it is bi-interpretable with a structure $\mathcal{M}$ which is (weakly) Lie coordinatised and has finitely many 1-types.

Back to RANKS

<u>Theorem.</u> If $\mathcal{M}$ is Lie coordinatisable then it has finite rank. ↜ at most the height of the tree.

Back to IMAGINARY ELEMENTS

<u>Lemma.</u> The following are equivalent for a $\emptyset$-definable $D \subseteq \mathcal{M}$:

1. D is stably embedded in $\mathcal{M}$ and admits weak EI.
2. For any $a \in \mathcal{M}^{eq}$, $tp(a/acl^{eq}(a) \cap D) \vdash tp(a/D)$.

Using this lemma together with weak EI for basic linear geometries and expanded affine geometries we may prove:

<u>Theorem.</u> A Lie coordinatisable structure is $\aleph_0$-categorical.

2.4. Orthogonality

Proposition. Let J_a and J_b be normal geometries which are a-definable and b-definable in a structure $\mathcal{M}$. Suppose that J_a and J_b are fully embedded in $\mathcal{M}$ over ab. Then one of the following occurs:

(i) J_a and J_b are <u>orthogonal</u>: Every ab-definable relation on $J_a \sqcup J_b$ is a Boolean combination of rectangles $R_a \times R_b$ where

notation $J_a \perp J_b$

- $R_a \subseteq J_a$ is $\mathrm{acl}(a)$-definable;
- $R_b \subseteq J_b$ is $\mathrm{acl}(b)$-definable.

(ii) J_a and J_b are ab-linked: There is an ab-definable bijection between them.

Lemma. Let J_1 and J_2 be basic linear geometries canonically embedded in a structure $\mathcal{M}$.

this lemma also makes sense over parameters

Suppose that in $\mathcal{M}$ there is a $\emptyset$-definable bijection

$$f : PJ_1 \longrightarrow PJ_2$$

between their projectivisations. Then, there is a $\emptyset$-definable bijection

$$\hat{f} : J_1 \longrightarrow J_2$$

which induces f and such that $\hat{f}$ is an isomorphism of weak geometries.

Lemma. Let $\mathcal{M}$ be a structure and P, Q basic projective geometries defined and fully embedded over $A \subseteq M$. Suppose that $P \perp_A Q$. Let $\hat{P}, \hat{Q}$ be localisations of P and Q respectively both definable over some set B. Then $\hat{P} \perp_B \hat{Q}$.

Lemma. Let P, Q_1, Q_2 be stably embedded in $\mathcal{M}$. If $P \perp_B Q_i$ then $P \perp_B Q_1 \cup Q_2$.

2.5. Canonical Projective Geometries