Definition 1. Let K be any field. We consider K-vector spaces in the language $\{+, (\lambda_k)_{k\in K}, 0\}$. Let $\mathrm{VEC}(K)$ denote the theory of infinite vector spaces over K.

EXERCISE 1. Show that $\mathrm{VEC}(K)$ is categorical in every infinite power $\kappa > |K|$ and deduce that $\mathrm{VEC}(K)$ is complete. Prove that $\mathrm{VEC}(K)$ has quantifier elimination.

In the exercise below you may use the following theorems:

Theorem 2 (Artin-Schreier). *Let $(F, <)$ be an ordered field (i.e. F is a field and $<$ is an order on the domain of F). The following are equivalent:*

1. *F is **real closed**, i.e. every positive element is a square;*
2. *$F(i)$ is algebraically closed (where $i = \sqrt{-1}$);*
3. *(intermediate value theorem) If $p(X) \in F[X]$ and $a, b \in F$ are such that $a < b$ and $p(a)p(b) < 0$, then there is $c \in F$ such that $a < c < b$ and $p(c) = 0$;*
4. *For any $a \in F$ either a or $-a$ is a square and every polynomial of odd degree has a root.*

Definition 3. We say that the ordered field $(R, <)$ is the **real closure** of the subfield $(K, <)$ if it is real closed and algebraic over K.

Theorem 4. *Every ordered field $(K, <)$ has a real closure and this is uniquely determined up to isomorphism over $(K, <)$.*

Definition 5. The theory of **real closed ordered fields** ROCF in the language of ordered fields consists of the axioms of the theory of fields and axioms expressing the intermediate value theorem for polynomials (Theorem 2.3).

EXERCISE 2. Show that ROCF has quantifier elimination.

Definition 6. We say that a type p for a theory T is **principal** if there is a formula $\phi(\overline{x}) \in p$ such that $T \models \exists \overline{x}\phi(\overline{x})$ and for all $\psi(\overline{x}) \in p$ we have

$$T \models \forall \overline{x}(\phi(\overline{x}) \to \psi(\overline{x})).$$

Note that principal types are realised in every model of T.
We call $\mathcal{M} \models T$ a **prime model of** T if it only realises the principal types of T.

Below, assume that T is countable.

EXERCISE 3. Show that if $\mathcal{M}$ and $\mathcal{N}$ are countable, prime and elementary equivalent then they are isomorphic.

EXERCISE 4. Suppose that $\mathcal{M}$ is a countable prime model of T. Show that $\mathcal{M}$ can be elementarily embedded into any model of T.