# Model Theory II

Winter Semester 2024

Paolo Marimon

January 31, 2025

## Contents

# 1 Why Model Theory II?

A major theme of model theory is that certain combinatorial properties of definable sets in a first order theory yield a lot of structural information about it. They can imply that a theory has a good notion of independence, or dimension, like in vector spaces, or help us understand the behaviour of groups or geometries interpretable in it. Sometimes, these properties can determine important algebraic features. This course will focus mainly on some of the strongest model theoretic properties: stability, and its strengthenings $\omega$-stability and superstability.

Modern model theory begins with the work of Morley [15], and subsequently Shelah [18] on the spectrum problem: what are the possible behaviours of the function $I(\aleph_\alpha, T)$, counting the number of non-isomorphic models of $T$ of cardinality $\aleph_\alpha$? Morley showed that, for countable $T$, if $I(\aleph_\alpha, T) = 1$ for some uncountable cardinal, then this is the case for all uncountable cardinals. Shelah studied for which theories we can define a system of invariants under which sufficiently large models of $T$ can be classified. In this process he defined various properties that shaped the development of model theory. Two of the major results of [18] are that $I(\aleph_\alpha, T)$ is non-decreasing for uncountable cardinals and the Main Gap Theorem: for $\alpha > 0$, either $T$ has the maximum number of models in each uncountable cardinality, $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$, or it satisfies few model theoretic properties, including superstability, and it is bounded above by $\beth_{\omega_1}(|\alpha|)$.

Over the years, a very sophisticated theory of stability [17] was developed, with fruitful generalisations to NIP [19], simple [10], and $NSOP_1$ theories [7]. A map of the universe with more model theoretic properties and many examples is given on the website [forkinganddividing.com](forkinganddividing.com). Model theory has fruitful applications and interactions in algebraic geometry [3], differential algebra [14], group theory [5, 1], number theory [12], and combinatorics [13]. Some classical notions from model theory have been independently discovered many times, such as VC-dimension in probability and combinatorics [11], and PAC learning and Littlestone dimension in machine learning [6]. My own research at TU Wien focuses on the computational complexity of problems in model theoretic structures [2], and on interactions between model theory and probability [4]. Overall, model theory is a highly versatile subject with many beautiful results, some of which we will cover in this course. I hope you will enjoy it!

# 2 The monster model

*This first lecture will require some additional knowledge of set theory and cardinal arithmetics. If you are not familiar with set theory, Appendix A in [20] should contain most relevant facts. One of the advantages of the construction of the monster model in this lecture is preventing us from keeping track of issues of cardinal arithmetics later in the course.*

Almost every article or book in model theory begins with the convention that we are working in a monster model $\mathbb{M}$. These are very large models of $T$ distinguished by being highly saturated, strongly homogeneous, and universal in the following sense:

**Definition 2.1.** Let $\kappa$ be an infinite cardinal. We say that $\mathcal{M} \models T$ is:

- **$\kappa$-saturated** if it realises types (in finitely many variables) over sets of parameters of cardinality $< \kappa$;

- **$\kappa$-universal** if every model of $T$ of cardinality $< \kappa$ elementarily embeds into $\mathcal{M}$;

- **$\kappa$-homogeneous** if for all $A \subseteq M$ of size $< \kappa$ and $a \in M$, every elementary map $f : A \to \mathcal{M}$ can be extended to an elementary map $A \cup \{a\} \to M$;

- **strongly $\kappa$-homogeneous** if for all $A \subseteq M$ of size $< \kappa$, any elementary map $f : A \to M$ can be extended to an automorphism of $\mathcal{M}$.

We say that $\mathcal{M}$ is **saturated** if it is $|M|$-saturated.

*Remark* 2.2. Recall that $\mathcal{M}$ is $\kappa$-saturated if and only if it is $\kappa$-saturated over 1-types, i.e. if it realises 1-types over sets of parameters of cardinality $< \kappa$;

In the following exercises we work with $|\mathcal{L}| \leq \kappa$:

**Exercise 2.3.** Prove that if $\mathcal{M}$ is $\kappa$-saturated then it is $\kappa$-homogeneous.

**Exercise 2.4.** Show that if $\mathcal{M}$ is $\kappa$-saturated, then it is $\kappa^+$-universal.

**Exercise 2.5.** (a) Show that if $\mathcal{M}$ is $|M|$-homogeneous, then it is strongly $|M|$-homogeneous. (b) For each cardinal $\kappa$, give an example of a $\kappa$-saturated structure $\mathcal{M}$ which is not strongly $\omega$-homogeneous.

**Exercise 2.6.** Show that $\mathcal{M}$ is $\kappa$-saturated if and only if it is $\kappa$-homogeneous and $\kappa^+$-universal.

Ideally, we would like to work with a large saturated model since these models are universal, homogeneous and strongly homogeneous. We will see that for this we will need additional set theoretic assumptions.

*Remark* 2.7. Let $(X, \leq)$ be a linear order. We say that $Y \subseteq X$ is **cofinal** with $X$ if for each $x \in X$ there is some $y \in Y$ with $x \leq y$. The **cofinality** of $X$, $\mathrm{cf}(X)$ is the smallest cardinality of a cofinal subset of $X$. We say that an infinite cardinal is $\kappa$ is **regular** if $\mathrm{cf}(\kappa) = \kappa$. Any successor cardinal $\kappa^+$ is a regular cardinal, and so is $\omega$.

**Theorem 2.8.** *Let $|\mathcal{L}| \leq \kappa$ and $\mathcal{M}$ be a model of cardinality $\leq 2^\kappa$. Then, $\mathcal{M}$ has an elementary extension $\mathcal{N}$ which is $\kappa^+$-saturated and of size $\leq 2^\kappa$.*

*Proof.* We build an elementary chain $(\mathcal{M}_\lambda)_{\lambda < \kappa^+}$ such that

- $|\mathcal{M}_\lambda| \leq 2^\kappa$ for each $\lambda < \kappa$;

- $\mathcal{M}_{\lambda^+}$ realises all types over subsets of $\mathcal{M}_\lambda$ of cardinality $\leq \kappa$;

Firstly, we show such chain exists and then we will show that its union, $\mathcal{N}$ satisfies the requirements of the theorem. We show how to perform the successor step. Since by inductive hypothesis $|\mathcal{M}_\lambda| \leq 2^\kappa$, $\mathcal{M}_\lambda$ has $2^\kappa$ subsets of size $\leq \kappa$. Since $|\mathcal{L}| \leq \kappa$, over each $B \subseteq \mathcal{M}_\lambda$ with $|B| \leq \kappa$, there are at most $2^\kappa$-many 1-types. In particular, there are at most $2^\kappa$-many 1-types over sets of size $\leq \kappa$. All of these can be realised in a model $\mathcal{M}_{\lambda^+}$ of cardinality $\leq 2^\kappa$.

We show that $\mathcal{N} := \bigcup_{\lambda < \kappa^+} \mathcal{M}_\lambda$ is $\kappa^+$-saturated and of size $\leq 2^\kappa$. For $\kappa^+$-saturation, consider $B \subseteq \mathcal{N}$ of size $< \kappa^+$. Since $\kappa^+$ is regular there must be some $\lambda \leq \kappa^+$ such that $B \subseteq \mathcal{M}_\lambda$ (otherwise, there would be a cofinal subset with $\kappa^+$ of cardinality $\leq |B| < \kappa^+$). Hence, all 1-types over $B$ are realised in $\mathcal{M}_{\lambda^+}$. Finally, for the cardinality,

$$|\mathcal{N}| \leq \bigcup_{\lambda < \kappa^+} 2^\kappa \leq 2^\kappa,$$

where the last inequality holds since we are taking a union of sets of size $\leq \alpha$ over ordinals $< \alpha$, where $\alpha$ is an infinite cardinal. $\square$

**Definition 2.9.** Let $\kappa$ be an infinite cardinal. A cardinal $\alpha$ is called a **strong limit cardinal** if for all cardinals $\beta < \alpha$, we have $2^\beta < \alpha$. A regular strong limit cardinal is called a **strongly inaccessible cardinal**.

*Remark* 2.10. It is easy to construct strong limit cardinals within ZFC. Moreover, the **global continuum hypothesis**, (GCH) implies that every limit cardinal is a strong limit cardinal. However, ZFC is consistent with there being no strongly inaccessible cardinals apart from $\omega$.

**Corollary 2.11.** *Let $|\mathcal{L}| \leq \kappa$ and $T$ be an $\mathcal{L}$-theory (with infinite models).*

*(a) Assuming* (GCH), *T has a saturated model in each regular cardinal $\nu > \kappa$;*

*(b) T has a saturated model in each strongly inaccessible cardinal $\nu > \kappa$.*

*Proof.* (Omitted from lecture) The ideas are essentially the same of the previous proof. (a) If $\nu$ is a successor cardinal the argument is immediate. For a limit cardinal, one can use an analogue of the argument below. (b) Starting from a model $\mathcal{M}_0$ of cardinality $\kappa$, using Theorem 2.8, we build an elementary chain $(\mathcal{M}_\lambda)_{\lambda < \nu}$, where $\mathcal{M}_{\lambda^+}$ is $\lambda^+$-saturated and of cardinality $\leq 2^\lambda$, and take $\mathcal{N}$ to be the union of this chain. Since $\nu$ is a strong limit cardinal,

$$|\mathcal{N}| \leq \bigcup_{\lambda < \nu} 2^\lambda \leq \nu.$$

Note that a $\beta$-saturated model must be of cardinality $\geq \beta$. Hence, $|\mathcal{N}| \geq \lambda^+$ for each $\lambda < \nu$, meaning that $|\mathcal{N}| = \nu$. Finally, we need to show saturation. Take $A \subset \mathcal{N}$ of cardinality $< \nu$. Since $\nu$ is regular, a set of cardinality $|A|$ cannot be cofinal with it, meaning that there is some $|A| \leq \lambda < \nu$ such that $\mathcal{M}_\lambda$ entirely contains $|A|$. Since $\mathcal{M}_\lambda$ is $\lambda^+$ saturated, it realised all types over $A$, and so does $\mathcal{N}$. $\qquad\square$

**Example 2.12.**
- $(\mathbb{C}; 0, 1; +, \cdot)$ is a saturated model of the theory of algebraically closed fields;

- In general, one can prove that stable theories have saturated models of arbitrarily large cardinalities;

- If the continuum hypothesis is false, the theory of $(\mathbb{N}; 0, 1; +, \cdot)$ has no saturated models of cardinality $\kappa$ for each $\aleph_0 < \kappa < 2^{\aleph_0}$.

**Convention 2.13.** From now on we will work with a **monster model** $\mathbb{M}$. which is $\kappa$-saturated, $\kappa$-universal and strongly $\kappa$-homogeneous for $\kappa$ a cardinal larger than all of the cardinalities of models and sets of parameters that we want to consider. Thus, all models $\mathcal{M}, \mathcal{N}, \dots$ we will consider will be elementarily embedded into this monster model, all sets of parameters $A, B, \dots$ will be subsets of the monster model of cardinality $< \kappa$, and a set of formulas will be consistent if it is realised in $\mathbb{M}$. Finally, for a formula $\phi$ or a type $p$, we write $\vDash \phi$ (or $\vDash p$) if $\mathbb{M} \vDash \phi$ (respectively, $\mathbb{M} \vDash p$).

*Remark* 2.14. There are several ways to achieve the above:

- Assume that strongly inaccessible cardinals exist and work in a sufficiently large one. *We will adopt this approach* since it allows us to move quickly to do more model theory;

- Work in BGC (Bernays-Gödel+Global Choice) set theory. This is a conservative extension of ZFC which allows working with classes. In this framework we can build the monster model as a class-size union of chains.

- Work with a special model (see Definition 2.16) of cardinality $\nu = \beth_\kappa(\aleph_0)$. This will be $\nu^+$-universal and strongly $\kappa$-homogeneous (add so $\kappa$-saturated by Exercise 2.6). This framework has the advantage of allowing us to work entirely within ZFC. If you are not comfortable with strongly inaccessible cardinals, you are welcome to read Subsection 2.1 and work with a large enough special model instead.

**Lemma 2.15.** *Let X be a definable subset of $\mathbb{M}$ and A a set of parameters (i.e. a set of size $< \kappa$ inside of $\mathbb{M}$). Then, the following are equivalent:*

*(a) X is definable over A;*

*(b) X is $\mathrm{Aut}(\mathbb{M}/A)$-invariant (i.e. invariant under automorphisms of $\mathbb{M}$ fixing A pointwise).*

*Proof.* ($\Rightarrow$) This direction works in every model $\mathcal{M}$. Suppose that $X := \phi(\mathcal{M}, b)$ for some $b \in A$. Then, for every $a \in \mathcal{M}$ and $\sigma \in \mathrm{Aut}(M/A)$, we have that

$$a \in X \Leftrightarrow \vDash \phi(a, b) \Leftrightarrow \phi(\sigma(a), \sigma(b)) \Leftrightarrow \phi(\sigma(a), b) \Leftrightarrow \sigma(a) \in X,$$

where the second last equivalence holds since $b \in A$ and $\sigma$ fixes $A$ pointwise.

($\Leftarrow$) Let $X = \phi(\mathbb{M}, b)$ and let $p(y) := \text{tp}(b/A)$.

**Claim 1:** $p(y) \vdash \forall x(\phi(x, y) \leftrightarrow \phi(x, b))$.

*Proof of Claim.* Take $b' \vdash p(y)$. By strong homogeneity there is some $\sigma \in \text{Aut}(\mathbb{M}/A)$ with $\sigma(b) = b'$. By assumption, $X = \sigma(X) = \phi(\mathbb{M}, b')$, yielding the desired formula is implied by $p(y)$. $\qquad\square$

By compactness, there is some $\psi(y) \in p(y)$ such that

$$\psi(y) \vDash \forall x(\phi(x, y) \leftrightarrow \phi(x, b)) \tag{1}$$

Take $\theta(x) := \exists y(\psi(y) \wedge \phi(x, y))$. This is an $\mathcal{L}_A$-formula. We claim $X = \theta(\mathbb{M})$. For ($\subseteq$) take $a \in X$. So $\vdash \phi(a, b)$. Since $\psi(y) \in \text{tp}(b/A)$, $\vDash \theta(a)$. For ($\supseteq$), if $\vDash \theta(a)$ there is some $b'$ such that $\vDash \psi(b') \wedge \phi(a, b')$. By $\vDash \psi(b')$ (1), we have $\vDash \phi(a, b)$, as desired. $\qquad\square$

## 2.1 Aside: special models

An issue with our definition of monster model being a saturated model of size a strongly inaccessible cardinal is that it makes it less transparent that our results are provable in ZFC. A more cautious reader might want to work with special models. An even more set theoretically oriented reader, might be interested in the approach of [8], which partially justifies the standard model theoretic practice of assuming we are working with a saturated model of large enough cardinality.

**Definition 2.16.** An infinite structure $\mathcal{M}$ of cardinality $\kappa$ is **special** if it is the union of an elementary chain $(\mathcal{M}_\lambda)_{\lambda < \kappa}$, where the $\lambda$ are *cardinals* of size $< \kappa$ and each $\mathcal{M}_\lambda$ is $\kappa^+$-saturated.

$\star\star$ **Exercise 2.17.** Let $|\mathcal{L}| \leq \kappa$. Show that the following hold:

(a) If $\mathcal{M}$ is saturated then it is special;

(b) A special structure of regular cardinality is saturated;

(c) Suppose that $\lambda < \nu$ implies $2^\lambda \leq \nu$. Then, $T$ has a special model of cardinality $\nu$;

(d) A special structure of cardinality $\kappa$ is $\kappa^+$-universal and strongly $\text{cf}(\kappa)$-homogeneous.

**Definition 2.18.** For every cardinal $\mu$, the **beth function** is defined as

$$\beth_\alpha(\mu) = \begin{cases} \mu & \text{if } \alpha = 0, \\ 2^{\beth_\beta(\mu)}, & \text{if } \alpha = \beta + 1, \\ \sup_{\beta < \alpha} \beth_\beta(\mu) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

**Fact 2.19.** *The cofinality of $\beth_\alpha(\mu)$ is $\text{cf}(\alpha)$. Moreover, for $\alpha$ a limit ordinal, $\beth_\alpha(\mu)$ is a strong limit cardinal.*

*Proof.* The first fact follows from a more general principle: let $f : \alpha \to \text{Ord}$ be a strictly increasing function which is continuous in the sense that for limit ordinals $\gamma$, $f(\gamma) = \bigcup_{\beta < \gamma} f(\beta)$. Then, $\text{cf}(f(\alpha)) = \text{cf}(\alpha)$. To see this, suppose that $(\beta_\gamma | \gamma < \text{cf}(\alpha))$ is an increasing sequence of ordinals cofinal with $\alpha$. Then, as $f(\beta_\gamma) < f(\alpha)$, we have that $(f(\beta_\gamma) | \gamma < \text{cf}(\alpha))$ is an increasing sequence of ordinals cofinal with $f(\alpha)$. So $\text{cf}(f(\alpha)) \leq \text{cf}(\alpha)$. We have equality as cf is non-decreasing.

The second statement just follows from definitions.

$\qquad\square$

*Remark* 2.20. As $\text{cf}(\beth_\kappa(\aleph_0)) = \kappa$, a special model of cardinality $\nu = \beth_\kappa(\aleph_0)$ is strongly $\kappa$-homogeneous, $\nu^+$-universal, and $\kappa$-saturated. There is no harm in working with a special model of such cardinality as the monster model (except from having to prove exercise 2.17).

# 3   Strong minimality and algebracity

*From now on it will be important to keep in mind the conventions that we set in the previous lecture (Convention 2.13). In particular, models are always taken to be elementary substructures of the monster model $\mathbb{M}$ and parameter sets $A, B, \ldots$ are always taken to be small enough and live in the monster model (which is why I don't specify every time where they come from).*

**Definition 3.1.** We say that a formula $\phi(x) \in \mathcal{L}(A)$ is **algebraic** (over $A$) if $\phi(\mathbb{M})$ is finite. $a \in \mathbb{M}$ is **algebraic** over $A$ if it realises an algebraic formula over $A$.
We denote by $\mathrm{acl}(A)$ the set of elements algebraic over $A$. For $\pi$ a partial type over $A$ (closed under conjunctions), we say that it is **algebraic** if it contains an algebraic formula.

❧ *Observation* 3.2. Note if a formula $\phi(x) \in \mathcal{L}(A)$ is **algebraic**, then it has the same set of realisations in every model containing $A$.

**Exercise 3.3.** Prove Neumann's Lemma: Let $A, B \subseteq \mathbb{M}$ and $(c_1, \ldots, c_n)$ a sequence of elements not algebraic over $A$. Show that $\mathrm{tp}(c_1, \ldots, c_n / A)$ has a realisation which is disjoint from $B$.

**Exercise 3.4.** Show that $\mathrm{acl}(A)$ is the intersection of all models containing $A$.

**Definition 3.5.** Let $\mathcal{M}$ be a model. Let $\phi(x) \in \mathcal{L}(M)$ be a non-algebraic formula. We say that $\phi$ is **minimal** in $\mathcal{M}$ if for all $\mathcal{L}(M)$-formulas $\psi(x)$,

$$\phi(M) \wedge \psi(M) \text{ is finite or cofinite in } \phi(M).$$

We say that $\phi(x) \in \mathcal{L}(M)$ is **strongly minimal** if it is minimal in the monster model $\mathbb{M}$. A theory $T$ is **strongly minimal** if $x = x$ is strongly minimal. A type $p \in S(A)$ is **strongly minimal** if it contains a strongly minimal formula.

**Examples 3.6** (Strongly minimal theories). It is easy to prove strong minimality of the following theories by quantifier eliminations:

- The theory $T_\infty$ of an infinite set with equality;

- The theory of infinite vector spaces over a field $K$, $(V; 0; +; (\lambda_k)_{k \in K})$;

- $\mathrm{ACF}_p$, the theory of algebraically closed fields in characteristic $p$.

**Example 3.7** (A minimal set which is not strongly minimal). Consider the structure $\mathcal{M}$ with an equivalence relation $E$ that has countably many equivalence classes, one of each finite size and no infinite classes. Note that each equivalence class is a definable subsets of $\mathcal{M}$ (using quantifiers). One can then show that adding predicates $P_n$ for each equivalence class to the language, the (new) theory of $\mathcal{M}$ has quantifier elimination (for example, by [20, Theorem 3.2.5]). From this it is easy to see that every definable subset of $\mathcal{M}$ is either finite or cofinite. However, $\mathbb{M} \succ \mathcal{M}$ has an infinite class (by $\omega$-saturation). So for $a$ in this class, $E(x, a)$ is infinite and coinfinite. Note that the fact $\mathcal{M}$ is not $\omega$-saturated plays an important role here (see Exercise 3.9 below).

**Non-examples 3.8.**   • The theory of two infinite predicates partitioning the domain is not strongly minimal. However, each predicate is;

- The theory of the random graph has no strongly minimal formula.

**Exercise 3.9.** Prove the following: let $\mathcal{M}$ be $\omega$-saturated. Suppose that $\phi \in \mathcal{L}(M)$ is minimal in $\mathcal{M}$. Then $\phi$ is strongly minimal.

**Exercise 3.10.**   (a) Consider the theory of $(\mathbb{Z}, s)$, the integers with the successor operation $s(x) = x + 1$. This theory has quantifier elimination. What is algebraic closure in this theory? Is this $x = x$ in $(\mathbb{Z}, s)$ minimal? is it strongly minimal?

(b) Consider the theory of $(\mathbb{N}, <)$. This theory has quantifier elimination if we add a function symbol for the successor and a constant symbol for 0 (both of which are definable in the original theory). Is $x = x$ in $(\mathbb{N}, <)$ minimal? is it strongly minimal?

The idea of the following lemma is that algebraic sets are very small (being finite), so it is possible to extend non-algebraic types to larger parameter sets whilst avoiding algebraic sets (over those parameters):

**Lemma 3.11** (Extension). *Let $\pi(x)$ be a partial type (closed under conjunctions) non-algebraic over $A$. Let $A \subseteq B$. Then, $\pi$ has a non-algebraic extension $a \in S(B)$.*

*Proof.* Consider

$$q_0(x) := \pi(x) \cup \{\neg\psi(x) | \psi(x) \in \mathcal{L}(B) \text{ is algebraic }\}.$$

We prove this is finitely satisfiable. Take $\phi(x) \in \pi(x)$ (note $\pi$ is closed under conjunctions) and $\psi_1(x), \ldots, \psi_n(x)$ algebraic. Then, since $\phi(\mathbb{M})$ is infinite and for each $i$ $\neg\psi_i(\mathbb{M})$ is cofinite,

$$\phi(x) \wedge \bigwedge_{i \leq n} \neg\psi_i(x)$$

has infinitely many realisations. This proves finite satisfiability, and by compacness satisfiability of $q_0$. Finally, take any completion $q \in S(B)$ of $q_0$. This will still be non-algebraic by construction of $q_0$, completing the proof. $\square$

One can actually prove a more general statement, where the "small" sets one is avoiding are charaterised from belonging to an ideal in the Boolean algebra of definable sets. This will be very important later.

**Definition 3.12.** A set of definable subsets of $\mathbb{M}$ in the variable $x$, $I \subseteq \text{Def}_x(\mathbb{M})$ is an **ideal** if it contains $\varnothing$, and it is closed under (definable) subsets and finite unions.

**Exercise 3.13.** Prove the following:
Let $I \subseteq \text{Def}_x(\mathbb{M})$ be an ideal. Let $\pi(x)$ be a partial type over $A$ (closed under conjunctions) such that $\pi(\mathbb{M})$ is not contained in any set in $I$. Then, for every $B \supseteq A$, there is a type $q \in S(B)$ extending $\pi$ and such that $q(\mathbb{M})$ is not contained in any set in $I$.

**Lemma 3.14.** *The $\mathcal{L}(M)$-formula $\phi(x)$ is minimal in $\mathcal{M}$ if and only if there is a unique non-algebraic type $p \in S(M)$ containing $\phi(x)$.*

*Proof.* ($\Rightarrow$) Assume $\phi$ is minimal in $\mathcal{M}$. Being non-algebraic, by extension (Lemma 3.11), it has a non-algebraic extension $p \in S(M)$. Note that if $\psi(x) \in p$, then $\phi(x) \wedge \psi(x)$ is infinite, and so by minimality of $\phi$, $\phi(x) \wedge \neg\psi(x)$ is finite. So any type containing $\phi$ and $\neg\psi$ is algebraic. This implies that $p$ is the unique non-algebraic type containing $\phi$.

($\Leftarrow$) By contrapositive. Suppose $\phi(x)$ is not minimal. If it is algebraic, then it cannot be contained in a non-algebraic type. So it is non-algebraic and by non-minimality there is some $\mathcal{L}(M)$-formula $\psi$ with both $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ non-algebraic. Hence, by extension (Lemma 3.11), each formula extends to a non-algebraic type in $S(M)$ containing $\phi$. Since the two types are clearly distinct (as one contains $\psi$ and the other $\neg\psi$), this completes the proof of the contrapositive. $\square$

**Corollary 3.15** (Stationarity). *Let $p \in S(A)$ be strongly minimal. Then*

(a) *$p$ has a unique non-algebraic extension to all $B \supseteq A$;*

(b) *If $a_1^0, \ldots, a_m^0$ and $a_1^1, \ldots, a_m^1$ are two sequences of realisations of $p$ of length $m$ which are algebraically independent in the sense that*

$$a_i^j \notin \text{acl}(Aa_1^j, \ldots a_{i-1}^j)$$

7

*for each $i \leq m$ and $j \in \{0, 1\}$. Then,*

$$a_1^0, \ldots, a_m^0 \equiv_A a_1^1, \ldots, a_m^1.$$

*So, the type over $A$ of an algebraically independent tuple of realisations of $p$ is entirely determined.*

*Proof.* (a) From Lemma 3.14, $p$ has a unique non-algebraic extension to $\mathbb{M}$, and so also to any set of parameters containing $A$.

(b) By induction. The base case is trivial. Suppose that $\bar{a}^0 \equiv_A \bar{a}^1$ for algebraically independent $m$-tuples of realisations of $p$. Let $a_{m+1}^0 \notin \text{acl}(A\bar{a}^0)$ and $a_{m+1}^1 \notin \text{acl}(A\bar{a}^1)$ be realisatinos of $p$. Take $\sigma \in \text{Aut}(\mathbb{M}/A)$ such that $\sigma(\bar{a}^0) = \bar{a}^1$. Since automorphisms preserve algebraicity, $\sigma(a_{m+1}^0)$ is non-algebraic over $A\bar{a}^1$. By Lemma 3.14, $\sigma(a_{m+1}^0) \equiv_{A\bar{a}^1} a_{m+1}^1$. So there is $\tau \in \text{Aut}(\mathbb{M}/A\bar{a}^1)$ such that $\tau\sigma(a_{m+1}^0) = a_{m+1}^1$. Since the composition of the two automorphisms fixes $A$, $\bar{a}^0 a_{m+1}^0 \equiv_A \bar{a}^1 a_{m+1}^1$, as desired. $\square$

# 4 Pregeometries

In this lecture we are going to use the results from the previous lecture to prove that algebraic independence behaves particularly well in strongly minimal sets (and theories). In particular, we will see that algebraic closure inside a strongly minimal set gives rise to a pregeometry: a structure whose behaviour of algebraic independence satisfies the axioms of linear independence, allowing us to talk about bases and dimensions.

The notion of a pregeometry (also known as matroid) originates from the work of Whitney [22] and Van de Waerden [21], both of whom gave axioms for linear independence in vector spaces. In particular, Whitney's work stemmed from applying notions from linear algebra to combinatorics after noticing various similarities between certain ideas of independence and ranks in graph theory and the behaviour of linear independence. Nowadays matroid theory is a branch of mathematics with several applications in combinatorics [16]. Our interests differ from standard matroid theory because we study infinite pregeometries, but we will make use of some basic facts about pregeometries in this section.

**Definition 4.1.** A **pregeometry** $(X, \text{cl})$ consists of a set $X$ with a closure operator

$$\text{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$$

such that for all $A \subseteq X$ and $a, b \in X$:

- (REFLEXIVITY) $A \subseteq \text{cl}(A)$;

- (FINITE CHARACTER) $\text{cl}(A) = \bigcup\{\text{cl}(A') | A' \subseteq A \text{ finite }\}$;

- (TRANSITIVITY) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$;

- (EXCHANGE) if $a \in \text{cl}(Ab) \setminus \text{cl}(A)$, then $b \in \text{cl}(Aa)$.

*Remark* 4.2. For any structure $\mathcal{M}$, $(\mathcal{M}, \text{acl})$ satisfies reflexivity, finite character, and transitivity.

**Theorem 4.3.** *Let $\phi$ be a strongly minimal $\mathcal{L}$-formula. Let $\text{cl} : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M})$ be defined by, for $A \subseteq \phi(\mathbb{M})$, $\text{cl}(A) := \text{acl}(A) \cap \phi(\mathbb{M})$. Then, $(\phi(\mathbb{M}), \text{cl})$ is a pregeometry.*

*Proof.* Reflexivity, finite character and transitivity are trivial. We only need to verify exchange. Without loss of generality (and to simplify notation), we assume that $A = \emptyset$. All elements we work with are inside of $\phi(\mathbb{M})$. Let $a \notin \text{acl}(\emptyset)$ and $b \notin \text{acl}(a)$. We need to prove $a \notin \text{acl}(b)$.

Firstly, note that $\phi(x)$ extends to a unique non-algebraic type $q(x)$ over $\emptyset$ (Lemma 3.14). By stationarity (Corollary 3.15 (b)), all pairs $a'b'$ satisfying $a' \notin \text{acl}(\emptyset)$ and $b' \notin \text{acl}(a')$ have

the same type $p(x,y)$.

Now, take $(a_i | i < \omega)$ an infinite sequence of realisations of $q(x)$ such that

$$a_i \notin \mathrm{acl}(a_0 \ldots a_{i-1}).$$

This can be done by induction iterating non-algebraic extensions (by Lemma 3.11). Using extension again, pick $b' \notin \mathrm{acl}((a_i | i < \omega))$ realising $q(x)$. Since $a_i \notin \mathrm{acl}(\varnothing)$ and $b' \notin \mathrm{acl}(a_i)$ for each $i < \omega$, we have $a_i b' \equiv ab$ for all $i < \omega$. So, $a_i \notin \mathrm{acl}(b')$, since $\mathrm{tp}(a_i/b') = p(x, b')$ has infinitely many realisations. But then, since $b \equiv b'$, $p(x, b)$ also has infinitely many realisations. So $a \notin \mathrm{acl}(b)$ as desired. $\qquad\square$

❧ *Observation* 4.4. The above proof actually works in any model: we may need to move outside of a given model $M$ to realise the $a_i$. However, the conclusion that $p(x, b)$ is non-algebraic, does tell us that $a \notin \mathrm{acl}(b)$ also in $M$.

**Definition 4.5.** Let $(X, \mathrm{cl})$ be a pregeometry. For $A \subseteq X$, we say that:

- $A$ is **independent** if $a \notin \mathrm{cl}(A \setminus \{a\})$ for each $a \in A$;

- $A$ is a **generating set** if $\mathrm{cl}(A) = X$;

- $A$ is a **basis** if it is an independent generating set for $X$.

**Fact 4.6.** *(a) Every pregeometry has a basis [to prove this you need the axiom of choice];*

*(b) Any two bases for a pregeometry have the same cardinality.*

**Definition 4.7.** For a pregeometry $(X, \mathrm{cl})$, we say that the **dimension** of $X$, $\dim(X)$ is the cardinality of a basis for $X$.

**Definition 4.8.** Given a pregeometry $(X, \mathrm{cl})$, for $S \subseteq X$, let

- $(S, \mathrm{cl})$ given by $\mathrm{cl}(A) = \mathrm{cl}(A) \cap S$ for all $A \subseteq S$ be the **restriction** of $(X, \mathrm{cl})$ to $S$;

- $(X, \mathrm{cl}_S)$ given by $\mathrm{cl}_S(A) = \mathrm{cl}(A \cup S)$ for all $A \subseteq S$ be the **relativisation** of $(X, \mathrm{cl})$ by $S$;

We write $\dim(S)$ for $\dim((S, \mathrm{cl}))$, and $\dim(X/S)$ for $\dim((X, \mathrm{cl}_S))$. It is easy to show both of these are also pregeometries.

Note that thinking with the restriction and relativisation allows us to speak of bases for subspaces of $X$, or of independence over some subset of $S \subseteq X$.

❧ *Observation* 4.9. Note that for strongly minimal $\phi$, the restriction $(\phi(M), \mathrm{cl})$ is well defined since $\phi(M) \subseteq \phi(\mathbb{M})$. Meanwhile, its relativisation by $A \subseteq M$, $(\phi(M), \mathrm{cl}_A)$ corresponds to the natural pregeometry on $(\phi(M_A), \mathrm{cl})$, where $M_A$ is the expansion of $M$ by constants naming the elements of $A$. We write $\dim_\phi(M)$ for the dimension $(\phi(M), \mathrm{cl})$, and $\dim_\phi(M/A)$ for the dimension of $(\phi(M_A), \mathrm{cl})$.

*Remark* 4.10. For a pregeometry $(X, \mathrm{cl})$ and $S \subseteq X$, we have

$$\dim(X) = \dim(S) + \dim(X/S).$$

**Exercise 4.11.** Let $f : A \to B$ be an elementary bijection between sets of parameters. Then, $f$ extends to an elementary bijection $f' : \mathrm{acl}(A) \to \mathrm{acl}(B)$.

**Lemma 4.12.** *Let $\phi \in \mathcal{L}(A)$ be strongly minimal. Let $A \subseteq \mathcal{M}, \mathcal{N} \models T$. Then, the following are equivalent:*

1. *there is an $A$-elementary bijection $f : \phi(M) \to \phi(N)$;*

2. *$\dim_\phi(M/A) = \dim_\phi(N/A)$.*

*Proof.* Without loss of generality we work over $\varnothing$ (we can always just work in $\text{Th}(\mathbb{M}_A)$).
($\Rightarrow$) We know there is an elementary bijection $f : \phi(M) \to \phi(N)$. Note that elementary bijections map bases to bases (since they preserve algebraic relations). Hence, $\dim_\phi(M) = \dim_\phi(N)$.
($\Leftarrow$) Take bases $U$ and $V$ for $\phi(M)$ and $\phi(N)$. Let $f : U \to V$ be a bijection. By independence of the bases and stationarity (Corollary 3.15 (b)), there is an elementary bijection between $U$ and $V$. Elementary bijection extend to algebraic closures (as noted in Exercise 4.11). So there is an elementary bijection $f' : \text{acl}(U) \to \text{acl}(V)$. Now $f'|_{\phi(M)}$ is an elementary bijection from $\phi(M)$ to $\phi(N)$. $\qquad\square$

👁 *Observation* 4.13. For any set of parameters $A$,

$$|\text{acl}(A)| \leq \max(|\mathcal{L}|, |A|),$$

where $|\mathcal{L}|$ is the size of the set of $\mathcal{L}$-formulas.

**Corollary 4.14.** *Let $T$ be a countable and strongly minimal theory. Then, it is categorical in all uncountable cardinals.*

*Proof.* Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ have cardinality $\kappa > \aleph_0$. Choose bases $B_1, B_2$ respectively. By Observation 4.13, for each $i \in \{1, 2\}$:

$$\kappa = |M_i| = \text{acl}(B_i) \leq \max(|\mathcal{L}|, |B_i|) = \max(\aleph_0, |B_i|) = |B_i|.$$

So $\dim(M_1) = \dim(M_2)$. So there is an elementary bijection $f : M_1 \to M_2$ by Lemma 4.12. $\qquad\square$

**Exercise 4.15.** Let $T$ be a strongly minimal theory (not necessarily countable). Show the following:

(a) Every infinite algebraically closed set of parameters $S$ is the universe of a model of $T$;

(b) A model $\mathcal{M}$ is $\omega$-saturated if and only if $\dim(M) \geq \aleph_0$;

(c) All models are $\omega$-homogeneous.

# 5 $\omega$-stability and the downwards Morley theorem

**Definition 5.1.** We say that $T$ is $\omega$-stable if for any $n \in \mathbb{N}$ and any set of parameters $A$ such that $|A| \leq \aleph_0$, $|S_n(A)| \leq \aleph_0$.

*Remark* 5.2. It is easy to prove that $T$ is $\omega$-stable if and only if for any set of parameters $A$ such that $|A| \leq \aleph_0$, $|S_1(A)| \leq \aleph_0$. We will generally use this characterisation of $\omega$-stability.

**Examples 5.3.** • if $T$ is strongly minimal, then it is $\omega$-stable. To see this note that if $|A| \leq \aleph_0$, there are only $\leq \aleph_0$ many algebraic types over $A$ (since $\mathcal{L}$ is countable) and there is a unique non-algebraic type over $A$, meaning that $|S_1(A)| \leq \aleph_0$;

• if $T$ is $\kappa$-categorical for $\kappa > \aleph_0$, then it is $\omega$-stable. This was proven in the previous model theory course and it is a strictly more general fact than the previous one;

• the theory of an infinitely branching infinite tree is $\omega$-stable (but not $\kappa$-categorical in any infinite $\kappa$).

**Definition 5.4.** We say that $T$ is **totally transcendental** if there is no binary tree of (consistent) $\mathcal{L}(\mathbb{M})$-formulas $(\phi_s(x) | s \in {}^{<\omega}2)$ such that

• $\vdash \forall x \neg(\phi_{s0}(x) \wedge \psi_{s1}(x))$;

• $\vdash \forall x((\phi_{s0}(x) \vee \psi_{s1}(x)) \to \phi_s(x))$.

That is, we ask that any two children of a common note are mutually inconsistent, but their union as a pair of definable sets is contained in the set defined by their parent.

**Lemma 5.5.** *A theory $T$ is $\omega$-stable if and only if it is totally transcendental.*

**Definition 5.6.** Let $A$ be a set of parameters and $x$ a tuple of variables. For an $\mathcal{L}(A)$-formula $\phi(x)$, we define
$$[\phi(x)] := \{p \in S_x(A) | \phi(x) \in p\}.$$
Sets of the form $[\phi(x)]$ form a basis of clopen sets for a topology on $S_x(A)$, which we call the **Stone topology**.
We say that a type $p \in S_x(A)$ is **isolated** if there is some $\mathcal{L}(A)$-formula $\psi(x)$ such that $[\psi(x)] = \{p\}$.

**Fact 5.7.** *The type space $S_x(A)$ with the Stone topology is compact, Hausdorff, and totally disconnected (i.e. for all $p, q \in S_x(A)$ there is a clopen set $X$ such that $p \in X$ and $q \notin X$.*

**Exercise 5.8.** Let $T$ be a countable complete theory. Let $A$ be a countable set of parameters and $x$ a finite tuple of variables. Suppose that $|S_x(A)| < 2^{\aleph_0}$. Prove the following:

- the isolated types in $S_n(A)$ are dense, i.e. for any $\mathcal{L}(A)$-formula $\phi(x)$, $[\phi]$ contains an isolated type;

- $|S_x(A)| \leq \aleph_0$.

[Hint: in both contexts, you need to build an adequate binary tree of $\mathcal{L}(A)$-formulas $(\phi_\sigma | \sigma \in 2^{<\omega})$ such that any finite branch is consistent but any two children of a common node are mutually inconsistent. Then, each infinite branch of the binary tree can be used to construct a type, giving $2^{\aleph_0}$-many.]

**Definition 5.9.** Let $A \subseteq \mathcal{M} \models T$. We say that $\mathcal{M}$ is **prime over** $A$ if for all $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ a partial elementary map, $f$ extends to an elementary $f' : \mathcal{M} \to \mathcal{N}$.

We say that $\mathcal{M}$ is **prime**, if it is prime over $\emptyset$.

**Example 5.10.** The theory of algebraically closed fields of characteristic zero $\mathrm{ACF}_0$ has $\overline{\mathbb{Q}}^{\mathrm{alg}}$, the algebraic closure of $(\mathbb{Q}; 0, 1; +, -, \times)$ as a prime model. In fact, $\overline{\mathbb{Q}}^{\mathrm{alg}}$ is a model of $\mathrm{ACF}_0$ and it embeds in every model of the theory. By model completeness of $\mathrm{ACF}_0$, such an embedding is elementary, yielding that $\overline{\mathbb{Q}}^{\mathrm{alg}}$ is prime. Indeed, we can prove that countable $\omega$-stable theories always have a prime model.

**Exercise 5.11.** Show the following: Let $T$ be a countable $\omega$-stable theory, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. Then, there is $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime model over $A$ and such that every $a \in M$ realises an isolated type over $A$.

**Theorem 5.12** (Lachlan). *Let $T$ be $\omega$-stable, $\mathcal{M} \models T$, $|M| \geq \aleph_1$. Then, for each $\kappa > |M|$ there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality $\kappa$ such that for any countable set of $\mathcal{L}(M)$-formulas $\Gamma(x)$ in a finite variable $x$, if $\mathcal{N}$ realises $\Gamma(x)$, then so does $\mathcal{M}$.*

**Exercise 5.13.** We shall prove Theorem 5.12 following the steps below. Consider an $\omega$-stable theory $T$ and $\mathcal{M} \models T$, such that $|M| \geq \aleph_1$. Say that an $\mathcal{L}(M)$-formula is **large** if $\phi(M)$ is uncountable.

- Prove that there is a large $\mathcal{L}(M)$-formula $\phi_0(x)$ such that for any other $\mathcal{L}(M)$-formula $\psi$, either $\phi_0(x) \wedge \psi(x)$ or $\phi_0(x) \wedge \neg\psi(x)$ has a countable set of realisations.

- Consider
$$p(x) := \{\psi(x) | \psi(x) \in \mathcal{L}(M) \text{ and } \phi_0(x) \wedge \psi(x) \text{ is large }\}.$$

  Show that $p$ is a complete type over $M$ which is not realised in $M$ but such that all of its countable subsets are realised in $M$. Take $\mathcal{N}' \succeq \mathcal{M}$ with a point $a$ realising $p$.

11

- By Exercise 5.11, take $\mathcal{N} \preceq \mathcal{N}'$ prime over $Ma$ and such that every $b \in \mathcal{N}$ realises an isolated type over $Ma$. Show that for every $b \in N$, every countable subset $\Gamma(x)$ of $\mathrm{tp}(b/M)$ is realised in $M$.

- Deduce Theorem 5.12.

*Remark* 5.14. Recall the two following facts from the model theory I course:

- any two saturated models of the same cardinality are isomorphic;

- if $T$ is $\kappa$-categorical, then all of its models of cardinality $\kappa$ are saturated.

**Theorem 5.15** (Downwards Morley Theorem). *Let $T$ be countable and $\kappa$-categorical in some uncountable $\kappa$. Then $T$ is $\aleph_1$-categorical.*

*Proof.* Suppose by contradiction that $T$ is $\kappa$-categorical (so $\omega$-stable) and not $\aleph_1$-categorical. Then, it has a non-saturated model $\mathcal{M}$ of cardinality $\aleph_1$. So there is some $p \in S_1(A)$ for $A \subseteq M$ countable which is not realised in $M$. By Theorem 5.12 and $\omega$-stability, there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality $\kappa$ and not realising $p$. But if $T$ is $\kappa$-categorical, all models of cardinality $\kappa$ are saturated and $\mathcal{N}$ is not saturated. Contradiction. $\qquad\square$

# 6 Vaughtian pairs

**Definition 6.1.** We say that $T$ has a **Vaughtian pair** if there are $\mathcal{M} \preceq \mathcal{N} \models T$ such that $M \subsetneq N$ and $\phi \in \mathcal{L}(M)$ non-algebraic such that $\phi(M) = \phi(N)$.

We will often write a Vaughtian pair as $(\mathcal{N}, \mathcal{M})$ since it is convenient to think about this as the expansion of $\mathcal{N}$ by a predicate $P$ naming the smaller model $\mathcal{M}$.

**Exercise 6.2.** Show that the theory of the random graph has a Vaughtian pair.

$\star\star$ **Exercise 6.3.** Show that there is no Vaughtian pair of real closed fields.

**Lemma 6.4.** *Suppose that $T$ has a Vaughtian pair. Then, $T$ has a Vaughtian pair $(\mathcal{N}, \mathcal{M})$ with $\mathcal{N}$ and $\mathcal{M}$ countable.*

*Proof.* This proof is essentially an application of the downwards Lowenheim-Skolem theorem. Let $(\mathcal{N}^\star, \mathcal{M}^\star)$ be a Vaughtian pair for $T$ as witnessed by the formula $\phi(x) \in \mathcal{L}(A)$ for $A \subseteq M^\star$ finite. Consider $(\mathcal{N}^\star, \mathcal{M}^\star)$ as $\mathcal{N}^\star$ expanded by a predicate $P$ naming $\mathcal{M}^\star$. Then, by the downwards Lowenheim-Skolem theorem, there is $(\mathcal{N}, P(N)) \preceq (\mathcal{N}^\star, \mathcal{M}^\star)$ countable and containing $A$, and so, in particular, $A \subseteq P(N)$.

It is easy to verify that $P(N) \preceq \mathcal{N}$ by the Tarski-Vaught test. Moreover, $\phi(P(N))$ is infinite and such that $\phi(P(N)) = \phi(N)$, $P(N) \subsetneq N$, since all of this is coded by the theory of $(\mathcal{N}^\star, \mathcal{M}^\star)$. Hence, $(\mathcal{N}, P(N))$ is a Vaughtian pair. $\qquad\square$

**Lemma 6.5** (Basic facts about $\omega$-homogeneous models). *Let $T$ be a countable theory.*

1. *Every countable model of $T$ has a countable $\omega$-homogeneous elementary extension;*

2. *the union of an elementary chain of $\omega$-homogeneous models is $\omega$-homogeneous;*

3. *two $\omega$-homogeneous countable models of $T$ realising the same $n$-types over $\varnothing$ for all $n \in \mathbb{N}$ are isomorphic.*

*Proof.* For (1), start with $\mathcal{M}_0 \models T$ countable. Build a countable elementary extension $\mathcal{M}_1 \succeq \mathcal{M}_0$ such that for all $a \in M_0$, $A \subseteq M_0$ finite, $p(x, A) := \mathrm{tp}(a/A)$ and $f : A \to \mathcal{M}_0$ elementary, $\mathcal{M}_1$ realises $p(x, f(A))$. This can be done since it requires realising only countably many types. Iterate this for a countable elementary chain $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \ldots$, and consider $\mathcal{M} := \bigcup_{i<\omega} \mathcal{M}_i$. By construction this is $\omega$-homogeneous (the argument is

essentially the same as the one below).

For (2), Consider $\mathcal{N} := \bigcup_{\beta < \lambda} \mathcal{N}_\beta$, the union of an elementary chain of $\omega$-homogeneous models. Take $a \in N, A \subseteq N$ finite, and $f : A \to \mathcal{N}$ elementary. Since $A$ is finite, $f : A \to \mathcal{N}_\beta$ containing both $A$ and $a$ for some $\beta < \lambda$. Since $\mathcal{N}_\beta$ is $\omega$-homogeneous $f$ can be extended to $f' : Aa \to \mathcal{N}_\beta \preceq \mathcal{N}$, yielding that $\mathcal{N}$ itseld is $\omega$-homogeneous.

The proof of (3) is a trivial back & forth argument. $\qquad\square$

**Example 6.6** (Thanks to R. Feller during the class)**.** Usually, we work with $\omega$-homogeneous models. For an example of a non $\omega$-homogeneous countable structure consider the model of $\mathrm{Th}(\mathbb{Z}, <)$ consisting of three disjoint copies of $(\mathbb{Z}, <)$ ordered one after the other $\mathcal{M} := \mathbb{Z}_{a1} \sqcup \mathbb{Z}_{a2} \sqcup \mathbb{Z}_{a3}$. Take $a \in \mathbb{Z}_{a1}, b \in \mathbb{Z}_{a3}$ and consider an elementary embedding $f : ab \to \mathcal{M}$ such that $a \mapsto a$ and $b \mapsto b'$ for $b' \in \mathbb{Z}_{a2}$. Take $c \in \mathbb{Z}_{a2}$. $f$ cannot be extended to $c$: we know that $c$ has infinite distance from both $a$ and $b$ and is between then in the order $<$. But every element between $a$ and $b'$ is in either $\mathbb{Z}_{a1}$ or $\mathbb{Z}_{a2}$ and so cannot have infinite distance from both of them.

**Corollary 6.7.** *Suppose $\mathcal{M}_0 \preceq \mathcal{N}_0$ are countable models of T. Then, there are $(\mathcal{N}, \mathcal{M}) \succeq (\mathcal{N}_0, \mathcal{M}_0)$ such that $\mathcal{M}$ and $\mathcal{N}$ are countable, $\omega$-homogeneous and satisfy the same n-types over $\varnothing$. In particular, $\mathcal{M} \cong \mathcal{N}$.*

*Proof.* We construct a countable elementary chain

$$(\mathcal{N}_0, \mathcal{M}_0) \preceq (\mathcal{N}_1, \mathcal{M}_1) \preceq \ldots$$

as follows: for $(\mathcal{N}_i, \mathcal{M}_i)$ take $(\mathcal{N}', \mathcal{M}') \succeq (\mathcal{N}_i, \mathcal{M}_i)$ such that $\mathcal{M}'$ realises all $n$-types over $\varnothing$ realised by $\mathcal{N}_i$. Then, by Lemma 6.5 (1) take a countable $\omega$-homogeneous elementary extension $(\mathcal{N}_{i+1}, \mathcal{M}_{i+1}) \succeq (\mathcal{N}', \mathcal{M}')$. Note that since $(\mathcal{N}_{i+1}, \mathcal{M}_{i+1})$, we also have that both $\mathcal{N}_{i+1}$ and $\mathcal{M}_{i+1}$ are $\omega$-homogeneous.

Now, consider the union of this elementary chain, $(\mathcal{N}, \mathcal{M})$. By construction and Lemma 6.5 (2) this is also $\omega$-homogeneous, and so such that both $\mathcal{N}$ and $\mathcal{M}$ are $\omega$-homogeneous. Furthermore, by construction $\mathcal{M}$ and $\mathcal{N}$ satisfy the same $n$-types over $\varnothing$. Hence, $(\mathcal{N}, \mathcal{M})$ satisfies all of the desired properties. $\qquad\square$

**Theorem 6.8** (Vaught's two cardinal theorem)**.** *Let T have a Vaughtian pair. Then, there is $\mathcal{N}^\star \models T$ of cardinality $\aleph_1$ and $\phi \in \mathcal{L}(N^\star)$ such that $|\phi(N^\star)| = \aleph_0$.*

*Proof.* By Lemma 6.4, $T$ has a countable Vaughtian pair and by Corollary 6.7 we can choose it so that $\mathcal{M}$ and $\mathcal{N}$ are countable, $\omega$-homogeneous and realising the same $n$-types over $\varnothing$. In particular, $\mathcal{M} \cong \mathcal{N}$. We build an elementary chain

$$(\mathcal{N}_\alpha | \alpha < \omega_1)$$

such that

- $\mathcal{N}_0 = \mathcal{M}, \mathcal{N}_1 = \mathcal{N}$;

- $\mathcal{N}_\alpha \cong \mathcal{N}$;

- $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$.

For the successor step, suppose that we have $\mathcal{N}_\alpha \cong \mathcal{N}$. Then, $\mathcal{N}_\alpha \cong \mathcal{M}$, and so it has an elementary extension $\mathcal{N}_{\alpha+1}$ such that $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$, and so we are done. For the limit step, for $\alpha < \omega_1$ consider $\mathcal{N}_\alpha = \bigcup_{\beta < \alpha} \mathcal{N}_\beta$. Since $\alpha < \omega_1$, this is a countable union of countable sets and so is $\mathcal{N}_\alpha$ countable. By Lemma 6.5 (2), this is $\omega$-homogeneous. Also, since any finite subset of $\mathcal{N}_\alpha$ is contained in some $\mathcal{N}_\beta \preceq \mathcal{N}_\alpha$ and $\mathcal{N}_\beta \cong \mathcal{N}$, $\mathcal{N}_\alpha$ realises all of the same $n$-types over $\varnothing$ as $\mathcal{N}$. Thus, $\mathcal{N}_\alpha \cong \mathcal{N}$ as desired. Hence, we can build the desired elementary chain $(\mathcal{N}_\alpha | \alpha < \omega_1)$.

Take $\mathcal{N}^\star = \bigcup_{\alpha < \omega_1} \mathcal{N}_\alpha$. Note that $|\mathcal{N}^\star| = \aleph_1$: it must have size at least $\aleph_1$ since we are taking an uncountable union, where at each stage $\mathcal{N}_{\alpha+1} \supsetneq \mathcal{N}_\alpha$. It has size at most $\aleph_1$ being an uncountable union of countable sets. However, since $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$, we have that

$$\phi(N_{\alpha+1}) = \phi(N_\alpha) = \cdots = \phi(N) = \phi(M),$$

and so $\phi(N^\star) = \phi(M)$, which is a countable set. So the conclusion of the theorem holds. $\square$

❧ *Observation* 6.9. Note that if $\mathcal{M} \models T$ is saturated and $\phi(x) \in \mathcal{L}(M)$ is non-algebraic, then $|\phi(M)| = |M|$. Suppose by contradiction $|\phi(M)| < |M|$ and consider the following partial type:

$$\{\phi(x)\} \cup \{x \neq c \mid c \in \phi(M)\}.$$

This is a finitely satisfiable and over a set of cardinality $< |M|$. So by saturation of $M$ it is realised in $M$, yielding a contradiction.

**Corollary 6.10.** *Let $T$ be $\kappa$-categorical in some uncountable $\kappa$. Then, $T$ has no Vaughtian pairs.*

*Proof.* Suppose that $T$ is $\kappa$-categorical. Since it is $\kappa$-categorical, it is $\aleph_1$-categorical by the downwards Morley Theorem 5.15. Hence, all of its models of cardinality $\aleph_1$ are saturated. Hence, by Observation, for every uncountable model $\mathcal{N}$ and $\mathcal{L}(N)$-formula $\phi(x)$, $|\phi(N)| = \aleph_1$. By Vaught's two cardinal Theorem 6.8, this implies that $T$ has no Vaughtian pair. $\square$

**Definition 6.11.** We say that $T$ **eliminates the quantifier** $\exists^\infty x$ if for every $\mathcal{L}$-formula $\phi(x, \overline{y})$ there is $n_\phi \in \mathbb{N}$ such for all tuples $\overline{a} \in \mathbb{M}^{|\overline{y}|}$, if $|\phi(\mathbb{M}, \overline{a})| \geq n_\phi$, then $\phi(\mathbb{M}, \overline{a})$ is infinite.

**Exercise 6.12.** Show that if $T$ has no Vaughtian pairs, then it eliminates the quantifier $\exists^\infty x$.

**Exercise 6.13.** Suppose that $T$ eliminates the quantifier $\exists^\infty x$. Let $\mathcal{M} \models T$ and let $\phi(x) \in \mathcal{L}(M)$ be minimal in $\mathcal{M}$. Show that $\phi(x)$ is strongly minimal.

**Definition 6.14.** For infinite cardinals $\kappa > \lambda$, we say that $T$ has has a $(\lambda, \kappa)$-model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

**Exercise 6.15.** Prove the following:

1. If $T$ has a $(\kappa, \lambda)$-model then it has a Vaughtian pair (and so an $(\aleph_1, \aleph_0)$-model) [Hint: this should be trivial];

2. Prove that if $T$ is $\omega$-stable and has an $(\aleph_1, \aleph_0)$-model, then for each $\kappa > \aleph_1$, $T$ has a $(\kappa, \aleph_0)$-model [Hint: you may need to use Theorem 5.12].

⋆⋆ **Exercise 6.16.** We show that in Exercise 6.15 (2), the assumption of $\omega$-stability is necessary. Let $\mathcal{L} = \{P_0, \ldots, P_n, E_1, \ldots, E_n\}$ for unary predicates $P_i$ and binary relations $E_i$. Consider the $\mathcal{L}$-theory $T$ stating that:

- the $P_i$ are infinite and partition the domain;

- for each $i \in \{1, \ldots, n\}$, $\forall xy(E_i(x, y) \to P_{i-1}(x) \land P_i(y))$;

- for each $i \in \{1, \ldots, n\}$, $\forall xy((P_i(x) \land P_i(y) \land \forall z(E_i(z, x) \leftrightarrow E_i(z, y))) \to x = y)$.

For example, for $X_0$ an infinite, take $X_{i+1} = \mathcal{P}(X_i)$ for $i \in \{1, \ldots, n\}$. Let $\mathcal{M}$ be the disjoint union of the $X_i$ with $P_i$ naming each of the $X_i$ and $E_i$ being the membership relation restricted to $X_i \times x_{i+1}$. Then, $\mathcal{M} \models T$. Show that if $\mathcal{M} \models T$ and $|P_0(M)| = \aleph_0$, then $|M| \leq \beth_n$. Hence, $\mathcal{M}$ has a $(\beth_n, \aleph_0)$-model but it does not have a $(\kappa, \aleph_0)$-model for arbitrarily large $\kappa$. [Hint: I would only do the case of $n = 1$. Recall that $\beth_0 = \aleph_0$ and $\beth_{\alpha+1} = 2^{\beth_\alpha}$.]

# 7 The Baldwin-Lachlan Theorem and Morley's Theorem

Recall the following exercise from the Model Theory I course:

**Fact 7.1.** *Let $T$ be $\omega$-stable and $\mathcal{M} \models T$. Then there is a $\mathcal{L}(M)$-formula minimal in $M$.*

**Theorem 7.2** (Baldwin & Lachlan)**.** *Let $T$ be a countable theory. Then, $T$ is $\kappa$-categorical in some uncountable cardinal $\kappa$ if and only if it is $\omega$-stable and has no Vaughtian pair.*

*Proof.* ($\Rightarrow$) We have already proven that $\kappa$-categoricity implies $\omega$-stability in the Model Theory I course. We proved that $\kappa$-categoricity implies having no Vaughtian pair in Corollary 6.10.

($\Leftarrow$) Since $T$ is $\omega$-stable, it has a prime model $\mathcal{M}_0$ over $\varnothing$ by Exercise 5.11. Note that by the downwards Lowenheim-Skolem Theorem and countability of $T$, $M_0$ must be countable. By $\omega$-stability, there is an $\mathcal{L}(M_0)$-formula $\phi(x)$ which is minimal in $\mathcal{M}_0$ by Fact 7.1. Now, since $T$ has no Vaughtian pair, $\phi(x)$ is strongly minimal (by Exercises 6.12 and 6.13).

Now, consider two models $\mathcal{M}_1, \mathcal{M}_2$ of cardinality $\kappa$. We need to show that they are isomorphic. Since $\mathcal{M}_0$ is prime over $\varnothing$, it elementary embeds into both of the $\mathcal{M}_i$ for $i \in \{1, 2\}$. We may assume without loss of generality that $\mathcal{M}_0$ is an elementary substructure of both $\mathcal{M}_1$ and $\mathcal{M}_2$.

❂ *Observation* 7.3. For $i \in \{1, 2\}$, $\mathcal{M}_i$ has no proper elementary substructure containing $M_0 \cup \phi(M_i)$.

If it did, we would have $M_0 \cup \phi(M_i) \subseteq \mathcal{N} \preceq \mathcal{M}_i$ with $N \subsetneq M_i$. Since $\mathcal{N} \preceq \mathcal{M}_i$, $\phi(N) = \phi(M_i)$. But then this would give a Vaughtian pair.

❂ *Observation* 7.4. For $i \in \{1, 2\}$, $\mathcal{M}_i$ is prime over $M_0 \cup \phi(M_i)$.

Since $T$ is $\omega$-stable, there is $\mathcal{K} \preceq \mathcal{M}_i$ prime over $M_0 \cup \phi(M_i)$. By Observation 7.3, $\mathcal{K} = \mathcal{M}_i$.

Now, from Observation 7.3, we can deduce that $|\phi(M_i)| = \kappa$. Otherwise, by the downwards Lowenheim-Skolem Theorem, and countability of $T$ and $M_0$, we could find a proper elementary substructure of $M_i$ containing $M_0 \cup \phi(M_i)$. Now, we know from Theorem 4.3 (where we added parameters for the elements of $M_0$), that $(\phi(\mathbb{M}), \mathrm{cl})$ forms a pregeometry, where

$$\mathrm{cl}(A) = \mathrm{acl}(A \cup M_0) \cap \phi(\mathbb{M}).$$

Hence,

$$\dim_\phi(M_1 / M_0) = \dim_\phi(M_2 / M_0) = \kappa.$$

This is because for any set $A$, $|\mathrm{acl}(A)| \leq \max(|A|, |\mathcal{L}|)$ and since $\dim_\phi(M_i / M_0)$ is the cardinality of a generating set for $\phi(M_i)$ (over $M_0$), where $M_0$ and $|\mathcal{L}|$ are countable.

From Lemma 4.12, there is an $M_0$-elementary bijection $f : \phi(M_1) \to \phi(M_2)$. I.e. there is an elementary bijection $f : M_0 \cup \phi(M_1) \to M_0 \cup \phi(M_2) \subseteq \mathcal{M}_2$ fixing $M_0$. Since $\mathcal{M}_1$ is prime over $M_0 \cup \phi(M_1)$, we can extend $f$ to $f' : \mathcal{M}_1 \to \mathcal{M}_2$. Now, $f'(M_1) \preceq \mathcal{M}_2$ and $M_0 \cup \phi(M_2) \subseteq f'(M_1)$. Hence, from Observation 7.4, $f'(M_1) = M_2$, and so $f'$ is an isomorphism between $M_1$ and $M_2$, as desired. $\qquad\square$

**Corollary 7.5** (Morley's Theorem)**.** *Let $T$ be a countable theory. Suppose that $T$ is categorical in some uncountable cardinal. Then, $T$ is categorical in all uncountable cardinals.*

*Proof.* If $T$ is $\kappa$-categorical for some uncountable $\kappa$, by the Baldwin-Lachlan Theorem it is $\omega$-stable with no Vaughtian pair. But again by the Baldwin-Lachlan Theorem it is categorical in all uncountable cardinals. $\qquad\square$

# 8 Getting familiar with more examples

**Exercise 8.1.** 1. Prove that $(\mathbb{N}; 0, 1, +, \times, <)$ is the prime model of $\mathrm{Th}(\mathbb{N}; 0, 1, +, \times, <)$;

2. Prove that $\mathrm{Th}(\mathbb{Z}, 0, +)$ does not have a prime model. [Hint: you may assume that augmenting $(\mathbb{Z}, 0, +)$ by predicates $P_n$ for each $n \geq 2$ naming the elements divisible by $n$ yields a theory with quantifier elimination];

**Definition 8.2.** Let $\kappa$ be an infinite cardinal. We say that $T$ is $\kappa$-**stable** if for all $A$ such that $|A| \leq \kappa$, $|S_x(A)| \leq \kappa$. We say that $T$ is **superstable** if it is $\kappa$-stable for all $\kappa \geq 2^{|T|}$. We say that $T$ is **stable** if it is $\kappa$-stable for some $\kappa$.

**Exercise 8.3.** Let $\mathcal{L} := \{U_i | i < \omega\}$ be such that each $U_i$ is a unary predicate. For $X$ and $Y$ disjoint finite subsets of $\mathbb{N}$, let $\phi_{X,Y}$ be the sentence

$$\exists x \bigwedge_{i \in X} U_i(x) \wedge \bigwedge_{i \in Y} \neg U_i(x).$$

Let $T := \{\phi_{X,Y} | X, Y \text{ disjoint finite subsets of } \mathbb{N}\}$. You may assume this theory is complete and has quantifier elimination.

1. Show that no type over $\varnothing$ is isolated. Deduce that $T$ has no prime models;

2. Show that $T$ is $\kappa$-stable for all $\kappa \geq 2^{\aleph_0}$.

## 8.0.1 Equivalence relations

**Example 8.4.** Consider the following theories of equivalence relations for $\alpha$ an ordinal and $\kappa$ a cardinal.

- **Refining equivalence relation with infinite splitting:** $\mathrm{REI}_\alpha$ has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, $E_j$ refines $E_i$ and each $E_i$ class is refined into infinitely many $E_{i+1}$-classes. For this and all other examples below we assume each equivalence class of each equivalence relation is infinite;

- **Refining equivalence relation with finite splitting:** $\mathrm{REF}_\alpha$ has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, $E_j$ refines $E_i$ and each $E_i$ class is refined into two $E_{i+1}$-classes;

- **Crosscutting Equivalence relation with finite splitting:** $\mathrm{CEF}_\kappa$ has equivalence relations $(E_i | i < \kappa)$ such that each $E_i$ has only two classes and for all $i < \kappa$, $E_{i+1}$ splits each equivalence class of $E_i$ into two classes;

- **Crosscutting Equivalence relation with infinite splitting:** $\mathrm{CEI}_\kappa$ has equivalence relations $(E_i | i < \kappa)$ such that each $E_i$ has infinitely many classes and for all $i < \kappa$, $E_{i+1}$ splits each equivalence class of $E_i$ into infinitely many classes. These theories are complete and have quantifier elimination.

**Exercise 8.5.** Show the following:

1. $\mathrm{REI}_\alpha$ is stable. $\mathrm{REI}_\omega$ is $\kappa$-stable if and only if $\kappa^{\aleph_0} = \kappa$. [More generally, if $\alpha$ is infinite, $\mathrm{REI}_\alpha$ is not superstable];

2. $\mathrm{REF}_\alpha$ is stable. If $\alpha \leq \omega$ it is superstable. If $\alpha \geq \omega \cdot \omega$ then it is not superstable;

3. if $\kappa \leq \omega$, then $\mathrm{CEF}_\kappa$ is superstable.

$\star\star$ **Exercise 8.6.** Compute the number of models of each of the examples in Example 8.4 in each cardinality.

### 8.0.2 Differentially closed fields

**Definition 8.7.** A **derivation** on a ring $R$ is an additive homomorphism $D : R \to R$ such that

$$D(xy) = xD(y) + yD(x).$$

A **differential ring** is a ring equipped with a derivation. Given a derivation $D_0 : R \to R$, we call the **ring of differential polynomials** the differential ring

$$R\{X\} := R[X, X^{(1)}, X^{(2)}, X^{(3)}, \dots],$$

with derivation $D$ extending $D_0$ by setting $D\left(X^{(n)}\right) = X^{(n+1)}$.

**Definition 8.8.** An ideal $I \subseteq R\{X\}$ is a **differential ideal** if for all $f \in I, D(f) \in I$.

**Definition 8.9.** If $f(X) \in R\{X\} \setminus R$, the **order** of $f$ is the largest $n$ such that $X^{(n)}$ occurs in $f$. We can write

$$f(X) = \sum_{i=0}^{m} g_i(X, X^{(1)}, \dots, X^{(n-1)})(X^{(n)})^i,$$

for $g_i \in R[X, X^{(1)}, \dots, X^{(n-1)}]$ and $g_m \neq 0$. We call $m$ the **degree** of $f$. We write $g \leq f$ if either the order of $g$ is strictly less than the order of $f$ or if the orders are the same but $g$ has lower degree than $f$.

**Definition 8.10.** A differential field $K$ (of characteristic zero) which is algebraically closed is called **differentially closed** if for any non-constant differential polynomials $f$ and $g$ where the order of $g$ is less than the order of $f$ there is an $x$ such that $f(x) = 0$ and $g(x) \neq 0$.

Note that the theory of differentially closed fields of characteristic zero is axiomatisable. We call it $\mathrm{DCF}_0$.

**Fact 8.11.** *Every differential field $k$ has an extension $K$ which is differentially closed.*

**Fact 8.12.** $\mathrm{DCF}_0$ *is complete and has quantifier elimination.*

$\star\star$ **Exercise 8.13.** Let $K \models \mathrm{DCF}_0$. Let $k \subseteq K$.

1. for $p \in S_1^K(k)$, the set of 1-types over $k$ realised in $K$, let $I_p := \{f \in k\{X\} | f(x) = 0 \in p\}$. Show that $I_p$ is a differential prime ideal [i.e. $I_p \subsetneq k\{X\}$ and if $f \cdot g \in I_p$, then either $f \in I_p$ or $g \in I_p$].

2. Show that if $I \subset k\{X\}$ is a differential prime ideal, then $I = I_p$ for some $p \in S_n^K$. So $p \mapsto I_p$ is a bijection between complete $n$-types and differential prime ideals in $k\{X\}$;

3. The Ritt-Raudenbusch Basis Theorem says that every differential prime ideal in $k\{X\}$ is finitely generated. Use this to show that $\mathrm{DCF}_0$ is $\omega$-stable;

4. Suppose that $k$ is a differential field of characteristic zero and $k \subseteq K \models \mathrm{DCF}_0$. We say that $K$ is the **differential closure** of $k$ if for all $L \supseteq k$ such that $K \models \mathrm{DCF}_0$ there is a differential field embedding of $K$ into $L$ fixing $k$. Show that every field has a differential closure.

# 9   Stability

**Definition 9.1.** We say that the formula $\phi(x; y)$ has the **order property** if there is an infinite sequence $(a_i, b_i | i < \omega)$ such that

$$\models \phi(a_i, bj) \text{ if and only if } i < j.$$

We say that $\phi(x; y)$ is **stable** if it does not have the order property.

**Exercise 9.2.** Let $\phi(x;y)$ and $\psi(x,z)$ be stable $\mathcal{L}$-formulas. Show the following:

1. $\phi^{\text{opp}}(y;x) := \phi(x;y)$ is stable;

2. $\neg\phi(x;y)$ is stable;

3. $\theta(x;yz) := \phi(x;y) \wedge \psi(x;z)$ is stable;

4. $\theta(x;yz) := \phi(x;y) \vee \psi(x;z)$ is stable;

5. For $y = uv$ and $c \in \mathbb{M}^{|v|}$, $\phi(x;u,c)$ is stable.

**Theorem 9.3** (Erdös-Makkai). *Let $B$ be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ with $|B| < |\mathcal{F}|$. Then, there are sequences $(b_i|i < \omega)$ of elements of $B$ and $(S_i|i < \omega)$ of elements of $\mathcal{F}$ such that for all $i, j \in \omega$, we have that*

- *EITHER $b_i \in S_j$ if and only if $j < i$;*

- *OR $b_i \in S_j$ if and only if $i < j$.*

**Exercise 9.4** (Proof of Erdös-Makkai). Note that there is $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| = |B|$ and for all $B_0, B_1 \subseteq B$ finite, if there is some $S \in \mathcal{F}$ such that $B_0 \subseteq S, B_1 \subseteq B \setminus S$, then there is some $S' \in \mathcal{F}$ with $B_0 \subseteq S', B_1 \subseteq B \setminus S'$. Note that there is $S^\star \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$. Now, prove Erdös-Makkai. [Hint: you need to construct appropriate sequences in $S^\star$, $B \setminus S^\star$ and $\mathcal{F}'$, and then use Ramsey's theorem.]

**Definition 9.5.** $\phi(x;y)$ does has the **binary tree property** if there is a binary tree of parameters $(b_s|s \in {}^{<\omega}2)$ such that for each branch $\sigma \in {}^\omega 2$,

$$\{\phi^{\sigma(n)}(x;b_{\sigma|n})|n < \omega\} \text{ is consistent,}$$

where $\phi^0 := \neg\phi$ and $\phi^1 := \phi$.

**Definition 9.6.** Let $\phi(x;y) \in \mathcal{L}$ and $B$ be a set of parameters. A $\phi$-type over $B$ is a maximally consistent set of formulas of the form $\phi(x;b), \neg\phi(x;b)$. We denote the set of all $\phi$-types over $B$ by $S_\phi(B)$.

**Theorem 9.7.** *Let $\phi(x;y) \in \mathcal{L}$. The following are equivalent:*

(a) *$\phi$ is stable;*

(b) *$|S_\phi(B)| \leq |B|$ for every infinite $B$;*

(c) *For some infinite cardinal $\lambda$ we have $|B| \leq \lambda \Rightarrow |S_\phi(B)| \leq \lambda$;*

(d) *$\phi$ does not have the binary tree property.*

*Proof.* Every direction that we prove for this theorem will be a proof by contrapositive. $(a) \Rightarrow (b)$ Suppose that $|S_\phi(B)| > |B|$ for some infinite $B$. For any $a \in \mathbb{M}^{|x|}$ let

$$S_a := \{b \in B^{|y|} \mid \vdash \phi(a;b)\}.$$

Note that $\text{tp}_\phi(a/B) = \text{tp}_\phi(a'/B)$ if and only if $S_a = S_{a'}$. Let $\mathcal{F} := (S_a|a \in \mathbb{M}^{|x|}$. By the observation above,

$$|\mathcal{F}| = |S_\phi(B)| > |B| = |B^{|y|}|,$$

where the inequality is our original assumption. Thus, we can apply Erdös-Makkai (Theorem 9.3) to see there are $(b_i|i < \omega)$ in $B^{|y|}$ and $(S_{a_i}|i < \omega)$ from $\mathcal{F}$ such that

- either $b_i \in S_{a_j}$ if and only if $j < i$;

- or $b_i \in S_{a_j}$ if and only if $i < j$.

Note that $b_i \in S_{a_j}$ if and only if $\vDash \phi(a_j; b_i)$. So the first case is equivalent to the statement of $\phi(x; y)$ having the order property. The second case is equivalent to $\phi^{opp}$ (obtained from swapping the variables of $\phi$) having the order property, which we know implies that $\phi$ also has the order property. So we conclude that $\phi$ has the order property as desired.

$(b) \Rightarrow (c)$ is trivial. We move to $(c) \Rightarrow (d)$. Pick an arbitrary infinite $\lambda$. First, choose $\mu$ minimal such that $2^\mu > \lambda$. Then, the tree $I =^{<\mu} 2$ has cardinality $\leq \mu$ since

$$|I| \leq \left| \bigcup_{\nu < \mu} 2^\nu \right| \leq \lambda,$$

where the last inequality holds since we are taking a unioin of $\leq \lambda$-many sets of cardinality $\leq \lambda$. Now, suppose by contrapositive that $\phi(x; y)$ has the binary tree property. Note that by compactness we can take the binary tree to be $I$-indexed[1] By definition, each branch yields a consistent set of formulas which can be completed to a $\phi$-type over $B$. By construction all these $\phi$-types are disjoint: if $\sigma, \nu \in^\mu 2$ have meet $\tau \in^{<\mu} 2$, then the sets of formulas associated with $\sigma$ and $\nu$ disagree on whether $\phi(x, b_\tau)$ is true. So

$$|B| \leq \lambda < 2^\mu \leq S_\phi(B),$$

yielding the negation of $(c)$.

Finally, we prove $(d) \Rightarrow (a)$. Say that $\phi(x; y)$ has the order property. By compactness, there are $(a_i b_i \vdash i \in [0, 1])$ such that

$$\phi(a_i, b_j) \text{ if and only if } i < j.$$

Pick $b'_\varnothing = b_{\frac{1}{2}}$. By construction, for $i < 1/2, \vDash \phi(a_i, b'_\varnothing)$ and for $i \geq 1/2, \vDash\neq \phi(a_i, b'_\varnothing)$. Now, pick $b'_0 = b_{\frac{1}{4}}$ and $b_2 = b'_{\frac{3}{4}}$ and keep building the tree of parameters $(b_s | s \in^{<\omega} 2)$ by splitting the intervals. Now, for each branch $\sigma \in^\omega 2$, it is easy to see that $\{\phi^{\sigma(n)}(x; b_{\sigma|n}) | n < \omega\}$ is finitely satisfiable. So it is satisfiable and gives the desired binary tree, completing the final part of the proof. $\square$

**Exercise 9.8.** Show that the following are equivalent:

1. $T$ is stable (in the sense of being $\kappa$-stable for some infinite $\kappa$);

2. every $\mathcal{L}$-formula $\phi(x; y)$ is stable for $T$;

3. $T$ is $\kappa$-stable for all $\kappa$ such that $\kappa^{|T|} = \kappa$.

**Definition 9.9.** For $\kappa$ an infinite cardinal, let

$$\mathrm{ded}(\kappa) := \sup\{|I| : I \text{ is a linear ordering with a dense subset of size } \kappa\}.$$

It is easy to see that $\kappa < \mathrm{ded}(\kappa) \leq 2^\kappa$.

**Definition 9.10.** Let $T$ be a countable theory. Write $f_T : \mathrm{Card} \to \mathrm{Card}$ for the function on cardinals given by

$$f_T(\kappa) := \sup\{|S_n(M)| : \mathcal{M} \vDash T, |M| = \kappa, n \in \omega\}.$$

[It is easy to see that if we fixed $n$ in the definition above, we would still get $f_T$.]

---

[1] At this stage you are supposed to know how to run a standard compactness proof. In this case, you should expand $T$ by constants $c_s$ for $s \in I$ and consider the sets of formulas

$$\bigcup_{\sigma \in^\mu 2} \{\phi^{\sigma(\nu)}(x; c_{\sigma|\nu}) | \nu < \mu\}.$$

Any finite subset is finitely satisfiable by assigning to the $c_s$ the appropriate $b_s$ from the original binary tree, thus completing the compactness proof.

**Exercise 9.11.** Prove that if $T$ is unstable, then $f_T(\kappa) \geq \text{ded}(\kappa)$ for all cardinals $\kappa \geq |T|$.

Recall the following definitions and lemmas from the Model Theory course:

**Definition 9.12.** Let $I$ be an infinite linear order and $A$ a set of parameters. We say that $(a_i | i \in I)$ is **indiscernible** over $A$ if for every $\mathcal{L}(A)$-formula $\phi(x_1, \ldots, x_n)$ and $i_1 < \cdots < i_n, j_1 < \cdots < j_n$ from $I$, we have that

$$\vDash \phi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n}). \tag{2}$$

We say that the sequence is **totally indiscernible** over $A$ if the condition 2 holds for any $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ from $I$ of size $n$.

For a sequence $(a_i | i \in I)$, its **EM-type** (i.e. Ehrenfeucht-Motowski type) over $A$ is given by

$$\text{EM}(a_i | i \in I) := \{\phi(x_1, \ldots, x_n) \in \mathcal{L}(A) \mid \vDash \phi(a_{i_1}, \ldots, a_{i_n}) \text{ for all } i_1 < \cdots < i_n, n < \omega\}.$$

**Lemma 9.13** (Extracting indiscernible sequences). *Let $A$ be a set of parameters and $(b_i | i \in I)$ a infinite sequence. Let $J$ be a linear order. Then, there is a sequence $(a_j | j \in J)$ which is indiscernible over $A$ and realising the EM-type of $(b_i | i \in I)$.*

**Exercise 9.14.**
- Show that $T$ is unstable if and only if there is an infinite sequence $(a_i | i < \omega)$ and a formula $\phi(x, y)$ such that $\vDash \phi(a_i, a_j)$ if and only if $i < j$;

- Show that if $T$ is unstable there is an indiscernible sequence which is not totally indiscernible;

- Show that if $T$ is stable then every indiscernible sequence is totally indiscernible. [Hint: Say that we have an indiscernible sequence $(a_i | i < \omega)$ which is not totally indiscernible. Show that there is a formula $\phi(x_1, \ldots, x_n)$ such that for some transposition $\tau$ switching only two consecutive variables

$$\vdash \phi(a_1, \ldots, a_n) \wedge \neg\phi(a_{\tau(1)}, \ldots, a_{\tau(n)}).$$

Use this formula to find an unstable formula in $T$.]

**Definition 9.15.** We say that $\phi(x, y)$ has the **independence property**, IP, if there are $(b_i | i < \omega)$ and $(a_s | s \subseteq \omega)$ such that
$$\vDash \phi(a_s, b_i) \Leftrightarrow i \in s.$$
We say that $\phi(x, y)$ is NIP if it does not have the independence property. A theory is NIP if all of its formulas are NIP.

**Example 9.16.** The random graph has the independence property.

**Exercise 9.17.** Prove that a formula $\phi(x, y)$ has IP if and only if there is an indiscernible sequence $(b_i | i \in I)$ and a parameter $a$ such that

$$\vDash \phi(a, b_i) \text{ if and only if } i \text{ is even}.$$

**Definition 9.18.** We say that $\phi(x, y)$ has the **strict order property**, SOP if there is an infinite sequence $(b_i)_{i < \omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j < \omega$. We say that $\phi(x, y)$ is NSOP if it does not have the strict order property, and we say that a theory $T$ is NSOP if all of its formulas are NSOP.

**Exercise 9.19.** Prove that $T$ is stable if and only if it is NIP and NSOP. [Hint: For the ($\Leftarrow$) direction, consider an unstable theory with NIP and prove it must have the SOP.]

**Definition 9.20.** Let $R$ be an associative ring with identity. Let $\mathcal{L}_R := \{+, 0, (\mu_r)_{r \in R)}\}$, where the $\mu_r$ are unary function symbols. The theory of (left) $R$-modules $T_R$ asserts of a model $M$ that $(M, +, 0)$ is an Abelian group, and each $f_r$ is an endomorphism of this group with $f_1$ being the identity map. Note that this theory is not complete.

**Definition 9.21.** An $\mathcal{L}$-formula is called **primitive positive** if it is an existential quantification of a conjunction of atomic formulas. That is, if it is of the form

$$\exists \overline{y} \bigwedge_{i \leq n} \psi_i(\overline{x}, \overline{y}),$$

where the formulas $\psi_i(\overline{x}, \overline{y})$ are atomic.

**Fact 9.22.** *The theory $T_R$ has elimination of quantifiers up to primitive positive formulas. That is, for every $\mathcal{L}_R$-sentence, $\phi(\overline{x})$ there is a Boolean combination of primitive positive formulas $\psi(\overline{x})$ such that $T_R \vDash \forall \overline{x} \phi(\overline{x}) \leftrightarrow \psi(\overline{x})$.*

**Exercise 9.23.** Consider a primitive positive $\mathcal{L}_R$-formula $\phi(\overline{x}, \overline{z})$ and let $M$ be an $R$-module. Show that $\phi(\overline{x}, \overline{0})$, where $\overline{0}$ is a string of 0's of length $|\overline{z}|$ defines either $\varnothing$ or a subgroup of $M^{|\overline{x}|}$. Show that for any tuple $\overline{a}$ such that $\phi(\overline{x}, \overline{a})$ is consistent, $\phi(M, \overline{a})$ is a coset of $\phi(\overline{x}, \overline{0})$. Deduce that that for any tuples $\overline{a}, \overline{b}$, $\phi(\overline{x}, \overline{a})$ and $\phi(\overline{x}, \overline{b})$ are either mutually inconsistent or equivalent. Using elimination of quantifiers up to pp-formulas, prove that for any $R$-module $M$, its theory is stable.

**Definition 9.24.** A group $G$ satisfies the $\omega$-**stable descending chain condition** if there is no infinite proper descending chain of definable subgroups of $G$,

$$\cdots < H_{i+1} < H_i < \cdots < H_1 < G.$$

**Exercise 9.25.** Show that if $G$ is an $\omega$-stable group (i.e. a group whose theory is $\omega$-stable), then $G$ satisfies the $\omega$-stable descending chain condition. Show that for an $R$-module $M$ with $R$ countable, the following are equivalent:

- $M$ is $\omega$-stable;

- $M$ has the $\omega$-stable descending chain condition;

- For any set of pp-formulas in a single variable $\{\phi_n(x) | n < \omega\}$ there is some $n \in \omega$ such that for all $m \in \omega$, $M \vDash \forall x (\bigwedge_{i < n} \phi_i(x) \rightarrow \phi_m(x))$.

# 10 Imaginaries

*Remark* 10.1. Today we work in a multisorted setting. By this, we mean that our theory will have a set $S$ of sorts indicating separate domains on which our structure lives, so that

- variables and quantifiers are always restricted to a fixed sort $s \in S$. In particular, we have variables $x^s$ and quantifiers $\forall x^s$ and $\exists x^s$ for each sort $s \in S$;

- Constants, domains and codomains of functions or relations are always restricted to some fixed sorts. (we do allow different arguments to come from different sorts, such as a relation $R(x^{s_1}, x^{s_2})$ for sorts $s_1, s_2 \in S$);

- Equality is also restricted to elements of a fixed sort. So we only have formulas $t_1 = t_2$ where $t_1$ and $t_2$ are terms in the same sort (but our language does not even contain the formulas $t_1 = t_2$ for $t_1$ and $t_2$ of different sorts).

An $S$-sorted structure will have (disjoint) domains $(M_s | s \in S)$, and symbols, terms, and formulas from the associated $S$-sorted language are interpreted in the obvious way (keeping in mind that variables and quantifiers are always restricted to a fixed sort).

**Example 10.2.** Consider the multi-sorted structure

$$M^{\infty} := (M_1, M_2, \dots)$$

consisting of countably many sorts where the structure on each sort is an infinite set with equality. Note a fundamental difference between the multi-sorted theory $T_S^{\infty}$ of $M^{\infty}$ and the theory $T_P^{\infty}$ of an infinite set partitioned by infinitely many infinite predicates $(P_i | i < \omega)$. In fact, by compactness, $T_P^{\infty}$ has models with an element not contained in any of the infinitely many predicates. Meanwhile, by nature of being a multisorted theory, any element in any model of $T_S^{\infty}$ will be inside one of the sorts.

## 10.1  Construction of $\mathcal{M}^{eq}$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with theory $T$.

The idea behind the construction of $\mathcal{M}^{eq}$ is that we add sorts for each quotient by a 0-definable equivalence relation and add projections to these quotients from $M$ to induce the natural structure on them.

**Definition 10.3.** Let $\mathrm{ER}(T)$ be the set of 0-definable equivalence relations $E(\overline{x}, \overline{y})$ on $n_E$-tuples on $M$. Note that this depends only on $T$ since the theory already tells us which definable relations are equivalence relations and which ones yield the same or distinct equivalence relations.

**Definition 10.4** ($\mathcal{M}^{eq}$)**.** Consider the multi-sorted structure

$$\mathcal{M}^{eq} := (M, (M^{n_E}/E)_{E \in \mathrm{ER}(T)})$$

with $\mathcal{M}$, the **real** (or home) **sort** being the original $\mathcal{L}$-structure $\mathcal{M}$ identified with $M/(x = x)$ (in the sort $S_=$) with variables and quantifiers restricted accordingly. Additionally, for each $E \in \mathrm{ER}(T)$ we have the projection to the quotient

$$\pi_E : M^{n_E} \to M/E.$$

We call elements in $\mathcal{M}^{eq}$ **imaginary**.

Note that in $\mathcal{L}^{eq}$ we added to $\mathcal{L}$ for each $E \in \mathrm{ER}(T)$ a sort $S_E$ with variables $x^E$ and quantifiers $\forall x^E$ and $\exists x^E$ and the $n_E$-ary function symbol $\pi_E$ with domain $(S_=)^{n_E}$ and codomain $S_E$. Note that since there are only $|\mathcal{L}|$-many definable equivalence relations, $|\mathcal{L}| = |\mathcal{L}^{eq}|$, where in the cardinality of a multi-sorted langlage we include the number of sorts.

We now prove that defining $T^{eq}$ as $\mathrm{Th}(M^{eq})$ does not depend on the choice of model, but only on $T$.

**Lemma 10.5.**    *1. Let $f : M \to N$ be an elementary embedding. Then, $f$ extends uniquely to an elementary embedding $\tilde{f} : M^{eq} \to N^{eq}$;*

  *2. $M \equiv N$ implies that $M^{eq} \equiv N^{eq}$.*
    *So $T^{eq}$ does not depend on the choice of model of $T$;*

  *3. If $M^\star \models T^{eq}$, $M^\star = M^{eq}$ for some $M \models T$;*

  *4. Any automorphism of $M$ extends uniquely to an automorphism of $M^{eq}$;*

  *5. Let $\mathcal{M} \equiv \mathcal{N}$ and $\overline{a} \in M, \overline{b} \in N$. Then, if $\mathrm{tp}_M(\overline{a}) = \mathrm{tp}_N(\overline{b})$, we also have $\mathrm{tp}_{M^{eq}}(\overline{a}) = \mathrm{tp}_{N^{eq}}(\overline{b})$.*

*Proof.* For (1) note that there is only one way in which we can extend $f$ to an elementary embedding, which is by setting $\tilde{f}(\overline{a}/E) = f(\overline{a})/E$ for each $n_E$-tuple $\overline{e}$ and $E \in \mathrm{ER}(T)$. It is easy to prove the map $\tilde{f}$ is elementary by the Tarski-Vaught test.
For (2), since $M \equiv N$, $N \preceq \mathbb{M}$ and $M \preceq \mathbb{M}$, where $\mathbb{M}$ is the monster model for $\mathrm{Th}(M)$. So by (1), $N^{eq} \preceq \mathbb{M}^{eq}$ and $M^{eq} \preceq \mathbb{M}^{eq}$, meaning that $M^{eq} \equiv N^{eq}$.
(3) is obvious by restricting $\mathcal{M}^\star$ to the home sort, and (5) is an obvious consequence of (4) by considering the monster model. For (4) note that by (1) any $\sigma \in \mathrm{Aut}(M)$ extends uniquely to an elementary embedding $\tilde{\sigma}$ of $M^{eq}$ into itself. This is surjective since every element of $M^{eq}$ is of the form $\overline{a}/E = \pi_E(\overline{a})$. Hence $\tilde{\sigma}$ is an automorphism.  $\square$

*Remark* 10.6. With the same ideas it is possible to prove that $T^{eq}$ is axiomatized by $T$ joined with, for each $E \in \mathrm{ER}(T)$, formulas asserting that $\pi_E$ is surjective and that it maps elements in the same $E$-class to the same element.

**Corollary 10.7.** *For $E_1, \ldots E_k \in \mathrm{ER}(T)$ and $\phi(x_1^{E_1}, \ldots x_k^{E_k})$ an $\mathcal{L}^{eq}$-formula, there is an $\mathcal{L}$-formula $\psi(\overline{y}_1, \ldots \overline{y}_k)$ such that*

$$T^{eq} \vdash \forall \overline{y}_1 \ldots \overline{y}_k (\psi(\overline{y}_1 \ldots \overline{y}_k) \leftrightarrow \phi(\pi_{E_1}(\overline{y}_1) \ldots \pi_{E_k}(\overline{y}_k))).$$

**Exercise 10.8.** Let $F$ be the forgetful map

$$F : S_{(S_=)^n}(T^{eq}) \to S_n(T)$$

sending types of real $n$-tuples in $T^{eq}$ to their restriction to an $n$-type in $T$. Show that $F$ is a homeomorphism. Prove Corollary 10.7.

**Exercise 10.9.** Let $\kappa$ be an infinite cardinal. Show the following:

- If $\mathcal{M}$ is $\kappa$-saturated, then $\mathcal{M}^{eq}$ is $\kappa$-saturated;

- If $\mathcal{M}$ is strongly $\kappa$-homogeneous, then $\mathcal{M}^{eq}$ is strongly $\kappa$-homogeneous;

- If $T$ is $\kappa$-categorical, then $T^{eq}$ is $\kappa$-categorical.

- If $T$ is $\kappa$-stable, then $T^{eq}$ is $\kappa$-stable;

- If $T$ is stable, then $T^{eq}$ is stable;

- If $T$ is NIP, then $T^{eq}$ is NIP;

- If $T$ is NSOP, then $T^{eq}$ is NSOP.

For $X$ a non-empty set and $G \curvearrowright X$ we can endow $G$ with the pointwise-convergence topology, where stabilizers of finite sets $G_A$ for $A \subseteq X$ finite form a basis of clopen neighbourhoods of the identity. Hence, the cosets of stabilizers of finite sets $gG_A$ for $A \subseteq X$ finite and $g \in G$ form a basis of clopen sets.

**Exercise 10.10.** Consider $\mathrm{Aut}(M) \curvearrowright M$ for $M$ countable and $\omega$-categorical. Note that $\mathrm{Aut}(M) = \mathrm{Aut}(M^{eq})$. Prove that the open subgroups of $\mathrm{Aut}(M)$ are precisely the stabilizers of imaginaries $\mathrm{Aut}(M/e)$ for $e \in M^{eq}$. [Hint: the ($\Leftarrow$) direction does not use $\omega$-categoricity. For the ($\Rightarrow$) direction, note that for any $H \leq \mathrm{Aut}(M)$ and $\overline{a} \in \mathrm{Aut}(M)$ the equivalence relation $E$ on the $\mathrm{Aut}(M)$-orbit of $\overline{a}$ $\mathrm{Orb}(\overline{a})$ given by

$$E(g_1 \overline{a}, g_2 \overline{a}) \text{ if and only if } g_2^{-1} g_1 \in H$$

is 0-definable.]

# 11 How to eliminate your new imaginary friends

Let $X := \phi(\mathbb{M}, b)$ be a definable set. We can code information about $X$ by an element in $\mathbb{M}^{eq}$ as we explain below.

**Definition 11.1.** Let $X \subseteq \mathbb{M}$ be definable (possibly with parameters). We say that a (possibly imaginary) tuple $d$ is a **canonical parameter** for $X$ if for all $\sigma \in \mathrm{Aut}(\mathbb{M})$,

$$\sigma(X) = X \text{ setwise if and only if} \sigma(d) = d.$$

Now, for $X := \phi(\mathbb{M}, b)$, consider the equivalence relation $\sim$ on tuples of the same arity as $b$ given by

$$a \sim c \text{ if and only if } \phi(\mathbb{M}, a) = \phi(\mathbb{M}, b).$$

Then, it is easy to see that $\tilde{b} := b/\sim$ is a canonical parameter for $X$.

👁 *Observation* 11.2. The canonical parameter $\tilde{b} := b/\sim$ we just constructed depends on the choice of formula $\phi$. However, it is easy to see that any two canonical parameters are interdefinable. That is, if $d$ and $e$ are canonical parameters for the same definable set $X$, then $d \in \mathrm{dcl}^{eq}(e)$ and $e \in \mathrm{dcl}^{eq}(d)$. In fact, from Lemma 2.15, $d \in \mathrm{dcl}^{eq}(e)$ if and only if

$$\text{for all } \sigma \in \mathrm{Aut}(\mathbb{M}), \sigma(d) = d \Leftrightarrow \sigma(e) = e,$$

and since both $e$ and $d$ are canonical parameters for the same set, they must be preserved by exactly the same automorphisms. Conversely, any tuple interdefinable with a canonical parameter is also a canonical parameter.

**Exercise 11.3.** Let $X$ be a definable subset of $\mathbb{M}^n$. Let $A$ be a set of parameters, and let $b$ be a canonical parameter for $X$.

- Show that $X$ is definable over $A$ if and only if $b \in \mathrm{dcl}^{eq}(A)$;

- Show that the following are equivalent:

   - $X$ is **almost $A$ definable**, i.e. there is an $A$-definable equivalence relation $E$ on $n$-tuples with finitely many classes and such that $X$ is a union of $E$-classes;
   - $\{\sigma(X)|\sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite;
   - $\{\sigma(X)|\sigma \in \mathrm{Aut}(\mathbb{M}/A)\} < |\mathbb{M}|$;
   - $b \in \mathrm{acl}^{eq}(A)$.

**Definition 11.4.** $T$ has **elimination of imaginaries** (EI) if for every $e \in \mathbb{M}^{eq}$ there is a real tuple $\overline{b}$ from $\mathbb{M}$ such that
$$e \in \mathrm{dcl}^{eq}(\overline{b}) \text{ and } \overline{b} \in \mathrm{dcl}^{eq}(e).$$

**Lemma 11.5.** *T has EI if and only if every definable $X \subseteq \mathbb{M}$ has a real (i.e. in $\mathbb{M}$) canonical parameter.*

*Proof.* ($\Rightarrow$) This direction is trivial since you can just pick a canonical parameter for $X$ and then choose the real tuple interdefinable with it, which will also be a canoncal parameter for $X$. ($\Leftarrow$) Consider the imaginary element $e$. We know $e = b/E$ for some real tuple $b$ and an $\varnothing$-definable equivalence relation $E$. Consider $X := E(\mathbb{M}, b)$ and let $d$ be a real canonical parameter for $X$. Since they are preserved by the same automorphisms $e \in \mathrm{dcl}^{eq}(d)$ and $d \in \mathrm{dcl}^{eq}(e)$. $\square$

**Exercise 11.6.** Show that $T$ has EI if and only if for each $\phi(\overline{x}) \in \mathcal{L}(\mathbb{M})$ there is $\psi(x,y) \in \mathcal{L}$ such that there is a unique $\overline{a} \in \mathbb{M}$ such that $\phi(\mathbb{M}) = \psi(\mathbb{M}, \overline{a})$.

**Exercise 11.7.** Show that $T^{eq}$ eliminates imaginaries. [Hint/thoughts: techically, there is no problem in defining imaginaries for a multi-sorted structure and so check the original definition. However, it is probably more convenient to show that definable subsets of $\mathbb{M}^{eq}$ have canonical parameters in $\mathbb{M}^{eq}$.]

**Definition 11.8.** $T$ has **weak elimination of imaginaries** (WEI) if for every $e \in \mathbb{M}^{eq}$ there is a real tuple $\overline{b}$ from $\mathbb{M}$ such that

$$e \in \mathrm{dcl}^{eq}(\overline{b}) \text{ and } \overline{b} \in \mathrm{acl}^{eq}(e).$$

**Exercise 11.9.** Show that the following are equivalent

1. $T$ has weak elimination of imaginaries;

2. For every definable $X$ there is a finite set $\{\overline{a}_1, \ldots, \overline{a}_n\}$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{M})$

$$\sigma(X) = X \text{ if and only if } \sigma \text{ permutes the } \overline{a}_i;$$

3. For each $\phi(\overline{x}) \in \mathcal{L}(\mathbb{M})$ there is $\psi(\overline{x}, \overline{y})$ such that there is only a finite number of $\overline{a}$ such that $\phi(\mathbb{M}) = \psi(\mathbb{M}, \overline{a})$;

24

4. For every definable $X$ there is a smallest (real) algebraically closed set $A$ such that $X$ is $A$-definable.

**Example 11.10.** The theory of an infinite set with equality $T_\infty$ has weak elimination of imaginaries but does not eliminate imaginaries. Let $e$ be a canonical parameter for the definable set $\{a, b\}$ where $a, b$ are real elements from $\mathbb{M}$. Note that

$$\mathrm{dcl}^{eq}(e) \cap \mathbb{M} = \varnothing$$

since there is an automorphism fixing $e$ but moving every element of $\mathbb{M}$ (which swaps $a$ and $b$). However,

$$\mathrm{acl}^{eq}(e) \cap \mathbb{M} = \{a, b\}$$

and $e \in \mathrm{dcl}^{eq}(ab)$. More generally, WEI can be deduced from the following exercise:

**Exercise 11.11.** Let $\mathcal{M}$ be the countable model of an $\omega$-categorical theory. The following are equivalent:

- $\mathrm{Th}(M)$ has WEI;

- for all $A, B \subseteq M$ algebraically closed, we have

$$\mathrm{Aut}(M/A \cap B) = \langle \mathrm{Aut}(M/A), \mathrm{Aut}(M/B) \rangle.$$

**Lemma 11.12.** *Let $T$ be strongly minimal and such that* $\mathrm{acl}^{eq} \cap \mathbb{M}$ *is infinite. Then, $T$ has WEI.*

*Proof.* Let $e \in \mathbb{M}^{eq}$. So, $e = \bar{b}/E$ for $E$ some $\varnothing$-definable equivalence relation. Let

$$M_0 := \mathrm{acl}^{eq}(e) \cap \mathbb{M}.$$

By assumption, $M_0$ is infinite. We wish to find $c_1, \ldots, c_n$ in $M_0$ such that $e = \pi_E(\bar{c})$, where $\pi_E$ is the projection to the sort of $E$-classes. We prove this by induction. For the inductive hypothesis, suppose that we have $c_1, \ldots, c_{i-1}$ in $M_0$ such that for some $d_i, \ldots, d_n$ in $\mathbb{M}$,

$$e = \pi_E(c_1, \ldots, c_{i-1}, d_i, \ldots, d_n).$$

For the inductive step we find $c_i$. First, let $X$ be defined by

$$\phi(x_i) := \exists x_{i+1} \ldots x_n (e = \pi_E(c_1, \ldots, c_{i-1}, x_i, x_{i+1}, \ldots, x_n)).$$

Whilst $\psi(x_i)$ has imaginary parameters, we know that $X$ is also $\mathcal{L}(\mathbb{M})$-definable by Corollary 10.7. So, since $T$ is strongly minimal we have that $X$ is either finite or cofinite. If $X$ is finite, we have

$$X \subseteq \mathrm{acl}^{eq}(ec_1 \ldots c_{i-1}) \cap \mathbb{M} = M_0,$$

where the last equality follows from the fact that the $c_j$ are already in $M_0$ by hypothesis. Hence, take $c_i$ to be any element in $X$, and it will already satisfy the assumption. Otherwise, suppose that $X$ is cofinite. Then, since $M_0$ is infinite, $X \cap M_0$ is non-empty and so pick $c_i \in X \cap M_0$. At the end of this inductive process, we found $\bar{c} \in \mathrm{acl}^{eq}(e) \cap \mathbb{M}$ such that $e = \pi_E(\bar{c})$ (and so $e \in \mathrm{dcl}^{eq}(\bar{c})$), proving WEI. $\square$

**Lemma 11.13** (Chow coordinates). *Let $F$ be a field. Let $\bar{b}_1, \ldots, \bar{b}_m$ be $n$-tuples from $F$. Then, there is a tuple $\bar{c}$ from $F$ such that for all $\sigma \in \mathrm{Aut}(F)$,*

$$\sigma(\bar{c}) = \bar{c} \text{ if and only if } \sigma \text{ permutes the } \bar{b}_i.$$

*Proof.* For $\bar{b}_i = (b_i^1 \ldots b_i^n)$ let $p(X_1, \ldots, X_n, Y)$ be the polynomial

$$\prod_{i=1}^m \left( Y - \sum_{j=1}^n b_i^j X_j \right).$$

Let $\bar{c}$ be the coefficient of $p$. Let $\sigma \in \mathrm{Aut}(F)$. Since $F[X_1, \ldots, X_n, Y]$ is a unique factorisation domain, if $\sigma$ fixes $\bar{c}$, then it must permute the $\bar{b}_i$. Conversely, every automorphism permuting the $\bar{b}_i$ must fix $\bar{c}$. Hence, $\bar{c}$ is the desired tuple. $\square$

**Corollary 11.14.** *The theory of algebraically closed fieds in characteristic $p$,* $\mathrm{ACF}_p$*, has EI.*

*Proof.* Let $X$ be definable in $\mathbb{M} \vDash \mathrm{ACF}_p$. We know this theory is strongly minimal and $\mathrm{acl}(\varnothing)$ is infinite. So, by Lemma 11.12, it has weak elimination of imaginaries. By Exercise 11.9, there are $\{\bar{b}_1, \ldots, \bar{b}_n\}$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{M})$

$$\sigma(X) = X \text{ if and only if } \sigma \text{ permutes the } \bar{b}_i.$$

But by Lemma 11.13, we can then find $\bar{c}$ that is fixed by an automorphism if and only if that automorphism permutes the $\bar{b}_i$, meaning that $\bar{c}$ is a canonical parameter for $X$. Thus, $\mathrm{ACF}_p$ has EI. $\qquad\square$

## 12 Definability of types

**Definition 12.1.** Let $\phi(x, y)$ be an $\mathcal{L}$-formula. For a $\phi$-type $p \in S_\phi(B)$, we say that $p$ is **definable over** $C$ if there is $\psi(y) \in \mathcal{L}(C)$ such that for all $b \in B$,

$$\phi(x, b) \in p \text{ if and only if } \vDash \psi(b).$$

We say that $p$ is definable if it is definable over $B$. For a type $p \in S_x(B)$, we say that $p$ is definable over $C$ if for each $\psi(x, y) \in \mathcal{L}$, $p_{\restriction \phi}$ is definable over $C$. And similarly $p$ is definable if it is definable over $B$.

**Exercise 12.2.** Let $p \in S_x(M)$ be a definable type. Show that for each $B \supseteq M$, $p$ has a unique extension $q \in S_x(B)$ which is definable over $M$.

**Lemma 12.3.** *$\phi(x, y)$ is stable if and only if all $\phi$-types (over arbitrary sets of parameters) are definable.*

*Proof.* ($\Leftarrow$) Take $|A| \geq |T|$. Then, since all $\phi$-types over $A$ are definable and there are at most $|A|$, many $\mathcal{L}(A)$-formulas,

$$|S_\phi(A)| \leq |A|,$$

and so $\phi(x, y)$ is stable.

($\Rightarrow$) Let $\theta(x) \in \mathcal{L}(\mathbb{M})$. Since $\phi$ does not have the binary tree property, there is a largest $n$ such that there is a binary tree $(b_s | s \in {}^{<n}2)$ such that for all $\delta \in {}^n 2$,

$$\{\theta(x)\} \cup \{\phi^{\sigma(i)}(x, b_{\sigma_{\restriction i}}) | i < n\} \text{ is consistent.}$$

Otherwise, by compactness, we would get a binary tree of parameters witnessing the binary tree property. Hence, set the **$\phi$-degree of $\theta$** $D_\phi(\theta(x))$ to be this maximal $n$. We make few observations about $D_\phi$:

1. for $\theta(x, y) \in \mathcal{L}$,
$$\{c \in \mathbb{M} | D_\phi(\theta(x, c) \geq n\} \text{ is definable .}$$

   To see this note that the definition is given by

   $$\exists (z_s)_{s \in {}^{<n}2} \bigwedge_{\sigma \in {}^n 2} \left( \exists x \left( \theta(x, y) \wedge \bigwedge_{i < n} \phi^{\sigma(i)}(x, z_{\sigma_{\restriction i}}) \right) \right).$$

2. If $\phi(\mathbb{M}) \subseteq \xi(\mathbb{M})$, then $D_\phi(\theta(x)) \leq D_\phi(\xi(x))$.

3. For $D_\phi(\theta(x)) = n$ if at most one of $\theta(x) \wedge \psi(x, a)$ and $\phi(x) \wedge \neg\phi(x, a)$ has $\phi$-degree $n$. Otherwise, we could build a binary tree of height $n + 1$ witnessing $D_\phi(\theta(x)) = n + 1$.

Now, for $p \in S_\phi(B)$ pick $\theta(x)$ to be a conjunction of formulas from $p$ with minimal $\phi$-degree $n$. Then,
$$\phi(x, b) \in p \text{ if and only if } D_\phi(\theta(x) \wedge \phi(x, b)) = n.$$
The left-to-right implication is trivial from (2) and minimality of $n$. The right-to-left implication follows since by (2) and (3) $D_\phi(\theta(x) \wedge \phi(x, b)) = n$ implies $D_\phi(\theta(x) \wedge \neg\phi(x, b)) < n$, which by minimality of $n$ implies that $\neg\phi(x, b) \notin p$, and so $\phi(x, b) \in p$. Finally, from (1), $D_\phi(\theta(x) \wedge \phi(x, y)) = n$ is definable by an $\mathcal{L}(B)$-formula $\phi(y)$. Hence, this gives definability of $p$. $\qquad\square$

**Corollary 12.4.** *$T$ is stable if and only if all types are definable.*

**Exercise 12.5.** Let $\phi(x, y)$ be stable and let $p \in S_\phi(M)$ for $M$ a model. Show that $p$ is definable by a Boolean combination of $\phi^{\mathrm{opp}}$-formulas with parameters from $M$. [Hint: imitate the proof of Erdös-Makkai, which you proved in a previous problem sheet. In particular, note that for all finite $c_0, \ldots, c_n \in M$, $p$ is not defined by a Boolean combination of $\phi^{\mathrm{opp}}$-formulas with parameters from the $c_i$. Hence, you can inductively build sequences $(b_i)_{i<\omega}$, $(b_i)'_{i<\omega}$, and $(c_j)_{j<\omega}$ such that either $(b_i c_i)_{i<\omega}$ or $(b'_i c_i)_{i<\omega}$ yield that $\phi$ has the order property.]

$\star\star$ **Exercise 12.6.** Show that in the exercise above you can choose the Boolean combination of $\phi^{\mathrm{opp}}$-formulas to be positive. [Hint: the proof is almost identical but you need to be slightly more careful in the construction of the sequences. In particular, keep in mind that $X \subsetneq A$ is a positive Boolean combination of $X_0, \ldots, x_n$ if and only if for all $x, y \in A$ if $x \in X$ and, for every $i \le n$, we have that if $x \in X_i$, then $y \in X_i$, then $y \in X$.]

**Definition 12.7.** Let $X$ be a topological space. The **derived set** $X'$ is the set of limit points of $X$ (i.e. the set of non-isolated points). For an ordinal, we define inductively the **Cantor-Bendixon derivative** $X^{(\alpha)}$: $X^{(0)} := X$, $X^{(\alpha^+)} := \left(X^{(\alpha)}\right)'$, and for $\lambda$ a limit ordinal, $X^{(\lambda)} := \bigcap_{\alpha<\lambda} X^{(\alpha)}$. We will need this only for finite ordinals.

**Exercise 12.8.** Let $\phi(x, y)$ be stable. Let $X \subseteq S_\phi(\mathbb{M})$ be closed and non-empty. Then, $X^{(n+1)} = \varnothing$ for some $n \ge 0$. Moreover, (choosing $n$ to be minimal such that $X^{(n+1)} = \varnothing$), $X^{(n)}$ is finite. [Hint: first show that if $X^{(n+1)} \ne \varnothing$, there is a binary tree of parameters $(b_s | s \in {}^{<n+1}2)$ witnessing the (finite) binary tree property of height $n+1$. For finiteness of $X^{(n)}$, observe that every $X^{(i)}$ is closed.]

# 13   Properties of definable extensions

*In this and every other class, when we speak about topologies on a space of types or partial types we always refer to the standard logic (or Stone) topology. The idea is that to every Boolean algebra $B$, we may associate the Stone space $S(B)$ of ultrafilters on $B$. The Stone (or logic) topology on $S(B)$ is given by taking as a basis of clopen sets for the topology sets of the form*
$$[b] := \{p \in S(B) | b \in p\},$$
*where $b$ is an element of the Boolean algebra $B$. It is also possible to prove that every clopen set has to be of the form $[b]$ for some $b \in B$.*

*Notation* 13.1. Given $\phi(x, y) \in \mathcal{L}$ and $p \in S_\phi(B)$ definable over some set of parameters by a formula $\psi(y)$, we write $\mathrm{d}_p x \phi(x, y)$ for $\psi(y)$, the $\phi$-definition of $p$. Similarly, for $p \in S_x(B)$, $\mathrm{d}_p x \phi(x, y)$ denotes the $\phi$-definition of $p_{\restriction\phi}$. Obviously, the same $p \in S_\phi(B)$ can have multiple definitions (e.g. over different sets of parameters), but they will all denote the same definable set.

*Remark* 13.2. Let $q \in S_\phi(\mathbb{M})$ be definable. Note that for $\sigma, \tau \in \mathrm{Aut}(\mathbb{M})$, we have
$$\sigma q = \tau q \text{ if and only if } \sigma(\mathrm{d}_q x \phi(x, \mathbb{M})) = \tau(\mathrm{d}_q x \phi(x, \mathbb{M})).$$

This is easy to see from the definition of definability of a type.

**Lemma 13.3** (Existence)**.** *Let $\phi(x, y)$ be stable, $p \in S_x(A)$. Then, there is $q \in S_\phi(\mathbb{M})$ such that*

- *$p(x) \cup q(x)$ is consistent; and*

- *$q$ is definable over $\mathrm{acl}^{eq}(A)$.*

*Proof.* Let $X := \{q(x) \in S_\phi(\mathbb{M}) | p(x) \cup q(x) \text{ is consistent }\}$. Firstly, note that $X$ is closed in $S_\phi(\mathbb{M})$. To prove this, note that $X$ is the image of $Y := \{q \in S_x(\mathbb{M}) | p \subseteq q\}$ under the restriction map $S_x(\mathbb{M}) \mapsto S_\phi(\mathbb{M})$. It is easy to see that $Y$ is closed, since $Y := \bigcap_{\psi(x,b) \in p} [\psi(x, b)]$. The restriction map is continuous, and so $X$ is closed being the continuous image of a compact set into a Hausdorff space.

Since $X$ is a closed subset of $S_\phi(\mathbb{M})$, by Exercise 12.8, $X^{(n+1)} = \varnothing$ for some (minimal) $n \in \omega$, where $X^{(n)}$ is the $n$th Cantor-Bendixon derivative of $X$. Moreover, we know that $X^{(n)}$ is finite. Also note that $X^{(i)}$ is $\mathrm{Aut}(\mathbb{M}/A)$-invariant for all $i$. This is easy to see by induction since $X$ is $\mathrm{Aut}(\mathbb{M}/A)$-invariant and $\mathrm{Aut}(\mathbb{M}/A)$ acts homeomorphically on $S_\phi(\mathbb{M})$. In particular, for $q \in X$, the $\mathrm{Aut}(\mathbb{M}/A)$-orbit of $q$ is finite. We know that $q$ is definable by Lemma 12.3 with $\mathrm{d}_q x \phi(x, y) \in \mathcal{L}(\mathbb{M})$. By Remark 13.2, the $\mathrm{Aut}(\mathbb{M}/A)$-orbit of $\mathrm{d}_q x \phi(x, \mathbb{M})$ is also finite. Hence, by Exercise 11.3 $\mathrm{d}_q x \phi(x, y) \in \mathcal{L}(\mathrm{acl}^{eq}(A))$. $\square$

**Lemma 13.4** (Harrington's Lemma/ Symmetry)**.** *Let $\phi(x, y)$ be stable. Let $p \in S_x(\mathbb{M}), q \in S_y(\mathbb{M})$. Then,*
$$\mathrm{d}_p x \phi(x, y) \in q \text{ if and only if } \mathrm{d}_q y \phi(x, y) \in p.$$

*Proof.* Suppose by contradiction that $\mathrm{d}_q y \phi(x, y) \in p$ but $\mathrm{d}_p x \phi(x, y) \notin q$. Say that both definitions are over $A$ (a small subset of $\mathbb{M}$). By $|A|$-saturation, we can construct recursively $(a_n b_n)_{n < \omega}$ such that

1. $b_n \vDash q_{\upharpoonright A(a_j | j < n)}$; and

2. $a_n \vDash p_{\upharpoonright A(b_j | j \leq n)}$.

For $i > j$, we get

$$
\begin{aligned}
\vDash \phi(a_i, b_j) &\Leftrightarrow & \phi(x, b_j) \in p && \text{by (2)} \\
&\Leftrightarrow & b_j \vDash \mathrm{d}_p x \phi(x, y) && \text{by Definition 12.1} \\
&\Leftrightarrow & \mathrm{d}_p x \phi(x, y) \in q && \text{by (1) and the fact that } \mathrm{d}_p x \phi(x, y) \in \mathcal{L}(A).
\end{aligned}
$$

Similarly, for $i \leq j$
$$\vDash \phi(a_i, b_j) \text{ if and only if } \mathrm{d}_q y \phi(x, y) \in p.$$

So by our assumption, $\vDash \phi(a_i, b_j)$ if and only if $i < j$, contradicting stability of $\phi$, and so yielding the desired statement. $\square$

**Definition 13.5.** A **generalised $\phi$-type over** $A$ is a maximally consistent set of formulas over $A$ which are equivalent to a Boolean combination of $\phi$-formulas with parameters from $\mathbb{M}$ (i.e. not just in $A$).

**Example 13.6.** Consider the theory of an equivalence relation $E(x, y)$ with two infinite equivalence classes. An $E$-type over $\mathrm{acl}^{eq}(\varnothing)$ does not contain any formula since there are no parameters that $E(x, y)$ can take as arguments. However, for $b$ some element in $\mathbb{M}$, $b/E \in \mathrm{acl}^{eq}(\varnothing)$, and the formula $\pi_E(x) = b/E$ is equivalent to $E(x, b)$. Hence, there are two generalised $E$-types over $\mathrm{acl}^{eq}(\varnothing)$ stating that $x$ is in one of the two $E$-classes.

**Exercise 13.7.** Let $\mathcal{M}$ be a model and $p \in S_\phi(M)$. Show that $p$ extends uniquely to a generalised $\phi$-type.

In particular, when working over models, we will speak interchangeably of $\phi$-types and generalised $\phi$-types since they are essentially the same objects.

**Lemma 13.8** (Extension). *Let $\phi(x,y)$ be stable. Every generalised $\phi$-type over $A = \operatorname{acl}^{eq}(A)$ has a unique global extension $q \in S_\phi(\mathbb{M})$ definable over $A$.*

*Proof.* Let $p$ be a generalised $\phi$-type over $A$. By existence (Lemma 13.3) applied to a completion of $p$, $p$ has at least one global extension $p_1 \in S_\phi(\mathbb{M})$ definable over $A$. Let $p_2$ be another such extension. Let $p_1'$ and $p_2'$ be completions. Consider $\phi(x,b) \in \mathcal{L}(\mathbb{M})$. We need to prove that $\phi(x,b) \in p_1$ if and only if $\phi(x,b) \in p_2$. Take $q := \operatorname{tp}(b/A)$. By existence (Lemma 13.3) again, there is $\overline{q}(y) \in S_{\phi^{\mathrm{opp}}}(\mathbb{M})$ definable over $A$ and such that $q(y) \cup \overline{q}(y)$ is consistent. Let $q' \in S_y(\mathbb{M})$ be a completion of $\overline{q}$ (noting that $q'_{\restriction \phi} = \overline{q}$. Now, for $i \in \{1, 2\}$

$$
\begin{aligned}
\phi(x,b) \in p_i &\Leftrightarrow & b &\vDash \mathrm{d}_{p_i} x \phi(x,y) & &\text{by Definition 12.1} \\
&\Leftrightarrow & \mathrm{d}_{p_i} x \phi(x,y) &\in q & &\text{since } p_i \text{ is } A\text{-definable and } q = \operatorname{tp}(b/A) \\
&\Leftrightarrow & \mathrm{d}_{p_i} x \phi(x,y) &\in q' & &\text{since } q' \text{ extends } q \\
&\Leftrightarrow & \mathrm{d}_{q'} y \phi(x,y) &\in p_i' & &\text{by Symmetry (Lemma 13.4).} \quad (3)
\end{aligned}
$$

But by Exercise 12.5, $\mathrm{d}_{q'} y \phi(x,y)$ is equivalent to a Boolean combination of $\phi$-formulas (and is $A$-definable). So, by definition of a generalised $\phi$-type

$$
d_{q'}\phi(x,y) \in p_i' \text{ if and only if } d_{q'}\phi(x,y) \in p \quad (4)
$$

Thus, combining the conclusions in 3 and 4, we get that

$$
\phi(x,b) \in p_1 \text{ if and only if } d_{q'}\phi(x,y) \in p \text{ if and only if } \phi(x,b) \in p_2,
$$

yielding the desired statement that $p_1 = p_2$. $\qquad \square$

**Definition 13.9.** Let $\phi(x,y)$ be an $\mathcal{L}$-formula. Let $A \subseteq \mathbb{M}$. We denote by $\mathrm{FER}_\phi(A)$ the collection of equivalence relations $E(x,y)$ on $\mathbb{M}$ with finitely many classes such that for each $a \in \mathbb{M}$ the equivalence class of $a$, $E(\mathbb{M}, a)$ is equivalent to a Boolean combination of $\phi$-formulas over $A$.

**Theorem 13.10** (Finite Equivalence Relations Theorem). *Let $\phi(x,y)$ be a stable. Let $p$ be a generalised $\phi$-type over $A \subseteq \mathbb{M}$. Let*

$$
Y := \{q(x) \in S_\phi(\mathbb{M}) \mid q(x) \text{ is an extension of } p \text{ definable over } \operatorname{acl}^{eq}(A)\}.
$$

*Then, $Y$ is finite, $\operatorname{Aut}(\mathbb{M}/A)$ acts transitively on $Y$, and there is an equivalence relation $E \in \mathrm{FER}_\phi(A)$ such that for all $q_1, q_2 \in Y$, $q_1 = q_2$ if and only if $q_1(x) \cup q_2(y) \vdash E(x,y)$.*

**Exercise 13.11.** Prove the Finite Equivalence Relations Theorem.

**Definition 13.12.** Let $X$ be an $\varnothing$-definable subset of $\mathbb{M}$. We say that $X$ is **stably embedded** if every $\mathcal{L}(M)$-definable $Y \subseteq X$ is $\mathcal{L}(X)$-definable (i.e. definable already with parameters from $X$).

**Exercise 13.13.** Let $X$ be an $\varnothing$-definable subset of $\mathbb{M}$. Show that the following are equivalent:

- $X$ is stably embedded;

- Every type $\operatorname{tp}(a/X)$ is definable over some $C \subseteq X$;

- For every $a$ there is a subset $C \subseteq X$ such that $\operatorname{tp}(a/C) \vdash \operatorname{tp}(a/X)$;

- Every automorphism of $X$ extends to an automorphism of $\mathbb{M}$.

**Exercise 13.14.** Let $T$ be stable. Let $X$ be an $\varnothing$-definable subset of $\mathbb{M}$. Show that $X$ is stably embedded.

# 14 Dividing

*Note that our definition of dividing takes place in the monster model* $\mathbb{M}$.

**Definition 14.1.** We say that $\phi(x, b)$ **divides over** $A$ if there is some $k \in \mathbb{N}$ and $(b_i)_{i<\omega}$ such that $b_i \equiv_A b$ for all $i < \omega$ and $\{\phi(x, b_i) | i < \omega\}$ is $k$-inconsistent (i.e. any $k$-many formulas in the set are inconsistent). If we want to keep track of $k$, we sometimes say that $\phi(x, b)$ $k$-divides over $A$.

A set of formulas $\pi(x)$ divides over $A$ if it implies a formula dividing over $A$.

**Examples 14.2.** In the theory DLO of dense linear orders, $a < x < b$ 2-divides over $\varnothing$, but $a < x$ does not divide over $\varnothing$. In the theory of the generic triangle-free graph, $E(x, a) \wedge E(x, b)$ 2-divides over $\varnothing$. To see this, consider a sequence $(a_i b_i)_{i<\omega}$ forming a copy of the half-graph.

**Exercise 14.3.** Prove that $\phi(x, b)$ divides over $A$ if and only if there is $(b_i)_{i<\omega}$ indiscernible over $A$ with $b_0 \equiv_A b$ and $\{\phi(x, b_i) | i < k\}$ is inconsistent.

**Exercise 14.4.** Prove that $\phi(x, b)$ divides over $A$ if and only if $\phi(x, b)$ divides over $\mathrm{acl}^{eq}(A)$.

**Exercise 14.5.** Show the following are equivalent:

- $\mathrm{tp}(a/Ab)$ does not divide over $A$;

- For every infinite $A$-indiscernible sequence $I$ such that $b \in I$, there is some $a' \equiv_{Ab} a$ such that $I$ is $Aa'$-indiscernible;

- For every infinite $A$-indiscernible sequence $I$ such that $b \in I$ there is some $J \equiv_{Ab} I$ such that $J$ is $Aa$-indiscernible.

**Exercise 14.6.** Use the above exercise to show the following property of dividing: Suppose $\mathrm{tp}(a/B)$ does not divide over $A \subseteq B$ and $\mathrm{tp}(b/Ba)$ does not divide over $Aa$. Then, $\mathrm{tp}(ab/B)$ does not divide over $A$.

**Definition 14.7.** A partial type $\pi$ is finitely satisfiable over $A$ if every finite conjunction of formulas from $\pi$ is realised by a tuple in $A$.

Note that for a model $M$, $p \in S_x(M)$ is always finitely satisfiable in $M$.

👁 *Observation* 14.8. Definable sets not satisfied in $A$ form an ideal in $\mathrm{Def}_x(\mathbb{M})$ in the sense of Definition 3.12. Hence, for $\pi(x)$ over $A$ finitely satisfiable in $A$ and $B \supseteq A$, there is $p \in S_x(B)$ finitely satisfiable in $A$ (by Exercise 3.13).

**Exercise 14.9.** Prove the following

- If $p \in S_x(\mathbb{M})$ is finitely satisfiable in $A$, then $p$ is $\mathrm{Aut}(\mathbb{M}/A)$-invariant;

- If $p$ is $\mathrm{Aut}(\mathbb{M}/A)$-invariant and $(b_i)_{i<\omega}$ is such that $b_i \vDash p_{\restriction A(b_j | j < i)}$, then $(b_i)_{i<\omega}$ is an $A$-indiscernible sequence.

**Exercise 14.10.** Let $\phi(x, y)$ be stable. Let $p \in S_\phi(\mathbb{M})$ be definable over $M$ and consistent with a partial type $\pi(x)$ over $M$. Then, $\pi(x) \cup p(x)$ is finitely satisfiable in $M$.

**Theorem 14.11.** *Let* $\phi(x, y)$ *be stable. The following are equivalent:*

1. *There is* $p \in S_\phi(\mathbb{M})$ *definable over* $\mathrm{acl}^{eq}(A)$ *with* $\phi(x, b) \in p$;

2. $\phi(x, b)$ *does not divide over* $A$.

*Proof.* Firstly, by Exercise 14.4, without loss of generality, we may assume $A = \text{acl}^{eq}(A)$. For $(1) \Rightarrow (2)$ consider $(b_i)_{i<\omega}$ in $\text{tp}(b/A)$. By assumption, $\phi(x,b) \in p$, which is $A$-definable. So $b \vDash \text{d}_p x \phi(x,y)$. Since $\text{d}_p x \phi(x,y) \in \mathcal{L}(A)$ and $b_i \equiv_A b$, we also have that $b_i \vDash \text{d}_p x \phi(x,y)$. This implies that

$$\{\phi(x,b_i) \mid i < \omega\} \subseteq p,$$

and so this set is consistent and $\phi(x,b)$ does not divide over $A$ by Exercise 14.3.

Now, for $(2) \Rightarrow (1)$, suppose that $\phi(x,b)$ does not divide over $A$. Let $\overline{q} := \text{tp}(b/A)$. By Existence (Lemma 13.3, there is $q' \in S_{\phi^{\text{opp}}}(\mathbb{M})$ such that $\overline{q}(y) \cup q'(y)$ is consistent and $q'$ is $A$-definable. So it is $M$-definable for some $M \supseteq A$. So, $\overline{q}(y) \cup q'(y)$ is finitely satisfiable in $M$ by Exercise 14.10. So, there is $q \in S_y(\mathbb{M}$ extending $\overline{q} \cup q'$ and finitely satisfiable in $M$ by Observation 14.8. Consider $(b_i)_{i<\omega}$ such that $b_i \vDash q_{\restriction \text{acl}^{eq}(M(b_j \mid j<i))}$. This sequence is $M$-indiscernible, and so $A$-indiscernible by Exercise 14.9. Since we are assuming that $\phi(x,b)$ does not divide over $A$, by Exercise 14.3, we have that $\{\phi(x,b_i) \mid i < \omega\}$ is consistent. Take $a \vDash \{\phi(x,b_i) \mid i < \omega\}$ and $B := \text{acl}^{eq}(A(b_i \mid i < \omega))$. Now, by Existence (Lemma 13.3), take

- $p \in S_\phi(\mathbb{M})$ extending $\text{tp}(a/A)$ and $A$-definable;

- $r \in S_\phi(\mathbb{M})$ extending $\text{tp}(a/B)$ and $B$-definable.

We want to show that $\phi(x,b) \in p$ by relating the $\phi$-definitions of $p$ and $r$. To do this, note that

$$
\begin{array}{llll}
\text{d}_p x \phi(x,y) \in q & \text{if and only if} & \text{d}_q y \phi(x,y) \in p & \text{by Symmetry (Lemma 13.4)} \quad (5) \\
& \text{if and only if} & \text{d}_q y \phi(x,y) \in r & \text{since } p_{\restriction A} = \text{tp}(a/A) = r_{\restriction A} \quad (6) \\
& \text{if and only if} & \text{d}_r x \phi(x,y) \in q & \text{by Symmetry (Lemma 13.4).} \quad (7)
\end{array}
$$

Now, $\text{d}_r x \phi(x,y) \in q$ only uses finitely many parameters from $B$ and so it must be over $\text{acl}^{eq}(A(b_i \mid i < k))$ for some $k \in \mathbb{N}$. Note that $\phi(x,b_k) \in r$ since $r$ extends $\text{tp}(a/B)$. Hence, $b_k \vDash \text{d}_r x \phi(x,y)$. Hence, by the equivalence from 7, $\text{d}_p x \phi(x,y) \in q$. Since $q_{\restriction A} = \text{tp}(b/A)$, $b \vDash \text{d}_p x \phi(x,y)$. Thus, $\phi(x,b) \in p$, completing the proof. $\qquad \square$

# 15 Forking

To some extent, for a definable set $X$ to divide over $A$ means that $X$ is "tiny", since it can be moved around by automorphisms whilst avoiding too much overlapping. However, sets in $\text{Def}_x(\mathbb{M})$ dividing over $A$ do not necessarily form an ideal, meaning that we cannot *a priori* extend partial types whilst avoiding dividing formulas using Exercise 3.13. For this reason, we need to define the notion of forking.

**Definition 15.1.** The formula $\phi(x,b)$ **forks over** $A$ if it belongs to the ideal generated by formulas dividing over $A$ in $\text{Def}_x(\mathbb{M})$. Equivalently, if there are $\phi_i(x,b_i)$ for $i < n$ dividing over $A$ and such that

$$\phi(\mathbb{M},b) \subseteq \bigvee_{i<n} \phi_i(\mathbb{M},b_i).$$

**Exercise 15.2.** Consider $\text{Th}(\mathbb{Q}, \text{cyc})$, where

$$\mathbb{Q} \vDash \text{cyc}(a,b,c) \Leftrightarrow (a < b < c) \vee (b < c < a) \vee (c < a < b).$$

Show that for $a \neq b$, $\text{cyc}(a,x,b)$ divides over $\emptyset$. Show that $x = x$ forks over $\emptyset$, but does not divide over $\emptyset$.

We now proceed to prove that forking and dividing agree in stable theories. We begin by generalising the definition of a $\phi$-type to arbitrary (finite) sets of formulas. We will immediately show that such types can be already described by talking about a single formula:

**Definition 15.3.** Let $\Delta = \{\phi_i(x,y) | i \le n\}$. A $\Delta$-formula over $A$ is of the form $\phi_i(x,a)$ or $\neg\phi_i(x,a)$ for $a \in A$. A complete $\Delta$-type over $A$ is a maximally consistent set of $\Delta$-formulas over $A$.

👁 *Observation* 15.4. We can prove that $\Delta$-types can be coded as $\psi_\Delta$-types for appropriate choice of formula $\psi_\Delta$:

**Exercise 15.5.** Let $\Delta$ be as above. Show that there is $\psi_\Delta(x, y_0, \ldots, y_n, z, z_0, \ldots, z_{2n})$ such that

- if $|A| \ge 2$, each $\Delta$-formula over $A$ is equivalent to some $\psi_\Delta(x, \bar{a})$ for $\bar{a}$ a tuple from $A$;

- any consistent formula of the form $\psi_\Delta(x, \bar{a})$ for $\bar{a}$ a tuple from $A$ is equivalent to a $\Delta$-formula over $A$;

- if all formulas in $\Delta$ are stable, then so is $\psi_\Delta$.

**Lemma 15.6.** *Let $\phi(x,y)$ and $\psi(x,z)$ be stable. Suppose that $\phi(x,a)$ and $\psi(x,b)$ divide over $A$. Then, $\phi(x,a) \vee \psi(x,b)$ divides over $A$.*

*Proof.* Without loss of generality, $A = \mathrm{acl}^{eq}(A)$ (by Exercise 14.4). Let $\delta(x; y, z) := \phi(x,y) \wedge \psi(x,z)$. We know this is stable (Exercise 9.2). Suppose by contrapositive that $\delta(x, ab)$ does not divide over $A$. Then, by Theorem 14.11, there is some $p \in S_\delta(\mathbb{M})$ definable over $A$ and such that $\delta(x, ab) \in p$. We want to prove that either $\phi(x,a)$ and $\psi(x,b)$ does not divide over $A$, which, by Theorem 14.11 means that we want to find either a global $\phi$-type or a global $\psi$-type definable over $A$ and containing, respectively, $\phi(x,a)$, or $\psi(x,b)$. We will use Exercise 15.5 to achieve this.

Let $p_0$ be the restriction of $p$ to a generalised $\delta$-type over $A$ (note this makes sense since generalised $\delta$-types and $\delta$-types agree over models). Let

$$\Delta := \{\phi(x,y), \psi(x,z), \delta(x,y,z)\}.$$

Let $p_1$ be a generalised $\psi_\Delta$-type over $A$ extending $p_0$, where $\psi_\Delta$ is as in Exercise 15.5. Then by Lemma 13.3 and Lemma 3.14, $p_1$ has a unique $A$-definable global extension $q' \in S_{\psi_\Delta}(\mathbb{M})$. Let $q \in S_\Delta(\mathbb{M})$ be the corresponding $\Delta$-type (by Exercise 15.5). Now, by construction and uniqueness (Lemma 3.14), $q_{\restriction \delta} = p$ since they are both global $\delta$-types $A$-definable over $A$ extending the generalised $\delta$-type $p_0$. So $\delta(x, ab) \in q$, meaning that either $\phi(x,a)$ ot $\psi(x,b)$ is in $q$. But $q_{\restriction \phi}$ and $q_{\restriction \psi}$ are both $A$-definable, meaning that either $\phi(x,a)$ or $\psi(x,b)$ does not divide over $A$ by Theorem 14.11. $\square$

The following is immediate:

**Corollary 15.7.** *Let $T$ be stable. Then $\phi(x,b)$ forks over $A$ if and only if $\phi(x,b)$ divides over $A$.*

**Exercise 15.8.** Let $\phi(x,y)$ be stable. Show that $\phi(x,b)$ does not divide over $A$ if and only if it is satisfiable in every model containing $A$.
Now, consider a stable theory $T$. Let $p \in S_x(B)$ and let $A \subseteq B$. Show that the following are equivalent:

- $p$ does not fork over $A$;

- there is a global type extending $p$ which is $\mathrm{acl}^{eq}(A)$-invariant;

- there is a global type extending $p$ which is $\mathrm{acl}^{eq}(A)$-definable.

# 16 Independence in Stable Theories

**Definition 16.1.** Let $p \in S(A)$ and $B \supseteq A$. Let $q \in S(B)$ be such that $q \supseteq p$. We say that $q$ is a **non-forking extension** of $p$ if it does not fork over $A$ (i.e., the set of parameters of $p$).

**Lemma 16.2** (Stationarity). *Let $T$ be stable. $p \in S(\mathrm{acl}^{eq}(A))$. Then, $p$ has a unique global non-forking extension.*

*Proof.* Firstly, note that (in a stable theory) any type does not fork over its own set of parameters. Hence, by Exercise 3.13 and the fact that forking formulas form an ideal, $p$ has a global non-forking extension. Then, Lemma 3.14 implies uniqueness. □

**Lemma 16.3** (Conjugacy). *Let $T$ be stable. Let $p \in S(A)$. Then, any two global non-forking extensions of $p$ are $A$-conjugates.*

*Proof.* Write $B := \mathrm{acl}^{eq}(A)$. From stationarity (Lemma 16.2), we know that once we moved to extensions of $p$ to $B$, these have a unique non-forking extension (and non-forking over $B$ is equivalent to non-forking over $A$ by Exercise 14.4). We will exploit this for our proof.

Take $q$ and $r$ global non-forking extensions of $p$. Take $a \vDash q$, $b \vDash r$. Since $q$ and $r$ are both extensions of $p$, which is over $A$, $a \equiv_A b$, meaning that there is some $\sigma \in \mathrm{Aut}(\mathbb{M})$ such that $\sigma a = b$. Now, note that since $q$ and $r$ do not fork over $A$, they also do not fork over $B$ (by Exercise 14.4 and definition of forking). Moreover, since forking over $A$ is invariant under automorphisms fixing $A$, $\sigma q$ does not fork over $\sigma B = B$, where the latter equality follows since $\mathrm{acl}^{eq}(A)$ is fixed setwise by $\mathrm{Aut}(\mathbb{M}/A)$. Now,

$$(\sigma q)_{\restriction B} = \sigma(q_{\restriction B}) = r_{\restriction B},$$

where the first equality holds since $B$ is fixed setwise by $\sigma$, and the second is just the fact that $\sigma a = b$. Hence $\sigma q = r$ by stationarity (Lemma 16.2), since they are both global non-forking extensions of $r_{\restriction B}$. □

**Definition 16.4.** We say that $A$ is **independent from** $B$ **over** $C$, and write

$$A \underset{C}{\downharpoonright} B$$

if for all finite tuples $a$ from $A$, $\mathrm{tp}(a/BC)$ does not fork over $C$.

**Theorem 16.5** (Independence in Stable Theories). *Let $T$ be stable. Then, $\downharpoonright$ (considered as a ternary relation on small subsets of $\mathbb{M}$) satisfies the following properties:*

1. *(Invariance) If $A \downharpoonright_C B$ and $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $\sigma A \downharpoonright_{\sigma C} \sigma B$;*

2. *(Finite Character) $A \downharpoonright_C B$ if and only if $A_0 \downharpoonright_C B_0$ for all finite $A_0 \subseteq A, B_0 \subseteq B$;*

3. *(Extension) if $A \downharpoonright_C B$, then for any $D$ there is $A' \equiv_{CB} A$ such that $A' \downharpoonright_C BD$;*

4. *(Algebraicty) If $A \downharpoonright_C A$, then $A \subseteq \mathrm{acl}^{eq}(C)$;*

5. *(Local Character) For $A$ finite and for any $B$ there is $C \subseteq B$ such that $|C| \leq |T|$ and $A \downharpoonright_C B$;*

6. *(Transitivity and Monotonicity) $A \downharpoonright_C B$ and $A \downharpoonright_{CB} D$ if and only if $A \downharpoonright_C BD$ (with transitivity being the $\Rightarrow$ implication and transitivity being the $\Leftarrow$ implication);*

7. *(Symmetry) $A \downharpoonright_C B$ if and only if $B \downharpoonright_C A$;*

8. *(Stationarity) Let $a$ and $b$ be finite. Suppose that $a \equiv_{\mathrm{acl}^{eq}(C)} b$ and $a \downharpoonright_C D$ and $b \downharpoonright_C D$. Then, $a \equiv_{\mathrm{acl}^{eq}(AC)} b$.*

**Exercise 16.6.** Prove Theorem 16.5.[Hint: Invariance, Finite Character, Extension, and Algebraicity all follow (trivially) from the definition of forking and independence and should not require stability. Local Character uses Exercise 15.8. Transitivity and Monotonicity uses Lemma 16.3 and may use Exercise 14.6. Symmetry should use Lemma 13.4 and should be the only longer proof. Finally, Stationarity is already essentially Lemma 16.2, which you may want to write out formally on your own.]

*Remark* 16.7. A converse also holds: if $T$ has a ternary relation $\downharpoonright^\star$ on small subsets of $\mathbb{M}$ that satisfies all of the axioms of Theorem 16.5, then $T$ is stable and $\downharpoonright^\star$ is indeed non-forking independence. See [20, Theorem 8.5.10] for a proof of essentially this fact.

## 17 Ranks

**Definition 17.1.** Let $p$ be a type over a small set of parameters. We define the **Lascar rank** of $p$, $U(p)$, by recursion on ordinals as follows:

- $U(p) \geq 0$ always;

- $U(p) \geq \alpha + 1$ if $p$ has a forking extension $q$ with $U(q) \geq \alpha$;

- for a limit ordinal $\gamma$, $U(p) \geq \gamma$ if $U(p) \geq \alpha$ for all $\alpha < \gamma$.

We set $U(p) = \alpha$ if $U(p) \geq \alpha$ and $U(p) \not\geq \alpha + 1$. If $U(p) \geq \alpha$ for all ordinals $\alpha$, we write $U(p) = \infty$. For $p = \mathrm{tp}(a/A)$, we also write $U(a/A)$ instead of $U(p)$.

👁 *Observation* 17.2. Note that $U(p) = 0$ if and only if $p$ is algebraic.

👁 *Observation* 17.3. Note that by definition of $U$-rank, $p \subseteq q$ implies that $U(p) \geq U(q)$.

**Lemma 17.4.** *Let $T$ be stable. Let $p \in S(A) \subseteq a \in S(B)$. Then,*

- *If $q$ is a non-forking extension of $p$, then $U(p) = U(q)$;*

- *If $U(p) = U(q) < \infty$, then, $q$ is a non-forking extension of $p$.*

*Proof.* Note that the second point is a trivial consequence of the definition of $U$-rank. Hence, we need to prove the first point. Given Observation 17.3, it is sufficient to prove by induction on $\alpha$ that for types $p$ and $q$ such that $q$ is a non-forking extension of $p$, if $U(p) \geq \alpha$, then $U(q) \geq \alpha$. The base case and the limit case are trivial, so we need to deal with the successor step. Suppose that $U(p) \geq \alpha + 1$. Let $r$ be a forking extension of $p$ with $U(r) \geq \alpha$ (this exists by definition). Since $r_{\restriction A} = q_{\restriction A}$ for $a \vDash r$ and $b \vDash q$, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma a = b$. Now, $b \vdash \sigma r \cup q$. Say that $q$ is over $B$ and $\sigma r$ is over $C$. Note that by invariance, $\sigma r$ is a forking extension of $p$. Hence, $b \underset{A}{\downarrow} B$ and $b \underset{A}{\not\downarrow} C$. By extension and $C \underset{Ab}{\downarrow} \varnothing$, there is $C' \equiv AbC$ with $C' \underset{Ab}{\downarrow}^B$. By invariance, we get that $b \underset{A}{\not\downarrow} C'$ and $U(b/C') = U(b/C) = U(r) \geq \alpha$. Now

$$C' \underset{Ab}{\downarrow} B \Rightarrow B \underset{Ab}{\downarrow} C' \qquad\qquad \text{by Symmetry}$$
$$\Rightarrow B \underset{A}{\downarrow} C'B \qquad\qquad \text{by transitivity and } B \underset{A}{\downarrow} b$$
$$\Rightarrow B \underset{C'}{\downarrow} b \qquad\qquad \text{by monotonicity}$$
$$\Rightarrow b \underset{C'}{\downarrow} B \qquad\qquad \text{by Symmetry.}$$

Hence $\mathrm{tp}(b/BC')$ is a non-forking extension of $\mathrm{tp}(b/C')$, meaning that $U(b/BC') \geq \alpha$ by inductive hypothesis and the fact that $U(b/C') \geq \alpha$. Now suppose that $b \underset{B}{\downarrow} C'$. Then, we would have,

$$b \underset{B}{\downarrow} C' \Rightarrow n \underset{A}{\downarrow} BC' \qquad\qquad \text{by transitivity and } b \underset{A}{\downarrow} B$$
$$\Rightarrow b \underset{A}{\downarrow} C' \qquad\qquad \text{by monotonicity.}$$

But we know that $b \underset{A}{\not\downarrow} C'$. Hence, we must have that $b \underset{B}{\not\downarrow} C'$. But then by definition of the Lascar rank, $U(b/B) \geq \alpha + 1$ since $\mathrm{tp}(b/B)$ has a forking extension, $\mathrm{tp}(b/BC')$ such that $U(b/BC') \geq \alpha$. $\qquad\square$

## 18 Superstability

**Exercise 18.1.** Let $T$ be an arbitrary theory. Suppose that $p \in S(\mathbb{M})$ forks over $A$ with $|A| \leq \kappa$. Then $p$ has $\geq \kappa$ many $\mathrm{Aut}(\mathbb{M}/A)$-conjugates.

**Exercise 18.2.** Let $T$ be stable. Show there is an ordinal $\alpha$ such that if $U(p) \geq \alpha$, then $U(p) = \infty$. [Hint: use local character.]

Below, we include a couple of basic facts about cardinal arithmetics. For the purposes of this course, it is perfectly fine to know the proof of Theorem 18.5 only for countable theories (in which case none of these facts are needed). However, I thought it worth it to include the fully general proof since it is quite beautiful and uses cardinal arithmetics in a very clever way.

**Fact 18.3.** *There are arbitrarily large cardinals $\kappa$ such that $\beth_\kappa(\aleph_0) = \kappa$.*

*Proof.* This is just a consequence of the (more general) fixed point lemma for normal functions. $\square$

**Fact 18.4.** *Let $\kappa$ be $\beth_\kappa(\aleph_0) = \kappa$. Then,*

$$\kappa^\kappa > \kappa \text{ and for } \gamma < \kappa, \kappa^\gamma = \kappa.$$

*Proof.* Firstly, it is well-known that $\lambda < \lambda^{\mathrm{cf}(\lambda)}$ for any infinite cardinal [9, Theorem 3.11], and $\mathrm{cf}(\kappa) = \kappa$ by Fact 2.19, and so is regular. We prove that for $\gamma < \kappa, \kappa^\gamma = \kappa$ by transfinite induction on $\gamma$. The base case and successor case are trivial. For the limit case,

$$\kappa^\gamma \leq \bigcup_{\beta < \kappa} \beta^\gamma \leq \bigcup_{\beta < \kappa} 2^\beta \leq \kappa,$$

where the second equality follows from $\beta^\gamma < (2^\beta)^\gamma = 2^\gamma$ when $\beta \geq \gamma$. Finally, the last step follows from $2^\beta \leq \beth_\beta(\aleph_0)$ and the definition of $\beth_\kappa(\aleph_0)$. $\square$

**Theorem 18.5** (Equivalents to superstability). *Let $T$ be stable. The following are equivalent:*

1. *$T$ is superstable (in the sense of Definition 8.2);*

2. *Every type has ordinal-valued U-rank;*

3. *For every finite tuple $a$ and every $B$ there is some finite $C \subseteq B$ such that $a \underset{C}{\downarrow} B$.*

*Proof.* We will prove (1)⇒(2) and (2)⇒(3), and leave as an exercise (3)⇒(1). Firstly, we prove by contrapositive (1)⇒(2), which is the hardest implication. Suppose by contrapositive that (2) fails.

**Claim 1:** there is $p \in S_x(\varnothing)$ with a forking extensions $p'$ such that $U(p) = U(p') = \infty$.

*Proof of claim.* Take $r$ be some type with $U(r) = \infty$. Then, $r$ has a forking extension $p'$ with $U(p') \geq \alpha$, where $\alpha$ is as in Exercise 18.2. Let $p = p'_{\restriction \varnothing}$. Since $p'$ is a forking extension of $r$, it also forks over $\varnothing$ (by monotonicity), and so $p'$ is a forking extension of $p$. Hence, $U(p) = U(p') = \infty$. $\square$

Now, we need some $\kappa$ such that for some $\lambda \geq |T|$, we have that $\kappa^\lambda > \kappa$, but $\kappa^\gamma = \kappa$ for all $\gamma < \lambda$. If $T$ is countable, this is the case for all $\kappa$ (with $\lambda = \aleph_0$). Otherwise, we can take some large enough $\kappa$ such that $\beth_\kappa(\aleph_0) = \kappa$ as in Fact 18.4 (cf. Fact 18.3), and set $\lambda = \kappa$.

What we want to do is show that $\kappa$-stability fails. To achieve this, we need, as usual, to build an adequate tree of types such that each branch in the tree specifies a distinct complete type over a set of parameters of size $\leq \kappa$. Compared to earlier proofs in this style, we will need much more sophistication.

**Claim 2:** There is a tree of types $(p_s | s \in {}^{<\lambda}\kappa)$ and a sequence[2] of parameters $(A_\gamma | \gamma < \lambda)$ satisfying the following properties:

---

[2]Note the very subtle move of taking the types to form a tree but the set of parameters to form a sequence. If we just had all types in the tree over different sets of parameters of size $\kappa$, the cardinal arithmetic would not work out since the union of all of them would also have size $> \kappa$.

- $p_\varnothing = p$;

- For each $\gamma < \lambda$ and $s \in {}^\gamma \kappa$, $p_s$ has set of parameters $A_\gamma$;

- For each $\gamma < \lambda$ and $s \in {}^\gamma \kappa$, all of $(p_{s\alpha}|\alpha < \kappa)$ are pairwise inconsistent;

- For each $s \in {}^{<\lambda} \kappa$, $U(p_s) = \infty$;

- For each $\gamma < \lambda$, $|A_\gamma| \leq \kappa$.

*Proof.* We build by induction on $\gamma < \lambda$ our sequences $(p_s|s \in {}^\gamma \kappa)$ and $(A_\gamma|\gamma < \lambda)$. We deal with the base case in the same way as the successor case. Suppose we have built $p_s$ with set of parameters $A_\gamma$ of size $\leq \kappa$. We want to build its children $(p_{s\alpha}|\alpha < \kappa)$. Firstly, again by Exercise 18.2, as $U(p_s) = \infty$, there is some forking extension $p'_s$ of $p_s$ also with $U$-rank $\infty$. Take $q$ to be a global non-forking extension of $p'_s$. As $p'_s \subseteq q$, $q$ still forks over $B$. Moreover, by Lemma 17.4, since $q$ is a non-forking extension of $p'_s$, $U(q) = U(p'_s) = \infty$. So, by Exercise 18.1, there are $\geq \kappa$ many $\mathrm{Aut}(\mathbb{M}/B)$-conjugates of $q$, $(q_\alpha|\alpha < \kappa)$. Let $A^s_{\gamma+1} \supseteq A_\gamma$ of size $\leq \kappa$ be such that $r_{s\alpha} := q_\alpha \restriction_{A^s_{\gamma+1}}$ are all distinct. This can be done by induction on cardinals since at each successor step we are adding at most one new element to $A_\gamma$ (and at limit stages we just take unions). Now, note that the $r_{s\alpha}$ all have $U$-rank $\infty$ (being contained in $q_\alpha$, which has $U$-rank $\infty$ by invariance). Now, having found $r_{s\alpha}$ and $A^s_{\gamma+1}$ for all $s \in {}^\gamma \kappa$, take $A_{\gamma+1} := \bigcup_{s \in {}^\gamma \kappa} A^s_{\gamma+1}$. Note that

$$\left| A_{\gamma+1} \right| \leq \sum_{s \in {}^\gamma \kappa} \left| A^s_{\gamma+1} \right| = \kappa^\gamma \times \kappa = \kappa^\gamma = \kappa,$$

where the last equality follows from our choice of $\kappa$. Thus, we extend each $r_{s\alpha}$ to $p_{s\alpha}$ over $A_{\gamma+1}$ and non-forking over $A^s_{\gamma+1}$, so to ensure that $U(p_{s\alpha}) = \infty$ by Lemma 17.4 and extension (and the fact a type in a stable theory never forks over its own set of parameters). Hence, all of our requirements are satisfied for the successor step. At limit stages, we can simply take unions, noting that the $p_s$ will be over $A_\gamma$, which is of size $\kappa \times \gamma = \kappa$. Thus, we can build out tree of types and sequence of sets of parameters. $\square$

At this point we can contradict $\kappa$-stability by type-counting, and having chosen $k$ such that $k > |T|$. Take $A := \bigcup_{\gamma < \lambda} A_\gamma$, which still has size $\kappa$. The tree of types $(p_s|s \in {}^{<\lambda} \kappa)$ yields $\geq |\kappa^\lambda| > 2^\kappa$ many types over $A$, by choice of $\kappa$.

Now, we prove (2)$\Rightarrow$(3). For $a$ finite, consider $\mathrm{tp}(a/B)$. Let $C \subseteq B$ be finite such that $U(a/C)$ has minimal $U$-rank. We can do this since all types have ordinal-valued $U$-rank and the ordinals are well-ordered. By minimality and definition of $U$-rank, for any finite $C' \subseteq B$, $U(a/CC') = U(a/C)$. Thus, by Lemma 17.4, $a \underset{C}{\mathop{\smile}\limits^{\displaystyle |}} C'$ for all finite $C' \subseteq B$. By finite character, $a \underset{C}{\mathop{\smile}\limits^{\displaystyle |}} B$. $\square$

**Exercise 18.6.** Prove the implication (3) $\Rightarrow$ (1) in Theorem 18.5.

**Corollary 18.7.** *Let $T$ be a countable theory. Consider the **Stability spectrum***

$$\mathrm{StabSpec}_T := \{\kappa|\ T \text{ is } \kappa\text{-stable}\}.$$

*There are only four stability spectra:*

1. *$T$ is $\omega$-stable, in which case $T$ is $\kappa$-stable for all $\kappa \geq \aleph_0$;*

2. *$T$ is superstable but not $\omega$-stable, in which case $T$ is $\kappa$-stable if and only if $\kappa \geq 2^{\aleph_0}$;*

3. *$T$ is stable but not superstable, in which case $T$ is $\kappa$-stable if and only if $\kappa^{\aleph_0} = \kappa$;*

4. *$T$ is not stable, meaning it is never $\kappa$-stable.*

*Proof.* Note that you proved that being stable is equivalent of being $\kappa$-stable for all $\kappa$ such that $\kappa^{\aleph_0} = \kappa$ in Exercise 9.8. And in (3)$\Rightarrow$(1) of Theorem 18.5, we showed (for $|T|$ countable) that stable but non-superstable theories are not $\kappa$-stable for all $\kappa$ such that $\kappa^{\aleph_0} > \kappa$. Now, if $T$ is not $\omega$-stable, there is a countable set of parameters $A$ such that $|S_x(A)| \geq 2^{\aleph_0}$ by Exercise 5.8. Hence, $T$ cannot be $\kappa$-stable in any $\kappa \leq 2^{\aleph_0}$. Finally, it $T$ is $\omega$-stable it is $\kappa$-stable for all $\kappa \geq \aleph_0$. $\qquad\square$

# 19 Lachlan's theorem on superstable $\omega$-categorical theories

**Definition 19.1.** Let $\mathrm{FE}^n(A)$ denote the class of $A$-definable equivalence relations on $n$-tuples from $\mathbb{M}$ with finitely many classes. Write $\mathrm{FE}(A)$ for $\bigcup_{n<\omega} \mathrm{FE}^n(A)$.

**Lemma 19.2.** *Let $X$ be a definable subset of $\mathbb{M}^n$. The following are equivalent.*

1. *$X$ is $\mathrm{acl}^{eq}(A)$-definable;*

2. *$X$ is almost $A$-definable, i.e., there is $E \in \mathrm{FE}^n(A)$ and $\overline{b}_0, \ldots, \overline{b}_n \in \mathbb{M}^n$ such that*

$$X = \bigvee_{i<k} E(\mathbb{M}^n, \overline{b}_i).$$

*Proof.* By Exercise 11.3 being almost $A$-definable is equivalent to the $\mathrm{Aut}(\mathbb{M}/A)$-orbit of $X$ being finite. The latter is equivalent to $X$ being $\mathrm{acl}^{eq}(A)$-definable. $\qquad\square$

**Theorem 19.3** (global Finite Equivalence Relations Theorem). *Let $T$ be stable. Let $A \subseteq B$ and $a, a'$ be finite tuples such that $a \mathrel{\smile\hspace{-0.9em}\vert}_A B$, $a' \mathrel{\smile\hspace{-0.9em}\vert}_A B$ $a \equiv_A a'$ but $a \not\equiv_B a'$. Then, there is some $E \in \mathrm{FE}(A)$ such that $\vdash \neg E(a.a')$.*

*Proof.* The main idea is that by stationarity we know that differences of types over $B$ must already be witnessed by differences in types over $\mathrm{acl}^{eq}(A)$. By stationarity (in Theorem 16.5), $a \equiv_{\mathrm{acl}^{eq}(A)} a'$. So, there is $\phi(x,c) \in \mathrm{tp}(a/\mathrm{acl}^{eq}(A))$ but not in $\mathrm{tp}(a'/\mathrm{acl}^{eq}(A))$. As $\mathbb{M}$ is stably embedded in $\mathbb{M}^{eq}$ (i.e., $\mathbb{M}^{eq}$ does not give rise to new definable subsets of $\mathbb{M}$), the set $X$ defined by $\phi(x,c)$ is $\mathrm{acl}^{eq}(A)$-definable and so almost $A$-definable. Thus, choose $E \in \mathrm{FE}(A)$ such that $X$ is a union of $E$-classes. Then, clearly, as $a \in X$ and $a' \notin X$, $\vdash \neg E(a,a')$. $\qquad\square$

**Theorem 19.4.** *A countable $\omega$-categorical superstable theory is $\omega$-stable.*

*Proof.* Consider some countable $B$. We need to show that $S_1(B)$ is also countable. As $p \in S_1(B)$ does not fork over $B$ (by stability), by superstability (Theorem 18.5), there is some $A \subseteq B$ finite such that $p$ does not fork over $A$. Let

$$\mathrm{N}(B,A) := \{p \in S_1(B) \mid p \text{ does not fork over } A\}.$$

**Claim:** For $A$ finite, $\mathrm{N}(B,A)$ is finite.

*Proof of claim.* Firstly, by Ryll-Nardzewski, $S_1(A)$ is finite, meaning that we need to prove that each $r \in S_1(A)$ only has finitely many non-forking extensions. To see this, note that by the Finite Equivalence Relations Theorem 19.3, any for any two distinct non-forking extensions $q$ and $q'$ to $B$ of $r$, there must be some $E \in \mathrm{FE}^1(A)$ such that $q(x) \cup q(y) \vdash \neg E(x,y)$. Hence,

$$\text{\# of non-forking extensions of } r \leq \sum \{\text{\# of equivalence classes of } E \mid E \in \mathrm{FE}^1(E)\}$$

But the latter number is finite as by Ryll-Nardzewski there are only finitely many equivalence relations over a finite set, and since each of these equivalence relations has finitely many classes. This completes the proof that $\mathrm{N}(B,A)$ is finite. $\qquad\square$

We can now compute the size of $S_1(B)$:

$$|S_1(B) \leq \sum \{|\mathrm{N}(B,A)| \mid A \subseteq B \text{ finite}\} \leq \aleph_0,$$

where the last step follows since $B$ is countable, and so it has only countably many finitely many finite subsets. $\qquad\square$

# References

[1] Tuna Altınel, Alexandre Borovik, and Gregory L Cherlin. *Simple groups of finite Morley rank*. Vol. 145. American Mathematical Society Providence, RI, 2008.

[2] Manuel Bodirsky. *Complexity of infinite-domain constraint satisfaction*. Vol. 52. Cambridge University Press, 2021.

[3] Elisabeth Bouscaren. *Model theory and algebraic geometry: an introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture*. Springer, 2009.

[4] Samuel Braunfeld, Colin Jahel, and Paolo Marimon. "When invariance implies exchangeability (and applications to invariant Keisler measures)". In: *arXiv preprint arXiv:2408.08370* (2024).

[5] Peter J Cameron. "Oligomorphic permutation groups". In: *Perspectives in Mathematical Sciences II: Pure Mathematics*. World Scientific, 2009, pp. 37–61.

[6] Hunter Chase and James Freitag. "Model theory and machine learning". In: *Bulletin of Symbolic Logic* 25.3 (2019), pp. 319–332.

[7] Artem Chernikov and Nicholas Ramsey. "On model-theoretic tree properties". In: *Journal of Mathematical Logic* 16.02 (2016), p. 1650009.

[8] Yatir Halevi and Itay Kaplan. "Saturated models for the working model theorist". In: *Bulletin of Symbolic Logic* 29.2 (2023), pp. 163–169.

[9] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[10] Byunghan Kim. *Simplicity theory*. Vol. 53. OUP Oxford, 2013.

[11] Michael C Laskowski. "Vapnik-Chervonenkis classes of definable sets". In: *Journal of the London Mathematical Society* 2.2 (1992), pp. 377–384.

[12] Angus Macintyre. "On definable subsets of p-adic fields". In: *The Journal of Symbolic Logic* 41.3 (1976), pp. 605–610.

[13] Maryanthe Malliaris and Saharon Shelah. "Regularity lemmas for stable graphs". In: *Transactions of the American Mathematical Society* 366.3 (2014), pp. 1551–1585.

[14] David Marker et al. "Model theory of differential fields". In: *Model theory, algebra, and geometry* 39 (2000), pp. 53–63.

[15] Michael Morley. "Categoricity in power". In: *Transactions of the American Mathematical Society* 114.2 (1965), pp. 514–538.

[16] James G Oxley. *Matroid theory*. Vol. 3. Oxford University Press, USA, 2006.

[17] Anand Pillay. *Geometric stability theory*. Oxford University Press, 1996.

[18] Saharon Shelah. *Classification theory: and the number of non-isomorphic models*. Elsevier, 1990.

[19] Pierre Simon. *A guide to NIP theories*. Cambridge University Press, 2015.

[20] Katrin Tent and Martin Ziegler. *A course in model theory*. 40. Cambridge University Press, 2012.

[21] Bartel Leendert Van der Waerden. *Moderne Algebra*. Vol. 2. Springer, Berlin, 1937.

[22] Hassler Whitney. "On the Abstract Properties of Linear Dependence". In: *American Journal of Mathematics* 57.3 (1935), pp. 509–533. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2371182 (visited on 10/17/2024).