ω-categoricity of Lie co-ordinatisable structures.

RECAP.

Embeddings

- $\mathcal{M}$ is canonically embedded in $\mathcal{N}$ if $\emptyset$-def relations of $\mathcal{M}$ are the relations which are $\emptyset$-def in the sense of ~~M~~ $\mathcal{N}$

- $\mathcal{M}$ is stably embedded in $\mathcal{N}$ if every $\mathcal{N}$-def relation on $\mathcal{M}$ is $\mathcal{M}$-def uniformly.
- $\mathcal{M}$ is fully embedded if it is both canonically and stably embedded.

Lie co-ordinatised (2.1.10)

A structure $\mathcal{M}$ is Lie cordinatised by Lie geometries if it carries a tree structure of finite height with unique $\emptyset$-def root such that.

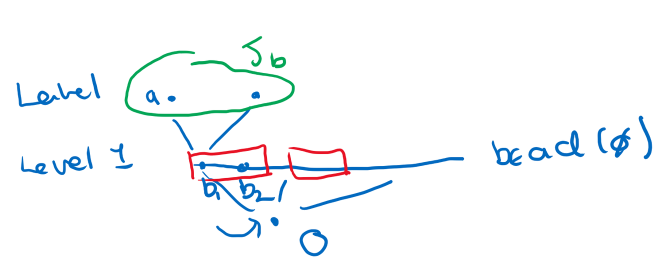
1. Co-ordinatisation For each $a \in M$ above the root either a is algebraic over its immediate predecessor or there exists $b < a$ and b-def proj geom J_b fully embedded in $\mathcal{M}$ such that either

   i) $a \in J_b$
   ii) there is a c with $b < c < a$ and c-def affine/quadratic geometry (J_c, A_c) with vector part J_c such that $a \in A_c$ and J_c projectivises to J_b

2. Orientation: only important in quadratic case - with defect [

$\mathcal{M}$ $\mathcal{M}$

lie-co-ordinatisable



bead $(\emptyset)$

2.1.12 Definition

The structure $\mathcal{M}$ is lie co-ordinatisable if

- bi-interpretable with a structure that is co-ordinatised by Lie geometries
- finitely many 1-types.

Consequences of 1-types:

→ Can't have geometries over ~~infinitely many~~ arbitrary large fields.

→

ω-categoricity

Remarks:

- Linear geometries are all ω-categorical. This follows from QE. Kernel field is finite
- Projective geometries are interpretable in $\mathcal{M}^{eq}$ of the corresponding linear geometry

  basic linear geom. are ω-cat $\Rightarrow$ proj geom ω-cat.

Lemma 2.3.2

If D is $\emptyset$-def in $\mathcal{M}$ then for $a \in M^{eq}$ TFAE

- D is stably embedded in $\mathcal{M}$ and admits WEI.
- For $a \in M^{eq}$ $tp(a/acl(a) \cap D)$ determines $tp(a/D)$.

2.3.5 Basic linear geometries have WEI.

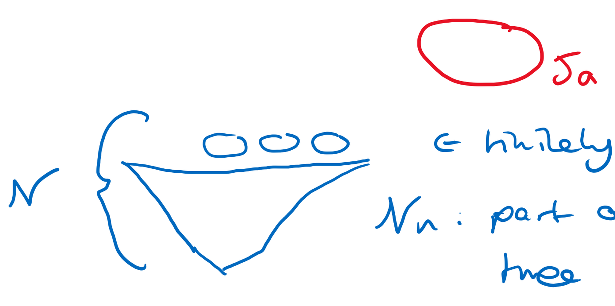
2.3.12 Affine " " "
(J, A, A^*)

2.3.19 LEMMA.

A Lie co-ordinatisable structure is $\aleph_0$-categorical.

Proof:

- Suffices to treat $\mathcal{M}$ equipped with a Lie-co-ordinatisation with finitely many 1-types.
- Use induction



$\in$ finitely many geometries at height $h+1$

N_h: part of $\mathcal{M}$ co-ord. by tree up to height h.

CLAIM: J_a realizes finitely many types over any finite subset of $N \cup \{a\}$.

WHY IS THIS ENOUGH?

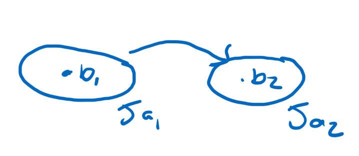
Aim: finitely many types over any finite set.

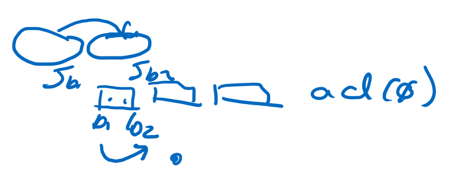
Use induction on height of the tree, ie

Ind hyp: there are only finitely many n-types over $\emptyset$
N_h

Consider types over subsets of N_{h+1}

- $b \in acl(A)$ $A \subseteq N_h$ OK.
- $b \in J_a$ $a \in N_h$





PROOF OF CLAIM

Fix J_a proceed inductively on h and the number of components at level $h+1$

→ Assume proj geometries are replaced by linear covers
 - affine spaces " " " (J, A, A^*) ?

The problem reduces to showing

for $A \subseteq N$ finite $acl(Aa) \cap J$ finite.

WHY:

Suppose $j, j' \in J_a$.

then $tp(j/Aa) = tp(j'/Aa)$

iff $tp^J(j/acl(Aa) \cap J) = tp(j'/acl(Aa) \cap J)$

J fully embedded 2.3.2

As J is ω-cat there are only finitely many types over finite subsets.

Suppose not then $acl(Aa) \cap J$ contains arbitrarily large finite dimensional subspaces V of J.

- Fix V subspace of J
- N is B-def for some set of parameters $B \subseteq N_h$

  by induction $acl(Ba) \cap J$ is finite.

- As A' varies over the set of realisations in N of $tp(A/Ba)$

corresponding V' varies " " " " " J of $tp(V/Ba)$

Let n_1 number of types of set AA' as A' varies
n_2 " " " " " VV' as V' " $\Big\}$ $n_1 \geq n_2$

Now n_1 is bounded by induction hyp. since $B \subseteq N_h$, $A \subseteq N$ ↑ ~~at~~ most level $n+1$

?

n_2 is finite ※ n_2 should be unbounded if we have arbitrarily large vector spaces V'