Homogeneity of Envelopes

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Let us begin with some preliminary reminders.

Definition 0.1. A **dimension function** μ maps equivalence classes of standard systems of geometries to isomorphism types of approximations to any of the projective geometries in the image of a representative.

Thus, μ gives us a finite or countable dimensional geometry of a fixed type.

Definition 0.2. Let μ be a dimension function. We say $E \subseteq M$ is a **μ-envelope** if it satisfies the following conditions:

(i) E is algebraically closed in M (not in M^{eq}).

(ii) For any $c \in M \setminus E$, there is a standard system of geometries $J \colon A \to M^{\mathrm{eq}}$ and some $b \in A \cap E$ such that

$$\mathrm{acl}^{\mathrm{eq}}(E, c) \cap J_b \supset \mathrm{acl}^{\mathrm{eq}}(E) \cap J_b.$$

(iii) For any standard system of geometries $J \colon A \to M^{\mathrm{eq}}$ and $b \in A \cap E$, $J_b \cap E$ has the isomorphism type given by $\mu(J)$.

We will also need one of the notions defined by Nick in last week's talk:

Definition 0.3. Suppose (V, A) is an affine space defined over B. We say A is **free over** B if there is no projective geometry J defined over B such that $A \subseteq \mathrm{acl}(B, J)$.

Nick also showed the following lemma:

Lemma 0.4 (3.2.3)**.** *Let $\mathcal{M}$ be a Lie coordinatised structure. Let (V, A) be an affine space defined and free over $B = \mathrm{acl}(B)$. Let $B \subseteq B' = \mathrm{acl}(B')$, where B' is finite, and let J be a canonical projective geometry associated with A. Suppose that:*

(i) $J \cap B' \subseteq B$.

(ii) $J \cap B$ is nondegenerate.

(iii) If J is a quadratic space, then the Q-sort of J meets B.

Then A either meets B' or is free over B'.

Recall we are trying to prove:

Lemma 0.5 (Main Theorem, 3.2.4)**.** *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatised structure, let μ be a dimension function, and let E and E' be μ-envelopes. If $A \subseteq E$, $A' \subseteq E'$ are finite, and $f\colon A \to A'$ is elementary in $\mathcal{M}$, then f extends to an elementary map from E to E'. In particular, μ-envelopes are unique and homogeneous.*

Proof. Recall the reductions Nick made last week:

- E and E' are finite.
- A and A' are algebraically closed.
- Suffices to show f extends to $\mathrm{acl}(A \cup \{b\})$ for some $b \in E \setminus A$.

Nick showed how to extend f when there are still points in E coming from a canonical projective geometry which are not contained in A. Now we will prove the second case, when there are no more such points left to add.

Case 2: For any standard system of geometries J, and any $a \in A$, $J_a \cap E \subseteq A$.

In particular, by elementarity, the same holds for A' relative to E'. We apply the same technique as last week: we will extend f to a $<$-minimal element a in the coordinatisation tree for E which is not already in A. Thus, the $<$-predecessor b of a is in A, and a is not algebraic over b (as A is algebraically closed).

Claim 1. a is contained in an affine space which is free over A.

Proof of Claim 1. Assume first, for contradiction, that $a \in J_b$, where J_b is the coordinatising projective geometry associated with b. Then, as b is the immediate $<$-predecessor of a, by definition of Lie coordinatisation it follows that $a \in J_b$. But this contradicts our assumption that $J_b \cap E \subseteq A$.

Therefore, there is $c < b$ such that (J_b, A_b) is an affine or quadratic geometry, $a \in A_b$, and J_c is the projectivisation of J_b. From now on, we denote by J_b the affine part A_b instead, and by V_b the linear space acting regularly on J_b.

It remains to prove now that J_b is free over A. (HOW?) $\square$

We may also assume J_b is basic. So, in E', by elementarity of f, we also get J_{fb} affine and free over A'.

Claim 2. J_{fb} is not free over E'.

Proof. Pick some $w \in J_{fb} \backslash E'$. By condition (ii) from the definition of an envelope, we can find some canonical projective geometry J' defined over E' such that $\mathrm{acl}^{\mathrm{eq}}(E, w) \cap J' \supset \mathrm{acl}^{\mathrm{eq}}(E) \cap J'$. We claim that $J_b \subseteq \mathrm{acl}^{\mathrm{eq}}(E, J')$. Indeed, we know that $w \in \mathrm{acl}^{\mathrm{eq}}(E, J')$. Now take some other $w' \in J_b$. As V_b acts regularly on J_b, there is some $v \in V_b$ such that $v = w' - w$. However, by choice of J', we know that PV_b (the projectivisation of V_b) and J' are nonorthogonal and projective, and thus, by Lemma 2.4.3, they are linked. Hence, we can use this bijection to obtain that $w' \in \mathrm{acl}(E, J')$, as required. $\square$

Thus, we can now apply Lemma 3.2.3 from Nick's talk with J_{fb} the affine geometry, and A' and E' the algebraically closed sets (using our reduction to finite E'), to conclude that $J_{fb} \cap E' \neq \varnothing$.

Claim 3. There is some $a' \in J_{fb} \cap E'$ such that, for all $\lambda \in J_b^* \cap A$,

$$\lambda(a) = (f\lambda)(a').$$

Proof of Claim 3. Using Lemma 2.5.2 (I assume), take a stably embedded canonical projective geometry P associated with J_b (i.e., non-orthogonal).[1] Then P is b-definable by Lemma 2.5.3, and PV_b is definably isomorphic to one of the sorts of P/b (using again Lemma 2.4.3 after localising to ensure we have the appropriate algebraic closure).

By our initial case assumption, we have $P \cap E \subseteq A$. In particular, $b \in A \cap E$, and thus, by (iii) from the definition of an envelope, it follows that the type of $P \cap E$ is given by μ. In particular,[2] $P \cap E$ is nondegenerate (if there are any forms around). The same applies to $P' \cap E'$ by our remarks above.

Hence, if β denotes the nondegenerate bilinear form that we have around, then, for any $\lambda \in (V_{fb} \cap E)^* = V_{fb}^*|_{V_{fb} \cap E'}$, there is some $v_\lambda \in V_{fb} \cap E'$ such that $\lambda(x) = \beta(v_\lambda, x)$. In particular, by the identification of V_{fb} with one of the sorts of P', it follows that $v_\lambda \in A'$. Hence, the action of V_{fb}^* on $V_{fb} \cap E'$ can be represented by elements of A'.

But, since $J_{fb} \cap E' \neq \varnothing$, we still get a surjection (cf. Lemma 2.3.9) which we can use to also represent by elements of A' the affine maps in E'.[3]

"Again by linear nondegeneracy and the fact that E' meets $J_{b'}$, the specified values for $(a', f\lambda)$ can be realized in $E' \cap J_{b'}$." $\square$

Now extend f to $A \cup \{a\}$ by $f(a) = a'$. The idea is that we want to conclude as in Case 1 by applying Lemma 2.3.2 (i.e., w.e.i. + stable embeddedness). However, in this case, J_b is affine, and by Lemma 2.3.11 (together with the remarks from 2.3.10), the relevant structure with w.e.i. here is (PV_b, J_b, J_b^*), where V_b is the linear space acting regularly on J_b. Therefore, in this case, the application of Lemma 2.3.2 tells us that

$$\mathrm{tp}(A/\, \mathrm{acl}^{\mathrm{eq}}_{(PV_b, J_b, J_b^*)}(A) \cap (PV_b, J_b, J_b^*)) \vdash \mathrm{tp}(A/(PV_b, J_b, J_b^*)),$$

[1]Note that, by adequacy, we may assume P lives in $\mathcal{M}$.

[2]In more detail: μ assigns approximations to canonical projective geometries, which in particular are projectivisations of weak Lie geometries. Since we may omit the "trivial" cases (degenerate and pure), it follows that we have a non-degenerate form, regardless of which case we actually are in.

[3]Note that, in this representation, we also refer to elements of the underlying field K. We can do this precisely because we assumed J_b to be basic.

and the same thing holds for the type of A over a. Therefore, to check elementarity, we need to check that the type of a over $A^{\mathrm{eq}} \cap (PV_b, J_b, J_b^*)$ corresponds to the type of a' over the f-image of the parameter set. As $a' \in J_{fb}$, the only case that needs more work is the type of a over J_b^*. This is precisely what Claim 3 gives us. Hence, f is elementary, as required. $\square$

Corollary 0.6 (3.2.5)**.** *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatised structure. Then a subset E of $\mathcal{M}$ is an envelope iff the following hold:*

(i) E is algebraically closed.

(ii) For any $c \in M \setminus E$, there is a projective geometry J defined over E, not quadratic, such that $(J \cap \mathrm{acl}(Ec)) \setminus E \neq \varnothing$.

(iii) If $c_1, c_2 \in E$ are conjugate in $\mathcal{M}$ and $D(c_1), D(c_2)$ are corresponding conjugate definable geometries, then $D(c_1) \cap E$ and $D(c_2) \cap E$ are isomorphic.

"This does not depend on a particular coordinatization of $\mathcal{M}$." I believe this just refers to the fact that, in the original definition, (ii) and (iii) appeal to standard systems of geometries, which by canonicity and linkage might be seen as coordinatisations of $\mathcal{M}$.