Finite Structures

with

Few Types

December 8, 2023

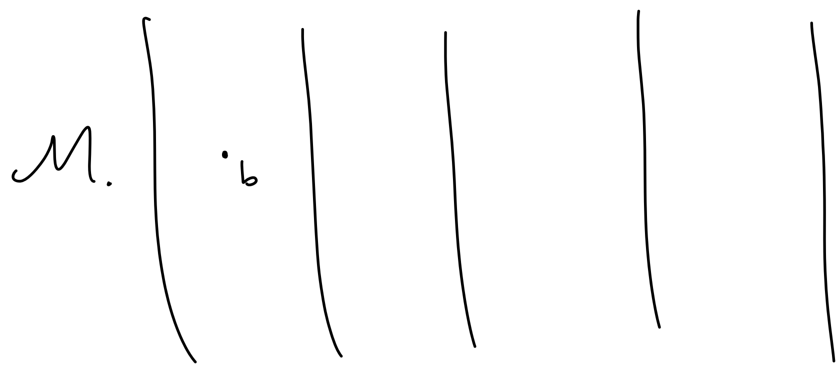
Weak EI

Definition $\mathcal{M}$ has weak elimination of imaginaries if for all $a \in \mathcal{M}^{eq}$,

$$a \in \operatorname{dcl}(\mathcal{M} \cap \operatorname{acl}(a)).$$

Non-example

T = theory of an equivalence relation with infinitely many classes, all of which are infinite.



$a = b/_E$ Then $\operatorname{acl}(a) \cap \mathcal{M} = \emptyset$ and $\operatorname{dcl}(\emptyset) = \emptyset$.

Lemma

If D is 0-definable in a structure $\mathcal{M}$, and $D(a) := \mathrm{acl}(a) \cap D$ for some $a \in \mathcal{M}^{eq}$.

Then TFAE:

(1) D is stably embedded in $\mathcal{M}$ and has weak EI.

(2) For all $a \in \mathcal{M}^{eq}$,

$$tp(a/D(a)) \vdash tp(a/D).$$

Proof (1) $\Rightarrow$ (2)

Take $\varphi(x;y)$ arbitrary w/ x in the sort of a and assume $\varphi(x;y) \vdash y \in D$. So $\varphi(a;y)$ is a relation on D. So it has a canonical parameter $d_0 \in D^{eq}$. Notice that

$$d_0 \in \mathrm{dcl}(a).$$

By weak EI, we know that

$$d_0 \in \mathrm{dcl}(\mathrm{acl}(d_0) \cap D) = \mathrm{dcl}(D(d_0)).$$

So there's some (finite) $B \subseteq D(d_0) \subseteq D(a)$ s.t. $d_0 \in \mathrm{dcl}(B)$. So $\varphi(a;y)$ is B-definable: so there's some φ^* s.t.

$$\varphi(a;y) \leftrightarrow \varphi^*(b;y) \quad \text{holds}$$

for $b \in B$. So

$$\varphi(x;y) \leftrightarrow \varphi^*(b,y) \in tp_x(a/B) \subseteq tp_x(a/D(a))$$

So $tp(a/D(a)) \vdash tp(a/D)$.

(2)⇒(1) Assume (2).

Weak EI: Take some $a \in D^{eq}$. Want that $a \in \mathrm{dcl}(D(a))$. Know that $a \in \mathrm{dcl}(D)$ and $tp(a/D(a)) \vdash tp(a/D)$ so

$a \in \mathrm{dcl}(D(a))$.

<u>Stably embedded</u>: Suppose $\varphi(x;a)$ defines a subset of D. [a need not be in D^{eq}].

$A = D(a)$. We know by (2) that if $a \equiv_A b$ then $a \equiv_D b$, in which case $\varphi(x;a)$ and $\varphi(x;b)$ define the same subset of D.

So since the set defined by $\varphi(x;a)$ is invariant under automorphisms fixing A it is A-def'l and since $A \subseteq D$ that's what we needed. $\boxtimes$

Lemma

Let J be a linear, projective, or affine geometry. Let $a \in J^{eq}$, and let $A = \mathrm{acl}(a) \cap J$. Then

$$\mathrm{acl}(a) = \mathrm{acl}(A).$$

Proof

We may assume J is basic.

Write $a = f(b)$ for $b \in J$ and f 0-def'l in J^{eq}. By extension, take $b' \equiv_{\mathrm{acl}(a)} b$ with $b' \mathop{\smile\!\!\!\!\!\!\mid}^{rk}_{\mathrm{acl}(a)} b$.

Now we recall Corollary 2.2.12: If J is a linear, projective, or affine geometry, and a, b are finite sequences with $\mathrm{acl}(a) \cap \mathrm{acl}(b) = C$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^{rk}_{C} b$.

Hence we get $a \mathop{\smile\!\!\!\!\!\!\mid}^{rk}_{A} b$, in our case.

Unravelling, we have

$$\mathrm{rk}(b'/Aab) = \mathrm{rk}(b'/Aa)$$

$$\mathrm{rk}(b'/Ab) = \mathrm{rk}(b'/A)$$

$f(b') = a$

But also

$$\mathrm{rk}(b'/Aa) = \mathrm{rk}(b'a/A) - \mathrm{rk}(a/A)$$

$$= \mathrm{rk}(b'/A) - \mathrm{rk}(a/A)$$

$$\Longrightarrow \mathrm{rk}(a/A) = 0 \Longrightarrow a \in \mathrm{acl}(A). \qquad \blacksquare$$

Main result (Lemma 2.3.5)

Let $\mathbb{J}$ be a basic linear geometry. Then $\mathbb{J}$ has weak EI.

Take $a \in \mathbb{J}^{eq}$, want to show that

$$a \in dcl(\underbrace{acl(a) \cap \mathbb{J}}_{A}).$$

Suffices to show, by the preceding lemma, that if $A \subseteq \mathbb{J}$ is alg closed, $a \in \mathbb{J}^{eq}$, and $a \in acl(A)$, then $a \in dcl(A)$. ← We will prove this.

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preceding lemma

$a \in \mathbb{J}^{eq}$, $acl(a) \cap \mathbb{J} = A$, $a \in acl(A)$

$$\Rightarrow a \in dcl(A) := a \in dcl(acl(a) \cap \mathbb{J}).$$

We will write $a = f(b)$ with f A-def'l and $b = (b_1, \dots, b_n) \in J^n$. We will take n minimal. Assume $a \notin \mathrm{dcl}(A)$ so $n \geq 1$.

By working over $Ab_{<n}$, we can assume $n = 1$ and $b = b_n$.

Let $D \subseteq J$ be the locus of b over A (i.e. the realizations in J of $tp(b/A)$).

We know $b \notin A$.

Let $I = \{ (x,y) \in D^2 \mid \langle xA \rangle \cap \langle yA \rangle = A \}$.

QE implies that, for $(x,y) \in I$,

$tp(x,y/A)$ is determined by

- $\beta(x,y)$ [non-deg or trivial, possibly coming from a quadratic form].
- $[x,y] := \chi(\sqrt{x+y})$ [in quadratic case when $x, y \in Q$].

So we will call these functions $x \cdot y$ to give it uniform notation. Note that in the first case, it takes values in K, in the second case, it takes values $T(K)$ where $x^2 + x = T(x)$.

Let $X \subseteq K$ be a subset such that, for all $(x,y) \in I$,

$$\underbrace{f(x) = f(y)} \iff x \cdot y \in X.$$

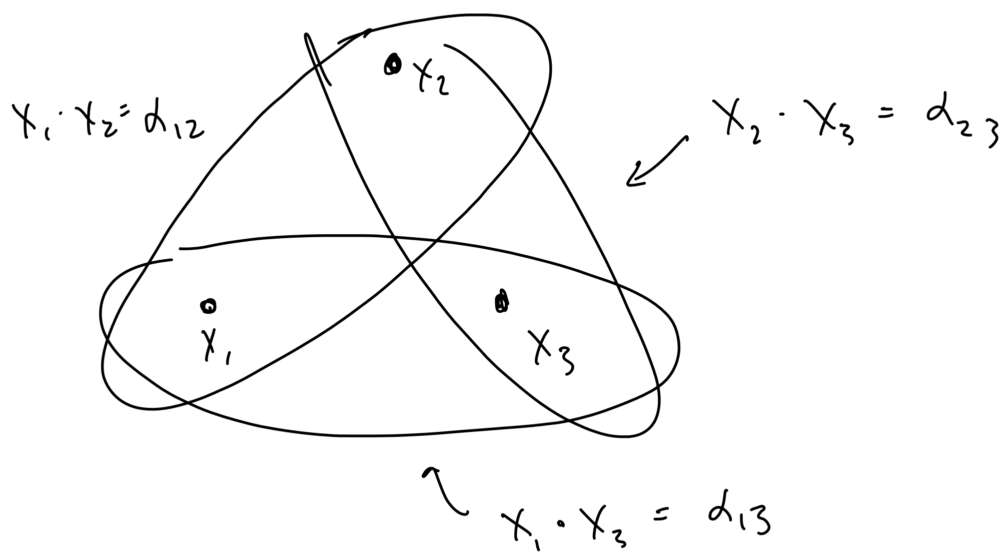
Let $X_0 = K$ in the nondeg. bilinear form case and let $X_0 = T(K)$ when $D \subseteq Q$.

Goal now is to prove that $X = X_0$. If we knew that, then f would be constant on D, so instead of writing $a = f(b)$, can write

"a is the unique x st $\exists y \in D$ w/ $f(y) = x$" $\Rightarrow a \in \mathrm{dcl}(A)$

Suffices to check that, for $\alpha_{12}, \alpha_{13}, \alpha_{23} \in X_0$, there are $x_1, x_2, x_3 \in D$ indep over A w/

$$x_i \circ x_j = \alpha_{ij}.$$



This follows from the proof of QE.

Bilinear form case:

Fix $x_1 \in D$ arbitrary. Consider a type $p(y)$ consisting of

- $y \in D$ ←
- $y \notin \langle A x_1 \rangle$
- $\beta(x_1, y) = \alpha_{12}$ ←

A neutralization gives x_2. Do same thing:

- $y \in D$
- $y \notin \langle A x_1, x_2 \rangle$
- $\beta(x_1, y) = \alpha_{13}$

  $\beta(x_2, y) = \alpha_{23}$

<u>Punchline:</u> Taken $\alpha_{12} = \alpha_{23} \in X$, take $\underbrace{\alpha_{13} \in X_0 \text{ arbitrary.}}$

Get the associated x_1, x_2, x_3.

$f(x_1) = f(x_2)$ since $\alpha_{12} = x_1 \cdot x_2 \in X$

$f(x_2) = f(x_3)$ since $\alpha_{23} = x_2 \cdot x_3 \in X$

$\Longrightarrow f(x_1) = f(x_3) \Rightarrow \alpha_{13} \in X$

$\Longrightarrow X_0 = X.$ $\boxtimes$

Now we include some details about how to choose x_1, x_2, x_3 in the case that $D \leq Q$, $d_{ij} \in T(K)$ are arbitrary.

Pick $q \in D$ arbitrary and set $x_2 = q$. By the proof of QE in the orthogonal case, we can choose $v \notin \mathrm{acl}(A x_2)$ s.t. $q(v) = \alpha_{12}$, hence

$$[q + \lambda^2 v, q] = q(v) = \alpha_{12}.$$

Next, again by the proof of QE

in the orthogonal case, we can choose $w \in V$ s.t.

- $w \notin \langle A, q, v \rangle$
- $\beta(v, w) = \gamma$ for some γ with $T(\gamma) = \alpha_{13} - \alpha_{12} - \alpha_{23}$
- $q(w) = \alpha_{23}$.

Note that there is such a γ since $T(K)$ is a subgroup of $(K, +)$ so $\alpha_{13} - \alpha_{12} - \alpha_{23} \in T(K)$.

Now we consider

$$X_1 = q + \lambda_v^2, \quad X_2 = q, \quad X_3 = q + \lambda_w^?.$$

By construction, they are independent over A. Also, we have

$$[x_1, x_2] = [q + \lambda_v^2, q] = q(v) = \alpha_{12}$$

$$[x_2, x_3] = [q, q + \lambda_w^2] = q(w) = \alpha_{23}.$$

So we are left with checking that

$$[x_1, x_3] = [q + \lambda_v^2, q + \lambda_w^2] = \alpha_{13}.$$

Note

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$$[q+\lambda_v^2,\, q+\lambda_w^2] = [q+\lambda_v^2,\, q+\lambda_v^2+\lambda_v^2+\lambda_w^2]$$

$$= [q+\lambda_v^2,\, q+\lambda_v^2+\lambda_{v+w}^2]$$

$$= (q+\lambda_v^2)(v+w)$$

$$= q(v+w) + \left(\beta(v, v+w)\right)^2$$

$$= q(v)+q(w)+\beta(v,w) + \beta(v,w)^2$$

$$= q(v)+q(w)+T(\beta(v,w))$$

$$= \alpha_{12}+\alpha_{23}+T(\gamma)$$

$$= \alpha_{12}+\alpha_{23}+(\alpha_{13}-\alpha_{12}-\alpha_{23}) = \alpha_{13}. \quad \blacksquare$$