Weak linear geometry : K finite field; V K-space $\cdot : K \times V \to V$; L 1-dim K space

1. Degenerate $(V; =)$
2. Pure v.space $(V, K; (+,-,0_V); (+,-,\cdot,0,1); \cdot)$
3. Polar space $(V \cup W, K, L; \beta)$ $\beta : V \times W \to L$ non-degenerate pairing (symmetrized).
4. Inner product space $(V, K, L; \beta)$ $\beta : V \times V \to L$ non-deg. sesquilinear form wrt σ, $\sigma^2 = 1$. $\sigma = 1$: β symplectic $\sigma \neq 1$: hermitian (unitary).
5. Orthogonal $(V, K, L; q)$ $q : V \to L$ quadratic form assoc. bilinear form β non-deg.

6. Quadratic geometry. $(V\,Q, K; \beta_V, +_Q, -_Q, \beta_Q, \omega)$ char $K = 2$; β_V symplectic; Q V-orbit on q.forms $\rightsquigarrow \beta_V$; ω Witt defect.

BASIC linear geometry : elts. of K & L are named; in (3) name V, W.

Remark: If $\dim V = \aleph_0$ ($= \dim W$ in Case (3)) each of these strc. is ω-categorical.

(2.2.8) Lemma. Suppose J is a (inf. dimensional) basic linear geometry. (in the indicated language).
Then J has QE

Pf: Use b & f. Suppose $\bar{a}, \bar{b}$ are tuples in J with same q.f. type & $c \in J$. Find $d \in J$ with $\mathrm{qftp}(\bar{a}^\wedge c) = \mathrm{qftp}(\bar{b}^\wedge d)$.

Case 0 Degenerate. Ex.
Case 1 J 'classical' $J = (V; \beta, q)$.
(2), (4), (5)

WLOG $\bar{a}, \bar{b}$ enumerate K-subspaces of V; A, B. $c \notin A$ the q.f. type to be realised by d (over B) is of the form

$$\begin{cases} x \notin B \\ \beta(b, x) = \lambda(b) & (b \in B, \text{some } \lambda \in B^*) \\ q(x) = \alpha & (\text{some } \alpha \in K) \end{cases}$$

Clear if (2). So assume β is non-degenerate.
Enlarge B so that $B^\perp \cap B = \{0\}$. (extend λ arbitrarily).

So $V = B \perp B^\perp$. As $\lambda \in B^*$ so there is $b' \in B$ st. $\lambda(b) = \beta(b, b')$ for all $b \in B$.

Replacing x by $x - b'$ we can assume $\lambda(b) = 0 \ \forall b \in B$.

ie. $x \in B^\perp$. The only condition left is $q(x) = \alpha$.

But if $q \neq 0$ then $q(B^\perp \setminus \{0\}) = K$ //.

(as $\dim B^\perp \geq 3$)

[If char $K = 2$, ~~Every~~ elt of $K^\times$ is a square so only issue is to find a non-zero $v \in B^\perp$ with $q(v) = 0$. No anisotropic orthog. space of dimension > 2. Modify this to get all elts. of $K^\times$ if char $K \neq 2$. See ⊛ below]

<u>Case 2</u> Polar space (3) Ex.

<u>Case 3</u> Quadratic geometry. If $\bar{a}$ (and $\therefore \bar{b}$) intersects Q non-trivially, this reduces to the orthogonal case (5).

So suppose $\bar{a}, \bar{b}$ are in V. If $c \in V$, this is the symplectic case (4). We can assume β is non-deg. on A and B (subspaces enumerated by $\bar{a}, \bar{b}$).

Assume $c \in Q$. The type to be ~~dr~~ realized by d is a q.f. $d \in Q$ with $d|B$ & $\omega(d)$ fixed.

All possibilities can be realized: all possibilities for $d|B$ can be achieved as $B^* \underset{\beta}{\cong} B$. Need to show extensions of $d|B$ in Q include forms with both defects.

$$v + q = q + \lambda_v^2 \qquad [q, v+q] = q(v)$$

$$\tau(K) = \{\alpha^2 + \alpha : \alpha \in K\}$$

$v \in B^\perp$ $\quad q(B^\perp) = K$, so any Witt defect arises. #

<u>Remarks</u> (1) In (6) also get QE without ω for inf. dim. case.

(2) In f.d. case + [4].[5] QE is equivalent to Witt's Theorem (Aschbacher §20).

(*): Find $v_1, v_2 \in B^\perp$ s.t.h. $\alpha_i = q(v_i) \neq 0$ and $B(v_1, v_2) = 0$.

Then $q(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1^2 \alpha_1 + \lambda_2^2 \alpha_2$.

To show $E = q(\langle v_1, v_2 \rangle \setminus \{0\}) \supseteq K^\times$, we show this contains both a non-zero square and a non-square. If not, then either E consists of the squares & is closed under addition or E consists of the non-squares / and possibly 0 and $E \cup \{0\}$ is closed under +.

In either case we have an additive subgp. of K of size $\frac{r+1}{2}$ where $r = |K|$ (using char. $K \neq 2$). This is impossible. ✓

§2.1.3 Coordinatization

(2.1.9) Def. Let $\mathcal{M} \subseteq \mathcal{N}$ be structures with $\mathcal{M}$ (parameter) definable in $\mathcal{N}$.
Let $a \in \mathcal{N}^{eq}$ represent the set $\mathcal{M}$ (canonical parameter).

1. $\mathcal{M}$ is canonically embedded in $\mathcal{N}$ if the $\emptyset$-def. relations in $\mathcal{M}$ are precisely the relations in $\mathcal{M}$ which are a-definable in $\mathcal{N}$.
2. $\mathcal{M}$ is stably embedded in $\mathcal{N}$ if every $\mathcal{N}$-definable relation in $\mathcal{M}$ is $\mathcal{M}$-definable, uniformly.
3. Fully embedded : both.

Remark: ① If $\mathcal{N}$ is ω-cat, $\mathcal{M}$ fully embedded $\iff$ $\mathrm{Aut}(\mathcal{N}/a)|\mathcal{M} = \mathrm{Aut}(\mathcal{M})$.

② In the polar space, the v. space V is canonically, but not stably embedded.

(2.1.10) Def. A structure $\mathcal{M}$ is coordinatized by Lie geometries (Lie coordinatized)
if it has a tree structure $<$ of finite height (invariant p.o.)
with a unique $\emptyset$-def. root such that:

1. (Coordinatization) For all $a \in \mathcal{M}$ above the root one of the following holds:

(0) a is algebraic over its $<$-predecessor;

OR there is $b < a$ & a b-def. proj. geometry J_b fully embedded in $\mathcal{M}$ st either

(i) $a \in J_b$; or

(ii) there is $b < c < a$ and a c-definable affine or quadratic geometry (J_c, A_c) st. $a \in A_c$ and the projectivization of J_c is J_b.

2. (Orientation) If $a, b \in \mathcal{M}$ have the same type (over $\emptyset$) & are associated with coordinatizing quadratic geometries J_a, J_b, then any definable map between them which preserves everything other than ω also preserves ω.

(2.1.12) Def. A str. $\mathcal{N}$ is Lie coordinatizable if it is biinterpretable with a structure $\mathcal{M}$ with finitely many 1-types (over $\emptyset$) which is Lie coordinatized.

Weak Lie coordinatized/able: without the orientation condition.

THM 2 ((1)$\Leftrightarrow$(5)) $\mathcal{M}$ smoothly approximable $\Leftrightarrow$ $\mathcal{M}$ Lie coordinatizable.

THM 3 $\mathcal{N}$ is a reduct of a s.a. str. $\Leftrightarrow$ $\mathcal{N}$ is weak Lie coordinatizable.

Example 2.1.11 Let $A = \left(\bigoplus_{i \in \omega} (\mathbb{Z}/p^2\mathbb{Z}); +\right)$ p prime.

$\cdot pu$

$u + A[p] = \{ v \in A : pv = pu \}$

$pA = A[p] = \{ a \in A : pa = 0 \}$

Coordinatization

$px = c$

$A[p] \setminus \{0\}$

$u + A[p]$

Root: O $\quad J_0$

$A[p] \setminus \{0\}$

scalars

$c/\sim$ $\quad c$

$(A[p], A_c)$

$A_c = \{ a \in A : pa = c \}$

Finite Proj. Finite Affine.