Model Theory II Problem Sheet 4

Extra exercises are marked with a $\star\star$. I DO <u>NOT</u> EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $A \subseteq \mathcal{M} \models T$. We say that \mathcal{M} is **prime over** A if for all $\mathcal{N} \models T$ and $f : A \to \mathcal{M}$ a partial elementary map, f extends to an elementary $f' : \mathcal{M} \to \mathcal{N}$.

EXERCISE 1. Show the following: Let T be a countable ω -stable theory, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. Then, there is $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime model over A and such that every $a \in M$ realises an isolated type over A.

Theorem 2 (Lachlan). Let T be ω -stable, $\mathcal{M} \models T$, $|M| \geq \aleph_1$. Then, for each $\kappa > |M|$ there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality κ such that for any countable set of $\mathcal{L}(M)$ -formulas $\Gamma(x)$ in a finite variable x, if \mathcal{N} realises $\Gamma(x)$, then so does \mathcal{M} .

EXERCISE 2. We shall prove Theorem 2 following the steps below. Consider an ω -stable theory T and $\mathcal{M} \models T$, such that $|M| \geq \aleph_1$. Say that an $\mathcal{L}(M)$ -formula is **large** if $\phi(M)$ is uncountable.

- Prove that there is a large $\mathcal{L}(M)$ -formula $\phi_0(x)$ such that for any other $\mathcal{L}(M)$ -formula ψ , either $\phi_0(x) \wedge \psi(x)$ or $\phi_0(x) \wedge \neg \psi(x)$ has a countable set of realisations.
- Consider

$$p(x) := \{ \psi(x) | \psi(x) \in \mathcal{L}(M) \text{ and } \phi_0(x) \land \psi(x) \text{ is large } \}.$$

Show that p is a complete type over M which is not realised in M but such that all of its countable subsets are realised in M. Take $\mathcal{N}' \succeq \mathcal{M}$ with a point a realising p.

- By Exercise 1, take $\mathcal{N} \preceq \mathcal{N}'$ prime over Ma and such that every $b \in \mathcal{N}$ realises an isolated type over Ma. Show that for every $b \in \mathcal{N}$, every countable subset $\Gamma(x)$ of $\operatorname{tp}(b/M)$ is realised in M.
- Deduce Theorem 2.

EXERCISE 3. Show that the theory of the random graph has a Vaughtian pair.

** **EXERCISE 4.** Show that there is no Vaughtian pair of real closed fields.

Definition 3. We say that T eliminates the quantifier $\exists^{\infty} x$ if for every \mathcal{L} -formula $\phi(x, \overline{y})$ there is $n_{\phi} \in \mathbb{N}$ such for all tuples $\overline{a} \in \mathbb{M}^{|\overline{y}|}$, if $|\phi(\mathbb{M}, \overline{a})| \geq n_{\phi}$, then $\phi(\mathbb{M}, \overline{a})$ is infinite.

EXERCISE 5. Show that if *T* has no Vaughtian pairs, then it eliminates the quantifier $\exists^{\infty} x$.

EXERCISE 6. Suppose that T eliminates the quantifier $\exists^{\infty} x$. Let $\mathcal{M} \models T$ and let $\phi(x) \in \mathcal{L}(M)$ be minimal in \mathcal{M} . Show that $\phi(x)$ is strongly minimal.

Definition 4. For infinite cardinals $\kappa > \lambda$, we say that T has has a (λ, κ) -model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

EXERCISE 7. Prove the following:

- 1. If *T* has a (κ, λ) -model then it has a Vaughtian pair (and so an (\aleph_1, \aleph_0) -model [Hint: this should be trivial];
- 2. Prove that if T is ω -stable and has an (\aleph_1, \aleph_0) -model, then for each $\kappa > \aleph_1$, T has a (κ, \aleph_0) -model [Hint: you may need to use Theorem 2].

** **EXERCISE 8.** We show that in Exercise 7 (2), the assumption of ω -stability is necessary. Let $\mathcal{L} = \{P_0, \dots, P_n, E_1, \dots, E_n\}$ for unary predicates P_i and binary relations E_i . Consider the \mathcal{L} -theory T stating that:

- the *P_i* are infinite and partition the domain;
- for each $i \in \{1, \ldots, n\}$, $\forall xy(E_i(x, y) \rightarrow P_{i-1}(x) \land P_i(y))$;

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• for each $i \in \{1, ..., n\}$, $\forall xy((P_i(x) \land P_i(y) \land \forall z(E_i(z, x) \leftrightarrow E_i(z, y)) \rightarrow x = y)$.

For example, for X_0 an infinite, take $X_{i+1} = \mathcal{P}(X_i)$ for $i \in \{1, \ldots, n\}$. Let \mathcal{M} be the disjoint union of the X_i with P_i naming each of the X_i and E_i being the membership relation restricted to $X_i \times x_{i+1}$. Then, $\mathcal{M} \models T$. Show that if $\mathcal{M} \models T$ and $|P_0(\mathcal{M})| = \aleph_0$, then $|\mathcal{M}| \leq \beth_n$. Hence, \mathcal{M} has a (\beth_n, \aleph_0) -model but it does not have a (κ, \aleph_0) -model for arbitrarily large κ . [Hint: I would only do the case of n=1. Recall that $\beth_0 = \aleph_0$ and $\beth_{\alpha+1} = 2^{\beth_\alpha}$.]