# Finite Structures with Few Types

(2/16/2024)

$\mathcal{M} \leq \mathcal{N}$ $\quad$ $\mathcal{M}$ def'l in $\mathcal{N}$, w/ canonical parameter $a \in \mathcal{N}^{eq}$.

We say that $\mathcal{M}$ is <u>canonically embedded</u> in $\mathcal{N}$ if the $a$-def'l relations on $M$ in $\mathcal{N}$ are exactly the $0$-def'l relations in $\mathcal{M}$.

<u>Lemma</u> 2.4.6

Let $J_1, J_2$ be basic linear geometries canonically embedded in a structure $\mathcal{M}$. Suppose that in $\mathcal{M}$ there's a $0$-def'l bijection $f: PJ_1 \to PJ_2$ between the projectivizations of $J_1, J_2$ resp.

Then, there is a 0-def'l isomorphism (of unoriented weak geometries) $\hat{f} : J_1 \to J_2$ which induces $f$.

$$\begin{tikzcd} J_1 \arrow[r, dashed, "\hat{f}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & J_2 \arrow[d] \\ PJ_1 \arrow[r, "f"] & PJ_2 \end{tikzcd}$$

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Proof: Wlog $M = J_1 \cup J_2$, so canonical parameter $= \emptyset$.

We can assume that $acl(\emptyset) = dcl(\emptyset)$.

and any forms are $K$-valued.

$J_i$ is one of:

- 1 vector space (w/ maybe additional structure)
- 2 paired vector spaces (in polar case)
- A quadratic geometry $(V, Q)$

$PJ_i$, then, is one of:

- 1 projective space
- 2 paired projective spaces
- $(PV, Q)$

(1) There's a linear lift. $\hat{f}$

$f$ preserves algebraic closure:

$\mathrm{span}(w) \in \mathrm{cl}_{J_2}(\mathrm{span}(v_1), \ldots, \mathrm{span}(v_{l_2}))$

$\Uparrow$

$\hat{f}$

$$\Downarrow$$

$f(\mathrm{span}(w)) \in \mathrm{aul}_{J_2}(f(\mathrm{span}(v_1)), \ldots, f(\mathrm{span}(v_k)))$

which follows by canonical embedding.

Recall an <u>isomorphism of projective spaces</u> is a bijection that maps a subspace to a subspace (in both directions).

## <u>The Fundamental Theorem of Projective Geometry</u>

Any isomorphism between projective spaces of dim $\geq 2$ is induced by a semi linear transformation between the underlying vector spaces, unique up to scalar multiplication.

(semi-linear = composition of a linear transformation + a field automorphism of the target space).

In our context, we get a linear lift $\hat{f}$, relative to an isomorphism between the field of $J_1$ and the field of $J_2$, so we will identify them and call them $K$.

② There's a $\emptyset$-definable linear lift

Cherlin-Hrushovski: "There are finitely many such maps and the set of them is implicitly definable, so by Beth's theorem, they are definable over $acl(\emptyset) = dcl(\emptyset)$ or, in other words they are $\emptyset$-definable."

???

How to understand this?

One way maybe: introduce a new sort $\mathcal{F}$ and functions $\mathrm{eval} : \mathcal{F} \times J_1 \longrightarrow J_2$

$\bullet : K^\times \times \mathcal{F} \to \mathcal{F}$

$T' = Th(M) +$

- $\forall \hat{f} \in \mathcal{F}$, $\mathrm{eval}(\hat{f}, -)$ is a $K$-linear map that induces $f$
- $\forall \hat{f}, \hat{f}'$, $\mathrm{eval}(\hat{f}, -) = \mathrm{eval}(\hat{f}', -) \Rightarrow \hat{f} = \hat{f}'$
- $\bullet : K^\times \times \mathcal{F} \to \mathcal{F}$ is a regular action and $\mathrm{eval}(\alpha \bullet \hat{f}, -) = \alpha\, \mathrm{eval}(\hat{f}, -)$.

Now if $M' \models T'$, $M = M' \upharpoonright J_1 \cup J_2$ then every automorphism $\sigma \in \mathrm{Aut}(M)$ lifts uniquely to $\tilde{\sigma} \in \mathrm{Aut}(M')$.

Just choose some $v_1 \neq 0$ in $J_1$,

pick some non-zero $v_2 \in \mathrm{span}(f(\mathrm{span}(v_1)))$.

Then if $\hat{f}(v_1) = \alpha v_2$ then there

is a unique $\hat{f}'$ st $\hat{f}'(\sigma(v_1)) = \alpha\, \sigma(v_2)$.

Put $\tilde{\sigma}(\hat{f}) = \hat{f}'$. Check that it

doesn't depend on the choices.

So res: $\mathrm{Aut}(M') \longrightarrow \mathrm{Aut}(M)$

an isomorphism. At least in the case that

$M$ is $\aleph_0$-categorical, this entails $M'$ and

$M$ are biinterpretable so we may regard

$$J \subseteq M^{eq} = (J_1 \cup J_2)^{eq}.$$

And since $J$ is finite, $J \subseteq \mathrm{acl}(\emptyset)^{eq}$.

$$\mathrm{acl}^{eq}(\emptyset) = (\mathrm{acl}^{eq}(\emptyset) \cap J_1) \cup (\mathrm{acl}^{eq}(\emptyset) \cap J_2)$$

Canonical embedding $= (\mathrm{acl}^{eq}_{J_1}(\emptyset)) \cup (\mathrm{acl}^{eq}_{J_2}(\emptyset))$

$\mathrm{Wd} \in I \Rightarrow a \in \mathrm{acl}^{eq}_{J_i}(\emptyset) \Rightarrow a \in \mathrm{dcl}^{eq}(\mathrm{acl}^{eq}(a) \cap J_i)$

$\Rightarrow a \in \mathrm{dcl}^{eq}(\underbrace{\mathrm{acl}^{eq}(\emptyset) \cap J_i}_{\mathrm{acl}_{J_i}(\emptyset)})$

$\Rightarrow a \in \mathrm{dcl}^{eq}(\mathrm{dcl}_{J_i}(\emptyset))$

$\Rightarrow a \in \mathrm{dcl}^{eq}(\emptyset)$

$J \subseteq \mathrm{dcl}^{eq}(\emptyset)$.

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Fix an $\hat{f} \in \mathcal{F}$.

(3) Preserves orthogonality

If we have a quadratic form is present, a totally isotropic space is one when only one non-trivial one type is realized — namely

$$p(v) = \{ v \neq 0, q(v) = 0 \}.$$

In the polar case, a totally isotropic space is one that consists of a pair of orthogonal (in the sense of $\perp$) one non-trivial $\perp$-type is realized in each factor.

Note that $tp_{J_2}(\hat{f}(a))$ is determined by $tp_{J_1}(a)$. by 0-definability.

$\Longrightarrow$ $\hat{f}$ takes totally isotropic spaces to totally isotropic spaces.

$\Rightarrow$ orthogonality is preserved in the polar case.

$\Rightarrow$ orthogonality is always preserved.

Consider the case $\mathrm{Char}(K) = 2$ and we're in a orthogonal space. Pick $x, y \in J_1$ st $\beta_{J_1}(x,y) = 0$.

Can choose isotropic vectors $v_1, v_2, w_1, w_2$ st $x \in \mathrm{Span}(v_1, v_2)$, $y \in \mathrm{span}(w_1, w_2)$. and $\mathrm{span}(v_1, v_2, w_1, w_2)$ is an orthogonal direct sum of the hyperbolic planes $\langle v_1, v_2 \rangle$, $\langle w_1, w_2 \rangle$. $\Rightarrow$ $\langle v_i, w_j \rangle$ is totally isotropic for all $i, j$. Say $x = \alpha_1 v_1 + \alpha_2 v_2$

$y = \gamma_1 w_1 + \gamma_2 w_2$.

Because $\hat{f}$ is linear, we get

$$\hat{f}(x) = \alpha_1 \hat{f}(v_1) + \alpha_2 \hat{f}(v_2)$$

$$\hat{f}(y) = \gamma_1 \hat{f}(w_1) + \gamma_2 \hat{f}(w_2)$$

$$\beta_{J_2}(\hat{f}(x), \hat{f}(y)) = \sum_{i,j} \alpha_i \gamma_i \beta_{J_2}(\hat{f}(v_i), \hat{f}(w_j))$$

$$= 0$$

because $\langle \hat{f}(v_i), \hat{f}(w_j) \rangle$ is totally isotropic.

$$\hat{f}(\langle v_i, w_j \rangle)$$

4 Preserving the other structure

If quadratic or skew quadratic forms $Q_1, Q_2$ are present there's a function $F$ st

$$Q_2(\hat{f}(x)) = F(Q_1(x)).$$

where $F: K_0 \to K_0$ where

$$K_0 = \begin{cases} K^\sigma & \sigma \text{ order 2 in the Hermitian case} \\ k & \text{o/w.} \end{cases}$$

This $F$ is additive: Consider $x \perp y$

$$F(Q_1(x) + Q_1(y)) = F(Q(x+y))$$

$$= Q_2(\hat{f}(x) + \hat{f}(y))$$

$$= (Q_2(\hat{f}(x)) + Q_2(\hat{f}(y))).$$

Also linear with respect to squares (or with respect to $K_0$ in the Hermitian case)

So $F$ is equal to scalar multiplication by some $\alpha$. $\Rightarrow Q_2 = \alpha Q_1$

So we're done except in the polar, symplectic, and quadratic cases.

No nontrivial 1-types in polar & symplectic cases other than $\{x \neq 0\}$.

Have a function $F : K \to K$ st

$$\beta_{J_2}(\hat{f}(v), \hat{f}(w)) = F(\beta_{J_1}(v, w))$$

Clearly linear, so again we get

$$\beta_{J_2} = \alpha \beta_{J_1}, \quad \text{So we're}$$

happy.

Slight complication in the quadratic case, but basically same story. $\boxtimes$