WEI Notes

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Many of the details of the proofs here come from some old notes by David Evans.

1 Projective and semiprojective geometries

For $\Gamma \subset \mathrm{Aut}(M)$, write $x^\Gamma := \{x^\gamma : \gamma \in \Gamma\}$. Recall the following definitions:

Definition 1.1. (i) A **projective geometry** J is obtained from a linear geometry by factoring out the equivalence relation $\mathrm{acl}(x) = \mathrm{acl}(y)$.

(ii) A **semiprojective geometry** J is obtained from a basic linear geometry by factoring out the equivalence relation $x^Z = y^Z$, where Z is the centre of the automorphism group.

Remark 1.2. Note that, since we start with a basic linear geometry, the elements of the underlying field K are fixed. Recall that, for any K-vector space V, we have $\mathrm{Aut}(V/K) \cong K^\times$.[1] Thus, we get that x^Z is the set of αx for $\alpha \in K^\times$ respecting all of the additional structure of the basic linear geometry.

Example 1.3. Let $M = (V, K, L, \beta)$, where V is a K-vector space, L is a K-line, and β is a symplectic form with respect to some σ. Take $f \in \mathrm{Aut}(M)$. Since in particular f is an automorphism of V fixing K, it sends $v \mapsto \alpha v$ for some fixed $\alpha \in K^\times$. Moreover, $f(\beta(x,y)) = \beta(f(x), f(y))$. But $\beta(f(x), f(y)) = \beta(\alpha x, \alpha y) = \alpha^2\beta(x,y)$, and $f(\beta(x,y)) = \beta(x,y)$, since f fixes K. Therefore, $\alpha \in \{\pm 1\}$. In this case, for $x \in V$, we get $x^Z = \{x, -x\}$.

Lemma 1.4 (2.3.6)**.** *Let J be a basic semiprojective geometry. Then J has weak elimination of imaginaries.*

Proof. Let $a \in J^{\mathrm{eq}}$, write $J = V/\!\sim$ where V is a basic linear geometry, and let $A := \mathrm{acl}(a) \cap V$. Note that by Remark 1.2, we have $J \subseteq V^{\mathrm{eq}}$, and hence, $a \in V^{\mathrm{eq}}$. So, by Lemma 2.3.5, it follows that $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$. Write $B := \mathrm{acl}(a) \cap J$. For $g \in \mathrm{Aut}(J/B)$, we can find some $h \in \mathrm{Aut}(V/A)$ inducing g, and so, in particular, if $\bar{c}$ is a tuple in J such that a is $\bar{c}$-definable, the orbit of $\bar{c}$ in J over B is the same as that over A. Therefore, $a \in \mathrm{dcl}^{\mathrm{eq}}(B)$. □

[1] Indeed, if $f \in \mathrm{Aut}(V/K)$ is such that $v \mapsto \alpha v + \sum_i \alpha_i v_i$ (a linearly independent expression), then $f(0_V) = 0_V + \sum_i \alpha_i v_i = 0_V$, which implies that $\alpha_i = 0_V$ for all i. So any automorphism is given by multiplication by a non-zero scalar.

Remark 1.5 (cf. 2.3.7)**.** A projective geometry J does not have weak elimination of imaginaries. As an example, suppose J is an inner product space with a symplectic form β. First, let J_s be the semiprojective geometry associated to V, the basic linear geometry covering J. Then $J_s \subseteq J^{\mathrm{eq}}$. Let $a \in J_s$, so that $a = \{\pm v\}$ for some $v \in V$. We claim that $a \notin \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(a) \cap J)$: indeed, if $|K| \geq 4$, there is an automorphism sending $v \mapsto \alpha v$ for some $\alpha \notin \{0, \pm 1\}$, which clearly fixes $\langle v \rangle = \mathrm{acl}(a) \cap J$.

2 Affine geometries

Definition 2.1. Let V be a definable K-vector space and let A be a definable set on which V acts regularly. A K**-affine map** is a map $\lambda \colon A \to K$ such that

$$\lambda\Big(\sum_i \alpha_i a_i\Big) = \sum_i \alpha_i \lambda(a_i),$$

where $\alpha_i \in K$ such that $\sum_i \alpha_i = 1$.

Remark 2.2. Fix a base point $a \in A$. Since V acts regularly on A, we can write $a_i = a + v_i$ for some $v_i \in V$. Thus, $\sum_i \alpha_i a_i = a + (\sum_i \alpha_i v_i)$ and $\sum_i \alpha_i v_i \in V$, but we need to check that this is well-defined, i.e., that it does not depend on our choice of base point. So choose some other $a' \in A$ and write $a_i = a' + v_i'$. Once again, since V acts regularly on A, we can write $a' = a + v'$ for some $v' \in V$, and hence $v_i = v_i' + v'$. Therefore, the left-hand side is well-defined.

Definition 2.3. Let V^* be the **definable dual** of V (i.e., the set of definable linear functionals), and let A^* be the set of M-definable K-affine maps on A.

Lemma 2.4 (2.3.9)**.** *There is an exact sequence*

$$(0) \longrightarrow K \xrightarrow{f} A^* \xrightarrow{g} V^* \longrightarrow (0),$$

i.e., f is injective, g is surjective, and $\operatorname{im} f = \ker g$.

Proof. We first define f by $k \mapsto \lambda_k$, where $\lambda_k \colon A \to K$ is the constant map in k. This is clearly injective.

Now, define g by $\lambda \mapsto \lambda'$, where we define, for $v \in V$, $\lambda'(v) := \lambda(a+v) - \lambda(a)$ for some $a \in A$. We need to check that λ' is well-defined and linear, so that $\lambda' \in V^*$. We first show linearity with respect to a choice of base point a:

$$\begin{aligned}
\lambda'(v + v') &= \lambda(a + v + v') - \lambda(a) \\
&= \lambda((a + v) + (a + v') - a) - \lambda(a) \\
&= \lambda(a + v) + \lambda(a + v') - 2\lambda(a) \\
&= \lambda'(v) + \lambda'(v').
\end{aligned}$$

Having this, we can now show that λ' is well-defined: let $a' \in A$ be some other base point, and write $a' = a + v'$. Then:

$$\begin{aligned}\lambda(a'+v) - \lambda(a') &= \lambda(a+v'+v) - \lambda(a+v') \\ &= \lambda'(v'+v) + \lambda(a) - \lambda'(v) - \lambda(a) \\ &= \lambda'(v').\end{aligned}$$

To show that g is surjective, take some $\lambda^* \in V^*$. Choose a base point $a \in A$, set $\lambda(a) := 0$, and, for any other $a' \in A$, define $\lambda(a') := \lambda^*(v')$, where $a' = a + v'$.

Finally, we need to show that $\operatorname{im} f = \ker g$, i.e., that $g(\lambda_k)$ is the constant function at 0. But this is clear: $\lambda'(v) = \lambda(a+v) - \lambda(a) = k - k = 0$. □

Remark 2.5 (2.3.10)**.** (i) Recall the following corollary of quantifier elimination:

> **Corollary 2.6** (2.2.9)**.** *The definable linear functions on the vector space V in a linear geometry are those afforded either by the inner product (if one is given, or comes from a quadratic form), or by the dual in the polar case.*

In particular, in the case of a pure vector space, any linear form in V^* must be definable using only names for elements in K, i.e., must be constant. But, by the argument above, the only constant linear form in V^* is the 0 function. So, in this case, $A^* = K$ and $V^* = (0)$. Clearly this is also the case in the degenerate space.

(ii) We can code A^* in $(V, V^*, A)^{\mathrm{eq}}$: given some $\lambda' \in V^*$, $a \in A$, and $k \in K$, we can associate a unique $\lambda \in A^*$ to (λ, a, k) by declaring $\lambda(a+v) := \lambda'(v) + k$ for all $v \in V$. Thus, quotienting out by the $\varnothing$-definable relation $(\lambda_1, a_1, k_1) \sim (\lambda_2, a_2, k_2)$ iff the corresponding K-affine maps are equal, we see that A^* is identified with a sort in $(V, V^*, A)^{\mathrm{eq}}$. In particular, the algebraic closure of $\lambda \in A^*$ in (V, V^*, A) is the line $\langle \lambda' \rangle$ in V^* for $\lambda' \in V^*$ corresponding to it, and hence, (V, V^*, A) does not have weak elimination of imaginaries.

Moreover, if we have a non-degenerate bilinear form, we can write each $\lambda \in V^*$ as $\lambda(x) = \beta(v, x)$ for some fixed $v \in V$, and so we do not have to mention V^* explicitly.

(iii) We do have weak elimination of imaginaries in (V, V^*, A^*), but this is not stably embedded in (V, V^*, A, A^*). Indeed, note that, in (V, V^*, A, A^*), after choosing a base point for A, we can definably identify A^* with $K \oplus V^*$. If (V, V^*, A^*) is stably embedded in (V, V^*, A, A^*), then we must be able to define the isomorphism without invoking any elements from A. But this is impossible.

(iv) The surjective map from the previous lemma shows that V^* is definable in (V, A, A^*), so we do not need to include it in the geometry whenever A^* is already included (even in the polar case!).

Lemma 2.7 (2.3.11)**.** *Let V be a vector space, let J be a basic, linear, nonquadratic geometry covered by V (or with V one of the two vector spaces if J is polar), and let A be a definable subset on which V acts regularly. Then (J, A, A^*) admits quantifier elimination in its natural language.*

Proof. We first need to describe the language. This will include:

- The original language for J.
- Predicates for A and A^*.
- Maps $+\colon V \times A \to A$ to be interpreted as $(v, a) \mapsto a + v$ and $-\colon A \times A \to V$ as $(a_1, a_2) \mapsto v$ where $a_2 = a_1 + v$.
- A map $\mathrm{ev}\colon A \times A^* \to K$ to be interpreted as $(a, \lambda) \mapsto \lambda(a)$.
- A K-vector structure on A^*.
- Names for the constant functions in A^*.
- If V^* is explicitly included in J, a map $A^* \to V^*$ to be interpreted as λ to λ' as in the previous lemma; and if V^* is just definable (cf. (iv)) but not explicitly in the language, then a map $A^* \times V \to K$ to be interpreted as $(\lambda, v) \mapsto \lambda'(v)$.

The idea is to run again the same proof as we saw in David's talk for quantifier elimination. Let $\bar{b}$ and $\bar{c}$ be tuples from $J \cup A \cup A^*$ with the same quantifier-free type and $d \in J \cup A \cup A^*$. We want to find some $e \in J \cup A \cup A^*$ such that $\mathrm{qftp}(\bar{b}^\frown d) = \mathrm{qftp}(\bar{c}^\frown e)$. We can split the proof into several cases:

Case 1: $\bar{b} \subseteq J$.

Then, by the proof of quantifier elimination for basic linear geometries, there is some $g \in \mathrm{Aut}(J \cup A)$ with $g(\bar{b}) = \bar{c}$, and so we can take $e := g(d)$.

Case 2: There is some $a_0 \in \bar{b} \cap A$.

Let us write $\bar{b} = (a_0, \bar{b}_1)$. Applying a translation, write $\bar{c} = (a_0, \bar{c}_1)$. Then, using a_0 as a base point for A, any points from A and A^* are quantifier-free interdefinable with points in J (cf. Remark 2.5(ii)). So this reduces to the first case.

Case 3: $\bar{b} \subseteq J \cup A^*$.

Without loss of generality, we may assume $\bar{b}$ is quantifier-free definably closed. Let $\lambda_1, \ldots, \lambda_r$ be the elements of A^* that appear in $\bar{b}$. Then, for $a \in A$, $\mathrm{tp}(a/\bar{b})$ is determined by $\lambda_1(a), \ldots, \lambda_r(a)$, since, for any $v \in \bar{b} \cap J$, we have $\lambda_i(v + a) = \lambda_i'(v) + \lambda_i(a)$. Let $\lambda_1^*, \ldots, \lambda_r^*$ be the corresponding elements from A^* in $\bar{c}$. By an automorphism, we may assume that $\lambda_i' = (\lambda_i^*)'$, so that $\lambda_i - \lambda_i^* = k_i$ with k_i constant for all i. Remove any constant forms if necessary, and let $q \le r$ be maximal such that $\lambda_{i_1}', \ldots, \lambda_{i_q}'$ are linearly independent. Then choose $a' = v' + a$ such that $\lambda_{i_j}^*(a') = \lambda_{i_j}(a)$, so that $\lambda_{i_j}'(v') = k_{i_j}$ for all $j \le q$. Thus, we see that $\mathrm{qftp}(a/\bar{b}) = \mathrm{qftp}(a'/\bar{c})$. Hence, this reduces to the second case. □

Lemma 2.8 (2.3.12)**.** *Let J be a basic, nonquadratic, linear geometry and let (J, A) be a corresponding basic affine geometry. Then (J, A, A^*) has weak elimination of imaginaries.*

Proof. Since we had Lemma 2.3.3 for the affine case and A^* is algebraic over V^*, we can just mimic Nick's proof from last week. So it suffices to show that, if $B \subseteq (J, A, A^*)^{\mathrm{eq}}$ is algebraically closed and $f\colon (J, A, A^*) \to (J, A, A^*)^{\mathrm{eq}}$ is B-definable, then f is constant on each 1-type D over B.

As we did last time, let $I := \{(x, y) \in D^2 : \langle xB\rangle \cap \langle yB\rangle = B\}$, where the span is the algebraic closure in (J, A, A^*). We claim that, if $(x, y) \in I$, then $f(x) = f(y)$. Note that, if $B \cap A \neq \varnothing$, then we can apply the same identification of A with V and A^* with $K \oplus V^*$ as before and return to the linear case. So assume $B \cap A = \varnothing$.

<u>Case 1</u>: $D \subseteq J$.

Let d be a realisation of D. Then $D = \mathrm{tp}(d/B) = \mathrm{tp}(d/B \cap J)$ (by quantifier-elimination), and we have $\langle xB\rangle \cap \langle yB\rangle = B$ iff $\langle xB \cap J\rangle_J \cap \langle yB \cap J\rangle_J = B \cap J$. Moreover, since f is B-definable, $f(d)$ is $\langle B \cap J, d\rangle_J$-definable, and so we may take $\operatorname{im} f \subseteq J^{\mathrm{eq}}$. So this reduces to the linear case.

<u>Case 2</u>: $D \subseteq A$.

Now D is determined by the values that affine maps $\lambda \in B \cap A^*$ take on $d \in D$. Since linear maps in B are covered by affine maps, the relevant part of B for this information is $B \cap V$. Moreover, if $(x, y) \in D^2$, then, in particular, $x \equiv_B y$, and thus, $\beta(x - y, v) = \beta(x, v) - \beta(y, v) = 0$ for $v \in B \cap V$. Thus, $x - y$ is orthogonal to $B \cap V$. The only remaining information left concerns the values of $Q(x - y)$ for a nondegenerate quadratic form (assuming there is one). For this, we need to find $v, w \in B^\perp$ such that $Q(v)$, $Q(w)$ and $Q(v + w)$ take on arbitrary values. But this we can do by using similar arguments as in the previous talks.

<u>Case 3</u>: $D \subseteq A^*$.

For $(x, y) \in I$, $\mathrm{tp}(x, y/B)$ is determined by the type of the image in V^*. So once again we can run the same proof as in the linear case. □

The upshot of these results about imaginaries is the following:

Lemma 2.9 (2.3.19)**.** *A Lie coordinatisable structure is ω-categorical.*