Model Theory II

Winter Semester 2024

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November 14, 2024

Contents

1 Why Model Theory II?

A major theme of model theory is that certain combinatorial properties of definable sets in a first order theory yield a lot of structural information about it. They can imply that a theory has a good notion of independence, or dimension, like in vector spaces, or help us understand the behaviour of groups or geometries interpretable in it. Sometimes, these properties can determine important algebraic features. This course will focus mainly on some of the strongest model theoretic properties: stability, and its strengthenings ω-stability and superstability.

Modern model theory begins with the work of Morley [14], and subsequently Shelah [17] on the spectrum problem: what are the possible behaviours of the function $I(\aleph_\alpha, T)$, counting the number of non-isomorphic models of T of cardinality $\aleph_\alpha$? Morley showed that, for countable T, if $I(\aleph_\alpha, T) = 1$ for some uncountable cardinal, then this is the case for all uncountable cardinals. Shelah studied for which theories we can define a system of invariants under which sufficiently large models of T can be classified. In this process he defined various properties that shaped the development of model theory. Two of the major results of [17] are that $I(\aleph_\alpha, T)$ is non-decreasing for uncountable cardinals and the Main Gap Theorem: for $\alpha > 0$, either T has the maximum number of models in each uncountable cardinality, $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$, or it satisfies few model theoretic properties, including superstability, and it is bounded above by $\beth_{\omega_1}(|\alpha|)$.

Over the years, a very sophisticated theory of stability [16] was developed, with fruitful generalisations to NIP [18], simple [9], and NSOP$_1$ theories [7]. A map of the universe with more model theoretic properties and many examples is given on the website forkinganddividing.com. Model theory has fruitful applications and interactions in algebraic geometry [3], differential algebra [13], group theory [5, 1], number theory [11], and combinatorics [12]. Some classical notions from model theory have been independently discovered many times, such as VC-dimension in probability and combinatorics [10], and PAC learning and Littlestone dimension in machine learning [6]. My own research at TU Wien focuses on the computational complexity of problems in model theoretic structures [2], and on interactions between model theory and probability [4]. Overall, model theory is a highly versatile subject with many beautiful results, some of which we will cover in this course. I hope you will enjoy it!

2 The monster model

This first lecture will require some additional knowledge of set theory and cardinal arithmetics. If you are not familiar with set theory, Appendix A in [19] should contain most relevant facts. One of the advantages of the construction of the monster model in this lecture is preventing us from keeping track of issues of cardinal arithmetics later in the course.

Almost every article or book in model theory begins with the convention that we are working in a monster model $\mathbb{M}$. These are very large models of T distinguished by being highly saturated, strongly homogeneous, and universal in the following sense:

Definition 2.1. Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is:

- **κ-saturated** if it realises types (in finitely many variables) over sets of parameters of cardinality $< \kappa$;
- **κ-universal** if every model of T of cardinality $< \kappa$ elementarily embeds into $\mathcal{M}$;
- **κ-homogeneous** if for all $A \subseteq M$ of size $< \kappa$ and $a \in M$, every elementary map $f : A \to \mathcal{M}$ can be extended to an elementary map $A \cup \{a\} \to M$;
- **strongly κ-homogeneous** if for all $A \subseteq M$ of size $< \kappa$, any elementary map $f : A \to M$ can be extended to an automorphism of $\mathcal{M}$.

We say that $\mathcal{M}$ is **saturated** if it is $|M|$-saturated.

Remark 2.2. Recall that $\mathcal{M}$ is κ-saturated if and only if it is κ-saturated over 1-types, i.e. if it realises 1-types over sets of parameters of cardinality $< \kappa$;

In the following exercises we work with $|\mathcal{L}| \leq \kappa$:

Exercise 2.3. Prove that if $\mathcal{M}$ is κ-saturated then it is κ-homogeneous.

Exercise 2.4. Show that if $\mathcal{M}$ is κ-saturated, then it is κ^+-universal.

Exercise 2.5. (a) Show that if $\mathcal{M}$ is $|M|$-homogeneous, then it is strongly $|M|$-homogeneous. (b) For each cardinal κ, give an example of a κ-saturated structure $\mathcal{M}$ which is not strongly ω-homogeneous.

Exercise 2.6. Show that $\mathcal{M}$ is κ-saturated if and only if it is κ-homogeneous and κ^+-universal.

Ideally, we would like to work with a large saturated model since these models are universal, homogeneous and strongly homogeneous. We will see that for this we will need additional set theoretic assumptions.

Remark 2.7. Let $(X, \leq)$ be a linear order. We say that $Y \subseteq X$ is **cofinal** with X if for each $x \in X$ there is some $y \in Y$ with $x \leq y$. The **cofinality** of X, $\mathrm{cf}(X)$ is the smallest cardinality of a cofinal subset of X. We say that an infinite cardinal is κ is **regular** if $\mathrm{cf}(\kappa) = \kappa$. Any successor cardinal κ^+ is a regular cardinal, and so is ω.

Theorem 2.8. *Let $|\mathcal{L}| \leq \kappa$ and $\mathcal{M}$ be a model of cardinality $\leq 2^\kappa$. Then, $\mathcal{M}$ has an elementary extension $\mathcal{N}$ which is κ^+-saturated and of size $\leq 2^\kappa$.*

Proof. We build an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\kappa^+}$ such that

- $|\mathcal{M}_\lambda| \leq 2^\kappa$ for each $\lambda < \kappa$;
- $\mathcal{M}_{\lambda^+}$ realises all types over subsets of $\mathcal{M}_\lambda$ of cardinality $\leq \kappa$;

Firstly, we show such chain exists and then we will show that its union, $\mathcal{N}$ satisfies the requirements of the theorem. We show how to perform the successor step. Since by inductive hypothesis $|\mathcal{M}_\lambda| \leq 2^\kappa$, $\mathcal{M}_\lambda$ has 2^κ subsets of size $\leq \kappa$. Since $|\mathcal{L}| \leq \kappa$, over each $B \subseteq \mathcal{M}_\lambda$ with $|B| \leq \kappa$, there are at most 2^κ-many 1-types. In particular, there are at most 2^κ-many 1-types over sets of size $\leq \kappa$. All of these can be realised in a model $\mathcal{M}_{\lambda^+}$ of cardinality $\leq 2^\kappa$.

We show that $\mathcal{N} := \bigcup_{\lambda<\kappa^+} \mathcal{M}_\lambda$ is κ^+-saturated and of size $\leq 2^\kappa$. For κ^+-saturation, consider $B \subseteq \mathcal{N}$ of size $< \kappa^+$. Since κ^+ is regular there must be some $\lambda \leq \kappa^+$ such that $B \subseteq \mathcal{M}_\lambda$ (otherwise, there would be a cofinal subset with κ^+ of cardinality $\leq |B| < \kappa^+$). Hence, all 1-types over B are realised in $\mathcal{M}_{\lambda^+}$. Finally, for the cardinality,

$$|\mathcal{N}| \leq \bigcup_{\lambda<\kappa^+} 2^\kappa \leq 2^\kappa,$$

where the last inequality holds since we are taking a union of sets of size $\leq \alpha$ over ordinals $< \alpha$, where α is an infinite cardinal. ☐

Definition 2.9. Let κ be an infinite cardinal. A cardinal α is called a **strong limit cardinal** if for all cardinals $\beta < \alpha$, we have $2^\beta < \alpha$. A regular strong limit cardinal is called a **strongly inaccessible cardinal**.

Remark 2.10. It is easy to construct strong limit cardinals within ZFC. Moreover, the **global continuum hypothesis**, (GCH) implies that every limit cardinal is a strong limit cardinal. However, ZFC is consistent with there being no strongly inaccessible cardinals apart from ω.

Corollary 2.11. *Let $|\mathcal{L}| \leq \kappa$ and T be an $\mathcal{L}$-theory (with infinite models).*

(a) Assuming (GCH)*, T has a saturated model in each regular cardinal $\nu > \kappa$;*

(b) T has a saturated model in each strongly inaccessible cardinal $\nu > \kappa$.

Proof. (Omitted from lecture) The ideas are essentially the same of the previous proof. (a) If ν is a successor cardinal the argument is immediate. For a limit cardinal, one can use an analogue of the argument below. (b) Starting from a model $\mathcal{M}_0$ of cardinality κ, using Theorem 2.8, we build an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\nu}$, where $\mathcal{M}_{\lambda^+}$ is λ^+-saturated and of cardinality $\leq 2^\lambda$, and take $\mathcal{N}$ to be the union of this chain. Since ν is a strong limit cardinal,

$$|\mathcal{N}| \leq \bigcup_{\lambda<\nu} 2^\lambda \leq \nu.$$

Note that a β-saturated model must be of cardinality $\geq \beta$. Hence, $|\mathcal{N}| \geq \lambda^+$ for each $\lambda < \nu$, meaning that $|\mathcal{N}| = \nu$. Finally, we need to show saturation. Take $A \subset \mathcal{N}$ of cardinality $< \nu$. Since ν is regular, a set of cardinality $|A|$ cannot be cofinal with it, meaning that there is some $|A| \leq \lambda < \nu$ such that $\mathcal{M}_\lambda$ entirely contains $|A|$. Since $\mathcal{M}_\lambda$ is λ^+ saturated, it realised all types over A, and so does $\mathcal{N}$. ☐

Example 2.12.
- $(\mathbb{C}; 0, 1; +, \cdot)$ is a saturated model of the theory of algebraically closed fields;
- In general, one can prove that stable theories have saturated models of arbitrarily large cardinalities;
- If the continuum hypothesis is false, the theory of $(\mathbb{N}; 0, 1; +, \cdot)$ has no saturated models of cardinality κ for each $\aleph_0 < \kappa < 2^{\aleph_0}$.

Convention 2.13. From now on we will work with a **monster model** $\mathbb{M}$. which is κ-saturated, κ-universal and strongly κ-homogeneous for κ a cardinal larger than all of the cardinalities of models and sets of parameters that we want to consider. Thus, all models $\mathcal{M}, \mathcal{N}, \dots$ we will consider will be elementarily embedded into this monster model, all sets of parameters $A, B, \dots$ will be subsets of the monster model of cardinality $< \kappa$, and a set of formulas will be consistent if it is realised in $\mathbb{M}$. Finally, for a formula ϕ or a type p, we write $\vDash \phi$ (or $\vDash p$) if $\mathbb{M} \vDash \phi$ (respectively, $\mathbb{M} \vDash p$).

Remark 2.14. There are several ways to achieve the above:

- Assume that strongly inaccessible cardinals exist and work in a sufficiently large one. *We will adopt this approach* since it allows us to move quickly to do more model theory;
- Work in BGC (Bernays-Gödel+Global Choice) set theory. This is a conservative extension of ZFC which allows working with classes. In this framework we can build the monster model as a class-size union of chains.
- Work with a special model (see Definition 2.16) of cardinality $\nu = \beth_\kappa(\aleph_0)$. This will be ν^+-universal and strongly κ-homogeneous (add so κ-saturated by Exercise 2.6). This framework has the advantage of allowing us to work entirely within ZFC. If you are not comfortable with strongly inaccessible cardinals, you are welcome to read Subsection 2.1 and work with a large enough special model instead.

Lemma 2.15. *Let X be a definable subset of $\mathbb{M}$ and A a set of parameters (i.e. a set of size $< \kappa$ inside of $\mathbb{M}$). Then, the following are equivalent:*

(a) X is definable over A;

(b) X is $\mathrm{Aut}(\mathbb{M}/A)$*-invariant (i.e. invariant under automorphisms of $\mathbb{M}$ fixing A pointwise).*

Proof. $(\Rightarrow)$ This direction works in every model $\mathcal{M}$. Suppose that $X := \phi(\mathcal{M}, b)$ for some $b \in A$. Then, for every $a \in \mathcal{M}$ and $\sigma \in \mathrm{Aut}(M/A)$, we have that

$$a \in X \Leftrightarrow \vDash \phi(a, b) \Leftrightarrow \phi(\sigma(a), \sigma(b)) \Leftrightarrow \phi(\sigma(a), b) \Leftrightarrow \sigma(a) \in X,$$

where the second last equivalence holds since $b \in A$ and σ fixes A pointwise.

$(\Leftarrow)$ Let $X = \phi(\mathbb{M}, b)$ and let $p(y) := \mathrm{tp}(b/A)$.

Claim 1: $p(y) \vdash \forall x(\phi(x, y) \leftrightarrow \phi(x, b))$.

Proof of Claim. Take $b' \vdash p(y)$. By strong homogeneity there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(b) = b'$. By assumption, $X = \sigma(X) = \phi(\mathbb{M}, b')$, yielding the desired formula is implied by $p(y)$. □

By compactness, there is some $\psi(y) \in p(y)$ such that

$$\psi(y) \vDash \forall x(\phi(x, y) \leftrightarrow \phi(x, b)) \tag{1}$$

Take $\theta(x) := \exists y(\psi(y) \wedge \phi(x, y))$. This is an $\mathcal{L}_A$-formula. We claim $X = \theta(\mathbb{M})$. For $(\subseteq)$ take $a \in X$. So $\vdash \phi(a, b)$. Since $\psi(y) \in \mathrm{tp}(b/A)$, $\vDash \theta(a)$. For $(\supseteq)$, if $\vDash \theta(a)$ there is some b' such that $\vDash \psi(b') \wedge \phi(a, b')$. By $\vDash \psi(b')$ (1), we have $\vDash \phi(a, b)$, as desired. □

2.1 Aside: special models

An issue with our definition of monster model being a saturated model of size a strongly inaccessible cardinal is that it makes it less transparent that our results are provable in ZFC. A more cautious reader might want to work with special models. An even more set theoretically oriented reader, might be interested in the approach of [8], which partially justifies the standard model theoretic practice of assuming we are working with a saturated model of large enough cardinality.

Definition 2.16. An infinite structure $\mathcal{M}$ of cardinality κ is **special** if it is the union of an elementary chain $(\mathcal{M}_\lambda)_{\lambda<\kappa}$, where the λ are *cardinals* of size $< \kappa$ and each $\mathcal{M}_\lambda$ is κ^+-saturated.

$\star\star$ **Exercise 2.17.** Let $|\mathcal{L}| \leq \kappa$. Show that the following hold:

(a) If $\mathcal{M}$ is saturated then it is special;

(b) A special structure of regular cardinality is saturated;

(c) Suppose that $\lambda < \nu$ implies $2^\lambda \leq \nu$. Then, T has a special model of cardinality ν;

(d) A special structure of cardinality κ is κ^+-universal and strongly $\mathrm{cf}(\kappa)$-homogeneous.

Definition 2.18. For every cardinal μ, the **beth function** is defined as

$$\beth_\alpha(\mu) = \begin{cases} \mu & \text{if } \alpha = 0, \\ 2^{\beth_\beta(\mu)}, & \text{if } \alpha = \beta + 1, \\ \sup_{\beta<\alpha} \beth_\beta(\mu) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Remark 2.19. We have that $\mathrm{cf}(\beth_\kappa(\aleph_0)) = \kappa$, meaning that a special model of cardinality $\nu = \beth_\kappa(\aleph_0)$ is strongly κ-homogeneous, ν^+-universal, and κ-saturated. There is no harm in working with a special model of such cardinality as the monster model (except from having to prove exercise 2.17).

3 Strong minimality and algebracity

From now on it will be important to keep in mind the conventions that we set in the previous lecture (Convention 2.13). In particular, models are always taken to be elementary substructures of the monster model $\mathbb{M}$ and parameter sets $A, B, \ldots$ are always taken to be small enough and live in the monster model (which is why I don't specify every time where they come from).

Definition 3.1. We say that a formula $\phi(x) \in \mathcal{L}(A)$ is **algebraic** (over A) if $\phi(\mathbb{M})$ is finite.
$a \in \mathbb{M}$ is **algebraic** over A if it realises an algebraic formula over A.
We denote by $\mathrm{acl}(A)$ the set of elements algebraic over A. For π a partial type over A (closed under conjunctions), we say that it is **algebraic** if it contains an algebraic formula.

👁 *Observation* 3.2. Note if a formula $\phi(x) \in \mathcal{L}(A)$ is **algebraic**, then it has the same set of realisations in every model containing A.

Exercise 3.3. Prove Neumann's Lemma: Let $A, B \subseteq \mathbb{M}$ and $(c_1, \ldots, c_n)$ a sequence of elements not algebraic over A. Show that $\mathrm{tp}(c_1, \ldots, c_n/A)$ has a realisation which is disjoint from B.

Exercise 3.4. Show that $\mathrm{acl}(A)$ is the intersection of all models containing A.

Definition 3.5. Let $\mathcal{M}$ be a model. Let $\phi(x) \in \mathcal{L}(M)$ be a non-algebraic formula. We say that ϕ is **minimal** in $\mathcal{M}$ if for all $\mathcal{L}(M)$-formulas $\psi(x)$,

$$\phi(M) \wedge \psi(M) \text{ is finite or cofinite in } \phi(M).$$

We say that $\phi(x) \in \mathcal{L}(M)$ is **strongly minimal** if it is minimal in the monster model $\mathbb{M}$.
A theory T is **strongly minimal** if $x = x$ is strongly minimal. A type $p \in S(A)$ is **strongly minimal** if it contains a strongly minimal formula.

Examples 3.6 (Strongly minimal theories)**.** It is easy to prove strong minimality of the following theories by quantifier eliminations:

- The theory T_∞ of an infinite set with equality;
- The theory of infinite vector spaces over a field K, $(V; 0; +; (\lambda_k)_{k \in K})$;
- ACF_p, the theory of algebraically closed fields in characteristic p.

Example 3.7 (A minimal set which is not strongly minimal)**.** Consider the structure $\mathcal{M}$ with an equivalence relation E that has countably many equivalence classes, one of each finite size and no infinite classes. Note that each equivalence class is a definable subsets of $\mathcal{M}$ (using quantifiers). One can then show that adding predicates P_n for each equivalence class to the language, the (new) theory of $\mathcal{M}$ has quantifier elimination (for example, by [19, Theorem 3.2.5]). From this it is easy to see that every definable subset of $\mathcal{M}$ is either finite or cofinite. However, $\mathbb{M} \succ \mathcal{M}$ has an infinite class (by ω-saturation). So for a in this class, $E(x, a)$ is infinite and coinfinite. Note that the fact $\mathcal{M}$ is not ω-saturated plays an important role here (see Exercise 3.9 below).

Non-examples 3.8.
- The theory of two infinite predicates partitioning the domain is not strongly minimal. However, each predicate is;
- The theory of the random graph has no strongly minimal formula.

Exercise 3.9. Prove the following: let $\mathcal{M}$ be ω-saturated. Suppose that $\phi \in \mathcal{L}(M)$ is minimal in $\mathcal{M}$. Then ϕ is strongly minimal.

Exercise 3.10.
(a) Consider the theory of $(\mathbb{Z}, s)$, the integers with the successor operation $s(x) = x + 1$. This theory has quantifier elimination. What is algebraic closure in this theory? Is this $x = x$ in $(\mathbb{Z}, s)$ minimal? is it strongly minimal?

(b) Consider the theory of $(\mathbb{N}, <)$. This theory has quantifier elimination if we add a function symbol for the successor and a constant symbol for 0 (both of which are definable in the original theory). Is $x = x$ in $(\mathbb{N}, <)$ minimal? is it strongly minimal?

The idea of the following lemma is that algebraic sets are very small (being finite), so it is possible to extend non-algebraic types to larger parameter sets whilst avoiding algebraic sets (over those parameters):

Lemma 3.11 (Extension)**.** *Let $\pi(x)$ be a partial type (closed under conjunctions) non-algebraic over A. Let $A \subseteq B$. Then, π has a non-algebraic extension $a \in S(B)$.*

Proof. Consider

$$q_0(x) := \pi(x) \cup \{\neg\psi(x) | \psi(x) \in \mathcal{L}(B) \text{ is algebraic }\}.$$

We prove this is finitely satisfiable. Take $\phi(x) \in \pi(x)$ (note π is closed under conjunctions) and $\psi_1(x), \ldots, \psi_n(x)$ algebraic. Then, since $\phi(\mathbb{M})$ is infinite and for each i $\neg\psi_i(\mathbb{M})$ is cofinite,

$$\phi(x) \wedge \bigwedge_{i \leq n} \neg\psi_i(x)$$

has infinitely many realisations. This proves finite satisfiability, and by compacness satisfiability of q_0. Finally, take any completion $q \in S(B)$ of q_0. This will still be non-algebraic by construction of q_0, completing the proof. □

One can actually prove a more general statement, where the "small" sets one is avoiding are charaterised from belonging to an ideal in the Boolean algebra of definable sets. This will be very important later.

Definition 3.12. A set of definable subsets of $\mathbb{M}$ in the variable x, $I \subseteq \mathrm{Def}_x(\mathbb{M})$ is an **ideal** if it contains $\emptyset$, and it is closed under (definable) subsets and finite unions.

Exercise 3.13. Prove the following:
Let $I \subseteq \mathrm{Def}_x(\mathbb{M})$ be an ideal. Let $\pi(x)$ be a partial type over A (closed under conjunctions) such that $p(\mathbb{M})$ is not contained in any set in I. Then, for every $B \supseteq A$, there is a type $q \in S(B)$ extending p and such that $q(\mathbb{M})$ is not contained in any set in I.

Lemma 3.14. *The $\mathcal{L}(M)$-formula $\phi(x)$ is minimal in $\mathcal{M}$ if and only if there is a unique non-algebraic type $p \in S(M)$ containing $\phi(x)$.*

Proof. ($\Rightarrow$) Assume ϕ is minimal in $\mathcal{M}$. Being non-algebraic, by extension (Lemma 3.11), it has a non-algebraic extension $p \in S(M)$. Note that if $\psi(x) \in p$, then $\phi(x) \wedge \psi(x)$ is infinite, and so by minimality of ϕ, $\phi(x) \wedge \neg\psi(x)$ is finite. So any type containing ϕ and $\neg\psi$ is algebraic. This implies that p is the unique non-algebraic type containing ϕ.

($\Leftarrow$) By contrapositive. Suppose $\phi(x)$ is not minimal. If it is algebraic, then it cannot be contained in a non-algebraic type. So it is non-algebraic and by non-minimality there is some $\mathcal{L}(M)$-formula ψ with both $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ non-algebraic. Hence, by extension (Lemma 3.11), each formula extends to a non-algebraic type in $S(M)$ containing ϕ. Since the two types are clearly distinct (as one contains ψ and the other $\neg\psi$), this completes the proof of the contrapositive. $\square$

Corollary 3.15 (Stationarity)**.** *Let $p \in S(A)$ be strongly minimal. Then*

(a) p has a unique non-algebraic extension to all $B \supseteq A$;

(b) If $a_1^0, \dots, a_m^0$ and $a_1^1, \dots, a_m^1$ are two sequences of realisations of p of length m which are algebraically independent in the sense that

$$a_i^j \notin \mathrm{acl}(Aa_1^j, \dots a_{i-1}^j)$$

for each $i \leq m$ and $j \in \{0, 1\}$. Then,

$$a_1^0, \dots, a_m^0 \equiv_A a_1^1, \dots, a_m^1.$$

So, the type over A of an algebraically independent tuple of realisations of p is entirely determined.

Proof. (a) From Lemma 3.14, p has a unique non-algebraic extension to $\mathbb{M}$, and so also to any set of parameters containing A.
(b) By induction. The base case is trivial. Suppose that $\bar{a}^0 \equiv_A \bar{a}^1$ for algebraically independent m-tuples of realisations of p. Let $a_{m+1}^0 \notin \mathrm{acl}(A\bar{a}^0)$ and $a_{m+1}^1 \notin \mathrm{acl}(A\bar{a}^1)$ be realisatinos of p. Take $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(\bar{a}^0) = \bar{a}^1$. Since automorphisms preserve algebraicity, $\sigma(a_{m+1}^0)$ is non-algebraic over $A\bar{a}^1$. By Lemma 3.14, $\sigma(a_{m+1}^0) \equiv_{A\bar{a}^1} a_{m+1}^1$. So there is $\tau \in \mathrm{Aut}(\mathbb{M}/A\bar{a}^1)$ such that $\tau\sigma(a_{m+1}^0) = a_{m+1}^1$. Since the composition of the two automorphisms fixes A, $\bar{a}^0 a_{m+1}^0 \equiv_A \bar{a}^1 a_{m+1}^1$, as desired. $\square$

4 Pregeometries

In this lecture we are going to use the results from the previous lecture to prove that algebraic independence behaves particularly well in strongly minimal sets (and theories). In particular, we will see that algebraic closure inside a strongly minimal set gives rise to a pregeometry: a structure whose behaviour of algebraic independence satisfies the axioms of linear independence, allowing us to talk about bases and dimensions.

The notion of a pregeometry (also known as matroid) originates from the work of Whitney [21] and Van de Waerden [20], both of whom gave axioms for linear independence in vector spaces. In particular, Whitney's work stemmed from applying notions from linear algebra to combinatorics after noticing various similarities between certain ideas of independence and ranks in graph theory and the behaviour of linear independence. Nowadays matroid theory is a branch of mathematics with several applications in combinatorics [15]. Our interests differ from standard matroid theory because we study infinite pregeometries, but we will make use of some basic facts about pregeometries in this section.

Definition 4.1. A **pregeometry** (X, cl) consists of a set X with a closure operator

$$\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$$

such that for all $A \subseteq X$ and $a, b \in X$:

- (REFLEXIVITY) $A \subseteq \mathrm{cl}(A)$;
- (FINITE CHARACTER) $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A') | A' \subseteq A \text{ finite }\}$;
- (TRANSITIVITY) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- (EXCHANGE) if $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(Aa)$.

Remark 4.2. For any structure $\mathcal{M}$, $(\mathcal{M}, \mathrm{acl})$ satisfies reflexivity, finite character, and transitivity.

Theorem 4.3. *Let ϕ be a strongly minimal $\mathcal{L}$-formula. Let $\mathrm{cl} : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M})$ be defined by, for $A \subseteq \phi(\mathbb{M})$, $\mathrm{cl}(A) := \mathrm{acl}(A) \cap \phi(\mathbb{M})$. Then, $(\phi(\mathbb{M}), \mathrm{cl})$ is a pregeometry.*

Proof. Reflexivity, finite character and transitivity are trivial. We only need to verify exchange. Without loss of generality (and to simplify notation), we assume that $A = \emptyset$. All elements we work with are inside of $\phi(\mathbb{M})$. Let $a \notin \mathrm{acl}(\emptyset)$ and $b \notin \mathrm{acl}(a)$. We need to prove $a \notin \mathrm{acl}(b)$.
Firstly, note that $\phi(x)$ extends to a unique non-algebraic type $q(x)$ over $\emptyset$ (Lemma 3.14). By stationarity (Corollary 3.15 (b)), all pairs $a'b'$ satisfying $a' \notin \mathrm{acl}(\emptyset)$ and $b' \notin \mathrm{acl}(a')$ have the same type $p(x, y)$.
Now, take $(a_i | i < \omega)$ an infinite sequence of realisations of $q(x)$ such that

$$a_i \notin \mathrm{acl}(a_0 \dots a_{i-1}).$$

This can be done by induction iterating non-algebraic extensions (by Lemma 3.11). Using extension again, pick $b' \notin \mathrm{acl}((a_i | i < \omega))$ realising $q(x)$. Since $a_i \notin \mathrm{acl}(\emptyset)$ and $b' \notin \mathrm{acl}(a_i)$ for each $i < \omega$, we have $a_i b' \equiv ab$ for all $i < \omega$. So, $a_i \notin \mathrm{acl}(b')$, since $\mathrm{tp}(a_i / b') = p(x, b')$ has infinitely many realisations. But then, since $b \equiv b'$, $p(x, b)$ also has infinitely many realisations. So $a \notin \mathrm{acl}(b)$ as desired. □

👁 *Observation* 4.4. The above proof actually works in any model: we may need to move outside of a given model M to realise the a_i. However, the conclusion that $p(x, b)$ is non-algebraic, does tell us that $a \notin \mathrm{acl}(b)$ also in M.

Definition 4.5. Let (X, cl) be a pregeometry. For $A \subseteq X$, we say that:

- A is **independent** if $a \notin \mathrm{cl}(A \setminus \{a\})$ for each $a \in A$;
- A is a **generating set** if $\mathrm{cl}(A) = X$;
- A is a **basis** if it is an independent generating set for X.

Fact 4.6. *(a) Every pregeometry has a basis [to prove this you need the axiom of choice];*

(b) Any two bases for a pregeometry have the same cardinality.

Definition 4.7. For a pregeometry (X, cl), we say that the **dimension** of X, $\dim(X)$ is the cardinality of a basis for X.

Definition 4.8. Given a pregeometry (X, cl), for $S \subseteq X$, let

- (S, cl) given by $\mathrm{cl}(A) = \mathrm{cl}(A) \cap S$ for all $A \subseteq S$ be the **restriction** of (X, cl) to S;

- (X, cl_S) given by $\mathrm{cl}_S(A) = \mathrm{cl}(A \cup S)$ for all $A \subseteq S$ be the **relativisation** of (X, cl) by S;

We write $\dim(S)$ for $\dim((S, \mathrm{cl}))$, and $\dim(X/S)$ for $\dim((X, \mathrm{cl}_S))$. It is easy to show both of these are also pregeometries.

Note that thinking with the restriction and relativisation allows us to speak of bases for subspaces of X, or of independence over some subset of $S \subseteq X$.

👁 *Observation* 4.9. Note that for strongly minimal ϕ, the restriction $(\phi(M), \mathrm{cl})$ is well defined since $\phi(M) \subseteq \phi(\mathbb{M})$. Meanwhile, its relativisation by $A \subseteq M$, $(\phi(M), \mathrm{cl}_A)$ corresponds to the natural pregeometry on $(\phi(M_A), \mathrm{cl})$, where M_A is the expansion of M by constants naming the elements of A. We write $\dim_\phi(M)$ for the dimension $(\phi(M), \mathrm{cl})$, and $\dim_\phi(M/A)$ for the dimension of $(\phi(M_A), \mathrm{cl})$.

Remark 4.10. For a pregeometry (X, cl) and $S \subseteq X$, we have

$$\dim(X) = \dim(S) + \dim(X/S).$$

Exercise 4.11. Let $f : A \to B$ be an elementary bijection between sets of parameters. Then, f extends to an elementary bijection $f' : \mathrm{acl}(A) \to \mathrm{acl}(B)$.

Lemma 4.12. *Let $\phi \in \mathcal{L}(A)$ be strongly minimal. Let $A \subseteq \mathcal{M}, \mathcal{N} \models T$. Then, the following are equivalent:*

1. *there is an A-elementary bijection $f : \phi(M) \to \phi(N)$;*
2. $\dim_\phi(M/A) = \dim_\phi(N/A)$.

Proof. Without loss of generality we work over $\emptyset$ (we can always just work in $\mathrm{Th}(\mathbb{M}_A)$).
$(\Rightarrow)$ We know there is an elementary bijection $f : \phi(M) \to \phi(N)$. Note that elementary bijections map bases to bases (since they preserve algebraic relations). Hence, $\dim_\phi(M) = \dim_\phi(N)$.
$(\Leftarrow)$ Take bases U and V for $\phi(M)$ and $\phi(N)$. Let $f : U \to V$ be a bijection. By independence of the bases and stationarity (Corollary 3.15 (b)), there is an elementary bijection between U and V. Elementary bijection extend to algebraic closures (as noted in Exercise 4.11). So there is an elementary bijection $f' : \mathrm{acl}(U) \to \mathrm{acl}(V)$. Now $f'|_{\phi(M)}$ is an elementary bijection from $\phi(M)$ to $\phi(N)$. □

👁 *Observation* 4.13. For any set of parameters A,

$$|\mathrm{acl}(A)| \leq \max(|\mathcal{L}|, |A|),$$

where $|\mathcal{L}|$ is the size of the set of $\mathcal{L}$-formulas.

Corollary 4.14. *Let T be a countable and strongly minimal theory. Then, it is categorical in all uncountable cardinals.*

Proof. Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ have cardinality $\kappa > \aleph_0$. Choose bases B_1, B_2 respectively. By Observation 4.13, for each $i \in \{1, 2\}$:

$$\kappa = |M_i| = \mathrm{acl}(B_i) \leq \max(|\mathcal{L}|, |B_i|) = \max(\aleph_0, |B_i|) = |B_i|.$$

So $\dim(M_1) = \dim(M_2)$. So there is an elementary bijection $f : M_1 \to M_2$ by Lemma 4.12. □

Exercise 4.15. Let T be a strongly minimal theory (not necessarily countable). Show the following:

(a) Every infinite algebraically closed set of parameters S is the universe of a model of T;

(b) A model $\mathcal{M}$ is ω-saturated if and only if $\dim(M) \geq \aleph_0$;

(c) All models are ω-homogeneous.

5 ω-stability and the downwards Morley theorem

Definition 5.1. We say that T is ω-stable if for any $n \in \mathbb{N}$ and any set of parameters A such that $|A| \leq \aleph_0$, $|S_n(A)| \leq \aleph_0$.

Remark 5.2. It is easy to prove that T is ω-stable if and only if for set of parameters A such that $|A| \leq \aleph_0$, $|S_1(A)| \leq \aleph_0$. We will generally use this characterisation of ω-stability.

Examples 5.3.
- if T is strongly minimal, then it is ω-stable. To see this note that if $|A| \leq \aleph_0$, there are only $\leq \aleph_0$ many algebraic types over A (since $\mathcal{L}$ is countable) and there is a unique non-algebraic type over A, meaning that $|S_1(A)| \leq \aleph_0$;
- if T is κ-categorical for $\kappa > \aleph_0$, then it is ω-stable. This was proven in the previous model theory course and it is a strictly more general fact than the previous one;
- the theory of an infinitely branching infinite tree is ω-stable (but not κ-categorical in any infinite κ).

Definition 5.4. We say that T is **totally transcendental** if there is no binary tree of (consistent) $\mathcal{L}(\mathbb{M})$-formulas $(\phi_s(x) | s \in^{<\omega} 2)$ such that

- $\vdash \forall x \neg(\phi_{s0}(x) \wedge \psi_{s1}(x))$;
- $\vdash \forall x((\phi_{s0}(x) \vee \psi_{s1}(x)) \rightarrow \phi_s(x))$.

That is, we ask that any two children of a common note are mutually inconsistent, but their union as a pair of definable sets is contained in the set defined by their parent.

Lemma 5.5. *A theory T is ω-stable if and only if it is totally transcendental.*

Definition 5.6. Let A be a set of parameters and x a tuple of variables. For an $\mathcal{L}(A)$-formula $\phi(x)$, we define

$$[\phi(x)] := \{p \in S_x(A) | \phi(x) \in p\}.$$

Sets of the form $[\phi(x)]$ form a basis of clopen sets for a topology on $S_x(A)$, which we call the **Stone topology**.
We say that a type $p \in S_x(A)$ is **isolated** if there is some $\mathcal{L}(A)$-formula $\psi(x)$ such that $[\psi(x)] = \{p\}$.

Fact 5.7. *The type space $S_x(A)$ with the Stone topology is compact, Hausdorff, and totally disconnected (i.e. for all $p, q \in S_x(A)$ there is a clopen set X such that $p \in X$ and $q \notin X$.*

Exercise 5.8. Let T be a countable complete theory. Let A be a countable set of parameters and x a finite tuple of variables. Suppose that $|S_x(A)| < 2^{\aleph_0}$. Prove the following:

- the isolated types in $S_n(A)$ are dense, i.e. for any $\mathcal{L}(A)$-formula $\phi(x)$, $[\phi]$ contains an isolated type;
- $|S_x(A)| \leq \aleph_0$.

[Hint: in both contexts, you need to build an adequate binary tree of $\mathcal{L}(A)$-formulas $(\phi_\sigma | \sigma \in 2^{<\omega})$ such that any finite branch is consistent but any two children of a common node are mutually inconsistent. Then, each infinite branch of the binary tree can be used to construct a type, giving $2^{\aleph_0}$-many.]

Definition 5.9. Let $A \subseteq \mathcal{M} \models T$. We say that $\mathcal{M}$ is **prime over** A if for all $\mathcal{N} \models T$ and $f : A \rightarrow \mathcal{N}$ a partial elementary map, f extends to an elementary $f' : \mathcal{M} \rightarrow \mathcal{N}$.

We say that $\mathcal{M}$ is **prime**, if it is prime over $\emptyset$.

Example 5.10. The theory of algebraically closed fields of characteristic zero ACF_0 has $\overline{\mathbb{Q}}^{\mathrm{alg}}$, the algebraic closure of $(\mathbb{Q}; 0, 1; +, -, \times)$ as a prime model. In fact, $\overline{\mathbb{Q}}^{\mathrm{alg}}$ is a model of ACF_0 and it embeds in every model of the theory. By model completeness of ACF_0, such an embedding is elementary, yielding that $\overline{\mathbb{Q}}^{\mathrm{alg}}$ is prime. Indeed, we can prove that countable ω-stable theories always have a prime model.

Exercise 5.11. Show the following: Let T be a countable ω-stable theory, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. Then, there is $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime model over A and such that every $a \in M$ realises an isolated type over A.

Theorem 5.12 (Lachlan)**.** *Let T be ω-stable, $\mathcal{M} \models T$, $|M| \geq \aleph_1$. Then, for each $\kappa > |M|$ there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality κ such that for any countable set of $\mathcal{L}(M)$-formulas $\Gamma(x)$ in a finite variable x, if $\mathcal{N}$ realises $\Gamma(x)$, then so does $\mathcal{M}$.*

Exercise 5.13. We shall prove Theorem 5.12 following the steps below. Consider an ω-stable theory T and $\mathcal{M} \models T$, such that $|M| \geq \aleph_1$. Say that an $\mathcal{L}(M)$-formula is **large** if $\phi(M)$ is uncountable.

- Prove that there is a large $\mathcal{L}(M)$-formula $\phi_0(x)$ such that for any other $\mathcal{L}(M)$-formula ψ, either $\phi_0(x) \wedge \psi(x)$ or $\phi_0(x) \wedge \neg\psi(x)$ has a countable set of realisations.

- Consider
$$p(x) := \{\psi(x) | \psi(x) \in \mathcal{L}(M) \text{ and } \phi_0(x) \wedge \psi(x) \text{ is large }\}.$$
Show that p is a complete type over M which is not realised in M but such that all of its countable subsets are realised in M. Take $\mathcal{N}' \succeq \mathcal{M}$ with a point a realising p.

- By Exercise 5.11, take $\mathcal{N} \preceq \mathcal{N}'$ prime over Ma and such that every $b \in \mathcal{N}$ realises an isolated type over Ma. Show that for every $b \in N$, every countable subset $\Gamma(x)$ of $\mathrm{tp}(b/M)$ is realised in M.

- Deduce Theorem 5.12.

Remark 5.14. Recall the two following facts from the model theory I course:

- any two saturated models of the same cardinality are isomorphic;

- if T is κ-categorical, then all of its models of cardinality κ are saturated.

Theorem 5.15 (Downwards Morley Theorem)**.** *Let T be countable and κ-categorical in some uncountable κ. Then T is $\aleph_1$-categorical.*

Proof. Suppose by contradiction that T is κ-categorical (so ω-stable) and not $\aleph_1$-categorical. Then, it has a non-saturated model $\mathcal{M}$ of cardinality $\aleph_1$. So there is some $p \in S_1(A)$ for $A \subseteq M$ countable which is not realised in M. By Theorem 5.12 and ω-stability, there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality κ and not realising p. But if T is κ-categorical, all models of cardinality κ are saturated and $\mathcal{N}$ is not saturated. Contradiction. □

6 Vaughtian pairs

Definition 6.1. We say that T has a **Vaughtian pair** if there are $\mathcal{M} \preceq \mathcal{N} \models T$ such that $M \subsetneq N$ and $\phi \in \mathcal{L}(M)$ non-algebraic such that $\phi(M) = \phi(N)$.

We will often write a Vaughtian pair as $(\mathcal{N}, \mathcal{M})$ since it is convenient to think about this as the expansion of $\mathcal{N}$ by a predicate P naming the smaller model $\mathcal{M}$.

Exercise 6.2. Show that the theory of the random graph has a Vaughtian pair.

⋆⋆ **Exercise 6.3.** Show that there is no Vaughtian pair of real closed fields.

Lemma 6.4. *Suppose that T has a Vaughtian pair. Then, T has a Vaughtian pair $(\mathcal{N}, \mathcal{M})$ with $\mathcal{N}$ and $\mathcal{M}$ countable.*

Proof. This proof is essentially an application of the downwards Lowenheim-Skolem theorem. Let $(\mathcal{N}^\star, \mathcal{M}^\star)$ be a Vaughtian pair for T as witnessed by the formula $\phi(x) \in \mathcal{L}(A)$ for $A \subseteq M^\star$ finite. Consider $(\mathcal{N}^\star, \mathcal{M}^\star)$ as $\mathcal{N}^\star$ expanded by a predicate P naming $\mathcal{M}^\star$. Then, by the downwards Lowenheim-Skolem theorem, there is $(\mathcal{N}, P(N)) \preceq (\mathcal{N}^\star, \mathcal{M}^\star)$ countable and containing A, and so, in particular, $A \subseteq P(N)$.

It is easy to verify that $P(N) \preceq \mathcal{N}$ by the Tarski-Vaught test. Moreover, $\phi(P(N))$ is infinite and such that $\phi(P(N)) = \phi(N)$, $P(N) \subsetneq N$, since all of this is coded by the theory of $(\mathcal{N}^\star, \mathcal{M}^\star)$. Hence, $(\mathcal{N}, P(N))$ is a Vaughtian pair. $\square$

Lemma 6.5 (Basic facts about ω-homogeneous models)**.** *Let T be a countable theory.*

1. *Every countable model of T has a countable ω-homogeneous elementary extension;*
2. *the union of an elementary chain of ω-homogeneous models is ω-homogeneous;*
3. *two ω-homogeneous countable models of T realising the same n-types over $\emptyset$ for all $n \in \mathbb{N}$ are isomorphic.*

Proof. For (1), start with $\mathcal{M}_0 \models T$ countable. Build a countable elementary extension $\mathcal{M}_1 \succeq \mathcal{M}_0$ such that for all $a \in M_0$, $A \subseteq M_0$ finite, $p(x, A) := \mathrm{tp}(a/A)$ and $f : A \to \mathcal{M}_0$ elementary, $\mathcal{M}_1$ realises $p(x, f(A))$. This can be done since it requires realising only countably many types. Iterate this for a countable elementary chain $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \dots$, and consider $\mathcal{M} := \bigcup_{i<\omega} \mathcal{M}_i$. By construction this is ω-homogeneous (the argument is essentially the same as the one below).

For (2), Consider $\mathcal{N} := \bigcup_{\beta<\lambda} \mathcal{N}_\beta$, the union of an elementary chain of ω-homogeneous models. Take $a \in N$, $A \subseteq N$ finite, and $f : A \to \mathcal{N}$ elementary. Since A is finite, $f : A \to \mathcal{N}_\beta$ containing both A and a for some $\beta < \lambda$. Since $\mathcal{N}_\beta$ is ω-homogeneous f can be extended to $f' : Aa \to \mathcal{N}_\beta \preceq \mathcal{N}$, yielding that $\mathcal{N}$ itseld is ω-homogeneous.

The proof of (3) is a trivial back & forth argument. $\square$

Example 6.6 (Thanks to R. Feller during the class)**.** Usually, we work with ω-homogeneous models. For an example of a non ω-homogeneous countable structure consider the model of $\mathrm{Th}(\mathbb{Z}, <)$ consisting of three disjoint copies of $(\mathbb{Z}, <)$ ordered one after the other $\mathcal{M} := \mathbb{Z}_{a1} \sqcup \mathbb{Z}_{a2} \sqcup \mathbb{Z}_{a3}$. Take $a \in \mathbb{Z}_{a1}$, $b \in \mathbb{Z}_{a3}$ and consider an elementary embedding $f : ab \to \mathcal{M}$ such that $a \mapsto a$ and $b \mapsto b'$ for $b' \in \mathbb{Z}_{a2}$. Take $c \in \mathbb{Z}_{a2}$. f cannot be extended to c: we know that c has infinite distance from both a and b and is between then in the order $<$. But every element between a and b' is in either $\mathbb{Z}_{a1}$ or $\mathbb{Z}_{a2}$ and so cannot have infinite distance from both of them.

Corollary 6.7. *Suppose $\mathcal{M}_0 \preceq \mathcal{N}_0$ are countable models of T. Then, there are $(\mathcal{N}, \mathcal{M}) \succeq (\mathcal{N}_0, \mathcal{M}_0)$ such that $\mathcal{M}$ and $\mathcal{N}$ are countable, ω-homogeneous and satisfy the same n-types over $\emptyset$. In particular, $\mathcal{M} \cong \mathcal{N}$.*

Proof. We construct a countable elementary chain

$$(\mathcal{N}_0, \mathcal{M}_0) \preceq (\mathcal{N}_1, \mathcal{M}_1) \preceq \dots$$

as follows: for $(\mathcal{N}_i, \mathcal{M}_i)$ take $(\mathcal{N}', \mathcal{M}') \succeq (\mathcal{N}_i, \mathcal{M}_i)$ such that $\mathcal{M}'$ realises all n-types over $\emptyset$ realised by $\mathcal{N}_i$. Then, by Lemma 6.5 (1) take a countable ω-homogeneous elementary extension $(\mathcal{N}_{i+1}, \mathcal{M}_{i+1}) \succeq (\mathcal{N}', \mathcal{M}')$. Note that since $(\mathcal{N}_{i+1}, \mathcal{M}_{i+1})$, we also have that both $\mathcal{N}_{i+1}$ and $\mathcal{M}_{i+1}$ are ω-homogeneous.

Now, consider the union of this elementary chain, $(\mathcal{N}, \mathcal{M})$. By construction and Lemma 6.5 (2) this is also ω-homogeneous, and so such that both $\mathcal{N}$ and $\mathcal{M}$ are ω-homogeneous. Furthermore, by construction $\mathcal{M}$ and $\mathcal{N}$ satisfy the same n-types over $\emptyset$. Hence, $(\mathcal{N}, \mathcal{M})$ satisfies all of the desired properties. $\square$

Theorem 6.8 (Vaught's two cardinal theorem)**.** *Let T have a Vaughtian pair. Then, there is $\mathcal{N}^\star \models T$ of cardinality $\aleph_1$ and $\phi \in \mathcal{L}(N^\star)$ such that $|\phi(N^\star)| = \aleph_0$.*

Proof. By Lemma 6.4, T has a countable Vaughtian pair and by Corollary 6.7 we can choose it so that $\mathcal{M}$ and $\mathcal{N}$ are countable, ω-homogeneous and realising the same n-types over $\emptyset$. In particular, $\mathcal{M} \cong \mathcal{N}$. We build an elementary chain

$$(\mathcal{N}_\alpha | \alpha < \omega_1)$$

such that

- $\mathcal{N}_0 = \mathcal{M}, \mathcal{N}_1 = \mathcal{N}$;
- $\mathcal{N}_\alpha \cong \mathcal{N}$;
- $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$.

For the successor step, suppose that we have $\mathcal{N}_\alpha \cong \mathcal{N}$. Then, $\mathcal{N}_\alpha \cong \mathcal{M}$, and so it has an elementary extension $\mathcal{N}_{\alpha+1}$ such that $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$, and so we are done. For the limit step, for $\alpha < \omega_1$ consider $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$. Since $\alpha < \omega_1$, this is a countable union of countable sets and so is $\mathcal{N}_\alpha$ countable. By Lemma 6.5 (2), this is ω-homogeneous. Also, since any finite subset of $\mathcal{N}_\alpha$ is contained in some $\mathcal{N}_\beta \preceq \mathcal{N}_\alpha$ and $\mathcal{N}_\beta \cong \mathcal{N}$, $\mathcal{N}_\alpha$ realises all of the same n-types over $\emptyset$ as $\mathcal{N}$. Thus, $\mathcal{N}_\alpha \cong \mathcal{N}$ as desired. Hence, we can build the desired elementary chain $(\mathcal{N}_\alpha | \alpha < \omega_1)$.
Take $\mathcal{N}^\star = \bigcup_{\alpha<\omega_1} \mathcal{N}_\alpha$. Note that $|\mathcal{N}^\star| = \aleph_1$: it must have size at least $\aleph_1$ since we are taking an uncountable union, where at each stage $\mathcal{N}_{\alpha+1} \supsetneq \mathcal{N}_\alpha$. It has size at most $\aleph_1$ being an uncountable union of countable sets. However, since $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$, we have that

$$\phi(N_{\alpha+1}) = \phi(N_\alpha) = \cdots = \phi(N) = \phi(M),$$

and so $\phi(N^\star) = \phi(M)$, which is a countable set. So the conclusion of the theorem holds. □

Observation 6.9. Note that if $\mathcal{M} \models T$ is saturated and $\phi(x) \in \mathcal{L}(M)$ is non-algebraic, then $|\phi(M)| = |M|$. Suppose by contradiction $|\phi(M)| < |M|$ and consider the following partial type:

$$\{\phi(x)\} \cup \{x \neq c | c \in \phi(M)\}.$$

This is a finitely satisfiable and over a set of cardinality $< |M|$. So by saturation of M it is realised in M, yielding a contradiction.

Corollary 6.10. *Let T be κ-categorical in some uncountable κ. Then, T has no Vaughtian pairs.*

Proof. Suppose that T is κ-categorical. Since it is κ-categorical, it is $\aleph_1$-categorical by the downwards Morley Theorem 5.15. Hence, all of its models of cardinality $\aleph_1$ are saturated. Hence, by Observation, for every uncountable model $\mathcal{N}$ and $\mathcal{L}(N)$-formula $\phi(x)$, $|\phi(N)| = \aleph_1$. By Vaught's two cardinal Theorem 6.8, this implies that T has no Vaughtian pair. □

Definition 6.11. We say that T **eliminates the quantifier** $\exists^\infty x$ if for every $\mathcal{L}$-formula $\phi(x, \overline{y})$ there is $n_\phi \in \mathbb{N}$ such for all tuples $\overline{a} \in \mathbb{M}^{|\overline{y}|}$, if $|\phi(\mathbb{M}, \overline{a})| \geq n_\phi$, then $\phi(\mathbb{M}, \overline{a})$ is infinite.

Exercise 6.12. Show that if T has no Vaughtian pairs, then it eliminates the quantifier $\exists^\infty x$.

Exercise 6.13. Suppose that T eliminates the quantifier $\exists^\infty x$. Let $\mathcal{M} \models T$ and let $\phi(x) \in \mathcal{L}(M)$ be minimal in $\mathcal{M}$. Show that $\phi(x)$ is strongly minimal.

Definition 6.14. For infinite cardinals $\kappa > \lambda$, we say that T has has a (λ, κ)-model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

Exercise 6.15. Prove the following:

1. If T has a (κ, λ)-model then it has a Vaughtian pair (and so an $(\aleph_1, \aleph_0)$-model) [Hint: this should be trivial];
2. Prove that if T is ω-stable and has an $(\aleph_1, \aleph_0)$-model, then for each $\kappa > \aleph_1$, T has a $(\kappa, \aleph_0)$-model [Hint: you may need to use Theorem 5.12].

⋆⋆ **Exercise 6.16.** We show that in Exercise 6.15 (2), the assumption of ω-stability is necessary. Let $\mathcal{L} = \{P_0, \ldots, P_n, E_1, \ldots, E_n\}$ for unary predicates P_i and binary relations E_i. Consider the $\mathcal{L}$-theory T stating that:

- the P_i are infinite and partition the domain;
- for each $i \in \{1, \ldots, n\}$, $\forall xy(E_i(x, y) \to P_{i-1}(x) \wedge P_i(y))$;
- for each $i \in \{1, \ldots, n\}$, $\forall xy((P_i(x) \wedge P_i(y) \wedge \forall z(E_i(z, x) \leftrightarrow E_i(z, y)) \to x = y)$.

For example, for X_0 an infinite, take $X_{i+1} = \mathcal{P}(X_i)$ for $i \in \{1, \ldots, n\}$. Let $\mathcal{M}$ be the disjoint union of the X_i with P_i naming each of the X_i and E_i being the membership relation restricted to $X_i \times x_{i+1}$. Then, $\mathcal{M} \models T$. Show that if $\mathcal{M} \models T$ and $|P_0(M)| = \aleph_0$, then $|M| \leq \beth_n$. Hence, $\mathcal{M}$ has a $(\beth_n, \aleph_0)$-model but it does not have a $(\kappa, \aleph_0)$-model for arbitrarily large κ. [Hint: I would only do the case of $n = 1$. Recall that $\beth_0 = \aleph_0$ and $\beth_{\alpha+1} = 2^{\beth_\alpha}$.]

7 The Baldwin-Lachlan Theorem and Morley's Theorem

Recall the following exercise from the Model Theory I course:

Fact 7.1. *Let T be ω-stable and $\mathcal{M} \models T$. Then there is a $\mathcal{L}(M)$-formula minimal in M.*

Theorem 7.2 (Baldwin & Lachlan)**.** *Let T be a countable theory. Then, T is κ-categorical in some uncountable cardinal κ if and only if it is ω-stable and has no Vaughtian pair.*

Proof. ($\Rightarrow$) We have already proven that κ-categoricity implies ω-stability in the Model Theory I course. We proved that κ-categoricity implies having no Vaughtian pair in Corollary 6.10.

($\Leftarrow$) Since T is ω-stable, it has a prime model $\mathcal{M}_0$ over $\emptyset$ by Exercise 5.11. Note that by the downwards Lowenheim-Skolem Theorem and countability of T, M_0 must be countable. By ω-stability, there is an $\mathcal{L}(M_0)$-formula $\phi(x)$ which is minimal in $\mathcal{M}_0$ by Fact 7.1. Now, since T has no Vaughtian pair, $\phi(x)$ is strongly minimal (by Exercises 6.12 and 6.13).

Now, consider two models $\mathcal{M}_1, \mathcal{M}_2$ of cardinality κ. We need to show that they are isomorphic. Since $\mathcal{M}_0$ is prime over $\emptyset$, it elementary embeds into both of the $\mathcal{M}_i$ for $i \in \{1, 2\}$. We may assume without loss of generality that $\mathcal{M}_0$ is an elementary substructure of both $\mathcal{M}_1$ and $\mathcal{M}_2$.

👁 *Observation* 7.3. For $i \in \{1, 2\}$, $\mathcal{M}_i$ has no proper elementary substructure containing $M_0 \cup \phi(M_i)$.

If it did, we would have $M_0 \cup \phi(M_i) \subseteq \mathcal{N} \preceq \mathcal{M}_i$ with $N \subsetneq M_i$. Since $\mathcal{N} \preceq \mathcal{M}_i$, $\phi(N) = \phi(M_i)$. But then this would give a Vaughtian pair.

👁 *Observation* 7.4. For $i \in \{1, 2\}$, $\mathcal{M}_i$ is prime over $M_0 \cup \phi(M_i)$.

Since T is ω-stable, there is $\mathcal{K} \preceq \mathcal{M}_i$ prime over $M_0 \cup \phi(M_i)$. By Observation 7.3, $\mathcal{K} = \mathcal{M}_i$.

Now, from Observation 7.3, we can deduce that $|\phi(M_i)| = \kappa$. Otherwise, by the downwards Lowenheim-Skolem Theorem, and countability of T and M_0, we could find a proper elementary substructure of M_i containing $M_0 \cup \phi(M_i)$. Now, we know from Theorem 4.3 (where we added parameters for the elements of M_0), that $(\phi(\mathbb{M}), \mathrm{cl})$ forms a pregeometry, where

$$\mathrm{cl}(A) = \mathrm{acl}(A \cup M_0) \cap \phi(\mathbb{M}).$$

Hence,

$$\dim_\phi(M_1/M_0) = \dim_\phi(M_2/M_0) = \kappa.$$

This is because for any set A, $|\mathrm{acl}(A)| \leq \max(|A|, |\mathcal{L}|)$ and since $\dim_\phi(M_i/M_0)$ is the cardinality of a generating set for $\phi(M_i)$ (over M_0), where M_0 and $|\mathcal{L}|$ are countable.

From Lemma 4.12, there is an M_0-elementary bijection $f : \phi(M_1) \to \phi(M_2)$. I.e. there is an elementary bijection $f : M_0 \cup \phi(M_1) \to M_0 \cup \phi(M_2) \subseteq \mathcal{M}_2$ fixing M_0. Since $\mathcal{M}_1$ is prime over $M_0 \cup \phi(M_1)$, we can extend f to $f' : \mathcal{M}_1 \to \mathcal{M}_2$. Now, $f'(M_1) \preceq \mathcal{M}_2$ and $M_0 \cup \phi(M_2) \subseteq f'(M_1)$. Hence, from Observation 7.4, $f'(M_1) = M_2$, and so f' is an isomorphism between M_1 and M_2, as desired.

□

Corollary 7.5 (Morley's Theorem)**.** *Let T be a countable theory. Suppose that T is categorical in some uncountable cardinal. Then, T is categorical in all uncountable cardinals.*

Proof. If T is κ-categorical for some uncountable κ, by the Baldwin-Lachlan Theorem it is ω-stable with no Vaughtian pair. But again by the Baldwin-Lachlan Theorem it is categorical in all uncountable cardinals. □

8 Getting familiar with more examples

Exercise 8.1.

1. Prove that $(\mathbb{N}; 0, 1, +, \times, <)$ is the prime model of $\mathrm{Th}(\mathbb{N}; 0, 1, +, \times, <)$;
2. Prove that $\mathrm{Th}(\mathbb{Z}, 0, +)$ does not have a prime model. [Hint: you may assume that augmenting $(\mathbb{Z}, 0, +)$ by predicates P_n for each $n \geq 2$ naming the elements divisible by n yields a theory with quantifier elimination];

Definition 8.2. Let κ be an infinite cardinal. We say that T is **κ-stable** if for all A such that $|A| \leq \kappa$, $|S_x(A)| \leq \kappa$. We say that T is **superstable** if it is κ-stable for all $\kappa \geq 2^{|T|}$. We say that T is **stable** if it is κ-stable for some κ.

Exercise 8.3. Let $\mathcal{L} := \{U_i | i < \omega\}$ be such that each U_i is a unary predicate. For X and Y disjoint finite subsets of $\mathbb{N}$, let $\phi_{X,Y}$ be the sentence

$$\exists x \bigwedge_{i \in X} U_i(x) \wedge \bigwedge_{i \in Y} \neg U_i(x).$$

Let $T := \{\phi_{X,Y} | X, Y \text{ disjoint finite subsets of } \mathbb{N}\}$. You may assume this theory is complete and has quantifier elimination.

1. Show that no type over $\emptyset$ is isolated. Deduce that T has no prime models;
2. Show that T is κ-stable for all $\kappa \geq 2^{\aleph_0}$.

8.0.1 Equivalence relations

Example 8.4. Consider the following theories of equivalence relations for α an ordinal and κ a cardinal.

- **Refining equivalence relation with infinite splitting:** REI_α has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, E_j refines E_i and each E_i class is refined into infinitely many E_{i+1}-classes. For this and all other examples below we assume each equivalence class of each equivalence relation is infinite;
- **Refining equivalence relation with finite splitting:** REF_α has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, E_j refines E_i and each E_i class is refined into two E_{i+1}-classes;
- **Crosscutting Equivalence relation with finite splitting:** CEF_κ has equivalence relations $(E_i | i < \kappa)$ such that each E_i has only two classes and for all $i < \kappa$, E_{i+1} splits each equivalence class of E_i into two classes;
- **Crosscutting Equivalence relation with infinite splitting:** CEI_κ has equivalence relations $(E_i | i < \kappa)$ such that each E_i has infinitely many classes and for all $i < \kappa$, E_{i+1} splits each equivalence class of E_i into infinitely many classes. These theories are complete and have quantifier elimination.

Exercise 8.5. Show the following:

1. REI_α is stable. REI_ω is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$. [More generally, if α is infinite, REI_α is not superstable];

2. REF_α is stable. If $\alpha \leq \omega$ it is superstable. If $\alpha \geq \omega \cdot \omega$ then it is not superstable;

3. if $\kappa \leq \omega$, then CEF_κ is superstable.

$\star\star$ **Exercise 8.6.** Compute the number of models of each of the examples in Example 8.4 in each cardinality.

8.0.2 Differentially closed fields

Definition 8.7. A **derivation** on a ring R is an additive homomorphism $D : R \to R$ such that

$$D(xy) = xD(y) + yD(x).$$

A **differential ring** is a ring equipped with a derivation. Given a derivation $D_0 : R \to R$, we call the **ring of differential polynomials** the differential ring

$$R\{X\} := R[X, X^{(1)}, X^{(2)}, X^{(3)}, \ldots],$$

with derivation D extending D_0 by setting $D\left(X^{(n)}\right) = X^{(n+1)}$.

Definition 8.8. An ideal $I \subseteq R\{X\}$ is a **differential ideal** if for all $f \in I, D(f) \in I$.

Definition 8.9. If $f(X) \in R\{X\} \setminus R$, the **order** of f is the largest n such that $X^{(n)}$ occurs in f. We can write

$$f(X) = \sum_{i=0}^{m} g_i(X, X^{(1)}, \ldots, X^{(n-1)})(X^{(n)})^i,$$

for $g_i \in R[X, X^{(1)}, \ldots, X^{(n-1)}]$ and $g_m \neq 0$. We call m the **degree** of f. We write $g \leq f$ if either the order of g is strictly less than the order of f or if the orders are the same but g has lower degree than f.

Definition 8.10. A differential field K (of characteristic zero) which is algebraically closed is called **differentially closed** if for any non-constant differential polynomials f and g where the order of g is less than the order of f there is an x such that $f(x) = 0$ and $g(x) \neq 0$.

Note that the theory of differentially closed fields of characteristic zero is axiomatisable. We call it DCF_0.

Fact 8.11. *Every differential field k has an extension K which is differentially closed.*

Fact 8.12. DCF_0 *is complete and has quantifier elimination.*

$\star\star$ **Exercise 8.13.** Let $K \models \mathrm{DCF}_0$. Let $k \subseteq K$.

1. for $p \in S_1^K(k)$, the set of 1-types over k realised in K, let $I_p := \{f \in k\{X\} | f(x) = 0 \in p\}$. Show that I_p is a differential prime ideal [i.e. $I_p \subsetneq k\{X\}$ and if $f \cdot g \in I_p$, then either $f \in I_p$ or $g \in I_p$].

2. Show that if $I \subset k\{X\}$ is a differential prime ideal, then $I = I_p$ for some $p \in S_n^K$. So $p \mapsto I_p$ is a bijection between complete n-types and differential prime ideals in $k\{X\}$;

3. The Ritt-Raudenbusch Basis Theorem says that every differential prime ideal in $k\{X\}$ is finitely generated. Use this to show that DCF_0 is ω-stable;

4. Suppose that k is a differential field of characteristic zero and $k \subseteq K \models \mathrm{DCF}_0$. We say that K is the **differential closure** of k if for all $L \supseteq k$ such that $K \models \mathrm{DCF}_0$ there is a differential field embedding of K into L fixing k. Show that every field has a differential closure.

9 Stability

Definition 9.1. We say that the formula $\phi(x; y)$ has the **order property** if there is an infinite sequence $(a_i, b_i | i < \omega)$ such that

$$\vDash \phi(a_i, bj) \text{ if and only if } i < j.$$

We say that $\phi(x; y)$ is **stable** if it does not have the order property.

Exercise 9.2. Let $\phi(x; y)$ and $\psi(x, z)$ be stable $\mathcal{L}$-formulas. Show the following:

1. $\phi^{\text{opp}}(y; x) := \phi(x; y)$ is stable;
2. $\neg\phi(x; y)$ is stable;
3. $\theta(x; yz) := \phi(x; y) \wedge \psi(x; z)$ is stable;
4. $\theta(x; yz) := \phi(x; y) \vee \psi(x; z)$ is stable;
5. For $y = uv$ and $c \in \mathbb{M}^{|v|}$, $\phi(x; u, c)$ is stable.

Theorem 9.3 (Erdös-Makkai)**.** *Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ with $|B| < |\mathcal{F}|$. Then, there are sequences $(b_i | i < \omega)$ of elements of B and $(S_i | i < \omega)$ of elements of $\mathcal{F}$ such that for all $i, j \in \omega$, we have that*

- *EITHER $b_i \in S_j$ if and only if $j < i$;*
- *OR $b_i \in S_j$ if and only if $i < j$.*

Exercise 9.4 (Proof of Erdös-Makkai)**.** Note that there is $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| = |B|$ and for all $B_0, B_1 \subseteq B$ finite, if there is some $S \in \mathcal{F}$ such that $B_0 \subseteq S, B_1 \subseteq B \setminus S$, then there is some $S' \in \mathcal{F}$ with $B_0 \subseteq S', B_1 \subseteq B \setminus S'$. Note that there is $S^\star \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$. Now, prove Erdös-Makkai. [Hint: you need to construct appropriate sequences in $S^\star$, $B \setminus S^\star$ and $\mathcal{F}'$, and then use Ramsey's theorem.]

Definition 9.5. $\phi(x; y)$ does has the **binary tree property** if there is a binary tree of parameters $(b_s | s \in^{<\omega} 2)$ such that for each branch $\sigma \in^{\omega} 2$,

$$\{\phi^{\sigma(n)}(x; b_{\sigma|n}) | n < \omega\} \text{ is consistent,}$$

where $\phi^0 := \neg\phi$ and $\phi^1 := \phi$.

Definition 9.6. Let $\phi(x; y) \in \mathcal{L}$ and B be a set of parameters. A ϕ-type over B is a maximally consistent set of formulas of the form $\phi(x; b), \neg\phi(x; b)$. We denote the set of all ϕ-types over B by $S_\phi(B)$.

Theorem 9.7. *Let $\phi(x; y) \in \mathcal{L}$. The following are equivalent:*

(a) *ϕ is stable;*

(b) *$|S_\phi(B)| \leq |B|$ for every infinite B;*

(c) *For some infinite cardinal λ we have $|B| \leq \lambda \Rightarrow |S_\phi(B)| \leq \lambda$;*

(d) *ϕ does not have the binary tree property.*

Proof. Every direction that we prove for this theorem will be a proof by contrapositive. $(a) \Rightarrow (b)$ Suppose that $|S_\phi(B)| > |B|$ for some infinite B. For any $a \in \mathbb{M}^{|x|}$ let

$$S_a := \{b \in B^{|y|} \mid \vdash \phi(a; b)\}.$$

Note that $\text{tp}_\phi(a/B) = \text{tp}_\phi(a'/B)$ if and only if $S_a = S_{a'}$. Let $\mathcal{F} := (S_a | a \in \mathbb{M}^{|x|}$. By the observation above,

$$|\mathcal{F}| = |S_\phi(B)| > |B| = |B^{|y|}|,$$

where the inequality is our original assumption. Thus, we can apply Erdös-Makkai (Theorem 9.3) to see there are $(b_i | i < \omega)$ in $B^{|y|}$ and $(S_{a_i} | i < \omega)$ from $\mathcal{F}$ such that

- either $b_i \in S_{a_j}$ if and only if $j < i$;
- or $b_i \in S_{a_j}$ if and only if $i < j$.

Note that $b_i \in S_{a_j}$ if and only if $\models \phi(a_j; b_i)$. So the first case is equivalent to the statement of $\phi(x; y)$ having the order property. The second case is equivalent to ϕ^{opp} (obtained from swapping the variables of ϕ) having the order property, which we know implies that ϕ also has the order property. So we conclude that ϕ has the order property as desired.
$(b) \Rightarrow (c)$ is trivial. We move to $(c) \Rightarrow (d)$. Pick an arbitrary infinite λ. First, choose μ minimal such that $2^\mu > \lambda$. Then, the tree $I =^{<\mu} 2$ has cardinality $\leq \mu$ since

$$|I| \leq \left| \bigcup_{\nu < \mu} 2^\nu \right| \leq \lambda,$$

where the last inequality holds since we are taking a unioin of $\leq \lambda$-many sets of cardinality $\leq \lambda$. Now, suppose by contrapositive that $\phi(x; y)$ has the binary tree property. Note that by compactness we can take the binary tree to be I-indexed[1] By definition, each branch yields a consistent set of formulas which can be completed to a ϕ-type over B. By construction all these ϕ-types are disjoint: if $\sigma, \nu \in^\mu 2$ have meet $\tau \in^{<\mu} 2$, then the sets of formulas associated with σ and ν disagree on whether $\phi(x, b_\tau)$ is true. So

$$|B| \leq \lambda < 2^\mu \leq S_\phi(B),$$

yielding the negation of (c).

Finally, we prove $(d) \Rightarrow (a)$. Say that $\phi(x; y)$ has the order property. By compactness, there are $(a_i b_i \vdash i \in [0, 1])$ such that

$$\phi(a_i, b_j) \text{ if and only if } i < j.$$

Pick $b'_\emptyset = b_{\frac{1}{2}}$. By construction, for $i < 1/2, \models \phi(a_i, b'_\emptyset)$ and for $i \geq 1/2, \models \neq \phi(a_i, b'_\emptyset)$. Now, pick $b'_0 = b_{\frac{1}{4}}$ and $b_2 = b'_{\frac{3}{4}}$ and keep building the tree of parameters $(b_s | s \in^{<\omega} 2)$ by splitting the intervals. Now, for each branch $\sigma \in^\omega 2$, it is easy to see that $\{\phi^{\sigma(n)}(x; b_{\sigma|n}) | n < \omega\}$ is finitely satisfiable. So it is satisfiable and gives the desired binary tree, completing the final part of the proof. $\square$

Exercise 9.8. Show that the following are equivalent:

1. T is stable (in the sense of being κ-stable for some infinite κ);
2. every $\mathcal{L}$-formula $\phi(x; y)$ is stable for T;
3. T is κ-stable for all κ such that $\kappa^{|T|} = \kappa$.

Definition 9.9. For κ an infinite cardinal, let

$$\mathrm{ded}(\kappa) := \sup\{|I| \ : I \text{ is a linear ordering with a dense subset of size } \kappa\}.$$

It is easy to see that $\kappa < \mathrm{ded}(\kappa) \leq 2^\kappa$.

[1] At this stage you are supposed to know how to run a standard compactness proof. In this case, you should expand T by constants c_s for $s \in I$ and consider the sets of formulas

$$\bigcup_{\sigma \in^\mu 2} \{\phi^{\sigma(\nu)}(x; c_{\sigma|\nu}) | \nu < \mu\}.$$

Any finite subset is finitely satisfiable by assigning to the c_s the appropriate b_s from the original binary tree, thus completing the compactness proof.

Definition 9.10. Let T be a countable theory. Write f_T : Card $\to$ Card for the function on cardinals given by

$$f_T(\kappa) := \sup\{|S_n(M)| \ : \mathcal{M} \models T, |M| = \kappa, n \in \omega\}.$$

[It is easy to see that if we fixed n in the definition above, we would still get f_T.]

Exercise 9.11. Prove that if T is unstable, then $f_T(\kappa) \geq \mathrm{ded}(\kappa)$ for all cardinals $\kappa \geq |T|$.

Recall the following definitions and lemmas from the Model Theory course:

Definition 9.12. Let I be an infinite linear order and A a set of parameters. We say that $(a_i|i \in I)$ is **indiscernible** over A if for every $\mathcal{L}(A)$-formula $\phi(x_1, \ldots, x_n)$ and $i_1 < \cdots < i_n, j_1 < \cdots < j_n$ from I, we have that

$$\vDash \phi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n}). \tag{2}$$

We say that the sequence is **totally indiscernible** over A if the condition 2 holds for any $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\}$ from I of size n.

For a sequence $(a_i|i \in I)$, its **EM-type** (i.e. Ehrenfeucht-Motowski type) over A is given by

$$\mathrm{EM}(a_i|i \in I) := \{\phi(x_1, \ldots, x_n) \in \mathcal{L}(A) \mid \vDash \phi(a_{i_1}, \ldots, a_{i_n}) \text{ for all } i_1 < \cdots < i_n, n < \omega\}.$$

Lemma 9.13 (Extracting indiscernible sequences)**.** *Let A be a set of parameters and $(b_i|i \in I)$ a infinite sequence. Let J be a linear order. Then, there is a sequence $(a_j|j \in J)$ which is indiscernible over A and realising the EM-type of $(b_i|i \in I)$.*

Exercise 9.14.

- Show that T is unstable if and only if there is an infinite sequence $(a_i|i < \omega)$ and a formula $\phi(x, y)$ such that $\vDash \phi(a_i, a_j)$ if and only if $i < j$;
- Show that if T is unstable there is an indiscernible sequence which is not totally indiscernible;
- Show that if T is stable then every indiscernible sequence is totally indiscernible. [Hint: Say that we have an indiscernible sequence $(a_i|i < \omega)$ which is not totally indiscernible. Show that there is a formula $\phi(x_1, \ldots, x_n)$ such that for some transposition τ switching only two consecutive variables

  $$\vdash \phi(a_1, \ldots, a_n) \wedge \neg\phi(a_{\tau(1)}, \ldots, a_{\tau(n)}).$$

  Use this formula to find an unstable formula in T.]

Definition 9.15. We say that $\phi(x, y)$ has the **independence property**, IP, if there are $(b_i|i < \omega)$ and $(a_s|s \subseteq \omega)$ such that

$$\vDash \phi(a_s, b_i) \Leftrightarrow i \in s.$$

We say that $\phi(x, y)$ is NIP if it does not have the independence property. A theory is NIP if all of its formulas are NIP.

Example 9.16. The random graph has the independence property.

Exercise 9.17. Prove that a formula $\phi(x, y)$ has IP if and only if there is an indiscernible sequence $(b_i|i \in I)$ and a parameter a such that

$$\vDash \phi(a, b_i) \text{ if and only if } i \text{ is even.}$$

Definition 9.18. We say that $\phi(x, y)$ has the **strict order property**, SOP if there is an infinite sequence $(b_i)_{i<\omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j < \omega$. We say that $\phi(x, y)$ is NSOP if it does not have the strict order property, and we say that a theory T is NSOP if all of its formulas are NSOP.

Exercise 9.19. Prove that T is stable if and only if it is NIP and NSOP. [Hint: For the $(\Leftarrow)$ direction, consider an unstable theory with NIP and prove it must have the SOP.]

Definition 9.20. Let R be an associative ring with identity. Let $\mathcal{L}_R := \{+, 0, (\mu_r)_{r\in R))}\}$, where the μ_r are unary function symbols. The theory of (left) R-modules T_R asserts of a model M that $(M, +, 0)$ is an Abelian group, and each f_r is an endomorphism of this group with f_1 being the identity map. Note that this theory is not complete.

Definition 9.21. An $\mathcal{L}$-formula is called **primitive positive** if it is an existential quantification of a conjunction of atomic formulas. That is, if it is of the form

$$\exists \overline{y} \bigwedge_{i\leq n} \psi_i(\overline{x}, \overline{y}),$$

where the formulas $\psi_i(\overline{x}, \overline{y})$ are atomic.

Fact 9.22. *The theory T_R has elimination of quantifiers up to primitive positive formulas. That is, for every $\mathcal{L}_R$-sentence, $\phi(\overline{x})$ there is a Boolean combination of primitive positive formulas $\psi(\overline{x})$ such that $T_R \vDash \forall \overline{x} \phi(\overline{x}) \leftrightarrow \psi(\overline{x})$.*

Exercise 9.23. Consider a primitive positive $\mathcal{L}_R$-formula $\phi(\overline{x}, \overline{z})$ and let M be an R-module. Show that $\phi(\overline{x}, \overline{0})$, where $\overline{0}$ is a string of 0's of length $|\overline{z}|$ defines either $\emptyset$ or a subgroup of $M^{|\overline{x}|}$. Show that for any tuple $\overline{a}$ such that $\phi(\overline{x}, \overline{a})$ is consistent, $\phi(M, \overline{a})$ is a coset of $\phi(\overline{x}, \overline{0})$. Deduce that that for any tuples $\overline{a}, \overline{b}$, $\phi(\overline{x}, \overline{a})$ and $\phi(\overline{x}, \overline{b})$ are either mutually inconsistent or equivalent. Using elimination of quantifiers up to pp-formulas, prove that for any R-module M, its theory is stable.

Definition 9.24. A group G satisfies the **ω-stable descending chain condition** if there is no infinite proper descending chain of definable subgroups of G,

$$\cdots < H_{i+1} < H_i < \cdots < H_1 < G.$$

Exercise 9.25. Show that if G is an ω-stable group (i.e. a group whose theory is ω-stable), then G satisfies the ω-stable descending chain condition. Show that for an R-module M, the following are equivalent:

- M is ω-stable;
- M has the ω-stable descending chain condition;
- For any set of pp-formulas in a single variable $\{\phi_n(x) | n < \omega\}$ there is some $n \in \omega$ such that for all $m \in \omega$, $M \vDash \forall x (\bigwedge_{i<n} \phi_i(x) \to \phi_m(x))$.

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